Introduction to Business Math

INTRODUCTION TO BUSINESS MATH

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Fanshawe College Pressbooks
London Ontario

CONTENTS

Chapter 6: Compound Interest

INTRODUCTION

Introduction to Business Mathematics was designed specifically for first semester students in the Lawrence Kinlin School of Business and the first semester course for all students in the Business program, Math-1052.This resource was designed with a vision of hybrid learning, combining different learning environments of face-to-face and web-based, and with a clear focus of creating a world where everyone can “do” mathematics. The concepts covered throughout this resource should support students dealing with practical scenarios and such as repaying student loans, leasing a car, securing a mortgage, running a business, paying taxes, making investments and much more.

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ACKNOWLEDGEMENTS

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- Business Math: A Step-by-Step Handbook (2021B) by J. Olivier and Lyryx Learning Inc. through a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License, unless otherwise noted.
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- Mathematics for the Liberal Arts Corequisite by Deborah Devlin and Lumen Learning is licensed under a Creative Commons Attribution 4.0 license, unless otherwise noted.

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Collaborators

This project was a collaboration between the author and the team in the OER Design Studio at Fanshawe. The following staff and students were involved in the creation of this project:

- Jason Benoit – *Instructional Design Student*
- Melanie Mitchell Sparkes – *Instructional Design Student*

- Robert Armstrong – *Graphic Design*
- Shauna Roch – *Project Lead*
- Wilson Poulter – *Copyright*

ABOUT THIS BOOK

Overall significant work was done on this version of the text to improve the accessibility and format from the import from Lyryx.

MathJax

All mathematical notation was written in LaTeX to allow for MathJax functionality.

Please note that "some screen readers support MathML, MathJax's internal format. Screen readers like ChromeVox, JAWS (on IE), and TextHelp support MathJax directly (most only version 2); other screen readers are supported by the `assistive-mml` extension as of version 3.0.1." (MathJax Consortium, 2021). It is important to also note that the quality of screen reader support varies greatly with the software you are using and the various settings enabled. For more information on MathJax and screen reader support please visit the *MathJax Consortium Accessibility Features page*.

Formatting Changes

In addition the following formatting changes were made:

- The majority of images that included math where replaced with LaTeX
- Examples were reformatted into steps for ease of understanding. The PUPP method was replaced with a step by step method for examples.

Specific content changes from the adapted resources can be found in the table below.

Chapter 1	Content from Business Math: A Step-by-Step Handbook (2021B) Chapter 1
Chapter 2	Content from Business Math: A Step-by-Step Handbook (2021B) Chapter 2 except for 2.3 Percentages and 2.6 Natural Logarithms. 3.2 Averages added. Section 2.1 is from Rounding Whole Numbers in Mathematics for the Liberal Arts Corequisite.
Chapter 3	Content combined from Business Math: A Step-by-Step Handbook (2021B) 2.3 Percentages, 3.1 Percent change, 3.3 Ratios, proportions and prorating. 7.1 sales tax, 7.2 property tax, 7.3 exchange rates.
Chapter 4	Content combined from Business Math: A Step-by-Step Handbook (2021B) Chapter 5 & Chapter 6 as well as 7.4: Invoicing — Terms of Payment and Cash Discounts
Chapter 5	Content from Business Math: A Step-by-Step Handbook Abridged Chapter 8
Chapter 6	Content from Business Math: A Step-by-Step Handbook Abridged Chapter 9

Accessibility Statement

We are actively committed to increasing the accessibility and usability of the textbooks we produce. Every attempt has been made to make this OER accessible to all learners and is compatible with assistive and adaptive technologies. We have attempted to provide closed captions, alternative text, or multiple formats for on-screen and off-line access.

The web version of this resource has been designed to meet Web Content Accessibility Guidelines 2.0, level AA. In addition, it follows all guidelines in Appendix A: Checklist for Accessibility of the *Accessibility Toolkit – 2nd Edition*.

In addition to the web version, additional files are available in a number of file formats including PDF, EPUB (for eReaders), and MOBI (for Kindles).

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Please include the following information:

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- The computer, software, browser, and any assistive technology you are using that can help us diagnose and solve your issue (e.g., Windows 10, Google Chrome (Version 65.0.3325.181), NVDA screen reader)

Feedback

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CHAPTER 1: SUCCEEDING IN BUSINESS MATHEMATICS

Outline of Chapter Topics

Math! You may really dread the "M" word. Not only that, but this course is business mathematics, which some people say is even harder. Is that true, or is it just a rumor? Take a look at your world from a different perspective—notice that math is everywhere. Before you went to bed last night, you performed some math and solved a complex algebraic equation (and you probably didn't even realize it!). That equation helped you to figure out what time to get up this morning so that you could arrive at your destination on time. You factored in variables such as the required arrival time, commuting time (including the unknown variables of traffic and weather delays), morning routine time, and even snooze button time. When you solved this complex algebraic equation, you then set your alarm clock.

From the moment you woke up this morning, numbers have appeared everywhere. Perhaps you figured out how many calories are in your breakfast cereal. Maybe your car's gas tank was low, so you calculated whether you had enough gas for your commute today. You checked your bank account online to see if you had enough money to pay the bills. At Tim Hortons you figured out what size of coffee you could buy to go with your donut using only the $1.80 in change in your pocket. I hope you get my point.

Students ask, "why do I want to learn about business math?" The answer is simply this: money. Our world revolves around money. We all work to earn money. We need money to purchase our necessities, such as homes, transportation, food, and utilities. We also need money to enjoy the pleasures of life, including vacations, entertainment, and hobbies. We need money to retire. And businesses need money to survive and prosper.

Quite simply, business math teaches you monetary concepts and how to make smart decisions with your money.

Attribution

1.1: WHAT IS BUSINESS MATH?

Introduction

Business math is the study of mathematics required by the field of business. You must be interested in a business field such as accounting, marketing, human resources, or economics, simply because you are taking the course using this textbook. Regardless of your path, you cannot avoid dealing with money and numbers.

There is a whole field of mathematics that deals specifically with money. You will work with taxes, gross earnings, product prices, and currency exchange; you will be offered loans, lines of credit, mortgages, leases, savings bonds, and other financial tools. The goal in this course is to help you use financial tools and maximize your earnings and minimize your costs; to help makes smart monetary decisions both personally and for your business.

How Do I Learn about Business Math?

Let's be realistic. In some areas of life and business, you can achieve a reasonable degree of understanding just by reading. However, reading about business mathematics without doing it would be disastrous. To succeed you must used a structured approach.

HOW TO

Use a Structured Approach

- Always read the content prior to your professor covering the topic in class.
- Attend class, with your notes printed out and ask questions.
- Do the homework and assignments—you absolutely must practice, practice, and practice!
- Seek help immediately when you need it.

Learning mathematics is like constructing a building. Each floor of the building requires the floor below it to be completed first. In mathematics, each section of a textbook requires the concepts and techniques from the sections that preceded it. If you have trouble with a concept, you must fix it NOW before it causes a large ripple effect on your ability to succeed in subsequent topics.

So the bottom line is that you absolutely cannot replace this approach—you must follow it.

Attribution

1.2: HOW CAN I USE THE FEATURES OF THIS BOOK TO MY ADVANTAGE?

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $C/Y =$ Compounds per year, or compounding frequency
- $FV =$ Future value, or maturity value
- $i =$ Periodic interest rate
- $I/Y =$ Nominal interest rate per year
- $n =$ Number of compound periods
- PV Present value, or principal

Formulas Used

- Formula 4.2a – **Single Discount**

$$\text{Single Discount: Net Price} = \text{List Price} \times 1 - (\text{Discount Rate}) \;\rightarrow\; N = L \times (1 - d)$$

- Formula 6.1a – **Periodic Interest Rate**

\begin{align*}i=\frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}}\end{align*}

- Formula 6.2a – **Number of Compound Periods for Single**

$$n = \text{C/Y} \times \ (\text{Number of Years})$$

This textbook was written for students like you, designed to address common student needs in mathematics.

I Don't Know How to Solve a Math Problem

Do you find it difficult to start a math problem? What do you do first? What is the next step? The reason this is challenging for some students is that they have never been shown problem-solving techniques. This textbook will use a step by step approach to solving problems, as illustrated in the example below. Successfully working through a problem involves a careful approach and this model provides a structured problem-solving approach that will aid you not only in mathematics but in any problem-solving situation you may encounter.

Example 1.2.1

When Sandra borrowed $\$7,100$ from Sanchez, she agreed to reimburse him $\$8,615.19$ three years from now including interest compounded quarterly. What interest rate is being charged?

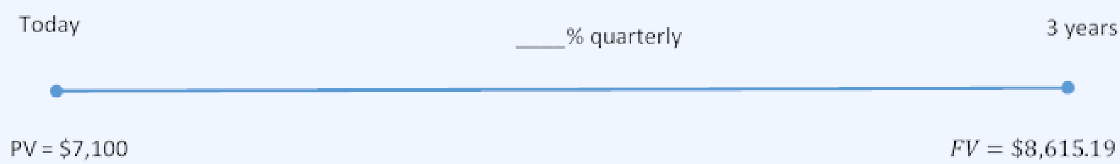

Figure 1.2.1

Solution

Step 1: **What are we looking for?**

Find the nominal quarterly compounded rate of interest (IY).

Step 2: **What do we already know from the question?**

The present value, future value, term, and compounding are known, as illustrated in the timeline above (Figure 1.2.1).

$$\begin{aligned} PV &= \$7,100 \\ FV &= \$8,615.19 \\ \text{C/Y} &= \text{quarterly} = 4 \\ \text{Term} &= 3 \text{ years} \end{aligned}$$

Step 3: **Make substitutions using the information known above.**

Calculate n using **Formula 6.2a** $n = \text{C/Y} \times \text{Number of Years}$.

$$n = C/Y \times (\text{Number of Years})$$
$$n = 4 \times 3$$
$$n = 12$$

Using the n value, we substitute it into the Compound Interest for Single Payments Formula. Rearrange and solve for the Periodic Rate value, i.

$$FV = PV(1+i)^n$$
$$\$8,615.19 = \$7,100(1+i)^{12}$$
$$1.213407 = (1+i)^{12}$$
$$1.213407^{\frac{1}{12}} = 1+i$$
$$1.01624996 = 1+i$$
$$i = 0.01624996$$

Substitute i value into **Formula 6.1a** $i = \frac{\text{I/Y}}{\text{C/Y}}$ and solve for I/Y.

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$0.01624996 = \frac{I/Y}{4}$$
$$I/Y = 0.06499985 \ = \ 0.065 \text{ or } 6.5\%$$

Note that this amount would be compounded quarterly.

Table 1.2.1

n	I/Y*	PV	PMT	FV	P/Y	C/Y
12	6.499985*	$-7,100$	0	$8,615.19$	4	4

*Answer provided by calculations above

Step 4: **Provide the information in a worded statement.**

Sanchez is charging an interest rate of 6.5% compounded quarterly on the loan to Sandra.

Try It

1) Use the technique from Example 1.2.1 to plan a vacation. Recall that some methods in this model may have multiple steps.

Solution

Step 1: If you are heading out on vacation, the first thing you want to know is where you are going! Otherwise, how would you get there? In mathematics, you need to figure out what variable you are solving for before you can proceed. This step focuses your problem solving on the end goal. Many students mistakenly rush through this step and start punching numbers on a calculator or plugging numbers into formulas without understanding what it is they are doing. Spend plenty of time on this step, because the remaining three steps do not help at all if you solve for the wrong variable!

Step 2: In planning your vacation, once you figure out where you want to go you need to gather information about your destination. There are things you already know, but there are also things that you do not know, so you need a plan to figure these out. This step helps your problem solving in two ways:

- *First*, you need to assess what information has already been provided to you. You need to assign values to variables. You may need to draw diagrams to understand how all the numbers fit together.
- *Second*, you create a plan by which you will solve your problem. Your roadmap identifies how you go from Point A (some variable is unknown) to Point B (knowing what the variable is).

Step 3: Once your vacation is planned, all that is left is to go on vacation! In other words, execute your plan. This step involves all of the mechanics and computations of your roadmap that you developed in the previous step.

Step 4: After you return from vacation, you tell everyone about what you just did. In mathematics, this means you must explain solutions in the context of the question asked. If you just calculated $x = \$700$, what exactly does that represent and how should it be interpreted? Calculating a solution without understanding it is not good enough. A further benefit of this last step is that you can detect errors more easily when you have to explain the solution. For example, read this presentation statement: “If an investor puts $\$1,000$ into a savings account that earns 10% interest, I have calculated that after one year the investor has $\$700$ in her savings account.” Did that make sense? If an account is earning interest, the balance should have become bigger, not smaller! Clearly, the solution of $x = \$700$ must be an error!

Once you repeatedly practice these steps and ingrain this practice in your thinking process, you’ll have a structured approach to solving not just math problems but any life problem. As you keep practicing, you’ll find that this becomes a way of thinking. You can never skip one of the steps, but you won’t have to write them all out anymore because you’ll execute some steps with rapid-fire thought!

I Can’t Find What I Want When I Need It

When practicing your business math through homework and assignments, you will need to go back to look up various formulas or techniques or to review concepts. The challenge of most textbooks is in finding what you need when you need it buried in the vast pages of the textbook. Browse the table of contents of this book and notice the functional layout. This textbook incorporates a handbook design that aids in finding information. When you need to look something up, you will notice every part of every section in every chapter is designed and laid out using an identical structure.

While not every part is required in every section, the order is always as follows:

HOW TO

Find Resources Inside of a Textbook Chapter

- Concept introduction
- Concept fundamentals and characteristics
- Formula development and explanation
- Steps and procedures required to solve problems
- Important concept elements and technology use
- Common pitfalls and shared tips
- Conceptual understanding questions
- Guided examples

- Homework exercises
- The start of every chapter includes a miniature table of contents allowing you to locate what you need in the chapter without having to flip back to the book's main table of contents.
- End-of-chapter summaries deliver exactly that—each chapter in a nutshell. They contain key concept reviews, key terms, algebraic symbol definitions, formula summaries, and technology discussions on calculator usage. Links to other chapters are also provided as needed. If you need to look something up, this is the place to start!

Are There Any Steps to Help Me Remember?

Students have always asked if they need to follow a logical sequence of steps to solve a problem. The answer, of course, is YES! A section called "HOW TO" helps you with this by providing step-by-step procedures, techniques, and formulas you need to solve the problem.

Can This Book Help Me Understand the Formulas Better?

You can't avoid formulas—this is math, after all. What helps, though, is to understand what the formulas do and how each component comes together to produce the solution. If you can understand, you do not need to memorize! This book lays formulas out visually, clearly labelling all symbols. In addition, each component is fully explained (see below). You are not required to cross-reference paragraph-style explanations to formulas, as the visual layout makes it all clear. The goal is for you to understand what exactly the formula does and how, so that you understand enough to recognize when solutions make sense or might be wrong.

$\boxed{4.1a}$ Single Discount

$$\Large\text{Single Discount:}\;{\color[rgb]{1.0, 0.0, 0.0}N}\;=\;{\color[rgb]{0.0, 0.5, 0.0}L}\;\times\;(1\;-\;{\color[rgb]{0.0, 0.0, 1.0}d})$$

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Additionally, you will not find any of those archaic algebraic symbols in this book. Representative symbols make the algebra easier to remember and understand. Those ancient mathematicians just loved to speak Greek (literally!), but this text speaks modern-day English and understands that we have technology here in the twenty-first century! We'll use symbols like N for net price, L for list price, P for profit, and E for expenses, letters that actually remind you of the variable they stand for!

Does Anyone Actually Do Any of This in the Real World?

As you read through the guided examples and do your homework, pause to consider how the question applies to you. Does it demonstrate a business application you can use to be a smarter business professional? How can you extend the question to similar situations? This textbook shows you examples from both business and personal situations that you either have already encountered or probably will encounter in the future. (Remember: Math is all around you!)

- **Business**

  Why do business programs require a business math course in the first term or two? How is it relevant to your future? This textbook provides numerous examples of applications in marketing, finance, economics, accounting, and more. Once you complete this textbook, you should see clearly how business mathematics is relevant and important to your chosen future career.

- **Personal**

  You will see companies and products from your everyday life. Real-world situations show you how to put a few extra dollars in your pocket by making smart mathematical choices. What if you could save $\$100$ per month simply by making smarter choices? Over 40 years at a very conservative rate of interest, that is approximately $\$100,000$ in your pocket. You will also find life lessons. Are you ready to start planning your RRSP? Would you like to know the basics on how to do this? If you are thinking of buying a home, do you know how the numbers work on your mortgage? To learn how to buy a car at the lowest cost, read on!

Are There Any Secrets?

There are never any secrets. However, I will point out some tricks of the trade and commonly made mistakes throughout this book.

Paths To Success

This feature is designed to make you feel like part of the "in" crowd by letting you take advantage of shortcuts or tricks of the trade, such as some easy way to remember a particular technique or formula. It helps you see how some of the math could be made a little simpler or performed in a different way.

Things to Watch Out For

Awareness of common pitfalls and sources of error in mathematics reduces your chances of making a mistake. A reminder also helps when important concepts are about to be used, reused, or combined in a novel way. It gives you a "heads up" when needed.

Is There Any Way to Know That I'm "Getting It"?

Mastering business math requires two key skills:

1. Executing the required techniques and calculations
2. Understanding and successfully integrating various mathematical concepts. (For example,

when a variable changes, can you estimate how this affects the final result?)

The second skill is much harder to acquire than the first. A feature called "Try It" poses scenarios and questions for which no calculations are required. You need to visualize and conceptualize various mathematical theorems. Ultimately, you perform a self-test to see if you are "getting it."

In Today's World, Who Does This By Hand Anymore?

That is a perfectly valid question. You are right: Most people do not do this work by hand and instead use mathematical tools such as calculators, spreadsheet software, and apps. However, as the saying goes, you must learn to walk before you can run. The learning of mathematics requires pencil and paper first. Once you master these, you can then employ technology to speed up the process. This basic version textbook incorporates one technology to aid you in your application of business mathematics:

- **The Calculator**

  Integrated throughout this book are instructions for one of the financial calculators most widely used in Canada—the Texas Instruments BAII Plus. Calculator functions with step-by-step button sequences are presented with guided examples illustrating how you solve the math problem not only by hand but also using the financial calculator.

Is There Any Way I Can Assess My Ability?

At the end of almost every section of this book you will find approximately 18 practice questions covering the concepts specifically introduced in that section, with final solutions listed to all questions at the end of this textbook. These questions are divided into three categories to help develop your mathematical skills and assess your abilities:

Mechanics

This group of questions focuses on fundamental mathematical skills and formula usage. They are your first taste of using the section formulas and working with the mathematical concepts at an introductory level.

HOW TO

Be Successful in Business Math Mechanics

Successful completion of these questions means that you:

- Have at least a basic understanding of the variables at play,
- Show rudimentary application of concepts, and
- Can calculate solutions using the formulas.

Do not stop here, though! You do not yet have enough skills to meet the basic requirements of most business math courses. Once you have mastered the Mechanics section, you must progress to the next section, where you learn to apply your knowledge.

Applications

These questions are more typical of business math course expectations (check with your instructor). The key difference from the Mechanics section is that you must now execute your problem-solving skills. These questions require you to determine the unknown variable and figure out how the various pieces of the puzzle come together in the solution.

HOW TO

Be Successful in Business Math Applications

Successful completion of these questions means that you:

- Understand concepts at a satisfactory level,
- Can problem solve typical business math applications, and
- Can successfully integrate concepts at an acceptable performance level.

Challenge, Critical Thinking, and Other Applications

These questions put your knowledge and understanding to the test. The difficulty bar is set high, as these questions commonly require multistep solutions with advanced problem-solving techniques. As well, success might require you to integrate formulas, concepts, or procedures in novel ways. You will be challenged!

HOW TO

Be Successful in Business Math Critical Thinking

Successful completion of these questions means that you:

- Understand concepts at an above-average level,
- Can problem solve in difficult situations, and
- Can successfully integrate concepts at a higher level of thinking.

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $n = \text{C/Y} \times \text{Number of Years}$

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

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1.3: WHERE DO WE GO FROM HERE?

At this point, I hope that you can see that all the features of this textbook are here to help you learn, understand, and perform math. This book gives you every opportunity to succeed!

Now that you know what business mathematics is and why it is important to you, let's get going! Complete the exercises below. Then look at your course syllabus and read your first section. Remember to take advantage of all the features of this textbook along the way.

Section 1.3 Exercises

a. Think about your current job. List five activities that you perform that involve mathematics.
b. Talk to family members. Note their individual occupations and ask them about what work activities they perform every day that involve mathematics.

For questions 1-5, gather a few of your fellow students to discuss business mathematics.

1. Identify five specific activities or actions that you need to perform to succeed in your business math course.
2. Consider how you will study for a math test. Develop three specific study strategies.
3. Many students find it beneficial to work in study groups. List three ways that a study group can benefit you in your business math course.
4. For those students who may experience high levels of stress or anxiety about mathematics, identify three coping strategies.
5. Go online and enter the phrase "business math" or "business math help" into a search engine. Look at the table of contents for this book and note any websites that may help you study, practice, or review each chapter.

Attribution

APPENDIX 1: ROUNDING RULES

How and when to round is often a source of confusion. This Appendix provides you with some rules for rounding your answers. We start with some general guidelines that apply globally throughout the entire textbook, then comment on the rules that apply to some specific sections of text.

Global Rounding Rules

- Interim solutions never get rounded unless there is a logical reason or business process forcing them to be rounded.
- When writing nonterminating decimals in this textbook, up to six decimals are written. The horizontal line format is used for repeating decimals. If the number is not a final solution, then it is assumed that all decimals or as many as possible are being carried forward.
- All final numbers are rounded to four decimals in decimal format and two decimals in percent format unless instructions indicate otherwise.
- Final solutions are rounded according to common business practices, practical limitations, or specific instructions.
- Zeroes not required at the end of decimals are generally not written unless required to meet a rounding standard or to visually line up a sequence of numbers.

Topic Specific Rounding Rules

Some specific topics have their own rounding standards. Until you learn about these topics, it does not make sense to put those standards here. However, the standards are introduced with the relevant topic. They are also summarized in Chapter 2.1: Rounding Whole Numbers and Decimals.

Attribution

CHAPTER 2: BACK TO BASICS

Outline of Chapter Topics

Where can you go in life and not be exposed to numbers and mathematics? Whether you are figuring out the price of a product (including shipping) on eBay, or balancing your bank accounts, you use your elementary mathematical skills from both your primary and secondary education.

Think about the math you perform every day:

- At the grocery store, you compare products to calculate the best value. One brand of potato chips retails for $\$3.99$ for 300 g, while the equally satisfying brand beside it is priced at $\$3.49$ for 250 g. Which offers the better value?
- As the host for a large gathering, you are preparing a homemade lasagna and need to triple the original recipe, which calls for $1\frac{2}{3}$ cups of tomato sauce. In the expanded recipe, how many cups of tomato sauce do you need?
- Many employers pay out bonuses. Perhaps in your company managers get twice as large a bonus as employees. Your company has five managers and 25 employees. If it announces a $\$35,000$ total bonus, what is your share as an employee?

Mathematics and numbers surround you in the business world, where you must read many numerical reports, interpret how the numbers fit together, and create your own reports showing such metrics as sales and profit projections.

Away from work, you must manage your income and pay your bills. This is a mathematical problem you likely solve on a daily basis, ensuring that the money flowing out of your bank account does not exceed the money flowing in. To purchase groceries, vacations, or entertainment, you need numbers.

This chapter gives you a refresher on your basic mathematical skills, which are essential for success in later chapters.

Attribution

2.1 ROUNDING WHOLE NUMBERS AND DECIMALS

Introduction

Throughout this text we will be deriving multiple formulas related to the mathematics of business and finance. Before we can understand how these formulas work and how to properly apply them it is essential that we gain confidence in performing arithmetic operations in the right order, using whole numbers, decimals, and fractions.

In this chapter you will review the basic arithmetic skills necessary for these business and finance applications.

Rounding Rules

One of the most common sources of difficulties in math is that different people sometimes use different standards for rounding. This seriously interferes with the consistency of final solutions and makes it hard to assess accuracy. So that everyone arrives at the same solution to the exercises/examples in this textbook, these rounding rules apply throughout the book:

- Never round an interim solution unless there is a logical reason or business process that forces the number to be rounded. Here are some examples of logical reasons or business processes indicating you should round:
    - You withdraw money or transfer it between different bank accounts. In doing so, you can only record two decimals and therefore any money moving between the financial tools must be rounded to two decimals.
    - You need to write the numbers in a financial statement or charge a price for a product. As our currency is in dollars and cents, only two decimals can appear.
- When you write nonterminating decimals, show only the first six (or up to six) decimals. Use the horizontal line format for repeating decimals. If the number is not a final solution, then assume that all decimals or as many as possible are being carried forward.
- Round all final numbers to six decimals in decimal format and four decimals in percentage format unless instructions indicate otherwise.
- Round final solutions according to common business practices, practical limitations, or specific

instructions.
    - For example, round any final answer involving dollar currency to two decimals. These types of common business practices and any exceptions are discussed as they arise at various points in this textbook.
- Generally avoid writing zeros, which are not required at the end of decimals, unless they are required to meet a rounding standard or to visually line up a sequence of numbers.
    - For example, write 6.340 as 6.34 since there is no difference in interpretation through dropping the zero.

Paths To Success

Does your final solution vary from the actual solution by a small amount? Did the question involve multiple steps or calculations to get the final answer? Were lots of decimals or fractions involved? If you answer yes to these questions, the most common source of error lies in rounding. Here are some quick error checks for answers that are "close":

- Did you remember to obey the rounding rules laid out above? Most importantly, are you carrying decimals for interim solutions and rounding only at final solutions?
- Did you resolve each fraction or step accurately? Check for incorrect calculations or easy-to-make errors, like transposed numbers.
- Did you break any rules of **BEDMAS**?

Rounding Whole Numbers and Decimal Numbers

Rounding numbers makes them easier to work with and easier to remember. Rounding changes some of the digits in a number but keeps its value close to the original. It is used in reporting large quantities or values that change often, such as in population, income, expenses, etc.

The process of approximating numbers is called **rounding**. Numbers are rounded to a specific place value depending on how much accuracy is needed. Saying that the population of Canada is approximately 37 million means we rounded to the millions place. The place value we round to depends on how we need to use the number.

HOW TO

Round Whole Numbers

Step 1: Identify the digit to be rounded (this is the place value for which the rounding is required)

Step 2: If the digit to the immediate right of the required rounding is less than 5 do not change the value of the rounding digit.

If the digit to the immediate right of the required rounding digit is 5 or greater than 5, increase the value of the rounding digit by one (Round the number up).

Step 3: Change the value of all the digits that are to the right of the rounding digit to 0.

Place Value of Whole Numbers

When reading whole numbers, the position of each digit in a whole number determines the **place value** for the digit. Table 2.1.1 below illustrates the place value of the ten digits in the whole number $3,957,261,840$
.

Table 2.1.1

3,	9	5	7,	2	6	1,	8
Billions	**Hundred million**	**Ten Million**	**Millions**	**Hundred Thousand**	**Ten Thousand**	**Thousands**	**Hundreds**

In this example, 2 is in the Hundred of Thousands place value and represents, $200,000$, whereas 9 is in the Hundred of Millions place value and represents $900,000,000$.

We read and write numbers from left to right. A comma or a space separates every three digits into groups, starting from the "ones" which makes the whole number easier to read.

Table 2.1.2

Billions	Hundred Million	Ten Million	Millions	Hundred Thousand	Ten Thousand	Thousands
1,000,000,000	100,000,000	10,000,000	1,000,000	100,000	10,000	1,000
10^9	10^8	10^7	10^6	10^5	10^4	10^3

The place value of "ones" is $1(=10^0)$ and each place has a value 10 times the place value to its right, as shown in Table 2.1.2 above.

Place Value of Decimal Numbers

The position of each digit in a decimal number determines the place value of the digit. Table 2.1.3 below illustrates the place value of the five digit decimal number of 0.75396.

Table 2.1.3

Ones		Tenths	Hundredths	Thousandths	Ten Thou
0	.	7	5	3	

Example 2.1.1

What is the place value of the digit $rgb]1.0, 0.0, 0.03$ in each of the following numbers and what amount does it represent?

a. $67rgb]1.0, 0.0, 0.03, 542$
b. $5rgb]1.0, 0.0, 0.03, 721, 890$
c. $251.rgb]1.0, 0.0, 0.0347$

d. $64.0 rgb]1.0, 0.0, 0.037$

Solution

a.

Step 1: **Find where the digit 3 is in the number.**

$$= 67 rgb]1.0, 0.0, 0.0 \overbrace{3}, 542$$
$$= rgb]1.0, 0.0, 0.03, \overset{rgb]0.0,0.0,1.01 rgb]0.0,0.0,1.02 rgb]0.0,0.0,1.03}{5 \quad 4 \quad 2}$$
$$= rgb]1.0, 0.0, 0.03 rgb]1.0, 0.0, 0.0, rgb]1.0, 0.0, 0.0000$$

Step 2: **Make a statement**

The 3 is in the thousandths place.

---

b.

Step 1: **Find where the digit 3 is in the number.**

$$= 5 rgb]1.0, 0.0, 0.0 \overbrace{3}, 721, 890$$
$$= rgb]1.0, 0.0, 0.03, \overset{rgb]0.0,0.0,1.01 rgb]0.0,0.0,1.02 rgb]0.0,0.0,1.03}{7 \quad 2 \quad 1}, \overset{rgb]0.0,0.0,1.04 rgb]0.0,0.0,1.05 rgb]0.0,0.0,1.06}{8 \quad 9 \quad 0}$$
$$= rgb]1.0, 0.0, 0.03 rgb]1.0, 0.0, 0.0, rgb]1.0, 0.0, 0.0000 rgb]1.0, 0.0, 0.0, rgb]1.0, 0.0, 0.0000$$

Step 2: **Make a statement.**

The 3 is in the millionths place.

---

c.

Step 1: **Find where the digit 3 is in the number.**

$$= 251. rgb]1.0, 0.0, 0.0 \overbrace{3} 47$$
$$= . rgb]1.0, 0.0, 0.0 \overset{rgb]0.0,0.0,1.01}{3} \quad 47$$
$$= rgb]1.0, 0.0, 0.00 rgb]1.0, 0.0, 0.0. rgb]1.0, 0.0, 0.03$$

Step 2: **Make a statement.**

The 3 is in the tenths place.

---

d.

Step 1: **Find where the digit 3 is in the number.**

$$\begin{aligned} &= 64.0 rgb]1.0,0.0,0.037 \\ &= .\overset{rgb]0.0,0.0,1.01}{0} rgb]1.0,0.0,0.0\overset{rgb]0.0,0.0,1.02}{3}7 \\ &= rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.003 \end{aligned}$$

Step 2: **Make a statement.**

The 3 is in the hundredths place.

Try It

1) Identify the digit that occupies the following place values in the number $320,948.751$:

a. Hundred thousandths
b. Hundredths
c. Tenths

Solution

a. $3\ (rgb]1.0,0.0,0.0320,948.751)$
b. $9\ (320,rgb]1.0,0.0,0.0948.751)$
c. $7\ (320,948.rgb]1.0,0.0,0.0751)$

Example 2.1.2

Round the following whole numbers to the indicated place value: $19,456$ to the nearest tenth.

Solution

Step 1: **Identify the rounding digit in the tens place.**

$$19,4rgb]1.0,0.0,0.056$$

Step 2: **Check the digit to the right. Do we increase?**

The digit to the immediate right is 6, which is greater than 5.

Therefore, we do increase the value of the rounding digit by one, from 5 to 6.

Step 3: **Do we change the value?**

Yes, we change the value of the digits to the right of the 5 to 0.

Step 4: **Write as a statement.**

Therefore the rounded number becomes $19,460$.

Attribution

2.2: FRACTIONS AND DECIMALS

Introduction

Your local newspaper quotes a political candidate as saying, "The top half of the students are well-educated, the bottom half receive extra help, but the middle half we are leaving out " (Neal, 2008). You stare at the sentence for a moment and then laugh. To halve something means to split it into two. However, there are three halves here! You conclude that the speaker was not thinking carefully.

In coming to this conclusion, you are applying your knowledge of fractions. In this section, you will review fraction types, convert fractions into decimals, perform operations on fractions, and also address rounding issues in business mathematics.

Types of Fractions

To understand the characteristics, rules, and procedures for working with fractions, you must become familiar with fraction terminology. First of all, what is a fraction? A **fraction** is a part of a whole. It is written in one of three formats:

$$1/2 \text{ or } {}^{1}\!/\!{}_{2} \text{ or } \frac{1}{2}$$

Each of these formats means exactly the same thing. The number on the top, side, or to the left of the line is known as the **numerator**. The number on the bottom, side, or to the right of the line is known as the **denominator**. The slash or line in the middle is the **divisor line**. In the above example, the numerator is 1 and the denominator is 2. There are five different types of fractions, as explained in the table below.

Table 2.2.1

Fraction	Terminology	Characteris
$\frac{2}{5}$	Proper	The numera
	Improper	The numera
$3\frac{2}{5}$	Mixed	A fraction th improper fra improper fra
$3\frac{\frac{2}{5}}{7}$	Complex	A fraction th of compoun important to
$\frac{1}{2}$ and $\frac{2}{4}$	Equivalent	Two or more value upon c examples wo

How It Works

First, focus on the correct identification of proper, improper, compound, equivalent, and complex fractions. In the next section, you will work through how to accurately convert these fractions into their decimal equivalents.

Equivalent fractions require you to either solve for an unknown term or express the fraction in larger or smaller terms.

HOW TO

Solve For An Unknown Term

These situations involve two fractions where only one of the numerators or denominators is missing. Follow this four-step procedure to solve for the unknown:

Step 1: Set up the two fractions.

Step 2: Note that your equation contains two numerators and two denominators. Pick the pair for which you know both values.

Step 3: Determine the multiplication or division relationship between the two numbers.

Step 4: Apply the same relationship to the pair of numerators or denominators containing the unknown.

Example 2.2.1

Assume you are having a party and one of your friends says he would like to eat one-third of the pizza. You notice the pizza has been cut into nine slices. How many slices would you give to your friend?

Solution

***Step 1:* Assign a meaningful variable to represent unknown.**

$s =$ the number of slices to give out

Your friend wants one out of three pieces. This is one-third. You want to know how many pieces out of nine to give him. There are a total of 9 pieces, so we are looking for $s/9$.

$$\frac{1}{3} = \frac{s}{9}$$

***Step 2:* Work with the denominators 3 and 9 since you know both of them.**

Step 3: **Take the larger number and divide it by the smaller number.**

$$9 \div 3 = 3$$

The denominator on the right is three times larger than the denominator on the left.

Step 4: **Take the 1 and multiply it by 3 to get the s.**

$$\frac{rgb]1.0, 0.0, 0.01 rgb]1.0, 0.0, 0.0 \times rgb]1.0, 0.0, 0.03}{3 \times 3} = \frac{rgb]1.0, 0.0, 0.03}{9}$$

$$rgb]1.0, 0.0, 0.0 s = rgb]1.0, 0.0, 0.03$$

Step 5: **Write as a statement.**

You should give your friend three slices of pizza.

Expressing The Fraction In Larger Or Smaller Terms

When you need to make a fraction easier to understand or you need to express it in a certain format, it helps to try to express it in larger or smaller terms:

To express a fraction in larger terms, multiply both the numerator and denominator by the same number.

- **Larger terms:** $\frac{2}{12}$ expressed with terms twice as large would be $\frac{2 \times 2}{12 \times 2} = \frac{4}{24}$

To express a fraction in smaller terms, divide both the numerator and denominator by the same number.

- **Smaller terms:** $\frac{2}{12}$ expressed with terms half as large would be $\frac{2 \div 2}{12 \div 2} = \frac{1}{6}$

When expressing fractions in higher or lower terms, you do not want to introduce decimals into the fraction unless there would be a specific reason for doing so. For example, if you divided 4 into both the numerator and denominator of $\frac{2}{12}$, you would have $\frac{0.5}{3}$, which is not a typical format.

HOW TO

Find numbers that divide evenly into the numerator or denominator (called factoring).

Step 1: Pick the smallest number in the fraction.

Step 2: Use your multiplication tables and start with $1\times$ before proceeding to $2\times$, $3\times$, and so on.

Step 3: When you find a number that works, check to see if it also divides evenly into the other number.

Example 2.2.2

Reduce the following fraction: $\frac{12}{18}$.

Solution

***Step 1:* Factor the numerator.**

$$1 \times 12 = 12$$

***Step 2:* Does it divide evenly?**

12 does not divide evenly into the denominator.

***Step 3:* Try another factor.**

$$2 \times 6 = 12$$

6 does divide evenly into the denominator.

***Step 4:* Reduce the fraction into smaller terms.**

Reduce the fraction to smaller terms by dividing by 6.

$$\frac{12 \div 6}{18 \div 6} = \frac{2}{3}$$

***Step 5:* Write as a statement.**

The reduced fraction is $\frac{2}{3}$.

Things To Watch Out For

With complex fractions, it is critical to obey the rules of **BEDMAS**.

Note in the following example that an addition sign and two sets of brackets were hidden:

You should rewrite $3\frac{\frac{2}{5}}{7}$ as $3 + \left[\frac{\left(\frac{2}{5}\right)}{7}\right]$ before you attempt to solve with **BEDMAS**.

Paths To Success

What do you do when there is a negative sign in front of a fraction, such as $-\frac{1}{2}$? Do you put the negative with the numerator or the denominator? The common solution is to multiply the numerator by negative 1, resulting in $\frac{(-1) \times 1}{2} = \frac{-1}{2}$.

In the special case of a compound fraction, multiply the entire fraction by -1. Thus:

$$\begin{aligned} &= -1\frac{1}{2} \\ &= (-1) \times (1 + \frac{1}{2}). \\ &= -1 - \frac{1}{2} \end{aligned}$$

Example 2.2.3

Identify the type of fraction represented by each of the following:

a. $\frac{2}{3}$

b. $6\frac{7}{8}$

c. $12\frac{\frac{4}{3}}{6\frac{4}{5}}$

d. $\frac{15}{11}$

e. $\frac{5}{6}$

f. $\frac{3}{4}$ & $\frac{9}{12}$

Solution

Step 1: **Identify what we are looking for.**

For each of these six fractions, identify the type of fraction.

Step 2: **State what we know.**

There are five types of fractions, including proper, improper, compound, complex, or equivalent.

Step 3: **Use the definition from the Types of Fractions table at the beginning of this section to identify the type.**

---

a. $\frac{2}{3}$

The numerator is smaller than the denominator. This matches the characteristics of a **proper fraction**.

---

b. $6\frac{7}{8}$

This fraction combines an integer with a proper fraction (since the numerator is smaller than the denominator). This matches the characteristics of a **compound fraction.**

---

c. $12\frac{\frac{4}{3}}{6\frac{4}{5}}$

There are lots of fractions involving fractions nested inside other fractions. The fraction as a whole is a compound fraction, containing an integer with a proper fraction (since the numerator is smaller than the denominator). Within the proper fraction, the numerator is an improper fraction $\left(\frac{4}{3}\right)$ and the denominator is a compound fraction containing an integer and a proper fraction $\left(6\frac{4}{5}\right)$.

This all matches the definition of a **complex fraction**: nested fractions combining elements of compound, proper, and improper fractions together.

---

d. $\frac{15}{11}$

The numerator is larger than the denominator. This matches the characteristics of an **improper fraction**.

---

e. $\frac{5}{6}$

The numerator is smaller than the denominator. This matches the characteristics of a **proper fraction**.

---

f. $\frac{3}{4} \& \frac{9}{12}$

There are two proper fractions here that are equal to each other. If you were to complete the division, both fractions calculate to 0.75. These are **equivalent fractions**.

Example 2.2.4

a. Solve for the unknown term: $x : \frac{7}{12} = \frac{49}{x}$
b. Express this fraction in lower terms: $\frac{5}{50}$

Solution

a.

***Step 1:* Is the fraction in a format I can solve in?**

Yes. You have both of the numerators, so work with that pair.

***Step 2:* Take the larger number and divide by the smaller number.**

$$49 \div 7 = 7$$

***Step 3:* Multiple the fraction on the left by 7 to get the fraction on the right. Applying the same relationship:**

$$12 \times 7 = 84$$

***Step 4:* Write as a statement.**

The unknown denominator on the right is 84, and therefore $\frac{7}{12} = \frac{49}{84}$.

---

b.

Step 1: **Find a common divisor that divides into the numerator and denominator evenly.**

As only 1 and 5 go into the number 5, it makes sense that you should choose 5 to divide into both the numerator and denominator.

Note that 5 factors evenly into the denominator, 50, meaning that no remainder or decimals are left over.

$$\frac{5 \div 5}{50 \div 5} = \frac{1}{10}$$

Step 2: **Write as a statement.**

In lower terms, $\frac{5}{50}$, is expressed as $\frac{1}{10}$.

Converting to Decimals

Although fractions are common, many people have trouble interpreting them. For example, in comparing $\frac{27}{37}$ to $\frac{57}{73}$, which is the larger number? The solution is not immediately apparent. As well, imagine a retail world where your local Walmart was having a $\frac{3}{20}$th off sale! It's not that easy to realize that this equates to 15% off. In other words, fractions are converted into decimals by performing the division to make them easier to understand and compare.

HOW TO

Convert fractions into decimals based on the fraction types and fraction rules

Proper and Improper Fractions

Resolve the division. For example, $\frac{3}{4}$ is the same as $3 \div 4 = 0.75$. As well:

$$\begin{aligned} &\frac{5}{4} \\ &= 5 \div 4 \\ &= 1.25 \end{aligned}$$

Compound Fractions

The decimal number and the fraction are joined by a hidden addition symbol. Therefore, to convert to a decimal you need to reinsert the addition symbol and apply **BEDMAS**:

$$\begin{aligned} &3\frac{4}{5} \\ &= 3 + 4 \div 5 \\ &= 3 + 0.8 \\ &= 3.8 \end{aligned}$$

Complex Fractions

The critical skill here is to reinsert all of the hidden symbols and then apply the rules of **BEDMAS**:

$$
\begin{aligned}
&2\frac{\frac{11}{4}}{1\frac{1}{4}} \\
&= \ 2+\left[\frac{(11 \div 4)}{(1+1 \div 4)}\right] \\
&= 2+\left[\frac{(11 \div 4)}{(1+0.25)}\right] \\
&= 2+\left[\frac{2.75}{1.25}\right] \\
&= 2+2.2 \\
&= 4.2
\end{aligned}
$$

Example 2.2.5

Convert the following fractions into decimals:

a. $\frac{2}{5}$
b. $6\frac{7}{8}$
c. $12\frac{\frac{9}{2}}{1\ \frac{2}{10}}$

Solution

a.

Step 1: **This is a proper fraction requiring you to complete the division.**

$$\frac{2}{5}$$

$$= 2 \div 5$$

$$= 0.4$$

Step 2: **Write as a statement.**

The decimal form is 0.4.

---

b.

Step 1: **This is a compound fraction requiring you to reinsert the hidden addition symbol and then apply BEDMAS.**

$$6\frac{7}{8}$$

$$= 6 + 7 \div 8$$
$$= 6 + 0.875$$
$$= 6.875$$

Step 2: **Write as a statement.**

The decimal form is 6.875.

---

c.

Step 1: **This is a complex fraction requiring you to reinsert all hidden symbols and apply BEDMAS.**

$$\begin{aligned} &12\frac{\frac{9}{2}}{1\frac{2}{10}} \\ &= 12 + \left[\frac{(9 \div 2)}{(1 + 2 \div 10)}\right] \\ &= 12 + \left[\frac{(9 \div 2)}{(1 + 0.2)}\right] \\ &= 12 + \left[\frac{4.5}{1.2}\right] \\ &= 12 + 3.75 \\ &= 15.75 \end{aligned}$$

Step 2: **Write as a statement.**

The decimal form is 15.75.

Rounding Principle

Your company needs to take out a loan to cover some short-term debt. The bank has a posted rate of 6.875%. Your bank officer tells you that, for simplicity, she will just round off your interest rate to 6.9%. Is that all right with you? It shouldn't be!

What this example illustrates is the importance of rounding. This is a slightly tricky concept that confuses most students to some degree. In business math, sometimes you should round your calculations off and sometimes you need to retain all of the digits to maintain accuracy.

HOW TO

Apply the Rounding Principle

To round a number off, you always look at the number to the right of the digit being rounded. If that number is 5 or higher, you add one to your digit; this is called **rounding up**. If that number is 4 or less, you leave your digit alone; this is called **rounding down**.

For example, if you are rounding 8.345 to two decimals, you need to examine the number in the third decimal place (the one to the right). It is a 5, so you add one to the second digit and the number becomes 8.35.

For a second example, let's round 3.6543 to the third decimal place. Therefore, you look at the fourth decimal position, which is a 3. As the rule says, you would leave the digit alone and the number becomes 3.654.

Nonterminating Decimals

What happens when you perform a calculation and the decimal doesn't terminate?

1. You need to assess if there is a pattern in the decimals:

    - **The Nonterminating Decimal without a Pattern:**

    For example, $\frac{6}{17} = 0.352941176$ with no apparent ending decimal and no pattern to the decimals.

    - **The Nonterminating Decimal with a Pattern:**

    For example, $\frac{2}{11} = 0.18181818$ endlessly. You can see that the numbers 1 and 8 repeat. A shorthand way of expressing this is to place a horizontal line above the digits that repeat. Thus, you can rewrite 0.18181818 as $0.\overline{18}$.

2. You need to know if the number represents an interim or final solution to a problem:

    - **Interim Solution**

    You must carry forward all of the decimals in your calculations, as the number should not be rounded until you arrive at a final answer. If you are completing the question by hand, write out as many decimals as possible; to save space and time, you can use the shorthand horizontal bar for repeating decimals. If you are completing the question by calculator, store the entire number in a memory cell.

- **Final Solution**

To round this number off, an industry protocol or other clear instruction must apply. If these do not exist, then you would make an arbitrary rounding choice, subject to the condition that you must maintain enough precision to allow for reasonable interpretation of the information.

Key Takeaways

To assist in your calculations, particularly those that involve multiple steps to resolve, your calculator has 10 memory cells. Your display is limited to 10 digits, but when you store a number into a memory cell the calculator retains all of the decimals associated with the number, not just those displaying on the screen. Your calculator can, in fact, carry up to 13 digit positions. It is strongly recommended that you take advantage of this feature where needed throughout this textbook.

Let's say that you just finished keying in $\frac{6}{17}$ on your calculator, and the resultant number is an interim solution that you need for another step. With 0.352941176 on your display, press **STO** followed by any numerical digit on the keypad of your calculator. **STO** stands for **store**.

To store the number into memory cell 1, for example, press **STO 1**. The number with 13 digits is now in permanent memory. If you clear your calculator (press CE/C) and press **RCL #** (where # is the memory cell number), you will bring the stored number back. **RCL** stands for **recall**. Press **RCL 1**. The stored number 0.352941176 reappears on the screen.

Example 2.2.6

Convert the following to decimals. Round each to four decimals or use the repeating decimal notation.

a. $\frac{6}{13}$
b. $\frac{4}{9}$
c. $\frac{4}{11}$
d. $\frac{3}{22}$
e. $5\frac{\frac{1}{7}}{\frac{10}{27}}$

Solution

a.

Step 1: **Divide to convert to decimal.**

$$\frac{6}{13} = 0.461538$$

Step 2: **Round and write as a statement.**

The fifth decimal is a 3, so round down.

Step 3: **Write as a statement.**

The answer is 0.4615.

---

b.

Step 1: **Divide to convert to decimal.**

$$\frac{4}{9} = 0.444444$$

Step 2: **Round and write as a statement.**

Note the repeating decimal of 4.

Step 3: **Write as a statement.**

Using the horizontal bar, write $0.\overline{4}$.

---

c.

Step 1: **Divide to convert to decimal.**

$$\frac{4}{11} = 0.363636$$

Step 2: **Round and write as a statement.**

Note the repeating decimals of 3 and 6.

Step 3: **Write as a statement.**

Using the horizontal bar, write $0.\overline{36}$.

---

d.

Step 1: **Divide to convert to decimal.**

$$\frac{3}{22} = 0.136363$$

Step 2: **Round and write as a statement.**

Note the repeating decimals of 3 and 6 after the 1.

Step 3: **Write as a statement.**

Using the horizontal bar, write $0.1\overline{36}$.

---

e.

Step 1: **Divide to convert to decimal.**

$$5\frac{\frac{1}{7}}{\frac{10}{27}}$$

$$= 5 + \frac{(1 \div 7)}{(10 \div 27)}$$

$$= 5 + \frac{0.142857}{0.\overline{370}}$$

$$= 5 + 0.385714$$

$$= 5.385714$$

Step 2: **Round and write as a statement.**

Since the fifth digit is a 1, round down.

Step 3: **Write as a statement.**

The answer is 5.3857.

Section 2.2 Exercises

Mechanics

1. For each of the following, identify the type of fraction presented.

   a. $\frac{1}{8}$

   b. $3\frac{3}{4}$

c. $\frac{10}{9}$
d. $\frac{34}{49}$
e. $1\frac{\frac{2}{3}}{9\frac{5}{3}}$
f. $\frac{56}{27}$
g. $\frac{10\frac{1}{5}}{9}$
h. $\frac{6}{11}$

2. In each of the following equations, identify the value of the unknown term.

a. $\frac{3}{4} = \frac{x}{36}$
b. $\frac{y}{8} = \frac{16}{64}$
c. $\frac{2}{z} = \frac{18}{45}$
d. $\frac{5}{6} = \frac{75}{p}$

3. Take each of the following fractions and provide one example of the fraction expressed in both higher and lower terms.

a. $\frac{5}{10}$
b. $\frac{6}{8}$

4. Convert each of the following fractions into decimal format.

a. $\frac{7}{8}$

b. $15\frac{5}{4}$
c. $\frac{13}{5}$
d. $133\frac{\frac{17}{2}}{3\frac{2}{5}}$

5. Convert each of the following fractions into decimal format and round to three decimals.

a. $\frac{7}{8}$
b. $15\frac{3}{4}$
c. $\frac{10}{9}$
d. $\frac{15}{32}$

6. Convert each of the following fractions into decimal format and express in repeating decimal notation.

a. $\frac{1}{12}$
b. $5\frac{8}{33}$
c. $\frac{4}{3}$
d. $\frac{-34}{110}$

Solutions

1a. proper

1b.compound

1c.improper

1d. proper

1e. complex

1f. improper

1g. complex

1h.proper

2a. $\frac{3}{4} = \frac{27}{36}$

2b. $\frac{2}{8} = \frac{16}{64}$

2c. $\frac{2}{5} = \frac{18}{45}$

2d. $\frac{5}{6} = \frac{75}{90}$

3a. $\frac{5 \times 2}{10 \times 2} = \frac{10}{20} \quad \frac{5 \div 5}{10 \div 5} = \frac{1}{2}$

3b. $\frac{6 \times 5}{8 \times 5} = \frac{30}{40} \quad \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$

4a. 0.875

4b. 16.25

4c. 2.6

4d. $137.7\overline{2}$

5a. 0.875

5b. 15.750

5c. 1.111

5d. 0.469

6a. $0.08\overline{3}$

6b. $5.\overline{24}$

6c. $1.\overline{3}$

6d. $-0.3\overline{09}$

Applications

7. Calculate the solution to each of the following expressions. Express your answer in decimal format.

   a. $\frac{1}{5}+3\frac{1}{4}+\frac{5}{2}$

   b. $1\frac{\frac{3}{8}}{2}-\frac{11}{40}+19\frac{1}{2}\times\frac{3}{4}$

8. Calculate the solution to each of the following expressions. Express your answer in decimal format with two decimals.

   a. $\left(1+\frac{0.11}{12}\right)^4$

   b. $1-0.05\times\frac{263}{365}$

   c. $200\left[1-\frac{1}{\left(1+\frac{0.10}{4}\right)^2}\right]$

9. Calculate the solution to each of the following expressions. Express your answer in repeating decimal notation as needed.

   a. $\frac{1}{11}+3\frac{1}{9}$

   b. $\frac{5}{3}-\frac{7}{6}$

Questions 10–14 involve fractions. For each, evaluate the expression and round your answer to the nearest cent.

10. $\$134,000(1+0.14\times 23/365)$
11. $\$10,000\left(1+\frac{0.0525}{2}\right)^{13}$

12. $\dfrac{\$535,000}{\left(1+\frac{0.07}{12}\right)^{3}}$

13. $\$2,995\left(1+0.13\times\dfrac{90}{365}\right)-\dfrac{\$400}{1+0.13\times\frac{15}{365}}$

14. $\dfrac{\$155,600}{\left(1+\frac{0.06}{12}\right)^{8}}$

Solutions

7a. 5.95

7b. 15.0375

8a. 1.04

8b. 0.96

8c. 9.64

9a. $3.\overline{20}$

9b. 0.5

10. $\$135,182.14$

11. $\$14,005.26$

12. $\$525,745.68$

13. $\$2,693.13$

14. $\$149,513.74$

Challenge, Critical Thinking, & Other Applications

Questions 15–20 involve more complex fractions and reflect business math equations encountered later in this textbook. For each, evaluate the expression and round your answer to the nearest cent.

15. $\frac{\$648}{0.0575/12}\left[1-\frac{1}{\left(1+\frac{0.0575}{12}\right)^{7}}\right]$

16. $\frac{\$10,000}{\left(1+\frac{0.115}{4}\right)^{2}}+\$68\frac{\left[1-\frac{1}{\left(1+\frac{0.115}{4}\right)^{2}}\right]}{\frac{0.115}{4}}$

17. $\frac{\$2,000,000}{\left[\frac{\left(1+\frac{0.065}{2}\right)^{12}-1}{\frac{0.065}{2}}\right]}$

18. $\$8,500\left[\frac{1-\left(\frac{1}{1.08}\right)^{4}}{1.08}\right]+\$19,750\left(\frac{1}{1.08}\right)^{4}-\$4,350$

19. $\$15,000\left[\frac{\left(1+\frac{0.058}{4}\right)^{16}-1}{\frac{0.058}{4}}\right]$

20. $\frac{0.08}{2}(\$1,000)\left[\frac{1-\frac{1}{\left(1+\frac{0.07}{2}\right)^{10}}}{\frac{0.07}{2}}\right]+\$1,000\frac{1}{\left(1+\frac{0.07}{2}\right)^{10}}$

Solutions

15. $\$4,450.29$
16. $\$9,579.23$
17. $\$138,934.38$
18. $\$12,252.25$
19. $\$267,952.30$
20. $\$1,041.58$

Attribution

2.3: ORDER OF OPERATIONS

Introduction

Try It

1) You have just won $\$50,000$ in a contest. Congratulations! But before you can claim it, you are required to answer a mathematical skill-testing question, and no calculators are permitted. After you hand over your winning ticket to the redemption agent, she hands you your time-limited skill-testing question: $2 \times 5 + 30 \div 5$. What does question this equal?

Solution

If you figured out the solution is 16, you are on the right path. If you thought it was something else, this is a great time to review order of operations.

The Symbols

While some mathematical operations such as addition use a singular symbol (+), there are other operations, like multiplication, for which multiple representations are acceptable. With the advent of computers, even more new symbols have crept into mathematical symbology. The table below lists the various mathematical operations and the corresponding mathematical symbols you can use for them.

Table 2.3.1

Mathematical Operation	**Symbol or Appearance**	
Brackets	( ) or [ ] or { }	In order, these are known as round, square, and curly brackets.
Exponents	2^3 or $2\hat{}3$	• In 2^3, 3 is the exponent; an exponent is always written as • On a computer, the exponent is recognized with the ^ sym
Multiplication	$\times$ or $*$ or $\cdot$ or $2(2)$ or $(2)(2)$	In order, these are known as times, star, or bullet. The last two i brackets.
Division	/ or $\div$ or $\frac{4}{2}$	In order, these are known as the slash, divisor, and divisor line.
Addition	$+$	There are no other symbols.
Subtraction	$-$	There are no other symbols.

You may wonder if the different types of brackets mean different things. Although mathematical fields like calculus use specialized interpretations for the different brackets, business math uses all the brackets to help the reader visually pair up the brackets. Consider the following two examples:

Example 1: $3 \times (4/(6-(2+2))+2)$

Example 2: $3 \times [4/\{6-(2+2)\}+2]$

Notice that in the second example you can pair up the brackets much more easily, but changing the shape of the brackets did not change the mathematical expression. This is important to understand when using a calculator, which usually has only round brackets. Since the shape of the bracket has no mathematical impact, solving example 1 or example 2 would involve the repeated usage of the round brackets.

BEDMAS

In the section opener, your skill-testing question was $2 \times 5 + 30 \div 5$. Do you just solve this expression from left to right, or should you start somewhere else? To prevent any confusion over how to resolve these mathematical operations, there is an agreed-upon sequence of mathematical steps commonly referred to as **BEDMAS**.

HOW TO

Use BEDMAS

BEDMAS is an acronym for **B**rackets, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, and **S**ubtraction.

Step 1: Brackets must be resolved first. As brackets can be nested inside of each other, you must resolve the innermost set of brackets first before proceeding outwards to the next set of brackets. When resolving a set of brackets, you must perform the mathematical operations within the brackets by following the remaining steps in this model (EDMAS). If there is more than one set of brackets but the sets are not nested, work from left to right and top to bottom.

Step 2: If the expression has any exponents, you must resolve these next. Remember that an exponent indicates how many times you need to multiply the base against itself. For example, $2^3 = 2 \times 2 \times 2$. More review of exponents is found in Section 2.5.

Step 3: The order of appearance for multiplication and division does not matter. However, you must resolve these operations in order from left to right and top to bottom as they appear in the expression.

Step 4: The last operations to be completed are addition and subtraction. The order of appearance doesn't matter; however, you must complete the operations working left to right through the expression.

Key Takeaway

Before proceeding with the Texas Instruments BAII Plus calculator, you must change some of the factory defaults, as explained in the table below. To change the defaults, open the Format window on your calculator. If for any reason your calculator is reset (either by removing the battery or pressing the reset button), you must perform this sequence again.

Table 2.3.2

Buttons Pushed	Calculator Display	What It Means
2nd Format	DEC=2.00	You have opened the Format window to its first setting. DEC tells your calculator how to round the calculations. In business math, it is important to be accurate. Therefore, we will set the calculator to what is called a floating display, which means your calculator will carry all of the decimals and display as many as possible on the screen.
9 Enter	DEC=9.	The floating decimal setting is now in place. Let us proceed.
↓	DEG	This setting has nothing to do with business math and is just left alone. If it does not read DEG, press 2nd Set to toggle it.
↓	US 12-31-1990	Dates can be entered into the calculator. North Americans and Europeans use slightly different formats for dates. Your display is indicating North American format and is acceptable for our purposes. If it does not read US, press 2nd Set to toggle it.
↓	US 1,000	In North America it is common to separate numbers into blocks of 3 using a comma. Europeans do it slightly differently. This setting is acceptable for our purposes. If your display does not read US, press 2nd Set to toggle it.
↓	Chn	There are two ways that calculators can solve equations. This is known as the Chain method, which means that your calculator will simply resolve equations as you punch it in without regard for the rules of BEDMAS. This is not acceptable and needs to be changed.
2nd Set	AOS	AOS stands for Algebraic Operating System. This means that the calculator is now programmed to use BEDMAS in solving equations.
2nd Quit	0.	Back to regular calculator usage.

Also note that on the BAII Plus calculator you have two ways to key in an exponent:

- If the exponent is squaring the base (e.g., 3^2), press $3\ x^2$. It calculates the solution of 9.
- If the exponent is anything other than a 2, you must use the y^x button. For 2^3, you press $2\ y^x\ 3\ =$. It calculates the solution of 8.

Things to Watch For

Negative Signs

Remember that mathematics use both positive numbers (such as $+3$) and negative numbers (such as -3). Positive numbers do not need to have the $+$ sign placed in front of them since it is implied. Thus $+3$ is written as just 3. Negative numbers, though, must have the negative sign placed in front of them. Be careful not to confuse the terminology of a negative number with a subtraction or minus sign. For example, $4+(-3)$ is read as "four plus negative three" and not "four plus minus three." To key a negative number on a calculator, enter the number first followed by the $\pm$ button, which switches the sign of the number.

Horizontal Divisor Line

One of the areas in which people make the most mistakes involves the "hidden brackets." This problem almost always occurs when the horizontal line is used to represent division. Consider the following mathematical expression:

$$(4+6)\div(2+3)$$

If you rewrite this expression using the horizontal line to represent the divisor, it looks like this:

$$\frac{4+6}{2+3}$$

Notice that the brackets disappear from the expression when you write it with the horizontal divisor line because they are implied by the manner in which the expression appears. Your best approach when working with a horizontal divisor line is to reinsert the brackets around the terms on both the top and bottom. Thus, the expression looks like this:

$$\frac{(4+6)}{(2+3)}$$

Employing this technique will ensure that you arrive at the correct solution, especially when using calculators.

Paths to Success

Hidden and Implied Symbols

If there are hidden or implied symbols in the expressions, your first step is to reinsert those hidden symbols in their correct locations. In the example below, note how the hidden multiplication and brackets are reinserted into the expression: $4\left[\dfrac{3+2^2\times 3}{(2+8)\div 2}\right]$ transforms into $4\times\left[\dfrac{\{3+2^2\times 3\}}{\{(2+8)\div 2\}}\right]$.

Once you have reinserted the symbols, you are ready to follow the **BEDMAS** model.

Calculators are not programmed to be capable of recognizing implied symbols. If you key in "$3(4+2)$" on your calculator, failing to input the multiplication sign between the "3" and the "$(4+2)$,“ you get a solution of 6. Some calculators ignore the "3" since they don't know what mathematical operation to perform on it. To have your calculator solve the expression correctly, you must punch the equation through as "$3\times(4+2)=$ ". This produces the correct answer of 18.

Simplifying Negatives

If your question involves positive and negative numbers, it is sometimes confusing to know what symbol to put when simplifying or solving. Remember these two rules:

Rule #1: A pair of the same symbols is always positive.

Thus "$4+(+3)$" and "$4-(-3)$" both become "$4+3$."

Rule #2: A pair of the opposite symbols is always negative.

Thus "$4+(-3)$" and "$4-(+3)$" both become "$4–3$."

A simple way to remember these rules is to count the total sticks involved, where a "$+$" sign has two sticks and a "$-$" sign has one stick. If you have an odd number of total sticks, the outcome is a negative sign. If you have an even number of total sticks, the outcome is a positive sign.

Note the following examples:

- $4+(-3)=3$ total sticks is odd and therefore simplifies to negative $4-3=1$

- $(-2) \times (-2) = 2$ total sticks is even and therefore simplifies to positive $(-2) \times (-2) = +4$

Example 2.3.1

Evaluate each of the following expressions:

a. $2 \times 5 + 30 \div 5$
b. $(6+3)^2 + 18 \div 2$
c. $4 \times \left[\dfrac{\{3 + 2^2 \times 3\}}{\{(2+8) \div 2\}} \right]$

Solution

a.

Step 1: **B – Are there brackets?** No.

Step 2: **E – Are there exponents?** No.

Step 3: **D – Resolve the division.**

$rgb]0.1, 0.1, 0.130 rgb]0.1, 0.1, 0.1 \div rgb]0.1, 0.1, 0.15 rgb]0.1, 0.1, 0.1 = rgb]1.0, 0.0, 0.06$

The expression now looks like this:

$$2 \times 5 + rgb]1.0, 0.0, 0.06$$

Step 4: **M – Resolve the multiplication.**

$$2 \times 5 = rgb]0.0, 0.0, 1.010$$

The expression becomes:

$$rgb]0.0, 0.0, 1.010 + rgb]1.0, 0.0, 0.06$$

Step 5: **A – Perform the remaining addition.**

$rgb]0.0, 0.0, 1.010 + rgb]1.0, 0.0, 0.06 rgb]1.0, 0.0, 0.0\ rgb]0.1, 0.1, 0.1 = rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.116$

Step 6: **S – There is no subtraction, so we state the final solution.**

$$16$$

---

b.

Step 1: **B – Are there brackets?** Yes, start with the brackets.

$$rgb]0.1, 0.1, 0.1(6+3)^{rgb]0.1,0.1,0.12} rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1 = rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1(rgb]1.0, 0.0, 0.09)^{rgb]0.1,0.1,0.12}$$

The expression now looks like this:

$$rgb]0.1, 0.1, 0.1(rgb]1.0, 0.0, 0.09)^{rgb]0.1,0.1,0.12} + 18 \div 2$$

Step 2: **E – Are there exponents?** Yes, resolve exponents next.

$$rgb]0.1, 0.1, 0.1(rgb]1.0, 0.0, 0.09)^{rgb]0.0,0.0,1.02} rgb]0.1, 0.1, 0.1 = rgb]0.1, 0.1, 0.1\ rgb]0.0, 0.0, 1.081$$

The expression now looks like this:

$$rgb]0.0, 0.0, 1.081 rgb]0.1, 0.1, 0.1 + rgb]0.1, 0.1, 0.118 rgb]0.1, 0.1, 0.1 \div rgb]0.1, 0.1, 0.12$$

Step 3: **D – Perform the division.**

$$rgb]0.0, 0.0, 1.081 rgb]0.1, 0.1, 0.1 + rgb]0.5, 0.0, 0.518 rgb]0.5, 0.0, 0.5 \div rgb]0.5, 0.0, 0.52 rgb]0.5, 0.0, 0.5\ rgb]0.1, 0.1, 0.1 = rgb]0.5, 0.0, 0.5\ rgb]0.5, 0.0, 0.59$$

The expression now looks like this:

$$rgb]0.0, 0.0, 1.081 rgb]0.1, 0.1, 0.1 + rgb]0.5, 0.0, 0.5\ rgb]0.5, 0.0, 0.59$$

Step 4: **M – Is there multiplication?** No.

Step 5: **A – Perform the remaining addition.**

$$rgb]0.0, 0.0, 1.081 rgb]0.1, 0.1, 0.1 + rgb]0.5, 0.0, 0.5\ rgb]0.5, 0.0, 0.59 rgb]0.5, 0.0, 0.5\ rgb]0.1, 0.1, 0.1 = rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.190$$

Step 6: **S – There is no subtraction, so we state the final solution.**

$$90$$

---

c.

Step 1: **B – Are there brackets?**

Yes, start with the innermost set of brackets and perform EDMAS.

$$4 \times \left[\frac{\{3+2^2 \times 3\}}{\{rgb]1.0, 0.0, 0.0(rgb]1.0, 0.0, 0.02 rgb]1.0, 0.0, 0.0 + rgb]1.0, 0.0, 0.08 rgb]1.0, 0.0, 0.0) \div 2\}}\right] = 4 \times \left[\frac{\{3+2^2 \times 3\}}{\{rgb]1.0, 0.0, 0.010 \div 2\}}\right]$$

Now solve the curly brackets, starting with the top. Perform EDMAS.

Step 2: **E – Are there exponents?** Yes, resolve exponents next.

$$4 \times \left[\frac{\{3 + rgb]0.0, 0.0, 1.02^{rgb]0.0,0.0,1.02} rgb]0.0, 0.0, 1.0 = rgb]0.0, 0.0, 1.04 \times 3\}}{\{rgb]1.0, 0.0, 0.010 \div 2\}} \right]$$

The expression now looks like this:

$$4 \times \left[\frac{\{3 + rgb]0.0, 0.0, 1.04 \times 3\}}{\{rgb]1.0, 0.0, 0.010 \div 2\}} \right]$$

Step 3: **D – Perform the division.**

$$4 \times \left[\frac{\{3 + rgb]0.0, 0.0, 1.04 \times 3\}}{\{rgb]0.5, 0.0, 0.510 rgb]0.5, 0.0, 0.5 \div rgb]0.5, 0.0, 0.52 rgb]0.1, 0.1, 0.1 = rgb]0.5, 0.0, 0.55\}} \right]$$

We no longer need the curly brackets on the bottom, so they are dropped.

The expression now looks like this:

$$4 \times \left[\frac{\{3 + rgb]0.0, 0.0, 1.04 \times 3\}}{rgb]0.5, 0.0, 0.55} \right]$$

Step 4: **M – Perform multiplication on the top.**

$$4 \times \left[\frac{\{3 + rgb]0.0, 0.5, 0.04 rgb]0.0, 0.5, 0.0 \times rgb]0.0, 0.5, 0.03 rgb]0.1, 0.1, 0.1 = rgb]0.0, 0.5, 0.012\}}{rgb]0.5, 0.0, 0.55} \right]$$

The expression now looks like this:

$$4 \times \left[\frac{\{3 + rgb]0.0, 0.5, 0.012\}}{rgb]0.5, 0.0, 0.55} \right]$$

Step 5: **A – Perform the remaining addition.**

We no longer need the curly brackets on the top, so they are dropped.

$$4 \times \left[\frac{\{rgb]0.68, 0.46, 0.123 rgb]0.68, 0.46, 0.12 + rgb]0.68, 0.46, 0.1212 rgb]0.1, 0.1, 0.1 = rgb]0.68, 0.46, 0.1215\}}{rgb]0.5, 0.0, 0.55} \right]$$

The expression now looks like this:

$$4 \times \left[\frac{rgb]0.68, 0.46, 0.1215}{rgb]0.5, 0.0, 0.55} \right]$$

Step 6: **Repeating the division, we solve the fraction.**

$$4 \ \times \ \left[rgb]1.0,0.0,1.0\frac{15}{5}\right]$$

$$= 4 \ \times \ rgb]1.0,0.0,1.03$$

We no longer need the square brackets so they are dropped here.

Step 7: **The last step is to perform multiplication.**

$4 \ \times \ rgb]1.0,0.0,1.03rgb]1.0,0.0,1.0 \ rgb]0.1,0.1,0.1 = rgb]0.1,0.1,0.1 \ rgb]0.1,0.1,0.112$

Step 6: **S – There is no subtraction, so we state the final solution.**

$$12$$

Section 2.3 Exercises

Solve the following.

Hint: If a question involves money, round the answers to the nearest cent.

Mechanics

1. $81 \div 27 + 3 \times 4$
2. $100 \div (5 \times 4 – 5 \times 2)$
3. $3^3 – 9 + (1 + 7 \times 3)$
4. $(6 + 3)^2 – 17 \times 3 + 70$
5. $100 – (4^2 + 3) – (3 + 9 \times 3 – 4)$

Solutions

1. 15
2. 10

3. 40
4. 100
5. 55

Applications

6. $[(7^2 - \{-41\}) - 5 \times 2] \div (80 \div 10)$
7. $\$1,000\left(1 + 0.09 \times \dfrac{88}{365}\right)$
8. $3[\$2,000(1 + 0.003)^8] + \$1,500$
9. $\dfrac{\$20,000}{1 + 0.07 \times \frac{7}{12}}$
10. $4 \times [(5^2 + 15)^2 \div (13^2 - 9)]^2$
11. $\$500\left[\dfrac{(1 + 0.00875)^{43} - 1}{0.00875}\right]$
12. $\$1,000\left(1 + \dfrac{0.12}{6}\right)^{15}$

Solutions

6. 10
7. $\$1,021.70$
8. $\$7,645.52$
9. $\$19,215.37$
10. 400
11. $\$25,967.38$
12. $\$1,345.87$

Challenge, Critical Thinking, & Other Applications

13. $\left(\dfrac{\$2,500}{1 + 0.10}\right) + \left(\dfrac{\$7,500}{(1 + 0.10)^2}\right) + \left(\dfrac{-\$1,500}{(1 + 0.10)^3}\right) + \left(\dfrac{-\$2,000}{(1 + 0.10)^4}\right)$

14. $\$175,000(1+0.07)^{15}+\$14,000\left[\frac{(1+0.07)^{20}-1}{0.07}\right]$
15. $\$5,000\left[\frac{\left\{1+\left[(1+0.08)^{0.5}-1\right]\right\}^{75}-1}{(1+0.08)^{0.5}-1}\right]$
16. $\$800\left[\frac{(1+0.07)^{20}-(1+0.03)^{20}}{0.07-0.03}\right]$
17. $\$60,000(1+0.0058)^{80}-\$450\left[\frac{(1+0.0058)^{25}-1}{0.0058}\right]$
18. $\left(\frac{0.08}{2}\right)\$1,000\left[\frac{1-\frac{1}{\left\{1+\left(\frac{0.08}{2}\right)\right\}^{16}}}{\left(\frac{0.08}{2}\right)}\right]+\$1,000\left[1+\left(\frac{0.08}{2}\right)\right]^{16}$
19. $\$1,475\left[\frac{\left(1+\frac{0.06}{4}\right)^{16}-1}{\frac{0.06}{4}}\right]$
20. $\$6,250(1+0.0525)^{10}+\$325\left[\frac{(1+0.0525)^{10}-1}{0.0525}\right]$

Solutions

13. $\$5,978.08$
14. $\$1,056,767.41$
15. $\$2,156,764.06$
16. $\$41,271.46$
17. $\$83,228.60$
18. $\$2,339.07$
19. $\$26,450.25$
20. $\$14,561.43$

Attribution

2.4: AVERAGES

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $\sum$ = Summation
- $\%C$ = Percent change
- GAvg = Geometric average
- n = Number of pieces of data
- SAvg = Simple average
- w = Weight factor for a piece of data
- WAvg = Weighted average
- x = A piece of data

Formulas Used

- Formula 2.4a – **Simple Average**

$$\text{SAvg} = \frac{\sum x}{n}$$

- Formula 2.4b – **Weighted Average**

$$\text{WAvg} = \frac{\sum wx}{\sum w}$$

- Formula 2.4c – **Geometric Average**

$$\text{GAvg} = \left(\left[(1+\%C_1)\times(1+\%C_2)\times\ \ldots\ \times(1+\%C_n)\right]^{\frac{1}{n}}-1\right)\times 100$$

- Formula 3.1b – **Rate, Portion, Base** (see Section 3.1)

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

Introduction

No matter where you go or what you do, averages are everywhere. Let's look at some examples:

- Three-quarters of your student loan is spent. Unfortunately, only half of the first semester has passed, so you resolve to squeeze the most value out of the money that remains. But have you noticed that many grocery products are difficult to compare in terms of value because they are packaged in different sized containers with different price points?
  - For example, one tube of toothpaste sells in a 125 mL size for $\$1.99$ while a comparable brand sells for $\$1.89$ for 110 mL. Which is the better deal? A fair comparison requires you to calculate the average price per millilitre.
- Your local transit system charges $\$2.25$ for an adult fare, $\$1.75$ for students and seniors, and $\$1.25$ for children. Is this enough information for you to calculate the average fare, or do you need to know how many riders of each kind there are?
- Five years ago you invested $\$8,000$ in Roller Coasters Inc. The stock value has changed by 9%, -7%, 13%, 4%, and -2% over these years, and you wonder what the average annual change is and whether your investment kept up with inflation.
- If you participate in any sport, you have an average of some sort: bowlers have bowling averages; hockey or soccer goalies have a goals against average (GAA); and baseball pitchers have an earned run average (ERA).

Averages generally fall into three categories. This section explores simple, weighted, and geometric averages.

Simple Averages

An **average** is a single number that represents the middle of a data set. It is commonly interpreted to mean the "typical value." Calculating averages facilitates easier comprehension of and comparison between different data sets, particularly if there is a large amount of data. For example, what if you want to compare year-over-year sales? One approach would involve taking company sales for each of the 52 weeks in the current year and comparing these with the sales of all 52 weeks from last year. This involves 104 weekly sales figures with 52 points of comparison. From this analysis, could you concisely and confidently determine whether sales are up or down? Probably not. An alternative approach involves comparing last year's average weekly sales against this year's average weekly sales. This involves the direct comparison of only two numbers, and the determination of whether sales are up or down is very clear.

In a **simple average**, all individual data share the same level of importance in determining the typical value. Each individual data point also has the same frequency, meaning that no one piece of data occurs more frequently than another. Also, the data do not represent a percent change. To calculate a simple average, you require two components:

- The data itself—you need the value for each piece of data.
- The quantity of data—you need to know how many pieces of data are involved (the count), or the total quantity used in the calculation.

2.4a Simple Average

Formula does not parse

$rgb]0.0, 0.0, 1.0 \sum$ $rgb]0.1, 0.1, 0.1$is Summation: This symbol is known as the Greek capital letter sigma. In mathematics it denotes that all values written after it (to the right) are summed.

{\color[rgb]{1.0, 0.0, 0.0}\text{SAvg }}{\color[rgb]{0.1, 0.1, 0.1}\text{is Simple Average:}} A simple average for a data set in which all data has the same level of importance and the same frequency.

$rgb]0.0, 0.5, 0.0n$ $rgb]0.1, 0.1, 0.1$is Total Quantity: This is the physical total count of the number of pieces of data or the total quantity being used in the average calculation. In business, the symbol n is a common standard for representing counts.

$rgb]0.5, 0.0, 0.5x$ $rgb]0.1, 0.1, 0.1$is Any Piece of Data: In mathematics this symbol is used to represent an individual piece of data.

As expressed in Formula 2.4a, you calculate a simple average by adding together all of the pieces of data then taking that total and dividing it by the quantity.

HOW TO

Calculate a Simple Average

The steps required to calculate a simple average are as follows:

Step 1: Sum every piece of data.

Step 2: Determine the total quantity involved.

Step 3: Calculate the simple average using **Formula 2.4a** $\text{SAvg} = \dfrac{\sum x}{n}$.

Assume you want to calculate an average on three pieces of data: 95, 108, and 97. Note that the data are equally important and each appears only once, thus having the same frequency. You require a simple average.

Step 1: Sum all data:

$$\sum x = 95 + 108 + 97 = 300$$

Step 2: There are three pieces of data, or $n = 3$.

Step 3: Apply Formula 2.4a $\text{SAvg} = \dfrac{\sum x}{n}$:

$$\text{SAvg} = \frac{300}{3} = 100$$

The simple average of the data set is 100.

Key Takeaway

Although mentioned earlier, it is critical to stress that a simple average is calculated only when all of the following conditions are met:

- All of the data shares the same level of importance toward the calculation.
- All of the data appear the same number of times.
- The data does not represent percent changes or a series of numbers intended to be multiplied with each other.

If any of these three conditions are not met, then either a weighted or geometric average is used depending on which of the above criteria failed. We discuss this later when each average is introduced.

Try It

It is critical to recognize if you have potentially made any errors in calculating a simple average. Review the following situations and, without making any calculations, determine the best answer.

1) The simple average of 15, 30, 40, and 45 is:

a. lower than 20
b. between 20 and 40, inclusive
c. higher than 40

Solution

The best answer is b. because a simple average should fall in the middle of the data set, which appears spread out between 15 and 45, so the middle would be around 30).

Try It

2) If the simple average of three pieces of data is 20, which of the following data do not belong in the data set? Data set: $10, 20, 30, 40$

a. 10
b. 20
c. 30
d. 40

Solution
The data set that does not belong is d. If the number 40 is included in any average calculation involving the other numbers, it is impossible to get a low average of 20.

Example 2.4.1

First quarter sales for Buzz Electronics are as indicated in the table below.

Table 2.4.1

Month	2013 Sales	2014 Sales
January	$413,200	$455,875
February	$328,987	$334,582
March	$359,003	$312,777

Martha needs to prepare a report for the board of directors comparing year-over-year quarterly performance. To do this, she needs you to do the following:

a. Calculate the average sales in the quarter for each year.
b. Express the 2014 sales as a percentage of the 2013 sales, rounding your answer to two decimals.

Solution

***Step 1:* What are we looking for?**

You need to calculate a simple average, or $SAvg$, for the first quarter in each of 2013 and 2014. Then convert the numbers into a percentage.

***Step 2:* What do we already know?**

You know the three quarters annually:

$$2013: \quad x_1 = \$413,000 \;\; x_2 = \$328,986 \;\; x_3 = \$350,003$$

$$2014: \quad x_1 = \$455,876 \;\; x_2 = \$334,582 \;\; x_3 = \$312,777$$

Additionally, you know that the simple average can be obtained using **Formula 2.4a** $\text{SAvg} = \dfrac{\sum x}{n}$ and that using **Formula 3.1b** $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$ you can calculate 2014 sales as a percentage of 2013 sales by treating 2013 average sales as the base and 2014 average sales as the portion.

***Step 3:* Make substitutions using the information known above.**

Calculate simple averages for 2013 and 2014 using Formula 2.4a:

$$\text{SAvg} = \frac{\sum x}{n}$$

Simple average for 2013:

$$\text{SAvg}_{2013} = \frac{\$413,200 + \$328,986 + \$350,003}{3}$$

$$\text{SAvg}_{2013} = \frac{\$1,092,189}{3}$$

$$\text{SAvg}_{2013} = \$364,063$$

Simple average for 2014:

$$\text{SAvg}_{2014} = \frac{\$455,876 + \$334,582 + \$312,777}{3}$$

$$\text{SAvg}_{2014} = \frac{\$1,103,235}{3}$$

$$\text{SAvg}_{2014} = \$367,745$$

Finally, apply , substituting SAvg_{2013} for Base and SAvg_{2014} for Portion and multiplying by 100 to obtain percentage. Round the result to two decimal places:

$$\% = \frac{\text{Portion}}{\text{Base}} \times 100$$

$$\% = \frac{\$367,745}{\$364,063} \times 100$$

$$\% = 101.01\%$$

***Step 4:* Provide the information in a worded statement.**

The average monthly sales in 2013 were $\$364,063$ compared to sales in 2014 of $\$367,745$. This means that 2014 sales are 101.01% of 2013 sales.

Weighted Averages

Have you considered how your grade point average (GPA) is calculated? Your business program requires the successful completion of many courses. Your grades in each course combine to determine your GPA; however, not every course necessarily has the same level of importance as measured by your course credits.

Perhaps your math course takes one hour daily while your communications course is only delivered in one-hour sessions three times per week. Consequently, the college assigns the math course five credit hours and the communications course three credit hours. If you want an average, these different credit hours mean that the two courses do not share the same level of importance, and therefore a simple average cannot be calculated.

In a **weighted average**, not all pieces of data share the same level of importance or they do not occur with the same frequency. The data cannot represent a percent change or a series of numbers intended to be multiplied with each other. To calculate a weighted average, you require two components:

- The data itself—you need the value for each piece of data.
- The weight of the data—you need to know how important each piece of data is to the average. This is either an assigned value or a reflection of the number of times each piece of data occurs (the frequency).

2.4b Weighted Average

Formula does not parse

{\color[rgb]{1.0, 0.0, 0.0}\text{WAvg}}{\color[rgb]{0.1, 0.1, 0.1}\text{ is Weighted Average:}} An average for a data set where the data points may not all have the same level of importance or they may occur at different frequencies.

\color[rgb]{0.0, 0.0, 1.0}\sum{\color[rgb]{0.1, 0.1, 0.1}\text{ is Summation:}} This symbol is known as the Greek capital letter sigma. In mathematics it denotes that all values written after it (to the right) are summed.

\color[rgb]{0.0, 0.5, 0.0}w{\color[rgb]{0.1, 0.1, 0.1}\text{ is Weighting Factor:}} A number that represents the level of importance for each piece of data in a particular data set. It is either predetermined or reflective of the frequency for the data.

$rgb]0.5, 0.0, 0.5x$ $rgb]0.0, 0.0, 0.0$is Any Piece of Data: In mathematics this symbol is used to represent an individual piece of data.

As expressed in Formula 2.4b, calculate a weighted average by adding the products of the weights and data for the entire data set and then dividing this total by the total of the weights.

HOW TO

Calculate a Weighted Average

The steps required to calculate a weighted average are:

Step 1: Sum every piece of data multiplied by its associated weight.

Step 2: Sum the total weight.

Step 3: Calculate the weighted average using **Formula 2.4b** $\text{WAvg} = \dfrac{\sum wx}{\sum w}$.

Let's stay with the illustration of the math and communications courses and your GPA. Assume that these are the only two courses you are taking. You finish the math course with an A, translating into a grade point of 4.0 . In the communications course, your C+ translates into a 2.5 grade point. These courses have five and three credit hours, respectively. Since they are not equally important, you use a weighted average.

Step 1: In the numerator, sum the products of each course's credit hours (the weight) and your grade point (the data). This means:

$$(\text{math credit hours} \times \text{math grade point}) + (\text{communications credit hours} \times \text{communications grade point})$$

.

Numerically, this is:

$$\sum wx = (5 \times 4) + (3 \times 2.5) = 27.5$$

Step 2: In the denominator, sum the weights. These are the credit hours. You have:

$$\sum w = 5 + 3 = 8$$

Step 3: Apply Formula 2.4b $\text{WAvg} = \dfrac{\sum wx}{\sum w}$ to calculate your GPA.

$$\text{WAvg} = \frac{\sum wx}{\sum w}$$

$$\text{WAvg} = \frac{27.5}{8}$$

$$\text{WAvg} = 3.44 \text{ (GPAs have two decimals)}$$

Note that your GPA is higher than if you had just calculated a simple average:

$$\text{SAvg} = \frac{\sum x}{n}$$

$$\text{SAvg} = \frac{4 + 2.4}{2}$$

$$\text{SAvg} = 2.25$$

This happens because your math course, in which you scored a higher grade, was more important in the calculation.

Things To Watch Out For

The most common error in weighted averages is to confuse the data with the weight. If you have the two backwards, your numerator is still correct; however, your denominator is incorrect. To distinguish the data from the weight, notice that the data forms a part of the question. In the above example, you were looking to calculate your **grade point** average; therefore, grade point is the data. The other information, the credit hours, must be the weight.

Paths To Success

The formula used for calculating a simple average is a simplification of the weighted average formula. In a simple average, every piece of data is equally important. Therefore, you assign a value of 1 to the weight for each piece of data. Since any number multiplied by 1 is the same number, the simple average formula omits the weighting in the numerator as it would have produced unnecessary calculations. In the denominator, the sum of the weights of 1 is no different from counting the total number of pieces of data. In essence, you can use a weighted average formula to solve simple averages.

Try It

Determine which information is the data and which is the weight.

3) Rafiki operates a lemonade stand during his garage sale today. He has sold 13 small drinks for $\$0.50$, 29 medium drinks for $\$0.90$, and 21 large drinks for $\$1.25$. What is the average price of the lemonade sold?

Solution
The price of the drinks is the data, and the number of drinks is the weight.

Try It

4) Natalie received the results of a market research study. In the study, respondents identified how many times per week they purchased a bottle of Coca-Cola. Calculate the average number of purchases made per week.

Table 2.4.2

Purchases per Week	# of People
1	302
2	167
3	488
4	256

Solution
The purchases per week is the data, and the number of people is the weight.

Example 2.4.2

A mark transcript received by a student at a local college:

Table 2.4.3

Course	Grade	Credit Hours
Economics 100	B	4
Math 100	A	5
Marketing 100	B+	3
Communications 100	C	4
Computing 100	A+	3
Accounting 100	D	4

This chart shows how each grade translates into a grade point:

Table 2.4.4

Grade	Grade Point
A+	4.5
A	4.0
B+	3.5
B	3.0
C+	2.5
C	2.0
D	1.0
F	0.0

Calculate the student's grade point average (GPA). Round your final answer to two decimals.

Solution

Step 1: **What are we looking for?**

The courses do not carry equal weights as they have different credit hours. Therefore, to calculate the GPA you must find a weighted average, or **WAvg**.

Step 2: **What do we already know?**

Since the question asked for the grade point average, the grade points for each course are the data, or x. The corresponding credit hours are the weights, or w. This information can be substituted into **Formula 2.4b** $\text{WAvg} = \dfrac{\sum wx}{\sum w}$ to find the weighted average.

Step 3: **Make substitutions using the information known above.**

Use the secondary table above to convert each course grade into its grade point:

Table 2.4.5

Course	Grade	Grade Point	Credit Hours
Economics 100	B	3.0	4
Math 100	A	4.0	5
Marketing 100	B+	3.5	3
Communications 100	C	2.0	4
Computing 100	A+	4.5	3
Accounting 100	D	1.0	4

Sum every piece of data multiplied by its associated weight:

$$\sum wx = \sum(\text{Credit Hours} \times \text{Grade Point})$$
$$\sum wx = (4 \times 3.0) + (5 \times 4.0) + (3 \times 3.5) + (4 \times 2.0) + (4 \times 4.5) + (4 \times 1.0)$$
$$\sum wx = 68$$

Sum the total weight:

$$\sum w = 4 + 5 + 3 + 4 + 3 + 4$$
$$\sum w = 23$$

Substitute into Formula 2.4b:

$$\text{WAvg} = \frac{\sum wx}{\sum w}$$

$$\text{WAvg} = \frac{68}{23}$$

$$\text{WAvg} = 2.96$$

Step 4: **Provide the information in a worded statement.**

The student's GPA is 2.96. Note that math contributed substantially (almost one-third) to the student's grade point because this course was weighted heavily and the student performed well.

Example 2.4.3

Angelika started the month of March owing $\$20,000$ on her home equity line of credit (HELOC). She made a payment of $\$5,000$ on the fifth, borrowed $\$15,000$ on the nineteenth, and made another payment of $\$5,000$ on the twenty-sixth. Using each day's closing balance for your calculations, what was the average balance in the HELOC for the month of March?

Solution

Step 1: **What are we looking for?**

The balance owing in Angelika's HELOC is not equal across all days in March. Some balances were carried for more days than others. This means you will need to use the weighted average technique and find WAvg.

Step 2: **What do we already know?**

You know the following:

Table 2.4.6

Dates	**Number of Days (w)**	**Balance in HELOC (x)**
March 1 - March 4	4	$20,000
March 5 - March 18	14	$20,000 — $5,000 = $15,000
March 19 - March 25	7	$15,000 + $15,000 = $30,000
March 26 - March 31	6	$30,000 — $5,000 = $25,000

Step 3: **Make substitutions using the information known above.**

Sum every piece of data multiplied by its associated weight:

$$\sum wx = (4 \times \$20,000) + (14 \times \$15,000) + (7 \times \$30,000) + (6 \times \$25,000)$$
$$\sum wx = \$650,000$$

Sum the total weight:

$$\sum w = 4 + 14 + 7 + 6$$
$$\sum w = 31$$

Calculate the weighted average using Formula 2.4b:

$$\text{WAvg} = \frac{\sum wx}{\sum w}$$
$$\text{WAvg} = \frac{\$650,000}{331}$$
$$\text{WAvg} = \$20,967.74$$

Step 4: **Provide the information in a worded statement.**

Over the entire month of March, the average balance owing in the HELOC was $\$20,967.74$. Note that the balance with the largest weight (March 5 to March 18) and the largest balance owing (March 19 to March 25) account for almost two-thirds of the calculated average.

Geometric Averages

How do you average a percent change? If sales increase 100% this year and decrease 50% next year, is the average change in sales an increase of

$$\frac{100\% - 50\%}{2} = 25\%$$

per year? The answer is clearly "no." If sales last year were $\$100$ and they increased by 100%, that results in a $\$100$ increase. The total sales are now $\$200$. If sales then decreased by 50%, you have a $\$100$ decrease. The total sales are now $\$100$ once again. In other words, you started with $\$100$ and finished with $\$100$. That is an average change of nothing, or 0% per year! Notice that the second percent change is, in fact, multiplied by the result of the first percent change. A **geometric average** finds the typical value for a set of numbers that are meant to be multiplied together or are exponential in nature.

In business mathematics, you most commonly use a geometric average to average a series of percent changes. Formula 2.4c is specifically written to address this situation.

2.4c Geometric Average

Formula does not parse

$rgb]0.0, 0.5, 0.0n$ $rgb]0.1, 0.1, 0.1$is Total Quantity: The physical total count of how many percent changes are involved in the calculation.

{\color[rgb]{1.0, 0.0, 0.0}\text{GAvg}}{\color[rgb]{0.1, 0.1, 0.1}\text{ is Geometric Average:}} The average of a series of percent changes expressed in percent format. Every percent change involved in the calculation requires an additional $(1 + \%C)$ to be multiplied under the radical. The formula accommodates as many percent changes as needed.

$rgb]0.5, 0.0, 0.5 \times 100$ $rgb]0.1, 0.1, 0.1$is Percent Conversion: Because you are averaging percent changes, convert the final result from decimal form into a percentage.

$rgb]0.0, 0.0, 1.0\%C$ $rgb]0.1, 0.1, 0.1$is Percent Change: The value of each percent change in the series from which the average is calculated. You need to express the percent changes in decimal format.

HOW TO

Calculate a Geometric Average

To calculate a geometric average follow these steps:

Step 1: Identify the series of percent changes to be multiplied.

Step 2: Count the total number of percent changes involved in the calculation.

Step 3: Calculate the geometric average using **Formula 2.4c**

$$\text{GAvg} = \left([(1+\%C_1)\times(1+\%C_2)\times \ldots \times(1+\%C_n)]^{\frac{1}{n}} - 1\right)\times 100.$$

Let's use the sales data presented above, according to which sales increase 100% in the first year and decrease 50% in the second year. What is the average percent change per year?

Step 1: The changes are $\%C_1 = +100\%$ and $\%C_2 = -50\%$.

Step 2: Two changes are involved, or $n = 2$.

Step 3: Apply Formula 2.4c:

$$\text{GAvg} = \left([(1+\%C_1)\times\ \ldots\ \times(1+\%C_n)]^{\frac{1}{n}} - 1\right)\times 100$$

$$\text{GAvg} = \left([(1+100\%)\times(1-50\%)]^{\frac{1}{2}} - 1\right)\times 100$$

$$\text{GAvg} = \left([2\times 0.50]^{\frac{1}{2}} - 1\right)\times 100$$

$$\text{GAvg} = \left([1]^{\frac{1}{2}} - 1\right)\times 100$$

$$\text{GAvg} = 0\%$$

The average percent change per year is 0% because an increase of 100% and a decrease of 50% cancel each other out.

Thing To Watch Out For

A critical requirement of the geometric average formula is that every $(1 + \%C)$ expression must result in a number that is positive. This means that the $\%C$ cannot be a value less than 100% else **Formula 2.4c**

$$\text{GAvg} = \left([(1 + \%C_1) \times (1 + \%C_2) \times \ \ldots \ \times (1 + \%C_n)]^{\frac{1}{n}} - 1 \right) \times 100$$

cannot be used.

Paths To Success

An interesting characteristic of the geometric average is that it will always produce a number that is either smaller than (closer to zero) or equal to the simple average. In the example, the simple average of $+100\%$ and 50% is 25%, and the geometric average is 0%. This characteristic can be used as an error check when you perform these types of calculations.

Try It

Determine whether you should calculate a simple, weighted, or geometric average.

5) Randall bowled 213, 245, and 187 in his Thursday night bowling league and wants to know his average.

Solution
Simple; each item has equal importance and frequency.

Try It

Determine whether you should calculate a simple, weighted, or geometric average.

6) Cindy invested in a stock that increased in value annually by 5%, 6%, 3%, and 5%. She wants to know her average increase.

Solution
Geometric; these are a series of percent changes on the price of stock.

Try It

Determine whether you should calculate a simple, weighted, or geometric average.

7) A retail store sold 150 bicycles at the regular price of $\$300$ and 50 bicycles at a sale price of $\$200$. The manager wants to know the average selling price.

Solution
Weighted; each item has a different frequency.

Try It

8) Gonzalez has calculated a simple average of 50% and a geometric average of 60%. He believes his numbers are correct. What do you think?

Solution

At least one of the numbers is wrong since a geometric average is always smaller than or equal to the simple average.

Examples 2.4.4

From 2006 to 2010, WestJet's year-over-year annual revenues changed by $+21.47\%$, $+19.89\%$, 10.55%, and $+14.38\%$. This reflects growth from sales of $\$1.751$ billion in 2006 to $\$2.609$ billion in 2010.[1] What is the average percent growth in revenue for WestJet during this time frame?

Solution

Step 1: **What are we looking for?**

Note that these numbers reflect percent changes in revenue. Year-over-year changes are multiplied together, so you would calculate a geometric average, or GAvg.

Step 2: **What do we already know?**

You know the four percent changes:

$$\begin{aligned}\%C_1 &= 21.47\% \\ \%C_2 &= 19.89\% \\ \%C_3 &= 10.55\%\end{aligned}$$

$$\%C_4 = 14.38\%$$

You also know that four changes are involved, or $n = 4$.

Step 3: **Make substitutions using the information known above.**

Express the percent changes in decimal format and substitute into Formula 2.4c:

$$\text{GAvg} = \left([(1+\%C_1) \times \ . \ . \ . \ \times (1+\%C_n)]^{\frac{1}{n}} - 1\right) \times 100$$

$$\text{GAvg} = \left([(1+0.2147) \times (1+0.1989) \times (1-0.1055) \times (1+0.1438)]^{\frac{1}{4}} - 1\right) \times 100$$

$$\text{GAvg} = 10.483\%$$

Step 4: **Provide the information in a worded statement.**

On average, WestJet revenues have grown 10.483% each year from 2006 to 2010.

Section 2.4 Exercises

Mechanics

Calculate a simple average for questions 1 and 2.

1. 8, 17, 6, 33, 15, 12, 13, 16
2. $\$1,500$, $\$2,000$, $\$1,750$, $\$1,435$, $\$2,210$

Calculate a weighted average for questions 3 and 4.

3. 4, 4, 4, 4, 12, 12, 12, 12, 12, 12, 12, 15, 15
4.

Table 2.4.7

Data	\$3,600	\$3,300	\$3,800	\$2,800	\$5,800
Weight	2	5	3	6	4

Calculate a geometric average for exercises 5 and 6. Round all percentages to four decimals.

Table 2.4.8

5.	5.4%	8.7%	6.3%	
6.	10%	4%	17%	10%

Solutions

1. 15
2. $\$1,779$
3. 10
4. $\$3,795$
5. 6.7910%
6. 2.6888%

Applications

7. If a 298 mL can of soup costs $\$2.39$, what is the average price per millilitre?
8. Kerry participated in a fundraiser for the Children's Wish Foundation yesterday. She sold 115 pins for $\$3$ each, 214 ribbons for $\$4$ each, 85 coffee mugs for $\$7$ each, and 347 baseball hats for $\$9$ each. Calculate the average amount Kerry raised per item.
9. Stephanie's mutual funds have had yearly changes of 9.63%, 2.45%, and 8.5%. Calculate the annual average change in her investment.
10. In determining the hourly wages of its employees, a company uses a weighted system that factors in local, regional, and national competitor wages. Local wages are considered most important and have been assigned a weight of 5. Regional and national wages are

not as important and have been assigned weights of 3 and 2, respectively. If the hourly wages for local, regional, and national competitors are $\$16.35$, $\$15.85$, and $\$14.75$, what hourly wage does the company pay?

11. Canadian Tire is having an end-of-season sale on barbecues, and only four floor models remain, priced at $\$299.97$, $\$345.49$, $\$188.88$, and $\$424.97$. What is the average price for the barbecues?
12. Calculate the grade point average (GPA) for the following student. Round your answer to two decimals.

Table 2.4.9

Course	Grade	Credit Hours
Economics 100	D	5
Math 100	B	3
Marketing 100	C	4
Communications 100	A	2
Computing 100	A+	3
Accounting 100	B+	4

Grade	Grade Point	Grade	Grade Point
A+	4.5	C+	2.5
A	4.0	C	2.0
B+	3.5	D	1.0
B	3.0	F	0.0

13. An accountant needs to report the annual average age (the length of time) of accounts receivable (AR) for her corporation. This requires averaging the monthly AR averages, which are listed below. Calculate the annual AR average.

Table 2.4.10

Month	Monthly AR Average	Month	Monthly AR Average	Month	Monthly AR Average
January	$45,000	May	$145,000	September	$185,000
February	$70,000	June	$180,000	October	$93,000
March	$85,000	July	$260,000	November	$60,000
April	$97,000	August	$230,000	December	$50,000

14. From January 2007 to January 2011, the annual rate of inflation has been 2.194%, 1.073%, 1.858%, and 2.346%. Calculate the average rate of inflation during this period.

Solutions

7. $\$0.00802/\mathrm{ml}$
8. $\$6.46$
9. 5.0821%
10. $\$15.88$
11. $\$314.83$
12. 2.74
13. $\$125,000$
14. 1.8666%

Challenge, Critical Thinking, & Other Applications

15. Gabrielle is famous for her trail mix recipe. By weight, the recipe calls for 50% pretzels, 30% Cheerios, and 20% peanuts. She wants to make a 2 kg container of her mix. If pretzels cost $\$9.99/\mathrm{kg}$, Cheerios cost $\$6.99/\mathrm{kg}$, and peanuts cost $\$4.95/\mathrm{kg}$, what is the average cost per 100 g rounded to four decimals?
16. Caruso is the marketing manager for a local John Deere franchise. He needs to compare his average farm equipment sales against his local Case IH competitor's sales. In the past three months, his franchise has sold six $\$375,000$ combines, eighteen $\$210,000$ tractors, and fifteen $\$120,000$ air seeders. His sales force estimates that the Case IH

dealer has sold four $\$320,000$ combines, twenty-four $\$225,000$ tractors, and eleven $\$98,000$ air seeders. Express the Case IH dealer's average sales as a percentage of the John Deere dealer's average sales.

17. You are shopping for shampoo and consider two brands. Pert is sold in a bundle package of two 940 mL bottles plus a bonus bottle of 400 mL for $\$13.49$. Head & Shoulders is sold in a bulk package of three 470 mL bottles plus a bonus bottle of 280 mL for $\$11.29$.
    a. Which package offers the best value?
    b. If the Head & Shoulders increases its package size to match Pert at the same price per mL, how much money do you save by choosing the lowest priced package?

18. The following are annual net profits (in millions of dollars) over the past four years for three divisions of Randy's Wholesale:

    Cosmetics: $\$4.5$, $\$5.5$, $\$5.65$, $\$5.9$

    Pharmaceutical: $\$15.4$, $\$17.6$, $\$18.5$, $\$19.9$

    Grocery: $\$7.8$, $\$6.7$, $\$9.87$, $\$10.75$

    Rank the three divisions from best performing to worst performing based on average annual percent change.

19. You are shopping for a Nintendo Wii gaming console and visit www.shop.com, which finds online sellers and lists their prices for comparison. Based on the following list, what is the average price for a gaming console (rounded to two decimals)?

Table 2.4.11

NothingButSoftware.com	$274.99
eComElectronics	$241.79
NextDayPC	$241.00
Ecost.com	$249.99
Amazon	$169.99
eBay	$165.00
Buy.com	$199.99
HSN	$299.95
Gizmos for Life	$252.90
Toys 'R' Us	$169.99
Best Buy	$169.99
The Bay	$172.69
Walmart	$169.00

20. Juanita receives her investment statement from her financial adviser at Great-West Life. Based on the information below, what is Juanita's average rate of return on her investments?

Table 2.4.12

Investment Fund	Proportion of Entire Portfolio Invested in Fund	Fund Rate of Return
Real Estate	0.176	8.5%
Equity Index	0.073	36.2%
Mid Cap Canada	0.100	−1.5%
Canadian Equity	0.169	8.3%
US Equity	0.099	−4.7%
US Mid Cap	0.091	−5.7%
North American Opportunity	0.063	2.5%
American Growth	0.075	−5.8%
Growth Equity	0.085	26.4%
International Equity	0.069	−6.7%

Solutions

15. $\$0.8082$

16. 99.0805%

17a. Pert better; Pert=$\$0.005916/\text{ml}$; H&S=$\$0.006680/\text{ml}$

17b. Pert saves $\$1.74$

18. Grocery 11.2853%; Cosmetics 9.4493%; Pharmaceuticals 8.9208%
19. $\$213.64$
20. 5.9115%

1 WestJet, *WestJet Fact Sheet*.

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $\text{SAvg} = \frac{\sum x}{n}$

$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$ $\text{WAvg} = \frac{\sum wx}{\sum w}$

$\text{GAvg} = \left([(1+\%C_1) \times (1+\%C_2) \times \ \ldots \ \times (1+\%C_n)]^{\frac{1}{n}} - 1 \right) \times 100$

Attribution

2.5: ALGEBRAIC EXPRESSIONS

The Pieces of the Puzzle

If you are like most Canadians, your employer pays you biweekly. Assume you earn $\$12.00$ per hour. How do you calculate your pay cheque every pay period? Your earnings are calculated as follows:

$$\$12.00\;\times\;\text{(Hours worked during the biweekly pay period)}$$

The hours worked during the biweekly pay period is the unknown variable. Notice that the expression appears lengthy when you write out the explanation for the variable. Algebra is a way of making such expressions more convenient to manipulate. To shorten the expression, making it easier to read, algebra assigns a letter or group of letters to represent the variable. In this case, you might choose h to represent "hours worked during the biweekly pay period." This rewrites the above expression as follows:

$$\$12.00 \times h \text{ or } \$12h$$

Unfortunately, the word **algebra** makes many people's eyes glaze over. But remember that algebra is just a way of solving a numerical problem. It demonstrates how the pieces of a puzzle fit together to arrive at a solution.

For example, you have used your algebraic skills if you have ever programmed a formula into Microsoft Excel. You told Excel there was a relationship between cells in your spreadsheet. Perhaps your calculation required cell A3 to be divided by cell B6 and then multiplied by cell F2. This is an algebraic equation. Excel then took your algebraic equation and calculated a solution by automatically substituting in the appropriate values from the referenced cells (your variables).

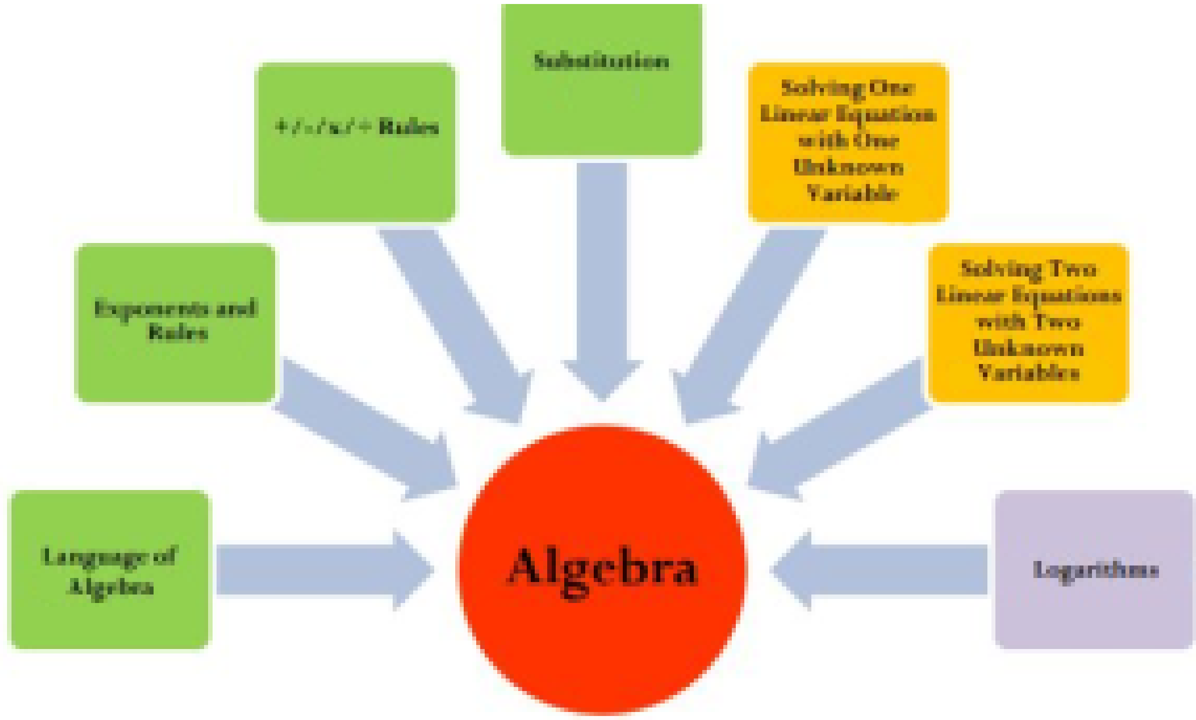

Figure 2.5.1

As illustrated in the figure above, algebra involves integrating many interrelated concepts. This figure shows only the concepts that are important to business mathematics, which this textbook will present piece by piece. Your understanding of algebra will become more complete as more concepts are covered over the course of this book.

This section reviews the language of algebra, exponent rules, basic operation rules, and substitution. In Section 2.6 you will put these concepts to work in solving one linear equation for one unknown variable along with two linear equations with two unknown variables.

The Language of Algebra

Understanding the rules of algebra requires familiarity with four key definitions.

Algebraic Expression

A mathematical **algebraic expression** indicates the relationship between mathematical operations that must be conducted on a series of numbers or variables. For example, the expression $\$12h$ says that you must take the hourly wage of $\$12$ and multiply it by the hours worked. Note that the expression does not include an equal sign, or "$=$". It only tells you what to do and requires that you substitute a value in for the unknown variable(s) to solve. There is no one definable solution to the expression.

Algebraic Equation

rgb]1.0, 0.0, 0.06rgb]1.0, 0.0, 0.0xrgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0y rgb]0.0, 0.5, 0.0 = rgb]0.0, 0.0, 1.04rgb]0.0, 0.0, 1.0xrgb]0.0, 0.0, 1.0+rgb]0.0, 0.0, 1.03

A mathematical **algebraic equation** takes two algebraic expressions and equates them. This equation can be solved to find a solution for the unknown variables. Examine the equation above to see how algebraic expressions and algebraic equations are interrelated.

rgb]1.0, 0.0, 0.06rgb]1.0, 0.0, 0.0xrgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0yrgb]1.0, 0.0, 0.0 is the First Algebraic Expression: This expression on the left-hand side of the equation expresses a relationship between the variables x and y. By itself, any value can be substituted for x and y and any solution can be calculated.

{\color[rgb]{0.0, 0.0, 1.0}4}{\color[rgb]{0.0, 0.0, 1.0}x}{\color[rgb]{0.0, 0.0, 1.0}+}{\color[rgb]{0.0, 0.0, 1.0}3}{\color[rgb]{0.0, 0.0, 1.0}\;}\text{is the Second Algebraic Expression:} This expression on the right-hand side of the equation expresses a relationship between x and a number. By itself, any value can be substituted for x and any solution can be generated.

{\color[rgb]{0.0, 0.5, 0.0}=}{\color[rgb]{0.0, 0.5, 0.0}\;}\text{is the Equality:} By having the equal sign placed in between the two algebraic expressions, they now become an algebraic equation. Now there would be a particular value for x and a particular value for y that makes both expressions equal to each other (try $x = -0.75$ and $y = 1.125$).

Term

In any algebraic expression, **terms** are the components that are separated by addition and subtraction. In looking at the example above, the expression $6x + 3y$ is composed of two terms. These terms are "$6x$" and "$3y$". A **nomial** refers to how many terms appear in an algebraic expression. If an algebraic expression contains only one term, like "$\$12.00h$," it is called a **monomial**. If the expression contains two terms or more, such as "$6x + 3y$," it is called a **polynomial**.

Factor

Terms may consist of one or more **factors** that are separated by multiplication or division signs. Using the $6x$ from above, it consists of two factors. These factors are "6" and "x"; they are joined by multiplication.

- If the factor is numerical, it is called the **numerical coefficient**.

- If the factor is one or more variables, it is called the **literal coefficient**.

The equation and info below shows how algebraic expressions, algebraic equations, terms, and factors all interrelate within an equation.

$$rgb]1.0, 0.0, 0.0\frac{7x}{3} + rgb]0.0, 0.0, 1.04rgb]0.0, 0.0, 1.0xrgb]0.0, 0.0, 1.0y^{rgb]0.0,0.0,1.02}rgb]0.0, 0.5, 0.0 = rgb]1.0, 0.0, 1.0x^{rgb]1.0,0.0,1.03}rgb]0.68, 0.46, 0.12 - rgb]0.68, 0.46, 0.122rgb]0.68, 0.46, 0.12y$$

$rgb]1.0, 0.0, 0.0\frac{7x}{3}$ First Term: This term consists of two factors, where the numerical coefficient is $\frac{7}{3}$ and the literal coefficient is x.

\underbrace{{\color[rgb]{1.0, 0.0, 0.0}\frac{7x}3}+{\color[rgb]{0.0, 0.0, 1.0}4}{\color[rgb]{0.0, 0.0, 1.0}x}{\color[rgb]{0.0, 0.0, 1.0}y}^{\color[rgb]{0.0, 0.0, 1.0}2}}\;\text{First Algebraic Expression:}

This first polynomial expression is composed of the two terms $\frac{7x}{3}$ and $4xy^2$.

$rgb]0.0, 0.0, 1.04rgb]0.0, 0.0, 1.0xrgb]0.0, 0.0, 1.0y^{rgb]0.0,0.0,1.02}$ Second Term: This term consists of three factors where the numerical coefficient is 4 and the literal coefficients are x and y^2.

{\color[rgb]{0.0, 0.5, 0.0}=}\;\text{Equal Sign:} The equal sign joins the two algebraic expressions, turning this into an algebraic equation.

{\color[rgb]{0.1, 0.1, 0.1}\underbrace{{\color[rgb]{1.0, 0.0, 1.0}x}^{\color[rgb]{1.0, 0.0, 1.0}3}{\color[rgb]{0.68, 0.46, 0.12}-}{\color[rgb]{0.68, 0.46, 0.12}2}{\color[rgb]{0.68, 0.46, 0.12}y}}}{\color[rgb]{0.68, 0.46, 0.12}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{Second Algebraic Expression:}}

This second polynomial expression consists of the two terms x^3 and $-2y$.

{\color[rgb]{1.0, 0.0, 1.0}x}^{\color[rgb]{1.0, 0.0, 1.0}3}\;\text{First Term:} This term consists of two factors, which are the literal coefficient x^3 and a numerical coefficient of 1 (recall that in math you do not write the 1 when writing $1x^3$ since it doesn't change anything).

{\color[rgb]{0.68, 0.46, 0.12}-}{\color[rgb]{0.68, 0.46, 0.12}2}{\color[rgb]{0.68, 0.46, 0.12}y}\;\text{Second Term:}

This term consists of two factors, where the numerical coefficient is -2 and the literal coefficient is y.

Exponents

Exponents are widely used in business mathematics and are integral to financial mathematics. When applying compounding interest rates to any investment or loan, you must use exponents. **Exponents** are a mathematical shorthand notation that indicates how many times a quantity is multiplied by itself. The format of an exponent is illustrated below.

$$rgb]0.0, 0.0, 1.0\text{Base}^{rgb]1.0,0.0,0.0\text{Exponent}} rgb]0.1, 0.1, 0.1 = rgb]0.0, 0.5, 0.0\text{Power}$$

{\color[rgb]{0.0, 0.0, 1.0}\text{Base }}{\color[rgb]{0.1, 0.1, 0.1}\text{is the quantity:}} This is the quantity that is to be multiplied by itself.

{\color[rgb]{1.0, 0.0, 0.0}\text{Exponent }}{\color[rgb]{0.1, 0.1, 0.1}\text{is the multiplication factor:}}

This is the number that indicates how many times the base is to be multiplied by itself.

$rgb]0.0, 0.5, 0.0\text{Power } rgb]0.1, 0.1, 0.1\text{is the product:}$ This is the result of performing the multiplication indicated, or the answer.

Assume you have $2^3 = 8$. The exponent of 3 says to take the base of 2 multiplied by itself three times, or $2 \times 2 \times 2$. The power is 8. The proper way to state this expression is "2 to the exponent of 3 results in a power of 8."

HOW TO

Simplify Exponents

Apply the rules for simplifying exponents as shown in the table below.

Table 2.

Rule	**Formula**
1. Multiplication	$y^a \times y^b = y^{a+b}$
2. Division	$\frac{y^a}{y^b} = y^{a-b}$
3. Raising powers to exponents	$(y^b z^c)^a = y^{b \times a} z^{c \times a}$ or $\left(\frac{y^b}{z^c}\right)^a = \frac{y^{b \times a}}{z^{c \times a}}$
4. Zero exponents	$y^0 = 1$
5. Negative exponents	$y^{-a} = \frac{1}{y^a}$
6. Fractional exponents	$y^{\frac{a}{b}} = \sqrt[b]{y^a}$

Key Takeaway

Recall that mathematicians do not normally write the number 1 when it is multiplied by another factor because it doesn't change the result. The same applies to exponents. If the exponent is a 1, it is generally not written because any number multiplied by itself only once is the same number. For example, the number 2 could be written as 2^1, but the power is still 2. Or take the case of $(yz)^a$. This could be written as $(y^1 z^1)^a$, which when simplified becomes $y^{1\times a} y^{1\times a}$ or $y^a z^a$. Thus, even if you don't see an exponent written, you know that the value is 1.

Example 2.5.1

Simplify the following expressions:

a. $h^3 \times h^6$
b. $\dfrac{h^{14}}{h^8}$
c. $\left[\dfrac{hk^5m^3}{n^4}\right]^3$
d. 1.49268^0
e. $\dfrac{x^2y^4}{xy^{-2}}$
f. $6^{\frac{3}{5}}$

Solution

a.

***Step 1:* Figure out the rule needed from the table above to answer the question.**

This expression involves multiplying two powers with the same base.

Apply Rule #1

***Step 2:* Solve using Rule #1 – Multiplication.**

$$\begin{aligned} &= h^3 \times h^6 \\ &= h^{3+6} \\ &= h^9 \end{aligned}$$

---

b.

***Step 1:* Figure out the rule needed from the table above to answer the question.**

This expression involves dividing two powers with the same base.

Apply Rule #2

***Step 2:* Solve using Rule #2 – Division.**

$$\begin{aligned} &= \frac{h^{14}}{h^8} \\ &= h^{14-8} \\ &= h^6 \end{aligned}$$

---

c.

***Step 1:* Figure out the rule needed from the table above to answer the question.**

This expression involves a single term, with products and a quotient all raised to an exponent.

Apply Rule #3

***Step 2:* Solve using Rule #3 – Raising Powers to Exponents.**

$$= \left[\frac{hk^5m^3}{n^4}\right]^3$$

$$= \frac{h^{1\times3}k^{5\times3}m^{3\times3}}{n^{4\times3}}$$

$$= \frac{h^3k^{15}m^9}{n^{12}}$$

---

d.

Step 1: **Figure out the rule needed from the table above to answer the question.**

This power involves a zero exponent.

Apply Rule #4

Step 2: **Solve using Rule #4 – Zero exponents.**

$$1.49268^0 = 1$$

---

e.

Step 1: **Figure out the rule needed from the table above to answer the question.**

This expression involves multiplication, division, and negative exponents.

Apply Rules #1, #2, and #5

Step 2: **Solve using Rule #1, #2, #5 – Multiplication, Division and Negative exponents**

$$= \frac{x^2y^4}{xy^{-2}}$$

$$= \frac{x^2y^4y^2}{x^1} \text{ Rule \#5}$$

$$= \frac{x^2y^{4+2}}{x^1} \text{ Rule \#1}$$

$$= \frac{x^2y^6}{x^1}$$

$$= x^{2-1}y^6 \text{ Rule \#2}$$

$$= xy^6$$

---

f.

Step 1: **Figure out the rule needed from the table above to answer the question.**

This power involves a fractional exponent.

Apply Rule #6

Step 2: **Solve using Rule #6 – Fractional Exponents.**

$$= 6^{\frac{3}{5}}$$
$$= 2.930156$$

Addition and Subtraction

Simplification of unnecessarily long or complex algebraic expressions is always preferable to increase understanding and reduce the chances of error. For example, assume you are a production manager looking to order bolts for a product that you make. Your company makes three products, all in equal quantity. Product A requires seven bolts, Product B requires four bolts, and Product C requires fourteen bolts. If q represents the quantity of products required, you need to order $7q + 4q + 14q$ bolts. This expression requires four calculations to solve every time (each term needs to be multiplied by q and you then need to add everything together). With the algebra rules that follow, you can simplify this expression to $25q$. This requires only one calculation to solve. So what are the rules?

HOW TO

Add or Subtract like terms

In math, terms with the same literal coefficients are called **like terms**. Only terms with the identical literal coefficients may be added or subtracted through the following procedure:

Step 1: Simplify any numerical coefficients by performing any needed mathematical operation or converting fractions to decimals.

For example, terms such as $\frac{1}{2}y$ should become $0.5y$.

Step 2: Add or subtract the numerical coefficients of like terms as indicated by the operation while obeying the rules of **BEDMAS**.

Step 3: Retain and do not change the common literal coefficients. Write the new numerical coefficient in front of the retained literal coefficients.

From the previous example, you require $7q + 4q + 14q$ bolts. Notice that there are three terms, each of which has the same literal coefficient. Therefore, you can perform the required addition.

Step 1: All numerical coefficients are simplified already. Skip to ***Step 2***.

Step 2: Take the numerical coefficients and add the numbers: $7 + 4 + 14$ equals 25.

Step 3: Retain the literal coefficient of q. Put the new numerical coefficient and literal coefficient together. Thus, $25q$. Therefore $7q + 4q + 14q$ is the same as $25q$.

Things To Watch Out For

A common mistake in addition and subtraction is combining terms that do not have the same literal coefficient. You need to remember that the literal coefficient *must* be identical. For example, $7q$ and $4q$ have the identical literal coefficient of q. However, $7q$ and $4q^2$ have different literal coefficients, q and q^2, and cannot be added or subtracted.

Paths To Success

Remember that if you come across a literal coefficient with no number in front of it, that number is assumed to be a 1. For example, x has no written numerical coefficient, but it is the same as $1x$. Another example would be $\frac{x}{4}$ is the same as $\frac{1x}{4}$ or $\frac{1}{4}x$. On a similar note, mathematicians also don't write out literal coefficients that have an exponent of zero. For example, 7x^0 is just $7(1)$ or 7. Thus, the literal coefficient is always there; however, it has an exponent of zero. Remembering this will help you later when you multiply and divide in algebra.

Try it

1) Examine the following algebraic expressions and indicate how many terms can be combined through addition and subtraction. No calculations are necessary. Do not attempt to simplify.

a. $32x + 4x^2 - 10x - 2y + x^3$
b. $23g^2 - 17g^2 5 + g^4 + g^2 - 23g^2 - 0.15g + g^3$

Solution

a. Three terms (all of the x)
b. Four terms (all of the g^2)

Example 2.5.2

Simplify the following three algebraic expressions.

a. $9x+3y-\dfrac{7}{2}x+4y$

b. $P\left(1+0.11\times\dfrac{121}{365}\right)+\dfrac{15P}{1+0.11\times\dfrac{36}{365}}$

c. $x\left(1+\dfrac{0.1}{4}\right)^{3}+\dfrac{x}{\left(1+\dfrac{0.1}{4}\right)^{4}}-\dfrac{3x}{\left(1+\dfrac{0.1}{4}\right)^{2}}$

Solution

a.

Step 1: **Simplify the numerical coefficients.**

Eliminate the fraction.

$$9x+3y-rgb]1.0,0.0,0.03rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.05x+4y$$

Step 2: **Combine the numerical coefficients of like terms and solve.**

You have two terms with x and two terms with y from which you need combine and simplify.

$$\begin{aligned}
&= rgb]1.0,0.0,0.09rgb]1.0,0.0,0.0x+rgb]0.0,0.0,1.03rgb]0.0,0.0,1.0yrgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0x+rgb]0.0,0.0,1.04rgb]0.0,0.0,1.0y\\
&= (rgb]1.0,0.0,0.09rgb]1.0,0.0,0.0xrgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0x)+(rgb]0.0,0.0,1.03rgb]0.0,0.0,1.0yrgb]0.0,0.0,1.0+rgb]0.0,0.0,1.04rgb]0.0,0.0,1.0y)\\
&= rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0x+rgb]0.0,0.0,1.07rgb]0.0,0.0,1.0y
\end{aligned}$$

Step 3: **Write as a statement.**

The simplified expression is $5.5x\ +7y$.

---

b.

Step 1: **Simplify the numerical coefficients.**

Eliminate the first fraction.

$$(rgb]1.0,0.0,0.01rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.0036465)P+rgb]0.0,0.0,1.0\frac{15}{1.010849}P$$

Step 2: **Continue to simplify the second term.**

Eliminate the second fraction.

$$rgb]1.0,0.0,0.01rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.0036465rgb]0.1,0.1,0.1P+rgb]0.0,0.0,1.014rgb]0.0,0.0,1.0.rgb]0.0,0.0,1.0839006P$$

Step 3: Combine the numerical coefficients by performing addition.

Rewrite the numerical coefficient in front of the unchanged literal coefficient.

$$15.875472P$$

Step 4: **Write as a statement.**

The simplified expression is $15.875472P$.

---

c.

Step 1: **Simplify the numerical coefficients.**

Eliminate the first fraction.

$$rgb]1.0,0.0,0.01rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.0076890x+rgb]0.0,0.0,1.0\frac{1}{1.103812}x-rgb]0.0,0.5,0.0\frac{3}{1.050625}x$$

Step 2: **Continue to simplify the second term.**

Eliminate the second fraction.

$$rgb]1.0,0.0,0.01rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.0076890x+rgb]0.0,0.0,1.00rgb]0.0,0.0,1.0.rgb]0.0,0.0,1.0905950x-rgb]0.0,0.5,0.0\frac{3}{1.050625}x$$

Step 3: **Complete the last simplification.**

Eliminate the third fraction.

$$rgb]1.0,0.0,0.01rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.0076890x+rgb]0.0,0.0,1.00rgb]0.0,0.0,1.0.rgb]0.0,0.0,1.0905950x-rgb]0.0,0.5,0.02rgb]0.0,0.5,0.0.rgb]0.0,0.5,0.0855443x$$

Step 4: **Combine and solve the numerical coefficients through the specified operation.**

$$\begin{aligned}&=1.98284x-2.855443x\\&=-0.872603x\end{aligned}$$

Step 5: **Write as a statement.**

The simplified expression is $-0.872603x$.

Notice how much easier these are to work with than the original expressions.

Multiplication

Whether you are multiplying a monomial by another monomial, a monomial by a polynomial, or a polynomial by another polynomial, the rules for multiplication remain the same.

HOW TO

Solve multiple algebraic expressions

Step 1: Check to see if there is any way to simplify the algebraic expression first. Are there any like terms you can combine?

For example, you can simplify $(3x + 2 + 1)(x + x + 4)$ to $(3x + 3)(2x + 4)$ before attempting the multiplication.

Step 2: Take *every* term in the first algebraic expression and multiply it by *every* term in the second algebraic expression. This means that numerical coefficients in both terms are multiplied by each other, and the literal coefficients in both terms are multiplied by each other. It is best to work methodically from left to right so that you do not miss anything. Working with the example, in $(3x + 3)(2x + 4)$ take the first term of the first expression, $3x$, and multiply it by $2x$ and then by 4. Then move to the second term of the first expression, 3, and multiply it by $2x$ and then by 4 (See Figure 2.5.2).

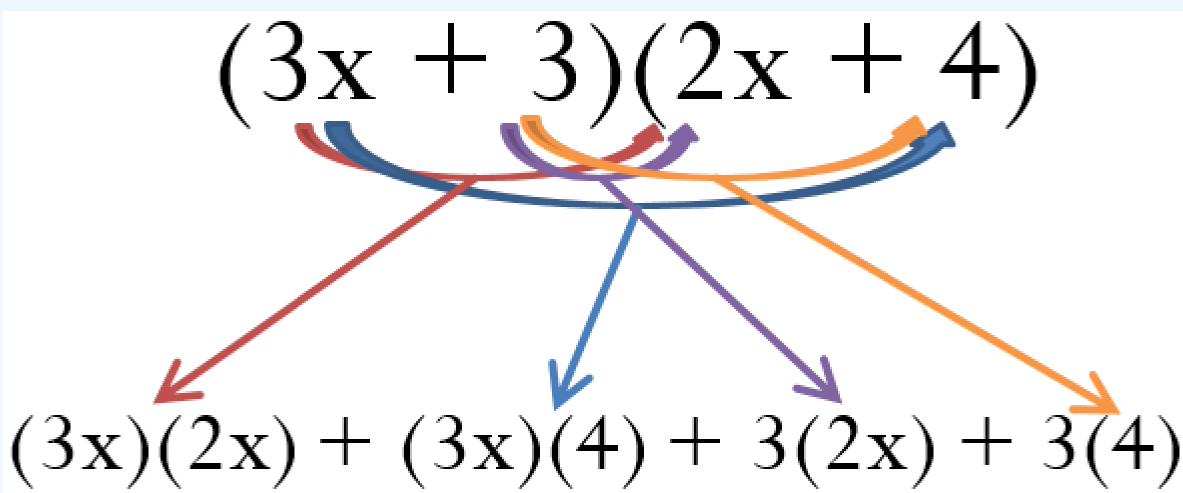

Figure 2.5.2

It becomes: 6x^2 + 12x + 6x + 12

Step 3: Perform any final steps of simplification by adding or subtracting like terms as needed. In the example, two terms contain the literal coefficient x, so you simplify the expression to $6x^2 + 18x + 12$.

Key Takeaway

If the multiplication involves more than two expressions being multiplied by each other, it is easiest to work with only one pair of expressions at a time starting with the leftmost pair. For example, if you are multiplying $(4x + 3)(3x)(9y + 5x)$, resolve $(4x + 3)(3x)$ first. Then take the solution, keeping it in brackets since you have not completed the math operation, and multiply it by $(9y + 5x)$. This means you are required to repeat Step 2 in the multiplication procedure until you have resolved all the multiplications.

Things To Watch Out For

The negative sign causes no end of grief for a lot of people when working with multiplication. First, if a numerical coefficient is not written explicitly, it is assumed to be 1.

For example, look at $2(4a + 6b) - (2a - 3b)$. This is the same as $2(4a + 6b) + (-1)(2a - 3b)$.

When you multiply a negative through an expression, all signs in the brackets will change.

Continuing with the second term in the above example, $-(2a - 3b)$ becomes $-2a + 3b$. The expression then looks like $2(4a + 6b) - 2a + 3b$.

Paths To Success

The order in which you write the terms of an algebraic expression does not matter so long as you follow all the rules of BEDMAS. For example, whether you write 3×4 or 4×3, the answer is the same because you can do multiplication in any order. The same applies to $4 + 3 - 1$ or $3 - 1 + 4$. Now let's get more complex. Whether you write $3x^2 + 5x4$ or $5x4 + 3x^2$, the answer is the same as you have not violated any rules of **BEDMAS**. You still multiply first and add last from left to right.

Although the descending exponential format for writing expressions is generally preferred, for example, 3x^2 + 5x + 4, which lists literal coefficients with higher exponents first, it does not matter if you do this or not. When checking your solutions against those given in this textbook, you only need to ensure that each of your terms matches the terms in the solution provided.

Example 2.5.3

Simplify the following algebraic expression: $(6x + 2 + 2)(3x – 2)$

Solution

Step 1: **Simplify the expression.**

$$(6x + rgb]1.0, 0.0, 0.04)(3x - 2)$$

Step 2: **Multiply all terms in each expression with all the terms in the other expression.**

$$(6x)(3x) + (6x)(-2) + (4)(3x) + (4)(-2)$$

Step 3: **Resolve the multiplication.**

$$18x^2 - 12x + 12x - 8$$

Step 4: **Perform final simplification.**

$$18x^2 - 8$$

Step 5: **Write as a final statement.**

The simplified algebraic expression is $18x^2 - 8$.

Example 2.5.4

Simplify the following algebraic expression: –(3ab)(a^2 + 4b – 2a) – 4(3a + 6)

Solution

Step 1: **Simplify.**

You cannot simplify anything. Be careful with the negatives and write them out.

$$(rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.01)(3ab)(a^2 + 4b - 2a) + (rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.04)(3a + 6)$$

Step 2: **Work with the first pair of expressions in the first term and multiply.**

$$(rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0argb]1.0, 0.0, 0.0b)(a^2 + 4b - 2a) + (rgb]0.1, 0.1, 0.1-rgb]0.1, 0.1, 0.14)(3a + 6)$$

Step 3: **Multiply the resulting pair of expressions in the first term.**

$$rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0a^{rgb]1.0,0.0,0.03}rgb]1.0, 0.0, 0.0brgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.012rgb]1.0, 0.0, 0.0argb]1.0, 0.0, 0.0b^{rgb]1.0,0.0,0.02}rgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.06rgb]1.0, 0.0, 0.0a^{rgb]1.0,0.0,0.02}rgb]1.0, 0.0, 0.0b + (rgb]0.1, 0.1, 0.1-rgb]0.1, 0.1, 0.14)(3a + 6)$$

Step 4: **Work with the pair of expressions in the original second term and multiply.**

$$rgb]0.1, 0.1, 0.1-rgb]0.1, 0.1, 0.13rgb]0.1, 0.1, 0.1a^{rgb]0.1,0.1,0.13}rgb]0.1, 0.1, 0.1brgb]0.1, 0.1, 0.1-rgb]0.1, 0.1, 0.112rgb]0.1, 0.1, 0.1argb]0.1, 0.1, 0.1b^{rgb]0.1,0.1,0.12}rgb]0.1, 0.1, 0.1+rgb]0.1, 0.1, 0.16rgb]0.1, 0.1, 0.1a^{rgb]0.1,0.1,0.12}rgb]0.1, 0.1, 0.1b + [rgb]0.0, 0.0, 1.0-rgb]0.0, 0.0, 1.012rgb]0.0, 0.0, 1.0argb]0.0, 0.0, 1.0-rgb]0.0, 0.0, 1.024]$$

Step 5: **Drop the brackets to simplify.**

$$-3a^3b - 12ab^2 + 6a^2b + -12a - 24$$

Step 6: **There are no like terms. You cannot simplify this expression any further.**

Step 7: **Write a statement.**

The simplified algebraic expression is $-3a^3b - 12ab^2 + 6a^2b + -12a - 24$.

Division

You are often required to divide a monomial into either a monomial or a polynomial. In instances where the denominator consists of a polynomial, it is either not possible or extremely difficult to simplify the expression algebraically. Only division where denominators are monomials are discussed here.

HOW TO

Simplify an expression when its denominator is a monomial

Step 1: Just as in multiplication, determine if there is any way to combine like terms before completing the division.

For example, with:

$$\frac{3ab + 3ab - 3a^2b + 9ab^2}{3ab}$$

you can simplify the numerator to:

$$\frac{6ab - 3a^2b + 9ab^2}{3ab}.$$

Step 2: Take *every* term in the numerator and divide it by the term in the denominator. This means that you must divide both the numerical and the literal coefficients. As with multiplication, it is usually best to work methodically from left to right so that you do not miss anything.

So in our example we get:

$$\begin{aligned} &= \frac{6\cancel{ab}}{3\cancel{ab}} - \frac{3rgb]1.0,0.0,0.0a^{\cancel{2}}\cancel{b}}{3\cancel{ab}} + \frac{9\cancel{a}rgb]0.0,0.0,1.0b^{\cancel{2}}}{3\cancel{ab}} \\ &= 2 - 1rgb]1.0,0.0,0.0a + 3rgb]0.0,0.0,1.0b \\ &= 2 - a + 3b \end{aligned}$$

Step 3: Perform any final simplification by adding or subtracting the like terms as needed.

As there are no more like terms, the final expression remains $2 - a + 3b$.

Things To Watch Out For

You may have heard of an outcome called "cancelling each other out." For example, in resolving the division $\frac{4a}{4a}$ many people would say that the terms cancel each other out. Many people will also mistakenly interpret this to mean that the quotient is zero and say that $\frac{4a}{4a} = 0$. In fact, when terms cancel each other out the quotient is one, not zero. The numerical coefficient is $\frac{4}{4} = 1$. The literal coefficient is $\frac{a}{a} = 1$. Thus, $\frac{4a}{4a} = \frac{1}{1} = 1$. This also explains why a zero exponent equals one: $\frac{a^1}{a^1} = a^{1-1} = a^0 = 1$.

Paths To Success

A lot of people dislike fractions and find them hard to work with. Remember that when you are simplifying any algebraic expression you can transform any fraction into a decimal. For example, if your expression is $\frac{2x}{5} + \frac{3x}{4}$, you can convert the fraction into decimals: $0.4x + 0.75x$. In this format, it is easier to solve.

Example 2.5.5

Simplify the following algebraic expression: $\dfrac{30x^6 + 5x^3 + 10x^3}{5x}$

Solution

Step 1: **The numerator has two terms with the same literal coefficient (x^3).**

Combine these using the rules of addition.

$$\frac{30x^6 + rgb]1.0, 0.0, 0.015rgb]1.0, 0.0, 0.0x^{rgb]1.0,0.0,0.03}}{5x}$$

Step 2: **Now that the numerator is simplified, divide each of its terms by the denominator.**

$$\frac{30x^6}{5x} + \frac{rgb]1.0, 0.0, 0.015rgb]1.0, 0.0, 0.0x^{rgb]1.0,0.0,0.03}}{5x}$$

Step 3: **Resolve the divisions by dividing both numerical and literal coefficients.**

$$= \frac{30x^{\cancel{6}^5}}{5\cancel{x}} + \frac{rgb]0.1, 0.1, 0.115rgb]0.1, 0.1, 0.1x^{rgb]0.1,0.1,0.1\cancel{3}^2}}{5\cancel{x}}$$

$$= 6x^5 + 3x^2$$

Step 4: **There are no like terms, so this is the final solution.**

Step 5: **Write a statement.**

The simplified algebraic expression is $6x^5 + 3x^2$.

Example 2.5.6

Simplify the following algebraic expression: $\dfrac{15x^2y^3 + 25xy^2 - xy + 10x^4y + 5xy^2}{5xy}$

Solution

***Step 1:* The numerator has two terms with the same literal coefficient (xy^2).**

Combine these by the rules of addition.

$$\frac{15x^2y^3 + rgb]1.0, 0.0, 0.030rgb]1.0, 0.0, 0.0xrgb]1.0, 0.0, 0.0y^{rgb]1.0,0.0,0.0,0.02} - xy + 10x^4y}{5xy}$$

***Step 2:* Now that the numerator is simplified, divide each of its terms by the denominator.**

$$\frac{15x^2y^3}{5xy} + \frac{rgb]0.1, 0.1, 0.130rgb]0.1, 0.1, 0.1xrgb]0.1, 0.1, 0.1y^{rgb]0.1,0.1,0.12}}{5xy} - \frac{xy}{5xy}\frac{10x^4y}{5xy}$$

***Step 3:* Resolve the divisions by dividing both numerical and literal coefficients.**

$$\frac{15x\cancel{^2}\,y\cancel{^3}{}^2}{5\,\cancel{xy}} + \frac{rgb]0.1, 0.1, 0.130rgb]0.1, 0.1, 0.1\,\cancel{x}\,rgb]0.1, 0.1, 0.1y^{rgb]0.1,0.1,0.1\cancel{2}}}{5\,\cancel{xy}} - \frac{\cancel{xy}}{5\,\cancel{xy}}\frac{10x\cancel{^4}{}^3\,y}{5\,\cancel{xy}}$$

$$= 3xy^2 + 6y - \frac{1}{5} + 2x^3$$

***Step 4:* Simplify and combine any like terms.**

$$3xy^2 + 6y - rgb]0.0, 0.0, 1.00rgb]0.0, 0.0, 1.0.rgb]0.0, 0.0, 1.02 + 2x^3$$

***Step 5:* There are no like terms, so this is the final solution.**

***Step 6:* Write as a statement.**

The simplified algebraic expression is $3xy^2 + 6y - 0.2 + 2x^3$.

Substitution

The ultimate goal of algebra is to represent a relationship between various variables. Although it is beneficial to simplify these relationships where possible and shorten the algebraic expressions, in the end you want to calculate a solution. **Substitution** involves replacing the literal coefficients of an algebraic expression with known numerical values. Once the substitution has taken place, you solve the expression for a final value.

HOW TO

Perform algebraic substitution

Step 1: Identify the value of your variables.

Assume the algebraic equation is $PV = \frac{\mathrm{FV}}{1+rt}$. You need to calculate the value of PV. It is known that:

$$\begin{aligned} FV &= \$5,443.82 \\ r &= 0.12 \\ t &= \frac{270}{365} \end{aligned}$$

Step 2: Take the known values and insert them into the equation where their respective variables are located, resulting in:

$$PV = \frac{\$5,443.82}{1+(0.12)\left(\frac{270}{365}\right)}.$$

Step 3: Resolve the equation to solve for the variable.

Calculate.

$$\begin{aligned} PV &= \frac{\$5,443.82}{1.088767} \\ PV &= \$5,000.00 \end{aligned}$$

Things To Watch Out For

It is common in algebra to represent a variable with more than just one letter. As you can see from the example above, FV is a variable, and it represents **future value**. This should not be interpreted to be two variables, F and V. Similarly, PMT represents **annuity payment**. When you learn new formulas and variables, take careful note of how a variable is represented.

As well, some literal coefficients have subscripts. For example, you could see d_1 and d_2 in the same formula. What sometimes happens is that there is more than one value for the same variable. As you will learn in Chapter 4 about merchandising, when you buy an item you may receive more than one discount rate (what d stands for). Therefore, the first discount gets a subscript of 1, or d_1, and the second discount gets a subscript of 2, or d_2. This allows you to distinguish between the two values in the equation and substitute the correct value in the correct place.

Paths To Success

If you are unsure whether you have simplified an expression appropriately, remember that you can make up your own values for any literal coefficient and substitute those values into both the original and the simplified expressions. If you have obeyed all the rules and simplified appropriately, both expressions will produce the same answer.

For example, assume you simplified $2x + 5x$ into $7x$, but you are not sure if you are right. You decide to let $x = 2$. Substituting into $2x + 5x$, you get $2(2) + 5(2) = 14$. Substituting into your simplified expression you get $7(2) = 14$. Since both expressions produced the same answer, you have direct confirmation that you have simplified correctly.

Example 2.5.7

Substitute and solve the following equation:

$$N = L \times (1 - d_1) \times (1 - d_2) \times (1 - d_3).$$

Where:

$$L = \$1,999.99$$

$$d_1 = 35\%$$

$$d_2 = 15\%$$

$$d_3 = 5\%$$

Solution

Step 1: **Substitute known values into the equation.**

$$N = \$1,999.99 \times (1 - 0.35) \times (1 - 0.15) \times (1 - 0.05)$$

Step 2: **Solve for N.**

$$N = \$1,999.99 \times 0.65 \times 0.85 \times 0.95$$
$$N = \$1,049.74$$

Step 3: **Write as a statement.**

The value of N is $\$1,049.74$.

Section 2.5 Exercises

Mechanics

For questions 1–4, simplify the algebraic expressions.

1. $2a - 3a + 4 + 6a - 3$
2. $5b(4b + 2)$
3. $\dfrac{6x^3 + 12x^2 + 13.5x}{3x}$
4. $(1+i)^3 \times (1+i)^{14}$
5. Evaluate the power $8^{\frac{2}{3}}$
6. Substitute the known variables and solve for the unknown variable:

$$I = Prt \text{ where } P = \$2,500,\ r = 0.06, \text{ and } t = \frac{135}{365}$$

Solutions

1. $5a + 1$
2. $20b^2 + 10b$
3. $2x^2 + 4x + 4.5$
4. $(1+i)^{17}$
5. 4
6. $I = \$55.48$

Applications

For questions 7-11, simplify the algebraic expressions.

7. $(6r^2 + 10 - 6r + 4r^2 - 3) - (-12r - 5r^2 + 2 + 3r)$
8. $\left[\dfrac{5x^9 + 3x^9}{2x}\right]^5$
9. $\dfrac{t}{2} + 0.75t - t^3 + \dfrac{5t^4}{t} - \dfrac{2\left(t + t^3\right)}{4}$
10. $\dfrac{14\,(1+i) + 21(1+i)^4 - 35(1+i)^7}{7\,(1+i)}$

11. $\dfrac{R}{1+0.08\times\frac{183}{365}}+3R\left(1+0.08\times\dfrac{52}{365}\right)$
12. Evaluate the power: $\left[\left(\dfrac{2}{5}\right)^2\right]^2$

For 13 and 14 substitute the known variables and solve for the unknown variable.

13. $\mathrm{PV}=\dfrac{\mathrm{FV}}{(1+i)^{\mathrm{N}}}$ where $\mathrm{FV}=\$3,417.24$, $i=0.05$, and $\mathrm{N}=6$
14. $\mathrm{PMT}=\dfrac{FV}{\left[\frac{(1+i)^{\mathrm{N}}-1}{i}\right]}$ where $\mathrm{FV}=\$10,000$, $\mathrm{N}=17$, and $i=0.10$

Solutions

7. $15r^2+3r+5$
8. $1,024x^{40}$
9. $0.75t+3.5t^3$
10. $2+3(1+i)^3-5(1+i)^6$
11. $3.995628\mathrm{R}$
12. 0.0256
13. $\mathrm{PV}=\$2,550$
14. $\mathrm{PMT}=\$246.64$

Challenge, Critical Thinking, & Other Applications

For questions 15-17, simplify the algebraic expressions.

15. $\left[\dfrac{10a^2b^3c^4}{5b^3c^4}\right]^2+6\left(a^8\right)^{\frac{1}{2}}-\left(3a^2+6\right)\left(3a^2-3\right)$
16. $\dfrac{-(5x+4y+3)(2x-5y)-(10x-2y)(2y+3)}{5}$
17. $\dfrac{(-3z)^3\left(3z^2\right)^2}{\left(2z^3\right)^{-4}}$

18. Substitute the known variables and solve for the unknown variable.

$$\mathrm{FV_{ORD}} = \mathrm{PMT}(1+\%C)^{\mathrm{N}-1}\left[\frac{\left[\frac{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{(1+\%C)}\right]^{\mathrm{N}}-1}{\frac{(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}}{(1+\%C)}-1}\right]$$

where

$$\mathrm{PMT} = \$500, \quad i = 0.05, \quad \%C = 0.02, \quad \mathrm{CY} = 2, \quad \mathrm{PY} = 4, \quad \text{and } \mathrm{N} = 20$$

For questions 19-20, evaluate the expression.

19. $\$50,000 \times \left(1+\frac{0.10}{12}\right)^{-27}$
20. $\$995\left[\frac{1-(1+0.02)^{13}\left(1+\frac{0.09}{4}\right)^{-13}}{\frac{0.09}{4}-0.02}\right]$

Solutions

19. $a^4 - 9a^2 + 18$
20. $-2x^2 - 0.6xy + 4.8y^2 - 7.2x + 4.2y$
21. $-3,888z^{19}$
22. $\$15,223.10$
23. $\$39,963.05$
24. $\$12,466.44$

Attribution

2.6: LINEAR EQUATIONS: MANIPULATING AND SOLVING

Introduction

You are shopping at Old Navy for seven new outfits. The price points are $\$10$ and $\$30$. You really like the $\$30$ outfits; however, your total budget can't exceed $\$110$. How do you spend $\$110$ to acquire all the needed outfits without exceeding your budget while getting as many $\$30$ items as possible?

This is a problem of linear equations, and it illustrates how you can use them to make an optimal decision. Let L represent the quantity of clothing at the low price point of $\$10$, and H represent the quantity of clothing at the high price point of $\$30$. This results in the following algebraic equations:

$$L \; + \; H \;=\; 7 \text{ (the total number of outfits you need)}$$
$$\$10L + \$30H \;=\; \$110 \text{ (your total budget)}$$

By simultaneously solving these equations you can determine how many outfits at each price point you can purchase.

You will encounter many situations like this in your business career, for example, in making the best use of a manufacturer's production capacity. Assume your company makes two products on the same production line and sells all its output. Each product contributes differently to your profitability, and each product takes a different amount of time to manufacture. What combination of each of these products should you make such that you operate your production line at capacity while also maximizing the profits earned? This section explores how to solve linear equations for unknown variables.

Understanding Equations

To manipulate algebraic equations and solve for unknown variables, you must first become familiar with some important language, including linear versus nonlinear equations and sides of the equation.

$$rgb]0.0, 0.0, 1.04rgb]0.0, 0.5, 0.0xrgb]0.0, 0.0, 1.0+rgb]0.0, 0.0, 1.03rgb]0.1, 0.1, 0.1 =rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.02rgb]0.0, 0.5, 0.0xrgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.03$$

$rgb]0.0, 0.0, 1.04rgb]0.0, 0.5, 0.0xrgb]0.0, 0.0, 1.0+rgb]0.0, 0.0, 1.03rgb]0.1, 0.1, 0.1$ is the Left side:

Every equation has two sides. Everything to the left side of the equal sign is known as the left side of the equation.

$rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.02rgb]0.0, 0.5, 0.0xrgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.03rgb]0.1, 0.1, 0.1$ is the Right side: Everything to the right side of the equal sign is known as the right side of the equation.

$rgb]0.0, 0.5, 0.0xrgb]0.1, 0.1, 0.1$ $rgb]0.1, 0.1, 0.1$is the Variable x: Observe two important features of this variable:

1. Note that the variable has an exponent of 1. So if you were to plot each algebraic expression onto a Cartesian coordinate system, the expressions would form straight lines (see Figure 2.6.1a). Such equations are called **linear equations**. Because each equation has only one variable, there is only one solution that makes the equation true. In business mathematics, this is the most common situation you will encounter.
2. If the exponent is anything but a one, the equation is a nonlinear equation since the graph of the expression does not form a straight line. Figure 2.5.1b illustrates two algebraic expressions where the exponent is other than a 1.

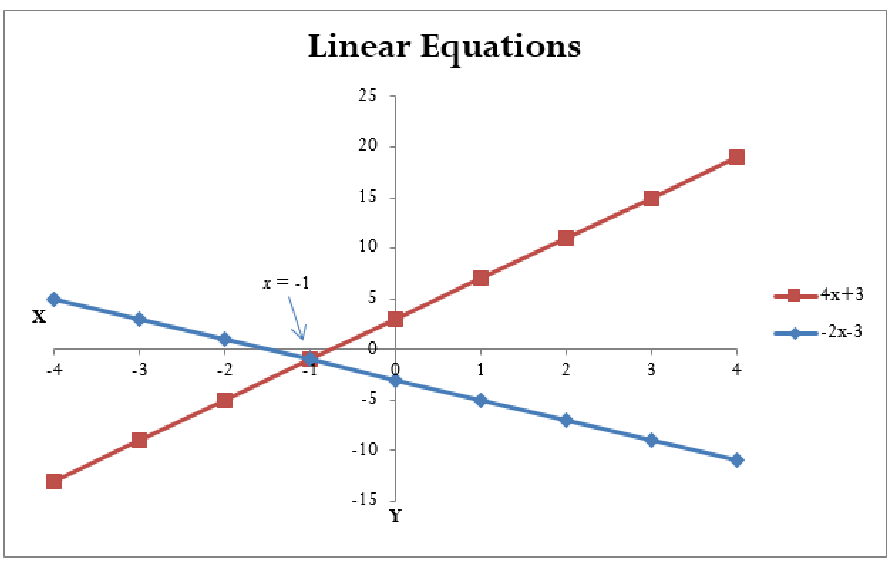

Figure 2.6.1a

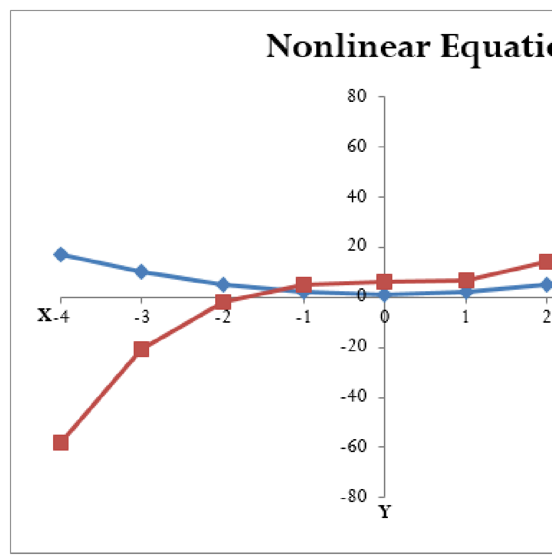

Figure 2.6.1b

The goal in manipulating and solving a linear equation is to find a value for the unknown variable that makes the equation true. If you substitute a value of $x = -1$ into the above example, the left-hand side of the equation equals the right-hand side of the equation (see Figure 2.6a). The value of $x = -1$ is known as the **root**, or solution, to the linear equation.

Solving One Linear Equation with One Unknown Variable

In your study of solving linear equations, you need to start by manipulating a single equation to solve for a single unknown variable. Later in this section you will extend from this foundation to the solution of two linear equations with two unknowns.

HOW TO

Determine the root of a linear equation with only one unknown variable

Apply the following steps:

Step 1: Your first goal is to separate the terms containing the literal coefficient from the terms that only have numerical coefficients. Collect all of the terms with literal coefficients on only one side of the equation and collect all of the terms with only numerical coefficients on the other side of the equation. It does not matter which terms go on which side of the equation, so long as you separate them.

Step 2: To move a term from one side of an equation to another, take the mathematical opposite of the term being moved and add it to both sides.

For example, if you want to move the $+3$ in $4x + 3 = -2x - 3$ from the left-hand side to the right-hand side, the mathematical opposite of $+3$ is -3. When you move a term, remember the cardinal rule: *What you do to one side of an equation you must also do to the other side of the equation. Breaking this rule breaks the equality in the equation.*

Step 3: Combine all like terms on each side and simplify the equation according to the rules of algebra.

Step 4: In the term containing the literal coefficient, reduce the numerical coefficient to a 1 by dividing both sides of the equation by the numerical coefficient.

Key Takeaway

When you are unsure whether your calculated root is accurate, an easy way to verify your answer is to take the original equation and substitute your root in place of the variable. If you have the correct root, the left-hand side of the equation equals the right-hand side of the equation. If you have an incorrect root, the two sides will be unequal. The inequality typically results from one of the three most common errors in algebraic manipulation:

1. The rules of **BEDMAS** have been broken.
2. The rules of algebra have been violated.
3. What was done to one side of the equation was not done to the other side of the equation.

Things To Watch Out For

When you move a term from one side of the equation to another using multiplication or division, remember that this affects *each and every term on both sides of the equation*. To remove the x from the denominator in the following equation, multiply both sides of the equation by x:

$$\frac{5}{x}+\frac{1}{x}=\frac{2}{x}+2$$

becomes:

$$x\left(\frac{5}{x}+\frac{1}{x}\right)=\left(\frac{2}{x}+2\right)x$$

which then becomes:

$$5+1=2+2x$$

Multiplying every term on both sides by x maintains the equality.

Paths To Success

Negative numbers can cause some people a lot of grief. In moving terms from a particular side of the equation, many people prefer to avoid negative numerical coefficients in front of literal coefficients. Revisiting $4x+3=-2x-3$, you could move the $4x$ from the left side to the right side by subtracting $4x$ from both sides. However, on the right side this results in $-6x$. The negative is easily overlooked or accidentally dropped in future steps. Instead, move the variable to the left side of the equation, yielding a positive coefficient of $6x$.

Example 2.6.1

Take the ongoing example in this section and solve it for x:

$$4x+3=-2x-3.$$

Solution

***Step 1:* Move terms with literal coefficients to one side and terms with only numerical coefficients to the other side.** Let's collect the literal coefficient on the left-hand side of the equation. Move $-2x$ to the left-hand side by placing $+2x$ on both sides.

On the right-hand side, the $-2x$ and $+2x$ cancel out to zero.

$4x+3rgb]1.0,0.0,0.0{+}rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0x=-2x-3rgb]1.0,0.0,0.0{+}rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0x$

***Step 2:* All of the terms with the literal coefficient are now on the left.** Let's move all of the terms containing only numerical coefficients to the right-hand side. Move the $+3$ to the right-hand side by placing -3 on both sides.

On the left-hand side, the $+3$ and -3 cancel out to zero.

$$4x+3+2x rgb]0.0,0.0,1.0-rgb]0.0,0.0,1.03=-3rgb]0.0,0.0,1.0-rgb]0.0,0.0,1.03$$

***Step 3:* The terms are now separated. Combine like terms according to the rules of algebra.**

$$6x=-6$$

***Step 4:* The term with the literal coefficient is being multiplied by the numerical coefficient of 6.** Therefore, divide both sides by 6.

The left-hand side numerical coefficients will divide to 1. Resolve the numerical coefficients on the right-hand side.

$$\begin{aligned} rgb]0.1,0.1,0.1\frac{\cancel{6}\,x}{rgb]0.0,0.5,0.0\cancel{6}}rgb]0.1,0.1,0.1 &= rgb]0.1,0.1,0.1\frac{-\cancel{6}}{rgb]0.0,0.5,0.0\cancel{6}} \\ rgb]0.1,0.1,0.1x rgb]0.1,0.1,0.1 &= rgb]0.1,0.1,0.1-rgb]0.1,0.1,0.11 \end{aligned}$$

***Step 5:* Write a statement.**

The root of the equation is $x=-1$.

***Step 6:* Substitute back into the equation to check.**

$$\begin{aligned} 4(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.01)+3 &= -2(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.01)-3 \\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.04+3 &= rgb]1.0,0.0,0.02-3 \\ -1 &= -1\ \checkmark \end{aligned}$$

Example 2.6.2

Solve the following equation for m:

$$\frac{3m}{4}+2m=4m-15\,.$$

Solution

***Step 1:* Simplify all fractions to make the equation easier to work with.**

Still simplifying, collect like terms where possible.

$$rgb]1.0, 0.0, 0.00rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.075m \ + \ 2m \ = \ 4m \ - \ 15$$

***Step 2:* Collect all terms with the literal coefficient on one side of the equation.** Move all terms with literal coefficients to the right-hand side.

$$rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.075m \ = \ 4m \ - \ 15$$

***Step 3:* Combine like terms and move all terms with only numerical coefficients to the left-hand side.**

$$2.75m \ rgb]1.0, 0.0, 0.0- \ rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.075m \ = \ 4m \ - \ 15 \ rgb]1.0, 0.0, 0.0- \ rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.075m$$

On the left-hand side, the $+2.75m$ and $-2.75m$ cancel each other out. Now move the numerical coefficients to the left-hand side.

$$rgb]0.0, 0.0, 1.00 \ = \ 4m \ - \ 15 \ rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0 \ rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.075m$$

On the right-hand side, the -15 and $+15$ cancel each other out.

$$0 \ rgb]0.0, 0.0, 1.0+rgb]0.0, 0.0, 1.0 \ rgb]0.0, 0.0, 1.015 \ = \ 4m \ - \ 15 \ - \ 2.75m \ rgb]0.0, 0.0, 1.0+rgb]0.0, 0.0, 1.0 \ rgb]0.0, 0.0, 1.015$$

***Step 4:* Combine like terms on each side.**

$$rgb]0.1, 0.1, 0.10rgb]0.1, 0.1, 0.1 \ rgb]0.1, 0.1, 0.1+rgb]0.0, 0.0, 1.0 \ rgb]0.0, 0.0, 1.015rgb]0.1, 0.1, 0.1 \ rgb]0.1, 0.1, 0.1 =rgb]0.1, 0.1, 0.1 \ rgb]0.1, 0.1, 0.14rgb]0.1, 0.1, 0.1mrgb]0.1, 0.1, 0.1 \ rgb]0.1, 0.1, 0.1-rgb]0.1, 0.1, 0.1 \ rgb]0.1, 0.1, 0.12rgb]0.1, 0.1, 0.1.rgb]0.1, 0.1, 0.175rgb]0.1, 0.1, 0.1m$$

***Step 5:* Divide both sides by the numerical coefficient that accompanies the literal coefficient.**

Simplify.

$$15 \ = \ 1.25m$$

$$\frac{15}{rgb]0.0, 0.0, 1.01rgb]0.0, 0.0, 1.0.rgb]0.0, 0.0, 1.025} \ = \ \frac{1.25m}{rgb]0.0, 0.0, 1.01rgb]0.0, 0.0, 1.0.rgb]0.0, 0.0, 1.025}$$

$$12 = m$$

***Step 6:* Write a statement.**

The root of the equation is $m = 12$.

This makes both sides of the equation ($\frac{3m}{4} + 2m$ and $4m - 15$) equal 33.

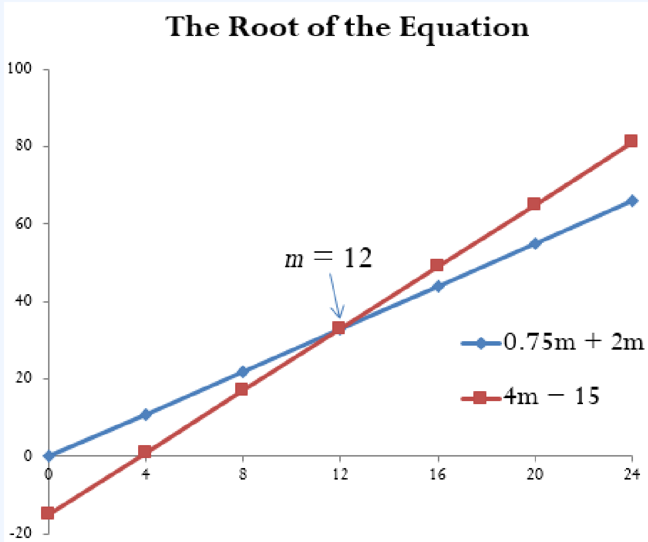

Figure 2.6.2

Example 2.6.3

Solve the following equation for *b* and round your answer to four decimals:

$$\frac{5}{8}b+\frac{2}{5}=\frac{17}{20}-\frac{b}{4}.$$

Solution

***Step 1:* Simplify the fractions and convert to decimals.**

$$rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.0625b+rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.04=rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.085-rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.025b$$

***Step 2:* Move the literal coefficient terms to the left-hand side.**

The literal coefficients on the right-hand side cancel each other out.

$$0.625b+0.4+rgb]0.0,0.0,1.00rgb]0.0,0.0,1.0.rgb]0.0,0.0,1.025rgb]0.0,0.0,1.0b=0.85rgb]0.0,0.0,1.0-rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.125rgb]0.1,0.1,0.1b+0.25b$$

$$rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.1625rgb]0.1,0.1,0.1brgb]0.1,0.1,0.1+rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.14rgb]0.1,0.1,0.1+rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.125rgb]0.1,0.1,0.1brgb]0.1,0.1,0.1=rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.185$$

***Step 3:* Move the numerical coefficient terms to the right-hand side.**

The numerical coefficients on the left-hand side cancel each other out.

$$rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.1625rgb]0.1,0.1,0.1brgb]0.1,0.1,0.1 +0.4 rgb]0.1,0.1,0.1+rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.125rgb]0.1,0.1,0.1brgb]1.0,0.0,0.0 -0.4 rgb]0.1,0.1,0.1 = rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.185rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.04$$
$$rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.1625rgb]0.1,0.1,0.1brgb]0.1,0.1,0.1+rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.125rgb]0.1,0.1,0.1brgb]0.1,0.1,0.1 = rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.185rgb]0.1,0.1,0.1-rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.14$$

Step 4: **Combine like terms on each side.**

$$0.875b = 0.45$$

Step 5: **Divide both sides by the numerical coefficient that accompanies the literal coefficient.**

$$\frac{0.875b}{rgb]1.0,0.0,1.00rgb]1.0,0.0,1.0.rgb]1.0,0.0,1.0875} = \frac{0.45}{rgb]1.0,0.0,1.00rgb]1.0,0.0,1.0.rgb]1.0,0.0,1.0875}$$

Step 6: **Simplify.**

$$b = 0.514285$$

Step 7: **Round to four decimals as instructed and write a statement.**

$$b = 0.5143$$

Step 8: **Write as a statement.**

The root is $b = 0.5143$.

Solving Two Linear Equations with Two Unknown Variables

The manipulation process you have just practiced works well for solving one linear equation with one variable. But what happens if you need to solve two linear equations with two variables simultaneously? Remember when you were at Old Navy purchasing seven outfits earlier in this chapter? You needed to stay within a pricing budget. Each equation had two unknown variables representing the number of lower-priced outfits (L) and higher-priced outfits (H).

$$\begin{aligned} L + H &= 7 \text{ (the total number of outfits you need)} \\ \$10L + \$30H &= \$110 \text{ (your total budget)} \end{aligned}$$

The goal is to reduce two equations with two unknowns into a single linear equation with one unknown. Once this transformation is complete, you then identify the unknown variable by applying the three-step procedure for solving one linear equation, as just discussed.

When you work with two linear equations with two unknowns, the rules of algebra permit the following two manipulations:

- What you do to one side of the equation must be done to the other side of the equation to maintain the equality. Therefore, you can multiply or divide any equation by any number without changing the root of the equation.

- For example, if you multiply all terms of $x + y = 2$ by 2 on both sides, resulting in $2x + 2y = 4$, the equality of the equation remains unchanged and the same roots exist.

- Terms that are on the same side of an equation can be added and subtracted between equations by combining like terms. Each of the two equations has a left side and right side. This rule permits taking the left side of the first equation and either adding or subtracting like terms on the left side of the second equation. When you perform this action, remember the first rule above. If you add the left sides of the equations together, you then must add the right side of both equations together to maintain equality.

HOW TO

Solve two linear equations with two unknown variables

Step 1: Write the two equations one above the other, vertically lining up terms that have the same literal coefficients and terms that have only the numerical coefficient. If necessary, the equations may need to be manipulated such that all of the literal coefficients are on one side with the numerical coefficients on the other side.

Step 2: Examine your two equations. Through multiplication or division, make the numerical coefficient on one of the terms containing a literal coefficient exactly equal to its counterpart in the other equation.

Step 3: Add or subtract the two equations as needed so as to eliminate the identical term from both equations.

Step 4: In the new equation, solve for the last literal coefficient.

Step 5: Substitute the root of the known literal coefficient into either of the two original equations. If one of the equations takes on a simpler structure, pick that equation.

Step 6: Solve your chosen equation for the other literal coefficient.

Paths To Success

Sometimes it is unclear exactly how you need to multiply or divide the equations to make two of the terms identical. For example, assume the following two equations:

$$4.9x + 1.5y = 38.3$$
$$2.7x - 8.6y = 17.8$$

If the goal is to make the terms containing the literal coefficient x identical, there are two alternative solutions:

- Take the larger numerical coefficient for x and divide it by the smaller numerical coefficient. The resulting number is the factor for multiplying the equation containing the smaller numerical coefficient. In this case, $4.9 \div 2.7 = 1.\overline{814}$. Multiply all terms in the second equation by $1.\overline{814}$ to make the numerical coefficients for x equal to each other, resulting in this pair of equations:

$$\begin{aligned} 4.9x + 1.5y &= 38.3 \\ &\text{and} \\ 4.9x - 15.6\overline{074}y &= 32.3\overline{037} \text{ (every term multiplied by } 1.\overline{814}) \end{aligned}$$

- Take the first equation and multiply it by the numerical coefficient in the second equation. Then take the second equation and multiply it by the numerical coefficient in the first equation. In this case, multiply all terms in the first equation by 2.7. Then multiply all terms in the second equation by 4.9.

13.23x+4.05y=103.41\;\text{(every term multiplied by 2.7)}

13.23x-\;42.14y=87.22\;\text{(every term multiplied by 4.9)}

Note that both approaches successfully result in both equations having the same numerical coefficient in front of the literal coefficient x.

Paths To Success

Ultimately, every pairing of linear equations with two unknowns can be converted into a single equation through substitution. To make the conversion, do the following:

1. Solve either equation for one of the unknown variables.
2. Take the resulting algebraic expression and substitute it into the other equation. This new equation is solvable for one of the unknown variables.
3. Substitute your newfound variable into one of the original equations to determine the value for the other unknown variable.

Take the following two equations:

$$a+b=4\;\;\text{and }\;2a+b=6$$

1. Solving the first equation for a results in $a = 4 - b$.
2. Substituting the expression for a into the second equation and solving for b results in $2(4 - b)b = 6$, which solves as $b = 2$.
3. Finally, substituting the root of b into the first equation to calculate a gives $a + 2 = 4$ resulting in $a = 2$. Therefore, the roots of these two equations are $a = 2$ and $b = 2$.

Example 2.6.4

Recall from the section opener that in shopping for outfits there are two price points of $\$10$ and $\$30$, your budget is $\$110$, and that you need seven articles of clothing. The equations below represent these conditions. Identify how many low-priced outfits (L) and high-priced outfits (H) you can purchase.

$$L + H = 7$$
$$\$10L + \$30H = \$110$$

Solution

Step 1: **Write the equations one above the other and line them up.**

$$L + H = 7$$
$$\$10L + \$30H = \$110$$

Step 2: **Multiply all terms in the first equation by 10 so that L has the same numerical coefficient in both equations.**

$$rgb]1.0,0.0,0.010L + rgb]1.0,0.0,0.010H = rgb]1.0,0.0,0.070$$
$$\$10L + \$30H = \$110$$

Step 3: **Subtract the equations by subtracting all terms on both sides.**

$$10L + 10H = 70$$
$$\$10L + 30\$H = \$110$$
$$rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.0\$rgb]0.0,0.0,1.020rgb]0.0,0.0,1.0Hrgb]0.0,0.0,1.0 = rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.0\$rgb]0.0,0.0,1.040$$

Step 4: **Solve for H by dividing both sides by -20.**

$$rgb]0.1,0.1,0.1\frac{-\$20H}{rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.0\$rgb]1.0,0.0,0.020}rgb]0.1,0.1,0.1 = rgb]0.1,0.1,0.1\frac{-\$40}{rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.0\$rgb]1.0,0.0,0.020}$$
$$rgb]1.0,0.0,0.0Hrgb]1.0,0.0,0.0 = rgb]1.0,0.0,0.02$$

Step 5: **Substitute the known value for H into one of the original equations. The first equation is simple, so choose that one.**

$$rgb]0.1,0.1,0.1Lrgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.1{+}rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.1Hrgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.1 = rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.17rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.1$$
$$rgb]0.1,0.1,0.1Lrgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.1{+}rgb]0.1,0.1,0.1\ rgb]1.0,0.0,0.02rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.1 = rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.17$$

Step 6: **Solve for L by subtracting 2 from both sides. You now have the roots for L and H.**

$$L\ +\ 2\ rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.02\ =\ 7\ rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.02$$
$$rgb]0.0,0.0,1.0Lrgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.0 = rgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.05$$

Paths To Success

One of the most difficult areas of mathematics involves translating words into mathematical symbols and operations. To assist in this translation, the table below lists some common language and the mathematical symbol that is typically associated with the word or phrase.

Table 2.6.1

Language			Math Symbol	Language		
Sum Addition	In addition to In excess	Increased by Plus	$+$	Subtract Decreased by Diminished by	Less Minus	Difference Reduce by
Divide Division	Divisible Quotient	Per	$\div$	Multiplied by Times	Percentage of	Produc of Of
Becomes Is/Was/Were	Will be	Results in Totals	$=$	More than Greater than		
Less than	Lower than		$<$	Greater than or equal to		
Less than or equal to			$\leq$	Not equal to		

Example 2.6.5

Tinkertown Family Fun Park charges $\$15$ for a child wrist band and $\$10.50$ for an adult wrist band. On a warm summer day, the amusement park had total wrist band revenue of $\$15,783$ from sales of $1,279$ wrist bands. How many adult and child wrist bands did the park sell that day?

Solution

***Step 1:* Write out what you know.**

The price of the wrist bands, total quantity, and sales are known.

$$\begin{aligned}\text{Child wrist band price} &= \$15\\ \text{Adult wrist band price} &= \$10.50\\ \text{Total revenue} &= \$15,783\\ \text{Total unit sales} &= 1,279\end{aligned}$$

The quantity of adult wrist bands sold and the quantity of child wrist bands sold are unknown:

$$\begin{aligned}\text{Adult wrist bands quantity} &= a\\ \text{Child wrist bands quantity} &= c\end{aligned}$$

Step 2: **Write out how you will get to the solution.**

Work with the quantities first. Calculate the total unit sales by adding the number of adult wrist bands to the number of child wrist bands:

$$\begin{aligned}rgb]1.0,0.0,0.0\#\text{ of adult wrist bands} + rgb]0.0,0.0,1.0\#\text{ of child wrist bands } &= \text{total unit sales}\\ rgb]1.0,0.0,0.0a \;+\; rgb]0.0,0.0,1.0c \;&=\; 1,279\end{aligned}$$

Now consider the dollar figures. Total revenue for any company is calculated as unit price multiplied by units sold. In this case, you must sum the revenue from two products to get the total revenue.

$$\begin{aligned}rgb]1.0,0.0,0.0\text{Total adult revenue} + rgb]0.0,0.0,1.0\text{Total child revenue } &= \text{ Total revenue}\\ (rgb]1.0,0.0,0.0\text{Adult price}\times\text{Adult quantity})+(rgb]0.0,0.0,1.0\text{Child price}\times\text{Child quantity}) &= \text{Total revenue}\\ rgb]1.0,0.0,0.0\$rgb]1.0,0.0,0.010rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.050rgb]1.0,0.0,0.0a \;+\; rgb]0.0,0.0,1.0\$rgb]0.0,0.0,1.015rgb]0.0,0.0,1.0c \;&=\; \$15,783\end{aligned}$$

Step 3: **Write the equations one above the other and line them up.**

$$\begin{aligned}a +\ \ c &= 1,279\\ \$10.50a + \$15c &= \$15,783\end{aligned}$$

Step 4: **Multiply all terms in the first equation by 10.5, resulting in *a* having the same numerical coefficient in both equations.**

$$\begin{aligned}rgb]1.0,0.0,0.010rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.050a + rgb]1.0,0.0,0.010rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.050c &= 13,429.50\\ \$rgb]1.0,0.0,0.010rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.050a + \$15c &= \$15,783\end{aligned}$$

Step 5: **Subtract the equations by subtracting all terms on both sides.**

$$\begin{aligned}10.5a + 10.5c &= 13,429.50\\ \$10.50a + \$15c &= \$15,783 \ \text{ Subract.}\\ rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.04rgb]0.0,0.0,1.0.rgb]0.0,0.0,1.05rgb]0.0,0.0,1.0crgb]0.0,0.0,1.0 &= rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0,rgb]0.0,0.0,1.0353rgb]0.0,0.0,1.0.rgb]0.0,0.0,1.050\end{aligned}$$

Step 6: **Solve for c by dividing both sides by -4.5.**

$$\begin{aligned}rgb]0.1,0.1,0.1\frac{-4.5c}{rgb]1.0,0.0,1.0{-}rgb]1.0,0.0,1.04rgb]1.0,0.0,1.0.rgb]1.0,0.0,1.05}rgb]0.1,0.1,0.1 &= rgb]0.1,0.1,0.1\frac{-2,353.50}{rgb]1.0,0.0,1.0{-}rgb]1.0,0.0,1.04rgb]1.0,0.0,1.0.rgb]1.0,0.0,1.05}\\ rgb]1.0,0.0,1.0crgb]1.0,0.0,1.0\ rgb]1.0,0.0,1.0 &= rgb]1.0,0.0,1.0\ rgb]1.0,0.0,1.0523\end{aligned}$$

Step 7: **Substitute the known value for c into one of the original equations. The first equation is simple, so choose that one.**

Solve for a by subtracting 523 from both sides. You now have the roots for a and c.

$$\begin{aligned} a + c &= 1,279 \\ a + rgb]1.0,0.0,1.0523 &= 1,279 \\ a + rgb]1.0,0.0,1.0\,\cancel{523 - 523} &= 1,279 - rgb]1.0,0.0,1.0523 \\ a &= 756 \end{aligned}$$

Step 8: **Write as a statement.**

Tinkertown Family Fun Park sold 523 child wrist bands and 756 adult wrist bands.

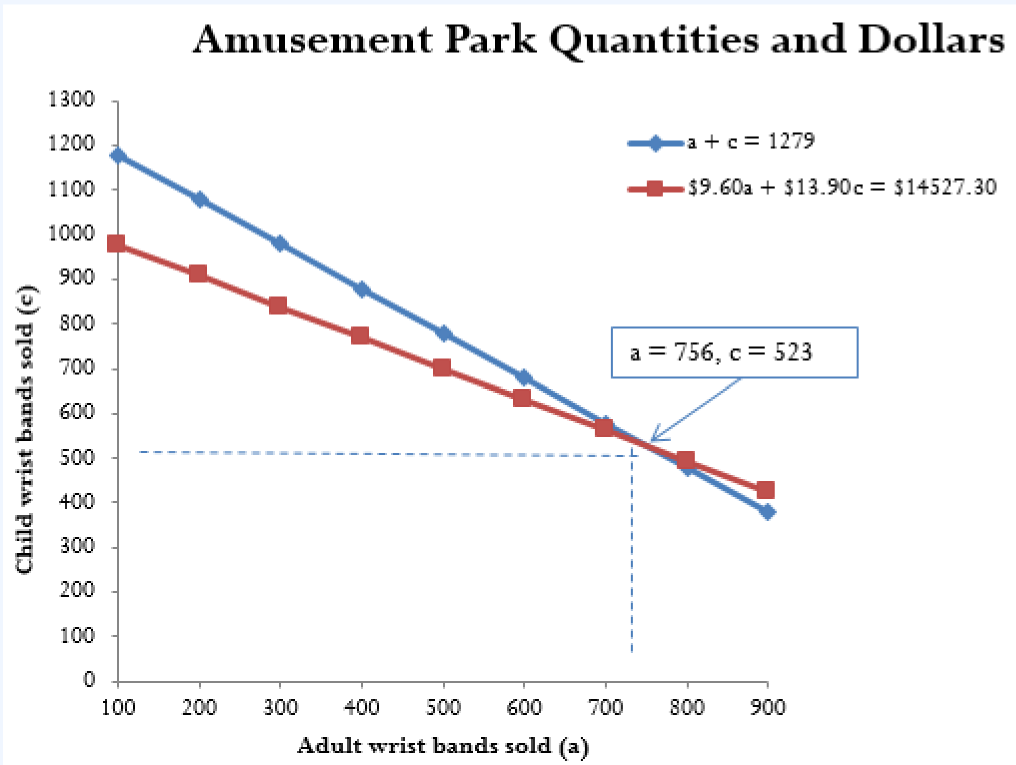

Figure 2.6.3

Section 2.6 Exercises

Mechanics

Solve the following equations for the unknown variable.

1. $3(x-5)=15$
2. $12b-3=4+5b$
3. $0.75(4m+12)+15-3(2m+6)=5(-3m+1)+25$

Solve each of the following pairs of equations for both unknown variables.

4. $$\begin{aligned} x+y&=6 \\ 3x-2y&=8 \end{aligned}$$
5. $$\begin{aligned} 4h-7q&=13 \\ 6h+3q&=33 \end{aligned}$$
6. $$\begin{aligned} 0.25a+\frac{5b}{2}&=3.5 \\ \frac{3a}{4}-\frac{b}{0.2}&=3 \end{aligned}$$

Solutions

1. $x=1.0$
2. $b=1$
3. $m=2$
4. $x=4$, $y=2$
5. $h=5$, $q=1$
6. $a=8$, $b=0.6$

Applications

In questions 7 and 8, solve the equation for the unknown variable.

7. $\frac{4y}{1.025^4}+y-2y(1.05)^2=\$1,500$
8. $\$2,500(1+0.06t)+\$1,000(1+0.04t)=\$3,553.62$

For exercises 9–14, read each question carefully and solve for the unknown variable(s).

1. Pamela is cooking a roast for a 5:30 p.m. dinner tonight. She needs to set a delay timer on her oven. The roast takes 1 hour and 40 minutes to cook. The time right now is 2:20 p.m. How long of a delay must she set the oven for (before it automatically turns on and starts to cook the roast)?
2. In 2010, 266 million North Americans were using the Internet, which represented a 146.3% increase in Internet users over the year 2000. How many North American Internet users were there in 2000?
3. A human resource manager is trying to estimate the cost of a workforce accident. These costs usually consist of direct costs (such as medical bills, equipment damage, and legal expenses) and indirect costs (such as decreased output, production delays, and fines). From past experience, she knows that indirect costs average six times as much as direct costs. If she estimates the cost of an accident to be $\$21,000$, what is the direct cost of the accident?
4. In 2011, Canadian federal tax rates were 0% on the first $\$10,527$ of gross income earned, 15% on the next $\$31,017$, 22% on the next $\$41,544$, 26% on the next $\$45,712$, and 29% on anything more. If a taxpayer paid $\$28,925.35$ in federal tax, what was her gross annual income for 2011?
5. St. Boniface Hospital raises funds for research through its Mega Lottery program. In this program, $16,000$ tickets are available for purchase at a price of one for $\$100$ or three for $\$250$. This year, the lottery sold out with sales of $\$1,506,050$. To better plan next year's lottery, the marketing manager wants to know how many tickets were purchased under each option this year.
6. An accountant is trying to allocate production costs from two different products to their appropriate ledgers. Unfortunately, the production log sheet for last week has gone missing. However, from other documents he was able to figure out that $1,250$ units in total were produced last week. The production machinery was run for $2,562.5$ minutes, and he knows that Product A takes 1.5 minutes to manufacture while Product B takes 2.75 minutes to manufacture. How many units of each product were produced last week?

Solutions

1. $y = \$620.14$
2. $t = 0.282211$
3. $\text{Delay} = 1.5 \text{ hours}$

4. $\text{Users in 2000} = 107,998,376$
5. $\text{Direct cost} = \$3,000$
6. $\text{Gross income} = \$140,000$
7. $\text{Single tickets} = 10,363$, $\text{3-pack Tickets} = 1,879$
8. $\text{A} = 700 \text{ units}$, $\text{B} = 550 \text{ units}$

Challenge, Critical Thinking, & Other Applications

15. Jacob owns $15,000$ shares in a corporation, which represents 2% of all issued shares for the company. He sold 25 of his shares to another investor for $\$7,800$. What is the total value for all of the shares issued by the company?
16. Two cellphone companies are offering different rate plans. Rogers is offering $\$19.99$ per month, which includes a maximum of 200 weekday minutes plus $\$0.35$ for every minute above the maximum. TELUS is offering $\$39.99$ for a maximum 300 weekday minutes, but it charges $\$0.10$ for every minute above the maximum. Above how many minutes would TELUS be the better choice?
17. Marianne, William, Hendrick, and Charlotte have all decided to go into business together. They need $\$175,000$ in initial capital funding. William was able to contribute 20% less than Marianne, Hendrick contributed 62.5% more than William, and Charlotte contributed $\$5,000$ less than half as much as Marianne. How much did each partner contribute to the initial funds?
18. A mall is being constructed and needs to meet the legal requirements for parking availability. Parking laws require one parking stall for every 100 square feet of retail space. The mall is designed to have $1,200,000$ square feet of retail space. Of the total parking stalls available, 2% need to be handicap accessible, there need to be three times as many small car spaces as handicap spaces, RV spaces need to be one-quarter of the number of small car spaces, and the rest of the spaces are for regular stalls. How many of each type of parking space does the mall require?
19. Simplify the following equation into the format of "$z =$" and find the root. Verify the solution through substitution.
$$z\left(1 + 0.073 \times \frac{280}{365}\right) - \frac{z}{1 + 0.073 \times \frac{74}{365}} + \$1,000 = \$2,764.60$$
20. Find the roots for the following pair of equations. Verify the solution through substitution into both equations.

$$3\frac{4}{5}q + 0.18r = 12.2398$$

$$-5.13q - \frac{13r}{5} = -38.4674$$

Solutions

15. $\text{Company shares} = \$975,000$
16. 240 minutes
17. $\text{Marianne} = \$50,000$; $\text{William} = \$40,000$; $\text{Hendrick} = \$65,000$; $\text{Charlotte} = \$20,000$
18. $\text{Regular stalls} = 10,860$; $\text{Handicap stalls} = 240$; $\text{Small car stalls} = 720$; $\text{RV stalls} = 180$
19. $z = \$25,000$
20. $q = 2.78$; $r = 9.31$

Attribution

CHAPTER 2 SUMMARY

Formula & Symbol Hub Summary

For this chapter you used the following:

Symbols Used

- $\sum$ = Summation
- $\%C$ = Percent change
- GAvg = Geometric average
- n = Number of pieces of data
- SAvg = Simple average
- w = Weight factor for a piece of data
- WAvg = Weighted average
- x = A piece of data

Formulas Used

- Formula 2.4a – **Simple Average**

$$\text{SAvg} = \frac{\sum x}{n}$$

- Formula 2.4b – **Weighted Average**

$$\text{WAvg} = \frac{\sum wx}{\sum w}$$

- Formula 2.4c – **Geometric Average**

$$\text{GAvg} = \left([(1 + \%C_1) \times (1 + \%C_2) \times \ \ldots \ \times (1 + \%C_n)]^{\frac{1}{n}} - 1 \right) \times 100$$

- Formula 3.1b – **Rate** (see Section 3.1)

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

Key Concepts Summary

Section 2.1: Rounding of Whole Numbers and Decimals

- Procedures for proper rounding
- The rounding rules that are used throughout this textbook

Section 2.2: Fractions and Decimals (Just One Slice of Pie, Please)

- The language and types of fractions
- Working with equivalent fractions by either solving for an unknown or increasing/reducing the fraction
- Converting any fraction to a decimal format

Section 2.3: Order of Operations (Proceed in an Orderly Manner)

- A review of key mathematical operator symbols
- Rules for order of operations are known as BEDMAS

Section 2.4: Averages (What Is Typical?)

- The calculation of simple averages when everything is equal

- The calculation of weighted averages when not everything is equal
- The calculation of geometric averages when everything is multiplied together

Section 2.5: Algebraic Expressions (The Pieces of the Puzzle)

- Learning about the language of algebra
- The rules for manipulating exponents
- The rules of algebra for addition and subtraction
- The rules of algebra for multiplication
- The rules of algebra for division
- What is substitution and how it is performed?

Section 2.6: Linear Equations: Manipulating and Solving (Solving the Puzzle)

- A review of key concepts about equations
- The procedures required to solve one linear equation for one unknown variable
- The procedures required to solve two linear equations for two unknown variables

The Language of Business Mathematics

algebraic equation An equation that takes two algebraic expressions and makes them equal to each other.

algebraic expression Indicates the relationship between and mathematical operations that must be conducted on a series of numbers or variables.

base The entire amount or quantity of concern.

BEDMAS An order of operations acronym standing for **B**rackets, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, and **S**ubtraction.

common logarithm A logarithm with a base value of 10.

complex fraction A fraction that has fractions within fractions, combining elements of compound, proper, and/or improper fractions together.

compound fraction A fraction that combines an integer with either a proper or improper fraction.

denominator Any term by which some other term is divided; commonly the number on the bottom of a fraction.

divisor line The line that separates the numerator and the denominator in a fraction.

equivalent fractions Two or more fractions of any type that have the same numerical value upon completion of the division.

exponent A mathematical shorthand notation that indicates how many times a quantity is multiplied by itself

factor Components of terms that are separated from by multiplication or division signs.

fraction A part of a whole.

improper fraction A fraction in which the numerator is larger than the denominator.

left side of the equation The part of an equation that is to the left of the equal sign.

like terms Terms that have the same literal coefficient.

linear equation An algebraic expression in which the variable has an exponent of 1; when plotted, it will form a straight line.

literal coefficient A factor that is a variable.

logarithm The exponent to which a base must be raised to produce a particular power.

monomial An algebraic expression with only one term.

nomial The number of terms that appear in an algebraic expression.

nonlinear equation An algebraic expression in which the variable has an exponent other than 1; when plotted, it will not form a straight line.

numerator Any term into which some other term is being divided; commonly the number on the top in a fraction.

numerical coefficient A factor that is numerical.

percentage A part of a whole expressed in hundredths.

polynomial An algebraic expression with two or more terms.

portion Represents part of a whole or base.

proper fraction A fraction in which the numerator is smaller than the denominator.

rate Expresses a relationship between a portion and a base.

right side of the equation The part of an equation that is to the right of the equal sign.

root The value of the unknown variable that will make a linear equation true.

substitution Replacing the literal coefficients of an algebraic expression with their numerical values.

term In any algebraic expression, the components that are separated by addition and subtraction.

triangle technique A memorization technique that displays simple multiplication formulae in the form of a triangle. Anything on the same line is multiplied, and items above or below each other are divided to arrive at various solutions.

Technology

Calculator Formatting Instructions

Buttons Pushed	Calculator Display	What It Means
2nd Format	DEC=2.00	You have opened the Format window to its first setting. DEC tells your calculator how to round the calculations. In business math, it is important to be accurate. Therefore, we will set the calculator to what is called a floating display, which means your calculator will carry all of the decimals and display as many as possible on the screen.
9 Enter	DEC=9.	The floating decimal setting is now in place. Let us proceed.
↓	DEG	This setting has nothing to do with business math and is just left alone. If it does not read DEG, press 2nd Set to toggle it.
↓	US 12-31-1990	Dates can be entered into the calculator. North Americans and Europeans use slightly different formats for dates. Your display is indicating North American format and is acceptable for our purposes. If it does not read US, press 2nd Set to toggle it.
↓	US 1,000	In North America it is common to separate numbers into blocks of 3 using a comma. Europeans do it slightly differently. This setting is acceptable for our purposes. If your display does not read US, press 2nd Set to toggle it.
↓	Chn	There are two ways that calculators can solve equations. This is known as the Chain method, which means that your calculator will simply resolve equations as you punch it in without regard for the rules of BEDMAS. This is not acceptable and needs to be changed.
2nd Set	AOS	AOS stands for Algebraic Operating System. This means that the calculator is now programmed to use BEDMAS in solving equations.
2nd Quit	0.	Back to regular calculator usage.

Exponents and Signs

x^2 is used for exponents that are squares. 2^2 is keyed in as $2\ x^2$.

y^x is used for exponents that are not squares. 2^3 is keyed in as $2\ y^x\ 3\ =$.

$\pm$ is used to change the sign of a number. To use, key in the number first and then press the $\pm$ key.

Memory

STO = Store

RCL = Recall

$0\text{-}9$ = Memory cell numbers (10 in total)

To store a number on the display, press STO # (where # is a memory cell number).

To recall a number in the memory, press RCL # (where # is the memory cell where the number is stored).

Attribution

CHAPTER 3: GENERAL BUSINESS MANAGEMENT APPLICATIONS

Outline of Chapter Topics

Basic calculations are so much a part of your everyday life that you could not escape them if you tried. While driving to school, you may hear on the radio that today's forecasted temperature of 27°C is 3°C warmer than the historical average. In the news, consumers complain that gas prices rose 11% in three months. A commercial tells you that the Toyota Prius you are driving uses almost 50% less gas than the Honda Civic Coupe Si. After class you head to Anytime Fitness to cancel a gym membership because you are too busy to work out. You have used only four months of an annual membership for which you paid $449, and now you want to know how large a prorated refund you are eligible for.

On a daily basis you use basic calculations such as averages (the temperature), percent change (gas prices), ratios and proportions (energy efficiency), and prorating (the membership refund). To invest successfully, you must also apply these basic concepts. Or if you are an avid sports fan, you need basic calculations to understand your favorite players' statistics.

And it's not hard to see how these calculations would be used in the business world. Retail management examines historical average daily sales to predict future sales and to schedule the employees who will service those sales. Human resource managers continually calculate ratios between accounting and performance data to assess how efficiently labor is being used. Managers proration a company's budget across its various departments.

This chapter covers universal business mathematics you will use whether your chosen business profession is marketing, accounting, production, human resources, economics, finance, or something else altogether. To be a successful manager you need to understand percent changes, averages, ratios, proportions, and prorating.

Attribution

3.1: PERCENTAGES

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $\times 100 =$ percentage conversion factor
- $\text{Base} =$ the whole quantity
- $\text{dec} =$ decimal number to be converted to percentage
- $\text{Portion} =$ a part of the whole quantity
- $\text{Rate} =$ the relationship between the Portion and the Base

Formulas Used

- Formula 3.1a – **Percentage**

$$\% = \text{dec} \times 100$$

- Formula 3.1b – **Rate, Portion, Base**

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

Introduction

Your class just wrote its first math quiz. You got 13 out of 19 questions correct, or $\frac{13}{19}$.

In speaking with your friends Sandhu and Illija, who are in other classes, you find out that they also wrote math quizzes; however, theirs were different. Sandhu scored 16 out of 23, or $\frac{16}{23}$ while Illija got 11 out of 16, or $\frac{11}{16}$.

Who achieved the highest grade? Who had the lowest? The answers are not readily apparent, because fractions are difficult to compare.

Now express your grades in percentages rather than fractions. You scored 68%, Sandhu scored 70%, and Illija scored 69%. Notice you can easily answer the questions now. The advantage of percentages is that they facilitate comparison and comprehension.

Converting Decimals to Percentages

A **percentage** is a part of a whole expressed in hundredths. In other words, it is a value out of 100. For example, 93% means 93 out of 100, or $\frac{93}{100}$.

3.1a Percentage

Formula does not parse

\color[rgb]{1.0, 0.0, 0.0}\%\;{\color[rgb]{0.1, 0.1, 0.1}\text{is Percentage:}} This is the decimal expressed as a percentage. It must always be written with the percent ($\%$) symbol immediately following the number.

\color[rgb]{0.0, 0.5, 0.0}\times100{\color[rgb]{0.1, 0.1, 0.1}\text{ is Conversion Factor:}} A percentage is always expressed in hundredths.

{\color[rgb]{0.0, 0.0, 1.0}\text{dec}}{\color[rgb]{0.1, 0.1, 0.1}\text{ is Decimal Number:}} This is the decimal number needing to be converted into a percentage.

HOW TO

Covert a Decimal to a Percentage

Assume you want to convert the decimal number 0.0875 into a percentage. This number represents the dec variable in the formula. Substitute into Formula 3.1a:

$$\begin{aligned}\% &= \text{dec} \times 100\\ \% &= 0.0875 \times 100\\ \% &= 8.75\%\end{aligned}$$

Key Takeaways

You can also solve this formula for the decimal number. To convert any percentage back into its decimal form, you need to perform a mathematical opposite. Since a percentage is a result of multiplying by 100, the mathematical opposite is achieved by dividing by 100. Therefore, to convert 81% back into decimal form:

$$\frac{81\%}{100} = 0.81$$

Things To Watch Out For

Your Texas Instruments BAII Plus calculator has a $\%$ key that can be used to convert any percentage number into its decimal format. For example, if you press 81 and then $\%$, your calculator displays 0.81.

While this function works well when dealing with a single percentage, it causes problems when your math problem involves multiple percentages. For example, try keying $4\% + 3\% =$ into the calculator using the Formula does not parse key. This should be the same as $0.04 + 0.03 = 0.07$. Notice, however, that your calculator has 0.0412 on the display.

Why is this? As a business calculator, you BAII Plus is programmed to take portions of a whole. When you key 3% into the calculator, it takes 3% of the first number keyed in, which was 4%. As a formula, the calculator sees $4\% + (3\% \times 4\%)$. This works out to $0.04 + 0.0012 = 0.0412$.

To prevent this from happening, your best course of action is not to use the $\%$ key on your calculator. It is best to key all percentages as decimal numbers whenever possible, thus eliminating any chance that the $\%$ key takes a portion of your whole. Throughout this textbook, all percentages are converted to decimals before calculations take place.

Paths To Success

When working with percentages, you can use some tricks for remembering the formula and moving the decimal point.

Remembering the Formula

When an equation involves only multiplication of all terms on one side with an isolated solution on the other side, use a mnemonic called the triangle technique. In this technique, draw a triangle with a horizontal line through its middle. Above the line goes the solution, and below the line, separated by vertical lines, goes each of the terms involved in the multiplication. The figure to the right shows how **Formula 3.1a**$\% = \text{dec} \times 100$ would be drawn using the triangle technique.

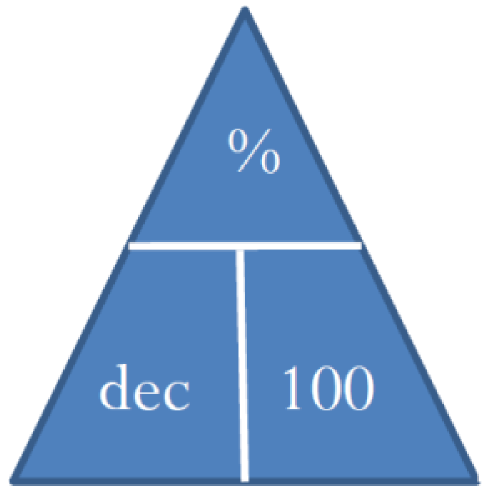

Figure 3.1.1

To use this triangle, identify the unknown variable, which you then calculate from the other variables in the triangle:

- Anything on the same line gets multiplied together. If solving for $\%$, then the other variables are on the same line and multiplied as $\text{dec} \times 100$.
- Any pair of items with one above the other is treated like a fraction and divided. If solving for dec, then the other variables are above/below each other and are divided as $\frac{\%}{100}$.

Moving the Decimal

Another easy way to work with percentages is to remember that multiplying or dividing by 100 moves the decimal over two places.

- If you are multiplying by 100, the decimal position moves two positions to the right.

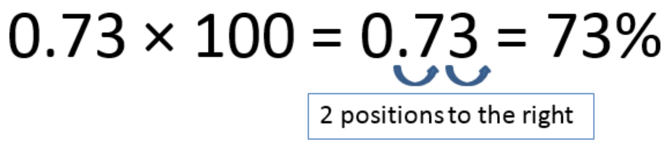

Figure 3.1.2

- If you are dividing by 100, the decimal position moves two positions to the left.

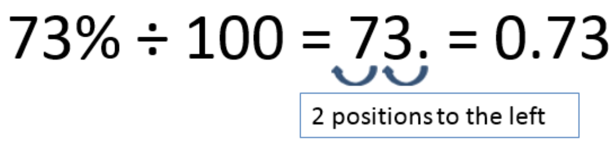

Figure 3.1.3

Example 3.1.1

Convert (a) and (b) into percentages. Convert (c) back into decimal format.

a. $\frac{3}{8}$
b. 1.3187
c. 12.399%

Solution

***Step 1:* What are we looking for?**

For questions (a) and (b), you need to convert these into percentage format. For question (c), you need to convert it back to decimal format.

***Step 2:* What do we already know?**

a. This is a fraction to be converted into a decimal, or **dec**.

b. This is **dec**.
c. This is **%**.

Step 3: **Make substitutions using the information known above.**

a. Convert the fraction into a decimal to have **dec**.

$$\text{dec} = \frac{3}{8}$$

$$\text{dec} = 0.35$$

Then apply Formula 3.1a to get the percentage:

$$\% = \text{dec} \times 100$$
$$\% = 0.35 \times 100$$
$$\% = 35\%$$

b. As this term is already in decimal format, apply Formula 3.1a to get the percentage.

$$\% = \text{dec} \times 100$$
$$\% = 1.3187 \times 100$$
$$\% = 131.87\%$$

c. This term is already in percentage format. Using the triangle technique, calculate the decimal number through

$$\text{dec} = \frac{\%}{100}$$

$$\text{dec} = \frac{12.399\%}{100}$$

$$\text{dec} = 0.12399$$

Step 4: **Provide the information in a worded statement.**

In percentage format, the first two numbers are 37.5% and 131.87%. In decimal format, the last number is 0.12399.

Rate, Portion, Base

In your personal life and career, you will often need to either calculate or compare various quantities involving fractions. For example, if your income is $\$3,000$ per month and you can't spend more than 30% on housing, what is your maximum housing dollar amount? Or perhaps your manager tells you that this year's sales of $\$1,487,003$ are 102% of last year's sales. What were your sales last year?

3.1b Rate, Portion, Base:

Formula does not parse

Formula does not parse The portion represents the part of the whole. Compare it against the base to assess the rate.

Formula does not parse The rate is the decimal form expressing the relationship between the portion and the base. Convert it to a percentage if needed by applying **Formula 3.1a**$\% = \text{dec} \times 100$. This variable can take on any value, whether positive or negative.

Formula does not parse The base is the entire amount or quantity that is of concern. It represents a whole, standard, or benchmark that you assess the portion against.

HOW TO

Calculate Percentage of a Part to the Whole

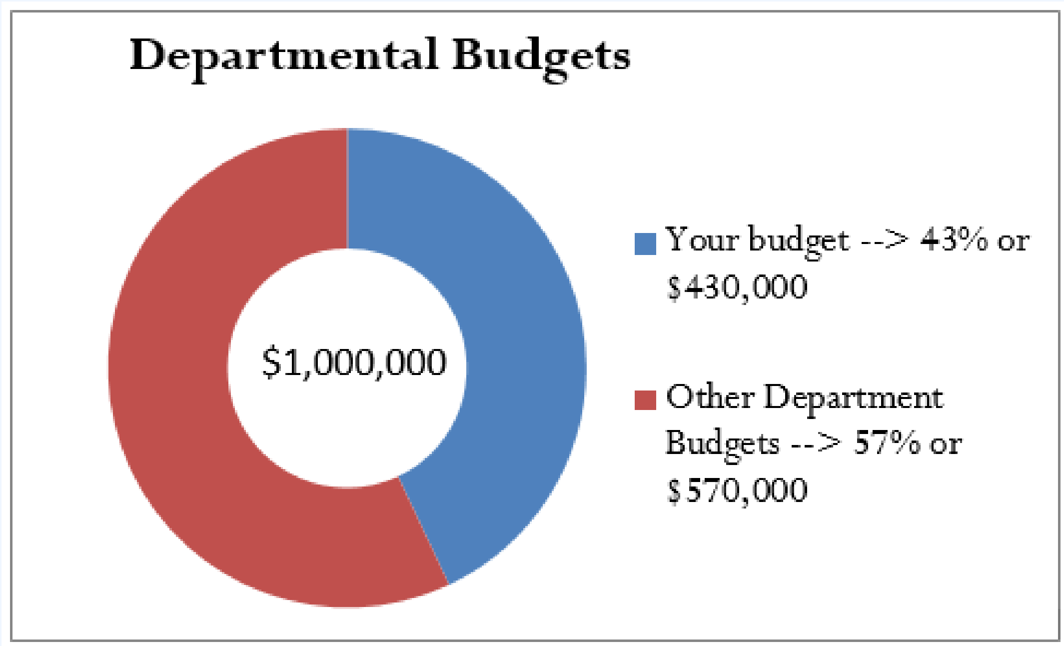

Figure 3.1.4

Assume that your company has set a budget of $\$1,000,000$. This is the entire amount of the budget and represents your *base*. Your department gets $\$430,000$ of the budget—this is your department's part of the whole and represents the *portion*. You want to know the relationship between your budget and the company's budget. In other words, you are looking for the *rate*.

Apply Formula 3.1b:

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

$$\text{Rate} = \frac{\$430,000}{\$1,000,000}$$

$$\text{Rate} = 0.43$$

Your budget is 0.43, or 43%, of the company's budget.

Key Takeaways

There are three parts to this formula. Mistakes commonly occur through incorrect assignment of a quantity to its associated variable. The table below provides some tips and clue words to help you make the correct assignment.

Table 3.1.1

Variable	Key Words	Example
Base	of	If your department can spend 43% of the company's total budget ***of $1,000,000***, what is your maximum departmental spending?
Portion	is, are	If your department can spend 43% of the company's total budget of $1,000,000, ***what is your maximum departmental spending***?
Rate	%, percent, rate	If your department can spend ***43%*** of the company's total budget of $1,000,000, what is your maximum departmental spending?

Things To Watch Out For

In resolving the rate, you must express all numbers in the same units—you cannot have apples and oranges in the same sequence of calculations. In the above example, both the company's budget and the departmental budget are in the units of dollars. Alternatively, you would not be able to calculate the rate if you had a base expressed in kilometres and a portion expressed in metres. Before you perform the rate calculation, express both in kilometres or both in metres.

Paths To Success

Formula 3.1b $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$ is another formula you can use the triangle technique for. You do not need to memorize multiple versions of the formula for each of the variables. The triangle appears to the right.

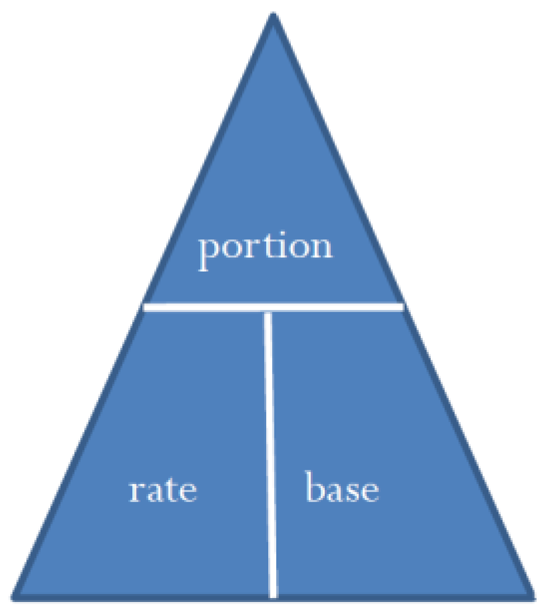

Figure 3.1.5

Be very careful when performing operations involving rates, particularly in summing or averaging rates.

Summing Rates

Summing rates requires each rate to be a part of the same whole or base. If Bob has 5% of the kilometres travelled and Sheila has 6% of the oranges, these are not part of the same whole and cannot be added. If you did, what does the 11% represent? The result has no interpretation. However, if there are 100 oranges of which Bob has 5% and Sheila has 6%, the rates can be added and you can say that in total they have 11% of the oranges.

Averaging Rates

Simple averaging of rates requires each rate to be a measure of the same variable with the same base.

If 36% of your customers are female and 54% have high income, the average of 45% is meaningless since each rate measures a different variable. Recall that earlier in this chapter you achieved 68% on your test and Sandhu scored 70%. However, your test involved 19 questions and Sandhu's involved 23 questions. These rates also cannot be simply averaged to 69% on the reasoning that $\frac{(68\% + 70\%)}{2} = 69\%$, since the bases are not the same. When two variables measure the same characteristic but have different bases (such as the math quizzes), you must use a weighted-average technique, which is discussed in Section 2.4.

When can you average rates? Hypothetically, assume Sandhu achieved his 70% by writing the same test with 19 questions. Since both rates measure the same variable and have the same base, the simple average of 69% is now calculable.

Try It

Consider the following situations and select the best answer without performing any calculations.

1) If the rate is 0.25%, in comparison to the base the portion is:

a. just a little bit smaller than the base.
b. a lot smaller than the base.
c. just a little bit bigger than the base.
d. a lot bigger than the base.

Solution

b. (0.25% is 0.0025, resulting in a very small portion)

Try It

Consider the following situations and select the best answer without performing any calculations.

2) If the portion is $\$44,931$ and the base is $\$30,000$, the rate is:

a. smaller than 100%.
b. equal to 100%.
c. larger than 100% but less than 200%.
d. larger than 200%.

Solution
c. (the portion is larger than the base, but not twice as large)

Try It

Consider the following situations and select the best answer without performing any calculations.

3) If the rate is 75% and the portion is $\$50,000$, the base is:

a. smaller than $\$50,000$.
b. larger than $\$50,000$.
c. the same as the portion and equal to $\$50,000$.

Solution
b. (the portion represents 75% of the base, meaning the base must be larger)

Example 3.1.2

Solve for the unknown in the following three scenarios.

a. If your total income is $\$3,000$ per month and you can't spend more than 30% on housing, what is the maximum amount of your total income that can be spent on housing?
b. Your manager tells you that 2014 sales are 102% of 2013 sales. The sales for 2014 are $\$1,487,003$. What were the sales in 2013?
c. In Calgary, total commercial real estate sales in the first quarter of 2008 were $\$1.28\text{ billion}$. The industrial, commercial, and institutional (ICI) land sector in Calgary had sales of $\$409.6\text{ million}$. What percentage of commercial real estate sales is accounted for by the ICI land sector?

Solution
Step 1: **What are we looking for?**

a. You are looking for the maximum amount of your income that can be spent on housing.
b. You need to figure out the sales for 2013.
c. You must determine the percentage of commercial real estate sales accounted for by the ICI land sector in Calgary.

Step 2: **What do we already know?**

a. Look for key words in the question: "what ***is*** the maximum amount" and "***of*** your total income." The total income is the base, and the maximum amount is the portion.

$$\begin{aligned}\text{Base} &= \$3,000\\ \text{Rate} &= 30\%\\ \text{Portion} &= \text{maximum amount}\end{aligned}$$

b. Look for key words in the question: "sales for 2014 ***are*** $1,487,003" and "***of*** 2013 sales." The 2014 sales is the portion, and the 2013 sales is the base.

$$\text{Portion} = \$1,487,003$$
$$\text{Rate} = 102\%$$
$$\text{Base} = 2013 \text{ sales}$$

c. Look for key words in the question: "***of*** commercial real estate sales" and "***are*** accounted for by the ICI land sector." The commercial real estate sales are the base, and the ICI land sector sales are the portion.

$$\text{Base} = \$1.28 \text{ billion}$$
$$\text{Portion} = \$409.6 \text{ million}$$
$$\text{Rate} = \text{percentage}$$

***Step 3:* Make substitutions using the information known above.**

a. Apply Formula 3.1b, but rearrange using the triangle technique to have:

$$\text{Portion} = \text{Rate} \times \text{Base}$$
$$\text{Portion} = 30\% \times \$3,000$$
$$\text{Portion} = 0.3 \times \$3,000$$
$$\text{Portion} = \$900$$

b. Apply Formula 3.1b, but rearrange using the triangle technique to have:

$$\text{Base} = \frac{\text{Portion}}{\text{Rate}}$$
$$\text{Base} = \frac{\$1,487,003}{102\%}$$
$$\text{Base} = \frac{\$1,487,003}{1.02}$$
$$\text{Base} = \$1,457,846.08$$

c. Apply Formula 3.1b:

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$
$$\text{Rate} = \frac{\$409.6 \text{ million}}{\$1.28 \text{ billion}}$$
$$\text{Rate} = \frac{\$409,600,000}{\$1,280,000,000}$$
$$\text{Rate} = 0.32$$
$$\text{Rate} = 32\%$$

Step 4: **Provide the information in a worded statement.**

a. The maximum you can spend on housing is $\$900$ per month.
b. 2013 sales were $\$1,457,846.08$.
c. The ICI land sector accounted for 32% of commercial real estate sales in Calgary for the first quarter of 2008.

Section 3.1 Exercises

Mechanics

1. Convert the following decimals into percentages.

   a. 0.4638
   b. 3.1579

c. 0.000138
d. 0.015

2. Convert the following fractions into percentages.

a. $\frac{3}{8}$
b. $\frac{17}{32}$
c. $\frac{42}{12}$
d. $2\frac{4}{5}$

3. Convert the following fractions into percentages. Round to four decimals or express in repeating decimal format as needed.

a. $\frac{46}{12}$
b. $\frac{2}{9}$
c. $\frac{3}{11}$
d. $\frac{48}{93}$

4. Convert the following percentages into decimal form.

a. 15.3%
b. 0.03%
c. 153.987%
d. d. 14.0005%

5. What percentage of $\$40,000$ is $\$27,000$?
6. What is $\frac{1}{2}\%$ of $\$500,000$?
7. $\$0.15$ is $4,900\%$ of what number?

Solutions

1a. 46.38%

1b. 315.79%

1c. 0.0138%

1d. 1.5%

2a. 37.5%

2b. 53.125%

2c. 350%

2d. 280%

3a. $383.\overline{3}\%$

3b. $22.\overline{2}\%$

3c. $27.\overline{27}\%$

3d. 51.6129%

4a. 0.153

4b. 0.0003

4c. 1.53987

4d. 0.140005

5. 67.5%

6. $\$2,500$

7. $\$0.003061$

Applications

8. In February 2009, $14,676$ mortgages were in arrears in Canada, which represented 0.38% of all mortgages. How many total mortgages were in the Canadian market at that time?
9. In 2009, medical experts predicted that one out of two Manitobans would contract some form of the H1N1 virus. If the population of Manitoba in 2009 was $1,217,200$, how many Manitobans were predicted to become ill?
10. In August 2004, Google Inc. offered its stocks to the public at $\$85$ per share. In October 2007, the share price had climbed to $\$700.04$. Express the 2007 share price as a percentage of the 2004 price.

11. During Michael Jordan's NBA career (1984–2003), he averaged a free throw completion percentage of 83.5% in regular season play. If Jordan threw $8,772$ free throws in his career, how many completed free throws did he make?
12. If total advertising expenditures on television advertising declined 4.1% to $\$141.7$ billion in the current year, how much was spent on television advertising in the previous year? Round your answer to one decimal.
13. If the new minimum wage of $\$8.75$ per hour is 102.9412% of the old minimum wage, what was the old minimum wage?
14. A machine can produce $2,500$ products per hour. If 37 of those products were defective, what is the defect rate per hour for the machine?

Solutions

8. $3,862,105$
9. $\$608,600$
10. 823.5765%
11. $7,325$
12. $\$147.8$ billion
13. $\$8.50$
14. 1.48%

Challenge, Critical Thinking, & Other Applications

15. In 2011, Manitoba progressive income tax rates were 10.8% on the first$\$31,000$, 12.75% on the next $\$36,000$, and 17.4% on any additional income. If your gross taxable earnings for the year were $\$85,000$, what percentage of your earnings did you pay in taxes?
16. In 2011, the maximum amount that you could have contributed to your RRSP (Registered Retirement Savings Plan) was the lesser of $\$22,450$ or 18% of your earned income from the previous year. How much income do you need to claim a $\$22,450$ contribution in 2011?
17. 17. Maria, a sales representative for a large consumer goods company, is paid 3% of the total profits earned by her company. Her company averages 10% profit on sales. If Maria's total income for the year was $\$60,000$, what total sales did her company realize?

18. A house was purchased six years ago for $\$214,000$. Today it lists at a price that is 159.8131% of the original purchase price. In dollars, how much has the price of the house increased over the six years?
19. An investor buys $1,000$ shares of WestJet Airlines at $\$10.30$ per share. A few months later, the investor sells the shares when their value hits 120% of the original share price. What is the price of a WestJet share when the investor sells these shares? How much money did the investor make?
20. A Honda Insight has fuel economy of 3.2 litres consumed per 100 kilometres driven. It has a fuel tank capacity of 40 litres. A Toyota Prius is rated at 4.2 L per 100 km driven. It has a fuel tank capacity of 45 L. What percentage is the total distance drivable (rounded to the nearest kilometre before calculating) of a Honda Insight compared to that of a Toyota Prius?

Solutions

15. 13.0235%
16. $\$124,722.22$
17. $\$20,000,000$
18. $\$128,000.03$
19. $\text{Share Price} = \$12.36$; $\text{Money made} = \$2,060$
20. $116.\overline{6}\%$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $\% = \text{dec} \times 100$ $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$

Attribution

3.2: PERCENT CHANGE

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $\%C =$ percent change
- $V_{\text{i}} =$ original value
- $V_{\text{f}} =$ updated value
- $RoC =$ rate of change
- $n =$ total number of time periods

Formulas Introduced

- Formula 3.2a – **Percent Change**

$$\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$$

- Formula 3.2b – **Rate of Change Over Time**

$$RoC = \left(\left(\frac{V_{\text{f}}}{V_{\text{i}}} \right)^{\frac{1}{n}} \right) \times 100$$

Introduction

On your way to work, you notice that the price of gasoline is about 10% higher than it was last month. At the office, reports indicate that input costs are down 5.4% and sales are up 3.6% over last year. Your boss asks you to analyze the year-over-year change in industry sales and submit a report. During your coffee break, you look through the day's e-flyers in your inbox. Home Depot is advertising that all garden furniture is 40% off this week; Safeway's ad says that Tuesday is 10% off day; and a *Globe and Mail* story informs you that the cost of living has risen by 3% since last year. You then log in to your financial services account, where you are happy to find that the mutual funds in your RRSP are up 6.7% from last year. What are you going to do with all this information?

Understanding how data changes from one period to the next is a critical business skill. It allows for quick assessment as to whether matters are improving or getting worse. It also allows for easy comparison of changes in different types of data over time. In this section, the concept of *percent change* is explored, which allows for the calculation of change between two points in time. Then a rate of change over time is introduced, which allows you to determine the change per period when multiple points in time are involved in the calculation.

Percent Change

It can be difficult to understand a change when it is expressed in absolute terms. Can you tell at a glance how good a deal it is to buy a $\$359$ futon for $\$215.40$? It can also be difficult to understand a change when it is expressed as a percentage of its end result. Are you getting a good deal if that $\$215.40$ futon is 60% of the regular price? What most people do find easier to understand is a change expressed as a percentage of its starting amount. Are you getting a good deal if that $\$359$ futon is on sale at 40% off? A **percent change** expresses in percentage form how much any quantity changes from a starting period to an ending period.

3.2a Percent Change

$$rgb]1.0,0.0,0.0\%C rgb]0.0,0.0,0.0 = rgb]0.0,0.0,0.0 \frac{rgb]0.0,0.0,1.0 V_f rgb]0.0,0.0,0.0 - rgb]0.0,0.5,0.0 V_{rgb]0.0,0.0,0.5,0.0i}}{rgb]0.0,0.5,0.0 V_i} rgb]0.68,0.46,0.12 \times rgb]0.68,0.46,0.12 100$$

Formula does not parse The change in the quantity is always expressed in percent format.

Formula does not parse This is the value that the quantity has become, or the number that you want to compare against a starting point.

Formula does not parse This is the value that the quantity used to be, or the number that you want all others to be compared against. Notice how the formula is structured:

1. First calculate "$V_f - V_i$," the change in the quantity. This is the numerator.
2. Divide the change by "V_i" to express the quantity change first as a fraction, then convert it into its decimal format by performing the division.

{\color[rgb]{0.68, 0.46, 0.12}\times}{\color[rgb]{0.68, 0.46, 0.12}100}\text{ is Percent Conversion:}

Recall from Section 3.1 that you convert a decimal to a percentage by multiplying it by 100. As the language suggests, percent changes are always percentages; therefore, you must include this component in the formula.

—

To calculate the percent change in a variable, you need to know the starting number and the ending number. Once you know these, Formula 3.2a represents how to express the change as a percentage. Remember two critical concepts about percent change:

Do Not Include the Original Quantity in the Change

Percent change represents the changes in the quantities, not the values of the quantities themselves. The original quantity does not count toward the percent change. Therefore, if any quantity doubles, its percent change is 100%, not 200%. For example, if the old quantity was 25 and the new quantity is 50, note that the quantity has doubled. However, 25 of the final 50 comes from the original amount and therefore does not count toward the change. We subtract it out of the numerator through calculating $50-25=25$. Therefore, the change as a percentage is:

$$\frac{50-25}{25}\times 100=100\%\,.$$

The same applies to a tripling of quantity. If our new quantity is 75 (triple the old quantity of 25), then:

$$\frac{75-25}{25}\times 100=200\%\,.$$

The original value of 25 is once again subtracted out of the numerator. The original 100% is always removed from the formula.

Negative Changes

A negative change must be expressed with a negative sign or equivalent wording. For example, if the old quantity was 20 and the new quantity is 15, this is a decrease of 5, or a change of $15-20=-5$. The percent change is

$$\frac{15-20}{20}\times 100=-25\%\,.$$

Be careful in expressing a negative percent change. There are two correct ways to do this properly:

1. "The change is −25%."
2. "It has decreased by 25%."

Note in the second statement that the word "decreased" replaces the negative sign. To avoid confusion, do not combine the negative sign with the word "decreased" — recall that two negatives make a positive, so "It has decreased by -25%" would actually mean the quantity has *increased* by 25.

HOW TO

Solve Any Question About Percent Change

Step 1: Notice that there are three variables in the formula. Identify the two known variables and the one unknown variable.

Step 2: Solve for the unknown variable using **Formula 3.2a** $\%C = \dfrac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$.

Assume last month your sales were $\$10,000$, and they have risen to $\$15,000$ this month. You want to express the percent change in sales.

The known variables are $V_{\text{i}} = \$10,000$ and $V_{\text{f}} = \$15,000$. The unknown variable is percent change, or $\%C$.

Using **Formula 3.2a** $\%C = \dfrac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$:

$$\%C = \frac{\$15,000 - \$10,000}{\$10,000} \times 100 = 0.5 \times 100 = 50\%$$

Observe that the change in sales is $\$5,000$ month-over-month. Relative to sales of $\$10,000$ last month, this month's sales have risen by 50%.

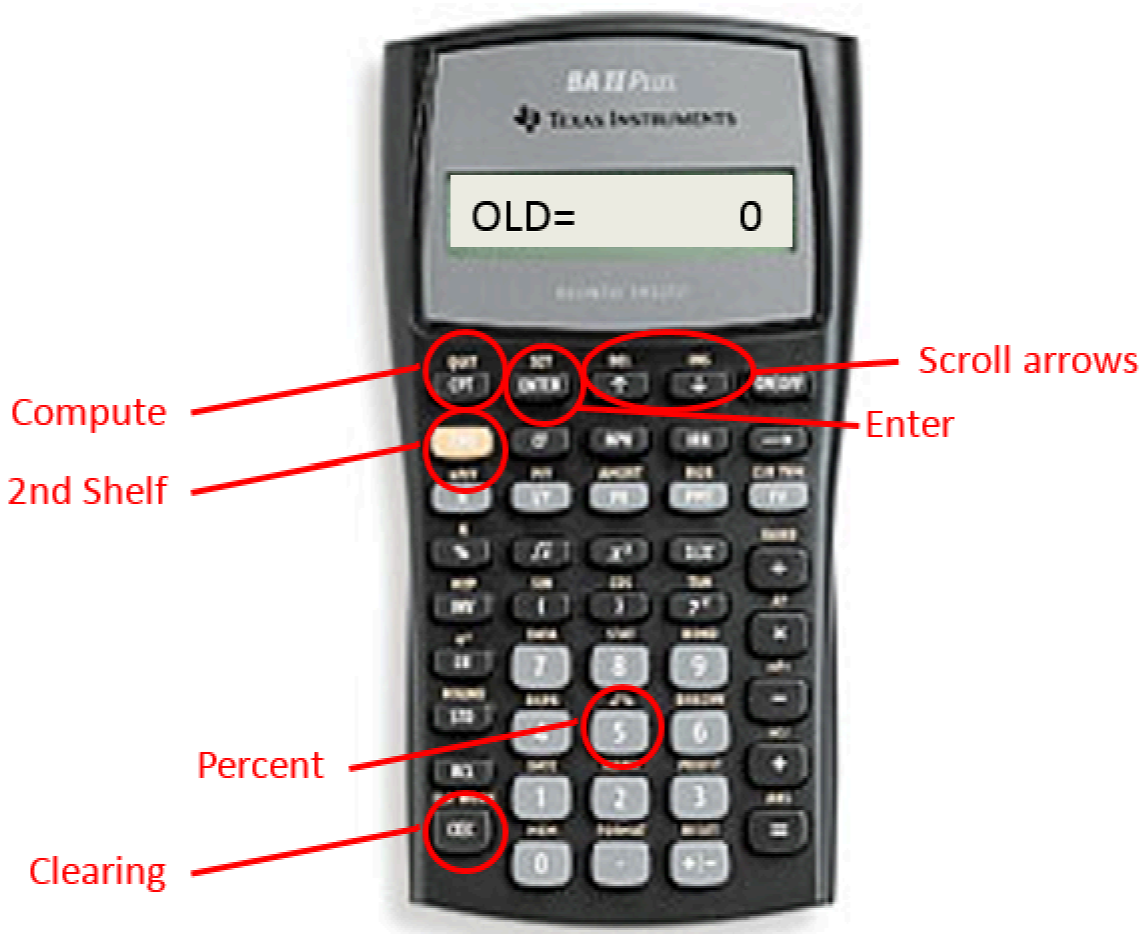

Figure 3.2.1

Key Takeaways

To access the percent change function on your BAII Plus calculator, press $2\text{nd}\Delta\%$ located above the number 5 on your keypad. Always clear the memory of any previous question by pressing **2nd CLR Work** once the function is open. Use the ↑ **(up)** and ↓ **(down)** arrows to scroll through the four lines of this function. To solve for an unknown variable, key in three of the four variables and then press **Enter**. With the unknown variable on your display, press **CPT**. Each variable in the calculator is as follows:

- **OLD** = The old or original quantity; the number that represents the starting point
- **NEW** = The new or current quantity; the number to compare against the starting point
- **%CH** = The percent change, or $\%C$ in its percent format without the Formula does not parse sign
- **#PD** = Number of consecutive periods for change. By default, it is set to 1. For now, do not touch this variable. Later in this section, when we cover rate of change over time, this variable will be explained.

Things To Watch Out For

Watch out for two common difficulties involving percent changes.

1. **Rates versus Percent Changes**. Sometimes you may be confused about whether questions involve rates (Section 3.1) or percent changes. Recall that a *rate* expresses the relationship between a portion and a base. Look for some key identifiers, such as "is/are" (the portion) and "of" (the base). For percent change, key identifiers are "by" or "than." For example, "x has increased *by* $y\%$" and "x is $y\%$ more *than* last year" both represent a percent change.
2. **Mathematical Operations**. You may imagine that you are supposed to add or subtract percent changes, but you cannot do this. Remember that percentages come from fractions. According to the rules of algebra, you can add or subtract fractions only if they share the same base (denominator). For example, if an investment increases in value in the first year by 10% and then declines in the second year by 6%, this is not an overall increase of $10\% - 6\% = 4\%$. Why? If you originally had $\$100$, an increase of 10% (which is $\$100 \times 10\% = \$10$) results in $\$110$ at the end of the first year. You must calculate the 6% decline in the second year using the $\$110$ balance, not the original $\$100$. This is a decline of $\$110 \times (-6\%) = -\$6.60$, resulting in a final balance of $\$103.40$. Overall, the percent change is 3.4%.

Paths To Success

A percent change extends the rate, portion, and base calculations introduced in Section 3.1. The primary difference lies in the portion. Instead of the portion being a part of a whole, the portion represents the change of the whole. Putting the two formulas side by side, you can calculate the rate using either approach.

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}} = \frac{V_\text{f} - V_\text{i}}{V_\text{i}}$$

Try It

1) It has been five years since Juan went shopping for a new car. On his first visit to a car lot, he had sticker shock when he realized that new car prices had risen by about 20%. What does this situation involve?

a. Percent Change
b. Rate, portion, Base

Solution

a. (the question involves how car prices have changed; note the key word “by”)

Try It

2) Manuel had his home custom built in 2006 for $\$300,000$. In 2014 he had it professionally appraised at $\$440,000$. He wants to figure out how much his home has appreciated. How would he do so?

a. The 2006 price is V_{f} and the 2014 price is V_{i}.
b. The 2006 price is V_{i} and the 2014 price is the V_{f}.

Solution

b. (the 2006 price is what the house used to be worth, which is V_{i}; the 2014 price represents the new value of the home, or V_{f}).

Example 3.2.1

In 1982, the average price of a new car sold in Canada was $\$10,668$. By 2009, the average price of a new car had increased to $\$25,683$. By what percentage has the price of a new car changed over these years?

Solution

Step 1: **What are you looking for?**

You are trying to find the percent change in the price of the new car, or $\%C$.

Step 2: **What do you already know?**

You know the old and new prices for the cars: $V_{=}\$10,668$ and $V_{\text{f}} = \$25,683$. You also know that you can apply **Formula 3.2a** $\%C = \dfrac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$ to find $\%C$.

Step 3: **Make substitutions using the information known above.**

$$\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$$

$$\%C = \frac{\$25,683 - \$10,668}{\$10,668} \times 100$$

$$\%C = \frac{\$15,015}{\$10,668} \times 100$$

$$\%C = 140.748\%$$

Step 4: **Provide the information in a worded statement.**

From 1982 to 2009, average new car prices in Canada have increased by 140.748%.

Example 3.2.2

When you purchase a retail item, the tax increases the price of the product. If you buy a $\$799.00$ Bowflex Classic Home Gym machine in Ontario, it is subject to 13% HST. What amount do you pay for the Bowflex including taxes?

Solution

Step 1: **What are you looking for?**

You are looking for the price of the Bowflex after increasing it by the sales tax. This is a "Final" price for the Bowflex.

Step 2: **What do you already know?**

You know the original price of the machine and how much to increase it by: $V_{\text{i}} = \$799.00$ and $\%C = 13\%$. You also know that, given these values, you can apply **Formula 3.2a** $\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$ to find the value of V_{f}.

Step 3: **Make substitutions using the information known above.**

$$\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$$
$$13\% = \frac{V_{\text{f}} - \$799}{\$799} \times 100$$
$$0.13 = \frac{V_{\text{f}} - \$799}{\$799}$$
$$\$103.87 = V_{\text{f}} - \$799$$
$$\$902.87 = V_{\text{f}}$$

Step 4: **Provide the information in a worded statement.**

The price of a Bowflex, after increasing the price by the taxes of 13%, is $\$902.87$.

Example 3.2.3

Consumers often object to price changes in many daily products, even though inflation and other cost changes may justify these increases. To reduce the resistance to a price increase, many manufacturers adjust both prices and product sizes at the same time. For example, the regular selling price for a bottle of shampoo was $\$5.99$ for 240 mL. To account for cost changes, the manufacturer decided to change the price to $\$5.79$, but also reduce the bottle size to 220 mL. What was the percent change in the price per millilitre?

Solution

Step 1: **What are you looking for?**

You need to find the percent change in the price per millilitre, or $\%C$.

Step 2: **What do you already know?**

You know the old price and bottle size, as well as the planned price and bottle size:

$$\text{Old price} = \$5.99 \qquad \text{Old size} = 240mL$$
$$\text{New price} = \$5.79 \qquad \text{New size} = 220mL$$

You can convert the price and size to a price per millilitre by taking the price and dividing by the size, and then find the percent change per millilitre by applying **Formula 3.2a** $\%C = \dfrac{V_\text{f} - V_\text{i}}{V_\text{i}} \times 100$.

Step 3: **Make substitutions using the information known above.**

First, find price per millilitre for Old and New prices / sizes:

$$\frac{\text{Old price}}{\text{Old size}} = \frac{\$5.99}{240mL} = \$0.024958/mL$$
$$\frac{\text{New price}}{\text{New size}} = \frac{\$5.79}{220mL} = \$0.026318/mL$$

Next, substitute into Formula 3.2a:

$$\%C = \frac{V_\text{f} - V_\text{i}}{V_\text{i}} \times 100$$
$$\%C = \frac{\$0.026318 - \$0.024958}{\$0.024958} \times 100$$
$$\%C = \frac{\$0.001359}{\$0.024958} \times 100$$
$$\%C = 5.4485\%$$

Step 4: **Provide the information in a worded statement.**

The price per mL has increased by 5.4485%. Note that to the consumer, it would appear as if the price has been lowered from $\$5.99$ to $\$5.79$, as most consumers would not notice the change in the bottle size.

Rate of Change over Time

The percent change measures the change in a variable from start to end overall. It is based on the assumption that only a single change occurs. What happens when the ending number results from multiple changes and you want to know the typical value of each change? For example, the population of the Toronto census metropolitan area (CMA) has grown from $4,263,759$ in 1996 to $5,113,149$ in 2006. What annual percentage growth in population does this reflect? Notice that we are not interested in calculating the change in population over the 10 years; instead we want to determine the percentage change in *each* of the 10 years . The **rate of change over time** measures the percent change in a variable per time period.

3.2b Rate of Change Over Time

Formula does not parse

Formula does not parse The total number of periods reflects the number of periods of change that have occurred between the Old and New quantities.

Formula does not parse This is a percentage that expresses how the quantity is changing per time period. It recognizes that any change in one period affects the change in the next period.

Formula does not parse What the quantity has become.

Formula does not parse What the quantity used to be.

Formula does not parse Rates of change over time are always expressed as percentages.

Calculating the rate of change over time is not as simple as dividing the percent change by the number of time periods involved, because you must consider the change for each time period relative to a different starting quantity. For example, in the Toronto census example, the percent change from 1996 to 1997 is based on the original population size of $4,263,759$. However, the percent change from 1997 to 1998 is based on the new population figure for 1997. Thus, even if the same number of people were added to the city in both years, the percent change in the second year is smaller because the population base became larger after the first year. As a result, when you need the percent change per time period, you must use Formula 3.2b.

HOW TO

Work With Rate of Change Over Time

When you work with any rate of change over time, follow these steps:

Step 1: Identify the three known variables and the one unknown variable.

Step 2: Solve for the unknown variable using **Formula 3.2b**

$$RoC = \left(\left(\frac{V_{\mathrm{f}}}{V_{\mathrm{i}}}\right)^{\frac{1}{n}}\right) \times 100\,.$$

Key Takeaways

On your calculator, calculate the rate of change over time using the percent change ($\%C$) function. Previously, we had ignored the **#PD** variable in the function and it was always assigned a value of 1. In rate of change, this variable is the same as n in our equation. Therefore, if our question involved 10 changes, such as the annual population change of the Toronto CMA from 1996 to 2006, then this variable is set to 10.

Paths To Success

You may find it difficult to choose which formula to use: percent change or rate of change over time. To distinguish between the two, consider the following:

- If you are looking for the **overall** rate of change from beginning to end, you need to calculate the percent change.
- If you are looking for the rate of change **per interval**, you need to calculate the rate of change over time.

Ultimately, the percent change formula is a simplified version of the rate of change over time formula where

$n = 1$. Thus you can solve any percent change question using **Formula 3.2b** $RoC = \left(\left(\frac{V_{\mathrm{f}}}{V_{\mathrm{i}}}\right)^{\frac{1}{n}}\right) \times 100$ instead of **Formula 3.2a** $\%C = \frac{V_{\mathrm{f}} - V_{\mathrm{i}}}{V_{\mathrm{i}}} \times 100$.

Try It

For each of the following, distinguish whether you should solve the question by the percent change formula or the rate of change over time formula.

3) When Peewee started five-pin bowling with the Youth Bowling Canada (YBC) in 1997, his average was 53. In 2011, he finished his last year of the YBC with an average of 248. How did his average change from 1997 to 2011?

Solution

Percent change (looking for overall change)

Try It

4) A stock was priced at $\$4.34$ per share in 2006 and reached $\$7.15$ per share in 2012. What annual return did a shareholder realize?

Solution

Rate of change over time (looking for change per year)

Try It

5) In 2004, total sales reached $\$1.2\text{ million}$. By 2010, sales had climbed to $\$4.25\text{ million}$. What is the growth in sales per year?

Solution

Rate of change over time (looking for change per year)

Example 3.2.4

Using the Toronto CMA information, where the population grew from $4,263,759$ in 1996 to $5,113,149$ in 2006, calculate the annual percent growth in the population.

Solution

Step 1: **What are you looking for?**

We need to calculate the percent change per year, which is the rate of change over time, or RoC.

Step 2: **What do you already know?**

We know the starting and ending numbers for the population along with the time frame involved.

$$\begin{aligned} V_{\text{i}} &= 4,263,759 \\ V_{\text{f}} &= 5,113,149 \\ n &= 2006 - 1996 = 10\text{ years} \end{aligned}$$

Step 3: **Make substitutions using the information known above.**

$$RoC = \left(\left(\frac{V_{\text{f}}}{V_{\text{i}}}\right)^{\frac{1}{n}} - 1\right) \times 100$$

$$RoC = \left(\left(\frac{5,113,149}{4,263,759}\right)^{\frac{1}{10}} - 1\right) \times 100$$

$$RoC = \left(1.199211^{\frac{1}{10}} - 1\right) \times 100$$

$$RoC = (1.018332 - 1) \times 100$$

$$RoC = 0.018332 \times 100$$

$$RoC = 1.8332\%$$

Step 4: **Provide the information in a worded statement.**

Over the 10 year span from 1996 to 2006, the CMA of Toronto grew by an average of 1.8332% per year.

Example 3.2.5

Kendra collects hockey cards. In her collection, she has a rookie year Wayne Gretzky card in mint condition. The book value of the card varies depending on demand for the card and its condition. If the estimated book value of the card fell by $84 in the first year and then rose by $\$113$ in the second year, determine the following:

a. What is the percent change in each year if the card is valued at $\$1,003.33$ at the end of the first year?
b. Over the course of the two years, what was the overall percent change in the value of the card?
c. What was the rate of change per year?

Solution

Step 1: **What are you looking for?**

We need to provide four answers to the questions and find the percent change in Year 1, or $\%C_1$, then the percent change in Year 2, or $\%C_2$. Using these first two solutions, we calculate both the overall percent change across both years, or $\%C_{\text{overall}}$, and the rate of change per year, or RoC.

Step 2: **What do you already know?**

We know the price of the card at the end of the first year as well as how it has changed each year.

$$\begin{aligned}\text{Price at end of first year} &= \$1,003.33\\ \text{Price change in first year} &= -\$84\\ \text{Price change in second year} &= \$113\end{aligned}$$

Furthermore, we know that we can find the percent change using **Formula 3.2a** $\%C = \frac{V_\text{f} - V_\text{i}}{V_\text{i}} \times 100$ to answer questions a and b, and use **Formula 3.2b** $RoC = \left(\left(\frac{V_\text{f}}{V_\text{i}}\right)^{\frac{1}{n}}\right) \times 100$ to find the rate of change over time for question c.

Step 3: **Make substitutions using the information known above.**

First, calculate the price at the beginning of the first year:

$$\begin{aligned}V_{\text{f}1} &= V_{\text{i}1} + \text{Change}_1\\ \$1,003.33 &= V_{\text{i}1} - \$84.00\\ \$1,087.33 &= V_{\text{i}1}\end{aligned}$$

For the percent change in Year 1, apply **Formula 3.2a** $\%C = \frac{V_\text{f} - V_\text{i}}{V_\text{i}} \times 100$:

$$\begin{aligned}\%C_1 &= \frac{V_{\text{f}1} - V_{\text{i}1}}{V_{\text{i}1}} \times 100\\ \%C_1 &= \frac{\$1,003.33 - \$1,087.33}{\$1,087.33} \times 100\\ \%C_1 &= -7.7253\%\end{aligned}$$

Calculate the price at the end of the second year:

$$V_{\text{f2}} = V_{\text{i2}} + \text{Change}_2$$
$$V_{\text{f2}} = \$1,003.33 + \$113.00$$
$$V_{\text{f2}} = \$1,116.33$$

For the percent change in $\text{Year } 2$, apply **Formula 3.2a** $\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$:

$$\%C_2 = \frac{V_{\text{f2}} - V_{\text{i2}}}{V_{\text{i2}}} \times 100$$
$$\%C_2 = \frac{\$1,116.33 - \$1,003.33}{\$1,003.33} \times 100$$
$$\%C_2 = 11.2625\%$$

For the overall percent change, take the old price at the beginning of the first year and compare it to the new price at the end of the second year. Apply **Formula 3.2a** $\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$:

$$\%C_{\text{overall}} = \frac{V_{\text{f2}} - V_{\text{i1}}}{V_{\text{i1}}} \times 100$$
$$\%C_{\text{overall}} = \frac{\$1,116.33 - \$1,087.33}{\$1,087.33} \times 100$$
$$\%C_{\text{overall}} = 2.6671\%$$

Calculate the rate of change over the two years using the same two prices, but apply **Formula 3.2b** $RoC = \left(\left(\frac{V_{\text{f}}}{V_{\text{i}}}\right)^{\frac{1}{n}}\right) \times 100$.

$$RoC = \left(\left(\frac{V_{\text{f2}}}{V_{\text{i1}}}\right)^{\frac{1}{n}} - 1\right) \times 100$$
$$RoC = \left(\left(\frac{\$1,116.33}{\$1,087.33}\right)^{\frac{1}{2}} - 1\right) \times 100$$
$$RoC = 1.3248\%$$

Step 4: **Provide the information in a worded statement.**

The value of the hockey card dropped 7.7253% in the first year and rose 11.2625% in the second year. Overall, the card rose by 2.6671% across both years, which represents a growth of 1.3248% in each year.

Section 3.2 Exercises

Mechanics

For questions 1–3, solve for the unknown (?) using **Formula 3.2a** $\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$ (percent change).

Table 3.2.1

	Old	New	Δ%
1.	\$109.95	\$115.45	?
2.	?	\$622.03	13.25%
3.	5.94%	?	−10%

4. If $\$9.99$ is changed to $\$10.49$, what is the percent change?
5. $\$19.99$ lowered by 10% is what dollar amount?
6. What amount when increased by 40% is$\$3,500$?
7. If $10,000$ grows to $20,000$ over a period of 10 years, what is the annual rate of change?

Solutions

1. 5.0023%

2. $\$549.25$
3. 5.346%
4. 5.005%
5. $\$17.99$
6. $\$2,500$
7. 7.1773%

Applications

8. How much, including taxes of 12%, would you pay for an item with a retail price of $\$194.95$?
9. From September 8, 2007 to November 7, 2007, the Canadian dollar experienced a rapid appreciation against the US dollar, going from $\$0.9482$ to $\$1.1024$. What was the percent increase in the Canadian dollar?
10. From 1996 to 2006, the "big three" automakers in North America (General Motors, Ford, and Chrysler) saw their market share drop from 71.5% to 52.7%. What is the overall change and the rate of change per year?
11. The average price of homes in Calgary fell by $\$10,000$ to $\$357,000$ from June 2009 to July 2009. The June 2009 price was 49% higher than the June 2005 price.
    a. What was the percent change from June 2009 to July 2009?
    b. What was the average price of a home in June 2005?
    c. What was the annual rate of change from June 2005 to June 2009?
12. On October 28, 2006, Saskatchewan lowered its provincial sales tax (PST) from 7% to 5%. What percent reduction does this represent?
13. A local Superstore sold $21,983$ cases of its President's Choice cola at $\$2.50$ per case. In the following year, it sold $19,877$ cases at $\$2.75$ per case.
    a. What is the percent change in price year-over-year?
    b. What is the percent change in quantity year-over-year?
    c. What is the percent change in total revenue year-over-year? (Hint: revenue = price × quantity)
14. A bottle of liquid laundry detergent priced at $\$16.99$ for a 52-load bottle has been changed to $\$16.49$ for a 48-load bottle. By what percentage has the price per load changed?

Solutions

8. $\$218.34$
9. 16.2624%
10. $\%C = -26.2937\%$; $\text{RoC} = -3.0048\%$
11a. -2.7248%
11b. $\$246,308.72$
11c. 10.4833%
12. -28.5714%
13a. 10%
13b. -9.5801%
13c. -0.5381%
14. 5.1452%

Challenge, Critical Thinking, & Other Applications

15. At a boardroom meeting, the sales manager is happy to announce that sales have risen from $\$850,000$ to $\$1,750,000$ at a rate of 4.931998% per year. How many years did it take for the sales to reach $\$1,750,000$?
16. The Nova Scotia Pension Agency needs to determine the annual cost of living adjustment (COLA) for the pension payments made to its members. To do this, it averages the consumer price index (CPI) for both the previous fiscal year and the current fiscal year. It then calculates the percent change between the two years to arrive at the COLA. If CPI information is as follows, determine the COLA that the pensioners will receive.

Table 3.2.2

Previous Fiscal Year				Current Fiscal Year			
Nov.	109.2	May	112.1	Nov.	111.9	May	114.6
Dec.	109.4	June	111.9	Dec.	112.0	June	115.4
Jan.	109.4	July	112.0	Jan.	111.8	July	115.8
Feb.	110.2	Aug.	111.7	Feb.	112.2	Aug.	115.6
Mar.	111.1	Sep.	111.9	Mar.	112.6	Sep.	115.7
Apr.	111.6	Oct.	111.6	Apr.	113.5	Oct.	114.5

17. During The Bay's warehouse clearance days, it has reduced merchandise by 60%. As a bonus, today is Scratch 'n' Save day, where you can receive up to an additional 25% off the reduced price. If you scratched the maximum of 25% off, how many dollars would you save off an item that is regularly priced at $\$275.97$? What percent savings does this represent?
18. Federal Canadian tax rates for 2010 and 2011 are listed below. For example, you pay no tax on income within the first bracket, 15% on income within the next bracket, and so on. If you earned $\$130,000$ in each year, by what percentage did your federal tax rate change? In dollars, what was the difference?

Table 3.2.3

2010 Tax Brackets	Taxed at	2011 Tax Brackets
\$0–\$10,382	0%	\$0–\$10,527
\$10,383–\$40,970	15%	\$10,528–\$41,544
\$40,971–\$81,941	22%	\$41,545–\$83,088
\$81,942–\$127,021	26%	\$83,089–\$128,800
\$127,022+	29%	\$128,801+

19. Melina is evaluating two colour laser printers for her small business. A Brother model is capable of printing 21 colour pages per minute and operates 162.5% faster than a similar Hewlett-Packard model. She needs to print $15,000$ pages for a promotion. How much less time (stated as a percentage) will it take on the Brother model?
20. A chocolate bar has been priced at $\$1.25$ for a 52 gram bar. Due to vending machine restrictions, the manufacturer needs to keep the price the same. To adjust for rising costs, it lowers the weight of the bar to 48 grams.

    a. By what percentage has the price per gram changed?
    b. If this plan is implemented over two periods, what rate of change occurs in each period?

Solutions

15. 15 years
16. 2.5148%
17. $\text{Amount saved} = \$193.18$; $\%C = -70.0004\%$
18. -0.6155%
19. $61.9048\% \text{ less time}$

20a. $8.\overline{3}\%$

20b. $4.\overline{083}\%$ per period

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$

$$RoC = \left(\left(\frac{V_{\text{f}}}{V_{\text{i}}}\right)^{\frac{1}{n}}\right) \times 100$$

Attribution

3.3: PAYROLL

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- GE = gross earnings

Formulas Used

- Formula 3.3 – **Salary & Hourly Gross Earnings**

$$\text{GE} = \text{Regular Earnings} + \text{Overtime Earnings} + \text{Holiday Earnings} + \text{Stat Holidays Worked Earnings}$$

- Formula 3.1b – **Rate, Portion, Base**

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

Introduction

You work hard at your job, and you want to be compensated properly for all the hours you put in. Assume you work full time with an hourly rate of pay of $\$10$. Last week you worked eight hours on Sunday and eight hours on Monday, which was a statutory holiday. Then you took Tuesday off, worked eight hours on each of Wednesday and Thursday, took Friday off, and worked 10 hours on Saturday. That's a total of 42 hours of work for the week. What is your gross pay? Give or take a small amount depending on provincial employment standards, it should be about $\$570$. But if you don't understand how to calculate gross earnings, you could be underpaid without ever realizing it.

Here are some notes about the content in this chapter: About 10% of Canadian workers fall under federal employment standards, which are not discussed here. This textbook generalizes the most common provincial employment standards; however, to calculate your earnings accurately requires you to apply your own provincial employment standards legislation. Part-time employment laws are extremely complex, so this textbook assumes in all examples that the employee is full time.

This section addresses the calculation of **gross earnings**, which is the amount of money earned before any deductions from your paycheque. The four most common methods of employee remuneration include salaries, hourly wages, commissions, and piecework wages.

Salary and Hourly Wages

One ad in the employment classifieds indicates compensation of $\$1,270$ biweekly, while a similar competing ad promotes wages of $\$1,400$ semi-monthly. If both job ads are similar in every other way, which job has the higher annual gross earnings? To make this assessment, you must understand how salaries work. A **salary** is a fixed compensation paid to a person on a regular basis for services rendered. Most employers pay employees by salary in occupations where the employee's work schedule generally remains constant.

In contrast, an **hourly wage** is a variable compensation based on the time an employee has worked. In contrast to a salary, this form of compensation generally appears in occupations where the number of hours is unpredictable or continually varies from period to period.

Employment Contract Characteristics

Salaried and hourly full-time employees are similar with regard to their gross earnings. The major earnings issues in an employment contract involve regular earnings structure, overtime, and holidays.

Regular Earnings Structure

An agreement with your employer outlines the terms of your employment, including the time frame and frequency of pay.

- ### Time Frame

For salaried employees, the time frame that the salary covers must be clearly stated. For example, you could receive a salary of $\$2,000$ monthly or $\$50,000$ annually. Notice that each of these salaries is followed by the specific time frame for the compensation. For hourly employees, the time frame requires identification of the wage earned per hour.

- **Frequency**

How often the gross earnings are paid out to the employee must be defined.

- **Monthly:** Earnings are paid once per month. By law, employees must receive compensation from their employer at least once per month, which equals 12 times per year.
- **Daily:** Earnings are paid at the end of every day. This results in about 260 paydays per year (5 days per week multiplied by 52 weeks per year). In a leap year, there might be one additional payday.
- **Weekly:** Earnings are paid once every week. This results in 52 paydays in any given year since there are 52 weeks per year.
- **Biweekly:** Earnings are paid once every two weeks. This results in 26 paydays in any given year since there are $52 \div 2 = 26$ biweekly periods per year.
- **Semi-monthly:** Earnings are paid twice a month, usually every half month (meaning on the $15th$ and last day of the month). This results in 24 paydays per year.

Thus, the earnings structure specifies both the time frame and the frequency of earnings. For a salaried employee, this may appear as "$\$2,000$ monthly paid semi-monthly" or "$\$50,000$ annually paid biweekly." For an hourly employee, this may appear as "$\$10$ per hour paid weekly." No matter whether you are salaried or hourly, earnings determined by your regular rate of pay are called your **regular earnings.**

* Special thanks to Steven Van Alstine (CPM, CAE), Vice-President of Education, the Canadian Payroll Association, for assistance in summarizing Canadian payroll legislation and jurisdictions.

Overtime

Overtime is work time in excess of your regular workday, regular workweek, or both. In most jurisdictions it is paid at 1.5 times your regular hourly rate (called **time-and-a-half**), though your company may voluntarily pay more or a union may have negotiated a more favourable rate such as two times your regular hourly rate (called **double time**). A contract with an employer will specify your regular workday and workweek, and some occupations are exempt from overtime. Due to the diversity of occupations, there is no set rule on what constitutes a regular workday or workweek. In most jurisdictions, a regular workweek is eight hours per day and 40 hours per week. Once you exceed these regular hours, you are eligible to receive **overtime or premium earnings**, which are based on your overtime rate of pay.

Holidays

A **statutory holiday** is a legislated day of rest with pay. Five statutory holidays are recognized throughout Canada, namely, New Year's Day, Good Friday (or Easter Monday in Quebec), Canada Day, Labour Day, and

Christmas Day. Each province or territory has an additional four to six **public holidays** (or general holidays), which may include Family Day (known as Louis Riel Day in Manitoba and Islander Day in PEI) in February, Victoria Day in May, the Civic Holiday in August, Thanksgiving Day in October, Remembrance Day in November, and Boxing Day in December. These public holidays may or may not be treated the same as statutory holidays, depending on provincial laws.

Statutory and public holidays generally require employees to receive the day off with pay. If the holiday falls on a nonworking day, it is usually the next working day that is given off instead. For example, if Christmas Day falls on a Saturday, typically the following Monday is given off with pay. Here's how holidays generally work (though you should always consult legislation for your specific jurisdiction):

- You should be given the day off with pay, called **holiday earnings**. The holiday earnings are in the amount of a regular day's earnings, and the hours involved count toward your weekly hourly totals for overtime purposes (preventing employers from shifting your work schedule that week).
- If you are required to work, the employer must offer another day off in lieu with pay. Your work on the statutory holiday is then paid at *regular earnings* and the hours involved contribute toward your weekly hourly totals for overtime purposes. You are paid holiday earnings on your future day off.
- If you are required to work and no day or rest is offered in lieu, this poses the most complex situation. Under these conditions:
- You are entitled to the holiday earnings you normally would have received for the day off. The hours that make up your holiday earnings contribute toward your weekly hourly totals for overtime purposes (again, consult your local jurisdiction).
- In addition, for the hours you worked on the statutory holiday you are entitled to overtime earnings known as **statutory holiday worked earnings**. These hours do not contribute toward your weekly hourly totals for overtime purposes since you are already compensated at a premium rate of pay. For example, assume you work eight hours on Labour Day, your normal day is eight hours, and you won't get another day off in lieu. Your employer owes you the eight hours of holiday earnings you should have received for getting the day off *plus* the eight hours of statutory holiday worked earnings for working on Labour Day.

The four forms of compensation consist of regular earnings, overtime earnings, holiday earnings, and statutory holiday worked earnings. Add these four elements together to determine total gross earnings. Formula 3.3 shows the relationship.

3.3 Salary & Hourly Gross Earnings

rgb]1.0, 0.0, 0.0GErgb]0.0, 0.0, 0.0 = rgb]0.0, 0.0, 1.0Regular Earningsrgb]0.0, 0.0, 0.0 + rgb]0.0, 0.5, 0.0Overtime Earningsrgb]0.0, 0.0, 0.0 + rgb]0.5, 0.0, 0.5Holiday Earnings
rgb]0.0, 0.0, 0.0+ rgb]0.68, 0.46, 0.12Stat Holidays Worked Earnings

rgb]0.68, 0.46, 0.12Statutory Holiday Worked Earnings: This pay shows up only if a statutory holiday is worked and the employee will not receive another paid day off in lieu. It is received in addition to the holiday pay and must be paid at a premium rate

Formula does not parse Gross earnings are earning before any deductions and represent the amount owed to the employee for services rendered. This is commonly called the gross amount of the paycheque.

Formula does not parse If a statutory holiday occurs during a pay period, this is holiday pay in an amount that represents a regular shift.

Formula does not parse Unless the employees have exceeded their daily or weekly thresholds or a holiday is involved, all hours worked are considered regular earnings

Formula does not parse Any hours worked that exceed daily or weekly thresholds fall under overtime earnings. For most individuals, this is calculated at 1.5 times their regular hourly rate.

HOW TO

Calculate Gross Earnings for Salaried Employees

To calculate the total gross earnings for a salaried employee, follow these steps:

Step 1: Analyze the employee's work performed and assign hours as needed into each of the four categories of pay. If the employee has only regular hours of pay, skip to Step 6.

Step 2: Calculate the employee's equivalent hourly rate of pay. This means converting the salary into an equivalent hourly rate:

$$\text{Equivalent Hourly Rate} = \frac{\text{Annual Salary}}{\text{Annual Hours Worked}}$$

For example, use a $\$2,000$ monthly salary requiring 40 hours of work per week. Express the salary annually by multiplying it by 12 months, yielding $\$24,000$. Express the $40 \text{ hours per week}$ annually by multiplying by $52 \text{ weeks per year}$, yielding $2,080 \text{ hours}$ worked. The equivalent hourly rate is $\$24,000 \div 2,080 = \$11.538461$.

Step 3: Calculate any holiday earnings. Take the unrounded hourly rate and multiply it by the number of hours in a regular shift, or

$$\text{Holiday Earnings} = \text{Unrounded Hourly Rate} \times \text{Hours in a Regular Shift}$$

A salaried employee earning $\$11.538461 \text{ per hour}$ having a daily eight-hour shift receives $\$11.538461 \times 8 = \$92.31$ in holiday earnings.

Step 4: Calculate any overtime earnings.

- Determine the overtime hourly rate of pay by multiplying the unrounded hourly rate by the minimum standard overtime factor of 1.5 (this could be higher if the company pays a better overtime rate than this):

$$\text{Overtime Hourly Rate} = \text{Unrounded Hourly Rate} \times 1.5$$

- Round the final result to two decimals. For the salaried worker:

$$\$11.538461 \times 1.5 = \$17.31 \text{ per overtime hour}.$$

- Multiply the overtime hourly rate by the overtime hours worked.

Step 5: Calculate any statutory holiday worked earnings which is similar to calculating overtime earnings:

$$\text{Stat Holiday Worked Earnings} = \text{Statutory Hourly Rate} \times \text{Statutory Hours Worked}$$

The statutory hourly rate is at minimum 1.5 times the unrounded hourly rate of pay. The salaried employee working eight hours on a statutory holiday receives $\$17.31 \times 8 = \$138.48$.

Step 6: Calculate the gross earnings paid at the regular rate of pay. Take the amount of the salary and divide it by the number of pay periods involved, then subtract any holiday earnings:

$$\text{Regular Earnings} = \frac{\text{Salary}}{\text{Salary Pay Periods}} - \text{Holiday Earnings}$$

You need to calculate the number of pay periods based on the regular earnings structure. For example, an annual $\$52,000$ salary paid biweekly would have 26 pay periods annually. Therefore, a regular paycheque is $\$52,000 \div 26 = \$2,000$ per paycheque.

As another example, a $\$2,000$ monthly salary paid semi-monthly has two pay periods in a single month, resulting in regular earnings of $\$2,000 \div 2 = \$1,000$ per paycheque. If a holiday is involved in the pay period, you must deduct the holiday earnings from these amounts.

Step 7: Calculate the total gross earnings by applying **Formula 3.3**
$\text{GE} = \text{Regular} + \text{OT} + \text{Holiday} + \text{Stat Worked}$.

HOW TO

Calculate Gross Earnings for Hourly Employees

To calculate the total gross earnings for an hourly employee, follow steps similar to those for the salaried employee:

Step 1: Analyze the employee's work performed and assign hours as needed into each of the four categories of pay. It is usually best to set up a table similar to the one below. This table allows you to visualize the employee's week at a glance along with totals, enabling proper assessment of their hours worked.

This table separates the four types of earnings into different rows. Enter the information into the table about the employee's workweek, placing it in the correct day and on the correct row. If any daily thresholds are exceeded, place the appropriate hours into the overtime row. Once you have completed this, total the regular hours and holiday hours for the week and check to see if they exceed any regular weekly threshold. If so, starting with the last workday of the week and working backwards, convert regular hours into overtime hours until you have reduced the regular hours to the regular weekly total.

Table 3.3.1

Type of Pay	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Total Hours	Rate of Pay	Earnings
Regular										
Holiday										
Overtime										
Holiday Worked										
								TOTAL EARNINGS		

Once you have completed the table, if the employee has only regular hours of pay, skip to step 5. Otherwise, proceed with the next step.

Step 2: Calculate any holiday earnings. Take the hourly rate and multiply it by the number of hours in a regular shift:

$$\text{Holiday Earnings} = \text{Hourly Rate} \times \text{Hours in a Regular Shift}$$

Step 3: Calculate any overtime earnings.

- Determine the overtime hourly rate of pay rounded to two decimals by multiplying the hourly rate by the minimum standard overtime factor of 1.5 (or higher).

$$\text{Overtime Hourly Rate} = \text{Hourly Rate} \times 1.5$$

- Multiply the overtime hourly rate by the overtime hours worked.

Step 4: Calculate any statutory holiday worked earnings. This is the same procedure as for a salaried employee.

Step 5: Calculate the gross earnings paid at the regular rate of pay. Take the number of hours worked and multiply it by the hourly rate of pay:

$$\text{Regular Earnings} = \text{Hours Worked} \times \text{Hourly Rate}$$

For example, 20 hours worked at $\$10 \text{ per hour}$ with no holiday earnings is $20 \times \$10 = \$200$.

Step 6: Calculate the total gross earnings by applying **Formula 3.3** $\text{GE} = \text{Regular} + \text{OT} + \text{Holiday} + \text{Stat Worked}$.

Things To Watch Out For

Be careful about the language of the payment frequency. It is very common to confuse **semi** and **bi**, and sometimes businesses use the terms incorrectly. The term **semi** generally means half. Therefore, to be paid semi-monthly means to be paid every half month. The term **bi** means two. Therefore, to be paid biweekly means to be paid every two weeks. Some companies that pay semi-monthly mistakenly state that they pay bimonthly, which in fact would mean they paid every two months.

Paths To Success

In calculating the pay for a salaried employee, this textbook assumes for simplicity that a year has exactly 52 weeks. In reality, there are 52 weeks plus one day in any given year. In a leap year, there are 52 weeks plus two days. This extra day or two has no impact on semi-monthly or monthly pay, since there are always 24 semi-months and 12 months in every year. However, weekly and biweekly earners are impacted as follows:

- If employees are paid weekly, approximately once every six years there are 53 pay periods in a single year. This would "reduce" the employees' weekly paycheque in that year. For example, assume they earn $\$52,000$ per year paid weekly. Normally, they are paid $\$52,000 \div 52 = \$1,000 \text{ per week}$. However, since there are 53 pay periods approximately every sixth year, this results in $\$52,000 \div 53 = \$981.13 \text{ per week}$ for that year.
- If employees are paid biweekly, approximately once every 12 years there are 27 pay periods in a single year. This has the same effect as the extra pay period above. For example, if they are paid $\$52,000$ per

year biweekly they normally receive $\$52,000 \div 26 = \$2,000$ per biweekly cheque. Approximately every twelfth year, they are paid $\$52,000 \div 27 = \$1,925.93$ per biweekly cheque for that year.

Many employers ignore these technical nuances in pay structure since the extra costs incurred to modify payroll combined with the effort required to calm down employees who don't understand the smaller paycheque are not worth the savings in labour. Therefore, most employers treat every year as if it has 52 weeks (26 biweeks) regardless of the reality. In essence, employees receive a bonus paycheque approximately once every six or twelve years!

Try It

1) A salaried employee whose normal workweek is 8 hours per day and $40 \text{ hours per week}$ works 8 hours each day from Monday to Saturday inclusive, where Monday was a statutory holiday. Which of the following statements is correct (assuming she will not get another day off in lieu of the holiday)?

a. The employee receives only her regular weekly earnings for 40 hours.
b. The employee receives 32 hours of regular earnings, 8 hours of holiday earnings, 8 hours of overtime earnings, and 8 hours of statutory holiday worked earnings.
c. The employee receives 40 hours of regular earnings, 8 hours of overtime earnings, and 8 hours of statutory holiday worked earnings.
d. The employee receives 40 hours of regular earnings and 8 hours of overtime earnings.

Solution

The correct answer is b. When working on the statutory holiday and not getting another day off in lieu, the salaried employee is eligible for eight hours of holiday earnings plus eight hours of statutory holiday worked earnings. The holiday earnings count toward the weekly total, but not the statutory holiday worked earnings. Thus the employee from Tuesday to Friday inclusive worked an additional 32 regular hours, bringing her weekly total to 40 hours. The work on Saturday exceeds her 40 hour workweek, and therefore all eight hours are paid as overtime earnings.

Example 3.3.1

Tristan is compensated with an annual salary of $\$65,000$ paid biweekly. His regular workweek consists of four 10-hour days, and he is eligible for overtime at 1.5 times pay for any work in excess of his regular requirements. Tristan worked regular hours for the first two weeks. Over the next two weeks, Tristan worked his regular hours and became eligible for 11 hours of overtime. During these two weeks, he worked his regular shift on Good Friday but his employer has agreed to give him another day off with pay in the future.

a. Determine Tristan's gross earnings for the first two-week pay period.
b. Determine Tristan's gross earnings for the second two-week pay period.

Solution

Step 1: **What are you looking for?**

You have been asked to calculate Tristan's gross earnings, or GE, for two consecutive pay periods.

Step 2: **What do you already know?**

You know Tristan's compensation:

$$\begin{aligned} \text{Annual Salary} &= \$65,000 \\ \text{Pay Periods} &= \text{biweekly} = 26 \text{ times per year} \\ \text{Annual Hours} &= 10/\text{day} \times 4 \text{ days/week} \times 52 \text{ weeks} = 2,080 \end{aligned}$$

You also know his work schedule:

$$\begin{aligned} \text{First Two Weeks (P1)} &= \text{regular pay} \\ \text{Overtime in P1} &= \$0 \\ \text{Second Two Weeks (P2)} &= \text{regular pay} \\ \text{Overtime in P2} &= 11 \text{ hours} \end{aligned}$$

There is a holiday in the second two weeks, but he will receive another day off in lieu.

Step 3: **Make substitutions using the information known above.**

For each biweekly pay period, apply the following steps:

Step 1: Calculate Tristan's equivalent hourly rate of pay.

$$\text{Equivalent Hourly Rate}_{\text{P1}} = \text{Only regular earnings - skip to Step 5}$$

$$\text{Equivalent Hourly Rate}_{\text{P2}} = \frac{\text{Annual Salary}}{\text{Annual Hours Worked}}$$

$$\text{Equivalent Hourly Rate}_{\text{P2}} = \frac{\$65,000}{2080}$$

$$\text{Equivalent Hourly Rate}_{\text{P2}} = \$31.25/\text{hour}$$

Step 2: Calculate holiday earnings using the equivalent hourly rate.

$$\text{Holiday Earnings}_{\text{P1}} = \$0$$

$$\text{Holiday Earnings}_{\text{P2}} = \$0$$

Tristan will take his holiday pay during $P2$ another day. Therefore, he is not eligible for this pay, and his work counts as regular hours.

Step 3: Calculate overtime earnings by taking the overtime hourly pay rate multiplied by hours worked.

$$\text{Overtime Earnings}_{\text{P1}} = \$0$$

$$\begin{aligned}
\text{Overtime Earnings}_{\text{P2}} &= \text{Overtime Hourly Rate} \times \text{Overtime Hours} \\
\text{Overtime Earnings}_{\text{P2}} &= (\text{Equivalent Hourly Rate} \times 1.5) \times 11 \\
\text{Overtime Earnings}_{\text{P2}} &= (\$31.25 \times 1.5) \times 11 \\
\text{Overtime Earnings}_{\text{P2}} &= \$46.88 \times 11 \\
\text{Overtime Earnings}_{\text{P2}} &= \$515.68
\end{aligned}$$

Step 4: Calculate statutory holiday worked earnings at the premium rate of pay.

$$\text{Statutory Worked}_{\text{P1}} = \$0$$

$$\text{Statutory Worked}_{\text{P2}} = \$0$$

Since Tristan is receiving another day off in lieu of the holiday he worked during $P2$, he is not eligible for this pay.

Step 5: Calculate regular earnings.

$$\text{Regular Earnings}_{\text{P1}} = \frac{\text{Salary}}{\text{Salary Pay Periods}} - \text{Holiday Earnings}$$

$$\text{Regular Earnings}_{\text{P1}} = \frac{\$65,000}{26} - \$0$$

$$\text{Regular Earnings}_{\text{P1}} = \$2,500$$

$$\text{Regular Earnings}_{\text{P2}} = \frac{\text{Salary}}{\text{Salary Pay Periods}} - \text{Holiday Earnings}$$

$$\text{Regular Earnings}_{\text{P2}} = \frac{\$65,000}{26} - \$0$$

$$\text{Regular Earnings}_{\text{P2}} = \$2,500$$

*Step 6: Determine total gross earnings using **Formula 3.3***

$\text{GE} = \text{Regular} + \text{OT} + \text{Holiday} + \text{Stat Worked}$.

$$\text{GE}_{\text{P1}} = \$2,500 + \$0 + \$0 + \$0 + \$2,500$$
$$\text{GE}_{\text{P1}} = \$5,000$$

$$\text{GE}_{\text{P2}} = \$2,500 + \$515.68 + \$0 + \$0$$
$$\text{GE}_{\text{P2}} = \$3,015.68$$

***Step 4:* Provide the information in a worded statement.**

For the first two-week pay period, Tristan worked only his regular hours and therefore is compensated $\$2,500$ as per his salary. For the second two-week pay period, Tristan is eligible to receive his regular hours plus his overtime, but he receives no additional pay for the worked holiday since he will receive another day off in lieu. His total gross earnings are $\$3,015.68$.

Example 3.3.2

Marcia receives an hourly wage of $\$32.16$ working on an automotive production line. Her union has negotiated a regular work day of 7.25 hours for five days totaling 36.25 hours for the week. Overtime is paid at 1.5 times her regular rate for any work that exceeds the daily or weekly limits. If work is required on a statutory holiday, her company does not give a day off in lieu and pays a premium rate of 2.5 times her regular rate. Last week, Marcia worked nine hours on Monday, her regular hours on Tuesday through Friday inclusive, and three hours on Saturday. Friday was a statutory holiday. Calculate Marcia's gross earnings for the week.

Solution

Step 1: **What are you looking for?**

You need to calculate Marcia's gross earnings, or GE, for the week.

Step 2: **What do you already know?**

Step 1: You know Marcia's pay structure and workweek:

$$\begin{aligned}\text{Regular Hourly Rate} &= \$32.16\\ \text{Overtime Hourly Pay} &= \times 1.5\\ \text{Statutory Holiday Worked Rate} &= \times 2.5\end{aligned}$$

Exceeding 7.25 hours daily or 36.25 hours weekly is overtime.

Table 3.3.2

Sunday	0
Monday	9
Tuesday	7.25
Wednesday	7.25
Thursday	7.25
Friday (statutory holiday)	7.25
Saturday	3

Step 3: **Make substitutions using the information known above.**

Take Marcia's hours and place them into the table. Assess whether any daily or weekly totals are considered overtime and make any necessary adjustments.

Table 3.3.3

Type	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total	Rate	Earnings
Regular	0	7.25	7.25	7.25	7.25		3	39.25	$32.16	
Holiday						7.25				
Overtime		1.75						1.75		
Holiday Worked						7.25		7.25		
								TOTAL EARNINGS		

She worked nine hours. Therefore, the first 7.25 hours are regular pay and the last 1.75 are overtime pay.

Friday was a statutory holiday, and she will not receive another day off in lieu. She must receive statutory holiday worked pay in addition to her hours worked.

Note the weekly total of 36.25 has been exceeded by three hours. Move Saturday's hours into overtime.

The following table is the final layout of her workweek:

Table 3.3.4

Type	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total	Rate	Earnings
Regular	0	7.25	7.25	7.25	7.25			36.25	$32.16	
Holiday						7.25				
Overtime		1.75					3	4.75		
Holiday Worked						7.25		7.25		
								TOTAL EARNINGS		

Perform necessary calculations to obtain her Gross Earnings using **Formula 3.3** $\text{GE} = \text{Regular} + \text{OT} + \text{Holiday} + \text{Stat Worked}$.

$$\begin{aligned}\text{Holiday Earnings } &= 7.25 \times \$32.16\\ \text{Holiday Earnings } &= \$233.16\end{aligned}$$

$$\begin{aligned}\text{Overtime Hourly Rate} &= \$32.16 \times 1.5\\ \text{Overtime Hourly Rate} &= \$48.24\end{aligned}$$

$$\begin{aligned}\text{Overtime Earnings} &= 4.75 \times \$48.24\\ \text{Overtime Earnings} &= \$229.14\end{aligned}$$

$$\begin{aligned}\text{Statutory Holiday Worked Rate} &= \$32.16 \times 2.5\\ \text{Statutory Holiday Worked Rate} &= \$80.40\end{aligned}$$

$$\begin{aligned}\text{Statutory Worked Earnings} &= 7.25 \times \$80.40\\ \text{Statutory Worked Earnings} &= \$582.90\end{aligned}$$

$$\begin{aligned}\text{Regular Earnings} &= 29 \times \$32.16\\ \text{Regular Earnings} &= \$932.64\end{aligned}$$

$$\begin{aligned}\text{GE} &= \$932.64 + \$229.14 + \$233.16 + \$582.90\\ \text{GE} &= \$1,977.84\end{aligned}$$

Table 3.3.5

<table>
<tr><th>Type</th><th>Sun</th><th>Mon</th><th>Tue</th><th>Wed</th><th>Thu</th><th>Fri</th><th>Sat</th><th>Total</th><th>Rate</th><th>Earnings</th></tr>
<tr><td>Regular</td><td>0</td><td>7.25</td><td>7.25</td><td>7.25</td><td>7.25</td><td></td><td></td><td rowspan="2">36.25</td><td rowspan="2">$32.16</td><td>$932.64</td></tr>
<tr><td>Holiday</td><td></td><td></td><td></td><td></td><td></td><td>7.25</td><td></td><td>$233.16</td></tr>
<tr><td>Overtime</td><td></td><td>1.75</td><td></td><td></td><td></td><td></td><td>3</td><td>4.75</td><td>$48.24</td><td>$229.14</td></tr>
<tr><td>Holiday Worked</td><td></td><td></td><td></td><td></td><td></td><td>7.25</td><td></td><td>7.25</td><td>$80.40</td><td>$582.90</td></tr>
<tr><td colspan="8"></td><td colspan="2">TOTAL EARNINGS</td><td>$1,977.84</td></tr>
</table>

***Step 4:* Provide the information in a worded statement.**

Marcia will receive total gross earnings of $\$1,977.84$ for the week.

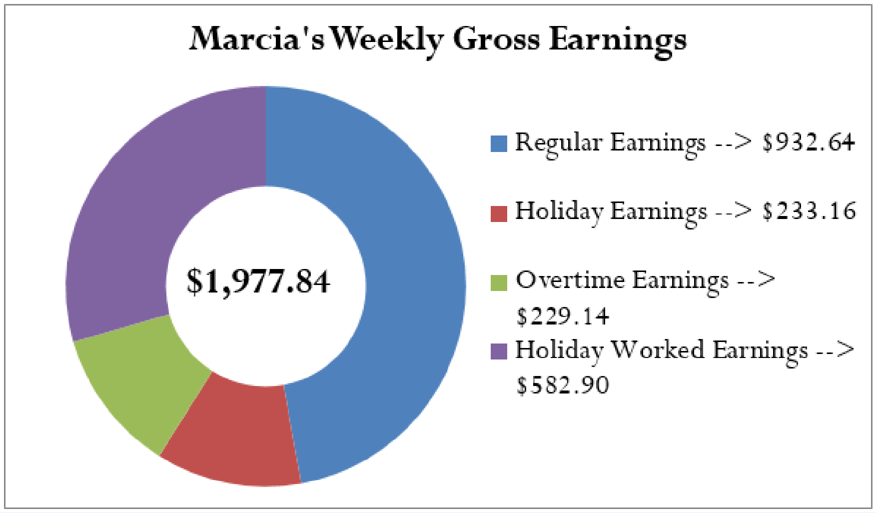

Figure 3.3.1

Commission

Over the last two weeks you sold $\$50,000$ worth of machinery as a sales representative for IKON Office Solutions Canada. IKON's compensation plan involves a straight commission rate of 3.5%. What are your gross earnings? If you sold an additional $\$12,000$ in machinery, how much more would you earn?

Particularly in the fields of marketing and customer service, many workers are paid on a commission basis. A **commission** is an amount or a fee paid to an employee for performing or completing some form of transaction. The commission typically takes the form of a percentage of the dollar amount of the transaction. Marketing and customer service industries use this form of compensation as an incentive to perform: If the representative doesn't sell anything then the representative does not get paid. Issues to be discussed about commission include what constitutes regular earnings, how to handle holidays and overtime, and the three different types of commission structures.

- ***Regular Earnings***. All commissions are considered to be regular earnings. To calculate the gross earnings for an employee, take the total amount of the transactions and multiply it by the commission rate:

$$\text{Gross Earnings} = \text{Total Transaction Amount} \times \text{Commission Rate}$$

This is not a new formula. It is a specific application of **Formula 3.1b** $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$ **: Rate,**

Portion, Base. In this case, the Base is the total amount of the transactions, the Rate is the commission rate, and the Portion is the gross earnings for the employee.

- ***Holidays and Overtime.*** Commission earners are eligible to receive overtime earnings, holiday earnings, and statutory holiday worked earnings. However, the provincial standards on these matters vary widely and the mathematics involved do not necessarily follow any one procedure or calculation. As such, this textbook leaves these issues to be covered in a payroll administration course.
- ***Types of Commission.*** Commission earnings typically follow one of the following three structures:
    - *Straight Commission*. If your entire earnings are based on your dollar transactions and calculated strictly as a percentage of the total, you are on **straight commission**. An application of **Formula 3.1b** $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$ (Rate, Portion, Base) calculates your gross earnings under this structure.
    - *Graduated Commission*. Within a **graduated commission** structure, you are offered increasing rates of commission for higher levels of performance. The theory behind this method of compensation is that the higher rewards motivate employees to perform better. An example of a graduated commission scale is found in the table below.

Table 3.3.6

Transaction Level	Commission Rate
\$0–\$100,000.00	3%
\$100,000.01–\$250,000.00	4.5%
\$250,000.01–\$500,000.00	6%
Over \$500,000.00	7.5%

Recognize that the commission rates are applied against the portion of the sales falling strictly into the particular category, not the entire balance. Thus, if the total sales equal $\$150,000$, then the first $\$100,000$ is paid at 3% while the next $\$50,000$ is paid at 4.5%.

    - *Salary Plus Commission*. If your earnings combine a basic salary together with commissions on your dollar transactions, you have a **salary plus commission** structure. No new mathematics are required for this commission type. You must combine salary calculations, discussed earlier in this section, with either a straight commission or graduated commission, as discussed above. Usually this form of compensation pays the lowest commission rate since a basic salary is already provided.

HOW TO

Calculate Commission Earnings

Follow these steps to calculate commission earnings:

Step 1: Determine which commission structure is used to pay the employee. Identify information on commission rates, graduated scales, and any salary.

Step 2: Determine the dollar amounts that are eligible for any particular commission rate and calculate commissions.

Step 3: Sum all earnings from every eligible commission rate plus any salary.

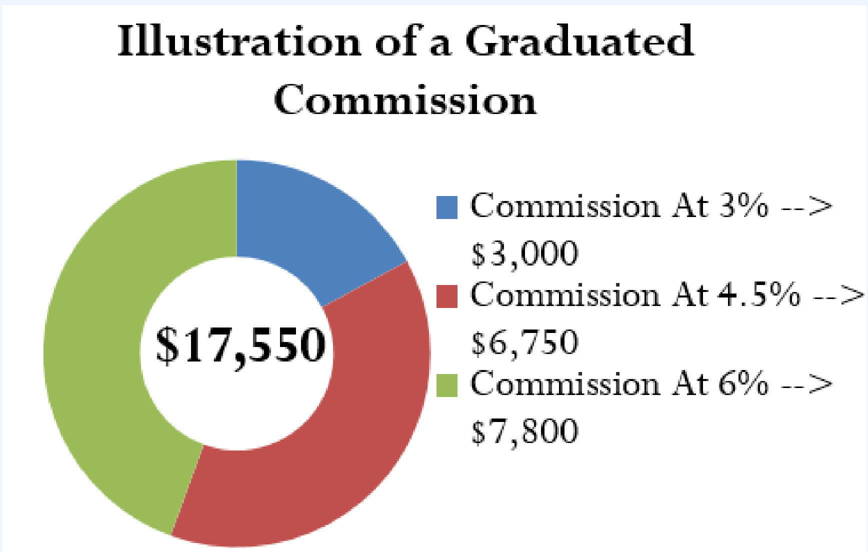

Figure 3.3.2

Table 3.3.7 Data Table for Figure 3.3.2

Commission Scale	Amount Earned
3%	$3,000
4.5%	$6,750
6%	$7,800
Total	$17,550

Let's assume $\$380,000$ of merchandise is sold. Using the previous table as our graduated commission scale, calculate commission earnings.

Step 1: Sales total $\$380,000$ and all commission rates and scales are found in the table.

Step 2: The first $\$100,000$ is compensated at 3%, equaling $\$3,000$. The next $\$150,000$ is compensated at 4.5%, equaling $\$6,750$. The last $\$130,000$ is compensated at 6%, equaling $\$7,800$. There is no compensation at the 7.5% level since sales did not exceed $\$500,000$.

Step 3: The total commission on sales of $\$380,000$ is $\$3,000 + \$6,750 + \$7,800 = \$17,550$.

Example 3.3.3

Josephine is a sales representative for Kraft Foods Canada. Over the past two weeks, she closed $\$325,000$ in retail distribution contracts. Calculate the total gross earnings that Josephine earns if

a. She is paid a straight commission of 3.45%.
b. She is paid 2% for sales on the first $\$100,000$, 3% on the next $\$100,000$, and 4% on all remaining sales.
c. She is paid a base salary of $\$2,000$ plus a commission of 3.5% on all sales above $\$100,000$.

Solution

***Step 1:* What are we looking for?**

You need to calculate the gross earnings, or GE, for Josephine under various commission structures.

***Step 2:* What do we already know?**

For all three options, $\text{total sales} = \$325,000$

a. This is a straight commission where Formula does not parse.

b. This is a graduated commission, structured as follows:

Table 3.3.8

Sales Level	Rate
$\$0$-$\$100,000.00$	2%
$\$100,000.01$-$\$200,000.00$	3%
$\$200,000.01$ and above	4%

c. This is a salary plus graduated commission, with $\text{Base Salary} = \$2,000$ and a graduated commission structure as follows:

Table 3.3.9

Sales Level	Rate
$\$0$-$\$100,000.00$	0%
$\$100,000.01$ and above	3.5%

Step 3: **Make substitutions using the information known above.**

Determine the sales eligible for each commission and calculate total commissions, then sum all commissions plus any salary to calculate total gross earnings.

a.

$$\begin{aligned}\text{Total Gross Earnings} &= \text{Rate} \times \text{Base}\\ \text{Total Gross Earnings} &= 3.45\% \times \$325,000\\ \text{Total Gross Earnings} &= 0.0345 \times \$325,000\\ \text{Total Gross Earnings} &= \$11,212.50\end{aligned}$$

b.

Table 3.3.10

Sales Level	**Rate**	**Eligible Sales at Each Rate (Base)**	**Commission Earned** $(\text{Rate} \times \text{Base})$
$\$0–\$100,000.00$	2%	$\$100,000 - \$0 = \$100,000$	$0.02 \times \$100,000 = \$2,000$
$\$100,000.01–\$200,000.00$	3%	$\$200,000 - \$100,000 = \$100,000$	$0.03 \times \$100,000 = \$3,000$
$\$200,000.01$ and above	4%	$\$325,000–\$200,000 = \$125,000$	$0.04 \times \$125,000 = \$5,000$

$$\text{Total Gross Earnings} = \$2,000 + \$3,000 + \$5,000$$
$$\text{Total Gross Earnings} = \$10,000$$

c. Recall $\text{Base Salary} = \$2,000$

Table 3.3.11

Sales Level	**Rate**	**Base**	**Commission Earned** $(\text{Rate} \times \text{Base})$
$\$0–\$100,000.00$	0%	$\$100,000 - \$0 = \$100,000$	$0 \times \$100,000 = \$0$
$\$100,000.01$ and above	3.5%	$\$325,000–\$100,000 = \$225,000$	$0.035 \times \$225,000 = \$7,875$

$$\text{Total Gross Earnings} = \$0 + \$7,875 + \$2,000$$
$$\text{Total Gross Earnings} = \$9,875$$

Step 4: **Provide the information in a worded statement.**

If Josephine is paid under straight commission, her total gross earnings will be $\$11,212.50$. Under the graduated commission, she will receive $\$10,000$ in total gross earnings. For the salary plus commission, she will receive $\$9,875$ in total gross earnings.

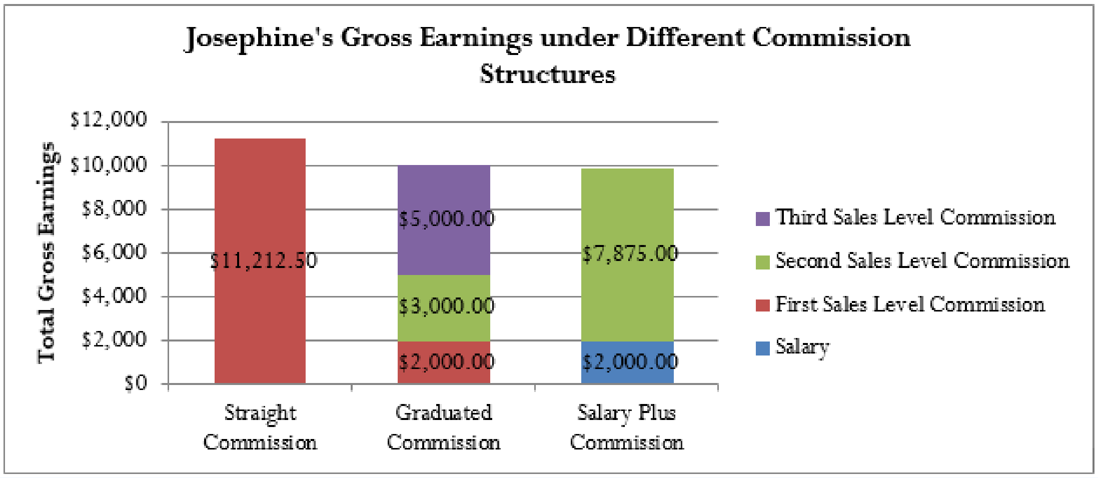

Figure 3.3.3

Piecework

Have you ever heard the phrase "pay-for-performance"? Although this phrase has many interpretations in different industries, for some people this phrase means that they get paid based on the quantity of work that they do. For example, many workers in clothing manufacturing are paid a flat rate for each article of clothing they produce. As another example, employees in fruit orchards may get paid by the number of pieces of fruit that they harvest, or simply by the kilogram. As you can see, these workers are neither salaried nor paid hourly, nor are they on commission. They earn their paycheque for performing a specific task. Therefore, a **piecework** wage compensates such employees on a per-unit basis.

This section focuses on the regular earnings only for piecework wage earners. Similar to workers on commission, piecework earners are eligible to receive overtime earnings, holiday earnings, and statutory holiday worked earnings. However, the standards vary widely from province to province, and there is not necessarily any one formula to calculate these earnings. As with commissions, this textbook leaves those calculations for a payroll administration course.

To calculate the regular gross earnings for a worker paid on a piecework wage, you require the piecework rate and how many units they are to be paid for:

$$\text{Gross Earnings} = \text{Piecework Rate} \times \text{Eligible Units}$$

This is not a new formula but another application of **Formula 3.1b** $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$ (Rate, Portion, Base). The Piecework Rate is the Rate, the Eligible Units are the Base, and the Gross Earnings are the Portion.

HOW TO

Calculate Earnings for Piecework

To calculate an employee's gross earnings for piecework, follow these steps:

Step 1: Identify the piecework rate and the level of production or units.

Step 2: Perform any necessary modifications on the production or units to match how the piecework is paid.

Step 3: Calculate the commission gross earnings by multiplying the rate by the eligible units.

Assume that Juanita is a piecework earner at a blue jean manufacturer. She is paid daily and earns $\$1.25$ for every pair of jeans that she sews. On a given day, Juanita sewed 93 pairs of jeans. Her gross earnings are calculated as follows:

Step 1: Her $\text{Piecework Rate} = \$1.25 \text{ per pair}$ with production of 93 units.

Step 2: The rate and production are both expressed by the pair of jeans. No modification is necessary.

Step 3: Her gross piecework earnings are the product of the rate and units produced, or $\$1.25 \times 93 = \$116.25$.

Things To Watch Out For

Pay careful attention to Step 2 in the procedure. In some industries, the piecework rate and the units of production do not match. For example, a company could pay a piecework rate per kilogram, but a single unit may not represent a kilogram. This is typical in some canning industries, where workers are paid per kilogram for canning the products, but the cans may only be 200 grams in size. Therefore, if workers produce five cans, they are not paid for five units produced. Rather, they are paid for only one unit produced since $\text{five cans} \times 200 \text{ g} = 1,000 \text{ g} = 1 \text{ kg}$. Before calculating piecework earnings, ensure that both the piecework rate and the eligible units are in the same terms, whether it be metric tonnes, kilograms, or otherwise.

Example 3.3.4

In outbound telemarketing, some telemarketers are paid on the basis of "completed calls." This is not commission since their pay is not based on actually selling anything. Rather, a completed call is defined as simply any phone call for which the agent speaks with the customer and a decision is reached, regardless of whether the decision was to accept, reject, or request further information. If a telemarketer produces five completed calls per hour and works $7\frac{1}{2}$-hour shifts five times per week, what are the total gross earnings she can earn over a biweekly pay period if her piecework wage is $\$3.25$ per completed call?

Solution

***Step 1:* What are you looking for?**

We are looking for the total gross earnings, or GE, for the telemarketer over the biweekly pay period.

***Step 2:* What do you already know?**

The frequency of the telemarketer's pay, along with her hours of work, piecework wage, and unit of production are known:

$$\begin{aligned}\text{Piecework Rate} &= \$3.25 \text{ per completed call}\\ \text{Hourly Units Produced} &= 5\\ \text{Hours of Work} &= 7\frac{1}{2} \text{ hours per day, five days per week}\\ \text{Frequency of Pay} &= \text{biweekly}\end{aligned}$$

***Step 3:* Make substitutions using the information known above.**

You must determine the telemarketer's production level. Calculate how many completed calls she achieves per biweekly pay period:

$$\begin{aligned}&\text{Eligible Units} = \text{Units Produced per Hour} \times \text{Hours per Day} \times \text{Days per Week} \times \text{Weeks}\\ &\text{Eligible Units} = 1.5 \times 7.5 \times 5 \times 2\\ &\text{Eligible Units} = 375\end{aligned}$$

Apply **Formula 3.1b** $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$ (adapted for piecework wages) to get the portion owing.

$$\begin{aligned} \text{GE} &= \text{Piecework Rate} \times \text{Eligible Units} \\ \text{GE} &= \$3.25 \times 375 \\ \text{GE} &= \$1,218.75 \end{aligned}$$

Step 4: **Provide the information in a worded statement.**

Over a biweekly period, the telemarketer completes 375 calls. At her piecework wage, this results in total gross earnings of $\$1,218.75$.

Section 3.3 Exercises

Mechanics

1. Laars earns an annual salary of$\$60,000$. Determine his gross earnings per pay period under each of the following payment frequencies:
   a. Monthly
   b. Semi-monthly
   c. Biweekly
   d. Weekly
2. A worker earning $\$13.66$ per hour works 47 hours in the first week and 42 hours in the second week. What are his total biweekly earnings if his regular workweek is 40 hours and all overtime is paid at 1.5 times his regular hourly rate?
3. Marley is an independent sales agent. He receives a straight commission of 15% on all sales from his suppliers. If Marley averages semi-monthly sales of $\$16,000$, what are his total annual gross earnings?

4. Sheila is a life insurance agent. Her company pays her based on the annual premiums of the customers that purchase life insurance policies. In the last month, Sheila's new customers purchased policies worth $\$35,550$ annually. If she receives 10% commission on the first $\$10,000$ of premiums and 20% on the rest, what are her total gross earnings for the month?
5. Tuan is a telemarketer who earns $\$9.00$ per hour plus 3.25% on any sales above $\$1,000$ in any given week. If Tuan works 35 regular hours and sells $\$5,715$, what are his gross earnings for the week?
6. Adolfo packs fruit in cans on a production line. He is paid a minimum wage of $\$9.10$ per hour and earns $\$0.09$ for every can packed. If Adolfo manages to average 160 cans per hour, what are his total gross earnings daily for an eight-hour shift?

Solutions

1a. $\$5,000$

1b. $\$2,5001$

1c. $\$2,307.69$

1d. $\$1,153.85$

2. $\$1,277.21$

3. $\$57,600$

4. $\$6,110$

5. $\$468.24$

6. $\$188$

Applications

7. Charles earns an annual salary of $\$72,100$ paid biweekly based on a regular workweek of 36.25 hours. His company generously pays all overtime at twice his regular wage. If Charles worked 85.5 hours over the course of two weeks, what are his gross earnings?
8. Armin is the payroll administrator for his company. In looking over the payroll, he notices the following workweek (from Sunday to Saturday) for one of the company's employees: 0, 6, 8, 10, 9, 8, and 9 hours, respectively. Monday was a statutory holiday, and with business booming the employee will not be given another day off in lieu. Company policy

pays all overtime at time-and-a-half, and all hours worked on a statutory holiday are paid at twice the regular rate. A normal workweek consists of five, eight-hour days. If the employee receives $\$22.20$ per hour, what are her total weekly gross earnings?

9. In order to motivate a manufacturer's agent to increase his sales, a manufacturer offers monthly commissions of 1.2% on the first $\$125,000$, 1.6% on the next$\$150,000$, 2.25% on the next $\$125,000$, and 3.75% on anything above. If the agent managed to sell $\$732,000$ in a single month, what commission is he owed?
10. Humphrey and Charlotte are both sales representatives for a pharmaceutical company. In a single month, Humphrey received $\$5,545$ in total gross earnings while Charlotte received $\$6,388$ in total gross earnings. In sales dollars, how much more did Charlotte sell if they both received 5% straight commission on their sales?
11. Mayabel is a cherry picker working in the Okanagan Valley. She can pick 17 kg of cherries every hour. The cherries are placed in pails that can hold 13.6 kg of cherries. If she works 40 hours in a single week, what are her total gross earnings if her piecework rate is $\$17.00$ per pail?
12. Miranda is considering three relatively equal job offers and wants to pick the one with the highest gross earnings. The first job is offering a base salary of $\$1,200$ semi-monthly plus 2% commission on monthly sales. The second job offer consists of a 9.75% straight commission. Her final job offer consists of monthly salary of $\$1,620$ plus 2.25% commission on her first $\$10,000$ in monthly sales and 6% on any monthly sales above that amount. From industry publications, she knows that a typical worker can sell $\$35,000$ per month. Which job offer should she choose, and how much better is it than the other job offers?
13. A Canadian travel agent is paid a flat rate of $\$37.50$ for every vacation booked through a certain airline. If the vacation is in North America, the agent also receives a commission of 2.45%. If the vacation is international, the commission is 4.68%. What are the total monthly gross earnings for the agent if she booked 29 North American vacations worth $\$53,125$ and 17 international vacations worth $\$61,460$?
14. Vladimir's employer has just been purchased by another organization. In the past, he has earned $\$17.90$ per hour and had a normal workweek of 37.5 hours. However, his new company only pays its employees a salary semi-monthly. How much does Vladimir need to earn each paycheque to be in the same financial position?

Solutions

7. $\$3,767.58$
8. $\$1,554$
9. $\$19,162.50$
10. $\$16,860$
11. $\$850$
12. **Best is** $\text{Offer }\#2 = \$3,412.50$; **Exceeds**$\text{Offer }\#1 = \$312.50$; **Exceeds** $\text{Offer }\#3 = \$67.50$
13. $\$5,902.89$
14. $\$1,454.38$

Challenge, Critical Thinking, & Other Applications

15. An employee on salary just received his biweekly paycheque in the amount of $\$1,832.05$, which included pay for five hours of overtime at time-and-a-half. If a normal workweek is 40 hours, what is the employee's annual salary?
16. A graduated commission scale pays 1.5% on the first $\$50,000$, 2.5% on the next $\$75,000$, and 3.5% on anything above. What level of sales would it take for an employee to receive total gross earnings of $\$4,130$?
17. A sales organization pays a base commission on the first $\$75,000$ in sales, base $+2\%$ on the next $\$75,000$ in sales, and base $+4\%$ on anything above. What is the base commission if an employee received total gross earnings of $\$7,500$ on $\$200,000$ in sales?
18. A typical sales agent for a company has annual sales of $\$4,560,000$, equally spread throughout the year, and receives a straight commission of 2%. As the new human resource specialist, to improve employee morale you have been assigned the task of developing different pay options of equivalent value to offer to the employees. Your first option is to pay them a base salary of $\$2,000$ per month plus commission. Your second option is to pay a base commission monthly on their first $\$100,000$ in sales, and a base $+2.01\%$ on anything over $\$200,000$ per month. In order to equate all the plans, determine the required commission rates, rounded to two decimals in percent format, in both options.
19. Shaquille earns an annual salary of $\$28,840.50$ paid biweekly. His normal workweek is 36.25 hours and overtime is paid at twice the regular rate. In addition, he is paid a commission of 3% of sales on the first $\$25,000$ and 4% on sales above that amount.

What are his total gross earnings during a pay period if he worked 86 hours and had sales of $\$51,750$?

20. Mandy is paid $\$9.50$ per hour and also receives a piecework wage of $\$0.30$ per kilogram, or portion thereof. A regular workday is 7.5 hours and 37.5 hours per week. Overtime is paid at time-and-a-half, and any work on a statutory holiday is paid at twice the regular rate. There is no premium piecework wage. Mandy's work record for a two-week period is listed below. Determine her total gross earnings.

Table 3.3.12

Week		Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
1	Hours worked	7.5	7.5	9	8	7.5	3
	250 g items produced	1,100	1,075	1,225	1,150	1,025	450
2	Hours worked	4 Statutory holiday, no day off in lieu	7.5	10	7.5	8	
	250 g items produced	575	1,060	1,415	1,115	1,180	

Solutions

15. $\$43,550.45$
16. $\$168,000$
17. 2%
18. $\text{Option } 1 = 1.47\%$; $\text{Option } 2 = 1.05\%$
19. $\$3,342.35$
20. $\$1,755.25$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE

CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER.

$\text{GE} = \text{Regular} + \text{OT} + \text{Holiday} + \text{Stat Worked}$ $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$

Attribution

3.4: SALES TAXES

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $\%C$ = percent change
- V_{i} = initial value
- V_{f} = final value
- S = price before taxes
- S_{tax} = price including taxes

Formulas Used

- Formula 3.1b- **Rate, Portion, Base**

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

- Formula 3.2a- **Percent Change**

$$\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$$

- Formula 3.4a- **Selling Price Including Tax**

$$S_{\text{tax}} = S + (S \times \text{Rate})$$

- Formula 3.4b – **GST/HST Remittance**

$$\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$$

Introduction

On your recent cross-Canada road trip, you purchased from many different Tim Hortons' stores. At each store, your products retailed for $\$6.99$. When you review your credit card receipts after returning home from your trip, you notice that you paid different totals everywhere. In Alberta, they only added GST and your combo cost $\$7.34$. In British Columbia, they added both PST and GST, resulting in total cost of $\$7.83$. In Ontario, they added something called HST, resulting in a total cost of $\$7.90$. You find it interesting that the same combo came to different totals as you travelled across Canada.

Three Sales Taxes

A **sales tax** is a percent fee levied by a government on the supply of products. In Canada, there are three types of sales taxes: the goods and services tax (GST), provincial sales tax (PST), and the harmonized sales tax (HST). In this section you will learn the characteristics of each of these taxes and then the mathematics for calculating any sales tax.

Goods & Services Tax (GST)

The goods and services tax, better known as **GST**, is a national federal tax of 5% that applies to the purchase of most goods and services in Canada. Every province and territory has GST. The consumer ultimately bears the burden of this sales tax.

Businesses must collect GST on most of their sales and pay GST on most purchases in the daily course of operations. However, when remitting these taxes, businesses claim a credit with the federal government to recover the GST they paid on eligible purchases. The net result is that businesses do not pay the GST on these eligible purchases. While this may outrage some people, the logic is simple. If a business pays the GST, it becomes a cost of the business, which is then passed on to consumers as it is incorporated into retail prices. When the consumer purchases the product, the consumer would be charged the GST again! In essence, a consumer would be double-taxed on all purchases if businesses paid the GST.

Some goods and services are exempt from GST. While there are many complexities and nuances to the exemptions, generally items that are deemed necessities (such as basic groceries), essential services (such as health, legal aid, and childcare), and charitable activities are nontaxable. You can find a complete listing of exemptions on the Canada Revenue Agency website at www.cra.gc.ca.

Provincial Sales Tax (PST)

Provincial sales taxes, or **PST**, are provincially administered sales taxes that are determined by each individual provincial or territorial government in Canada. The table here lists the current PST rates in Canada.

Table 3.4.1

Province/Territory	PST
Alberta	0%
British Columbia	7%
Manitoba	7%
Northwest Territories	0%
Nunavut	0%
Quebec	9.975%
Saskatchewan	6%
Yukon Territory	0%

Similar to GST, PST applies to the purchase of most goods and services in the province, and consumers bear the burden. For the same reasons as with GST, businesses typically pay the PST on purchases for non-resale items (such as equipment and machinery) and do not pay the PST on resale items. Businesses are responsible for collecting PST on sales and remitting the tax to the provincial government. Individual provincial websites list the items and services that are exempt from PST.

Harmonized Sales Tax (HST)

The harmonized sales tax, or HST, is a combination of GST and PST into a single number. Since most goods and services are subjected to both taxes anyway, HST offers a simpler method of collecting and remitting the sales tax—a business has to collect and remit only one tax instead of two. Because there are pros and cons to HST, not all provinces use this method of collection, as summarized in the table below.

Table 3.4.2

Pros	**Cons**
Items that are previously PST payable to a business are now refunded, lowering input costs and lowering consumer prices	Many items such as utilities, services, and children's clothing that are ineligible for PST become taxed at the full HST rate
Results in overall lower corporate taxes paid	Consumer cost of living increases
Increases the competitiveness of businesses and results in job creation	Tax-exempt items see prices rise because HST is being applied to services and goods such as transportation and gasoline
Businesses only remit one tax and not two, resulting in financial and auditing savings	

To understand HST you can separate it into its GST and PST components. For example, Ontario has an HST of 13%. If GST is 5% and the HST is 13%, then it is clear that the province has a PST of 8%. HST operates mostly in the same manner as GST, in that consumers generally bear the burden for paying this tax, and businesses both collect and pay the HST but have the HST reimbursed on eligible purchases when remitting taxes to the government.

Table 3.4.3

Province/Territory	**HST**
New Brunswick	15%
Newfoundland and Labrador	15%
Nova Scotia	15%
Ontario	13%
Prince Edward Island	15%

Calculating the Sales Tax Amount

With respect to sales taxes, you usually calculate two things:

1. The dollar amount of the sales tax.
2. The price of a product including the sales tax.

A sales tax is a percent rate calculated on the base selling price of the product. Therefore, if you are interested solely in the amount of the sales tax (the portion owing), applying Formula 3.1b on Rate, Portion, Base:

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}} \quad \rightarrow \quad \text{Tax Rate} = \frac{\text{Tax Amount}}{\text{Price Before Taxes}}$$

Rearranging this formula to solve for the tax amount gives the following:

$$\text{Portion} = \text{Rate} \times \text{Base}$$

$$(\text{which is the same as Tax Amount} = \text{Tax Rate} \times \text{Price Before Taxes})$$

This is not a new formula. It applies the existing formula from Chapter 3.

Calculating a Price Including Tax

When calculating a selling price including the tax, you take the regular selling price and increase it by the sales tax percentage. This is a percent change calculation using Formula 3.2a:

$$\%C = \frac{V_f - V_i}{V_i} \times 100$$

The V_i price is the price before taxes, the $\%C$ is the sales tax percentage, and you need to calculate the V_f price including the tax. Rearranging the formula for V_f, you have

$$V_f = V_i + V_i\left(\frac{\%C}{100}\right)$$

which you factor and rewrite as

$$V_f = V_i \times \left(\frac{\%C}{100}\right) \qquad or \qquad V_f = V_i + \left(V_i \times \frac{\%C}{100}\right)$$

This is a widespread application of the percent change formula. Since it can be tedious to keep rearranging **Formula 3.2a** $\%C = \dfrac{V_f - V_i}{V_i} \times 100$, Formula 3.4a expresses this relationship.

3.4a Selling Price Including Tax

Formula does not parse

Formula does not parse A subscript is added to the regular symbol of S for selling price to denote that this is the price that sums the selling price and sales tax amount together.

Formula does not parse The regular selling price before taxes.

Formula does not parse This is the sales tax in any form of GST, HST, or PST expressed in its decimal format.

Round the result of $(S \times \text{Rate})$ to two decimals. If two taxes are involved in the tax-inclusive price (such as GST and PST), you cannot combine the rates together into a single rate. For example, Manitoba has 5% GST and 7% PST. This is not necessarily equivalent to 12% tax since each tax is rounded to two decimals separately and then summed. If you use a single rate of 12%, you may miscalculate by a penny. Instead, expand the formula to include two separate $(S \times \text{Rate})$ calculations:

$$S_{\text{tax}} = S + (S \times \text{GST Rate}) + (S \times \text{PST Rate})$$

Ensure that you round off each calculation in brackets to two decimals before adding.

HOW TO

Perform Calculations Involving Sales Tax

Follow these steps to perform calculations involving sales taxes:

Step 1: Identify the pricing information. In particular, pay careful attention to distinguish whether the price is before taxes (S) or inclusive of taxes (S_{tax}). Also identify all applicable sales taxes, including GST, PST, and HST.

Step 2: Apply **Formula 3.4a** $S_{\text{tax}} = S + (S \times \text{Rate})$ to solve for the unknown variable.

Step 3: If you need to find sales tax amounts, apply **Formula 3.1b** $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$ and rearrange for portion. Ensure that for the base you use the price before taxes.

Assume a $\$549.99$ product is sold in British Columbia. Calculate the amount of the sales taxes and the price including the sales taxes.

Step 1: The price before taxes is $= \$549.99$. In British Columbia, GST is 5% and PST is 7% (from the PST Table).

Step 2: To calculate the price including the sales taxes, apply Formula 3.4a $S_{\text{tax}} = S + (S \times \text{Rate})$:

$$S_{\text{tax}} = S + (S \times \text{GST Rate}) + (S \times \text{PST Rate})$$
$$S_{\text{tax}} = \$549.99 + (\$549.99 \times 5\%) + (\$549.99 \times 7\%)$$
$$S_{\text{tax}} = \$615.99$$

Step 3: Applying the rearranged Formula 3.1b $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$ for the GST, you calculate:

$$\begin{aligned}\text{GST Tax Amount} &= \text{GST Rate} \times \text{Price Before Taxes)}\\ \text{GST Tax Amount} &= 5\% \times \$549.99\\ \text{GST Tax Amount} &= \$27.50\end{aligned}$$

Applying the same formula for the PST, you calculate:

$$\begin{aligned}\text{PST Tax Amount} &= \text{PST Rate} \times \text{Price Before Taxes)}\\ \text{PST Tax Amount} &= 7\% \times \$549.99\\ \text{PST Tax Amount} &= \$38.50\end{aligned}$$

(Note: You may notice that you could just pull these amounts from the interim calculations in Step 2). Therefore, on a $\$549.99$ item in British Columbia, $\$27.50$ in GST and $\$38.50$ in PST are owing, resulting in a price including sales taxes of $\$615.99$.

Key Takeaways

To calculate a price including a single sales tax rate, use the percent change function ($\%\mathrm{CH}$) on the calculator. You can review the full instructions for this function at the end of Chapter 3. When using this function, OLD is the price before taxes, NEW is the price after taxes, and $\%\mathrm{CH}$ is the single tax rate in its percentage format. You do not use the $\#\mathrm{PD}$ variable, which therefore defaults to 1. Note that if more than one tax rate applies on the same base, you cannot use this calculator function because of the possibility of a rounding error, as explained above.

Paths To Success

You will often need to manipulate **Formula 3.4a**$S_{\text{tax}} = S + (S \times \text{Rate})$. Most of the time, prices are advertised without taxes and you need to calculate the price including the taxes. However, sometimes prices are advertised including the taxes and you must calculate the original price of the product before taxes. When only one tax is involved, this poses no problem, but when two taxes are involved (GST and PST), combine the taxes into a single amount before you solve for S.

Try It

1) On any given product selling for the same price, put the following provinces in order from highest price to lowest price including taxes (GST and PST, or HST): Alberta, Saskatchewan, British Columbia, Ontario, Prince Edward Island.

Solution
PEI (15% HST), Ontario (13% HST), British Columbia (5% GST + 7% PST), Saskatchewan (5% GST + 6% PST), Alberta (5% GST, no PST)

Example 3.4.1

Dell Canada lists a complete computer system on its Canadian website for $\$1,999.99$. Calculate the price including taxes if the Canadian buyer is located in:

a. Alberta
b. Ontario
c. Quebec
d. British Columbia (BC)

Solution
***Step 1:* What are you looking for?**

Four answers are required. For each of the provinces listed, calculate the appropriate GST/PST or HST to add onto the price and arrive at the selling price including taxes (S_{tax}).

***Step 2:* What do you already know?**

The price of the computer and tax rates are known:

$$\begin{aligned} S &= \$1,999.99 \\ \text{Alberta sales tax} &= 5\% \text{ GST} \\ \text{Ontario sales tax} &= 13\% \text{ HST} \\ \text{Quebec sales tax} &= 5\% \text{ GST \& } 9.975\% \text{ PST} \\ \text{BC sales tax} &= 5\% \text{ GST \& } 7\% \text{ PST} \end{aligned}$$

***Step 3:* Make substitutions using the information known above.**

For all provinces, apply Formula 3.4a:

$$S_{\text{tax}} = S + (S \times \text{Rate})$$

a. For Alberta, Rate is the 5% GST. Substitute the S and Rate to solve for S_{tax}:

$$\begin{aligned} S_{\text{tax}} &= \$1,999.99 + (\$1,999.99 \times 5\%) \\ S_{\text{tax}} &= \$2,099.99 \end{aligned}$$

---

b. For Ontario, Rate is the 13% HST. Substitute the S and Rate to solve for S_{tax}:

$$\begin{aligned} S_{\text{tax}} &= \$1,999.99 + (\$1,999.99 \times 13\%) \\ S_{\text{tax}} &= \$2,259.99 \end{aligned}$$

---

c. For Quebec, both the 5% GST and 9.975% PST are based on the S. Expand the formula for each tax. Substitute these Rates with the S to arrive at S_{tax}:

$$\begin{aligned} S_{\text{tax}} &= \$1,999.99 + (\$1,999.99 \times 5\%) + (\$1,999.99 \times 9.975\%) \\ S_{\text{tax}} &= \$2,299.49 \end{aligned}$$

---

d. For British Columbia, both the 5% GST and 7% PST are based on the S. Expand the formula for each tax. Substitute these Rates with the S to arrive at S_{tax}:

$$\begin{aligned} S_{\text{tax}} &= \$1,999.99 + (\$1,999.99 \times 5\%) + (\$1,999.99 \times 7\%) \\ S_{\text{tax}} &= \$2,239.99 \end{aligned}$$

Step 4: **Provide the information in a worded statement.**

Depending on which province you reside in, the price for the computer from lowest to highest is Alberta at $\$2,099.99$, British Columbia at $\$2,239.99$, Ontario at $\$2,259.99$, and Quebec at $\$2,299.49$.

Example 3.4.2

"The Brick is having its Midnight Madness sale! Pay no taxes on products purchased during this event!" While this is good marketing, it probably goes without saying that governments do not give up the sales taxes. Essentially The Brick is advertising a tax-inclusive price. Calculate GST and PST amounts for a product advertised at $\$729.95$, including GST and PST, in Saskatchewan.

Solution

Step 1: **What are you looking for?**

To calculate the PST and GST for Saskatchewan, you must calculate the price before taxes (S). Then calculate each tax amount based on the S.

Step 2: **What do you already know?**

The price after taxes and the tax rates are as follows:

$$S_{\text{tax}} = \$729.95$$
$$\text{Tax rates} = 5\% \text{ GST and } 5\% \text{ PST}$$

Step 3: **Make substitutions using the information known above.**

Apply Formula 3.4a using the combined PST and GST as the Rate to calculate the S (remove the taxes):

$$\begin{aligned} S_{\text{tax}} &= S + (S \times \text{Rate}) \\ \$729.95 &= S + (S \times 10\%) \\ \$729.95 &= 1.1S \\ S &= \$663.59 \end{aligned}$$

Apply **Formula 3.1b** $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$ rearranged for Portion to calculate the tax amounts:

$$\text{GST Rate} = \frac{\text{GST Portion}}{S}$$

$$\begin{aligned} \text{GST Portion} &= S \times \text{GST Rate} \\ \text{GST Portion} &= \$663.59 \times 5\% \\ \text{GST Portion} &= \$33.18 \end{aligned}$$

Since PST is also 5%, $\text{PST Portion} = \$33.18$.

Step 4: **Provide the information in a worded statement.**

When paying a tax-inclusive price of $\$729.95$ in Saskatchewan, the regular selling price of the item is $\$663.59$ with $\$33.18$ in each of GST and PST included.

The GST/HST Remittance

When a business collects sales taxes, it is a go-between in the transaction. These sales tax monies do not belong to the business. On a regular basis, the business must forward this money to the government. This payment is known as a **tax remittance**.

Generally speaking, a business does not pay sales taxes. As a result, the government permits a business to take all eligible sales taxes that it paid through its acquisitions and net them against all sales taxes collected from sales. The end result is that the business is reimbursed for any eligible out-of-pocket sales tax that it paid. Formula 3.4b expresses this relationship.

$\boxed{3.4\text{b}}$ GST/HST Remittance

Formula does not parse

Formula does not parse This is the dollar amount of the remittance.

- If this amount is positive, it means that the business collected more tax than it paid out; the company must remit this balance to the government.
- If this amount is negative, it means that the business paid out more taxes than it collected; the government must refund this balance to the company.

Formula does not parse Both of these parts of the formula apply **Formula 3.1b** $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$, representing the total amount of sales tax (Portion) from all taxable amounts (Base) at the appropriate sales tax rate (Rate). The taxes collected are based on total tax-eligible revenues. The taxes paid are based on the total tax-eligible acquisitions.

HOW TO

Complete a GST/HST Remittance

Follow these steps to complete a GST/HST remittance:

Step 1: Identify the total amounts of tax-eligible revenues and acquisitions upon which the sales tax is collected or paid, respectively. Identify the applicable sales tax rate of the GST or HST.

Step 2: Calculate the total taxes collected by applying **Formula 3.1b** $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$, where the sales tax is the rate and the total revenue is the base. Solve for portion.

Step 3: Calculate the total taxes paid by applying **Formula 3.1b** $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$, where the sales tax is the rate and the total acquisitions are the base. Solve for portion.

Step 4: Apply **Formula 3.4b** $\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$ to calculate the tax remittance.

Assume a business has paid GST on purchases of $\$153,000$. It has also collected GST on sales of $\$358,440$. Calculate the GST remittance.

Step 1: Identifying the variables, you have:

$$\text{Total Revenue} = \$358,440$$
$$\text{Total Acquisitions} = \$153,000$$
$$\text{GST Rate} = 5\%$$

Step 2: Calculate taxes collected by applying Formula 3.1b $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$, where:

$$\text{GST collected} = 5\% \times \$358,440$$
$$\text{GST collected} = \$17,922$$

Step 3: Calculate taxes paid by applying Formula 3.1b $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$, where:

$$\text{GST paid} = 5\% \times \$153,000$$
$$\text{GST paid} = \$7,650$$

Step 4: To calculate the remittance, apply **Formula 3.4b** $\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$ and calculate:

$$\text{Remit} = \$17,922 - \$7,650$$
$$\text{Remit} = \$10,272$$

The business should remit a cheque for $\$10,272$ to the government.

Paths To Success

A shortcut can help you calculate the GST/HST Remittance using **Formula 3.4b** $\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$. If you do not need to know the actual amounts of the tax paid and collected, you can net GST/HST–eligible revenues minus acquisitions and multiply the difference by the tax rate:

$$\text{Remit} = (\text{Revenues} - \text{Acquisitions}) \times \text{Rate}$$

In the example above,

$$\text{Remit} = (\$358,440 - \$153,000) \times 5\%$$
$$\text{Remit} = \$10,272$$

If this calculation produces a negative number, then the business receives a refund instead of making a remittance.

Example 3.4.3

An Albertan lumber company reported the following quarterly purchases and sales in its 2013 operating year:

Table 3.4.4

Quarter	Purchases	Sales
Winter	$1,316,000	$911,500
Spring	$1,266,200	$1,473,300
Summer	$1,053,700	$1,381,700
Fall	$878,100	$1,124,600

Assuming all purchases and sales are eligible and subject to GST, calculate the GST remittance or refund for each quarter.

Solution

Step 1: **What are you looking for?**

To calculate the GST tax remittance, or Tax Remit, for each of the quarters.

Step 2: **What do you already know?**

From the information provided, the total purchases and sales for each quarter along with the $\text{GST} = 5\%$, which is the Tax Rate, are known.

Step 3: **Make substitutions using the information known above.**

For each quarter, calculate the GST collected by rearranging and applying **Formula 3.1b**

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}.$$

$$\text{GST Collected} = \text{Sales} \times \text{GST Rate}$$

$$\begin{aligned}\text{GST Collected}_{\text{Winter}} &= \$911,500 \times 5\% \\ \text{GST Collected}_{\text{Winter}} &= \$45,575\end{aligned}$$

$$\begin{aligned}\text{GST Collected}_{\text{Spring}} &= \$1,473,300 \times 5\% \\ \text{GST Collected}_{\text{Spring}} &= \$73,650\end{aligned}$$

$$\begin{aligned}\text{GST Collected}_{\text{Summer}} &= \$1,381,700 \times 5\% \\ \text{GST Collected}_{\text{Summer}} &= \$69,085\end{aligned}$$

$$\begin{aligned}\text{GST Collected}_{\text{Fall}} &= \$1,124,600 \times 5\% \\ \text{GST Collected}_{\text{Fall}} &= \$56,230\end{aligned}$$

Calculate the tax paid for each quarter.

$$\text{GST Paid} = \text{Purchases} \times \text{GST Rate}$$

$$\begin{aligned}\text{GST Paid}_{\text{Winter}} &= \$1,316,00 \times 5\% \\ \text{GST Paid}_{\text{Winter}} &= \$65,800\end{aligned}$$

$$\begin{aligned}\text{GST Paid}_{\text{Spring}} &= \$1,266,200 \times 5\% \\ \text{GST Paid}_{\text{Spring}} &= \$63,310\end{aligned}$$

$$\begin{aligned}\text{GST Paid}_{\text{Summer}} &= \$1,053,700 \times 5\% \\ \text{GST Paid}_{\text{Summer}} &= \$52,645\end{aligned}$$

$$\begin{aligned}\text{GST Paid}_{\text{Summer}} &= \$818,100 \times 5\% \\ \text{GST Paid}_{\text{Summer}} &= \$43,905\end{aligned}$$

Apply Formula 3.4b for each quarter.

$$\text{Remit} = \text{GST Collected} - \text{GSTPaid}$$

$$\begin{aligned} \text{Remit}_{\text{Winter}} &= \$45,575 - \$65,800 \\ \text{Remit}_{\text{Winter}} &= -\$20,225 \end{aligned}$$

$$\begin{aligned} \text{Remit}_{\text{Spring}} &= \$73,650 - \$63,310 \\ \text{Remit}_{\text{Spring}} &= \$10,340 \end{aligned}$$

$$\begin{aligned} \text{Remit}_{\text{Summer}} &= \$69,085 - \$52,685 \\ \text{Remit}_{\text{Summer}} &= \$16,400 \end{aligned}$$

$$\begin{aligned} \text{Remit}_{\text{Fall}} &= \$56,230 - \$43,905 \\ \text{Remit}_{\text{Fall}} &= \$12,325 \end{aligned}$$

Step 4: **Provide the information in a worded statement.**

In the Winter quarter, the company gets a GST refund of $\$20,225$. In the other three quarters, the company remits to the government payments of $\$10,340$, $\$16,400$, and $\$12,325$, respectively.

Section 3.4 Exercises

Mechanics

You are purchasing a new BlackBerry at the MSRP of $\$649.99$. Calculate the price including taxes in the following provinces or territories:

1. Northwest Territories
2. New Brunswick
3. Nova Scotia

4. 4. British Columbia

The Brick is advertising a new Serta mattress nationally for a price of $\$899.99$ including taxes. What is the price before taxes and the sales tax amounts in each of the following provinces?

5. Ontario
6. Saskatchewan
7. Audiophonic Electronics is calculating its HST remittance in Prince Edward Island. For each of the following months, calculate the HST remittance or refund on these HST-eligible amounts.

Table 3.4.5

Month	Purchases	Sales
January	$48,693	$94,288
February	$71,997	$53,639

8. Airwaves Mobility is calculating its GST remittance in Alberta. For each of the following quarters, calculate the GST remittance or refund on these GST-eligible amounts.

Table 3.4.6

Quarter	Purchases	Sales
Winter	$123,698	$267,122
Spring	$179,410	$158,905
Summer	$216,045	$412,111
Fall	$198,836	$175,003

Solutions

1. $S_{\text{tax}} = \$682.49$
2. $S_{\text{tax}} = \$747.49$
3. $S_{\text{tax}} = \$747.33$
4. $S_{\text{tax}} = \$727.99$
5. $S = \$796.45$; $\text{HST Tax Amount} = \$103.54$
6. $S = \$810.80$; $\text{GST Tax Amount} = \$40.54$;

$\text{PST Tax Amount} = \$48.65$

7. $\text{January Remit} = \$6,839.25$; $\text{February Refund} = \$2,753.70$
8. $\text{Winter Remit} = \$7,171.20$; $\text{Spring Refund} = \$1,025.25$;
   $\text{Summer Remit} = \$9,803.30$; $\text{Fall Refund} = \$1,191.65$

Applications

9. Elena lives in Nova Scotia and has relatives in Alberta, Saskatchewan, and Quebec. She gets together with them often. She wants to purchase a new aerobic trainer and would like to pay the lowest price. If a family member buys the item, Elena can pick it up at one of their regular family gatherings. The price of the trainer for each province is listed below:

Table 3.4.7

Province	Regular Selling Price before Taxes
Nova Scotia	$1,229.50
Alberta	$1,329.95
Saskatchewan	$1,274.25
Quebec	$1,219.75

   a. Where should Elena have the aerobic trainer purchased and how much would she pay?
   b. How much money would she save from her most expensive option?

10. Mary Lou just purchased a new digital camera in Nunavut for $\$556.49$ including taxes. What was the price of the camera before taxes? What amount of sales tax is paid?
11. Marley is at Peoples Jewellers in New Brunswick wanting to purchase an engagement ring for his girlfriend. The price of the ring is $\$2,699.95$. If the credit limit on his credit card is $\$3,000$, will he be able to purchase the ring on his credit card? If not, what is the minimum amount of cash that he must put down to use his credit card?
12. In the IKEA store in Vancouver, British Columbia, you are considering the purchase of a set of kitchen cabinets priced at $\$3,997.59$. Calculate the amount of GST and PST you must pay for the cabinets, along with the total price including taxes.
13. A company in Saskatchewan recorded the following GST-eligible purchases and sales throughout the year. Determine the GST remittance or refund per quarter.

Table 3.4.8

Quarter	Purchases	Sales
1st	$2,164,700	$2,522,000
2nd	$1,571,300	$2,278,700
3rd	$1,816,100	$1,654,000
4th	$2,395,900	$1,911,700

14. 14. A manufacturer in Nova Scotia recorded the following HST-eligible purchases and sales in its first three months of its fiscal year. Determine the HST remittance or refund per month.

Table 3.4.9

Month	Purchases	Sales
March	$20,209	$26,550
April	$28,861	$20,480
May	$22,649	$42,340

Solutions

9a. $\text{Alberta is best} = \$1,396.45$

9b. $\text{Savings} = \$17.97$

10. $S = \$529.99$; $\text{GST Tax Amount} = \$26.50$

11. $\text{Can't purchase at } S_{\text{tax}} = \$3,104.94$; $\text{Down Payment} = \$91.25$

12. $S_{\text{tax}} = \$4,477.30$; $\text{GST Tax Amount} = \$199.88$; $\text{PST Tax Amount} = \$279.83$

13. $\text{Q1 Remit} = \$17,865$; $\text{Q2 Remit} = \$35,370$; $\text{Q3 Refund} = \$8,105$; $\text{Q4 Refund} = \$24,210$

14. $\text{March Remit} = \$951.15$; $\text{April Refund} = \$1,257.15$; $\text{May Remit} = \$2,953.65$

Challenge, Critical Thinking, & Other Applications

15. If the selling price of an item is 6% higher in Yukon than in Ontario, will the price including taxes be higher in Yukon or Ontario? What percentage more?
16. Colin just travelled across the country on a road trip. He bought some skis in Alberta for $\$879.95$ plus tax, a boombox in British Columbia for $\$145.58$ including taxes, a Niagara Falls souvenir in Ontario for $99.97 plus tax, and some maple syrup in Quebec for $\$45.14$ including tax. Overall, how much GST, PST, and HST did Colin pay on his trip?
17. Cisco Enterprises in Ontario purchased the following in a single month:
    - $16,000$ units of network routers at $\$79.25$ each, priced at $\$97.97$ each
    - $12,000$ units of wireless LAN adapters at $\$129.95$ each, priced at $\$189.55$ each
    - $13,500$ units of computer boards at $\$229.15$ each, priced at $\$369.50$ each.

    Assuming that all units purchased are sold during the same month and that all purchases and sales are taxable, calculate the tax remittance or refund for the month.
18. In Quebec, the PST used to be calculated on the price including GST. When the PST was calculated in this manner, what PST rate did Quebec set to arrive at the same price including taxes?
19. For each of the following situations, compute the selling price of the product before taxes in the other province/territory that would result in the same selling price including taxes as the item listed.

Table 3.4.10

	Price before Tax	Sold In	Find Equivalent Price before Tax in This Province
a.	$363.75	British Columbia	Prince Edward Island
b.	$1,795.00	Alberta	Manitoba
c.	$19,995.95	Saskatchewan	Ontario
d.	$4,819.35	New Brunswick	Quebec

20. A company made the following taxable transactions in a single month. Compute the GST remittance on its operations assuming all sales and purchases are eligible for GST.

Solutions

15. $\text{Higher in Ontario by } 1.5274\%$

16. $\text{Total PST} = \$13.02$; $\text{Total GST} = \$52.46$; $\text{Total HST} = \$13$

17. $\text{Remit} = \$378,227.85$

18. $\text{PST Rate} = 9.5\%$

19a. $S = \$354.26$

19b. $S = \$1,682.81$

19c. $S = \$19,642.04$

19d. $S = \$4,820.40$

20. $\text{Remit} = \$31,331.89$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$ $S_{\text{tax}} = S + (S \times \text{Rate})$ $\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$ $\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$

Attribution

3.5: PROPERTY TAXES

Symbol & Formula Hub

For this section you will need the following:

Symbols Used

- $\sum$ = summation symbol
- $\%C$ = percent change
- V_{i} = initial value
- V_{f} = final value
- AV = assessed value
- PTR = property tax rate

Formulas Used

- Formula 3.2a – **Percent Change**

$$\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$$

- Formula 3.5 – **Property Taxes**

$$\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$$

Introduction

As you drive through your neighbourhood, you pass a city crew repairing the potholes in the road. Hearing sirens, you check your rear-view mirror and pull to the side of the road as a police car and fire engine race by, heading toward some emergency. Pulling back out, you drive slowly through a public school zone, where you smile as you watch children playing on the gigantic play structure. A city worker mows the lawn.

Where does the municipality get the money to pay for all you have seen? No one owns the roads, schools are free, fire crews and police do not charge for their services, play structures have no admission, and parks are open to everyone. These are just some examples of what your municipality does with the money it raises through property taxes.

Property Taxation

Property taxes are annual taxes paid by real estate owners to local levying authorities to pay for services such as roads, water, sewers, public schools, policing, fire departments, and other community services. Every individual and every business pays property taxes. Even if you don't own property, you pay property taxes that are included in your rental and leasing rates from your landlord.

Property taxes are imposed on real estate owners by their municipal government along with any other bodies authorized to levy taxes. For example, in Manitoba each divisional school board is authorized to levy property taxes within its local school division boundaries.

Since property taxes are administered at the municipal level and every municipality has different financial needs, there are a variety of ways to calculate a total property tax bill. Formula 3.5 is designed to be flexible to meet the varying needs of municipal tax calculations throughout Canada.

3.5 Property Taxes

Formula does not parse

Formula does not parse The amount of property taxes that are owing represents the total of all property taxes from all taxable services. For example, your property tax bill could consist of a municipal tax, a public school tax, a water tax, and a sewer/sanitation tax. Each of these taxes is levied at a specific rate. Therefore, the formula is designed to sum all applicable taxes, as represented by the summation symbol ($\sum$) on the right-hand side of the equation.

Formula does not parse Every piece of real estate has two valuations: the market value and the assessed value. Only the assessed value is relevant when computing property taxes.

- The market value of a property is a snapshot of the estimated selling price of your property. It is what you might have been able to sell your property for at a certain time period. For example, the City of Winnipeg updates the market value of all property in the city every two years.

- The assessed value of a property is the portion of the market value that is subjected to the property tax rate. It is calculated by taking the market value and multiplying it by a percentage determined by the municipality's tax policy:

$$\text{Market Value} \times \text{Tax Policy} = \text{Assessed Value}$$

In some municipalities, the tax policy is to tax 100% of the market value. In others, the tax policy can be substantially less. Continuing with the example, the tax policy for Winnipeg is to tax 45% of the market value. Therefore, a $\$200,000$ market value home in Winnipeg has a $\$90,000$ assessed value. This $\$90,000$ is the base for the tax.

Formula does not parse Two methods commonly express how assessed value is taxed, a tax rate and a mill rate.

- A tax rate is a tax per $\$100$ of assessed value. Most municipalities in Ontario and further east use this system. The mathematical expression for the tax rate is:

$$\frac{\text{Tax Rate}}{100}$$

- A mill rate is a tax per $\$1,000$ of assessed value. Most municipalities in Manitoba and further west use this system. The mathematical expression for the mill rate is:

$$\frac{\text{Mill Rate}}{1,000}$$

HOW TO

Work with Property Taxes

Follow these steps when working with calculations involving property taxes:

Step 1: Identify all known variables. This includes the market value, tax policy, assessed value, all property tax rates, and the total property taxes.

Step 2: If you know the assessed value, skip this step. Otherwise, calculate the assessed value of the property by multiplying the market value and the tax policy.

Step 3: Calculate either the tax amount for each property tax levy or the grand total of all property taxes by applying **Formula 3.5** $\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$ $\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$.

Continuing with the Winnipeg example in which a home has a market value of $\$200,000$, the tax policy of Winnipeg is to tax 45% of the market value. A Winnipegger receives a property tax levy from both the City of Winnipeg itself and the local school board. The mill rates are set at 14.6 and 16.724, respectively. Calculate the total property tax bill.

Step 1: The known variables are:

$$\begin{aligned}\text{Market Value} &= \$200,000\\ \text{Tax Policy} &= 45\%\\ \text{City of Winnipeg Mill Rate} &= 14.6\\ \text{School Board Mill Rate} &= 16.724\end{aligned}$$

Step 2: Calculate the assessed value by taking the market value of $\$200,000$ and multiplying by the tax policy of 45%, or

$$\begin{aligned}\text{Assessed Value} &= \$200,000 \times 45\%\\ \text{Assessed Value} &= \$90,000\end{aligned}$$

Step 3: To calculate each property tax, apply Formula 3.5

$$\text{Property Tax} = \sum(\text{AV} \times \text{PTR}) \text{Property Tax} = \sum(\text{AV} \times \text{PTR})$$

$$\begin{aligned}\text{City of Winnipeg Property Tax} &= \$90,000 \times \frac{14.6}{1,000}\\ \text{City of Winnipeg Property Tax} &= \$1,314\end{aligned}$$

$$\begin{aligned}\text{School Board Property Tax} &= \$90,000 \times \frac{16.724}{1,000}\\ \text{School Board Property Tax} &= \$1,505.16\end{aligned}$$

Add these separate taxes together to arrive at Total Property Tax.

$$\begin{aligned}\text{Total Property Tax} &= \$1,314 + \$1,505.16\\ \text{Total Property Tax} &= \$2,819.16\end{aligned}$$

Key Takeaways

Mill rates are commonly expressed with four decimals and tax rates are expressed with six decimals. Although some municipalities use other standards, this text uses these common formats in its rounding rules. In addition, each property tax levied against the property owner is a separate tax. Therefore, you must round each property tax to two decimals before summing the grand total property tax.

Things To Watch Out For

The most common mistake is to use the wrong denominator in the tax calculation. Ensure that you read the question accurately, noting which term it uses, tax rate or mill rate. If neither appears, remember that Ontario eastward uses tax rates and Manitoba westward uses mill rates.

A second common mistake is to add multiple property tax rates together when the assessed value remains constant across all taxable elements. For example, if the assessed value of $\$250,000$ is used for two tax rates of 2.168975 and 1.015566, you may be tempted to sum the rates, which would yield a rate of 3.184541. This does not always work and may produce a small error (a penny or two) since each tax is itemized on a tax bill. You must round each individual tax to two decimals before summing to the total property tax.

Example 3.5.1

A residence has a market value of $\$340,000$. The municipality's tax policy is set at 70%. Real estate owners have to pay three separate taxes: the municipality tax, a library tax, and an education tax. The tax rates for each are set at 1.311666, 0.007383, and 0.842988, respectively. Calculate the total property tax bill for the residence.

Solution

Step 1: **What are we looking for?**

You need to calculate the total property tax for the residence by summing the assessed value multiplied by the respective tax rates.

Step 2: **What do we already know?**

The house, tax policy, and tax rates are known:

$$\begin{aligned}\text{Market Value} &= \$340,000\\ \text{Tax Policy} &= 70\%\\ \text{Municipal PTR} &= 1.311666\\ \text{Library PTR} &= 0.007383\\ \text{Education PTR} &= 0.842988\end{aligned}$$

Step 3: **Make substitutions using the information known above.**

Calculate the assessed value (AV) by taking the market value and multiplying by the tax policy.

$$\begin{aligned}\text{AV} &= \$340,000 \times 70\%\\ \text{AV} &= \$340,000 \times 0.7\\ \text{AV} &= \$238,000\end{aligned}$$

Apply **Formula 3.5**$\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$

$\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$. Note that this municipality uses a tax rate, so the PTR is divided by 100.

$$\begin{aligned}\text{Municipal Tax} &= \$238,000 \times 1.311666100\\ \text{Municipal Tax} &= \$3,121.77\end{aligned}$$

$$\text{Library Tax} = \$238,000 \times 0.007383100$$
$$\text{Library Tax} = \$17.57$$

$$\text{Education Tax} = \$238,000 \times 0.842988100$$
$$\text{Education Tax} = \$2,006.31$$

$$\text{Property Tax} = \sum (\text{AV} \times \text{PTR})$$
$$\text{Property Tax} = \$3,121.77 + \$17.57 + \$2,006.31$$
$$\text{Property Tax} = \$5,145.65$$

Step 4: **Provide the information in a worded statement.**

The property owner owes the municipality $\$3,121.77$, the library $\$17.57$, and education $\$2,006.31$ for a total property tax bill of $\$5,145.65$.

Paths To Success

The collective property taxes paid by all of the property owners form either all or part of the operating budget for the municipality. Thus, if a municipality consisted of $1,000$ real estate owners each paying $\$2,000$ in property tax, the municipality's operating income from property taxes is

$$\$2,000 \times 1,000 = \$2,000,000.$$

If the municipality needs a larger budget from property owners, either the assessed values, the mill/tax rate, or some combination of the two needs to increase.

Example 3.5.2

A school board is determining next year's operating budget and calculates that it needs an additional $\$5 \text{ million}$. Properties in its municipality have an assessed value of $\$8.455 \text{ billion}$.

The current mill rate for the school board is set at 6.1998. If the assessed property values are forecasted to rise by 3% next year, what mill rate should the school set?

Solution

***Step 1:* What are we looking for?**

Aim to calculate the new property tax rate (PTR) for next year's mill rate.

***Step 2:* What do we already know?**

The information about the school board and its municipality are known:

$$\begin{aligned} \text{AV} &= \$8.455 \text{ billion} \\ \text{PTR (current mill rate)} &= 6.1998 \\ \%C \text{ (to AV next year)} &= 3\% \end{aligned}$$

Property tax increase needed is $\$5 \text{ million}$.

***Step 3:* Make substitutions using the information known above.**

Calculate the school board's current budget using Formula 3.5. The property tax collected is the board's operating budget.

$$\begin{aligned} \text{Property Tax} &= \sum(\text{AV} \times \text{PTR}) \\ \text{Property Tax} &= \$8,455,000,000 \times \frac{6.1998}{1,000} \\ \text{Property Tax} &= \$52,419,309 \end{aligned}$$

Increase the budget, or Property Tax, for next year.

$$\begin{aligned} \text{Next Year Property Tax} &= \$52,419,309 + \$5,000,000 \\ \text{Next Year Property Tax} &= \$57,419,309 \end{aligned}$$

Increase the current assessed value base by applying Formula 3.2a (percent change), rearranging to solve for V_f. The current AV is the V_i value.

$$\%C = \frac{V_\text{f} - V_\text{i}}{V_\text{i}} \times 100$$

$$3\% = \frac{V_\text{f} - \$8,455,000,000}{\$8,455,000,000} \times 100$$

$$\begin{aligned} \$253,650,000 &= V_\text{f} - \$8,455,000,000 \\ \$8,708,650,000 &= V_\text{f} \\ \text{Next Year Assessed Value} &= \$8,708,650,000 \end{aligned}$$

Recalculate the new mill rate using the Next Year Property Tax and Next Year Assessed Value. Apply Formula 3.5 and solve for the mill rate in the PTR.

$$\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$$
$$\$57,419,309 = \$8,708,650,000 \times \frac{\text{Mill Rate}}{1,000}$$
$$6.5934 = \text{Mill Rate}$$

Step 4: **Provide the information in a worded statement.**

The current budget for the school board is $\$52,419,309$, which will be increased to $\$57,419,309$ next year. After the board adjusts for the increased assessed values of the **properties,** it needs to set the mill rate at 6.5934 next year, which is an increase of 0.4736 mills.

Section 3.5 Exercises

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided.

Table 3.5.1

	Market Value	Tax Policy	Assessed Value	Rate	Type of Rate	Property Tax
1.	$320,000	55%	?	26.8145	Mill	?
2.	?	85%	$136,000	1.984561	Tax	?
3.	$500,000	?	?	9.1652	Mill	$3,666.08
4.	?	50%	$650,000	?	Tax	$4,392.91

Solutions

1. $\text{AV} = \$176,000$; $\text{Property Tax} = \$4,719.35$
2. $\text{Market Value} = \$160,000$; $\text{Property Tax} = \$2,699.00$
3. $\text{Assessed Value} = \$400,000$; $\text{Tax Policy} = 80\%$
4. $\text{Market Value} = \$1,300,000$; $\text{Tax Rate} = 0.675832$

Applications

5. A house with an assessed value of $\$375,000$ is subject to a tax rate of 1.397645. What is the property tax?
6. If a commercial railway property has a property tax bill of $\$166,950$ and the mill rate is 18.5500, what is the assessed value of the property?
7. A house in Calgary has a market value of $\$450,000$. The tax policy is 100%. The property is subject to a 2.6402 mill rate from the City of Calgary and a 2.3599 mill rate from the province of Alberta. What are the total property taxes?
8. A residential property in Regina has a market value of $\$210,000$. The Saskatchewan tax policy is 70%. The property is subject to three mill rates: 13.4420 in municipal taxes, 1.4967 in library taxes, and 10.0800 in school taxes. What amount of tax is collected for each, and what are the total property taxes?
9. A municipality needs to increase its operating budget. Currently, the assessed value of all properties in its municipality total $\$1.3555$ billion and the tax rate is set at 0.976513. If

the municipality needs an additional $\$1.8$ million next year, what tax rate should it set assuming the assessed values remain constant?

10. A municipality set its new mill rate to 10.2967, which increased its total operating budget by $\$10$ million on a constant assessed value of $\$7.67$ billion. What was last year's mill rate?

Solutions

5. $\text{Property Tax} = \$5,241.17$
6. $\text{AV} = \$9,000,000$
7. $\text{Property Tax} = \$2,250.05$
8. $\text{Municipal Property Tax} = \$1,975.97$;
   $\text{Library Property Tax} = \$220.01$; $\text{School Property Tax} = \$1,481.76$;
   $\text{Total Property Tax} = \$3,677.74$
9. $\text{Tax Rate} = 1.109305$
10. $\text{Mill Rate} = 8.9929$

Challenge, Critical Thinking, & Other Applications

11. A school board is determining the mill rate to set for next year. The assessed property values for next year total $\$5.782035$ billion, representing an increase of 5% over the current year. If the school board needs an additional $\$5.4$ million in funding next year, by what amount should it change its current year mill rate of 11.9985?
12. In the current year, the market value of properties totals $\$6.896$ billion. The current tax policy is 85% and the current mill rate is 15.6712. If the municipality requires an additional $\$2$ million in its operating budget next year, market values increase by 4%, and the tax policy changes to 90%, what mill rate should it set for next year?
13. A $\$600,000$ market value property is assessed with a tax policy of 75% and subject to two mill rates. If the total property taxes are $\$6,766.67$ and the second mill rate is half of the first tax rate, calculate each mill rate.
14. Two properties in different provinces pay the same property taxes of $\$2,840$. One province uses a mill rate of 24.6119 with a 60% tax policy, while the other province uses a tax rate of 1.977442 with an 80% tax policy. Compute the market values for each of these properties.
15. A water utility funded through property taxes requires $\$900$ million annually to operate.

It has forecasted increases in its operating costs of 7% and 3.5% over the next two years. Currently, properties in its area have a market value of $\$234.85$ billion, with projected annual increases of 3% and 5% over the next two years. The provincial government has tabled a bill that might change the tax policy from 70% to 75% effective next year, but it is unclear if the bill will pass at this point. For planning purposes, the utility wants to forecast its new mill rates for the next two years under either tax policy. Perform the necessary calculations for the utility.

Solutions

11. $\text{Increase the mill rate by } 0.3626$
12. $\text{Mill Rate} = 14.5412$
13. 10.0247 and 5.0124
14. $\$192,318.90$ and $\$179,524.86$
15. $\text{Mill Rates Under } 70\% \text{ Policy: } 5.6872 \text{ then } 5.6060$;
    $\text{Mill Rates Under } 75\% \text{ Policy: } 5.3081 \text{ then } 5.2322$

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$\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$ $\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$

Attribution

3.6: RATIOS, PROPORTIONS, AND PRORATING

Introduction

You and your business partner have a good problem: Consumers are snapping up packets of your new eucalyptus loganberry facial scrub as fast as you can produce them. Each packet of the scrub contains 600 mg of loganberry extract and 80 mg of eucalyptus oil, as well as water and clay and other ingredients. Ratios are invaluable in understanding the relationship among different quantities, such as how much of each ingredient you need.

You have no trouble obtaining the water and clay, but the loganberry extract and eucalyptus oil are in short supply because of poor weather. In any time period, your suppliers can provide seven times as much loganberry extract as eucalyptus oil. To figure out which ingredient is limiting your production, you need proportions.

Sometimes you need to relate a proportion to the total of the quantities. This requires prorating skills. For example, once your business has grown, you start using a production line, which follows the common practice of producing more than one product. The papaya facial scrub you have introduced recently and the eucalyptus loganberry scrub require the same amount of production time. Your equipment capacity is $1,000$ units per day. How many units of each type of scrub must you produce to meet market demand in the ratio of nine to two?

Years later, you are splitting the profits of your business partnership in proportion to each partner's total investment. You invested $\$73,000$, while your partner invested $\$46,000$. With total profits of $\$47,500$, what is your share?

Understanding the relationships among various quantities and how the components relate to an overall total emphasizes the need for understanding ratios, proportions, and prorating.

What Is a Ratio?

A **ratio** is a fixed relationship between two or more quantities, amounts, or sizes of a similar nature. For a ratio to exist, all terms involved in the ratio must be nonzero. Examine the criteria of this definition more closely:

1. There Must Be Two or More Quantities

A ratio does not exist if only one quantity is involved. For example, the fuel tank on your Mustang takes

60 litres of gasoline. This is not a ratio, as there is no relationship to any other quantity, amount, or size. On the other hand, if you compare your fuel tank to the fuel tank of your friend's Hummer, you now have two quantities involved and could say that her fuel tank has twice the capacity of yours.

2. All Terms Must Be of a Similar Nature

In all of the examples provided note that all quantities, amounts, or sizes are based on the same unit. In the cosmetics formulation it was milligrams to milligrams; in the production line, it was units to units; in the investment scenario, it was dollars to dollars. For a ratio to have meaning and to be properly interpreted, all terms of the ratio must be expressed in a similar nature. When you place different units such as kilometres and metres into the same ratio, the result is confusing and will lead to misinterpretation of the relationship.

3. All Terms Must Be Nonzero

The numbers that appear in a ratio are called the **terms of the ratio**. If we have a recipe with four cups of flour to one cup of sugar, there are two terms: four and one. If any term is zero, then the quantity, amount, or size does not exist. For example, if the recipe called for four cups of flour to zero cups of sugar, there is no sugar! Therefore, every term must have some value other than zero.

Let's continue using the example of four cups of flour to one cup of sugar. Business ratios are expressed in five common formats, as illustrated in the table below.

Table 3.6.1

Format	Ratio Example	Interpretation
To	4 to 1	Four cups of flour to one cup of sugar
: (colon)	$4:1$	Four cups of flour to one cup of sugar
Fraction	$\frac{4}{1}$	Four cups of flour per one cup of sugar
Decimal	4	Four times as much flour than sugar
Percent	400%	Flour is 400% of sugar (Hint: think rate, portion, base)

All of these formats work well when there are only two terms in the ratio. If there are three or more terms, ratios are best expressed in the colon format. For example, if the recipe called for four parts of flour to one part of sugar to two parts of chocolate chips, the ratio is $4:1:2$. The fraction, decimal, and percent forms do not work with three or more terms.

Simplification and Reduction of Ratios

When a ratio is used to express a relationship between different variables, it must be easy to understand and interpret. Sometimes when you set up a ratio initially, the terms are difficult to comprehend. For example, what if the recipe called for $62\frac{1}{2}$ parts flour to 25 parts sugar? That is not very clear. Expressing the same ratio another way, you can say the recipe requires 5 parts flour to 2 parts sugar. Note how the relationship is clearer in the latter expression. Either way, though, both ratios mean the same thing; in decimal format this ratio is expressed as a value of 2.5. Recall that Section 2.2 discussed how fractions are expressed in higher and lower terms. We now apply the same knowledge to ratios to make the relationship as clear as possible.

When you reduce ratios to lower terms, remember two important characteristics involving the cardinal rule and integers:

1. **The Cardinal Rule:** Recall from Section 2.6 that the cardinal rule of algebra states, "What you do to one you must do to the other." In other words, whatever mathematical operation is performed on a term in a ratio must be equally performed on every other term in the ratio. If this rule is violated then the relationship between the terms is broken.
2. **Maintaining Integers:** Integers are easier to understand than decimals and fractions. In reducing a ratio to lower terms, aim to maintain every term as an integer (much as in Section 2.2).

HOW TO

Reduce Ratios to Lower Terms

The steps involved in reducing ratios to lower terms are listed below. You may not need some steps, so skip them if the characteristic is not evident in the ratio.

Step 1: Clear any fractions that, when divided, produce a nonterminating decimal. Apply the rules of algebra and multiply each term by the denominator being cleared from the ratio. For example, if the ratio is $\frac{1}{3} : 2$, the first term when divided produces a nonterminating decimal. Clear the fraction by multiplying every term by the denominator of 3, resulting in a ratio of $1 : 6$.

Step 2: Perform division on all fractions that produce a terminating decimal. For example, if the ratio is $\frac{2}{5} : \frac{3}{10}$, both terms convert to terminating decimals, resulting in a ratio of $0.4 : 0.3$.

Step 3: Eliminate all decimals from the ratio through multiplication. In other words, express the ratio in higher terms by multiplying every term by a power of 10. The power of 10 you choose must be large enough to eliminate all decimals. For example, if the ratio is $0.2 : 0.25 : 0.125$, notice that the third term has the most decimal positions. A power of $1,000$ (10^3) is required to move the decimal three positions to the right. Multiply every term by a power of $1,000$, resulting in a ratio of$200 : 250 : 125$.

Step 4: Find a common factor that divides evenly into every term, thus producing integers. If you find no such factors, then the ratio is in its lowest terms. For example, if the ratio is $10 : 4 : 6$ it can be factored by dividing every term by 2, resulting in a ratio of $5 : 2 : 3$. There is no common factor that reduces this ratio further; therefore, it is in its lowest terms.

Key Takeaways

To factor any term, recall your multiplication tables. Assume the ratio is $36 : 24$. It is always best to pick the lowest term in the ratio when factoring. Looking at the ratio, you can see that the lowest term is 24. Recall what multiplies together to arrive at 24. In this case, your factors are 1×24, 2×12, 3×8, and 4×6. The goal is to find the largest value among these factors that also divides evenly into the other term of 36. The largest factor that makes this true is 12.

Therefore, perform Step 4 in our reduction process by dividing every term by a factor of 12, resulting in a ratio of $3 : 2$.

Things To Watch Out For

Always ensure that before you apply any of the reduction steps your relationship meets the requirements of being a ratio in the first place. For example, the expression of $10 \text{ km} : 500 \text{ m}$ is not a $1 : 50$ ratio since it violates the "similar nature" characteristic of the ratio definition. You need to convert the metres into kilometres, producing $10 \text{ km} : 0.5 \text{ km}$. Now that you have a ratio, the reduction to lowest terms results in a $20 : 1$ ratio. This is very different than $1 : 50$! The lesson learned is to make sure you are working with a proper ratio before you manipulate it.

Paths To Success

In the fourth step of the procedure, do not feel compelled to find what is called the "magic factor." This is the single factor that reduces the ratio to its lowest terms in a single calculation. Although this is nice if it happens, you may as well find any factor that will make the ratio smaller and easier to work with.

- For example, assume a ratio of $144 : 72 : 96$. It may not be very apparent what factor goes into each of these terms on first glance. However, they are all even numbers, meaning they can all be divided by 2. This produces $72 : 36 : 48$.
- Once again, the "magic factor" may not be clear, but every term is still even. Divide by 2 again, producing $36 : 18 : 24$.
- If no magic factor is apparent, again note that all numbers are even. Divide by 2 yet again, producing $18 : 9 : 12$.
- The common factor for these terms is 3. Divided into every term you have $6 : 3 : 4$. There is no common factor for these terms, and the ratio is now in its lowest terms.

In the above example it took four steps to arrive at the answer—and that is all right. Did you notice the "magic factor" that could have solved this in one step? You can find it if you multiply all of the factors you have: $2 \times 2 \times 2 \times 3 = 24$. Dividing every term in the original ratio by 24 produces the solution in one step. If you did not notice this "magic factor," there is nothing wrong with taking four steps (or more!) to get the answer. In the end, both methods produce the same solution.

Example 3.6.1

Reduce the following ratios to their lowest terms:

a. $49 : 21$
b. $0.33 : 0.066$
c. $\frac{9}{2} : \frac{3}{11}$
d. $5\frac{1}{8} : 6\frac{7}{8}$

Solution

***Step 1:* What are we looking for?**

You have been asked to reduce the ratios to their lowest terms.

***Step 2:* What do we already know?**

The four ratios to be reduced have been provided in colon format.

***Step 3:* Make substitutions using the information known above.**

Clear all fractions that produce nonterminating decimals.

a. No fractions

b. No fractions

c. Multiply both terms by 11:

$$(\frac{9}{2} \times 11) : (\frac{3}{11} \times 11) = \frac{99}{2} : 3$$

d. No fractions that produce nonterminating decimals.

Perform division on all fractions that produce terminating decimals.

a. No fractions

b. No fractions

c. Perform division:

$$\frac{99}{2} : 3 = 49.5 : 3$$

d. Perform divison:

$$5\frac{1}{8} : 6\frac{7}{8} = 5.125 : 6.875$$

Eliminate all decimals through multiplication.

a. No decimals

b. Multiply by $1,000$:

$$(0.33 \times 1,000) : (0.066 \times 1,000) = 330 : 66$$

c. Multiply by 10:

$$(49.5 \times 10) : (3 \times 10) = 495 : 30$$

d. Multiply by $1,000$:

$$(5.125 \times 1,000) : (6.875 \times 1,000) = 5,125 : 6,875$$

Divide every term by a common factor, reducing it to the lowest terms.

a. Divide by 7:

$$\frac{49}{7} : \frac{72}{7} = 7 : 3$$

b. Divide by 66:

$$\frac{330}{66} : \frac{66}{66} = 5 : 1$$

c. Divide by 15:

$$\frac{495}{15} : \frac{30}{15} = 33 : 2$$

d. Divide by 125:

$$\frac{5,125}{125} : \frac{6,875}{125} = 41 : 55$$

***Step 4:* Provide the information in a worded statement.**

In lowest terms, the ratios are $7 : 3$, $5 : 1$, $33 : 2$, and $41 : 55$, respectively.

Reducing a Ratio to the Smallest Term of One

Your goal in reducing a ratio is to make it easier to understand. Sometimes, even after you have applied the techniques for reducing the ratio, the end result is still hard to grasp. Look at part (d) of Example 3.6.1. You arrived at a final solution of $41 : 55$, with no further reduction possible. Think of this as 41 cups of flour to 55 cups of sugar. The relationship is not very clear.

In these circumstances, although integers are preferable in general, you must reduce the ratio further, reintroducing decimals to make the relationship more comprehensible. This means you apply a technique called "reducing the ratio to the smallest term of one." In this technique, the smallest term in the ratio will have a value of 1 once you perform the ratio reduction and simplification.

HOW TO

Reduce a Ratio to the Smallest Term of One

Follow these steps to reduce a ratio to the smallest term of one:

Step 1: Locate the smallest term in the ratio. (Do not just pick the first term.)

Step 2: Divide every term in the ratio by the selected smallest term. Every other term becomes a decimal number, for which either a clear rounding instruction is provided or you must obey the rounding rules used in this textbook. The smallest term by nature of the division equals one.

Let's continue with part (d) of Example 3.6.1, in which the reduced ratio is $41 : 55$.

Step 1: Locate the smallest term in the ratio. It is the first term and has a value of 41.

Step 2: Take every term and divide it by 41 to arrive at $1 : 1.\overline{34146}$. The decimal number allows you to roughly interpret the ratio as "one cup of flour to a touch over $1\dfrac{1}{3}$ cups of sugar." Although not perfect, this is more understandable than $41 : 55$.

Key Takeaways

You may have to make a judgment call when you decide whether to leave a reduced ratio alone or to reduce it to a ratio where the smallest term is one. There is no clear definitive rule; however, keep the following two thoughts in mind:

1. For purposes of this textbook, you are provided with clear instructions on how to handle the ratio, allowing everyone to arrive at the same solution. Your instructions will read either "Reduce the ratio to its smallest terms" or "Reduce the ratio to the smallest term of one."
2. In the real world, no such instructions exist. Therefore you should always base your decision on which format is easier for your audience to understand. If the reduced fraction leaves your audience unable to understand the relationship, then use the reduction to a ratio with the smallest term of one instead.

Example 3.6.2

The inventory management system at your company requires you to keep a ratio of 27 to 47 Nintendo Wiis to Sony PlayStations on the shelf. There are 38 Nintendo Wiis on the shelf.

a. Express the ratio where the smallest term is one. Round your solution to two decimals.
b. Using the ratio, determine how many PlayStations should be in your inventory.

Solution

***Step 1:* What are we looking for?**

These terms have no common factor (note that 47 is a prime number). The terms in this ratio cannot be reduced in any way, which makes the relationship difficult to assess. Therefore, reduce the ratio to the smallest term of one. Once the relationship is more evident, determine how many PlayStations need to be on the shelf.

***Step 2:* What do we already know?**

The ratio of Wiis to PlayStations is $27 : 47$. There are currently 38 Wiis on the shelf.

***Step 3:* Make substitutions using the information known above.**

Locate the smallest term.

$$\text{Smallest Term} = 27$$

Divide all terms by the smallest term to get the ratio.

$$\frac{27}{27} : \frac{47}{27} = 1 : 1.74$$

The inventory management requires 1.74 units of PlayStations to be on the shelf for every unit of Wiis.

Use the ratio to calculate the number of PlayStations needed on the shelf.

$$38 \times 1.74 = 66.12 \text{ units}$$

Since only whole units are possible, round to the nearest integer of 66.

***Step 4:* Provide the information in a worded statement.**

The inventory management system requires approximately 1.74 PlayStations for every Nintendo Wii on the shelf. Therefore, with 38 Wiis on the shelf you require 66 PlayStations to also be on the shelf.

What Are Proportions?

Knowing the relationship between specific quantities is helpful, but what if your quantity happens not to match the specific quantity expressed in the existing ratio? Example 3.6.2 illustrated that you need to learn how to apply the ratio to meet current conditions. The ratio in the inventory system was expressed in terms of 27 Nintendo Wiis; however, the shelf displayed 38 units. How can we relate an existing ratio to a needed ratio?

A **proportion** is a statement of equality between two ratios. Just as we have both algebraic expressions and algebraic equations, there are ratios and proportions. With algebraic expressions, only simplification was possible. When the expression was incorporated into an algebraic equation, you solved for an unknown. The same is true for ratios and proportions. With ratios, only simplification is possible. Proportions allow you to solve for any unknown variable. The Wii and PlayStation example could have been set up as follows:

$$27 : 47 = 48 : p$$

Formula does not parse The known ratio forms the left side of the proportion. It expresses the known relationship between the units of Wiis and PlayStations.

Formula does not parse The unknown ratio forms the right side of the proportion. It expresses the relationship between the variables in the same order, but uses the currently known information. Any unknown quantities are algebraically assigned to an unknown variable (in this case p for PlayStations).

$\text{Equal Sign } (=)$: By placing the equal sign between the two ratios, you create a proportion.

A proportion must adhere to three characteristics, including ratio criteria, order of terms, and number of terms.

- ***Characteristic #1: Ratio Criteria Must Be Met***. By definition, a proportion is the equality between two ratios. If either the left side or the right side of the proportion fails to meet the criteria for being a ratio, then a proportion cannot exist.
- ***Characteristic #2: Same Order of Terms***. The order of the terms on the left side of the proportion must be in the exact same order of terms on the right side of the proportion. For example, if your ratio is the number of MP3s to CDs to DVDs, then your proportion is set up as follows:

$$\text{MP3} : \text{CD} : \text{DVD} = \text{MP3} : \text{CD} : \text{DVD}$$

- ***Characteristic #3: Same Number of Terms***. The ratios on each side must have the same number of terms such that every term on the left side has a corresponding term on the right side. A proportion of $\text{MP3} : \text{CD} : \text{DVD} = \text{MP3} : \text{CD}$ is not valid since the DVD term on the left side does not have a corresponding term on the right side.

When you work with proportions, the mathematical goal is to solve for an unknown quantity or quantities. In order to solve any proportion, always obey the following four rules:

1. ***Rule #1: At Least One Value for Any Term Is Known.*** At least one of the left side or right side values for each term must be known. For example, $x : 5 = y : 10$ is not a solvable proportion since

the corresponding first terms on both sides are unknown. However, $15 : 5 = y : 10$ is solvable since at least one of the first corresponding terms (the 15 and the y) is known.

2. ***Rule #2: One Pair of Corresponding Terms Must Be Known.*** At least one pair of corresponding terms on the left side and right side must have both quantities known. For example, $3 : x : 6 = y : 4 : z$ is not a solvable proportion since there is no pair of first terms (3 and y), second terms (x and 4), or third terms (6 and z) that produces a pair of known values. However, $3 : x : 6 = 9 : 4 : z$ is a solvable proportion since the first terms on both sides (the 3 and 9) are known.
3. ***Rule #3: Obey BEDMAS and Perform Proper Algebraic Manipulation.*** To manipulate a proportion, you must satisfy the rules of BEDMAS (Section 2.3) and all of the rules of algebra (Sections 2.5 and 2.6). Violating any of these rules breaks the equality of the ratios and produces an incorrect proportion.
4. ***Rule #4: Use the Fractional Format.*** The fractional format for ratios is recommended for solving a proportion. The other four formats generally make solving the proportion much more difficult, and the mathematical operations required become unclear.

HOW TO

Solve Proportions for Unknown Variables

Follow these steps in solving any proportion for an unknown variable or variables:

Step 1: Set up the proportion with the known ratio on the left side. Place the ratio with any unknown variables on the right side.

Step 2: Work with only two terms at a time, and express the two terms in fractional format. This is not a problem if the proportion has only two terms on each side of the equation, for example, $27 : 47 = 38 : p$. This is expressed as:

$$\frac{27}{47} = \frac{38}{p}.$$

If the proportion has three or more terms on each side, you can pick any two terms from each side of the proportion so long as you pick the same two terms on each side. In making your selection, aim to have a pair of terms on one side of the equation where both values are known while the other side of the equation is made up of one known term and one unknown term. For example, assume the proportion

$6:5:4=18:15:y$. The selection of the first and third terms only on each side of the equation produces $6:4=18:y$. Notice that this is now a proportion with only two terms on each side, which you can express as:

$$\frac{6}{4}=\frac{18}{y}.$$

Step 3: Solve for the unknown variable. Obey the rules of **BEDMAS** and perform proper algebraic manipulation.

Step 4: If the proportion contained more than one unknown variable, go back to Step 2 and select another pairing that isolates one of the unknown variables. Although one of the unknown variables is now known as a result of Step 3, do not use this known value in making your selection. The danger in using a solved unknown variable is that if an error has occurred, the error will cascade through all other calculations. For example, assume the original proportion was $3:7:6:8=x:y:z:28$. From Step 2 you may have selected the pairing of $3:8=x:28$, and in Step 3 you may have erroneously calculated $x=11.5$ (the correct answer is $x=10.5$). In returning to Step 2 to solve for another unknown variable, do not involve x in your next pairing. To isolate y, use $7:8=y:28$ and not $3:7=11.5:y$, which will at least ensure that your calculation of y is not automatically wrong based on your previous error.

Things To Watch Out For

You must always pick terms from the same positions on both sides of the proportion. Otherwise, you will violate the equality of the proportion, since the terms are no longer in the same order on both sides and Characteristic #2 is not satisfied. For example, in working with the proportion $6:5:4=18:15:y$ from above, you cannot select the first and third terms on the left side and also select the second and third terms on the right side. In other words, $6:4\neq 15:y$ since $1^{\text{st}}\text{ term}:3^{\text{rd}}\text{ term}\neq 2^{nd}\text{ term}:3^{rd}\text{ term}$.

Paths To Success

It is always easiest to solve a proportion when the unknown variable is in the numerator. This characteristic requires minimal algebra and calculations to isolate the variable. If you find yourself with an unknown variable in the denominator, you can mathematically invert the fraction on both sides, since this obeys the cardinal rule of "what you do to one, you must do to the other." When you invert, the numerator becomes the denominator and vice versa. For example, if the proportion is $\frac{3}{4} = \frac{12}{b}$, inversion then produces a proportion of $\frac{4}{3} = \frac{b}{12}$. Notice that isolating the unknown variable in the inverted proportion requires only a multiplication of 12 on both sides. This is a lot less work!

Try It

1) Some of the following proportions violate the characteristics or rules of proportions. Examine each and determine if all the rules and characteristics are met. If not, identify the problem.

a. $4 : 7 = 6 : y$
b. $5 : 3 = 6 : a : b$
c. $6\text{km} : 3\text{m} = 2\text{m} : 4\text{km}$
d. $6 : k = 18 : 12$
e. $4 : 0 = 8 : z$
f. $9 = p$
g. $4 : 7 : 10 = d : e : f$
h. $y : 10 : 15 = x : 30 : z$

Solution

1a. OK

1b. Not the same number of terms on each side (Characteristic #3)

1c. Terms are not in same units; terms are not in the same order (Characteristics #1 and #2)

1d. OK

1e. Does not meet ratio criteria; there is a term of zero (Characteristic #1)

1f. Does not meet ratio criteria; must have at least two terms on each side (Characteristic #1)

1g. There is no corresponding known term on both sides; every pair of terms contains an unknown (Rule #2)

1h. The first corresponding term is not known on both sides; at least one needs to be known (Rule #1)

Try It

2) In the following solved proportions, which person properly executed Step 2 of the proportion steps?

$$6:5:4:3=x:y:z:9$$

Person A: $6:5=x:y$ Person B: $4:3=y:9$ Person C: $4:3=z:9$

Solution

Person C did it right. Person A failed to isolate a variable, and different terms were extracted by Person B (3^{rd} term $: 4^{\text{th}}$ term 2^{nd} term $: 4^{\text{th}}$ term).

Example 3.6.3

A recent article reported that companies in a certain industry were averaging an operating profit of $\$23,000$ per 10 full-time employees. A marketing manager wants to estimate the operating profitability for one of her company's competitors, which employs 87 full-time workers. What is the estimated operating profit for that competitor?

Solution

Step 1: **What are we looking for?**

There is a relationship between the number of full-time employees and the operating profit. This is a ratio. If you set up a similar ratio for the competitor, you create a proportion that you can solve for the competitor's estimated operating profit, or p.

Step 2: **What do we already know?**

You know the industry and competitor information:

$$\begin{aligned}\text{Industry operating profit} &= \$23,000\\ \text{Industry full-time employees} &= 10\\ \text{Competitor operating profit} &= p\\ \text{Competitor full-time employees} &= 87\end{aligned}$$

Step 3: **Make substitutions using the information known above.**

Set up the proportion.

$$\text{industry profit: industry employees} = \text{competitor profit} : \text{competitor employees}$$
$$\$23,000 : 10 = p : 87$$

Step 4: **Provide the information in a worded statement.**

If the industry is averaging $\$23,000$ in operating profits per 10 full-time employees, then the competitor with 87 full-time employees has an estimated $\$200,100$ in operating profits.

Example 3.6.4

In preparing for a party, Brendan needs to mix a punch consisting of fruit juice, vodka, and 7UP in the ratio of $5:3:2$, respectively. If Brendan has a 2 L bottle of vodka, how much fruit juice and 7UP must he mix with it?

Solution

Step 1: **What are we looking for?**

The relationship between the punch recipe ingredients of fruit juice, vodka, and 7UP forms a ratio. If you set up a similar ratio for what Brendan actually has and needs, you can calculate the exact amounts of fruit juice, vodka, and 7UP that are required.

Step 2: **What do we already know?**

You know the punch recipe ratio of $\text{fruit juice} : \text{vodka} : \text{7UP}$ is $5:3:2$. Brendan has 2 L of vodka.

Step 3: **Make substitutions using the information known above.**

Set up the proportion.

$$5:3:2=f:2:s$$

Pick two terms containing one unknown variable and express in fractional format. Isolate the unknown variable.

$$\frac{5}{3}=\frac{f}{2}$$

$$\frac{10}{3}=f$$

$$f=3.\bar{3}$$

Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable.

$$\frac{3}{2}=\frac{2}{s}$$

$$\frac{2}{3}=\frac{s}{2}$$

$$\frac{4}{3}=s$$

$$s=1.\bar{3}$$

Step 4: **Provide the information in a worded statement.**

Maintaining the ratio of $5 : 3 : 2$ between the fruit punch, vodka, and 7UP, Brendan should mix $3\frac{1}{3}$ L of fruit juice and $1\frac{1}{3}$ L of 7UP with his 2 L of vodka.

What Is Prorating?

Ratios and proportions are commonly used in various business applications. But there will be numerous situations where your business must allocate limited funds across various divisions, departments, budgets, individuals, and so on. In the opener to this section, one example discussed the splitting of profits with your business partner, where you must distribute profits in proportion to each partner's total investment. You invested $\$73,000$ while your partner invested $\$46,000$. How much of the total profits of $\$47,500$ should you receive?

The process of **prorating** is the taking of a total quantity and allocating or distributing it proportionally. In the above example, you must take the total profits of $\$47,500$ and distribute it proportionally with your business partner based on the investment of each partner. The proportion is:

$$\text{your investment} : \text{your partner's investment} = \text{your profit share} : \text{your partner's profit share}$$

This proportion has two major concerns:

1. You don't know either of the terms on the right side. As per the rules of proportions, this makes the proportion unsolvable.
2. There is a piece of information from the situation that you didn't use at all! What happened to the total profits of $\$47,500$?

Every prorating situation involves a **hidden term**. This hidden term is usually the sum of all the other terms on the same side of the proportion and represents a total. In our case, it is the $\$47,500$ of total profits. This quantity must be placed as an extra term on both sides of the proportion to create a proportion that can actually be solved.

HOW TO

Perform Prorating

Prorating represents a complex proportion. As such, the steps involved in prorating are similar to the steps for solving any proportion:

Step 1: Set up the proportion with the known ratio on the left side. Place the ratio with any unknown variables on the right side.

Step 2: Insert the hidden term on both sides of the proportion. Usually this term represents the total of all other terms on the same side of the proportion.

Step 3: Working with only two terms at a time, express the two terms in fractional format. Ensure that only one unknown variable appears in the resulting proportion.

Step 4: Solve for the unknown variable. Ensure that the rules of BEDMAS are obeyed and proper algebraic manipulation is performed.

Step 5: If the prorating contains more than one unknown variable, go back to Step 3 and select a pairing that isolates another one of the unknown variables.

To solve your profit-splitting scenario, let y represent your profit share and p represent your partner's share:

Step 1:

$$\textbf{your investment: your partner's investment} = \textbf{your profit share : your partner's profit share}$$

$$\$73,000 : \$46,000 = y : p$$

Step 2: Insert the hidden total term on both sides:

Table 3.6.2

your investment	:	your partner's investment	:	total investment	=	your profit share	:	your partner's profit share	:	total profits

$$\$73,000 : \$46,000 : \$119,000 = y : p : \$47,500$$

Step 3: Set up one proportion:

$$\frac{\$73,000}{\$119,000} = \frac{y}{\$47,500}$$

Step 4: Solve for y.

$$y = \frac{\$73,000 \times \$47,500}{\$119,000}$$

$$y = \$29,138.66$$

Step 5: Set up the other proportion:

$$\frac{\$46,000}{\$119,000} = \frac{p}{\$47,500}$$

Solve for p.

$$p = \frac{\$46,000 \times \$47,500}{\$119,000}$$

$$p = \$18,361.34$$

Your final proportion is:

$$\$73,000 : \$46,000 : \$119,000 = \$29,138.66 : \$18,361.34 : \$47,500.00$$

So you will receive $\$29,138.66$ of the total profits and your partner will receive $\$18,361.34$.

Key Takeaways

To make prorating situations easier to solve, it is always best to insert the hidden term as the *last term* on both sides of the equation. This forces the unknown variables into the numerator when you select pairs of terms. It then takes less algebra for you to isolate and solve for the unknown variable.

Things To Watch Out For

Ensure that when you insert the hidden term you put it in the same position on both sides of the proportion. A common error is to put the total on the “outsides” of the proportion:

$$A : B = C : D$$

when prorated becomes:

$$\text{Total} : A : B = C : D : \text{Total}$$

This violates the proportion characteristics, since the terms are not in the same order on both sides. The correct insertion of the hidden total term makes the proportion look like this:

$$A : B : \text{Total} = C : D : \text{Total}$$

Paths To Success

Another way of approaching prorating is to calculate a “per unit” basis and then multiply each term in the proportion by this base. For example, assume you are distributing $\$100$ across three people, who have shares of three, five, and two. This is a total of 10 shares (the hidden term). Therefore, $\$100$ divided by 10 shares is $\$10$ per share. If the first person has three shares, then $\$10 \times 3 = \$30$. The second and third people get $\$10 \times 5 = \$50$ and $\$10 \times 2 = \$20$, respectively. Therefore,

$$3 : 5 : 2 : 10 = \$30 : \$50 : \$20 : \$100.$$

Example 3.6.5

You paid your annual car insurance premium of $\$1,791$ on a Ford Mustang GT. After five complete months, you decide to sell your vehicle and use the money to cover your school expenses.

Assuming no fees or other deductions from your insurance agency, how much of your annual insurance premium should you receive as a refund?

Solution

Step 1: **What are we looking for?**

You need to figure out how much of your annual insurance premium should be refunded. This will be based on the amount of time that you did not require the insurance. Let's denote this amount as r, the refund.

Step 2: **What do we already know?**

You have some information on premiums and time frames:

$$\text{Total insurance paid} = \$1,791$$

$$\text{Total months paid for} = 12$$

$$\text{Months used} = 5$$

Step 3: **Make substitutions using the information known above.**

Set up the proportion.

$$\text{insurance used: refund} = \text{months used} : \text{months not used}$$
$$\text{insurance used} : r = 5 : \text{months not used}$$

Insert the hidden term.

$$\text{insurance used} : r : \$1,791 = 5 : \text{months not used} : 12$$

Two variables remain undefined. We deduce that you did not use your car for 7 months ($12-5$). We do not know the insurance used and assign it the variable u.

$$u : r : \$1,791 = 5 : 7 : 12$$

Pick two terms containing one unknown variable and express in fractional format.

$$\frac{r}{\$1,791} = \frac{7}{12}$$

Isolate the unknown variable.

$$r = \$1,044.75$$

Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable.

$$\frac{u}{\$1,791} = \frac{5}{12}$$

$$u = \$746.25$$

Step 4: **Provide the information in a worded statement.**

You have used $\$746.25$ of the $\$1,791$ annual insurance premium. With seven months left, you are entitled to a refund of $\$1,044.75$.

Example 3.6.6

An accountant is trying to determine the profitability of three different products manufactured by his company. Some information about each is below:

Table 3.6.3

	Chocolate	**Wafers**	**Nougat**
Direct Costs	$\$743,682$	$\$2,413,795$	$\$347,130$

Although each product has direct costs associated with its manufacturing and marketing, there are some overhead costs (costs that cannot be assigned to any one product) that must be distributed. These amount to $\$721,150$. A commonly used technique is to assign these overhead costs in proportion to the direct costs incurred by each product. What is the total cost (direct and overhead) for each product?

Solution

Step 1: **What are we looking for?**

You need to calculate the sum of the direct and overhead costs for each product. The direct costs are known. The individual overhead costs are unknown but can be calculated using prorating of the total overhead costs. You have three unknowns, namely, chocolate overhead (Co), wafers overhead (Wo), and nougat overhead (No).

Step 2: **What do we already know?**

You have some information on direct costs and overhead costs:

$$\text{Chocolate direct } (Cd) = \$743,682$$

$$\text{Wafers direct } (Wd) = \$2,413,795$$

$$\text{Nougat direct } (Nd) = \$347,130$$

$$\text{Total overhead costs} = \$721,150$$

Step 3: **Make substitutions using the information known above.**

Set up the proportion.

$$Cd : Wd : Nd = Co : Wo : No$$

$$\$743,682 : \$2,413,795 : \$347,130 = Co : Wo : No$$

Insert the hidden term.

$$\$743,682 : \$2,413,795 : \$347,130 : \text{Total direct costs} = Co : Wo : No : \$721,150$$

Calculate the total direct costs as a sum of all three direct costs:

$$\text{Total direct costs} = \$743,682 + \$2,413,795 + \$347,130 = \$3,504,607$$

The proportion is now:

$$\$743,682 : \$2,413,795 : \$347,130 : \$3,504,607 = Co : Wo : No : \$721,150$$

Pick two terms containing one unknown variable and express in fractional format. You can start with chocolate overhead, or Co.

$$\frac{\$743,682}{\$3,504,607} = \frac{Co}{\$721,150}$$

Isolate the unknown variable.

$$\$153,028.93 = Co$$

Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable. Proceed with the wafers overhead, or Wo.

$$\frac{\$2,413,795}{\$3,504,607} = \frac{Wo}{\$721,150}$$

$$\$496,691.43 = Wo$$

Pick another two terms containing the other unknown variable and express in fractional format before isolating the unknown variable. Proceed with the nougat overhead, or No.

$$\frac{\$347,130}{\$3,504,607} = \frac{No}{\$721,150}$$

$$\$71,429.64 = No$$

Sum the direct and overhead costs for each product.

$$\text{Chocolate total cost} = \$743,682.00 + \$153,028.93 = \$896,710.93$$

$$\text{Wafer total cost} = \$2,413,795.00 + \$496,691.43 = \$2,910,486.43$$

$$\text{Nougat total cost} = \$347,130.00 + \$71,429.64 = \$418,559.64$$

Step 4: **Provide the information in a worded statement.**

Putting together the direct costs with the allocated overhead costs, the total cost for chocolate is $\$896,710.93$, wafers is $\$2,910,486.43$, and nougat is $\$418,559.64$.

Section 3.6 Exercises

Mechanics

1. Reduce the following ratios to their smallest terms.

   a. $66 : 12$
   b. $48 : 112 : 80$
   c. $24 : 36 : 8 : 108$

2. Reduce the following ratios to their smallest terms.

   a. $7.2 : 6$

b. $0.03 : 0.035 : 0.02$
c. $0.27 : 0.18 : 0.51 : 0.15$

3. Reduce the following ratios to their smallest terms.

   a. $5\frac{1}{4} : 6$
   b. $\frac{3}{2} : \frac{9}{5} : \frac{21}{10}$
   c. $2\frac{1}{8} : \frac{9}{8} : 3\frac{3}{8} : 1$

4. Reduce the following ratios to their smallest terms.

   a. $\frac{8}{3} : \frac{5}{9}$
   b. $\frac{5}{6} : \frac{1}{3} : 2$
   c. $\frac{1}{7} : \frac{1}{3} : \frac{1}{4} : \frac{1}{6}$

5. Reduce the following ratios to the smallest term of one. Round to two decimals as needed.

   a. $48 : 53$
   b. $5\frac{7}{8} : 2\frac{3}{4} : 3\frac{1}{2}$
   c. $\frac{2}{3} : 11\frac{1}{11} : 5\frac{5}{9} : 2.08$

6. Solve the following proportions for the unknown variable. Round to two decimals as needed.

   a. $7 : 12 = 109 : y$
   b. $3.23 : 4.07 : 2.12 = x : 55.9625 : 29.15$
   c. $q : \frac{5}{3} = 13.75 : 25$

7. Solve the following proportions for all of the unknown variables.

   a. $\$12.15 : \$38.30 : r = x : \$59,184.53 : \$26,998.93$

b. $16:5:9:30=g:h:i:\$397,767$

For questions 8 and 9, prorate the total as indicated by the ratio.

8. $\text{Ratio}=3:2$, $\text{Total}=\$11,368.25$
9. $\text{Ratio}=7:5:3$, $\text{Total}=\$46,923.90$

Solutions

1a. $11:2$

1b. $3:7:5$

1c. $2:3:7:9$

2a. $6:5$

2b. $6:7:4$

2c. $9:6:17:5$

3a. $7:8$

3b. $5:6:7$

3c. $17:9:27:8$

4a. $24:5$

4b. $5:2:12$

4c. $12:28:21:14$

5a. $1:1.10$

5b. $2.14:1:1.12$

5c. $1:16.64:8.33:3.12$

6a. $y=186.86$

6b. $x=44.4125$

6c. $q=0.92$

7a. $r=\$17.47;x=\$18,775.25$

7b. $g=212,142.40;h=\$66,294.50;i=\$119,330.10$

8. $\$6,820.95:\$4,547.30$

9. $\$21,897.82 : \$15,641.30 : \$9,384.78$

Applications

10. EB Games sells gaming consoles. Last month, it sold $\$22,500$ worth of Nintendo Wii consoles, $\$31,500$ worth of Microsoft Xbox consoles, and $\$18,000$ worth of Sony PlayStation consoles. Express the ratio of Wii to Xbox to PlayStation sales in its lowest terms.
11. The manufacturing cost of a deluxe candle is made up of $\$2.40$ in paraffin, $\$1.20$ in dye, $\$1.60$ in overhead, and $\$4.40$ in direct labour. Express the ratio between these costs respectively in their lowest terms.
12. If one Canadian dollar buys $\$1.0385$ of US dollars, how much US money can you spend so that you do not exceed the maximum duty-free amount of $\$800$ Canadian on your next US vacation?
13. For every $\$100$ of retail spending, the average Canadian spends approximately $\$19.12$ at motor vehicle dealerships and $\$18.35$ at grocery stores. If the average Canadian spends $\$49,766$ per year on retail spending, how much more does a Canadian spend annually at motor vehicle dealerships than at grocery stores?
14. Marina has a three-sevenths interest in a partnership. what is the implied value of the partnership if she sells one-quarter of her interest for $\$8,250$?
15. You are making a punch for an upcoming party. The recipe calls for $2\frac{1}{2}$ parts ginger ale to $\frac{4}{5}$ parts grenadine to $1\frac{3}{8}$ parts vodka. If your punch bowl can hold 8.5 litres, how much (in litres) of each ingredient is needed? Round the answers to two decimals as needed.

Solutions

10. $5 : 7 : 4$
11. $6 : 3 : 4 : 11$
12. $\$830.80$
13. $\$383.20$
14. $\$77,000$
15. $\text{Gingerale} = 4.55\text{ L}$, $\text{Grenadine} = 1.45\text{ L}$, $\text{Vodka} = 2.5\text{ L}$

Challenge, Critical Thinking, & Other Applications

16. A manufacturing facility requires one member of the board of directors for every 10 executives. There are five managers for every executive and eight workers for every manager. The average salary paid to directors is $\$105,000$ along with $\$80,000$ to executives, $\$55,000$ to managers, and $\$30,000$ to workers. If the facility has $1,383$ employees, what is the total labour cost?
17. It is common for strip mall owners to allocate to their tenants general overhead costs such as snow clearing, security, and parking lot maintenance in one of two ways. The first method involves allocating these costs on the basis of the square metres each tenant operates. The second method involves allocating these costs on the basis of the number of annual transactions made by each tenant. At Charleswood Square, there are three tenants. The 7-Eleven takes up 150 square metres, Tim Hortons takes up 235 square metres, and Quiznos takes up 110 square metres. Last year, 7-Eleven had $350,400$ transactions, Tim Hortons had $657,000$ transactions, and Quiznos had $197,100$ transactions. Total overhead costs for Charleswood Square last year were $\$25,980$. Determine which method of allocation each tenant would prefer and how much of a savings in percentage its preferred method would represent relative to the other allocation method.
18. The average annual Canadian cellphone plan covers 780 minutes of voice time, 600 text messages, and 8 video messages with a total annual cost of $\$500.63$. However, some Canadians are high-usage consumers and average 439 minutes per month.
    a. Assuming the same proportion of usage and that each component shares the cost equally on a per-unit basis, how many text messages and video messages do these high-usage Canadian consumers average per year? How much should they pay annually for their cellphone usage?
    b. A similar typical cellphone plan in the Netherlands has a total annual cost of $\$131.44$. How much would a high-usage plan cost annually in the Netherlands?
19. Procter & Gamble (P&G) is analyzing its Canadian sales by region: 23% of sales were on the West coast; 18% were in the Prairies, 45% were in central Canada, and 14% were in Atlantic Canada. Express and reduce the ratio of P&G's respective sales to the smallest term of one. Round to two decimals.
20. A company has 14 managers, 63 supervisors, and 281 workers. After a record profit

year, the company wants to distribute an end-of-year bonus to all employees. Each worker is to get one share, each supervisor gets twice as many shares as workers, and each manager gets twice as many shares as supervisors. If the bonus to be distributed totals $\$1,275,000$, how much does each manager, supervisor, and worker receive?

Solutions

16. $\$46,965,000$
17. $\text{7-11: transactions, } 4\% \text{ savings}$;
$\text{Tim Horton's: square metres, } 12.963\% \text{ savings}$;
$\text{Quizno's: transactions, } 26.3636\% \text{ savings}$

18a. $\text{Texts} = 4,052$; $\text{Video} = 54$, $\text{Cost} = \$3,381.18$

18b. $\$887.73$

19. $1.64 : 1.29 : 3.21 : 1$
20. $\text{Manager} = \$11,015.12$; $\text{Supervisor} = \$5,507.56$; $\text{Worker} = \$2,753.78$

Attribution

3.7: EXCHANGE RATES AND CURRENCY EXCHANGE

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $S =$ Selling price
- $C =$ Cost
- $E =$ Expenses
- $P =$ Profit

Formulas Used

- Formula 3.7 – **Currency Exchange**

$$\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$$

- Formula 3.1b – **Rate, Portion, and Base**

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

- Formula 4.3a – **Selling Price of a Product** (see Section 4.3)

$$\text{S} = \text{C} + \text{E} + \text{P}$$

Introduction

You have set aside $\$6,000$ in Canadian funds toward hostel costs during a long backpacking trip through the United States, Mexico, and Europe. After searching on the Internet, you decide to use Hotwire.com to reserve your hostel rooms. The website quotes you the following amounts for each country: $1,980$ US dollars, $21,675$ Mexican pesos, and $1,400$ euros. Have you allocated enough money to cover these costs?

Whether you are a consumer backpacking around the world on distant vacations, investing in international securities, or shopping online at global Internet sites, you must pay for your purchases in local currencies out of your Canadian currency accounts. Businesses are no different as they export and import products to and from other countries. With outsourcing on the rise, it is also common for business services such as call centres and advertising agencies to be located abroad along with manufacturing facilities. Large-scale companies may have operations in several countries throughout the world.

All of these transactions and operations require the conversion of Canadian currency to a foreign currency or vice versa. This section shows you the basics of currency conversion rates and then explores finer details such as charges for currency conversion and what happens when one currency gets stronger or weaker relative to another.

Exchange Rates

An **exchange rate** between two currencies is defined as the number of units of a foreign currency that are bought with one unit of the domestic currency, or vice versa. Since two currencies are involved in every transaction, two published exchange rates are available. Let's use Canada and the United States to illustrate this concept.

- The first exchange rate indicates what one dollar of Canada's currency becomes in US currency.
- The second exchange rate indicates what one dollar of US currency becomes in Canadian currency.

These two exchange rates allow Canadians to determine how many US dollars their money can buy and vice versa. These currency rates have an inverse relationship to one another: if 1 Canadian dollar equals 0.80 US dollars, then 1 US dollar equals $\dfrac{1}{\$0.80} = 1.25$ Canadian dollars.

Most Canadian daily and business newspapers publish exchange rates in their financial sections. Although exchange rates are published in a variety of ways, a currency cross-rate table, like the table below, is most common. Note that exchange rates fluctuate all of the time as currencies are actively traded in exchange markets. Therefore, any published table needs to indicate the date and time at which the rates were determined. Also note that the cells where the same currency appears show no published rate as you never need to convert from Canadian dollars to Canadian dollars!

Table 3.7.1 Currency Cross-Rate Table

	Per C$	Per US$	Per €	Per ¥	Per MXN$	Per AU$
Canadian Dollar (C$)		0.9787	1.4012	0.0122	0.0823	1.0360
US Dollar (US$)	1.0218		1.4317	0.0125	0.0841	1.0585
Euro (€)	0.7137	0.6985		0.0087	0.0588	0.7394
Japanese Yen (¥)	82.0233	80.2765	114.9287		6.7540	84.9747
Mexican Peso (MXN$)	12.1445	11.8859	17.0165	0.1481		12.5814
Australian Dollar (AU$)	0.9652	0.9447	1.3525	0.0118	0.0795	

In the table, all exchange rates have been rounded to four decimals. In true exchange markets, most exchange rates are expressed in 10 decimals or more such that currency exchanges in larger denominations are precisely performed. For the purposes of this textbook, we will use a four decimal standard to simplify the calculations while still illustrating the principles of currency exchange.

Technically, only half of the table is needed, since one side of the diagonal line is nothing more than the inverse of the other. For example, the euro is 0.7137 per C$ on the bottom of the diagonal. The inverse, or $\dfrac{1}{0.7137}=1.4011$ per €, is what is listed on top of the diagonal (the difference of 0.0001 is due to rounding to four decimals).

The formula is yet again another adaptation of **Formula 3.1b** $\text{Rate}=\dfrac{\text{Portion}}{\text{Base}}$ on Rate, Portion, and Base. Formula 3.7 expresses this relationship in the language of currency exchange.

3.7 Currency Exchange

Formula does not parse

Formula does not parse In **Formula 3.1b** $\text{Rate}=\dfrac{\text{Portion}}{\text{Base}}$, the desired currency represents the portion. This is what your money is worth in the other currency.

Formula does not parse In **Formula 3.1b** $\text{Rate}=\dfrac{\text{Portion}}{\text{Base}}$, the exchange rate represents the rate, which is the current exchange rate per currency unit in which your money is currently valued. Looking at a

Cross-Rate table, find the column that represents your existing currency and then cross-tabulate this column with the row for the desired currency (so you have desired currency per current currency). For example, if you have one euro that you want to convert into yen, locate the euro column and the yen row. The exchange rate is 131.5789. It is critical that both the exchange rate and current currency always be expressed in the same unit of currency type.

Formula does not parse In **Formula 3.1b** $\text{Rate} = \dfrac{\text{Portion}}{\text{Base}}$, the current currency represents the base, which is the amount of money (number of units) in your existing currency.

Key Takeaways

Currencies, exchange rates, and currency cross-tables all raise issues regarding decimals and financial fees.

1. **Decimals in Currencies**. Not all currencies in the world have decimals. Here in North America, MXN$, US$, and C$ have two decimals. Mexicans call those decimals *centavos* while Canadians and Americans call them *cents*. Australia and the European countries using the euro also have cents. However, the Japanese yen does not have any decimals in its currency. If you are unsure about the usage of decimals, perform a quick Internet search to clarify the issue.
2. **Financial Fees**. Technically, the rates in a cross-rate table are known as mid-rates. A **mid-rate** is an exchange rate that does not involve or provide for any charges for currency conversion. When you convert currencies, you need to involve a financial organization, which will charge for its services.
3. **Sell Rates**. When you go to a bank and convert your domestic money to a foreign currency, the bank charges you a **sell rate**, which is the rate at which a foreign currency is sold. When you exchange your money, think of this much like a purchase at a store—the bank's product is the foreign currency and the price it charges is marked up to its selling price. The sell rate is

always higher than the mid-rate in terms of C$ per unit of foreign currency and thus is always lower than the mid-rate in terms of foreign currency per unit of C$. For example, the exchange rate of C$ per US$ is 0.9787 (you always look up the "per currency" column that you are purchasing). This means it will cost you $\text{C\$}0.9787$ to purchase $\text{US\$}1.00$. The bank, though, will sell you this money for a sell rate that is higher, say $\$0.9987$. This means it will cost you an extra $\text{C\$}0.02$ per US$ to exchange the money. That $\$0.02$ difference is the fee from the bank for its services, and it is how the bank makes a profit on the transaction.

4. **Buy Rates**. When you go to a bank and convert your foreign currency back into your domestic money, the bank charges you a **buy rate**, which is the rate at which a foreign currency is purchased. The buy rate is always lower than the mid-rate in terms of C$ per unit of foreign currency and thus is always higher than the mid-rate in terms of foreign currency per unit of C$. Using the same example as above, if you want to take your $\text{US\$}1.00$ to the bank and convert it back to Canadian funds, the bank charges you a buy rate that is lower, say $\$0.9587$. In other words, you receive $\text{C\$}0.02$ less per US$. Again, the $\$0.02$ difference is the bank's fee for making the currency exchange on your behalf.

HOW TO

Perform a Currency Exchange

Follow these steps when performing a currency exchange:

Step 1: Identify all known variables. Specifically, identify the currency associated with any amounts. You also require the mid-rate. If buy rates and sell rates are involved, identify how these rates are calculated.

Step 2: If there are no buy or sell rates, skip this step. If buy and sell rates are involved, calculate these rates in the manner specified by the financial institution.

Step 3: Apply **Formula 3.7**
$\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$ using the appropriate mid-rate, buy rate, or sell rate to convert currencies.

This section opened with your backpacking vacation to the United States, Mexico, and Europe, for which you were quoted prices of $\text{US}\$1,980$, $\text{MXN}\$21,675$, and $€1,400$ for hostels. Assume all purchases are made with your credit card and that your credit card company charges 2.5% on all currency exchanges. Can your $\text{C}\$6,000$ budget cover these costs?

Step 1: There are three currency amounts: $\text{US}\$1,980$, $\text{MXN}\$21,675$, and $€1,400$. Using the cross-rate table, the Canadian exchange mid-rate per unit of each of these currencies is $\text{US}\$0.9787$, $\text{MXN}\$0.0823$, and $€1.4012$.

Step 2: Calculate the buy rates (since you are converting foreign currency into domestic currency) for each currency:

$$\text{US}\$ = 0.9787(1.025) = 1.0032$$
$$\text{MXN}\$ = 0.0823(1.025) = 0.0844$$
$$€ = 1.4012(1.025) = 1.4362$$

Step 3: Apply Formula 3.7 ($\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$) to each of these currencies:

$$\text{US}\$\text{: Desired Currency} = \text{US}\$1,980 \times 1.0032 = \text{C}\$1,986.34$$
$$\text{MXN}\$\text{: Desired Currency} = \text{MXN}\$21,675 \times 0.0844 = \text{C}\$1,829.37$$
$$€\text{: Desired Currency} = €1,400 \times 1.4362 = \text{C}\$2,010.68$$

Putting the three amounts together, your total hostel bill is:

$$\$1,986.34 + \$1,829.37 + \$2,010.68 = \$5,826.39$$

Because this is under budget by $\$173.61$, all is well with your vacation plans.

Things To Watch Out For

When working with currency exchange, probably the trickiest element is that you have to choose one of two inverse exchange rates depending on which way the money conversion is taking place. In any currency situation, it is important that you take the time to understand the basis on which the currency rate is being expressed. Typically, exchange rates are expressed on a per-unit basis in the country's domestic currency. For example, Canadians express the US dollar exchange rate on a per C$ basis. From the cross-rate table, that exchange rate is 1.0218. In contrast, Americans express the Canadian dollar exchange rate on a per US$ basis, or 0.9787.

Paths To Success

Let the buyer beware when it comes to international transactions. If you have ever purchased and returned an item to an international seller, you may have noticed that you did not receive all of your money back. For most consumers, international purchases are made via credit cards. What most consumers do not know is that the credit card companies do in fact use buy and sell rates that typically charge 2.5% of the exchange rate when both buying and selling.

For example, if you purchase a $\text{US}\$2,000$ item at the rates listed in the cross-rate table, your credit card is charged $\$2,000 \times 0.9787(1.025) = \$2,006.40$. If you return an item you do not want, your credit card is refunded $\$2,000 \times 0.9787(0.975) = \$1,908.40$. In other words, you are out $\$2,006.40 - \$1,908.40 = \$98$! This amount represents your credit card company's charge for the currency conversion—a whopping 4.9% of your purchase price!

Try It

1) In each of the following situations and using the cross-rate table, determine on a strictly numerical basis whether you would have more or fewer units of the target currency than of the original currency. You want to convert:

a. C$ into US$
b. AU$ into MXN$
c. MXN$ into ¥
d. C$ into €

Solution

a. Less, since $\text{CAD}\$1$ becomes $\text{USD}\$0.8118$
b. More, since $\text{AUD}\$1$ becomes $\text{MXN}\$14.8810$
c. More, since $\text{MXN}\$1$ becomes $¥5.6180$
d. Less, since $\text{CAD}\$1$ becomes $€0.6841$

Example 3.7.1

Strictly using the mid-rates from the cross-rate table presented earlier, if you wanted to convert $\text{C\$}1,500$ into MXN$ for your spring break vacation in Cancun, Mexico, how many Mexican pesos would you have?

Solution

***Step 1:* What are we looking for?**

Take Canadian currency and convert it into the desired Mexican currency.

***Step 2:* What do we already know?**

The amount and the exchange rate are known:

$$\begin{aligned}\text{Current Currency} &= \$1,500 \\ \text{Exchange Rate} &= 12.1445\end{aligned}$$

***Step 3:* Make substitutions using the information known above.**

No buy or sell rates are involved. Apply Formula 3.7:

$$\begin{aligned}\text{Desired Currency} &= \text{Exchange Rate} \times \text{Current Currency} \\ \text{Desired Currency} &= 12.1445 \times \$1,500 \\ \text{Desired Currency} &= \$18,216.75\end{aligned}$$

***Step 4:* Provide the information in a worded statement.**

You have $\$18,216.75$ Mexican pesos available for your spring break vacation.

Example 3.7.2

A Mexican manufacturer imports its parts from Canada and assembles its product in Mexico, then exports some of the finished product back to Canada to sell at retail. Suppose the total cost of the imported parts, purchased and paid for in Canadian funds, is $\mathrm{C\$5.50}$ per unit, assembly costs are $\mathrm{MXN\$24.54}$ per unit, and expenses are $\mathrm{MXN\$4.38}$ per unit. If the product is exported back to Canada at a selling price of $\mathrm{C\$9.95}$ (for which its Canadian customers pay in Canadian funds), what is the total profit in Mexican pesos on $10,000$ units? The manufacturer's financial institution charges a sell rate 2% higher than the mid-rate and has a buy rate that is 3% lower than the mid-rate. Use the mid-rates from the cross-rate table.

Solution

Step 1: **What are we looking for?**

Calculate the total profit (P) in MXN$ for the Mexican manufacturer. However, since money is being expressed in different currencies, you must first convert all amounts into Mexican pesos.

Step 2: **What do we already know?**

You know the costs, expenses, price, and mid-rate:

$$\begin{aligned}\text{Cparts} &= 10,000 \times \mathrm{C\$5.50}\\ \text{Cparts} &= \mathrm{C\$55,000}\end{aligned}$$

$$\begin{aligned}\text{Cassembly} &= 10,000 \times \mathrm{MXN\$24.54}\\ \text{Cassembly} &= \mathrm{MXN\$245,400}\end{aligned}$$

$$\begin{aligned}E &= 10,000 \times \mathrm{MXN\$4.38}\\ E &= \mathrm{MXN\$43,800}\end{aligned}$$

$$\begin{aligned}S &= 10,000 \times \mathrm{C\$9.95}\\ S &= \mathrm{C\$99,500}\end{aligned}$$

$$\text{Mid-Rate} = 12.1445 \text{ per C}$$

$$\text{Sell Rate} = 2\% \text{ higher}$$

$$\text{Buy Rate} = 3\% \text{ lower}$$

Note the mid-rate used is "per C$" since the manufacturer needs to purchase Canadian funds to pay its suppliers and also needs to sell its Canadian revenues to the bank.

Step 3: **Make substitutions using the information known above.**

Calculate the buy and sell rates.

$$\text{Sell Rate} = 2\% \text{ higher than mid-rate}$$
$$\text{Sell Rate} = 12.1445(1 + 0.02)$$
$$\text{Sell Rate} = 12.3874$$

$$\text{Buy Rate} = 3\% \text{ lower than mid-rate}$$
$$\text{Buy Rate} = 12.1445(1 - 0.03)$$
$$\text{Buy Rate} = 11.7802$$

Convert the purchase of the parts in C$ into MXN$ using the sell rate. Apply and adapt **Formula 3.7**$\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$.

$$\text{Cparts in MXN\$} = 12.3874 \times \$55,000$$
$$\text{Cparts in MXN} = \$681,307$$

Convert the sale of the product in C$ into MXN$ using the buy rate. Apply and adapt **Formula 3.7**$\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$.

$$\text{S in MXN\$} = 11.7802 \times \$99,500$$
$$\text{S in MXN} = \$1,172,129.90$$

All amounts are in MXN$. Apply **Formula 4.3a**$\text{S} = \text{C} + \text{E} + \text{P}$ (selling price of a product):

$$S = \text{Cparts} + \text{Cassembly} + E + P$$
$$\$1,172,129.90 = \$681,307 + \$245,400 + \$43,800 + P$$
$$\$1,172,129.90 = \$970,507 + P$$
$$\$201,622.90 = P$$

Step 4: **Provide the information in a worded statement.**

After converting all Canadian currency into Mexican pesos and factoring in the financial institution's buy and sell rates, the manufacturer realizes a profit of $\$201,622.90$ in Mexican pesos.

Currency Appreciation and Depreciation

An analyst on *Global News* is discussing how the Canadian dollar has strengthened against the US dollar. Your first reaction is that a strong Canadian dollar ought to be good thing, so hearing that this change might hurt Canada's exports, you wonder how that could be.

Currencies are actively traded in the international marketplace, which means that exchange rates are changing all the time. As such, exchange rates rise and decline. A currency **appreciates** (or strengthens) relative to another currency when it is able to purchase *more* of that other currency than it could previously. A currency **depreciates** (or weakens) relative to another currency when it is able to purchase *less* of that other currency than it could previously. Take a look at two examples illustrating these concepts:

- EXAMPLE 1: If C$1 buys US$1.02 and the exchange rate rises to US$1.03, then your C$1 purchases an additional penny of the US dollars. Therefore, the Canadian currency appreciates, or strengthens, relative to the US dollar.
- EXAMPLE 2: Similarly, if the exchange rate drops to US$1.01, then your C$1 purchases one less penny of the US dollars. Therefore, the Canadian currency depreciates, or weakens, relative to the US dollar.

These concepts are particularly important to international business and global economies. Generally speaking, when a currency appreciates it has a positive effect on imports from the other country because it costs less money than it used to for domestic companies to purchase the same amount of products from the other country. However, the currency appreciation tends to also have a negative effect on exports to other countries because it costs the foreign companies more money to purchase the same amount of products from the domestic companies.

Key Takeaways

It is critical to observe that if currency A appreciates relative to currency B, then the opposite is true for currency B relative to currency A (currency B depreciates relative to currency A). The figure to the right illustrates this concept. Recalling Example 1 above, the Canadian currency appreciated, so the US currency depreciated. In Example 2, the Canadian currency depreciated, so the US currency appreciated.

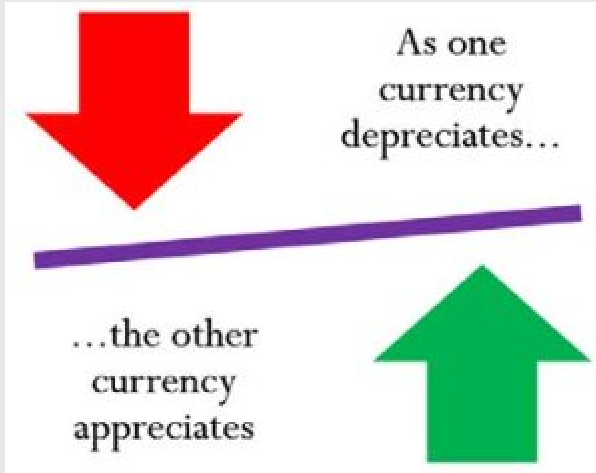

Figure 3.7.1

HOW TO

Work with Currency Appreciation and Depreciation

When you work with currency appreciation or depreciation, you still use the same basic steps as before. The additional skill you require in the first step is to adjust an exchange rate appropriately based on how it has appreciated or depreciated.

Things To Watch Out For

It is very easy to confuse the two relative currencies, their values, and the concepts of appreciation and depreciation. For example, if the exchange rate increases between currency A relative to a unit of currency B, which exchange rate appreciated? If currency A has increased per unit of currency B, then it takes more money of currency A to buy one unit of currency B. As a result, currency B appreciates because a single unit of currency B can now buy more of currency A. For example, if the exchange rate is $\text{US\$}1.0218$ relative to $\text{C\$}1$ and it increases to $\text{US\$}1.0318$ relative to $\text{C\$}1$, then the Canadian dollar purchases one penny more. The figure helps you understand the relationships involved and provides a visual reminder of which direction everything is moving.

If Currency A relative to one unit of Currency B...	Currency A	Currency B
Increases	Depreciates or weakens	Appreciates or strengthens
Decreases	Appreciates or strengthens	Depreciates or weakens

Figure 3.7.2

Try It

2) Answer the following:

a. If the exchange rate in terms of US dollars per unit of euros increases, which currency weakened?
b. If the Australian dollar weakens against the Canadian dollar, did the exchange rate increase or decrease in terms of Australian dollars per unit of Canadian dollars?
c. If the exchange rate in terms of yen per unit of the Mexican pesos decreases, which currency weakened?
d. If the British pound (£) appreciates against the US dollar, did the exchange rate increase or

decrease in terms of pounds per unit of US dollars?

Solution

a. US dollars
b. Increase
c. Mexican pesos
d. Decrease

Example 3.7.3

A Canadian manufacturer requires parts from the United States. It purchases from its supplier in lots of $100,000$ units at a price of $\mathrm{US}\$7.25$ per unit. Since the last time the manufacturer made a purchase, the Canadian dollar has appreciated 0.0178 from the previous mid-rate of $\mathrm{US}\$1.0218$ per C$. If the sell rate is 1.5% above the mid-rate, how have the manufacturer's costs changed?

Solution

Step 1: **What are we looking for?**

Calculate the cost of the product both before and after the currency appreciation. The difference between the two numbers is the change in the manufacturer's costs.

Step 2: **What do we already know?**

The following purchase and exchange rates are known:

$$\text{Current Currency} = \text{Total Purchase}$$
$$\text{Current Currency} = 100,000 \times \$7.25$$
$$\text{Current Currency} = \text{US}\$725,000$$

$$\text{Mid-Rate} = 1.0218 \text{ per C}\$$$

$$\text{Sell Rate} = 1.5\% \text{ higher}$$

$$\text{Canadian appreciation} = 0.0178$$

Step 3: **Make substitutions using the information known above.**

Calculate the old and new sell rates, factoring in the currency appreciation. Notice that the Current Currency is in US dollars but the sell rates are per Canadian dollar. You will need to invert the rates so that the exchange rate is expressed per US dollar to match the Current Currency.

$$\text{Previous Sell Rate} = 1.0218 \times (1 + 0.015)$$
$$\text{Previous Sell Rate} = 1.0371 \text{ per C}\$$$
$$\text{Previous Sell Rate} = \frac{1}{1.0371}$$
$$\text{Previous Sell Rate} = 0.9642 \text{ per US}\$$$

If the Canadian dollar has appreciated, it buys more US dollars per C$. Therefore,

$$\text{New Sell Rate} = (1.0218 + 0.0178) \times (1 + 0.015)$$
$$\text{New Sell Rate} = 1.0552 \text{ per C}\$$$
$$\text{New Sell Rate} = \frac{1}{1.0552}$$
$$\text{New Sell Rate} = 0.9477 \text{ per US}\$$$

Apply **Formula 3.7**$\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$ for each transaction.

$$\text{Previous C}\$ = 0.9642 \times \$725,000$$
$$\text{Previous C} = \$699,045$$

$$\text{New C}\$ = 0.9477 \times \$725,000$$
$$\text{New C} = \$687,082.50$$

Calculate the difference between the two numbers to determine the change in cost.

$$\text{Change in Cost} = \text{New C\$} - \text{Previous C\$}$$
$$\text{Change in Cost} = \$687,082.50 - \$699,045$$
$$\text{Change in Cost} = -\$11,962.50$$

Step 4: **Provide the information in a worded statement.**

The manufacturer has its input costs decrease by $\$11,962.50$ since $\text{C\$1}$ now purchases more US$.

Section 3.7 Exercises

Mechanics

For questions 1–6, use the mid-rates in the cross-rate table below to convert the current currency to the desired currency.

Table 3.7.2

	Per C$	Per US$	Per €	Per ¥	Per MXN$	Per AU$
Canadian Dollar (C$)		0.9787	1.4012	0.0122	0.0823	1.0360
US Dollar (US$)	1.0218		1.4317	0.0125	0.0841	1.0585
Euro (€)	0.7137	0.6985		0.0087	0.0588	0.7394
Japanese Yen (¥)	82.0233	80.2765	114.9287		6.7540	84.9747
Mexican Peso (MXN$)	12.1445	11.8859	17.0165	0.1481		12.5814
Australian Dollar (AU$)	0.9652	0.9447	1.3525	0.0118	0.0795	

Table 3.7.3 Exercise Questions 1-6

	Current Currency	Desired Currency
1.	C$68,000	US$
2.	¥15,000,000	€
3.	AU$3,000	MXN$
4.	US$180,000	AU$
5.	€230,500	C$
6.	MXN$1,300,000	¥

Solutions

1. $55,202.40\text{USD}$
2. $114,000€$
3. $44,643\text{MXN}$
4. $239,004\text{AUD}$
5. $336,944.90\text{CAD}$
6. $7,303,400¥$

Applications

7. If the exchange rate is $¥97.3422$ per C$, what is the exchange rate for C$ per ¥?
8. Procter & Gamble just received payment for a large export of Tide in the amount of $275,000$ Denmark kroner (DKK). If the exchange mid-rate is $\text{C}\$0.1992$ per DKK and the bank charges 3% on its buy rates, how many Canadian dollars will Procter & Gamble receive?
9. The exchange rate per US$ is $\text{C}\$0.9863$. If the Canadian dollar depreciates by $\$0.0421$ per US$, how many more or less US$ is $\text{C}\$12,500$ able to purchase?
10. Jack is heading home to visit his family in Great Britain and decided to stop at the airport kiosk to convert his money. He needs to convert $\text{C}\$5,000$ to British pounds (£). The exchange mid-rate per C$ is $£0.5841$. The kiosk charges a commission of 4.5% on the conversion, plus a flat fee of $£5.00$.

   a. How many pounds will Jack have?
   b. What is the percentage cost of his transaction?

11. Yarianni is heading on a vacation. She converts her $\text{C\$4,000}$ into Chinese yuan renminbi (CNY) at a sell rate of CNY6.3802 per C$. While in China, she spends CNY14,000 of her money. At the airport, she converts her remaining money into Indian rupees (INR) at a sell rate of INR6.7803 per CNY. In India, she spends INR50,000. When she returns home, she converts her INR back to C$ at a buy rate of $\text{C\$0.0231}$ per INR. How many Canadian dollars did she return with? Note that all currencies involved have two decimals.
12. Elena is an international investor. Four years ago she purchased $2,700$ shares of a US firm at a price of $\text{US\$23.11}$ per share when the exchange rate was $\text{US\$0.7536}$ per C$. Today, she sold those shares at a price of $\text{US\$19.87}$ per share when the exchange rate was $\text{US\$1.0063}$ per C$. In Canadian dollars, determine how much money Elena earned or lost on her investment.
13. International Traders regularly imports products from Hong Kong. If the exchange rate of C$ per Hong Kong dollar (HKD) is 0.1378 and the Canadian dollar appreciates by HKD0.0128, by what amount would the cost of a HKD1,000,000 purchase increase or decrease in Canadian dollars for International Traders?
14. Brian needs to purchase some Brazilian reals (BRL). He takes $\text{C\$7,500}$ to the bank and leaves the bank with BRL12,753.20. If the exchange mid-rate per C$ is 1.7621, determine the sell rate commission percentage (rounded to two decimals) charged by the bank.

Solutions

7. 0.0103 CAD per ¥

8. $53,130 \text{ CAD}$

9. Purchase $\$518.75$ less than before

10a. $2,784 \text{ £}$

10b. 4.6739%

11. 649.44 CAD

12. Elena lost $\$29,489.81$ on her investment

13. The products cost $\$12,800\ \text{CAD}$ less than before
14. 3.5%

Challenge, Critical Thinking, & Other Applications

15. Fernando could purchase a 55" Samsung HDTV in Winnipeg, Manitoba, for $\$2,999.99$ plus taxes. Alternatively, he could head across the border on Black Friday and shop in Grand Forks, North Dakota, where the same product is selling for $\text{US}\$2,499.99$ (plus 5% state sales tax and 1.75% local sales tax) at Best Buy. He estimates he would incur $\$65$ in gasoline and vehicle wear and tear, $\$130$ in accommodations, and $\$25$ in food (all money in US$). He would make all purchases on his credit card, which uses the mid-rate plus 2.5%. When returning across the border, he would have to pay in Canadian dollars the 5% GST on the Canadian value of the HDTV not including taxes. Once home, he can then have the North Dakota government refund all taxes paid on the HDTV through their Canadian sales tax rebate program. For all currency exchanges, assume a mid-rate of $\text{US}\$0.9222$ per C$. Which alternative is Fernando's better choice and by how much?
16. The current mid-rate is $\text{C}\$1.5832$ per €. Scotiabank has a sell rate of $\text{C}\$1.6196$ per € while an airport kiosk has a sell rate of $\backslash textC\$1.6544$ per € plus a service charge of $\text{C}\$4.75$. You need to purchase $€800$.
    a. Calculate the fee percentages charged by each financial institution. Round your answers to one decimal.
    b. Rounded to two decimals, what percentage more than Scotiabank is the airport kiosk charging on your purchase?
17. Henri and Fran have retired and are considering two options for a two-month vacation in Europe. Their local Lethbridge travel agent is offering them an all-inclusive package deal at $\text{C}\$7,975$ per person. Alternatively, they can book their own flights for $\text{C}\$1,425$ per person, stay in Britain at a small apartment averaging $£65$ per night for 30 days, and then in France for $€70$ average per night for 30 days. Estimated groceries cost a total of $£250$ in Britain and $€400$ in France. They will need to purchase a Eurail pass for $€986$ each while they are there. The exchange rates are $€0.6808$ per C$ and $£0.5062$ per C$. Which alternative is their cheapest option and by how much in Canadian dollars?
18. A Canadian manufacturer imports three parts from different countries. It assembles the

three parts into a finished item that is then exported to the United States. Every transaction always involves $25,000$ units. Expenses average $\$6.25$ per unit.

Table 3.7.4

Component	Price per unit	Exchange rate last month per C$	Exchange rate this month per C$
Part A	¥1,500	¥107.9420	¥108.9328
Part B	AU$14.38	AU$1.1319	AU$1.0928
Part C	€10.73	€0.6808	€0.6569
Finished product for export	US$59.45	US$1.0128	US$1.0243

Considering currency fluctuations, calculate the change in the profit month-over-month in Canadian dollars for the Canadian manufacturer.

19. In each of the following situations, convert the old amount to the new amount using the information provided.

Table 3.7.5 Exercises Question 19

	Old Amount	Old Exchange Rate per C$	Exchange Rate Change	New Amount
a.	US$625.00	US$0.9255	C$ appreciated by US$0.0213	? US$
b.	€16,232.00	€0.5839	C$ depreciated by €0.0388	? €
c.	¥156,500	¥93.4598	C$ depreciated by ¥6.2582	? ¥
d.	MXN$136,000	MXN$13.5869	C$ appreciated by MXN$0.4444	? MXN$

20. Compare the following four situations and determine which one would result in the largest sum of money expressed in Canadian funds.

Table 3.7.6 Exercises Question 20

Situation	Amount	Exchange Rate per C$
a.	65,204 Algerian dinars (DZD)	DZD65.5321
b.	1,807,852 Colombian pesos (COP)	COP1,781.1354
c.	3,692 Israeli new shekels (ILS)	ILS3.7672
d.	30,497 Thai baht (THB)	THB30.4208

Solutions

15. Buy in the U.S. to save $\$196.60$

16a.Scotiabank's sell rate fee is 2.3%; The kiosk's sell rate fee is 4.5%

16b. The airport kiosk charges 2.52% more than Scotiabank

17. Booking it themselves saves $\$1,824.98$

18. Currency fluctuations have decreased profits by $\$38,432.12$

19a. 639.38 USD

19b. $€15,153.19$

19c. $¥146,024$

19d. $140,447.70 \text{ MXN}$

20. $\text{COP} = \$1,084.71 \text{ CAD}$

$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$ THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$ $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$

$\text{S} = \text{C} + \text{E} + \text{P}$

Attribution

CHAPTER 3 SUMMARY

Formula & Symbol Hub Summary

For this chapter you used the following:

Symbols Used

- $\sum$ = summation symbol, which means you add everything up
- $\%C$ = percent change
- AV = assessed value of a property
- Current Currency = currency you are converting from
- Desired Currency = currency you are converting to
- Exchange Rate = per-unit conversion rate for current currency
- GE = gross earnings
- n = a count of something, whether a total number of periods or a total quantity
- Property Tax = property tax amount
- PTR = property tax rate, usually set on a per $\$100$ (tax rate) or per $\$1,000$ (mill rate) basis
- Remit = tax remittance
- RoC = = rate of change per time period
- S = selling price
- S_tax = selling price including taxes
- Tax Collected = total tax collected through sales
- Tax Paid = total tax paid through purchases
- V_{f} = the value that a quantity has become; the number that is being compared
- V_{i} = the value that a quantity used to be; the number to compare to
- x = any individual piece of data

Formulas Used

- Formula 3.1a – **Percentage**

$$\% = \text{dec} \times 100$$

Used to convert a decimal number into a percentage.

- Formula 3.1b – **Rate, Portion, Base**

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

- Formula 3.2a – **Percent Change**

$$\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$$

- Formula 3.2b – **Rate of Change Over Time**

$$RoC = \left(\left(\frac{V_{\text{f}}}{V_{\text{i}}}\right)^{\frac{1}{n}}\right) \times 100$$

- Formula 3.3 – **Salary & Hourly Gross Earnings**

$$\text{GE} = \text{Regular Earnings} + \text{Overtime Earnings} + \text{Holiday Earnings} + \text{Stat Holidays Worked Earnings}$$

- Formula 3.4a- **Selling Price Including Tax**

$$S_{\text{tax}} = S + (S \times \text{Rate})$$

- Formula 3.4b – **GST/HST Remittance**

$$\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$$

- Formula 3.5 – **Property Taxes**

$$\text{Property Tax} = \sum(\text{AV} \times \text{PTR})$$

- Formula 3.7 – **Currency Exchange**

$$\text{Desired Currency} = \text{Exchange Rate} \times \text{Current Currency}$$

- Formula 4.3a – **Selling Price of a Product** (see Section 4.3)

$$\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$$

Key Concepts Summary

Section 3.1: Percentages (How Does It All Relate?)

- Converting decimal numbers to percentages
- Working with percentages in the form of rates, portions, and bases

Section 3.2: Percent Change (Are We Up or Down?)

- Measuring the percent change in a quantity from one value to another
- Measuring the constant rate of change over time in a quantity

Section 3.3 Payroll

- Calculating salary and hourly gross earnings
- Discussion of employment contract characteristics for salary and hourly earners
- Calculating commission gross earnings
- Calculating piecework gross earnings

Section 3.4: Sales Tax

- The three sales taxes, current rates, and how to calculate prices including taxes
- How businesses complete GST/HST remittances

Section 3.5: Property Taxes

- How municipalities levy mill rates and tax rates against real estate owners

Section 3.6: Ratios, Proportions, and Prorating (It Is Only Fair)

- The characteristics of a ratio
- How to simplify and reduce a ratio to its lowest terms
- How to simplify a ratio by reducing its smallest term to a value of one
- How to equate two ratios into the form of a proportion
- How to use proportions to prorate

Section 3.7: Exchange Rates and Currency Exchange

- Converting currencies through exchange rates
- The rise and decline of exchange rates: currency appreciation and depreciation

The Language of Business Mathematics

assessed value The portion of the market value of a property that is subject to municipal taxes.

average A single number that represents the middle of a data set.

buy rate The rate at which a foreign currency is bought; it will always be lower than the mid-rate in terms of money per unit of foreign currency.

commission An amount or a fee paid to an employee for performing or completing some form of transaction.

currency appreciation When one currency strengthens relative to another currency, resulting in an ability to purchase more of that other currency than it could previously.

currency depreciation When one currency weakens relative to another currency, resulting in an ability to purchase less of that other currency than it could previously.

graduated commission A form of compensation where an employee is offered increasing rates of commission for higher levels of performance.

gross earnings The amount of money earned before any deductions are removed from a paycheque.

GST The goods and services tax is a Canadian federal sales tax on most products purchased by businesses and consumers alike.

hidden term In prorating, this is the sum of all the other terms on the same side of the proportion and represents a total.

holiday earnings Earnings paid to an employee on a statutory holiday for which no work is performed.

hourly wage A variable compensation based on the time an employee has worked.

HST The harmonized sales tax is a tax that combines the GST and PST into a single tax.

market value A snapshot of the estimated selling price of a property at some specified point in time.

mid-rate An exchange rate that does not involve or provide for any charges for currency conversion.

mill rate A tax per $1,000 of assessed value to determine property taxes.

percent change An expression in percent form of how much any quantity changes from a starting period to an ending period.

piecework A form of compensation where an employee is paid on a per-unit basis.

property taxes Annual taxes paid by the owners of real estate to local levying authorities to pay for services such as roads, water, sewers, public schools, policing, fire departments, and other community services.

PST The provincial sales tax is a consumer sales tax administered by Canadian provinces and territories.

proportion A statement of equality between two ratios.

prorating The process of taking a total quantity and allocating or distributing it proportionally.

public holiday A provincially recognized day for which employees may or may not get a day of rest and may or may not receive pay depending on provincial employment standards.

overtime Work time in excess of an employee's regular workday or regular workweek.

overtime or premium earnings Earnings determined by an employee's overtime rate of pay and that occur when regular hours are exceeded.

rate of change over time A measure of the percent change in a variable per time period.

ratio A fixed relationship between two or more quantities, amounts, or sizes of a similar nature where all terms are nonzero.

regular earnings Earnings determined by an employee's regular rate of pay.

salary A fixed compensation paid to a person on a regular basis for services rendered.

salary plus commission A form of compensation in which gross earnings combine a basic salary together with commissions on transactions.

sales tax A percent fee levied by a government on the supply of products.

sell rate The rate at which a foreign currency is sold. It is always higher than the mid-rate in terms of money per unit of foreign currency.

simple average An average where each piece of data shares the same level of importance and frequency but does not represent percent changes or numbers that are intended to be multiplied with each other.

statutory holiday A legislated day of rest with pay.

statutory holiday worked earnings Earnings paid to an employee on a statutory holiday at a premium rate for working on the statutory holiday.

straight commission A form of compensation where the employee's entire earnings are based on dollar transactions and calculated strictly as a percentage of the total transactions.

terms of the ratio The numbers appearing in a ratio.

tax policy A municipality-based percentage of the market value of a property that is used to convert the market value into an assessed value.

tax rate A tax per $100 of assessed value to determine property taxes.

tax remittance The fulfillment of a tax obligation.

weighted average An average where not all pieces of data share the same level of importance or they do not occur with the same frequency; the data cannot represent percent changes or numbers that are intended to be multiplied with each other.

Technology

Calculator

Percent Change

$\text{2nd } \Delta\%$ (located above the 5 key) to access and enter three of the following. Press CPT on the unknown to compute. It is recommended to clear a previous question from memory by then pressing 2nd CLR Work.

$\text{OLD} =$ The old or original quantity; the number to compare to

$\text{NEW} =$ The new or current quantity; the number wanting to be compared

$\%\text{CH} =$ The percent change; in percent format

$\#\text{PD} =$ Number of consecutive periods for the change. By default, it is set to 1. However, if you want to increase a number by the same percentage (a constant rate of change) consecutively, enter the number of times the change occurs. For example, if increasing a number by 10% three times in a row, set this variable to 3.

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CHAPTER 4: MARKETING APPLICATIONS

Outline of Chapter Topics

When a new retail store opens in your neighborhood strip mall, you wonder whether it will still be running six months or a year from now. Do its owners know how much merchandise they have to sell to cover their costs? Will any money be left over to help grow the business?

While the numbers vary, approximately 15% of new companies in Canada go out of business in their first year of operations. This number rises to 38% by the third year and to a staggering 49% over the first five years (Fisher & Rueber, 2010). Why do so many new businesses fail? Economic conditions, the fierceness of competition, changing consumer tastes, or even changes in taxes all contribute to the problem, but one of the most common reasons businesses fail is poor tracking of their basic financial numbers.

So if you want your own business to succeed, you need to understand its financial side. You need to be able to answer questions like the following:

- How does the money leave the business through costs and expenses?
- How does the money come into the business through revenue, which is determined by both the price tags on the merchandise and the volume at which that merchandise sells?
- What is the minimum amount of revenue that is needed to cover the costs?

- What is the resulting profitability of the business?

When you buy an iPod, it is very important that the right price is set. The price should

- be seen as fair by you the buyer,
- pay for the costs (plastics, battery, buttons, circuit boards, headset) and expenses (employees, factory, electricity, distribution) of making the iPod, and
- allow the seller's business to make some extra money as profit so that it can grow its business further.

If your business is selling a product, you will need to pay close attention to price adjustments because they affect profitability. Various discounts, like putting items on sale, may increase sales while lowering the amount of profit per transaction. How do you know where to set the balance to maximize profit overall?

As a student in a business program, consider this chapter essential to the success of any business. Whether your pricing strategy is high or low, the company must ensure that it can still pay its bills as a minimum requirement. And that requires careful juggling of many factors. If it fails to manage its pricing properly, the company will go bankrupt!

This chapter will make you a smarter business professional and a wiser consumer. You shop retail almost every day and regularly purchase goods and services. If you understand how product pricing works, you can make sense of "deals." You can easily explain why the same product sells for two different prices at two different stores.

In this chapter, you must learn the language of marketers to perform merchandising mathematics involving product costs, expenses, prices, markups, markdowns, and ultimately profitability. Once the study of the various pricing components is complete, we will see how the various pieces of the pricing puzzle fit together into a cohesive merchandising environment.

Attribution

4.1: FIGURING OUT THE COST: DISCOUNTS

Formula & Symbol Hub

In this section you will need the following:

Symbols Used

- d = Discount rate
- $D\$$ = Discount amount
- d_1 = First discount
- d_2 = Second discount
- d_{equiv} = Single equivalent discount rate
- d_n = nth discount
- L = List price
- N = Net price

Formulas Used

- Formula 4.1a – **Single Discount**

$$N = L \times (1 - d)$$

- Formula 4.1b – **Discount Amount**

$$D\$ = L \times d$$

- Formula 4.1c – **Discount Amount**

$$D\$ = L - N$$

- Formula 4.1d – **Multiple Discounts**

$$N = L \times (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$$

- Formula 4.1e – **Single Equivalent Discount**

$$d_{equiv} = 1 - (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$$

Introduction

You mutter in exasperation, "Why can't they just set one price and stick with it?" Your mind boggles at all the competing discounts you encounter at the mall in your search for that perfect Batman toy for your nephew. Walmart is running their Rollback promotion and is offering a Batmobile for 25% off, regularly priced at $\$49.99$. Toys R' Us has an outlet in the parking lot where the regular price for the same toy is $\$59.99$, but all Batman products are being cleared out at 40% off. You head over to The Bay for a warehouse clearance event that has the same toy priced at $\$64.99$ but at 35% off. It is also Bay Days, which means you can scratch and win a further 10% to 20% off the sale price. You go to Dairy Queen for a Blizzard to soothe your headache while you figure things out.

The **cost** of a product is the amount of money required to obtain the merchandise. If you are a consumer, the ticketed price tag on the product is your cost. If you are a reseller (also known as a *middleman* or *intermediary*), what you pay to your supplier for the product is your cost. If you are a manufacturer, then your cost equals all of the labor, materials, and production expenditures that went into creating the product.

A **discount** is a reduction in the price of a product. As a consumer, you are bombarded with discounts all the time. Retailers use various terms for discounts, including *sales* or *clearance*. If your business purchases a product from a supplier, any discount it receives lowers how much the business pays to acquire the product. When a business buys products, the price paid is the cost to the business. Therefore, a lower price means a lower cost.

If your business is the one selling the product, any discount offered lowers the selling price and reduces revenue per sale. Since the revenue must cover all costs and expenses associated with the product, the lower price means that the business reduces profits per sale. In business, it is common practice to express a discount as a percentage off the regular price.

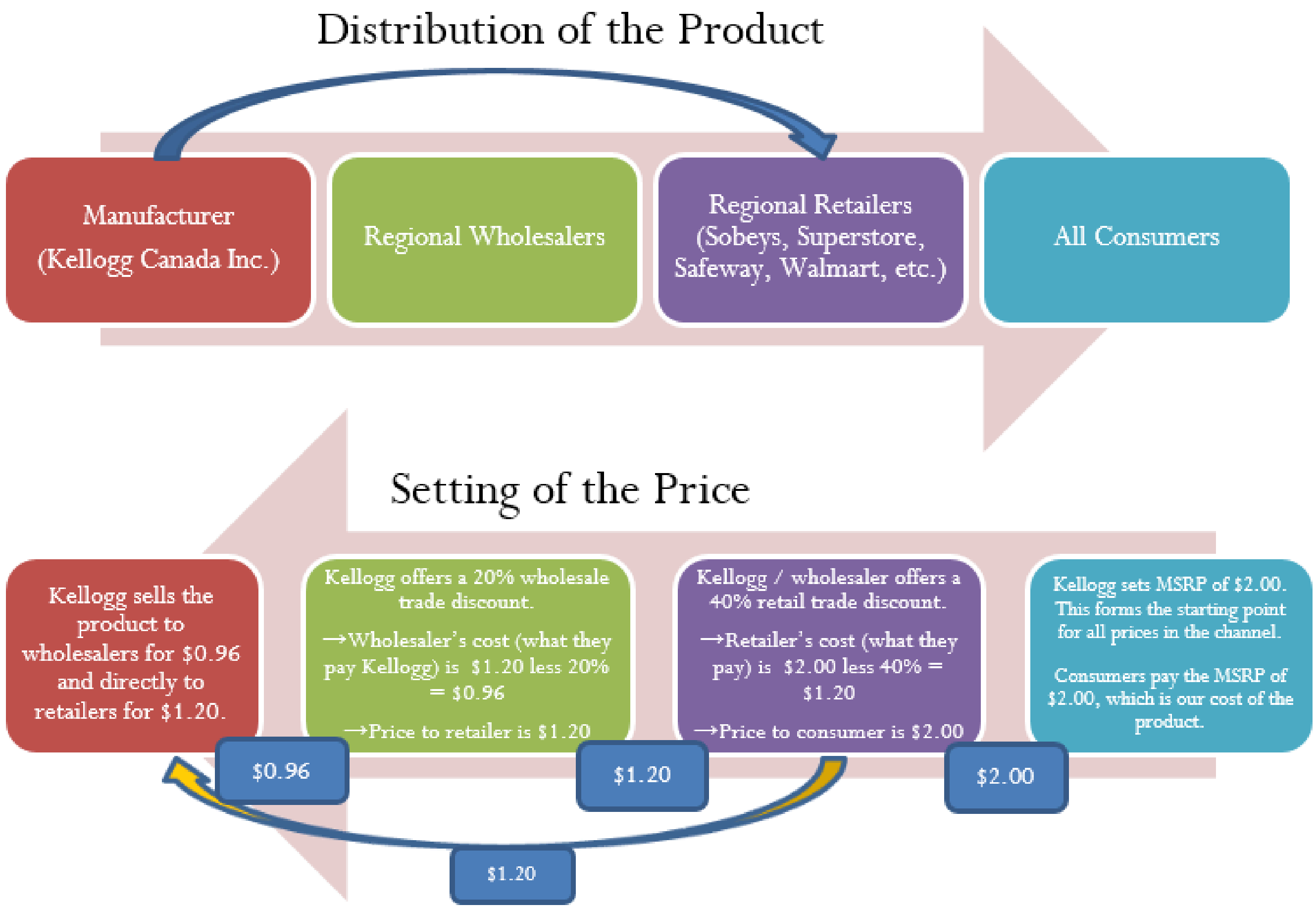


Figure 4.1.1

How Distribution and Pricing Work

Start with distribution in the top half of the figure and work left to right. As an example, let's look at a manufacturer such as Kellogg Canada Inc. (which makes such products as Pop-Tarts, Eggo Waffles, and Rice Krispies). Kellogg's Canadian production plant is located in London, Ontario. To distribute its products to the rest of Canada, Kellogg Canada uses various regional wholesalers. Each wholesaler then resells the product to retailers in its local trade area; however, some retailers (such as the Real Canadian Superstore) are very large, and Kellogg Canada distributes directly to these organizations, bypassing the wholesaler as represented by the blue arrow. Finally, consumers shop at these retailers and acquire Kellogg products.

The relationship of distribution to pricing is illustrated in the bottom half of the figure, working right to left. For now, focus on understanding how pricing works; the mathematics used in the figure will be explained later in this chapter. Kellogg Canada sets a **manufacturer's suggested retail price**, known as the MSRP. This is a recommended retail price based on consumer market research. Since grocery retailers commonly carry thousands or tens of thousands of products, the MSRP helps the retailer to determine the retail price at which the product should be listed. In this case, assume a $\$2.00$ MSRP, which is the price consumers will pay for the product.

The retailer must pay something less than $\$2.00$ to make money when selling the product. Kellogg Canada understands its distributors and calculates that to be profitable most retailers must pay approximately 40% less than the MSRP. Therefore, it offers a 40% discount. If the retailer purchases directly from Kellogg, as illustrated by the yellow arrow, the price paid by the retailer to acquire the product is $\$2.00$ less 40%, or $\$1.20$. Smaller retailers acquire the product from a wholesaler for the same price. Thus, the retailer's cost equals the wholesaler's price (or Kellogg Canada's price if the retailer purchases it directly from Kellogg).

The wholesaler's price is $\$1.20$. Again, Kellogg Canada, knowing that the wholesaler must pay something less than $\$1.20$ to be profitable, offers an additional 20% discount exclusively to the wholesaler. So the price paid by the wholesaler to acquire the product from Kellogg Canada is $\$1.20$ less 20%, or $\$0.96$. This $\$0.96$ forms Kellogg Canada's price to the wholesaler, which equals the wholesaler's cost.

In summary, this discussion illustrates two key pricing concepts:

- Companies higher up in the distribution channel pay lower prices than those farther down the channel. Companies receive discounts off the MSRP based on their level in the distribution system. This may result in multiple discounts, such as a wholesaler receiving both the retailer's discount and an additional discount for being a wholesaler.
- One organization's price becomes the next organization's cost (assuming the typical distribution channel structure):

$$\begin{aligned}\text{Manufacturer's Price} &= \text{Wholesaler's Cost}\\ \text{Wholesaler's Price} &= \text{Retailer's Cost}\\ \text{Retailer's Price} &= \text{Consumer's Cost}\end{aligned}$$

Types of Discounts

You will perform discount calculations more effectively if you understand how and why single pricing discounts and multiple pricing discounts occur. Businesses or consumers are offered numerous types of discounts, of which five of the most common are trade, quantity, loyalty, sale, and seasonal.

- ### Trade Discounts

A **trade discount** is a discount offered to businesses only based on the type of business and its position in the distribution system (e.g., as a retailer, wholesaler, or any other member of the distribution system that resells the product). Consumers are ineligible for trade discounts. In the discussion of the figure, two trade discounts are offered. The first is a 40% retail trade discount, and the second is a 20% wholesale trade discount.

Typically, a business that is higher up in the distribution system receives a combination of these trade discounts. For example, the wholesaler receives both the 40% retail trade discount and the 20% wholesale trade discount from the MSRP. The wholesaler's cost is calculated as an MSRP of $\$2.00$ less 40% less $20\% = \$0.96$.

- **Quantity Discounts**

A **quantity discount** (also called a *volume discount*) is a discount for purchasing larger quantities of a certain product. If you have ever walked down an aisle in a Real Canadian Superstore, you probably noticed many shelf tags that indicate quantity discounts, such as "Buy one product for $\$2$" or "Take two products for $\$3$." Many Shell gas stations offer a Thirst Buster program in which customers who purchase four Thirst Busters within a three-month period get the fifth one free. If the Thirst Busters are $\$2.00$ each, this is equivalent to buying five drinks for $\$10.00$ less a $\$2.00$ quantity discount.

- **Loyalty Discounts**

A **loyalty discount** is a discount that a seller gives to a purchaser for repeat business. Usually no time frame is specified; that is, the offer is continually available. As a consumer, you see this regularly in marketing programs such as Air Miles or with credit cards that offer cash back programs. For example, Co-op gas stations in Manitoba track consumer gasoline purchases through a loyalty program and mails an annual loyalty discount cheque to its customers, recently amounting to $12.5¢$ per liter purchased. In business-to-business circles, sellers typically reward loyal customers by deducting a loyalty discount percentage, commonly ranging from 1% to 5%, from the selling price.

- **Sale Discounts**

A **sale discount** is a temporary lowering of the price from a product's regular selling price. Businesses put items on sale for a variety of reasons, such as selling excess stock or attracting shoppers. You see such promotional events all the time: LED monitors are on sale at Best Buy; Blu-Ray discs are half off at Walmart; The Brick is having a door crasher event Saturday morning.

- **Seasonal Discounts**

A **seasonal discount** is a discount offered to consumers and businesses for purchasing products out of season. At the business level, manufacturers tend to offer seasonal discounts encouraging retailers, wholesalers, or distributors to purchase products before they are in season. Bombardier Inc. manufactures Ski-Doos, which are sold in Canada from approximately November through March—a time of year when most of the country has snow and consumers would want to buy one. To keep production running smoothly from April through October, Bombardier could offer seasonal discounts to its wholesalers and retailers for the coming winter season. At a retail level, the examples are plentiful. On November 1 most retailers place their Halloween merchandise on seasonal discount to clear out excess inventory, and many retailers use Boxing Day (or Boxing Week) to clear their out-of-season merchandise.

Single Discounts

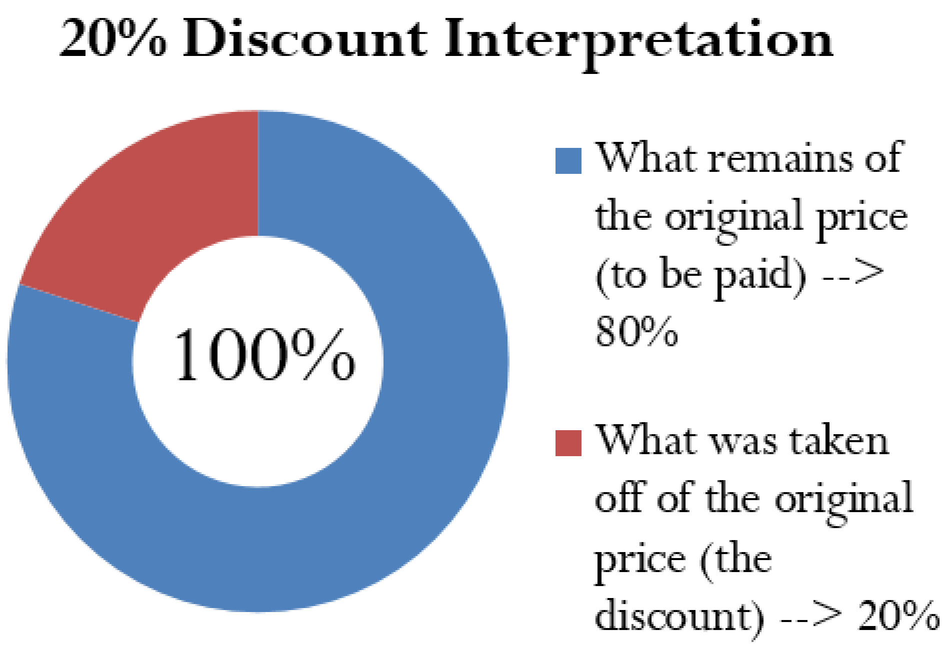


Figure 4.1.2

Let's start by calculating the cost when only one discount is offered. Later in this section you will learn how to calculate a cost involving multiple discounts.

$\boxed{4.1a}$ Single Discount

Figuring out the price after applying a single discount is called a net price calculation. When a business calculates the net price of a product, it is interested in what you still have to pay, not in what has been removed. Note in Formula 4.1a below that you take 1 and subtract the discount rate to determine the rate owing. If you are eligible for a 20% discount, then you must pay 80% of the list price, as illustrated in the figure to the right.

\Large\text{Single Discount:}\;{\color[rgb]{1.0, 0.0, 0.0}N}\;=\;{\color[rgb]{0.0, 0.5, 0.0}L}\;\times\;(1\;-\;{\color[rgb]{0.0, 0.0, 1.0}d})

$rgb]1.0, 0.0, 0.0N$ **is the Net Price:** The net price is the price of the product after the discount is removed from the list price. It is a dollar amount representing what price remains after you have applied the discount.

{\color[rgb]{0.0, 0.5, 0.0}L}\;\text{is List Price:} The list price is the normal or regular dollar price of the product before any discounts. It is the Manufacturer's Suggested Retail Price (MSRP – a price for a product that has been published or advertised in some way), or any dollar amount before you remove the discount.

$rgb]0.0, 0.0, 1.0d$ **is the Discount Rate:** The discount rate represents the percentage (in decimal format) of the list price that is deducted.

Formula 4.1a once again applies the formula on rate, portion, and base, where the list price is the base, the ($1 - d$) is the rate, and the net price represents the portion of the price to be paid.

Notice that Formula 4.1a requires the discount to be in a percentage (decimal) format; sometimes a discount is expressed as a dollar amount, though, such as "Save $\$5$ today." Formula 4.1a and **Formula 4.2b** $L = \frac{N}{(1-d)}$ relate the discount dollar amount to the list price, discount percent, and net price. Choose one formula or the other depending on which variables are known.

4.1b Discount Amount

Discount Amount: $rgb]1.0, 0.0, 0.0Drgb]1.0, 0.0, 0.0\$ = rgb]0.0, 0.5, 0.0L \times rgb]0.0, 0.0, 1.0d$

$rgb]1.0, 0.0, 0.0Drgb]1.0, 0.0, 0.0\$$ **is Discount Amount:** Determine the discount amount in one of two ways, depending on what information is known. If the list price and discount rate are known, apply Formula 4.1b. If the list price and net price are known, apply Formula 4.1c. Either of these formulas can be algebraically manipulated to solve for any other unknown variable as well.

{\color[rgb]{0.0, 0.5, 0.0}L}\;\text{is List Price:} The dollar amount of the price before any discounts.

{\color[rgb]{0.0, 0.0, 1.0}d}\;\text{is Discount Rate:} The percentage (in decimal format) of the list price to be deducted. This time you are interested in figuring out the amount of the discount, therefore you do not take it away from 1.

$\boxed{4.1c}$ Discount Amount

Discount Amount: $rgb]1.0, 0.0, 0.0Drgb]1.0, 0.0, 0.0\$ = rgb]0.0, 0.5, 0.0Lrgb]0.1, 0.1, 0.1-rgb]0.5, 0.0, 0.5N$

$rgb]1.0, 0.0, 0.0Drgb]1.0, 0.0, 0.0\$$ is Discount Amount: Determine the discount amount in one of two ways, depending on what information is known. If the list price and discount rate are known, apply **Formula 4.1b**$D\$ = L \times d$. If the list price and net price are known, apply Formula 4.1c. Either of these formulas can be algebraically manipulated to solve for any other unknown variable as well.

{\color[rgb]{0.0, 0.5, 0.0}L}\;\text{is List Price:} The dollar amount of the price before any discounts.

{\color[rgb]{0.5, 0.0, 0.5}N}\;\text{is Net Price:} The dollar amount of the price after you have deducted all discounts.

HOW TO

Calculate the net price involving a single discount

These steps are adaptable if the net price is a known variable and one of the other variables is unknown.

Step 1: Identify any known variables, including list price, discount rate, or discount amount.

Step 2: If the list price is known, skip this step. Otherwise, solve for list price using an appropriate formula.

Step 3: Calculate the net price.

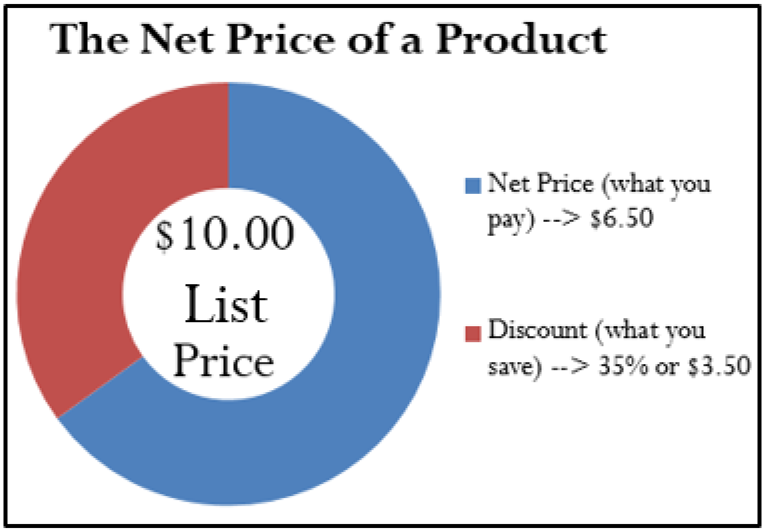


Figure 4.1.3

If the list price and discount are known, apply **Formula 4.1b**$D\$ = L \times d$.

- If the list price and discount amount are known, apply **Formula 4.1c**$D\$ = \text{L} - \text{N}$ and rearrange for N.

Assume a product sells for $\$10$ and is on sale at 35% off the regular price. Calculate the net price for the product.

Step 1: The list price of the product is $L = \$10$. It is on sale with a discount rate of $d = 0.35$.

Step 2: List price is known, so this step is not needed.

Step 3: Applying **Formula 4.1b**$D\$ = L \times d$ results in a new price of $N = \$10 \times (1 - 0.35) = \$6.50$. Note that if you are interested in learning the discount amount, you apply **Formula 4.1c**$D\$ = \text{L} - \text{N}$ to calculate $D\$ = \$10 - \$6.50 = \$3.50$.

Key Takeaways

You can combine **Formula 4.1a** $N = L \times (1 - d)$, **Formula 4.1b** $D\$ = L \times d$ and **Formula 4.1c** $D\$ = \mathrm{L} - \mathrm{N}$ in a variety of ways to solve any single discount situation for any of the three variables. As you deal with increasingly complicated pricing formulas, your algebraic skills in solving linear equations and substitution become very important.

Many of the pricing problems take multiple steps that combine various formulas, so you need to work through the problem systematically. In any pricing problem, you must understand which variables are provided and match them up to the known formulas. To get to your end goal, you must look for formulas in which you know all but one variable. In these cases, solving for variables will move you forward toward solving the overall pricing problem.

If you find you cannot produce a formula with only one unknown variable, can you find two formulas with the same two unknowns? If so, recall from Section 2.5 that you can use your algebraic skills to find the roots of the two equations simultaneously. Alternatively, you can solve one formula for a variable then substitute it into the other formula, allowing you to isolate the remaining variable. Throughout the examples in this chapter you will see many applications of these algebraic skills.

Things To Watch Out For

Remember to apply the rounding rules discussed in Chapter 2:

- Until you arrive at the final solution, avoid rounding any interim numbers unless you have some special reason to do so.

- Round all dollar amounts to the nearest cent. If the dollar amount has no cents, you may write it either without the cents or with the ".00" at the end.
- Round all percentages to four decimals when in percent format.

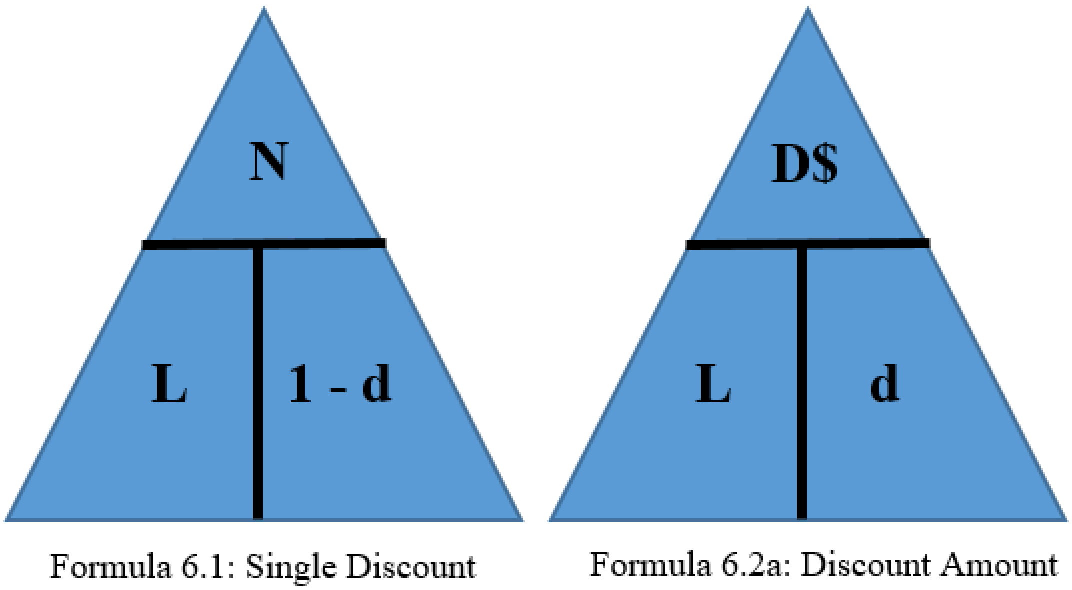


Figure 4.1.4

Paths To Success

When working with single discounts, you are not always solving for the net price. Sometimes you must calculate the discount percent or the list price. At other times you know information about the discount amount but need to solve for list price, net price, or the discount rate. The triangle technique discussed in Section 2 can remind you how to rearrange the formulas for each variable, as illustrated in the figure to the above.

Try It

1) Will you pay more than, less than, or exactly $\$10.00$ for a product if you are told that you are paying:

a. a net price of $\$10.00$ when there is a discount of 25%?
b. a list price of $\$10.00$ when there is a discount of 25%?

Solution
a. Exactly $\$10$. The net price is the price after the discount.
b. Less than $\$10$. The discount needs to be removed from the list price.

Try It

2) If an item is subject to a 40% discount, will the net price be more than or less than half of the list price of the product?

Solution
More than half. A 40% discount means that you will pay 60% of the list price.

Example 4.1.1

A manufacturer that sells jeans directly to its retailers uses market research to find out it needs to offer a 25% trade discount. In doing so, the retailers will then be able to price the product at the MSRP of $\$59.99$. What price should retailers pay for the jeans?

Solution

***Step 1:* List what you already know from the question.**

$$L = \$59.99$$
$$d = 0.25$$

***Step 2:* Apply Formula 4.1a** $N = L \times (1 - d)$ **and solve.**

$$N = \$59.99 \times (1 - 0.25)$$
$$N = \$59.99 \times 0.75$$
$$N = \$44.99$$

***Step 3:* Write as a statement.**

The manufacturer should sell the jeans to the retailers for $\$44.99$.

Example 4.1.2

Winners pays a net price of $\$27.50$ for a winter jacket after receiving a retail trade discount of 45%. What was the MSRP of the jacket?

Solution

***Step 1:* List what you already know from the question.**

$$N = \$27.50$$
$$d = 0.45$$

Step 2: **Apply Formula 4.1a** $N = L \times (1 - d)$**, rearranging for** L **and solve.**

$$\begin{aligned} \$27.50 &= L(1-0.45) \\ L &= \$27.50 \div (1-0.45) \\ L &= \$27.50 \div 0.55 \\ L &= \$50.00 \end{aligned}$$

Step 3: **Write as a statement.**

The MSRP, or list price, of the winter jacket is $\$50.00$.

Example 4.1.3

You are shopping at Mountain Equipment Co-op for a new environmentally friendly water bottle. The price tag reads $\$14.75$, which is $\$10.24$ off the regular price. Determine the discount rate applied.

Solution

Step 1: **List what you already know from the question.**

$$\begin{aligned} D\$ &= \$10.24 \\ N &= \$14.75 \end{aligned}$$

Step 2: **Use Formula 4.1c** $D\$ = \mathrm{L} - \mathrm{N}$ **to calculate the list price, rearranging for** L**.**

$$\begin{aligned} \text{List price: } \$10.24 &= L - \$14.75 \\ L &= \$10.24 + \$14.75 \\ L &= 24.99 \end{aligned}$$

Which results in $L = \$24.99$.

Step 3: **Convert the discount amount into a percentage by applying Formula 4.1b** $D\$ = L \times d$**, rearranging for** d**.**

$$\text{Discount rate: } d = D\$ \div L$$
$$d = \frac{\$10.24}{\$24.99}$$
$$d = 0.409764 \; or \; 40.9764\%$$

Step 4: **Write as a statement.**

The water bottle today has been reduced in price by the amount of $\$10.24$. This represents a sale discount of 40.9674%.

Multiple Discounts

You are driving down the street when you see a large sign at Old Navy that says, "Big sale, take an additional 25% off already reduced prices!" In other words, products on sale (the first discount) are being reduced by an additional 25% (the second discount). Because **Formula 4.1a**$N = L \times (1 - d)$ handles only a single discount, you must use an extended formula in this case.

4.1d Multiple Discounts

Businesses commonly receive more than one discount when they make a purchase. Consider a transaction in which a business receives a 30% trade discount as well as a 10% volume discount. First, you have to understand that this is not a $30\% + 10\% = 40\%$ discount. The second discount is always applied to the net price after the first discount is applied. Therefore, the second discount has a smaller base upon which it is calculated. If there are more than two discounts, you deduct each subsequent discount from continually smaller bases. Formula 4.1d expresses how to calculate the net price when multiple discounts apply.

$$\text{Multiple Discounts:}$$
$$rgb]1.0, 0.0, 0.0N = rgb]0.0, 0.5, 0.0L \times (1 - rgb]0.0, 0.0, 1.0d_{rgb]0.0,0.0,1.01}) \times (1 - rgb]0.0, 0.0, 1.0d_{rgb]0.0,0.0,1.02}) \times \ldots \times (1 - rgb]0.0, 0.0, 1.0d_{rgb]0.0,0.0,1.0n})$$

{\color[rgb]{1.0, 0.0, 0.0}N}\;\text{is Net Price:} The dollar amount of the price after all discounts have been deducted.

{\color[rgb]{0.0, 0.5, 0.0}L}\;\text{is List Price:} The dollar amount of the price before any discounts.

{\color[rgb]{0.0, 0.0, 1.0}d}_{\color[rgb]{0.0, 0.0, 1.0}1}\;\text{is First Discount,}\;{\color[rgb]{0.0, 0.0, 1.0}d}_{\color[rgb]{0.0, 0.0, 1.0}2}\;\text{is Second Discount,}\;{\color[rgb]{0.0, 0.0, 1.0}d}_{\color[rgb]{0.0, 0.0, 1.0}n}\;\text{is nth Discount:}

When there is more than one discount, you must extend beyond **Formula 4.1a**$N = L \times (1 - d)$ by multiplying another discount expression. These discounts are represented by the same d symbol; however, each discount receives a subscript to make its symbol unique. Therefore, the first discount receives the symbol of d_1, the second discount receives the symbol d_2, and so on. Recall that the symbol n represents the number of pieces of data (a count), so you can expand or contract this formula to the exact number of discounts being offered.

It is often difficult to understand exactly how much of a discount is being received when multiple discounts are involved. Often it is convenient to summarize the multiple discount percentages into a single percentage. This makes it easier to calculate the net price and aids in understanding the discount benefit. Simplifying multiple percent discounts into a single percent discount is called finding the **single equivalent discount**. Whether you apply the multiple discounts or just the single equivalent discount, you arrive at the same net price. The conversion of multiple discount percentages into a single equivalent discount percent is illustrated in Formula 4.1e.

4.1e Single Equivalent Discount

$$rgb]0.1, 0.1, 0.1\text{Single Equivalent Formula:}$$

$$rgb]1.0, 0.0, 0.0\, rgb]1.0, 0.0, 0.0d_{rgb]1.0,0.0,0.0equiv} = 1 - (1 - rgb]0.0, 0.0, 1.0d_{rgb]0.0,0.0,1.01}) \times (1 - rgb]0.0, 0.0, 1.0d_{rgb]0.0,0.0,1.02}) \times \ldots \times (1 - rgb]0.0, 0.0, 1.0d_{rgb]0.0,0.0,1.0n})$$

{\color[rgb]{1.0, 0.0, 0.0}d}_{\color[rgb]{1.0, 0.0, 0.0}{equiv}}\;\text{(or just d) is the single equivalent discount rate}

that is equal to the series of multiple discounts. Recall that taking $(1 - d)$ calculates what you pay. Therefore, if you take 1, which represents the entire amount, and reduce it by what you pay, the rate left over must be what you did not pay. In other words, it is the discount rate.

{\color[rgb]{0.0, 0.0, 1.0}d}_{\color[rgb]{0.0, 0.0, 1.0}1}\;\text{is First Discount,}\;{\color[rgb]{0.0, 0.0, 1.0}d}_{\color[rgb]{0.0, 0.0, 1.0}2}\;\text{is Second Discount,}\;{\color[rgb]{0.0, 0.0, 1.0}d}_{\color[rgb]{0.0, 0.0, 1.0}n}\;\text{is nth Discount:}

This is the same notation as in **Formula 4.1d**$N = L \times (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$. Since there are multiple discounts, each discount receives a numerical subscript to give it a unique identifier. You can expand or contract the formula to the exact number of discounts being offered.

HOW TO

Calculate Net Price involving Single Discount

Refer back to the steps in calculating net price. The procedure for calculating a net price involving a single discount extends to a more generic procedure involving multiple discounts. As with the single discount procedures, you can adapt the model if the net price is known and one of the other variables is unknown. Follow these steps to calculate the net price involving any number of discounts:

Step 1: Identify any known variables, including list price, discount rate(s), or discount amount.

Step 2: If the list price is known, skip this step. Otherwise, solve for list price.

- If only one discount is involved, apply **Formula 4.1a**$N = L \times (1 - d)$.
- If more than one discount is involved, the discount amount represents the total discount amount received from all of the discounts combined. This requires you first to convert the multiple discount rates into an equivalent single discount rate using **Formula 4.1e**$d_{\text{equiv}} = 1 - (1 - d_1) \times (1 - d_2) \times \ldots (1 - d_n)$ and then to apply **Formula 4.1a**$N = L \times (1 - d)$.

Step 3: Calculate the net price.

If the list price and only a single known discount rate are involved, apply **Formula 4.1a** $N = L \times (1 - d)$.

- If the list price and multiple discount rates are known and involved, apply **Formula 4.1e**

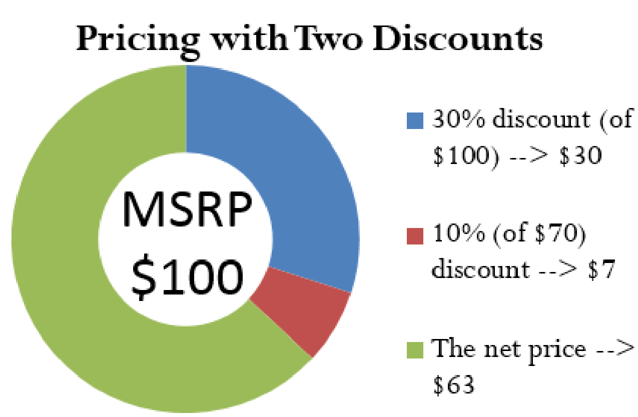


Figure 4.1.5

$$d_{\text{equiv}} = 1 - (1 - d_1) \times (1 - d_2) \times \ldots (1 - d_n).$$

- If the list price and the total discount amount are known, apply **Formula 4.1c**$D\$ = \mathrm{L} - \mathrm{N}$ and rearrange for N.

Assume a product with an MSRP of $\$100$ receives a trade discount of 30% and a volume discount of 10%. Calculate the net price.

Step 1: The list price and discounts are $L = \$100$, $d_1 = 0.30$ and $d_2 = 0.10$.

Step 2: List price is known, so skip this step.

Step 3: Apply **Formula 4.1d**$N = L \times (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$ to calculate the net price: $N = \$100 \times 1 - 0.30 \times 1 - 0.10 = \$63$

The net price is $\$63$, which is illustrated to the right.

If you are solely interested in converting multiple discounts into a single equivalent discount, you need only substitute into Formula 4.1e. In the above example, the product received a trade discount of 30% and a volume discount of 10%. To calculate the single equivalent discount, apply **Formula 4.1e** $d_{\text{equiv}} = 1 - (1 - d_1) \times (1 - d_2) \times \ldots (1 - d_n)$:

\begin{eqnarray*}d_{equiv}&=&1\;-\;(1\;–\;0.30)\;\times\;(1\;–\;0.10\
operatorname{) }\\d_{equiv}&=&1\;-\;(0.70)(0.90\
operatorname{) }\\d_{equiv}&=&1\;–\;0.63\;\\d_{equiv}&=&0.37\end{eqnarray*}

Therefore, whenever discounts of 30% and 10% are offered together, the single equivalent discount is 37%. Whether it is the multiple discounts or just the single equivalent discount that you apply to the list price, the net price calculated is always the same.

Key Takeaways

Order of Discounts: The order of the discounts *does not* matter in determining the net price. Remember from the rules of BEDMAS that you can complete multiplication in any order. Therefore, in the above example you could have arrived at the $\$63$ net price through the following calculation:

$$\$100 \times (1 - 0.10) \times (1 - 0.30) = \$63$$

The order of the discounts *does* matter if trying to interpret the value of any single discount. If the trade discount is applied before the quantity discount and you are wanting to know the quantity discount amount, then the quantity discount needs to be second. Thus, the amount of quantity discount is:

$$\$100 \times (1-0.30) = \$70$$
$$\$70 \times 0.10 = \$7$$

Price Does Not Affect Single Equivalent Discount: Notice in **Formula 4.1e** $d_{\text{equiv}} = 1-(1-d_1) \times (1-d_2) \times \ldots (1-d_n)$ that the list price and the net price are not involved in the calculation of the single equivalent discount. When working with percentages, whether you have a net price of $\$6.30$ and a list price of $\$10$, or a net price of $\$63$ and a list price of $\$100$, the equivalent percentage always remains constant at 37%.

Things To Watch Out For

A common mistake when working with multiple discounts is to add the discounts together to calculate the single equivalent discount. This mistaken single discount is then substituted into **Formula 4.1a**$N = L \times (1-d)$ to arrive at the wrong net price. Remember that if two discounts of 30% and 10% apply, you cannot sum these discounts. The second discount of 10% is applied on a smaller price tag, not the original price tag. To calculate the net price you must apply **Formula 4.1d**$N = L \times (1-d_1) \times (1-d_2) \times \ldots \times (1-d_n)$.

Paths To Success

If you happen to know any two of the net price (N), list price (L), or the total discount amount ($D\$$), then you could also use **Formula 4.1c**$D\$ = \text{L} - \text{N}$ to solve for the single equivalent discount, d_{equiv}. For example, if you know the net price is $\$63$ and the total discount amount for all discounts is $\$37$, you could use **Formula 4.1c**$D\$ = \text{L} - \text{N}$ to figure out that the list price is $\$100$, then convert the discount amount into a percentage using **Formula 4.1b** $D\$ = L \times d$. This method will also produce a single equivalent discount of 37%.

Another method of calculating the single equivalent discount is to recognize **Formula 4.1b** $D\$ = L \times d$ as an application involving percent change. The variable d is a discount rate, which you interpret as a negative percent change. The discount amount, $D\$$, is the difference between the list price (representing the Old price) and the net price (representing the New price after the discount). Therefore, **Formula 4.1b** $D\$ = L \times d$ can be rewritten as follows:

$$\begin{aligned} D\$ &= L \times d \\ New - Old &= Old \times \%C \end{aligned}$$

$$\frac{New - Old}{Old} = \%C$$

Therefore, any question about a single equivalent discount where net price and list price are known can be solved as a percent change. Using our ongoing net price example, you have:

$$\frac{\$63 - \$100}{\$100} = -0.37 \text{ or } -37\%\text{; this is a discount of } 37\%.$$

Try It

3) If you are offered discounts in the amount of 25%, 15%, 10%, and 5%, will your total discount percent be 55%, less than 55%, or more than 55%?

Solution

Less than 55%. Each percent discount is calculated from a smaller base.

Example 4.1.4

A retail dealership purchases some Expedition TUV Yeti II Ski-Doos to stock in its stores. Examining the merchandising terms of the manufacturer, Bombardier, the dealership notices that it would be eligible to receive a 35% trade discount, 15% volume discount, and 3% loyalty discount. Because it is June and Ski-Doos are out of season, Bombardier offers a seasonal discount of 12% for purchases made before June 30. If the MSRP for the Ski-Doo is $\$12,399.00$ and the dealership purchases this item on June 15, what price would it pay?

Solution

Step 1: **The retail dealership is eligible for all four discounts (it qualifies for the seasonal discount since it is purchasing before June 30). Therefore, we write our unknowns:**

$$\begin{aligned} L &= \$12,399.00 \\ d_1 &= 0.35 \\ d_2 &= 0.15 \\ d_3 &= 0.03 \\ d_4 &= 0.12 \end{aligned}$$

Step 2: **Apply Formula 4.1d**$N = L \times (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$ **and solve.**

$$\begin{aligned} N &= \$12,399 \times (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12) \\ N &= \$12,399 \times 0.65 \times 0.85 \times 0.97 \times 0.88 \\ N &= \$5,847.54 \end{aligned}$$

Step 3: **Write as a statement.**

After all four discounts, the retail dealership could purchase the SkiDoo for $\$5,847.54$.

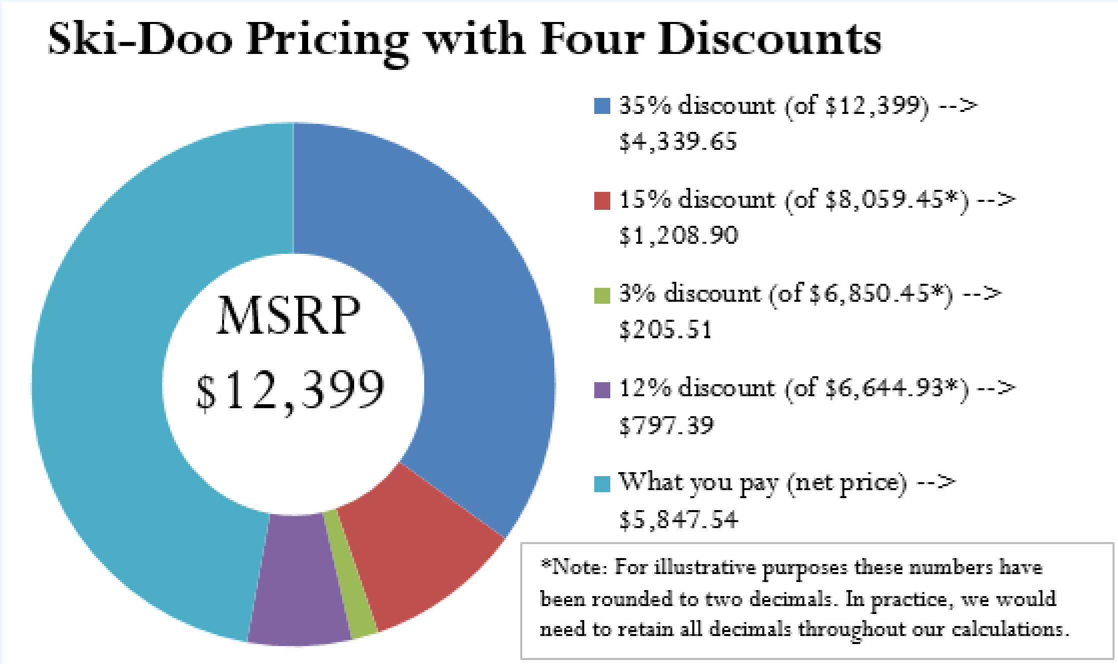


Figure 4.1.6

Example 4.1.5

The retail dealership in Example 4.1.4 purchases more products subject to the same discounts. It needs to simplify its calculations. Using the information from Example 4.1.4, what single equivalent discount is equal to the four specified discounts?

Solution

Step 1: **Write what you know from the question.**

$$d_1 = 0.35$$
$$d_2 = 0.15$$
$$d_3 = 0.03$$
$$d_4 = 0.12$$

Step 2: **Apply Formula 4.1e** $d_{\text{equiv}} = 1 - (1 - d_1) \times (1 - d_2) \times \ldots (1 - d_n)$ **and solve.**

$$\begin{aligned} d_{equiv} &= 1 - (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12) \\ d_{equiv} &= 1 - (0.65)(0.85)(0.97)(0.88) \\ d_{equiv} &= 1 - 0.471614 \\ d_{equiv} &= 0.528386 \text{ or } 52.8386\% \end{aligned}$$

Step 3: **Write as a statement.**

The retail dealership can apply a 52.8386% discount to all the products it purchases.

Example 4.1.6

You are shopping on Boxing Day for an 80" HDTV. You have just one credit card in your wallet, a cashback Visa card, which allows for a 1% cash rebate on all purchases. While scanning flyers for the best deal, you notice that Visions is selling the TV for $\$5,599.99$ including taxes, while Best Buy is selling it for $\$5,571.99$ including taxes. However, because of a computer glitch Best Buy is unable to accept Visa today. Where should you buy your television?

Solution

Step 1: **You know the list price for each of the stores. You also know the discount available from Visa. Thus, we state the known values:**

$$\begin{aligned} L_{Best\ Buy} &= \$5,571.99 \text{ with no discounts since Visa cannot be used there} \\ L_{Visions} &= \$5,599.99 \\ d &= 0.01 \text{ since you can use your Visa card there} \end{aligned}$$

Step 2: **Apply Formula 4.1a** $N = L \times (1 - d)$ **and solve.**

Best Buy: No discounts apply, so the list price equals the net price and $N = \$5,571.99$.

$$\begin{aligned}\text{Visions: } N &= \$5,599.99 \times (1 – 0.01\,) \\ N &= \$5,599.99 \times 0.99 \\ N &= \$5,543.99\end{aligned}$$

***Step 3:* Write as a statement.**

The net price for Visions is $\$5,543.99$. You save $\$5,571.99 – \$5,543.99 = \$28.00$ by purchasing your TV at Visions.

Example 4.1.7

An advertisement claims that at 60% off, you are saving $\$18$. However, today there is an additional 20% off. What price should you pay for this item? What percent savings does this represent?

Solution

***Step 1:* You know the discount amount for the first discount only, as well as the two discount rates:**

$$\begin{aligned}D\$_1 &= \$18 \\ d_1 &= 0.60 \\ d_2 &= 0.20\end{aligned}$$

***Step 2:* Calculate the list price by applying Formula 4.1b $D\$ = L \times d$ and rearranging for L.**

$$\begin{aligned}\$18.00 &= L \times 0.60 \\ L &= \$18.00 \div 0.6 \\ L &= \$30.00\end{aligned}$$

***Step 3:* To calculate the net price, apply Formula 4.1d**
$N = L \times (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$

$$N = \$30 \times (1 - 0.60) \times (1 - 0.20)$$
$$N = \$30 \times 0.40 \times 0.80$$
$$N = \$9.60$$

Step 4: **To calculate the single equivalent discount, apply Formula 4.1e**

$d_{\text{equiv}} = 1 - (1 - d_1) \times (1 - d_2) \times \ldots(1 - d_n)$

$$d_{equiv} = 1 - (1 - 0.60) \times (1 - 0.20)$$
$$d_{equiv} = 1 - (0.40)(0.80)$$
$$d_{equiv} = 1 - 0.32$$
$$d_{equiv} = 0.68 \text{ or } 68\%$$

Step 5: **Write as a statement.**

You should pay $\$9.60$ for the item, which represents a 68% savings.

Section 4.1 Exercises

Round all money to two decimals and percentages to four decimals in each of the following questions.

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided. “N/A” indicates that the particular variable is not applicable in the question.

	List Price or MSRP	First Discount	Second Discount	Third Discount	Net Price	Equivalent Single Discount Rate	Total Discount Amount
1.	$\$980.00$	42%	N/A	N/A	?	N/A	?
2.	?	25%	N/A	N/A	$\$600.00$	N/A	?
3.	$\$1,975.00$	25%	15%	10%	?	?	?
4.	?	18%	4%	7%	$\$366.05$	?	?

Solutions

1. $D\$ = \$411.60$; $N = \$568.40$
2. $L = \$800$; $D\$ = \$200$
3. $d = 42.625\%$; $N = \$1,133.16$; $D\$ = \$841.84$
4. $d = 26.7904\%$; $L = \$500$; $D\$ = \$133.95$

Applications

5. A wholesaler of stereos normally qualifies for a 35% trade discount on all electronic products purchased from its manufacturer. If the MSRP of a stereo is $\$399.95$, what net price will the wholesaler pay?
6. Mary is shopping at the mall where she sees a sign that reads, "Everything in the store is 30% off, including sale items!" She wanders in and finds a blouse on the clearance rack. A sign on the clearance rack states, "All clearance items are 50% off." If the blouse is normally priced at $\$69.49$, what price should Mary pay for it?
7. A distributor sells some shoes directly to a retailer. The retailer pays $\$16.31$ for a pair of shoes that has a list price of $\$23.98$. What trade discount percent is the distributor offering to its retailers?
8. A retailer purchases supplies for its head office. If the retailer pays $\$16.99$ for a box of paper and was eligible for a 15% volume discount, what was the original MSRP for the box of paper?

9. Mountain Equipment Co-op has purchased a college backpack for $\$29$ after discounts of 30%, 8%, and 13%. What is the MSRP for the backpack? What single discount is equivalent to the three discounts?
10. Walmart purchased the latest CD recorded by Selena Gomez. It received a total discount of $\$10.08$ off the MSRP for the CD, which represents a discount percent of 42%.
    a. What was the MSRP?
    b. What was the net price paid for the CD?
11. Best Buy just acquired an HP Pavilion computer for its electronics department. The net price on the computer is $\$260.40$ and Best Buy receives discounts of 40% and 38%.
    a. What single discount is equivalent to the two discounts?
    b. What is the list price?
    c. What is the total discount amount?
12. TELUS retails a Samsung cellphone at the MSRP of $\$399.99$. TELUS can purchase the phone from its supplier and receive a 20% trade discount along with a 5% volume discount.
    a. What is the single equivalent percent discount?
    b. What net price does TELUS pay for the phone?
    c. How much of a discount in dollars does this represent?
13. A wholesaler offers the following discounts: 10% seasonal discount for all purchases made between March 1 and May 1, 15% cumulative quantity discount whenever more than $5,000$ units are purchased in any month, 5% loyalty discount for customers who have made regular purchases every month for at least one year, and a 33% trade discount to any retailer. Ed's Retail Superstore makes a purchase of 200 watches, MSRP $\$10$, from the wholesaler on April 29. This month alone, Ed's has ordered more than $5,000$ watches. However, Ed's has purchased from the wholesaler for only the past six months. Determine the total price that Ed's should pay for the watches.
14. If a distributor is eligible for a 60% trade discount, 5% volume discount, and 3% seasonal discount, what single equivalent discount rate would it be eligible to receive? If the trade discount is applied first and equals a trade discount of $\$48$, calculate the net price for the item.

Solutions

5. $\$259.97$
6. $\$24.32$
7. 31.985%
8. $\$19.99$
9. $d = 43.972\%$; $L = \$51.76$
10. a. $L = \$24$; b. $N = \$13.92$
11. a. $d = 62.8\%$; b. $L = \$700$; c. $D\$ = \$439.60$
12. a. $d = 24\%$; b. $N = \$303.99$; c. $D\$ = \$96$
13. $\$1,026$
14. $d = 63.14\%$; $N = \$29.49$

Challenge, Critical Thinking, & Other Applications

As mentioned in one of the "Paths to Success" sections, discount percentages share a commonality with negative percent changes (Section 3.1). Use the formulas from this chapter to solve questions 15-17 involving percent change.

15. A human resource manager needs to trim labor costs in the following year by 3%. If current year labor costs are $\$1,231,498$, what are the labor costs next year?
16. At an accounting firm, the number of accountants employed is based on the ratio of $1 : 400$ daily manual journal entries. Because of ongoing increases in automation, the number of manual journal entries declines at a constant rate of 4% per year. If current entries are $4,000$ per year, how many years and days will it take until the firm needs to lay off one accountant? (Hint: An accountant is laid off when the number of journal entries drops below $3,600$.)
17. An economist is attempting to understand how Canada reduced its national debt from 1999 to 2008. In 1999, Canada's national debt was $\$554.143$ billion. In 2008, the national debt stood at $\$457.637$ billion. What percentage had the national debt been reduced by during this time period?
18. Sk8 is examining an invoice. The list price of a skateboard is $\$109.00$, and the invoice states it received a trade discount of 15% and quantity discount of 10% as well as a loyalty discount. However, the amount of the loyalty discount is unspecified.
    a. If Sk8 paid $\$80.88$ for the skateboard, what is the loyalty discount percent?
    b. If the loyalty discount is applied after all other discounts, what amount of loyalty dollars does Sk8 save per skateboard?

19. Currently, a student can qualify for up to six different tuition discounts at a local college based on such factors as financial need or corporate sponsorships. Mary Watson just applied to the college and qualifies for all six discounts: 20%, 15%, 23%, 5%, 3%, and 1%.
    a. She is confused and wants the college to tell her what single discount percent she is receiving. What should the college tell her?
    b. If her total list tuition comes to $\$6,435.00$, how much should she pay?
20. Sumandeep is very loyal to her local hairstylist. Because she is loyal, her hairstylist gives her three different discounts: 10%, 5%, and 5%. These discounts amount to $\$14.08$ in savings.
    a. What was the list price her hairstylist charged her?
    b. What amount did she pay her hairstylist?
    c. If her hairstylist increases prices by 5%, what are the list price, net price, and total discount amount?

Solutions

15. $\$1,194.553.06$
16. 2 years and 216 days
17. Reduced by 17.4154%
18. a. 3%; b. $\$2.50$
19. a. 52.2328%; b. $\$3,073.82$;
20. a. $\$74.99$; b. $\$60.91$; c. $L = \$78.74$; $N = \$63.96$; $D\$ = \$14.78$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $L = \dfrac{N}{(1-d)}$ $N = L \times (1-d)$
$D\$ = \text{L} - \text{N}$ $D\$ = L \times d$ $N = L \times (1-d_1) \times (1-d_2) \times \ldots \times (1-d_n)$
$d_{\text{equiv}} = 1 - (1-d_1) \times (1-d_2) \times \ldots (1-d_n)$

Attribution

4.2: INVOICING: TERMS OF PAYMENT AND CASH DISCOUNTS

Formula & Symbols Hub

For this section you will need the following:

Symbol Used

- d = Discount rate or Cash discount rate
- L = List price or Invoice amount
- N = Net price or Net payment amount

Formulas Used

- Formula 4.2a – **Single Discount Rearranged**

$$L = \frac{N}{(1-d)}$$

- Formula 4.2b – **Single Discount**

$$N = L \times (1-d)$$

Introduction

What a way to start your Monday morning! With dread you face a great stack of envelopes in your in-basket. The first envelope contains an invoice from your major supplier indicating an outstanding balance of $\$3,600$ with terms of $2/10$, $1/20$, net 30. In the second envelope, an office supplies bill totals $\$500$ with terms of $3/15$, net 45 EOM. The third envelope holds yet another invoice from your transportation company indicating an outstanding balance of $\$21,000$ with terms of $2/15, 1/25$, net 60 ROG. It also indicates that you have an overdue balance of $\$4,000$ subject to a 3% monthly penalty. Before opening any more envelopes that probably contain still more bills, you pour yourself another cup of coffee and settle down to figuring out how much you need to pay and when.

In the world of business, most purchases are not paid for in cash. Instead, businesses tend to work through an invoicing system in which they send out bills to their clients on a regular basis. In accounting, this means that the purchase is placed into accounts receivables until such time as a cheque arrives and the purchase is converted to cash. Invoices provide detailed transaction information, listing the amount owed and also indicating the terms on which payment is expected. Companies may offer what are known as cash discounts as incentives for early invoice payment. The rationale for these discounts is simple—a sale is not a sale until you have the cash in hand. The longer an invoice remains in accounts receivable, the less likely that it will be paid. Thus, it might turn into bad debt, which the creditor cannot collect at all.

Invoicing is less common in consumer purchases because of the sheer volume of transactions involved and the higher risk of nonpayment. Imagine purchasing items at Walmart and receiving an invoice to pay your bill next month instead of paying cash. It is hard to fathom the number of invoices Walmart would have to distribute monthly. How much would it cost to collect those debts? How many of those invoices would go unpaid? That is why consumer purchases typically do not involve invoicing.

Invoicing does commonly occur at a consumer level on credit card transactions along with many services where the business may not be able to assess the exact amount of the bill at the time of the transaction or until the service is delivered. Two examples illustrate this point:

1. Think of your MasterCard bill. You are able to make purchases, say, from March 9 to April 8. Then on April 9 the company sends out a statement saying you have until April 29 to pay your bill. If you do not, interest and late payment penalties are involved.
2. You have a dental visit for a regular cleaning. Before charging you, the dentist's office needs to determine how much your insurance will cover. It may not find out the answer for a day or two, so it sends you an invoice at a later date once it hears from the insurance company. The invoice terms indicate that payment is due upon receipt and you will incur late penalties if payment is not forthcoming.

This section explores the most common aspects of invoicing, including terms of payments and cash discounts. You will learn to calculate the amount required to pay invoices and how to reduce the outstanding balance of an invoice if a partial payment is received. Additionally, the calculation of late payments and penalties is introduced.

Invoice Terms and Invoice Dating

You must know how to read a business invoice. The figure on the next page (Kerry Mitchell) provides a sample invoice and highlights the following areas:

- **Invoicing Company:** The invoice must identify who is sending and issuing the invoice.
- **Invoice Date:** The date on which the invoice was printed, along with the invoice tracking number (in this case, the order date and shipping date are equivalent to the invoice date). When an invoice is paid, the cheque must reference the invoice number so that the invoicing company can identify which invoice to credit the payment to.
- **Transaction Details:** The details of the transaction might include number of units, unit prices, and any discounts for which the item is eligible.
- **Invoice Total:** The total amount owing is indicated, including any taxes or additional charges.
- **Terms of Payment:** The terms of payment include any cash discounts and due dates. The date of commencement (as discussed below) is determined from this part of the invoice.
- **Late Penalty:** A penalty, if any, for late payments is indicated on the invoice. Whether or not a company enforces these late penalties is up to the invoice-issuing organization.

Three Dates of Commencement

All invoice terms are affected by what is known as the **date of commencement**, which is the first day from which all due dates stated on the invoice will stem. The date of commencement is determined in one of three ways as illustrated below.

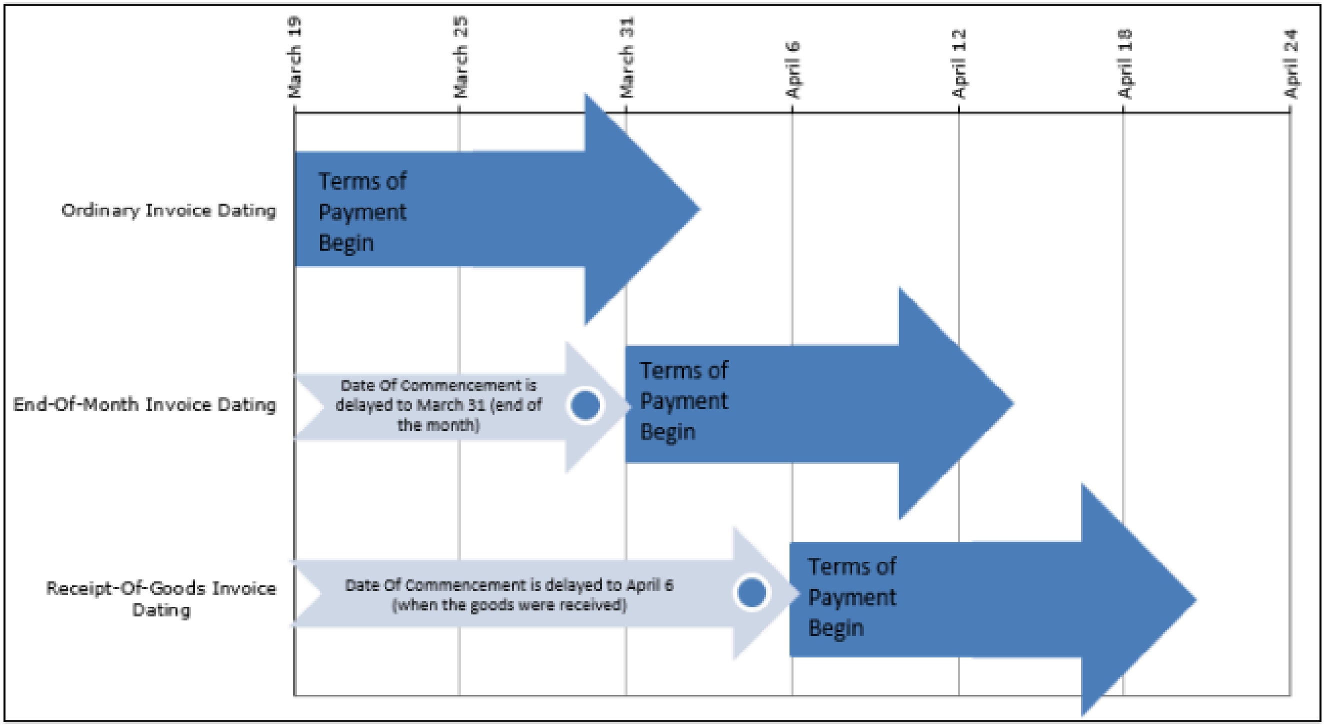


Figure 4.2.1

1. ## Ordinary Invoice Dating:

In **ordinary invoice dating**, or just invoice dating for short, the date of commencement is the same date as the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on March 19. This is the default manner in which most companies issue their invoices, so if an invoice does not specify any other date of commencement you can safely assume it is using ordinary dating.

2. ## End-of-Month Invoice Dating:

End-of-month invoice dating applies when the terms of payment include the wording "end-of-month" or the abbreviation "EOM" appears after the terms of payment. In end-of-month dating, the date of commencement is the last day of the same month as indicated by the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on the last day of March, or March 31. Many companies use this method of dating to simplify and standardize all of their due dates, in that if the date of commencement is the same for all invoices, then any terms of payment will also share the same dates.

3. ## Receipt-Of-Goods Invoice Dating:

Receipt-of-goods invoice dating applies when the terms of payment include the wording "receipt-of-goods" or the abbreviation "ROG" appears after the terms of payment. In receipt-of-goods dating, the date of commencement is the day on which the customer physically receives the goods. Therefore, if the invoice is printed on March 19 but the goods are not physically received until April 6, then all terms of payment commence on April 6. Companies with long shipping times involved in product distribution or long lead times in production commonly use this method of dating.

Terms of Payment

The most common format for the terms of paying an invoice is shown below and illustrated by an example.

$$rgb]0.0, 0.0, 1.03/rgb]1.0, 0.0, 0.010,\ rgb]0.0, 0.5, 0.0\text{net}rgb]0.0, 0.5, 0.0\ rgb]0.0, 0.5, 0.030$$

{\color[rgb]{0.0, 0.0, 1.0}3}\;\text{is Cash Discount:} A cash discount is the percentage of the balance owing on an invoice that can be deducted for payment received either in full or in part during the discount period (see below). In this case, 3% is deducted from the total invoice amount.

$rgb]1.0, 0.0, 0.010$ is Discount Period: The discount period is the number of days from the date of commencement in which the customer is eligible to take advantage of the cash discount. For this example, there is a period of 10 days during which a payment received in full or any portion thereof is eligible for the 3% cash discount.

{\color[rgb]{0.0, 0.5, 0.0}\text{net}}{\color[rgb]{0.0, 0.5, 0.0}\;}{\color[rgb]{0.0, 0.5, 0.0}30}\;\text{is Credit Period:}

The credit period is the number of interest-free days from the date of commencement that the customer has to pay the invoice in full before the creditor applies any penalties to the invoice.

—

The figure below plots the invoice term of "$3/10$, net 30" along a time diagram to illustrate the terms of payment. This term is combined with various dates of commencement using the examples from the "Date of Commencement" section.

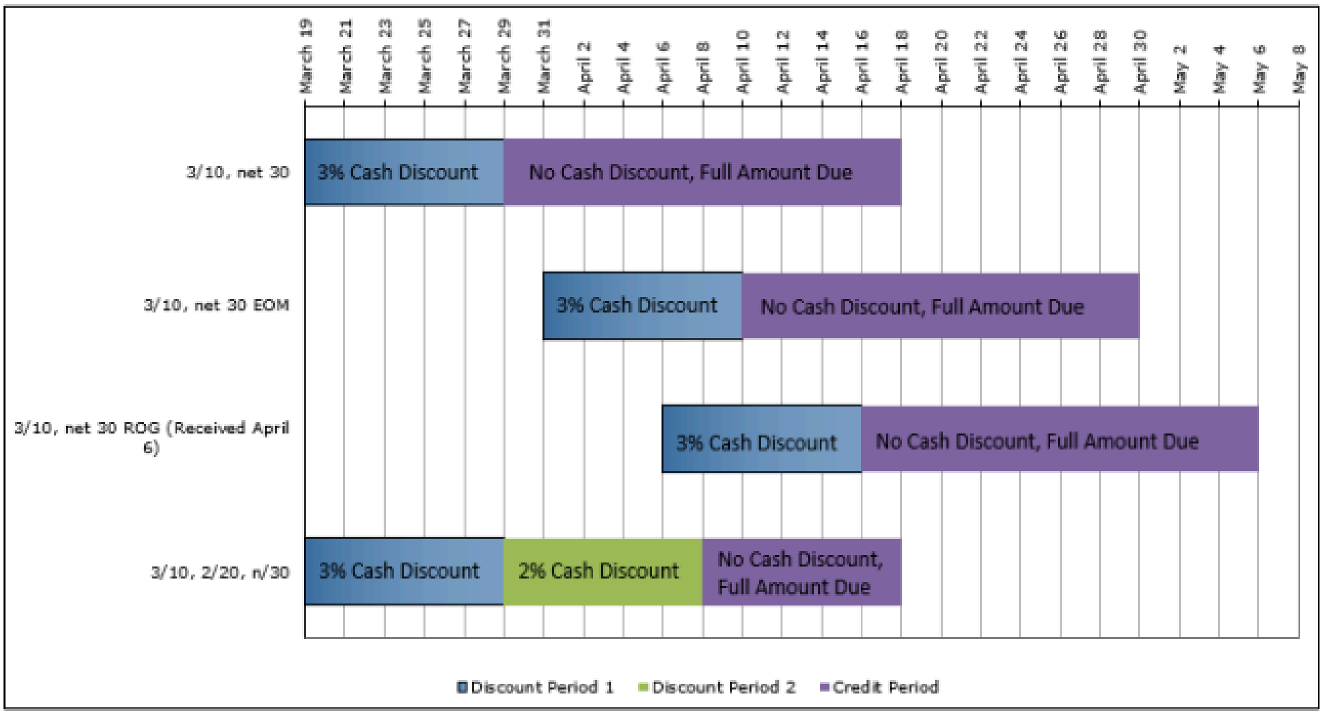


Figure 4.2.2

Note the following observations:

- In all four scenarios, the total length of all bars is the same, since the credit period is 30 days.
- Although all four invoices have the same invoice date of March 19, the date of commencement is modified in the ROG and EOM scenarios, which shift the discount and credit periods into the future. In these cases, any payment before the end of the discount period qualifies for the discount.
  - For example, in the EOM scenario a company could pay its bill early on March 27 and qualify for the 3% cash discount.

You may see invoice terms displayed in several other common formats:

1. $3/10, n/30$: In this case, the word "net" has been abbreviated to "$n/$". This is illustrated in the fourth scenario in the figure.
2. $3/10, 2/20, n/30$. In this case, multiple cash discounts are being offered, meaning 3% within 10 days from the date of commencement and 2% from the 11th to the 20th day from the date of commencement. This is also illustrated in the fourth scenario of the figure.
3. $n/30$. In this case, no cash discount is offered and only the credit period is identified.

Rules for Invoice Terms

Some common business practices are implemented across most industries. Note that you must always check with the invoicing organization to ensure it is applying these practices.

- **No Net Figure:** If no net figure is stated in the terms of payment, you should assume that the credit period is 20 days after the last discount period. If there are no discount periods, then the credit period is 20 days from the date of commencement.

  For example, "$3/10, 2/15$" means that the credit period ends 20 days after the second discount period of 15 days. Hence, the credit period is 35 days after the date of commencement.

- **No Cash Discount:** Not every invoice receives a cash discount. If no terms indicate a cash discount, then the invoicing company seeks full payment only.

  For example, "$n/30$" means that no cash discount applies and the credit period ends 30 days from the date of commencement.

- **Nonbusiness Days:** Most businesses operate Monday through Friday and are closed on weekends and holidays. As such, any date falling on a nonworking day is moved to the next business day.

  For example, if an invoice is ordinary dated December 21 and lists a cash discount of $2/10$, the end of the discount period falls on January 1. Since this is New Year's Day, the discount period extends to January 2. In this textbook, you should apply this practice to any of the five known statutory holidays as discussed previously. (New Year's Day, Good Friday or Easter Monday in Quebec only, Canada Day, Labor Day, and Christmas Day).

Three Types of Payments

When businesses pay invoices, three situations can occur:

1. **Full Payment:** A full payment means that the company wants to pay its invoice in full and reduce its balance owing to zero dollars. This is the most common practice.
2. **Partial Payment:** A partial payment means that the company wants to lower its balance owing but will not reduce that balance to zero. A company will generally employ this method when it wants to either lower its accounts payable or demonstrate good faith in paying its invoices. Partial payment may also occur if the company wants to take advantage of a cash discount but lacks the funds to clear the invoice

in its entirety.

3. **Late Payment:** A late payment occurs when the company pays its invoice either in full or partially after the credit period has elapsed. Late payments occur for a variety of reasons, but the most common are either insufficient funds to pay the invoice or a simple administrative oversight.

The rest of this section shows you how to handle each of these types of payments mathematically, and it also explores the implications of cash discounts.

Try It

1) In each of the following cases, determine which term of payment results in the longest credit period extending from the invoice date.

a. $2/10$, $n/30$ or $3/15$
b. $3/15$, $n/45$ or $2/10$, $1/20$
c. $2/10$, $n/60$ or $3/10$, $n/30$ EOM on an invoice dated January 7
d. $3/15$, $n/45$ or $2/10$, $n/35$ ROG, *or* $2/20$ EOM on an invoice dated September 22 and goods received on September 29

Solution

a. 30 days or 35 days
b. 45 days or 40 days
c. 60 days or 54 days (24 days left in January plus 30 more)
d. 45 days or 42 days (7 days until received plus 35) or 48 days (8 left in month plus 40 more)

Full Payments

In the opening scenario in this section, the first invoice on your desk was for $\$3,600$ with terms of $2/10$, $1/20$, net 30. Suppose that invoice was dated March 19. If you wanted to take advantage of the 2% cash discount, when is the last day that payment could be received, and in what amount would you need to write the cheque? In this section, we will look at how to clear an invoice in its entirety.

4.2a Single Discount

The good news is that a cash discount is just another type of discount similar to what you encountered in Section 4.1, and you do the calculations using the exact same formula. Apply Formula 4.1a on single discounts, which is reprinted below.

Single Discount: $rgb]1.0, 0.0, 0.0N rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1 = rgb]0.0, 0.5, 0.0\ rgb]0.0, 0.5, 0.0L rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1 \times rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1 (rgb]0.1, 0.1, 0.1 1 rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1 - rgb]0.1, 0.1, 0.1\ rgb]0.0, 0.0, 1.0 d rgb]0.1, 0.1, 0.1)$

$rgb]1.0, 0.0, 0.0N$ **is the Net Price:** The net price is the price of the product after the discount is removed from the list price. It is a dollar amount representing what price remains after you have applied the discount.

{\color[rgb]{0.0, 0.5, 0.0}L}\;\text{is List Price:} The list price is the normal or regular dollar price of the product before any discounts. It is the Manufacturer's Suggested Retail Price (MSRP – a price for a product that has been published or advertised in some way), or any dollar amount before you remove the discount.

$rgb]0.0, 0.0, 1.0d$ **is the Discount Rate:** The discount rate represents the percentage (in decimal format) of the list price that is deducted.

HOW TO

Work with payments and invoices

Step 1: Look for and identify key information such as the invoice date, date of commencement modifiers such as **EOM** or **ROG**, terms of payment, late penalty, and the invoice amount.

Step 2: Draw a timeline similar to the figure on the next page. On this timeline, chronologically arrange from left to right the invoice date and amount, the date of commencement, the end of any discount (if any) or credit periods, cash discounts (if any) that are being offered, and penalties (if any).

Step 3: Determine when payments are being made and in what amount. Locate them on the timeline.

Step 4: Apply the correct calculations for the payment, depending on whether the payment is a full payment, partial payment, or late payment.

Let's continue with the example of the first invoice on your desk. You mail in a cheque to be received by March 29. What amount is the cheque?

Step 1: The invoice amount is $L = \$3,600$, invoice date is March 19, and terms of payment are $2/10$, $1/20$, net 30.

Step 2: The figure on the next page displays the invoice timeline.

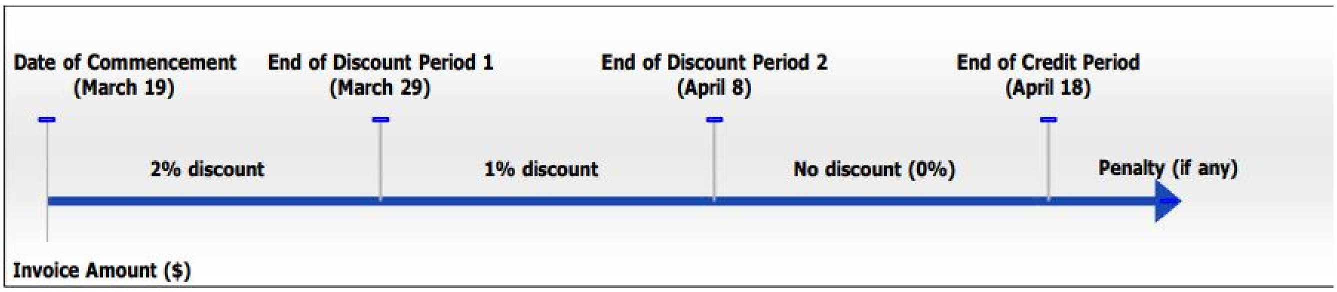


Figure 4.2.3

Step 3: Note on the timeline that a payment on March 29 is the last day of the 2% discount period.

Step 4: According to **Formula 4.2a**$N = L \times (1 - d)$, the amount to pay is:

$$N = \$3,600 \times (1 - 0.02)$$
$$N = \$3,600 \times 0.098$$
$$N = \$3,528$$

A cheque for $\$3,528$ pays your invoice in full. By taking advantage of the cash discount, you reduce your payment by $\$72$.

Things To Watch Out For

If there is one area of invoicing that causes the most confusion, it is assigning information to the net price and invoice amount variables. These are commonly assigned backwards. Remember these rules so that you always get the correct answer:

1. The invoice amount or any reference to the balance owing on an invoice is *always* a list price.

List Price = Invoice Amount

2. The payment of an invoice, whether in full, partial, or late, is *always* a net price.

Net Price = Payment Amount

Paths To Success

Who cares about 1% or 2% discounts when paying bills? While these percentages may not sound like a lot, remember that these discounts occur over a very short time frame. For example, assume you just received an invoice for $\$102.04$ with terms of $2/10, n/30$. When taking advantage of the 2% cash discount, you must pay the bill 20 days early, resulting in a $\$100$ payment. That is a $\$2.04$ savings over the course of 20 days. To understand the significance of that discount, imagine that you had $\$100$ sitting in a savings account at your bank. Your savings account must have a balance of $\$102.04$ twenty days later. This requires your savings account to earn an annual interest rate of 44.56%! Therefore, a 20-day 2% discount is the same thing as earning interest at 44.56%. Outrageously good!

Try It

2) In each of the following cases, determine the cash discount or penalty for which the payment qualifies.

Table 4.2.1

	Invoice Date	Terms of Payment	Payment Received on
a.	April 14	$2/10, n/30$	April 24
b.	July 7	$3/10, 2/20, n/30$ EOM, 1% per month penalty	August 12
c.	November 12 (goods received November 28)	$2/20$ ROG, 2% per month penalty	December 29
d.	February 27 (non–leap year)	$4/10, 2/15, 1/25$ EOM	March 25

Solution

a. 2% discount
b. 2% discount
c. No discount (0%) or penalty
d. 1% discount

Example 4.2.1

You receive an invoice for $\$35,545.50$ dated August 14. The terms of payment are listed as $3/10$, $2/20$, net 45 EOM. The accounting department is considering paying this debt on one of three days. Determine the full payment required if payment is received by the invoicing company on each of the following dates:

a. September 3
b. September 19
c. September 30

Solution
Step 1: **Write the known values.**

The invoice amount, terms of payment, and any payments are known:

$$\begin{aligned} L &= \$35,545.50 \\ \text{Invoice Date} &= \text{ August 14} \\ \text{Terms of Payment } &= \text{ 3/10, 2/20, net 45 EOM} \\ \text{Three payment date options are } &= \text{ Sept. 3, Sept. 19 and Sept. 30} \end{aligned}$$

***Step 2:* Look at the figure illustrated to see the timeline for the invoice and identification of payments.**

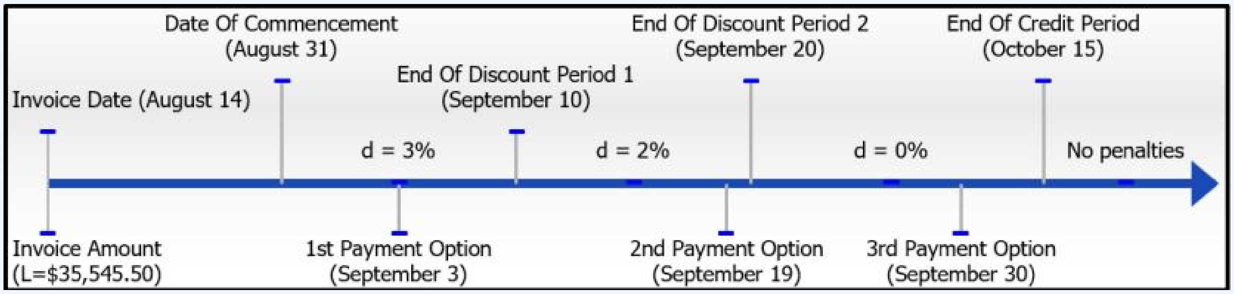


Figure 4.2.4

***Step 3:* Calculate the payment required by applying Formula 4.2a$N = L \times (1 - d)$. As noted from the timeline:**

a. The first payment option qualifies for $d = 3\%$
b. The second payment option qualifies for $d = 2\%$
c. The third payment option qualifies for $d = 0\%$ (no cash discount)

***Step 4:* Substitute values into Formula 4.2a$N = L \times (1 - d)$ and solve.**

a. $$\begin{aligned} N &= \$35,545.50 \times (1 - 0.03) \\ N &= \$35,545.50 \times 0.97 \\ N &= \$34,479.14 \end{aligned}$$

---

b. $$\begin{aligned} N &= \$35,545.50 \times (1 - 0.02) \\ N &= \$35,545.50 \times 0.98 \\ N &= \$34,834.59 \end{aligned}$$

---

c. $N = \$35,545.50 \times (1 - 0)$
$N = \$35,545.50 \times 1$
$N = \$35,545.50$ *No discount, so N=L

Step 5: **Write as a statement.**

If the invoicing company receives payment on September 3, a 3% discount is allowed and $\$34,479.14$ clears the invoice. If the payment is received on September 19, a 2% discount is allowed and $\$34,834.59$ pays it off. Finally, if the payment arrives on September 30, there is no discount so the full invoice amount of $\$35,545.50$ is due.

Partial Payments

In the section opener, your third invoice is for $\$21,000$ with terms of $2/15$, $1/25$, net 60 ROG. After you paid the $\$3,528$ to clear the first invoice, you realize that your company has insufficient funds to take full advantage of the 2% cash discount being offered by the transportation company. Not wanting to lose out entirely, you decide to submit a partial payment of $\$10,000$ before the first discount period elapses. What is the balance remaining on the invoice?

Unless you pay attention to invoicing concepts, it is easy to get confused. Perhaps you think that $\$10,000$ should be removed from the $\$21,000$ invoice total, thereby leaving a balance owing of $\$11,000$. Or maybe you think the payment should receive the discount of 2%, which would be $\$200$. Are you credited with $\$9,800$ off of your balance? Or maybe $\$10,200$ off of your balance? In all of these scenarios, you would be committing a serious mistake and miscalculating your balance owing. Let's look at the correct way of handling this payment.

4.2b Single Discount Rearranged

Recall that invoice amounts are gross amounts (amounts before discounts), or G, and that payment amounts are net amounts (amounts after discounts), or N. Also recall that an algebraic equation requires all terms to be expressed in the same unit.

In the case of invoice payments, you can think of the balance owing as being in the unit of "pre-discount" and any payment being in the unit of "post-discount." Therefore, a payment cannot be directly deducted from the invoice balance since it is in the wrong unit. You must convert the payment from a "post-discount" amount into a "pre-discount" amount using a rearranged version of **Formula 4.2a** $N = L \times (1 - d)$. You can then deduct it from the invoice total to calculate any balance remaining.

$$rgb]0.1, 0.1, 0.1\text{Single Discount Rearranged:}rgb]1.0, 0.0, 0.0 \; rgb]1.0, 0.0, 0.0L = \frac{rgb]0.0, 0.0, 1.0N}{(1 - rgb]0.0, 0.5, 0.0d)}$$

$rgb]1.0, 0.0, 0.0L$ **is Invoice Amount:** This is the amount deducted from the invoice balance as a result of the payment. It is what the payment is worth once the cash discount has been factored into the calculation.

$rgb]0.0, 0.0, 1.0N$ **is Net Payment Amount:** This is the actual payment amount made that must be converted into its value toward the invoice total. Since the payment represents an amount after the discount has been removed, you need to find out what it is worth before the cash discount was removed.

$rgb]0.0, 0.5, 0.0d$ **is Cash Discount Rate:** Any payment received within any discount period, whether in full or partial, is eligible for the specified cash discount indicated in the terms of payment.

HOW TO

Work with Partial Payments

Follow the same invoice payment steps even when working with partial payments. Commonly, once you calculate the gross amount of the payment in Step 4 you will also have to calculate the new invoice balance by deducting the gross payment amount.

Let's continue working with the $\$21,000$ invoice with terms of $2/15$, $1/25$, net 60 ROG. Assume the invoice is dated March 19 and the goods are received on April 6. If a $\$10,000$ payment is made on April 21, what balance remains on the invoice?

Step 1: The invoice amount is $\$21,000$, the invoice date is March 19, goods are received on April 6, and the terms of payment are $2/15$, $1/25$, net 60. A payment of $\$10,000$ is made on April 21.

Step 2: The figure below shows the invoice timeline.

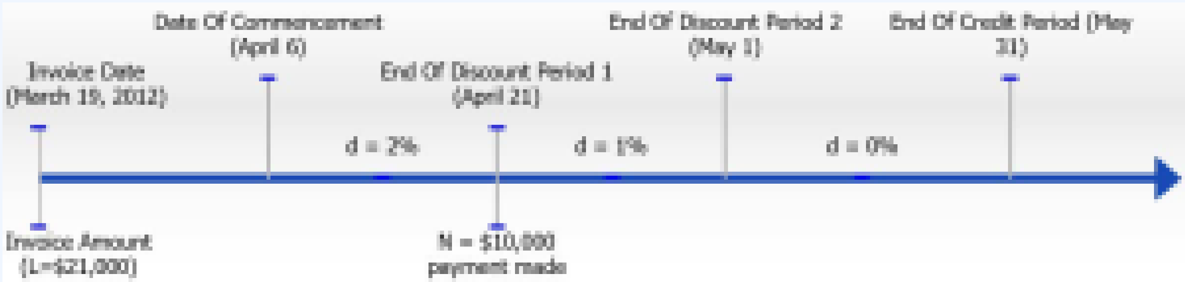


Figure 4.2.5

Step 3: The payment of $\$10,000$ on April 21 falls at the end of the first discount period and qualifies for a 2% discount.

Step 4: To credit the invoice, apply **Formula 4.2b** $L = \dfrac{N}{(1-d)}$:

$$L = \frac{\$10,000}{100\% - 2\%}$$

$$L = \frac{\$10,000}{98\%}$$

$$L = \frac{\$10,000}{0.98}$$

$$L = \$10,204.08$$

This means that before any cash discounts, the payment is worth $\$10,204.08$ toward the invoice total. The balance remaining is $\$21,000.00 - \$10,204.08 = \$10,795.92$.

Key Takeaway

In the case where a payment falls within a cash discount period, the amount credited toward an invoice total is always larger than the actual payment amount. If the payment does not fall within any discount period, then the amount credited toward an invoice total is equal to the actual payment amount (since there is no cash discount).

To help you understand why a partial payment works in this manner, assume an invoice is received in the amount of $103.09 and the customer pays the invoice in full during a 3% cash discount period. What amount is paid? The answer is $N = \$103.09(100\% - 3\%) = \$100$. Therefore, any payment of $\$100$ made during a 3% cash discount period is always equivalent to a pre-discount invoice credit of $\$103.09$. If the balance owing is more than $\$103.09$, that does not change the fact that the $\$100$ payment during the discount period is worth $\$103.09$ toward the invoice balance.

Try It

3) In each of the following situations, determine for the partial payment whether you would credit the invoice for an amount that is larger than, equal to, or less than the partial payment amount.

a. An invoice dated April 7 with terms $4/20$, $2/20$, $n/60$ EOM. The goods are received on April 9 and a partial payment is made on May 21.
b. An invoice dated July 26 with terms $3/30$, $n/45$ ROG. The goods are received on August 2

and a partial payment is made on September 3.

c. An invoice dated January 3 with terms $2\frac{1}{2}/10$, $1/20$. The goods are received on January 10 and a partial payment is made on January 24.

Solution

a. Since the payment falls within a discount period, the credited amount is larger than the payment.
b. Since the payment does not fall within a discount period, the credited amount is equal to the payment.
c. Since the payment falls within a discount period, the credited amount is larger than the payment.

Example 4.2.2

Heri just received an invoice from his supplier, R&B Foods. The invoice totaling $\$68,435.27$ is dated June 5 with terms of $2\frac{1}{2}/10$, $1/25$, $n/45$. Heri sent two partial payments in the amounts of $\$20,000$ and $\$30,000$ that R&B Foods received on June 15 and June 29, respectively. He wants to clear his invoice by making a final payment to be received by R&B Foods on July 18. What is the amount of the final payment?

Solution
Step 1: **Write what you know from the question.**

$$\begin{aligned} L &= \$68,435.27 \\ \text{Invoice Date} &= \text{June 5} \\ \text{Terms of Payment} &= \frac{2\frac{1}{2}}{10}, \frac{1}{25, \frac{n}{45}} \\ N_1 &= \$20,000 \text{ on June 15} \\ N_2 &= \$30,000 \text{ on June 29} \\ N_3 &= \text{? on July 18} \end{aligned}$$

Step 2: **Look at the figure illustrated to see the timeline for the invoice and identification of payments.**

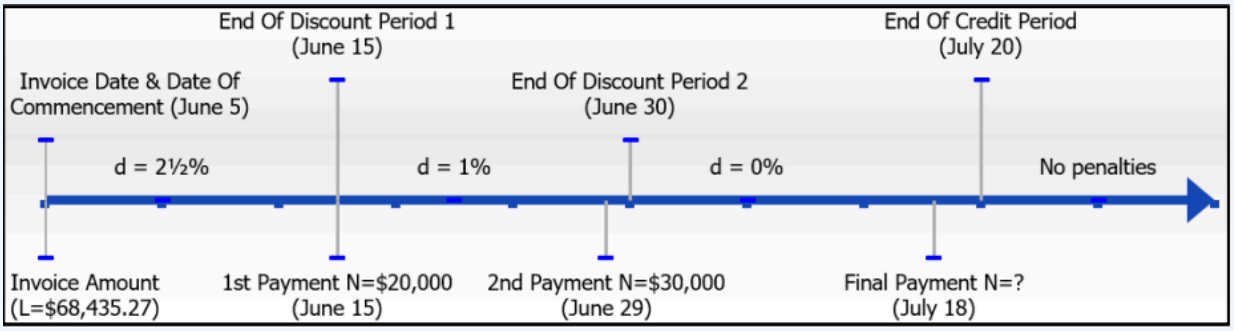


Figure 4.2.6

Step 3: **The first two payments are partial, reference the timeline.**

As noted on the timeline:

a. The first partial payment qualifies for $d = 2\frac{1}{2}\%$
b. The second partial payment qualifies for $d = 1\%$
c. The last payment is a full payment, where $d = 0\%$ (no cash discount)

Step 4: **Apply Formula 4.2b** $L = \dfrac{N}{(1-d)}$

a.

$$L = \frac{\$20,000}{(1-0.025)}$$

$$L = \frac{\$20,000}{0.975}$$

$$L = \$20,512.82$$

$$\text{Balance Owing} = \$68,435.27 - \$20,512.82$$
$$\text{Balance Owing} = \$47,922.45$$

---

b.

$$L = \frac{\$30,000}{(1-0.01)}$$

$$L = \frac{\$30,000}{0.99}$$

$$L = \$30,303.03$$

$$\text{Balance Owing} = \$47,922.45 \ - \$30,303.03$$
$$\text{Balance Owing} = \$17,619.42$$

---

c. Since there is no cash discount, $N_3 \ = \ L$. The balance owing is $L = \$17,619.42$.

Therefore, the payment is the same number and $N_3 = \$17,619.42$.

Step 5: **Write as a statement.**

The first two payments receive a credit of $\$20,512.82$ and $\$30,303.03$ toward the invoice total, respectively. This leaves a balance on the invoice of $\$17,619.42$. Since the final payment is made during the credit period, but not within any cash discount period, the final payment is the exact amount of the balance owing and equals $\$17,619.42$.

Late Payment Penalties

Among all of the invoices that you were paying in the section opener, you realize upon opening that third envelope that you somehow missed an invoice last month and failed to pay the $\$4,000$ owing, which is now overdue. In looking at the invoice, you notice that it states on the bottom that late payments are subject to a 3% per month penalty. You urgently want to pay this invoice to maintain good relations with your supplier but wonder in what amount you must issue the cheque?

Why Do Late Penalties Exist?

Suppliers are doing their business customers a favor when they use invoicing to seek payment. In essence, suppliers provide the products to the customer but are not paid for those products up front. This means customers are granted a period during which they "borrow" the products for free without any associated interest costs that are usually tied to borrowing. If the invoice is not paid in full by the time the credit period elapses, then the supplier starts treating any remaining balance like a loan and charges interest, which is called a late payment penalty.

The Formula

When a payment arrives after the credit period has expired, you need to adjust the balance of the invoice upward to account for the penalty. Continue to use **Formula 4.2a**$N = L \times (1 - d)$ on Single Discounts for this calculation. In this unique case, the discount rate represents a penalty. Instead of deducting money from the balance, you add a penalty to the balance. As a result, the late penalty is a *negative* discount.

For example, if the penalty is 3% then $d = -3\%$.

HOW TO

Work with Late Payments

Follow the same steps to solve the invoice payment once again when working with late payments. As an example, work with the $\$4,000$ overdue invoice that is subject to a 3% per month late penalty. If the invoice is paid within the first month of being overdue, what payment is made?

Step 1: The invoice balance is $L = \$4,000$, and the penalty percent is $d = -0.03$.

Steps 2 & 3: Since the focus is strictly on the late payment calculation, you do not need a timeline for the discount or credit periods. The late payment occurs within the first month past the expiry of the credit period.

Step 4: To determine the new invoice balance, apply **Formula 4.2a**$N = L \times (1 - d)$ to calculate:

$$N = \$4,000 \times (1 - (-0.03))$$
$$N = \$4,000 \times 1.03$$
$$N = \$4,120$$

Therefore, to clear the invoice you must issue a cheque in the amount of $\$4,120$. This covers the $\$4,000$ owing along with a late penalty of $\$120$.

Key Takeaways

Not all invoices have penalties, nor are they always enforced by the supplier. There are many reasons for this:

- Most businesses pay their invoices in a timely manner, which minimizes the need to institute penalties.
- Waiving a penalty maintains good customer relations and reinforces a positive, cooperative business partnership.
- Strict application of late penalties results in difficult situations. For example, if a payment was dropped in the mail on August 13 from your best customer and it arrives one day after the credit period expires on August 15, do you penalize that customer? Is it worth the hassle or the risk of upsetting the customer?

- The application of late penalties generally involves smaller sums of money that incur many administrative costs. The financial gains from applying penalties may be completely wiped out by the administrative expenses incurred.
- Businesses have other means at their disposal for dealing with delinquent accounts. For example, if a customer regularly pays its invoices in an untimely manner, the supplier may just to decide to withdraw the privilege of invoicing and have the customer always pay up front instead.

In this textbook, all penalties are strictly and rigidly applied. If a payment is late, the invoice balance has the appropriate late penalty applied.

Things To Watch Out For

When a late penalty is involved, you must resist the temptation to adjust the **payment** by the penalty percentage. The discussion and calculations in this section focus on adjusting the outstanding invoice **balance** to figure out the payment required. For example, assume there is a $\$500$ outstanding balance subject to a 2% penalty.

- This means that with the penalty, the invoice total is $\$510$, which determines that a payment of $\$510$ is required.
- If the customer makes a $\$200$ partial payment on this late debt, you cannot apply the partial payment procedure, which would give a credit of $\$204.08$ for the payment, resulting in a balance owing of $\$295.92$. If you make this mistake, you in fact reward the customer for being tardy!
- Nor can you deduct the 2% penalty from the payment and give credit for $\$200(1-0.02)=\$196$ (resulting in a balance owing of $\$304$) because the 2% penalty applies to the whole invoice amount and not just the partial payment.
- Instead, the $\$200$ payment must be deducted directly from the penalty-adjusted balance of $\$510$. In this case, the customer still owes $\$310$ to clear the invoice.

Paths To Success

An alternative method for applying a late penalty is to treat the penalty like a positive percent change (see Section 3.1). In this case, the penalty is how much you want to increase the balance owing. Thus, if an invoice for $\$500$ is subject to a 2% penalty, the $\%C = 2\%$ and $Old = \$500$. The goal is to look for the new balance (V_{f}) in **Formula 3.2b** $RoC = \left(\left(\frac{V_{\mathrm{f}}}{V_{\mathrm{i}}}\right)^{\frac{1}{n}}\right) \times 100 : V_{\mathrm{f}} = \$510$.

Example 4.2.3

You received an invoice dated December 17 in the amount of $\$53,455.55$ with terms of $4/15$, $2/30$, $n/60$ ROG, 2.75% penalty per month for late payments. The merchandise is received on January 24. You sent in a partial payment of $\$40,000$ on January 31 with intention to pay the remaining balance before the credit period expired. However, you forgot about the invoice and realized your mistake on March 30, when you submit payment in full for the invoice. What amount is the final payment? (Note: Assume this is not a leap year).

Solution

Step 1: **Write the known information.**

$$\begin{aligned} L &= \$53,455.55 \\ \text{Invoice Date} &= \text{December } 17 \\ \text{Goods Recieved} &= \text{January } 24 \\ \text{Terms of Payment} &= \frac{4}{15, \frac{2}{30}}, \frac{n}{60} \text{ ROG}, \frac{2.75\%}{\text{month}} \text{ late payment penalty} \\ N_1 &= \$40,000 \text{ on January } 31 \\ N_2 &= ? \text{ on March } 30 \end{aligned}$$

Step 2: **Look at the figure illustrated to see the timeline for the invoice and identification of payments.**

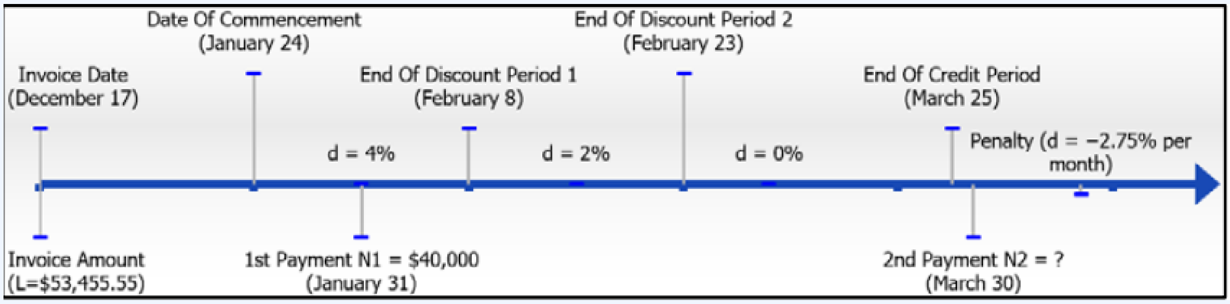


Figure 4.2.7

Step 3: **Credit the partial payment during the discount period** $(d = 4\%)$ **by applying the rearranged Formula 4.2a** $N = L \times (1 - d)$.

$$\begin{aligned}(\text{Partial Payment})\ L &= \frac{\$40,000}{(1-0.04)}\\ L &= \frac{\$40,000}{0.96}\\ L &= \$41,666.67\end{aligned}$$

Step 4: **Deduct the partial payment from the invoice total.**

$$\begin{aligned}(\text{New Invoice Balance})\ \text{Balancing Owing} &= \$53,455.55 - \$41,666.67\\ \text{Balance Owing} &= \$11,788.88\end{aligned}$$

Step 5: **The second payment occurs after the credit period elapses. Therefore the outstanding invoice balance is penalized by** $d = -0.0275$ **as per the policy. Apply Formula 4.2a** $N = L \times (1 - d)$

This calculates the outstanding balance including the penalty, which equals the final payment.

$$\begin{aligned}(\text{Late Payment}):\ \ N &= \$11,788.88 \times (1 - (-0.0275))\\ N &= \$11,788.88 \times 1.0275\\ N &= \$12,113.07\end{aligned}$$

Step 6: **Write as a statement.**

The first payment resulted in a $\$41,666.67$ deduction from the invoice. The overdue remaining balance of $\$11,788.88$ has a penalty of $\$324.19$ added to it, resulting in a final clearing payment of $\$12,113.07$.

Section 4.2 Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises. In all questions, assume that the dates indicated are the dates upon which the payment is received.

Mechanics

For questions 1–3, calculate the full payment required on the payment date that reduces the balance on the invoice to zero. Assume this is not a leap year.

Table 4.2.2

	Invoice Amount	Invoice Date	Invoice Terms	Receipt of Goods Date	Date of Full Payment
1.	$\$136,294.57$	January 14	$2/10, n/30$	January 10	January 22
2.	$\$98,482.75$	September 28	$3/10, 2/20, 1/30, n/50$ EOM	October 3	October 19
3.	$\$48,190.38$	February 21	$4/15, 3/40, n/60$ ROG	February 27	April 3

For questions 4–6, calculate the remaining invoice balance after crediting the invoice for the partial payments indicated.

Table 4.2.3

	Invoice Amount	Invoice Date	Invoice Terms	Receipt of Goods Date	Partial Payments and Dates
4.	$\$57,775.00$	June 16	$3/15$, $1\frac{1}{2}/25$	June 12	$\$20,000$ on July 2; $\$15,000$ on July 10
5.	$\$1,200,310.75$	August 25	$2\frac{1}{2}/10$, $1/20$, $n/30$ ROG	September 2	$\$500,000$ on September 4; \$250,000 on September 14
6.	$\$17,481.68$	December 3	$4/10$, $1/30$, $n/60$ EOM	December 20	$\$10,000$ on January 2

For questions 7–8, calculate the final payment required on the payment date that reduces the balance of the invoice to zero.

Table 4.2.4

	Invoice Amount	Invoice Date	Invoice Terms	Late Payment Penalty	Receipt of Goods Date	Partial Payments and Dates	Final Payment Date
7.	$\$23,694.50$	April 13	$3/15$, $2/25$, $1/40$, $n/60$	$1\frac{1}{2}\%$ per month	April 18	$\$10,000$ on April 30	June 15
8.	$\$332,053.85$	October 4	$4/10$ EOM	2% per month	October 30	$\$100,000$ on November 5	December 15

Solutions

1. \$133, 568.68[/latex] < /li >< li > [latex]\$96, 513.10[/latex] < /li >< li > [latex]\$46, 744.67[/latex] < /li >< li > [latex] Balance Remaining = \$21, 928.01[/latex] < /li >< li > [latex] Balance Remaining = \$434, 964.99[/latex] < /li >< li > [latex] Balance Remaining = \$7, 065.01[/latex] < /li >< li > [latex]\$13, 692.78[/latex] < /li >< li > [latex]\$232, 444.92[/latex] < /li >< /ol >< /details >< h4 >< strong > Applications < /strong >< /h4 >< olstart =" 9 ">< li > Hudson′sBayreceivedaninvoicedatedJuly13fromNygardInternationalintheamountof[latex]\$206, 731.75
   . The terms are $2/10$, $1/20$, $n/30$. If Nygard receives full payment on August 1, what amount is paid?
2. Time Bomb Traders Inc. in Burnaby just received an invoice dated February 17 (of a leap year) from UrbanEars headphones in the amount of $\$36,448.50$ with terms of $1½/15$, $½/30$, $n/45$ ROG. The items on the invoice are received on March 3. What amount is

the full payment if UrbanEars receives it on March 18?

3. Family Foods received an invoice dated July 2 from Kraft Canada in the amount of $\$13,002.96$ with terms of $2/15$, $1/30$, $n/45$ EOM and a late penalty of 2% per month. What amount is paid in full if Kraft Canada receives a cheque from Family Foods on August 17?
4. An invoice dated November 6 in the amount of $\$38,993.65$ with terms of $2½/10$, $1/15$ ROG, 2% penalty per month is received by Cargill Limited from Agricore United. The wheat shipment is received on December 2. Cargill Limited made a partial payment of $\$15,000$ on December 10. What amount should Cargill pay to clear its invoice on December 13?
5. Yamaha Music received two invoices from the same vendor. The first is for $\$1,260$ dated March 9 with terms $3/10$, net 30. The second is for $\$2,450$ dated March 12 with terms $2/10$, $1/20$, net 30. If a payment of $\$1,000$ is made on March 19, what payment amount on March 31 settles both invoices? (Note: Payments are applied to the earlier invoice first.)
6. An invoice is dated July 26 for $\$5,345.50$ with terms of $3¼/10$, net 45, $2½\%$ per month penalty. If the invoice is paid on September 10, what payment amount is required to pay the entire balance owing?
7. Mohawk College received an invoice dated August 20 from Office Depot for office supplies totalling $\$10,235.97$ with terms of $3¾/15$, $1½/30$, $n/45$, EOM, $2¾\%$ per month penalty. The college made three payments of $\$2,000$ dated September 4, September 29, and October 10. What payment on October 25 settles the invoice?

Solutions

9. $\$204,664.43$
10. $\$35,901.77$
11. $\$12,872.93$
12. $\$23,372.94$
13. $\$2,654.57$
14. $\$5,479.14$
15. $\$4,241.10$

Challenge, Critical Thinking, & Other Applications

16. An invoice for $\$100,000$ dated February 2 (of a non–leap year) with terms of $4/10$, $3/20$, $2/30$, $1/40$, net 60, ROG, $1\frac{3}{4}\%$ per month penalty. The merchandise is received on February 16. If four equal payments are made on February 20, March 17, April 1, and April 20 resulting in full payment of the invoice, calculate the amount of each payment.
17. An accounting department receives the following invoices from the same vendor and makes the indicated payments. The vendor always applies payments to the earliest invoices first and its late payment policy stands at $1\frac{1}{2}\%$ per month for any outstanding balance.

Table 4.2.5

Invoice #	Invoice Amount	Invoice Date	Invoice Terms	Receipt of Goods Date
3866	$\$47,690.11$	June 18	$2/10$, EOM	July 3
2928	$\$123,691.82$	June 26	$2\frac{1}{2}/15$, $1/25$	July 6
4133	$\$96,004.21$	June 30	$2/20$, ROG	July 7
6767	$\$16,927.50$	July 10	$2/10$, $1/20$	July 11

Table 4.2.6

Payment Received by Vendor on	Payment Amount
July 2	$\$40,000$
July 11	$\$75,000$
July 21	$\$100,000$
July 30	$\$25,000$

Calculate the amount of the final payment on August 17 that reduces the total balance owing to zero.

18. John's Home Hardware was invoiced on May 29 for the following non-taxable items with terms of $2/20$, $1/40$, 2% per month penalty.

Table 4.2.7

Item	Quantity	Unit Price
2″x 4″x 8′ framing studs	$13,000$	$\$0.96$
4″x 4″ x 8′ fence posts	$2,400$	$\$2.87$
2″x 8″x 16′ wood planks	480	$\$8.47$
1″x 6″x 5′ fence boards	$8,000$	$\$1.22$

If a partial payment of $\$20,000$ is made on July 7, what payment amount on August 30 reduces the balance owing to zero? What is the total dollar amount of the late penalty?

19. Your company receives a $\$138,175.00$ invoice dated April 12 with terms of $4/10$, $3/20$, $2/30$, $1/40$, $n/60$. Due to the large amount, the accounting department proposes two different ways to pay this invoice, depending on company income and financing alternatives. These plans are listed in the table below.

Table 4.2.8

Payment Date	Plan #1	Plan #2
April 21	$\$35,000$	$\$50,000$
May 2	$\$20,000$	$\$10,000$
May 10	$\$25,000$	$\$50,000$
May 13	$\$30,000$	$\$20,000$
June 11	Balance owing	Balance owing

a. Which alternative do you recommend?
b. If your recommendation is followed, how much money is saved over the other option?

Table 4.2.9

Payment Date	Payment Amount
August 14	$\$67,000.00$
August 28	$\$83,000.00$
September 18	$\$15,000.00$
September 30	$\$83,297.16$

Solutions

16. $\$24,772.64$
17. $\$40,608.80$
18. $N=\$13,511.24$; Late penalty amount = $\$519.66$
19. a. Choose Plan #2; b. Savings = $\$724.92$
20. $\$250,214.41$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $L=\frac{N}{(1-d)}$ $N=L\times(1-d)$ $N=L\times(1-d)$ $RoC=\left(\left(\frac{V_{\mathrm{f}}}{V_{\mathrm{i}}}\right)^{\frac{1}{n}}\right)\times 100$

Attribution

4.3: MARKUP: SETTING THE REGULAR PRICE

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- C = Cost
- E = Expenses
- $M\$$ = Markup amount
- $MoC\%$ = Markup on cost percentage
- $MoS\%$ = Markup on selling price Percentage
- P = Profit
- S = Selling price

Formulas Used

- Formula 4.3a – **The Selling Price of a Product**

$$S = C + E + P$$

- Formula 4.3b – **Markup Amount**

$$M\$ = E + P$$

- Formula 4.3c – **Selling Price Using Markup**

$$S = C + M\$$$

- Formula 4.3d – **Markup on Cost Percentage**

$$MoC\% = \frac{M\$}{C} \times 100$$

- Formula 4.3e – **Markup on Selling Price Percentage**

$$MoS\% = \frac{M\$}{S} \times 100$$

Introduction

As you wait in line to purchase your Iced Caramel Macchiato at Starbucks, you look at the pricing menu and think that $\$4.99$ seems like an awful lot of money for a frozen coffee beverage. Clearly, the coffee itself doesn't cost anywhere near that much. But then gazing around the café, you notice the carefully applied color scheme, the comfortable seating, the high-end machinery behind the counter, and a seemingly well-trained barista who answers customer questions knowledgeably. Where did the money to pay for all of this come from? You smile as you realize your $\$4.99$ pays not just for the macchiato, but for everything else that comes with it.

The process of taking a product's cost and increasing it by some amount to arrive at a selling price is called **markup**. This process is critical to business success because every business must ensure that it does not lose money when it makes a sale. From the consumer perspective, the concept of markup helps you make sense of the prices that businesses charge for their products or services. This in turn helps you to judge how reasonable some prices are (and hopefully to find better deals).

The Components in a Selling Price

Before you learn to calculate markup, you first have to understand the various components of a selling price. Then, in the next section, markup and its various methods of calculation will become much clearer.

When your business acquires merchandise for resale, this is a monetary outlay representing a cost. When you then resell the product, the price you charge must recover more than just the product cost. You must also recover all the selling and operating expenses associated with the product. Ultimately, you also need to make some money, or profit, as a result of the whole process.

4.3a The Selling Price of a Product

Most people think that marking up a product must be a fairly complex process. It is not. Formula 4.3a illustrates the relationship between the three components of cost, expenses, and profits in calculating the selling price.

$$rgb]1.0, 0.0, 0.0S = rgb]0.0, 0.0, 1.0C + rgb]0.0, 0.5, 0.0E + rgb]0.5, 0.0, 0.5P$$

{\color[rgb]{1.0, 0.0, 0.0}S}\;\text{is Selling Price:} Once you calculate what the business paid for the product (cost), the bills it needs to cover (expenses), and how much money it needs to earn (profit), you arrive at a selling price by summing the three components.

$rgb]0.0, 0.0, 1.0C$ is Cost: The cost is the amount of money that the business must pay to purchase or manufacture the product. If manufactured, the cost represents all costs incurred to make the product. If purchased, this number results from applying an appropriate discount formula from Section 4.1. There is a list price from which the business will deduct discounts to arrive at the net price. The net price paid for the product equals the cost of the product. If a business purchases or manufactures a product for $\$10$ then it must sell the product for at least $\$10$. Otherwise, it fails to recover what was paid to acquire or make the product in the first place—a path to sheer disaster!

{\color[rgb]{0.0, 0.5, 0.0}E}\;\text{is Expenses:} Expenses are the financial outlays involved in selling the product. Beyond just purchasing the product, the business has many more bills to pay, including wages, taxes, leases, equipment, electronics, insurance, utilities, fixtures, décor, and many more. These expenses must be recovered and may be calculated as:

- A fixed dollar amount per unit
- A percentage of the product cost

For example, if a business forecasts total merchandise costs of $\$100,000$ for the coming year and total business expenses of $\$50,000$, then it may set a general guideline of adding $50\%(\$50,000 \div \$100,000)$ to the cost of a product to cover expenses.

- A percentage of the product selling price based on a forecast of future sales.

For example, if a business forecasts total sales of $\$250,000$ and total business expenses of $\$50,000$

, then it may set a general guideline of adding $20\%(\$50,000 \div \$250,000)$ of the selling price to the cost of a product to cover expenses.

{\color[rgb]{0.5, 0.0, 0.5}P}\;\text{is Profit:} Profit is the amount of money that remains after a business pays all of its costs and expenses. A business needs to add an amount above its costs and expenses to allow it to grow. If it adds too much profit, though, the product's price will be too high, in which case the customer may refuse to purchase it. If it adds too little profit, the product's price may be too low, in which case the customer may perceive the product as shoddy and once again refuse to purchase it. Many businesses set general guidelines on how much profit to add to various products. As with expenses, this profit may be expressed as:

- A fixed dollar amount per unit
- A percentage of the product cost
- A percentage of the selling price

HOW TO

Solve Pricing Scenarios

Step 1: Four variables are involved in **Formula 4.3a**$\mathrm{S}=\mathrm{C}+\mathrm{E}+\mathrm{P}$. Identify the known variables. Note that you may have to calculate the product's cost by applying the single or multiple discount formulas. Pay careful attention to expenses and profits to capture how you calculate these amounts.

Step 2: Apply Formula 4.3a and solve for the unknown variable.

Assume a business pays a net price of $\$75$ to acquire a product. Through analyzing its finances, the business estimates expenses at $\$25$ per unit, and it figures it can add $\$50$ in profit. Calculate the selling price.

Step 1: The net price paid for the product is the product cost. The known variables are:

$$C=\$75, E=\$25, \text{ and } P=\$50.$$

Step 2: According to Formula 4.3a, the unit selling price is:

$$S=C+E+P=\$75+\$25+\$50=\$150.$$

Key Takeaways

In applying **Formula 4.3a** $\mathrm{S}=\mathrm{C}+\mathrm{E}+\mathrm{P}$ you must adhere to the basic rule of linear equations requiring all terms to be in the same unit. That is, you could use Formula 4.3a to solve for the selling price of an individual product, where the three components are the unit cost, unit expenses, and unit profit. When you add these, you calculate the unit selling price. Alternatively, you could use Formula 4.3a in an aggregate form where the three components are total cost, total expenses, and total profit. In this case, the selling price is a total selling price, which is more commonly known as total revenue. But you cannot mix individual components with aggregate components.

Things To Watch Out For

The most common mistake in working with pricing components occurs in identifying and labeling the information correctly. It is critical to identify and label information correctly. You have to pay attention to details such as whether you are expressing the expenses in dollar format or as a percentage of either cost or selling price. Systematically work your way through the information provided piece by piece to ensure that you do not miss an important detail.

Try It

1) Answer the following questions:

a. What three components make up a selling price? In what units are these components commonly expressed?
b. In what three ways are expenses and profits expressed?
c. What is the relationship between net price and cost?

Solution

a. Cost, expenses, and profit. They are expressed either per unit or as a total.
b. A specific dollar amount, a percentage of cost, or a percentage of the selling price.
c. The net price paid for a product is the same as the cost of the product.

Example 4.3.1

Mary's Boutique purchases a dress for resale at a cost of $\$23.67$. The owner determines that each dress must contribute $\$5.42$ to the expenses of the store. The owner also wants this dress to earn $\$6.90$ toward profit. What is the regular selling price for the dress?

Solution

***Step 1:* The unit cost of the dress and the unit expense and the unit profit are all known.**

$$C = \$23.67$$
$$E = \$5.42$$
$$P = \$6.90$$

***Step 2:* Apply Formula 4.3a** $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$ **and solve.**

$$S = \$23.67 + \$5.42 + \$6.90$$
$$S = \$35.99$$

Step 3: **Write as a statement.**

Mary's Boutique will set the regular price of the dress at $\$35.99$.

Example 4.3.2

John's Discount Store just completed a financial analysis. The company determined that expenses average 20% of the product cost and profit averages 15% of the product cost. John's Discount Store purchases Chia Pets from its supplier for an MSRP of $\$19.99$ less a trade discount of 45%. What will be the regular selling price for the Chia Pets?

Solution

Step 1: **The list price, discount rate, expenses, and profit are known:**

$$L = \$19.99$$
$$d = 0.45$$
$$E = 20\% \text{ of } \cos t \text{ or } 0.20C$$
$$P = 15\% \text{ of } \cos t \text{ or } 0.15C$$

Step 2: **Although the cost of the Chia Pets is not directly known, you do know the MSRP (list price) and the trade discount. The cost is equal to the net price. Apply Formula 4.1a.**

$N = L \times (1 - d)$

$$N = \$19.99 \times (1 - 0.45)$$
$$N = \$19.99 \times 0.55$$
$$N = \$10.99 = C$$

Step 3: **To calculate the selling price, apply Formula 4.3a.** $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$

$$S = \$10.99 + 0.20C + 0.15C$$
$$S = \$10.99 + 0.20(\$10.99) + 0.15(\$10.99)$$
$$S = \$10.99 + \$2.20 + \$1.65$$
$$S = \$14.84$$

Step 4: **Write as a statement.**

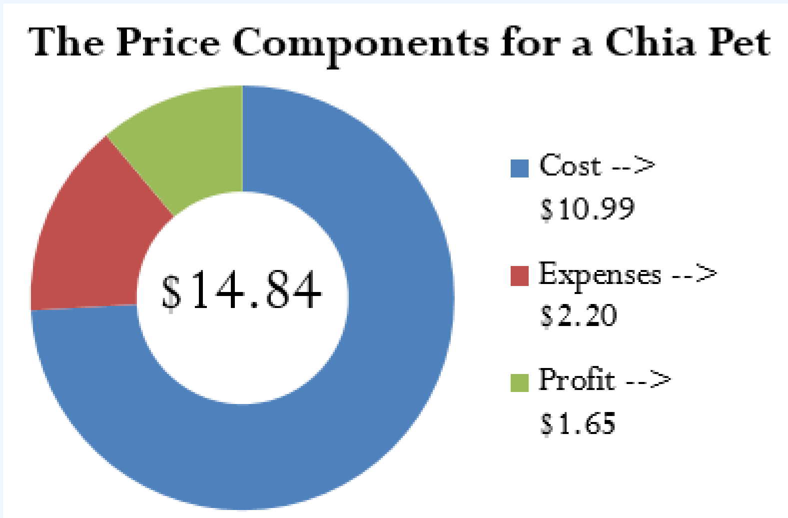


Figure 4.3.1

John's Discount Store will sell the Chia Pet for $\$14.84$.

Example 4.3.3

Based on last year's results, Benthal Appliance learned that its expenses average 30% of the regular selling price. It wants a 25% profit based on the selling price. If Benthal Appliance purchases a fridge for $\$1,200$, what is the regular unit selling price?

Solution

Step 1: **The cost, expenses, and profit for the fridge are known:**

$$E = 30\% \text{ of } S, \text{ or } 0.3S$$
$$P = 25\% \text{ of } S, \text{ or } 0.25S$$
$$C = \$1,200.00$$

***Step 2:* Apply Formula 4.3a.** $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$

$$\begin{aligned} S &= \$1,200.00 + 0.3S + 0.25S \\ S &= \$1,200.00 + 0.55S \\ S - 0.55S &= \$1,200.00 \\ 0.45S &= \$1,200.00 \\ S &= \$2,666.67 \end{aligned}$$

***Step 3:* Write as a statement.**

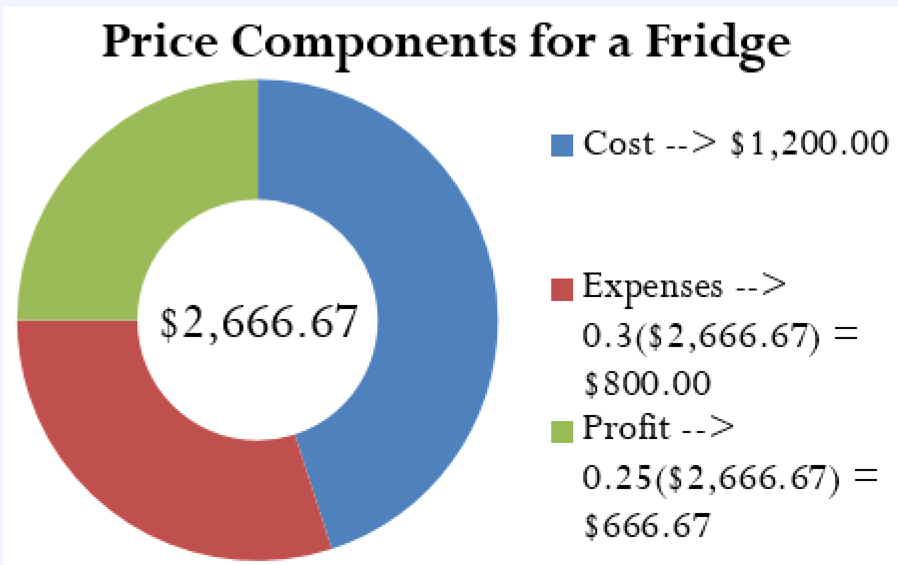


Figure 4.3.2

Benthal Appliance should set the regular selling price of the fridge at $\$2,666.67$.

Example 4.3.4

If a company knows that its profits are 15% of the selling price and expenses are 30% of cost, what is the cost of an MP3 player that has a regular selling price of $\$39.99$?

Solution

Step 1: **The expenses, profits, and the regular unit selling price are as follows:**

$$S = \$39.99$$
$$P = 15\% \; of \; S, \text{ or } 0.15S$$
$$E = 30\% \; of \; \cos t, \text{ or } 0.3C$$

Step 2: **Apply Formula 4.3a.** $\mathrm{S = C + E + P}$

$$\$39.99 = C + 0.3C + 0.15(\$39.99)$$
$$\$39.99 = 1.3C + \$6.00$$
$$\$33.99 = 1.3C$$
$$\$26.15 = C$$

Step 3: **Write a statement.**

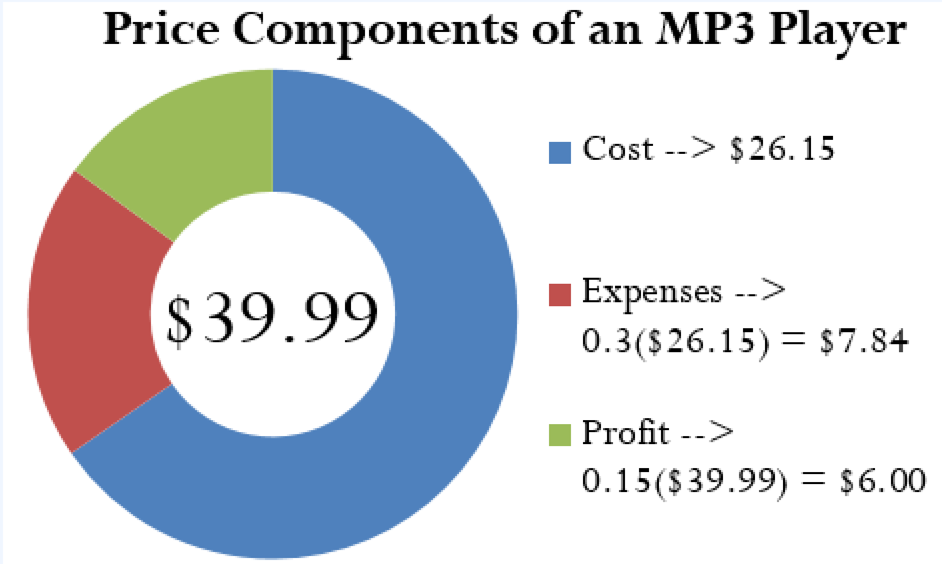


Figure 4.3.3

The cost of the MP3 Player is $\$26.15$.

Example 4.3.5

Peak of the Market considers setting the regular unit selling price of its strawberries at $\$3.99$ per kilogram. If it purchases these strawberries from the farmer for $\$2.99$ per kilogram and expenses average 40% of product cost, does Peak of the Market make any money?

Solution

***Step 1:* The cost, expenses, and proposed regular unit selling price for the strawberries are as follows:**

$$S = \$3.99$$
$$C = \$2.99$$
$$E = 40\% \text{ of } \cos t, \text{ or } 0.4C$$

***Step 2:* Apply Formula 4.3a.** $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$

$$\begin{aligned} \$3.99 &= \$2.99 + \ 0.4(\$2.99) + P \\ \$3.99 &= \$4.19 + P \\ -\$0.20 &= P \end{aligned}$$

Step 3: **Write as a statement.**

The negative sign on the profit means that Peak of the Market would take a loss of $\$0.20$ per kilogram if it sells the strawberries at $\$3.99$. Unless Peak of the Market has a marketing reason or sound business strategy for doing this, the company should reconsider its pricing.

Calculating the Markup Dollars

Most companies sell more than one product, each of which has different price components with varying costs, expenses, and profits. Can you imagine trying to compare 50 different products, each with three different components? You would have to juggle 150 numbers! To make merchandising decisions more manageable and comparable, many companies combine expenses and profit together into a single quantity, either as a dollar amount or a percentage. This section focuses on the markup as a dollar amount.

$\boxed{4.3\text{b}}$ Markup Amount

$$rgb]1.0, 0.0, 0.0M rgb]1.0, 0.0, 0.0\$ \ = \ rgb]0.0, 0.0, 1.0E + rgb]0.0, 0.5, 0.0P$$

One of the most basic ways a business simplifies its merchandising is by combining the dollar amounts of its expenses and profits together as expressed in Formula 4.3b.

{\color[rgb]{1.0, 0.0, 0.0}M}{\color[rgb]{1.0, 0.0, 0.0}\$}\text{ is Markup Amount:} Markup is taking the cost of a product and converting it into a selling price. The markup amount represents the dollar amount difference between the cost and the selling price.

{\color[rgb]{0.0, 0.0, 1.0}E}\;\text{is Expenses:} The expenses associated with the product.

{\color[rgb]{0.0, 0.5, 0.0}P}\;\text{is Profit:} The profit earned when the product sells.

Note that since the markup amount ($M\$$) represents the expenses (E) and profit (P) combined, you can substitute the variable for markup amount into **Formula 4.3a**$\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$ to create Formula 4.3c, which calculates the regular selling price.

4.3c Selling Price Using Markup

$$rgb]0.0,0.0,1.0S = rgb]0.0,0.5,0.0C + rgb]1.0,0.0,0.0Mrgb]1.0,0.0,0.0\$$$

{\color[rgb]{0.0, 0.0, 1.0}S}\;\text{is Selling Price:} The regular selling price of the product.

{\color[rgb]{0.0, 0.5, 0.0}C}\;\text{is Cost:} The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.

{\color[rgb]{1.0, 0.0, 0.0}M}{\color[rgb]{1.0, 0.0, 0.0}\$}\;\text{is Markup Amount:} From Formula 4.3c, this is the single number that represents the total of the expenses and profits.

HOW TO

Work with calculations involving the markup amount

Step 1: You require three variables in either **Formula 4.3b**$\mathrm{M\$} = \mathrm{E} + \mathrm{P}$ or **Formula 4.3c**$\mathrm{S} = \mathrm{C} + \mathrm{M\$}$. At least two of the variables must be known. If the amounts are not directly provided, you may need to calculate these amounts by applying other discount or markup formulas.

Step 2: Solve either Formula 4.3b or Formula 4.3c for the unknown variable.

Recall from the previous MP3 player's example, that the MP3 player's expenses are $\$7.84$, the profit is $\$6.00$, and the cost is $\$26.15$. Calculate the markup amount and the selling price.

Step 1: The known variables are:

$$E = \$7.84, P = \$6.00, \text{ and } C = \$26.15$$

Step 2: According to **Formula 4.3b**$\mathrm{M\$} = \mathrm{E} + \mathrm{P}$, the markup amount is the sum of the expenses and profit, or:

$$M\$ = \$7.84 + \$6.00 = \$13.84$$

Step 3: Applying **Formula 4.3c**$\mathrm{S} = \mathrm{C} + \mathrm{M\$}$, add the markup amount to the cost to arrive at the regular selling price, resulting in:

$$S = \$26.15 + \$13.84 = \$39.99$$

Paths To Success

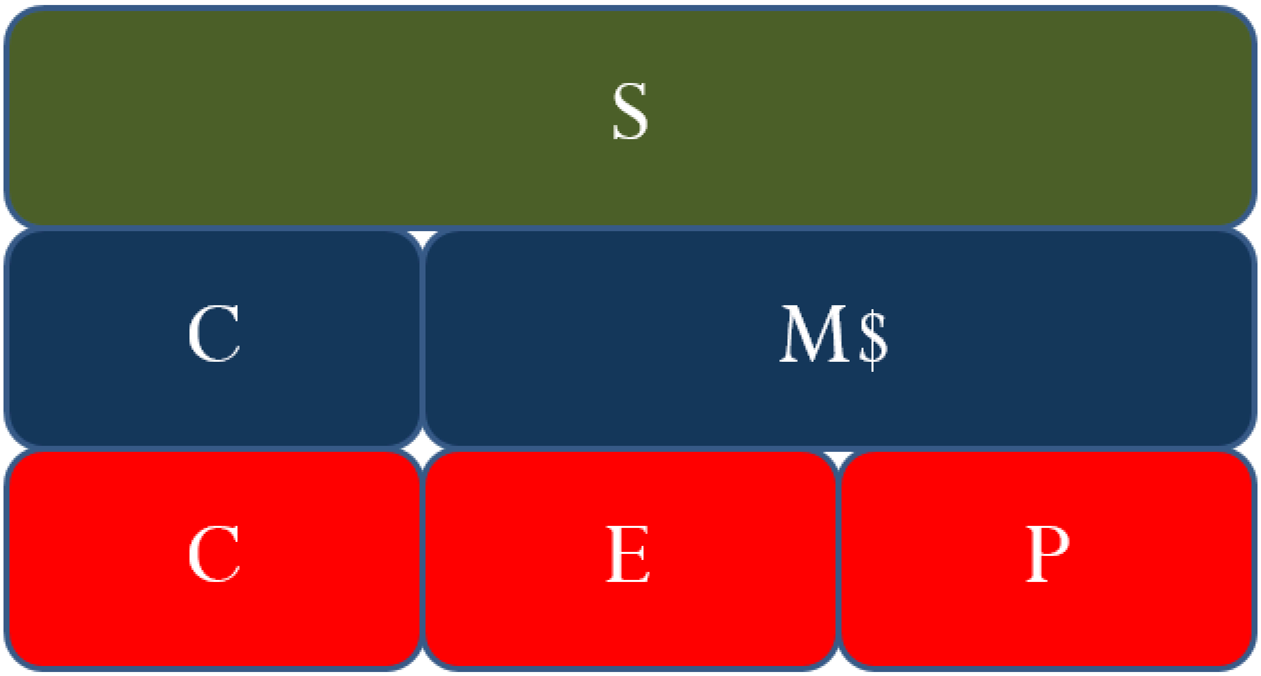


Figure 4.3.4

You might have already noticed that many of the formulas in this chapter are interrelated. The same variables appear numerous times but in different ways. To help visualize the relationship between the various formulas and variables, many students have found it helpful to create a markup chart, as shown to the right.

This chart demonstrates the relationships between **Formula 4.3a**$\mathrm{S}=\mathrm{C}+\mathrm{E}+\mathrm{P}$, **Formula 4.3b** $\mathrm{M\$}=\mathrm{E}+\mathrm{P}$, and **Formula 4.3c**$\mathrm{S}=\mathrm{C}+\mathrm{M\$}$. It is evident that the selling price (the green line) consists of cost, expenses, and profit (the red line representing **Formula 4.3a**$\mathrm{S}=\mathrm{C}+\mathrm{E}+\mathrm{P}$); or it can consist of cost and the markup amount (the blue line representing **Formula 4.3c**$\mathrm{S}=\mathrm{C}+\mathrm{M\$}$). The markup amount on the blue line consists of the expenses and profit on the red line (**Formula 4.3b**$\mathrm{M\$}=\mathrm{E}+\mathrm{P}$).

Example 4.3.6

A cellular retail store purchases an iPhone with an MSRP of $\$779$ less a trade discount of 35% and volume discount of 8%. The store sells the phone at the MSRP.

a. What is the markup amount?
b. If the store knows that its expenses are 20% of the cost, what is the store's profit?

Solution

***Step 1:* The smartphone MSRP and the two discounts are known, along with the expenses and selling price:**

$$\begin{aligned} L &= \$270 \\ d_1 &= 0.35 \\ d_2 &= 0.08 \\ E &= \ 20\% \text{ of } \cos t, \text{ or } 0.2C \\ S &= \$779 \end{aligned}$$

***Step 2:* Calculate the cost of the iPhone by applying Formula 4.1d.**

$N = L \times (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$

$$\begin{aligned} N &= \$779.00 \times (1 - 0.35) \times (1 - 0.08) \\ N &= \$779.00 \times 0.65 \times 0.92 \\ N &= \$465.84 = C \end{aligned}$$

***Step 3:* Calculate the markup amount using Formula 4.3c.** $\mathrm{S} = \mathrm{C} + \mathrm{M\$}$

$$\begin{aligned} \$779.00 &= \$465.84 + M\$ \\ \$313.16 &= \ M\$ \end{aligned}$$

***Step 4:* Calculate the profit by applying Formula 4.3b** $\mathrm{M\$} = \mathrm{E} + \mathrm{P}$**, rearranging for** P**.**

$$\begin{aligned} \$313.16 &= 0.2(\$465.84) + P \\ \$313.16 &= \$93.17P \\ \$219.99 &= P \end{aligned}$$

***Step 5:* Write a statement.**

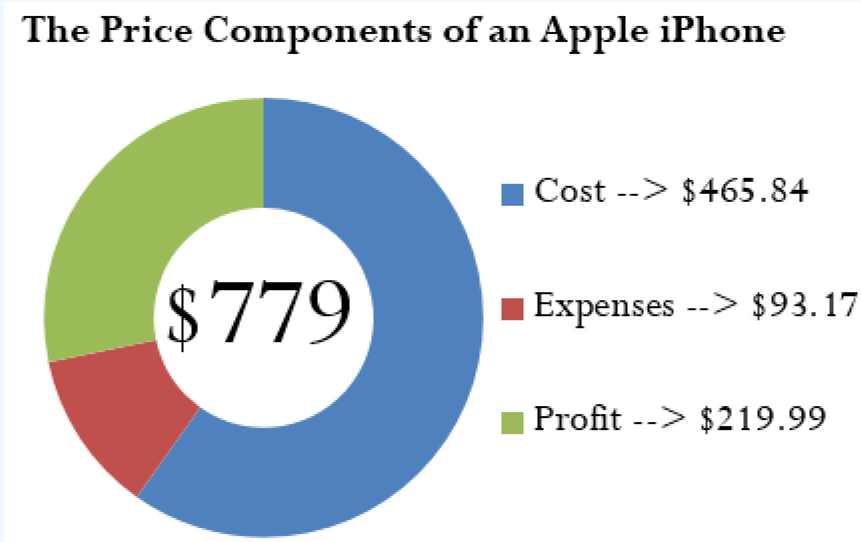


Figure 4.3.5

The markup amount for the iPhone is $\$313.16$. When the store sells the phone for $\$779.00$, its profit is $\$219.99$.

Calculating the Markup Percent

It is important to understand markup in terms of the actual dollar amount; however, it is more common in business practice to calculate the markup as a percentage. There are three benefits to converting the markup dollar amount into a percentage:

- Easy comparison of different products having vastly different price levels and costs, to help you see how each product contributes toward the financial success of the company.

  For example, if a chocolate bar has a $50¢$ markup included in a selling price of $\$1$, while a car has a $\$1,000$ markup included in a selling price of $\$20,000$, it is difficult to compare the profitability of these items. If these numbers were expressed as a percentage of the selling price such that the chocolate bar has a 50% markup and the car has a 5% markup, it is clear that more of every dollar sold for chocolate bars goes toward list profitability.

- Simplified translation of costs into a regular selling price—a task that must be done for each product, making it helpful to have an easy formula, especially when a company carries hundreds, thousands, or even tens of thousands of products.

> For example, if all products are to be marked up by 50% of cost, an item with a $\$100$ cost can be quickly converted into a selling price of $\$150$.

- An increased understanding of the relationship between costs, selling prices, and the list profitability for any given product.

> For example, if an item selling for $\$25$ includes a markup on selling price of 40% (which is $\$10$), then you can determine that the cost is 60% of the selling price ($\$15$) and that $\$10$ of every $\$25$ item sold goes toward list profits.

You can translate the markup dollars into a percentage using two methods, which express the amount either as a percentage of cost or as a percentage of selling price:

- ***Method 1: Markup as a Percentage of Cost.*** This method expresses the markup rate using cost as the base. Many companies use this technique internally because most accounting is based on cost information. The result, known as the **markup on cost percentage**, allows a reseller to convert easily from a product's cost to its regular unit selling price.
- ***Method 2: Markup as a Percentage of Selling Price.*** This method expresses the markup rate using the regular selling price as the base. Many other companies use this method, known as the **markup on selling price percentage**, since it allows for quick understanding of the portion of the selling price that remains after the cost of the product has been recovered. This percentage represents the list profits before the deduction of expenses and therefore is also referred to as the **list profit margin**.

4.3d Markup on Cost Percentage

$$rgb]0.0,0.0,1.0Mrgb]0.0,0.0,1.0orgb]0.0,0.0,1.0Crgb]0.0,0.0,1.0\% = \frac{rgb]1.0,0.0,0.0Mrgb]1.0,0.0,0.0\$}{rgb]0.0,0.5,0.0C} rgb]0.5,0.0,0.5\times rgb]0.5,0.0,0.5100$$

{\color[rgb]{0.0, 0.0, 1.0}M}{\color[rgb]{0.0, 0.0, 1.0}o}{\color[rgb]{0.0, 0.0, 1.0}C}{\color[rgb]{0.0, 0.0, 1.0}\%}{\color[rgb]{0.1, 0.1, 0.1}\text{ is Markup on Cost Percentage:}}

This is the percentage by which the cost of the product needs to be increased to arrive at the selling price for the product.

{\color[rgb]{1.0, 0.0, 0.0}M}{\color[rgb]{1.0, 0.0, 0.0}\$}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Markup Amount:}}

The total dollars of the expenses and the profits; this total is the difference between the cost and the selling price.

{\color[rgb]{0.0, 0.5, 0.0}C}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Cost:}}

The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.

$rgb]0.5, 0.0, 0.5\times rgb]0.5, 0.0, 0.5100rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1$is Percent Conversion:

The markup on cost is always a percentage.

The markup on cost percentage is expressed in Formula 4.3d, while the markup on selling price percentage is expressed in Formula 4.3e. Both formulas are versions of formulas on rate, portion, and base.

4.3e Markup on Selling Price Percentage

$$rgb]0.0,0.0,1.0Mrgb]0.0,0.0,1.0orgb]0.0,0.0,1.0Srgb]0.0,0.0,1.0\%rgb]0.1,0.1,0.1 = rgb]0.1,0.1,0.1\frac{rgb]1.0,0.0,0.0Mrgb]1.0,0.0,0.0\$}{rgb]0.68,0.46,0.12S}rgb]0.5,0.0,0.5\times rgb]0.5,0.0,0.5100$$

{\color[rgb]{0.0, 0.0, 1.0}M}{\color[rgb]{0.0, 0.0, 1.0}o}{\color[rgb]{0.0, 0.0, 1.0}S}{\color[rgb]{0.0, 0.0, 1.0}\%}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Markup on Selling Price Percentage:}}

This is the percentage of the selling price that remains available as list profits after the cost of the product is recovered.

{\color[rgb]{1.0, 0.0, 0.0}M}{\color[rgb]{1.0, 0.0, 0.0}\$}{\color[rgb]{1.0, 0.0, 0.0}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Markup Amount:}}

The total dollars of the expenses and the profits; this total is the difference between the cost and the selling price.

{\color[rgb]{0.68, 0.46, 0.12}S}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Selling Price:}}

The regular selling price of the product.

$rgb]0.5, 0.0, 0.5\times rgb]0.5, 0.0, 0.5100rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1$is Percent Conversion:

The markup on cost is always a percentage.

HOW TO

Solve calculations involving markup percent are almost identical to those for working with markup dollars

Step 1: Three variables are required in either **Formula 4.3d** $MoC\% = \frac{\text{M\$}}{\text{C}} \times 100$ or **Formula 4.3e** $\text{MoS\%} = \frac{\text{M\$}}{\text{S}} \times 100$. For either formula, at least two of the variables must be known. If the amounts are not directly provided, you may need to calculate these amounts by applying other discount or markup formulas.

Step 2: Solve either Formula 4.3d or Formula 4.3e for the unknown variable.

Continuing to work with the MP3 player example, recall that the cost of the MP3 player is $\$26.15$, the markup amount is $\$13.84$, and the selling price is $\$39.99$. Calculate both markup percentages.

Step 1: The known variables are $C = \$26.15$, $M\$ = \$13.84$, and $S = \$39.99$.

Step 2: To calculate the markup on cost percentage, apply **Formula 4.3d** $MoC\% = \frac{\text{M\$}}{\text{C}} \times 100$:

$$rgb]0.1,0.1,0.1Mrgb]0.1,0.1,0.1orgb]0.1,0.1,0.1Crgb]0.1,0.1,0.1\%rgb]0.1,0.1,0.1=rgb]0.1,0.1,0.1\frac{\$13.84}{\$26.15}rgb]0.1,0.1,0.1\times rgb]0.1,0.1,0.1100rgb]0.1,0.1,0.1=rgb]0.1,0.1,0.152rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.19254rgb]0.1,0.1,0.1\%$$

In other words, you must add52.9254% of the cost on top of the unit cost to arrive at the regular unit selling price of $\$39.99$.

Step 3: To calculate the markup on selling price percentage, apply **Formula 4.3e** $\text{MoS\%} = \frac{\text{M\$}}{\text{S}} \times 100$:

$$rgb]0.1,0.1,0.1Mrgb]0.1,0.1,0.1orgb]0.1,0.1,0.1Srgb]0.1,0.1,0.1\%rgb]0.1,0.1,0.1=rgb]0.1,0.1,0.1\frac{\$13.84}{\$39.99}rgb]0.1,0.1,0.1\times rgb]0.1,0.1,0.1100rgb]0.1,0.1,0.1=rgb]0.1,0.1,0.134rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.16087rgb]0.1,0.1,0.1\%$$

In other words, 34.6087% of the selling price represents list profits after the business recovers the $\$26.15$ cost of the MP3 player.

Key Takeaways

Businesses are very focused on profitability. Your Texas Instruments BAII Plus calculator is programmed with the markup on selling price percentage. The function is located on the second shelf above the number three. To use this function, open the window by pressing 2nd 3. You can scroll between lines using your ↑ and ↓ arrows. There are three variables:

- CST is the cost. Use the symbol C.
- SEL is the selling price. Use the symbol S.
- MAR is the markup on selling price percentage. Use the symbol $MoS\%$.

As long as you know any two of the variables, you can solve for the third. Enter any two of the three variables (you need to press ENTER after each), making sure the window shows the output you are seeking, and press CPT.

Things To Watch Out For

Merchandising involves many variables. Nine formulas have been established so far, and a few more are yet to be introduced. Though you may feel bogged down by all of these formulas, just remember that you have encountered most of these merchandising concepts since you were very young and that you interact with retailers and pricing every day. This chapter merely formalizes calculations you already perform on a daily basis, whether at work or at home. The calculation of discounts is no different than going to Walmart and finding your favorite CD on sale. You know that when a business sells a product, it has to recoup the cost of the product, pay its bills, and make some money. And you have worked with percentages since elementary school.

Do not get stuck in the formulas. Think about the concept presented in the question. Change the scenario of the question and put it in the context of something more familiar. Ultimately, if you really have difficulties then look at the variables provided and cross-reference them to the merchandising formulas. Your goal is to find formulas in which only one variable is unknown. These formulas are solvable. Then ask yourself, "How does knowing that new variable help solve any other formula?"

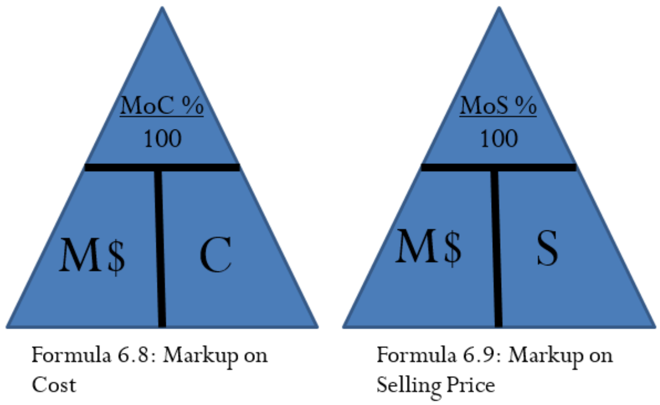


Figure 4.3.6

You do not need to get frustrated. Just be systematic and relate the question to what you already know.

Paths To Success

The triangle method simplifies rearranging both **Formula 4.3d** $MoC\% = \frac{\text{M\$}}{\text{C}} \times 100$ and **Formula 4.3e** $\text{MoS\%} = \frac{\text{M\$}}{\text{S}} \times 100$ to solve for other unknown variables as illustrated in the figure to the right.

Sometimes you need to convert the markup on cost percentage to a markup on selling price percentage, or vice versa. Two shortcuts allow you to convert easily from one to the other:

$$MoC\% = \frac{MoS\%}{1 - MoS\%}$$

$$MoS\% = \frac{MoC\%}{1 + MoC\%}$$

Notice that these formulas are very similar. How do you remember whether to add or subtract in the denominator? In normal business situations, the $MoC\%$ is always larger than the $MoS\%$. Therefore, if you are converting one to the other you need to identify whether you want the percentage to become larger or smaller.

- To calculate $MoC\%$, you want a larger percentage. Therefore, make the denominator smaller by subtracting $MoS\%$ from 1.
- To calculate $MoS\%$, you want a smaller percentage. Therefore, make the denominator larger by adding $MoC\%$ to 1.

Try It

2) Answer the following true/false questions:

a. The markup on selling price percentage can be higher than 100%.
b. The markup dollar amount can be more than the selling price.
c. The markup on cost percentage can be higher than 100%.
d. The markup on cost percentage in most business situations is higher than the markup on selling price percentage.
e. If you know the markup on cost percentage and the cost, you can calculate a selling price.
f. If you know the markup on selling price percentage and the cost, you can calculate a selling price.

Solution

a. False. The markup amount is a portion of the selling price and therefore is less than 100%.
b. False. The markup amount plus the cost equals the selling price. It must be less than the selling price.
c. True. A cost can be doubled or tripled (or increased even more) to reach the price.
d. True. The base for markup on cost percentage is smaller, which produces a larger percentage.
e. True. You could combine **Formula 4.3c**$\text{S} = \text{C} + \text{M\$}$ and **Formula 4.3d** $MoC\% = \dfrac{\text{M\$}}{\text{C}} \times 100$ to arrive at the selling price.
f. True. You could convert the $MoS\%$ to a $MoC\%$ and solve as in the previous question.

Example 4.3.7

A large national retailer wants to price a Texas Instruments BAII Plus calculator at the MSRP of $\$39.99$. The retailer can acquire the calculator for $\$17.23$.

a. What is the markup on cost percentage?
b. What is the markup on selling price percentage?

Solution

***Step 1:* The regular unit selling price and the cost are provided:**

$$S = \$39.99$$
$$C = \$17.23$$

***Step 2:* You need the markup dollars. Apply Formula 4.3c.$\mathrm{S} = \mathrm{C} + \mathrm{M\$}$, rearranging for $M\$$.**

$$\$39.99 = \$17.23 + M\$$$
$$\$22.76 = M\$$$

***Step 3:* To calculate markup on cost percentage, apply Formula 4.3d.**

$$MoC\% = \frac{\mathrm{M\$}}{\mathrm{C}} \times 100$$

$$MoC\% = \frac{\$22.76}{\$17.23} \times 100$$
$$MoC\% = 132.0952\%$$

***Step 4:* To calculate markup on selling price percentage, apply Formula 4.3e.**

$$\mathrm{MoS\%} = \frac{\mathrm{M\$}}{\mathrm{S}} \times 100$$

$$MoS\% = \frac{\$22.76}{\$39.99} \times 100$$
$$MoS\% = 56.9142\%$$

Calculator Instructions:

Table 4.3.1

CST	SEL	MAR
17.23	39.99	Answer: 56.9142

Step 5: **Write as a statement.**

The markup on cost percentage is 132.0952%. The markup on selling price percentage is 56.9142%.

Break-Even Pricing

In running a business, you must never forget the "bottom line." In other words, if you fully understand how your products are priced, you will know when you are making or losing money. Remember, if you keep losing money you will not stay in business for long! As previously stated, 15% of new businesses will not make it past their first year, and 49% fail in their first five years. This number becomes even more staggering with an 80% failure rate within the first decade (Statistics Canada, 2000). Do not be one of these statistics! With your understanding of markup, you now know what it takes to break even in your business. **Break-even** means that you are earning no profit, but you are not losing money either. Your profit is zero.

The Formula

If the regular unit selling price must cover three elements—cost, expenses, and profit—then the regular unit selling price must exactly cover your costs and expenses when the profit is zero. In other words, if **Formula 4.3a** $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$ is modified to calculate the selling price at the break-even point (S_{BE}) with $P = 0$, then

$$rgb]1.0,0.0,0.0S_{rgb]1.0,0.0,0.0BE} = rgb]0.0,0.5,0.0Crgb]0.0,0.0,1.0E$$

This is not a new formula. It summarizes that at break-even there is no profit or loss, so the profit (P) is eliminated from the formula.

HOW TO

Calculate break-even point

The steps you need to calculate the break-even point are no different from those you used to calculate the regular selling price. The only difference is that the profit is always set to zero.

Recall that the cost of the MP3 player is $\$26.15$ and expenses are $\$7.84$. The break-even price
(S_{BE}) is $\$26.15+\$7.84=\$33.99$. This means that if the MP3 player is sold for anything more than $\$33.99$, it is profitable; if it is sold for less, then the business does not cover its costs and expenses and takes a loss on the sale.

Example 4.3.8

John is trying to run an eBay business. His strategy has been to shop at local garage sales and find items of interest at a great price. He then resells these items on eBay. On John's last garage sale shopping spree, he only found one item—a Nintendo Wii that was sold to him for $\$100$. John's vehicle expenses (for gas, oil, wear/tear, and time) amounted to $\$40$. eBay charges a $\$2.00$ insertion fee, a flat fee of $\$2.19$, and a commission of 3.5% based on the selling price less $\$25$. What is John's minimum list price for his Nintendo Wii to ensure that he at least covers his expenses?

Solution

Step 1: **John's cost for the Nintendo Wii and all of his associated expenses are as follows:**

$$\begin{aligned} E &= \$40.00+\$2.00+\$2.19+3.5\%(S_{BE}-\$25.00) \\ E &= \$44.19+0.035(S_{BE}-\$25.00) \\ E &= \$44.19+0.035S_{BE}-\$0.875 \\ E &= \$43.315+0.035S_{BE} \end{aligned}$$

You have four expenses to add together that make up the E in the formula.

$$\begin{aligned} E &= \$40.00 + \$2.00 + \$2.193.5\%(S_{BE} - \$25.00) \\ E &= \$44.19 + 0.035(S_{BE} - \$25.00) \\ E &= \$44.19 + 0.035S_{BE} - \$0.875 \\ E &= \$43.315 + 0.035S_{BE} \end{aligned}$$

Step 2: Formula 4.3a states $S = C + E + P$. Since you are looking for the break-even point, then P is set to zero and $S_{BE} = C + E$.

$$\begin{aligned} S_{BE} &= \$100.00 + \$43.315 + 0.035S_{BE} \\ S_{BE} &= \$143.315 + 0.035S_{BE} \\ S_{BE} - 0.035S_{BE} &= \$143.315 \\ 0.965S_{BE} &= \$143.315 \\ S_{BE} &= \$148.51 \end{aligned}$$

***Step 3:* Write a statement.**

At a price of $\$148.51$ John would cover all of his costs and expenses but realize no profit or loss. Therefore, $\$148.51$ is his minimum price.

Section 4.3 Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

For questions 1–8, solve for the unknown variables (identified with a ?) based on the information provided.

Table 4.3.2

	Regular Unit Selling Price	Cost	Expenses	Profit	Markup Amount	Break-Even Price	Markup on Cost	Markup on Selling Price
1.	?	$\$188.42$	$\$48.53$	$\$85.00$	?	?	?	?
2.	$\$999.99$	?	30% of C	23% of C	?	?	?	?
3.	?	?	?	10% of S	$\$183.28$	?	155%	?
4.	$\$274.99$	?	20% of S	?	?	?	?	35%
5.	?	?	45% of C	?	$\$540.00$	$\$1,080.00$	?	?
6.	?	$\$200$ less 40%	?	15% of S	?	?	68%	?
7.	?	?	$\$100.00$	?	$\$275.00$	?	?	19%
8.	?	?	15% of C	12% of S	?	$\$253.00$	?	?

Solutions

1. $S = \$321.95$; $S_{BE} = \$236.95$; $M\$ = \$133.53$; $MoC\% = 70.8683\%$; $MoS\% = 41.4754\%$
2. $C = \$653.59$; $S_{BE} = \$849.67$; $M\$ = \$346.41$; $MoC\% = 53.0011\%$; $MoS\% = 34.6413\%$
3. $C = \$118.25$; $S = \$301.53$; $E = \$153.13$; $MoS\% = 60.7833\%$;

$S_{BE} = \$271.38$

4. $M\$ = \$96.25$; $P = \$41.25$; $C = \$178.74$; $S_{BE} = \$223.74$; $MoC\% = 53.8492\%$
5. $C = \$744.83$; $S = \$1,284.83$; $P = \$204.83$; $MoC\% = 72.4998\%$; $MoS\% = 42.0289\%$
6. $M\$ = 81.60$; $S = \$201.60$; $E = \$51.36$; $MoS\% = 40.4762\%$; $S_{BE} = \$171.36$
7. $P = \$175$; $S = \$1,447.37$; $C = \$1,172.37$; $MoC\% = 23.4568\%$; $S_{BE} = \$1,272.37$
8. $C = \$220$; $S = \$287.50$; $M\$ = \$67.50$; $MoS\% = 23.4783\%$; $MoC\% = 30.6818\%$

Applications

9. If a pair of sunglasses sells at a regular unit selling price of $\$249.99$ and the markup is always 55% of the regular unit selling price, what is the cost of the sunglasses?
10. A transit company wants to establish an easy way to calculate its transit fares. It has determined that the cost of a transit ride is $\$1.00$, with expenses of 50% of cost. It requires $\$0.75$ profit per ride. What is its markup on cost percentage?
11. Daisy is trying to figure out how much negotiating room she has in purchasing a new car. The car has an MSRP of $\$34,995.99$. She has learned from an industry insider that most car dealerships have a 20% markup on selling price. What does she estimate the dealership paid for the car?
12. The markup amount on an eMachines desktop computer is $\$131.64$. If the machine regularly retails for $\$497.25$ and expenses average 15% of the selling price, what profit will be earned?
13. Manitoba Telecom Services (MTS) purchases an iPhone for $\$749.99$ less discounts of 25% and 15%. MTS's expenses are known to average 30% of the regular unit selling price.

    a. What is the regular unit selling price if a profit of $\$35$ per iPhone is required?
    b. What are the expenses?
    c. What is the markup on cost percentage?
    d. What is the break-even selling price?

14. A snowboard has a cost of $\$79.10$, expenses of $\$22.85$, and profit of $\$18.00$.
    a. What is the regular unit selling price?
    b. What is the markup amount?
    c. What is the markup on cost percentage?
    d. What is the markup on selling price percentage?
    e. What is the break-even selling price? What is the markup on cost percentage at this break-even price?

Solutions

9. $\$112.50$
10. 125%
11. $\$27,996.79$
12. $\$57.05$
13. a. $\$733.03$; b. $\$219.91$; c. 53.3151%; d. $\$698.03$
14. a. $\$119.95$; b. $\$40.85$; c. 51.6435%; d. 34.0559%; e. 28.8887%

Challenge, Critical Thinking, & Other Applications

15. A waterpark wants to understand its pricing better. If the regular price of admission is $\$49.95$, expenses are 20% of cost, and the profit is 30% of the regular unit selling price, what is the markup amount?
16. Sally works for a skateboard shop. The company just purchased a skateboard for $\$89.00$ less discounts of 22%, 15%, and 5%. The company has standard expenses of 37% of cost and desires a profit of 25% of the regular unit selling price. What regular unit selling price should Sally set for the skateboard?
17. If an item has a 75% markup on cost, what is its markup on selling price percentage?
18. A product received discounts of 33%, 25%, and 5%. A markup on cost of 50% was then applied to arrive at the regular unit selling price of $\$349.50$. What was the original list price for the product?
19. Mountain Equipment Co-op (MEC) wants to price a new backpack. The backpack can be purchased for a list price of $\$59.95$ less a trade discount of 25% and a quantity discount of 10%. MEC estimates expenses to be 18% of cost and it must maintain a markup on selling price of 35%.
    a. What is the cost of backpack?

b. What is the markup amount?
c. What is the regular unit selling price for the backpack?
d. What profit will Mountain Equipment Co-op realize?
e. What happens to the profits if it sells the backpack at the MSRP instead?

20. Costco can purchase a bag of Starbucks coffee for $\$20.00$ less discounts of 20%, 15%, and 7%. It then adds a 40% markup on cost. Expenses are known to be 25% of the regular unit selling price.

    a. What is the cost of the coffee?
    b. What is the regular unit selling price?
    c. How much profit will Costco make on a bag of Starbucks coffee?
    d. What markup on selling price percentage does this represent?
    e. Repeat questions (a) through (d) if the list price changes to $\$24.00$.

Solutions

15. $\$20.82$
16. $\$102.40$
17. 42.8571%
18. $\$488.09$
19. a. $\$40.47$; b. $\$21.79$; c. $\$62.26$; d. $\$14.51$; e. $\$12.20$; $\$2.31$ reduction
20. a. $\$12.65$; b. $\$17.71$; c. $\$0.63$; d. $MoS\% = 28.5714\%$; e. $C = \$15.18$; $S = \$21.25$; $P = \$0.76$; $MoS\% = 28.5647\%$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$ $MoC\% = \frac{\mathrm{M\$}}{\mathrm{C}} \times 100$ $N = L \times (1 - d)$ Formula does not parse $\mathrm{M\$} = \mathrm{E} + \mathrm{P}$ $\mathrm{S} = \mathrm{C} + \mathrm{M\$}$ $N = L \times (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$

Attribution

4.4: MARKDOWN: SETTING THE SALE PRICE

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- C = Cost
- d = Markdown rate, discount rate
- $D\$$ = Markdown amount, discount amount
- E = Expenses
- L = List price
- N = Net price
- P = Profit
- P_{onsale} = Planned profit amount
- S = Selling price
- S_{onsale} = Sale price

Formulas Used

- Formula 4.2a – **Single Discount**

$$N = L \times (1 - d)$$

- Formula 4.2b – **Discount Amount**

$$D\$ = L \times d$$

- Formula 4.2c – **Discount Amount**

$$D\$ = L - N$$

- Formula 4.4a – **The Sale Price of a Product**

$$S_{onsale} = S \times (1 - d)$$

- Formula 4.4b – **Markdown Amount**

$$D\$ = S \times d$$

- Formula 4.4c – **Markdown Amount**

$$D\$ = S - S_{onsale}$$

- Formula 4.4d – **Markdown Percentage**

$$d = \frac{D\$}{S} \times 100$$

- Formula 4.4e – **The Selling Price of a Product Adapted**

$$S_{onsale} = C + E + P_{onsale}$$

Introduction

Flashy signs in a retail store announce, “40% off, today only!” Excitedly you purchase three tax-free products with regular price tags reading $\$100$, $\$250$, and $\$150$. The cashier processing the transaction informs you that your total is $\$325$. You are about to hand over your credit card when something about the total makes you pause. The regular total of all your items is $\$500$. If they are 40% off, you should receive a $\$200$ deduction and pay only $\$300$. The cashier apologizes for the mistake and corrects your total.
Although most retail stores use automated checkout systems, these systems are ultimately programmed by human beings. A computer system is only as accurate as the person keying in the data. A study by the Competition Bureau (1999). revealed that 6.3% of items at various retail stores scanned incorrectly. The average error spread is up to 13% around the actual product’s price. Clearly, it is important for you as a consumer to be able to calculate markdowns.

Businesses must also thoroughly understand markdowns so that customers are charged accurately for their purchases. Businesses must always comply with the Competition Act of Canada, which specifically defines legal pricing practices. If your business violates this law, it faces severe penalties.

The Importance of Markdowns

A **markdown** is a reduction from the regular selling price of a product resulting in a lower price. This lower price is called the **sale price** to distinguish it from the selling price.

Many people perceive markdowns as a sign of bad business management decisions. However, in most situations this is not true. Companies must always attempt to forecast the future. In order to stock products, a reseller must estimate the number of units that might sell in the near future for every product that it carries. This is both an art and a science. While businesses use statistical techniques that predict future sales with a relative degree of accuracy, consumers are fickle and regularly change shopping habits. Markdowns most commonly occur under four circumstances:

- **Clearing Out Excess or Unwanted Inventory:** In these situations, the business thought it could sell 100 units; however, consumers purchased only 20 units. In the case of seasonal inventory, such as Christmas items on Boxing Day, the retailer wishes to avoid packing up and storing the inventory until the next season.
- **Clearing Out Damaged or Discontinued Items:** Selling a damaged product at a discount is better than not selling it at all. When products are discontinued, this leaves shelf space underused, so it is better to clear the item out altogether to make room for profitable items that can keep the shelves fully stocked.
- **Increasing Sales Volumes:** Sales attract customers because almost everyone loves a deal. Though special marketing events such as a 48 hour sale reduce the profitability per unit, by increasing the volume sold these sales can lead to a greater profit overall.
- **Promoting Add-On Purchases:** Having items on sale attracts customers to the store. Many times customers will not only purchase the item on sale but also, as long as they are on the premises, grab a few other items, which are regularly priced and very profitable. Like many others, you may have walked into Target to buy one item but left with five instead.

—

Markdowns are no different from offering a discount. Recall from that one of the types of discounts is known as a sale discount. The only difference here lies in choice of language. Markdowns are common, so you will find it handy to adapt the discount formulas to the application of markdowns, replacing the symbols with ones that are meaningful in merchandising. **Formula 4.1a**$N = L \times (1 - d)$, introduced in Section 4.1, calculates the net price for a product after it receives a single discount:

4.1a

$$\text{Single Discount: } rgb]1.0,0.0,0.0N \;=\; rgb]0.0,0.5,0.0L \;\times\; (1 \;-\; rgb]0.0,0.0,1.0d)$$

Formula 4.4a adapts this formula for use in markdown situations.

4.4a The Sale Price Of A Product

$$\text{The Sale Price Of A Product: } rgb]0.68,0.46,0.12S_{rgb]0.68,0.46,0.12onsale} \;=\; rgb]0.0,0.5,0.0S \;\times\; (1 \;-\; rgb]0.0,0.0,1.0d)$$

{\color[rgb]{1.0, 0.0, 0.0}S}_{\color[rgb]{1.0, 0.0, 0.0}{onsale}}\;\text{is Sale Price:} The sale price is the price of the product after reduction by the markdown percent. Conceptually, the sale price is the same as the net price.

{\color[rgb]{0.0, 0.5, 0.0}S}\;\text{is Selling Price:} The regular selling price of the product before any discounts. The higher price is the list price. In merchandising questions, this dollar amount may or may not be a known variable. If the selling price is unknown, you must calculate it using an appropriate formula or combination of formulas from either Section 4.2 or Section 4.3.

{\color[rgb]{0.0, 0.0, 1.0}d}\;\text{is Markdown Rate:} A markdown rate is the same as a sale discount rate. Therefore, you use the same discount rate symbol (d) to represent the percentage (in decimal format) by which you reduce the selling price. Note that you are interested in calculating the sale price and not the amount saved. Thus, you take the markdown rate away from 1 to find out the rate owing.

—

In markdown situations, the selling price and the sale price are different variables. The sale price is always less than the selling price. In the event that a regular selling price has more than one markdown percent applied to it, you can extend Formula 4.4a to calculate multiple discounts.

If you are interested in the markdown amount in dollars, recall that **Formula 4.1b** $D\$ = L \times d$ calculates the discount amount in dollars. Depending on what information is known, the formula has two variations:

4.1b

$$\text{Discount Amount: } rgb]1.0,0.0,0.0D rgb]1.0,0.0,0.0\$ = rgb]0.0,0.5,0.0L \times rgb]0.0,0.0,1.0d$$

4.1c

$$\text{Discount Amount: } rgb]1.0,0.0,0.0D rgb]1.0,0.0,0.0\$ = rgb]0.0,0.5,0.0L rgb]0.1,0.1,0.1 - rgb]0.5,0.0,0.5N$$

—

Formula 4.4b and Formula 4.4c adapt these formulas to markdown situations.

4.4b Markdown Amount

$$\text{Markdown Amount: } rgb]1.0,0.0,0.0D rgb]1.0,0.0,0.0\$ = rgb]0.0,0.5,0.0S \times rgb]0.0,0.0,1.0d$$

{\color[rgb]{1.0, 0.0, 0.0}D}{\color[rgb]{1.0, 0.0, 0.0}\$}\;\text{is Markdown Amount:} You determine the markdown amount using either formula depending on what information is known. If you know the selling price and markdown percent, apply Formula 4.4b. If you know the selling price and sale price, apply Formula 4.4c.

{\color[rgb]{0.0, 0.5, 0.0}S}\;\text{is Selling Price:} The regular selling price before you apply any markdown percentages.

{\color[rgb]{0.0, 0.0, 1.0}d}\;\text{is Markdown Rate:} the percentage of the selling price to be deducted (in decimal format). In this case, because you are interested in figuring out how much the percentage is worth, you do not take it away from 1 as in **Formula 4.4a**$S_{\text{onsale}} = \text{S} \times (1 - d)$.

4.4c Markdown Amount

$$\text{Markdown Amount: } rgb]1.0,0.0,0.0D rgb]1.0,0.0,0.0\$ = rgb]0.0,0.5,0.0S - rgb]0.68,0.46,0.12S_{rgb]0.68,0.46,0.12onsale}$$

{\color[rgb]{1.0, 0.0, 0.0}D}{\color[rgb]{1.0, 0.0, 0.0}\$}\;\text{is Markdown Amount:} You determine the markdown amount using either formula depending on what information is known. If you know the selling price and markdown percent, apply Formula 4.4b. If you know the selling price and sale price, apply Formula 4.4c.

{\color[rgb]{0.0, 0.5, 0.0}S}\;\text{is Selling Price:} The regular selling price before you apply any markdown percentages.

$rgb]0.68, 0.46, 0.12S_{rgb]0.68,0.46,0.12onsale}$ is Sale Price: The price after you have deducted all markdown percentages from the regular selling price.

The final markdown formula reflects the tendency of businesses to express markdowns as percentages, facilitating easy comprehension and comparison. Recall the formula which calculated a markup on selling price percent: $MoS\% = \frac{M\$}{S} \times 100$

Formula 4.4d adapts this formula to markdown situations.

4.4d Markdown Percentage

$$\text{Markdown Percentage: } rgb]0.0, 0.0, 1.0d = \frac{rgb]1.0, 0.0, 0.0Drgb]1.0, 0.0, 0.0\$}{rgb]0.0, 0.5, 0.0S} rgb]0.5, 0.0, 0.5 \times rgb]0.5, 0.0, 0.5100$$

{\color[rgb]{0.0, 0.0, 1.0}d}\;\text{is Markdown Percentage:} You always deduct a markdown amount from the regular selling price of the product. Therefore, you always express the markdown percent as a percentage of the selling price. Use the same symbol for a discount rate, since markdown rates are synonymous with sale discounts.

{\color[rgb]{1.0, 0.0, 0.0}D}{\color[rgb]{1.0, 0.0, 0.0}\$}\;\text{is Markdown Amount:} The total dollar amount deducted from the regular selling price.

{\color[rgb]{0.0, 0.5, 0.0}S}\;\text{is Selling Price:} The regular selling price of the product before any discounts.

{\color[rgb]{0.5, 0.0, 0.5}\times}{\color[rgb]{0.5, 0.0, 0.5}\;}{\color[rgb]{0.5, 0.0, 0.5}100}\;\text{is Percent Conversion:}

The markdown is always expressed as a percentage.

HOW TO

Calculate a Markdown

Step 1: Across all three markdown formulas, the four variables consist of the selling price (S), sale price (S_{onsale}), markdown dollars ($D\$$), and markdown rate (d). Identify which variables are known. Depending on the known information, you may have to calculate the selling price using a combination of discount and markup formulas.

Step 2: Apply one or more of **Formula 4.4a**$S_{\text{onsale}} = \text{S} \times (1 - d)$, **Formula 4.4b** $\text{D\$} = \text{S} \times d$, **Formula 4.4c**$\text{D\$} = \text{S} - \text{S}_{\text{onsale}}$, and **Formula 4.4d** $d = \frac{\text{D\$}}{\text{S}} \times 100$ to calculate the unknown variable(s). In the event that multiple markdown rates apply, extend **Formula 4.4a**$S_{\text{onsale}} = \text{S} \times (1 - d)$ to accommodate as many markdown rates as required.

Recal the example of the MP3 player with a regular selling price of $\$39.99$. Assume the retailer has excess inventory and places the MP3 player on sale for 10% off. What is the sale price and markdown amount?

Step 1: The selling price and markdown percent are $S = \$39.99$ and $d = 0.10$, respectively.

Step 2: Apply **Formula 4.4a**$S_{\text{onsale}} = \text{S} \times (1 - d)$ to calculate the sale price, resulting in:

$$S_{onsale} = \$39.99 \times (1{-}0.10)$$

$$S_{onsale} = \$35.99$$

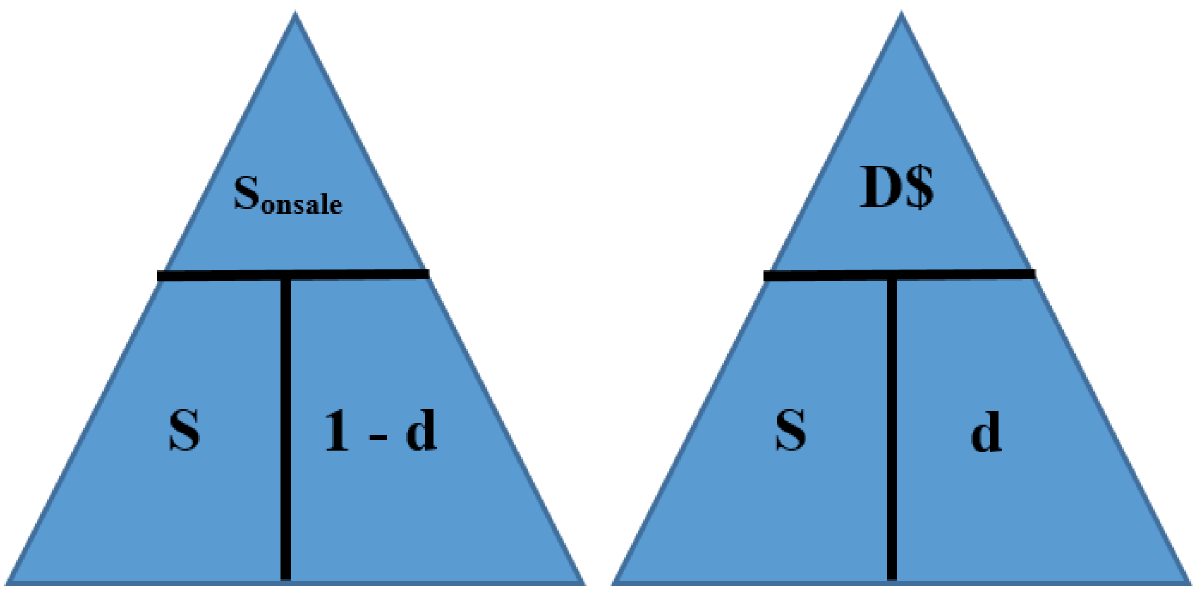
Step 3: You could use either of **Formula 4.4b**$\text{D\$} = \text{S} \times d$ or **Formula 4.4c**$\text{D\$} = \text{S} - \text{S}_{\text{onsale}}$ to calculate the markdown amount since the selling price, sale price, and markdown percent are all known. Arbitrarily choosing **Formula 4.4b**$\text{D\$} = \text{S} \times d$, you calculate a markdown amount of:

$$D\$ = \$39.99 \times 0.10$$

$$D\$ = \$4.00$$

Therefore, if the retailer has a 10% off sale on the MP3 players, it marks down the product by $\$4.00$ and retails it at a sale price of $\$35.99$.

Key Takeaways



Formula 6.10: The Sale Price of a Product

Formula 6.11a & 6.12: Markdown Amount and Rate

Figure 4.4.1

Avoid getting bogged down in formulas. Recall that the three formulas for markdowns are not new formulas, just adaptations of three previously introduced concepts. As a consumer, you are very experienced with endless examples of sales, bargains, discounts, blowouts, clearances, and the like. Every day you read ads in the newspaper and watch television commercials advertising percent savings. This section simply crystallizes your existing knowledge. If you are puzzled by questions involving markdowns, make use of your shopping experiences at the mall!

Paths To Success

Three of the formulas introduced in this section can be solved for any variable through algebraic manipulation when any two variables are known. Recall that the triangle technique helps you remember how to rearrange these formulas, as illustrated here.

Example 4.4.1

The MSRP for the "Guitar Hero: World Tour" video game is $\$189.99$. Most retail stores sell this product at a price in line with the MSRP. You have just learned that a local electronics retailer is selling the game for 45% off. What is the sale price for the video game and what dollar amount is saved?

Solution

Step 1: **Write what we know from the question.**

The regular selling price for the video game and the markdown rate are known:

$$S = \$189.99$$
$$d = 0.45$$

Step 2: **Calculate the sale price by applying Formula 4.4a.** $S_{\text{onsale}} = \text{S} \times (1 - d)$

$$S_{onsale} = \$189.99 \times (1 - 0.45)$$
$$S_{onsale} = \$189.99 \times 0.55$$
$$S_{onsale} = \$104.49$$

Step 3: **Calculate the markdown amount by applying Formula 4.4c.** $\text{D\$} = \text{S} - \text{S}_{\text{onsale}}$

$$D\$ = \$189.99 - \$104.49$$
$$D\$ = \$85.50$$

Step 4: **Write as a statement.**

The sale price for the video game is $\$104.49$. When purchased on sale, "Guitar Hero: World Tour" is $\$85.50$ off of its regular price.

Example 4.4.2

A reseller acquires an Apple iPad for $\$650$. Expenses are planned at 20% of the cost, and profits are set at $15\%$of the cost. During a special promotion, the iPad is advertised at $\$100$ off. What is the sale price and markdown percent?

Solution

Step 1: **Write what you know from the question.**

The pricing elements of the iPad along with the markdown dollars are known:

$$\begin{aligned} C &= \$650 \\ E &= 0.2C \\ P &= 0.15C \\ D\$ &= \$100 \end{aligned}$$

Step 2: **Calculate the selling price of the product by applying Formula 4.3a.**

$\mathrm{S = C + E + P}$

$$\begin{aligned} S &= \$650 + 0.2(\$650) + 0.15(\$650) \\ S &= \$877.50 \end{aligned}$$

Step 3: **Calculate the markdown percent by applying Formula 4.4d.** $d = \dfrac{\mathrm{D\$}}{\mathrm{S}} \times 100$

$$\begin{aligned} d &= \frac{\$100}{\$877.50} \times 100 \\ d &= 11.396\% \end{aligned}$$

Step 4: **Calculate the sale price by applying Formula 4.4c.**$\mathrm{D\$ = S - S_{onsale}}$.

$$\begin{aligned} \$100 &= \$877.50 - S_{onsale} \\ S_{onsale} &= \$777.50 \end{aligned}$$

Step 5: **Write as a statement.**

When the iPad is advertised at $\$100$ off, it receives an 11.396% markdown and it will retail at a sale price of $\$777.50$.

Never-Ending Sales

Have you noticed that some companies always seem to have the same item on sale all of the time? This is a common marketing practice. Recall the third and fourth circumstances for markdowns. Everybody loves a sale, so markdowns increase sales volumes for both the marked-down product and other regularly priced items.

Figure 4.4.2

For example, Michaels has a product line called the Lemax Village Collection, which has seasonal display villages for Christmas, Halloween, and other occasions. When these seasonal product lines come out, Michaels initially prices them at the regular unit selling price for a short period and then reduces their price. For Michaels, this markdown serves a strategic purpose. The company's weekly flyers advertising the Lemax Village Collection sale attract consumers who usually leave the store with other regularly priced items.

4.4e The Selling Price of a Product Adapted

If an item is on sale all the time, then businesses plan the pricing components with the sale price in mind. Companies using this technique determine the unit profitability of the product at the sale price and not the regular selling price. They adapt **Formula 4.3a**$\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$ as follows:

$$rgb]1.0,0.0,0.0S = rgb]0.0,0.0,1.0C + rgb]0.0,0.5,0.0E + rgb]0.5,0.0,0.5P \text{ becomes } rgb]0.68,0.46,0.12S_{onsale} = rgb]0.0,0.0,1.0C + rgb]0.0,0.5,0.0E + rgb]0.5,0.0,0.5P_{rgb]0.5,0.0,0.5onsale}$$

Where P_{onsale} represents the planned profit amount when the product is sold at the sale price. This is not a new formula, just a new application of **Formula 4.3a** $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$.

—

Under normal circumstances, when businesses set their selling and sale prices they follow a three-step procedure:

- Determine the product's cost, expenses, and profit amount.
- Set the regular selling price of the product.
- If a markdown is to be applied, determine an appropriate markdown rate or amount and set the sale price.

However, when a product is planned to always be on sale, businesses follow these steps instead to set the sale price and selling price:

HOW TO

Set the sale price and selling price

Step 1: Set the planned markdown rate or markdown dollars. Determine the pricing components such as cost and expenses. Set the profit so that when the product is marked down, the profit amount is achieved. Alternatively, a planned markup on cost, markup on selling price, or even markup dollars may be set for the sale price.

Step 2: Calculate the sale price of the product. If cost, expenses, and profit are known, apply **Formula 4.4e** $S_{onsale} = C + E + P_{onsale}$, the adapted version of **Formula 4.3a** $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$. Alternatively, adapt and apply any of the other markup formulas (**Formula 4.3b** $\mathrm{M\$} = \mathrm{E} + \mathrm{P}$ through **Formula 4.3e** $\mathrm{MoS\%} = \frac{\mathrm{M\$}}{\mathrm{S}} \times 100$) with the understanding that the result is the sale price of the product and not the regular selling price.

Step 3: Using the known markdown rate or markdown amount, set the regular selling price by applying any appropriate markdown formula (**Formula 4.4a** $S_{\mathrm{onsale}} = \mathrm{S} \times (1 - d)$ through **Formula 4.4d** $d = \frac{\mathrm{D\$}}{\mathrm{S}} \times 100$).

Assume for the Michael's Lemax Village Collection that most of the time these products are on sale for 40% off. A particular village item costs $\$29.99$, expenses are $\$10.00$, and a planned profit of $\$8.00$ is achieved at the sale price. Calculate the sale price and the selling price.

Step 1: The known variables at the sale price are:

$$C = \$29.99$$
$$E = \$10.00$$
$$P = \$8.00$$
$$d = 0.40$$

Step 2: Adapting **Formula 4.3a**$\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$, the sale price is:

$$S_{onsale} = C + \ E + P_{onsale}$$
$$S_{onsale} = \$29.99 + \$10.00 + \$8.00$$
$$S_{onsale} = \$47.99$$

This is the price at which Michael's plans to sell the product.

Step 3: However, to be on sale there must be a regular selling price. Therefore, if the 40% off results in a price of $\$47.99$, apply **Formula 4.4a**$S_{\mathrm{onsale}} = \mathrm{S} \times (1 - d)$ and rearrange to get the selling price:

$$S = \$47.99 \ \div (1 - \ 0.40)$$
$$S = \$79.98$$

Therefore, the product's selling price is $\$79.98$, which, always advertised at 40% off, results in a sale price of $\$47.99$. At this sale price, Michael's earns the planned $\$8.00$ profit.

Key Takeaways

You may ask, "If the product is always on sale, what is the importance of establishing the regular price?" While this textbook does not seek to explain the law in depth, it is worth mentioning that pricing decisions in Canada are regulated by the Competition Act. With respect to the discussion of never-ending sales, the Act does require that the product be sold at a regular selling price for a reasonable period of time or in reasonable quantity before it can be advertised as a sale price.

If you revisit the Michael's example, note in the discussion that the village initially needs to be listed at the regular selling price before being lowered to the sale price.

Try It

1) Answer the following questions:

a. If a product has a markup on cost of 40% and a markdown of 40%, will it sell above or below cost?
b. What happens to the profit if a product that is always on sale actually sells at the regular selling price?
c. Under normal circumstances, arrange from smallest to largest: regular selling price, cost, and sale price.

Solution

a. Below cost, since the 40% markdown is off of the selling price, which is a larger value.
b. The profit will be increased by the markdown amount.
c. Cost, sale price, regular selling price.

Example 4.4.3

An electronics retailer has 16GB USB sticks on sale at 50% off. It initially priced these USB sticks for a short period of time at regular price, but it planned at the outset to sell them at the sale price. The company plans on earning a profit of 20% of the cost when the product is on sale. The unit cost of the USB stick is $\$22.21$, and expenses are 15% of the cost.

a. At what price will the retailer sell the USB stick when it is on sale?
b. To place the USB stick on sale, it must have a regular selling price. Calculate this price.
c. If the USB stick is purchased at the regular selling price during the initial time period, how much profit is earned?

Solution

Step 1: **Write what you know from the question.**

You know the unit cost, the retailer's associated expenses, its planned profit at the sale price, and the markdown rate:

$$d = 0.50$$
$$C = \$22.21$$
$$E = 0.15C$$
$$P_{onsale} = 0.2C$$

Step 2: **To solve part a. apply Formula 4.4e.**

$$S_{onsale} = C + E + P_{onsale}$$
$$S_{onsale} = \$22.21 + 0.15(\$22.21) + 0.2(\$22.21)$$
$$S_{onsale} = \$22.21 + \$3.33 + \$4.44$$
$$S_{onsale} = \$29.98$$

Step 3: **After you know the sale price, solve part b. by applying Formula 4.4a.**

$S_{\text{onsale}} = \text{S} \times (1 - d)$

$$\$29.99 = S \times (1 - 0.5)$$
$$\$29.98 = S \times 0.5$$
$$\$59.96 = S$$

Step 4: **Solving part c. requires applying Formula 4.3a.** $\text{S} = \text{C} + \text{E} + \text{P}$

$$\$59.96 = \$22.21 + \$3.33 + P$$
$$\$59.96 = \$25.54 + P$$
$$\$34.42 = P$$

Step 5: **Write as a statement.**

The USB stick is on sale for $\$29.98$, letting the company achieve its profit of $\$4.44$ per unit. During the initial pricing period, the USB stick sells for $\$59.96$ (its regular selling price).

If a consumer actually purchases a USB stick during the initial pricing period, the electronics store earns a profit of $\$34.42$ per unit (which is a total of the $\$4.44$ planned profit plus the planned markdown amount of $\$29.96$).

Section 4.4 Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

For questions 1–6, solve for the unknown variables (identified with a ?) based on the information provided.

Table 4.4.1

	Regular Selling Price	Markdown Amount	Markdown Percent	Sale Price
1.	$\$439.85$	?	35%	?
2.	?	$\$100.00$	?	$\$199.95$
3.	$\$1,050.00$	?	?	$\$775.00$
4.	$\$28,775.00$	$\$3,250.00$	?	?
5.	?	?	33%	$\$13,199.95$
6.	?	$\$38.33$	12%	?

Solutions

1. .$D\$ = \$153.95$; $S_{onsale} = \$285.90$
2. $S = \$299.95$; $d = 33.3389\%$
3. $D\$ = \$275$; $d = 26.1905\%$
4. $S_{onsale} = \$25,525$; $d = 11.2945\%$
5. $S = \$19,701.42$; $D\$ = \$6,501.47$
6. $S = \$319.42$; $Sonsale = \$281.09$

Applications

7. A pair of Nike athletic shoes is listed at a regular selling price of $\$89.99$. If the shoes go on sale for 40% off, what is the sale price?
8. During its special Bay Days, The Bay advertises a Timex watch for $\$39.99$ with a regular price of $\$84.99$. Calculate the markdown percent and markdown amount.
9. For spring break you are thinking about heading to Tulum, Mexico. In planning ahead, you notice that a one-week stay at the Gran Bahia Principe Tulum, regularly priced at

$\$2,349$ for air and six nights all inclusive, offers an early-bird booking discount of $\$350$. What markdown percentage is being offered for booking early?

10. A Heritage Infusio deep frying pan is advertised at 70% off with a sale price of $\$39.99$. What is the frying pan's regular selling price, and what markdown amount does this represent?
11. A mass merchandiser uses its Lagostina cookware product line as a marketing tool. The cookware is always on sale at an advertised price of 45% off. The cost of the cookware is $\$199.99$, expenses are $\$75$, and the planned profit at the sale price is $\$110$. Calculate the sale price and selling price for the cookware.
12. Quicky Mart regularly sells its Red Bull sports drink for $\$2.99$ per can. Quicky Mart noticed that one of its competitors down the street sells Red Bull for $\$1.89$. What markdown percentage must Quicky Mart advertise if it wants to match its competitor?
13. A hardware store always advertises a Masterdesigner 75-piece screwdriver set at 80% off for a sale price of $\$17.99$.

    a. If the cost of the set is $\$10$ and expenses are 30% of the sale price, what is the planned profit when the product is on sale?
    b. What profit is earned if the product actually sells at its regular selling price?

14. A campus food outlet is advertising a "Buy one, get one 25% off" deal. The 25% off comes off the lower-priced item. If you purchase a chicken dinner for $\$8.99$ and your friend gets the burger combo for $\$6.99$, what is the markdown percentage on the total price?
15. Blast'em Stereos purchases a stereo system for $\$1,900$ less two discounts of 40% and 18%. The store uses this product to draw customers to the store and always offers the stereo on sale at 25% off. When the stereo is on sale, it plans on expenses equalling 30% of the cost and a profit of 20% of the sale price.

    a. What is the sale price for the stereo?
    b. How much profit does Blast'em make when the stereo sells at the sale price?
    c. By law, this stereo must sell at the regular selling price for a period of time before going on sale. What is the regular selling price?
    d. What profit does Blast'em earn if a customer purchases the stereo during this initial period?

Solutions

7. $S_{onsale} = \$53.99$
8. $D\$ = \$45$; $d = 52.9474\%$
9. $d = 14.9\%$
10. $S = \$133.30$; $D\$ = \$93.31$
11. $S_{onsale} = \$384.99$; $S = \$699.98$
12. $d = 36.7893$
13. a. $P_{onsale} = \$2.59$; b. $P = \$74.55$
14. $d = 10.9512\%$
15. a. $S_{onsale} = \$1,519.05$; b. $P_{onsale} = \$303.81$; c. $S = \$2,025.40$; d. $P = \$810.16$

Challenge, Critical Thinking, & Other Applications

16. Frigid Boards purchases one of its snowboards for $\$395$ less a retail trade discount of 15% and a loyalty discount of 4%. Its markup on selling price percentage on all snowboards is 21%. At the end of the season, any leftover snowboards are marked down by 10%. What is the sale price for the snowboard?
17. An HP LaserJet printer has an MSRP of $\$399.95$. It is subject to trade discounts of 30% and 23%. The LaserJet is a featured item for a computer store and is always on sale. The store plans to sell the LaserJet for a sale price that allows it to cover expenses equalling 15% of cost and realize a profit of $\$35.00$.

    a. What is the sale price?
    b. If the MSRP is the regular unit price of the printer, what rate of markdown can the computer store advertise?
    c. What markup on selling price percentage is realized at the sale price?

18. The Brick advertises that when you purchase a queen-size Tempur-Pedic mattress set for $\$2,499.97$ it will give you a 51" 3-D plasma television with a 3-D starter kit included. The value of this gift is $\$1,199.99$. What markdown percent does this represent?
19. A Maytag 27 cubic foot refrigerator retails for $\$2,400.00$ at Landover Appliance Centre. The company, which is celebrating its 30th anniversary this coming weekend, features the fridge for 30% off. The markup on selling price percentage on the fridge at the regular unit selling price is 53%.

    a. What is the sale price?

b. At the sale price, what is the markup on selling price percentage?
c. If the expenses are 15% of the regular selling price, what is the profit when the fridge is on sale?

20. Dreger Jewellers is selling a diamond bracelet. It uses this bracelet in its promotions and almost always has it on sale. The cost of the bracelet is $\$2,135$ less discounts of 20% and 30%. When the bracelet is on sale for 25% off, the expenses are 15% of cost and the profit is 20% of cost.
    a. What is the sale price?
    b. What is the bracelet's regular selling price?
    c. If the bracelet sells at the regular selling price, what are the markup amount and the markup on cost percent?

Solutions

16. $S_{onsale} = \$367.20$
17. a. $S_{onsale} = \$282.91$; b. $d = 29.2637\%$; c. $MoS\% = 23.8026\%$
18. $D = 32.4325\%$
19. a. $S_{onsale} = \$1,680$; b. $MoS\%$ when on sale = 32.8571%; c. $P_{onsale} = \$192.00$
20. a. $S_{onsale} = \$1,614.06$; b. $S = \$2,152.08$; c. $M\$ = \$956.48$; $MoC\% = 80\%$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $S_{onsale} = C + E + P_{onsale}$ $N = L \times (1 - d)$ $\mathrm{D\$} = \mathrm{S} \times d$ $D\$ = L \times d$ $S_{\mathrm{onsale}} = \mathrm{S} \times (1 - d)$ $\mathrm{D\$} = \mathrm{S} - \mathrm{S}_{\mathrm{onsale}}$ $d = \frac{\mathrm{D\$}}{\mathrm{S}} \times 100$ $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$ $\mathrm{M\$} = \mathrm{E} + \mathrm{P}$ Formula does not parse

---

Attribution

4.5: MERCHANDISING

Formula & Symbols Hub

In this section you will need the following:

Symbols Used

- $\times 100$ = Percentage multiplier
- C = Cost
- E = Expenses
- $\mathrm{M\$}$ = Markup amount
- $MoC\%$ = Markup on cost percentage
- $\mathrm{MoS\%}$ = Markup on selling price percentage
- P = Profit
- P_{red} = Reduced profit
- S_{red} = Reduced sale price

Formulas Used

- Formula 4.3a –**The Selling Price of a Product**

$$\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$$

- Formula 4.3b – **Markup Amount**

$$\mathrm{M\$} = \mathrm{E} + \mathrm{P}$$

- Formula 4.3d – **Markup on Cost Percentage**

$$MoC\% = \frac{\mathrm{M\$}}{\mathrm{C}} \times 100$$

- Formula 4.3e – **Markup on Selling Price Percentage**

$$\text{MoS\%} = \frac{\text{M\$}}{\text{S}} \times 100$$

- Formula 4.4b – **Markdown Amount**

$$\$M = S \times d$$

- Formula 4.4c – **Reduced Sale Price**

$$S_{red} = S - D$$

- Formula 4.5a – **Reduced Profit**

$$P_{red} = S_{red} - C - E$$

- Formula 4.5b – **Reduced Profit**

$$P_{red} = P - E$$

Introduction

Running a business requires you to integrate all of the concepts in this chapter. From discounts to markups and markdowns, all the numbers must fit together for you to earn profits in the long run. It is critical to understand how pricing decisions affect the various financial aspects of your business and to stay on top of your numbers.

Merchandising does not involve difficult concepts, but to do a good job of it you need to keep track of many variables and observe how they relate to one another. Merchandising situations in this section will apply all of the previously discussed pricing concepts. Next, the concept of maintained markup helps you understand the combined effect of markup and markdown decisions on list profitability. Finally, you will see that coupons and rebates influence expenses and profitability in several ways beyond the face value of the discounts offered.

The Complete Pricing Picture

Up to this point, you have examined the various components of merchandising as separate topics. Your study of basic product pricing has included the following:

- Taking an MSRP or list price and applying discounts to arrive at a product's cost
- Marking up a product by adding expenses and profits to the cost to arrive at a regular selling price
- Marking down a product by applying a discount and arriving at a sale price
- Working with various percentages in both markup and markdown situations to either simplify calculations or present a clearer pricing picture

Now is the time to tie all of these merchandising concepts together. Section 4.3 introduced the concept of a markup chart to understand the relationship between various markup components. The figure below extends this figure to include all of the core elements of merchandising.

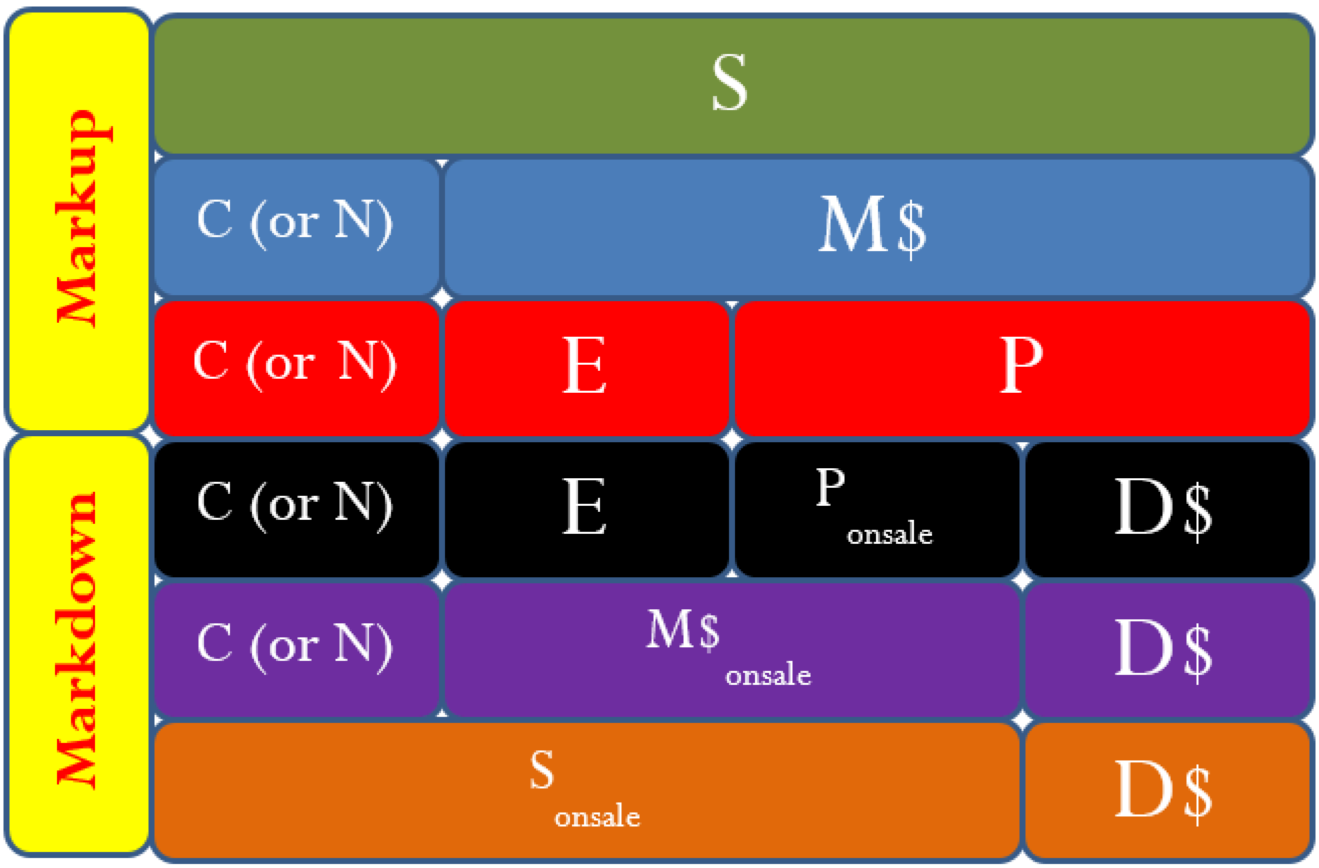


Figure 4.5.1

The figure reveals the following characteristics:

- You calculate the cost of a product by applying a discount formula. The net price, N, is synonymous with cost, C.
    - If Markdown (D) $<$ Profit (P) then the **reduced profit** (P_{red}) will be positive. Also, S_{red} will be greater than BE (breakeven). Therefore, there will be **reduced profit** at the sale price.
    - If Markdown (D) $<$ Profit (P) then the **reduced profit** (P_{red}) will be 0. Also, S_{red} will be

equal to BE (breakeven). Therefore, there will be **neither profit nor loss** at the sale price.

- If Markdown $(D) > \text{Profit}\,(P)$ then the **reduced profit** (P_{red}) will be negative. Also, S_{red} will be less than BE (breakeven). Therefore, there will be **loss** at the sale price.

- When you mark a product down, not only is the profit reduced but the markup dollars are also reduced. The following relationships exist when a product is on sale:

$$\begin{aligned} M\$_{onsale} &= \ E \ \ P_{onsale} \quad \text{OR} \\ M\$_{onsale} &= M\$ - D\$ \\ P_{onsale} &= P – D\$ \end{aligned}$$

- At break-even (remember, P or P_{onsale} is zero) a sale product has the following relationships:

$$\begin{aligned} M\$_{onsale} &= E \\ S_{onsale} &= CE \\ D\$ &= P \end{aligned}$$

None of these relationships represent new formulas. Instead, these reflect a deeper understanding of the relationships between the markup and markdown formulas.

4.5a Reduced Profit

So we can conclude that:

$$P_{red} = S_{red} – \text{ Breakeven } (BE)$$

$$rgb]0.0, 0.0, 1.0P_{rgb]0.0,0.0,1.0red} rgb]0.0, 0.0, 1.0\ rgb]0.0, 0.0, 1.0 = rgb]0.0, 0.0, 1.0S_{rgb]0.0,0.0,1.0red} rgb]0.0, 0.0, 1.0\ rgb]0.0, 0.0, 1.0 – rgb]0.0, 0.0, 1.0C rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.0E$$

4.5b Reduced Profit

We can also also know that:

$$
\begin{aligned}
S_{red} &= \text{Sale price } (S) - \text{discount } (D) \\
\text{substituting } P_{red} &= S - D - C - E \\
\text{regrouping } P_{red} &= (S - C - E) - D \\
P &= (S - C - E) \\
rgb]1.0,0.0,0.0P_{rgb]1.0,0.0,0.0red} rgb]1.0,0.0,0.0 &= rgb]1.0,0.0,0.0P rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0E
\end{aligned}
$$

The relationship of formulas 4.5a and 4.5b can also be represented using the Table below: a Merchandising Calculation Table.

Table 4.5.1

Merchandising Calculation Table			
	Amount	**Rate**	
		on C	On S
P			
$+E$			
M			
$+C$			
S			
$-D$			
S_{red}			
**			
**			
P_{red}			

**

$$rgb]0.0,0.0,1.0 - rgb]0.0,0.0,1.0B rgb]0.0,0.0,1.0E$$

HOW TO

Solve complete merchandising scenarios

Step 1: It is critically important to correctly identify both the known and unknown merchandising variables that you are asked to calculate.

Step 2: For each of the unknown variables in Step 1, examine Formula 4.1a through Formula 4.5b. Locate which formulas contain the unknown variable. You must solve one of these formulas to arrive at the answer. Based on the information provided, examine these formulas to determine which formula may be solvable. Write out this formula, identifying which components you know and which components remain unknown.

For example, assume that in Step 1 you were asked to solve for the expenses, or E. In looking at the formulas, you find this variable appears only in **Formula 4.3a** $\mathrm{S} = \mathrm{C} + \mathrm{E} + \mathrm{P}$ and **Formula 4.3b** $\mathrm{M\$} = \mathrm{E} + \mathrm{P}$, meaning that you must use one of these two formulas to calculate the expenses. Suppose that, in reviewing the known variables from Step 1, you already have the markup amount and the profit. In this case, **Formula 4.3b** $\mathrm{M\$} = \mathrm{E} + \mathrm{P}$ would be the right formula to calculate expenses with.

Step 3: Note the unknown variables among all the formulas written in Step 2. Are there common unknown variables among these formulas? These common variables are critical variables. Solving for these common unknowns is the key to completing the question. Note that these unknown variables may not directly point to the information you were requested to calculate, and they do not resolve the merchandising scenario in and of themselves. However, without these variables you cannot solve the scenario.

For example, perhaps in Step 1 you were requested to calculate both the markup on cost percent ($MoC\%$) and the markup on selling price percent ($MoS\%$). In Step 2, noting that these variables are found only in **Formula 4.3d** $MoC\% = \dfrac{\mathrm{M\$}}{\mathrm{C}} \times 100$ and **Formula 4.3e** $\mathrm{MoS\%} = \dfrac{\mathrm{M\$}}{\mathrm{S}} \times 100$, you wrote them down. Examination of these formulas revealed that you know the cost and selling price variables. However, in both formulas the markup dollars remain unknown.

Therefore, markup dollars is the critical unknown variable. You must solve for markup dollars using **Formula 4.3b** $\mathrm{M\$} = \mathrm{E} + \mathrm{P}$, which will then allow you to calculate the markup percentages.

Step 4: Apply any of the merchandising formulas from Formula 4.1a to Formula 4.5b to calculate the unknown variables required to solve the formulas. Your goal is to identify all required variables and then solve the original unknown variables from Step 1.

Things To Watch Out For

Before proceeding, take a few moments to review the various concepts and formulas covered earlier in this chapter. A critical and difficult skill is now at hand. As evident in Steps 2 through 4 of the solving process, you must use your problem-solving skills to figure out which formulas to use and in what order. Here are three suggestions to help you on your way:

- Analyze the question systematically.
- If you are unsure of how the pieces of the puzzle fit together, try substituting your known variables into the various formulas. You are looking for
- Any solvable formulas with only one unknown variable *or*
- Any pair of formulas with the same two unknowns, since you can solve this system using your algebraic skills of solving two linear equations with two unknowns (covered in Section 2.6).
- Merchandising has multiple steps. Think through the process. When you solve one equation for an unknown variable, determine how knowing that variable affects your ability to solve another formula. As mentioned in Step 3, there is usually a critical unknown variable. Once you determine the value of this variable, a domino effect allows you to solve any other remaining formulas

Example 4.5.1

A skateboard shop stocks a Tony Hawk Birdhouse Premium Complete Skateboard. Each skateboard costs $\$45.46$ and the store has overheads expenses of $\$25.17$ on each skateboard. The shop wants to sell each skateboard for an operating profit of $\$10.55$. During a sale, it offers a markdown rate of 10%. At the sale prices, calculate the store's profit or loss.

Solution

***Step 1:* List what you know.**

$$C = \$45.46$$
$$E = \$25.17$$
$$P = \$10.55$$
$$D = 25\%$$

Write the unknowns: $\$D,\ S,\ S_{red},\ P_{red}$

***Step 2:* Where can you find the information in known formulas?**

Using Formula 4.3a ($S = C + E + P$) will give us the selling price and the $\$D$.

$$\$D = 0.10S$$

Using Formula 4.4c, $S_{red} = S - D$, we can find the reduced sale price.

Using Formula 4.5a, $P_{red} = S_{red} - C - E$, we can find the reduced profit.

Using Formula 4.5b, $P_{red} = P - D$, we can find the reduced profit.

***Step 3:* Where do you find the unknown variables.**

Same as Step 2.

***Step 4:* Solve the Question: There are two methods to solve the question.**

Method 1

$$S = rgb]0.0, 0.5, 0.0C + rgb]0.0, 0.0, 1.0E + rgb]1.0, 0.0, 0.0P$$
$$S = rgb]0.0, 0.5, 0.0\$rgb]0.0, 0.5, 0.045rgb]0.0, 0.5, 0.0.rgb]0.0, 0.5, 0.046 + rgb]0.0, 0.0, 1.0\$rgb]0.0, 0.0, 1.025rgb]0.0, 0.0, 1.0.rgb]0.0, 0.0, 1.017 + rgb]1.0, 0.0, 0.0\$rgb]1.0, 0.0, 0.010rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.055$$
$$S = \$81.18$$

$$D = 0.10rgb]0.5, 0.0, 0.5S$$
$$D = 0.10 \times rgb]0.5, 0.0, 0.5\$rgb]0.5, 0.0, 0.581rgb]0.5, 0.0, 0.5.rgb]0.5, 0.0, 0.518$$
$$D = \$8.12$$

$$S_{red} = rgb]0.68, 0.46, 0.12S - rgb]1.0, 0.0, 1.0D$$
$$S_{red} = rgb]0.68, 0.46, 0.12\$rgb]0.68, 0.46, 0.1281rgb]0.68, 0.46, 0.12.rgb]0.68, 0.46, 0.1218 - rgb]1.0, 0.0, 1.0\$rgb]1.0, 0.0, 1.08rgb]1.0, 0.0, 1.0.rgb]1.0, 0.0, 1.012rgb]1.0, 0.0, 1.0$$
$$S_{red} = \$73.06$$

$$P_{red} = rgb]1.0, 0.0, 0.0S_{rgb]1.0,0.0,0.0red} - rgb]0.0, 0.5, 0.0C - rgb]0.0, 0.0, 1.0E$$
$$P_{red} = rgb]1.0, 0.0, 0.0\$rgb]1.0, 0.0, 0.073rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.006 - rgb]0.0, 0.5, 0.0\$rgb]0.0, 0.5, 0.045rgb]0.0, 0.5, 0.0.rgb]0.0, 0.5, 0.046 - rgb]0.0, 0.0, 1.025rgb]0.0, 0.0, 1.0.rgb]0.0, 0.0, 1.017rgb]0.0, 0.0, 1.0$$
$$P_{red} = \$2.43$$

Method 2

Table 4.5.2

Merchandising Calculation Table			
	Amount	**Rate**	
		on C	On S
P	\$10.55		
$+E$	\$25.17		
M	\$35.72		
$+C$	\$45.46		
S	\$81.18		
$-D$	**\$8.12**		0.10
S_{red}	\$73.06		
**	\$45.46		
**	\$25.17		
P_{red}	**\$2.43**		

**

$$rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.0Brgb]0.0, 0.0, 1.0E$$

Step 5: **Write as a statement.**

At the sale price of $\$73.06$ the skateboard shop will make a profit of $\$2.43$.

Example 4.5.2

The Cycle Corner bought a bicycle for $\$500$, the overhead expense were 20% on the cost, and the operating profit is 30% on the cost. The bicycle is marked down by 25% during a sale.

a. What is the profit or loss on the reduced sale price?
b. What is the amount of markup on sale price?

Solution

a.

***Step 1:* List what you know from the question.**

$$C = \$500$$
$$E = 0.20C$$
$$P = 0.30\ C$$
$$D = .25S$$

Write the unknowns: S,\;D,\;E,\;P,\;\text{Markup on Sale Price}

***Step 2:* Where can your find the information in known formulas?**

Using Formula 4.3a, $S = C + E + P$, will give us the selling price.

Using Formula 4.4c, $S_{red} = S - D$, will give us the reduced sale price.

Using Formula 4.5a, $P_{red} = S_{red} - C - E$, will give us the reduced profit.

Using Formula 4.5b, $P_{red} = P - D$, will give us the reduced profit.

***Step 3:* Where do you find the unknown variables:**

Same as Step 2.

***Step 4:* Solve the Question: There are two methods to solve the question.**

Method 1

$$rgb]0.0,0.5,0.0\text{Cost} = rgb]0.0,0.5,0.0\$rgb]0.0,0.5,0.0500$$
$$\$E = 0.20(\$500)$$
$$rgb]0.0,0.0,1.0\$rgb]0.0,0.0,1.0E = rgb]0.0,0.0,1.0\$rgb]0.0,0.0,1.0100$$

$$\$P = 0.30(\$500)$$
$$rgb]1.0,0.0,0.0\$rgb]1.0,0.0,0.0P = rgb]1.0,0.0,0.0\$rgb]1.0,0.0,0.0150rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.000$$

Using Formula 4.3a: $S = rgb]0.0,0.5,0.0C + rgb]0.0,0.0,1.0E + rgb]1.0,0.0,0.0P$
$$S = rgb]0.0,0.5,0.0\$rgb]0.0,0.5,0.0500 + rgb]0.0,0.0,1.0\$rgb]0.0,0.0,1.0100 + rgb]1.0,0.0,0.0\$rgb]1.0,0.0,0.0150rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.000$$
$$rgb]0.5,0.0,0.5S = rgb]0.5,0.0,0.5\$rgb]0.5,0.0,0.5750rgb]0.5,0.0,0.5.rgb]0.5,0.0,0.500$$

$$\$D = 0.25rgb]0.5,0.0,0.5S$$
$$\$D = 0.25(rgb]0.5,0.0,0.5\$rgb]0.5,0.0,0.5750)$$
$$rgb]1.0,0.0,1.0\$rgb]1.0,0.0,1.0D = rgb]1.0,0.0,1.0\$rgb]1.0,0.0,1.0187rgb]1.0,0.0,1.0.rgb]1.0,0.0,1.050$$

$$S_{red} = rgb]0.5,0.0,0.5S - rgb]1.0,0.0,1.0D$$
$$S_{red} = rgb]0.5,0.0,0.5\$rgb]0.5,0.0,0.5750rgb]0.5,0.0,0.5.rgb]0.5,0.0,0.500 - rgb]1.0,0.0,1.0187rgb]1.0,0.0,1.0.rgb]1.0,0.0,1.050$$
$$rgb]0.68,0.46,0.12S_{rgb]0.68,0.46,0.12red} = rgb]0.68,0.46,0.12\$rgb]0.68,0.46,0.12562rgb]0.68,0.46,0.12.rgb]0.68,0.46,0.1250$$

$$P_{red} = rgb]0.68,0.46,0.12S_{rgb]0.68,0.46,0.12red} - rgb]0.0,0.5,0.0C - rgb]0.0,0.0,1.0E$$
$$P_{red} = rgb]0.68,0.46,0.12\$rgb]0.68,0.46,0.12562rgb]0.68,0.46,0.12.rgb]0.68,0.46,0.1250 - rgb]0.0,0.5,0.0\$rgb]0.0,0.5,0.0500 - rgb]0.0,0.0,1.0\$rgb]0.0,0.0,1.0100$$
$$P_{red} = \$ - \$37.50$$

Method 2

Table 4.5.3

Merchandising Calculation Table

	Amount	**Rate**	
		on C	On S
P	\$150.00	30%	
$+E$	\$100.00	20%	
M	\$250.00		
$+C$	\$500		
S	\$750.00		
$-D$	**\$187.50**		0.25
S_{red}	**\$562.50**		
**	\$500		
**	\$100		
P_{red}	**-\$37.50**		

**

$$rgb]0.0,0.0,1.0 - rgb]0.0,0.0,1.0Brgb]0.0,0.0,1.0E$$

b.

$$\text{Markup on Sale Price} = S_{red} - C$$
$$\text{Markup on Sale Price} = \$562.50 - \$500$$
$$\text{Markup on Sale Price} = \$62.50$$

***Step 5:* Write as a statement:**

At the sale price of $\$562.50$ the Cycle Corner will have a loss of $\$37.50$ and the Markup amount on the sale price is $\$62.50$.

Example 4.5.3

A manufacturer makes laptop computers at a cost of $\$1794$ per machine. The Company's operating profit is 20% on the selling price and the markup is 35% on the selling price. During a computer trade show the company offered the computer at a discount of 17.5%.

a. What is the selling price of the machine?
b. At the 17.5% discount did the company have a profit or a loss and how much?
c. What should be the rate of markdown to sell the machines at the break-even price?

Solution

***Step 1:* List what you know.**

$$\text{Cost } (C) = \$1794$$
$$\text{Profit } (P) = 20\% \text{ on selling price}$$
$$\text{Markup } (M) = 35\% \text{ on Selling price}$$

Write the unknowns: $D,\ S,\ M,\ P,\ S_{red}\ ,\ P_{red}$

***Step 2:* Where can you find the information in known formulas?**

Using Formula 4.3a, $S = C + E + P$, will give us the selling price.

Using Formula 4.3b, $M = E + P$, will give us the Markup.

Using Formula 4.4c, $S_{red} = S - D$, will give us the reduced sale price.

Using Formula 4.5a, $P_{red} = S_{red} - C - E$, will give us the reduced profit.

Using Formula 4.5b, $P_{red} = P - D$, will give us the reduced profit.

Step 3: **Where do you find the unknown variables:**

Same as Step 2.

Step 4: **Solve the Question: There are two methods to solve the question.**

a.

Selling Price

$$S = rgb]0.0, 0.5, 0.0C + rgb]1.0, 0.0, 0.0Mrgb]1.0, 0.0, 0.0$$
$$S = rgb]0.0, 0.5, 0.0\$rgb]0.0, 0.5, 0.01794 + rgb]1.0, 0.0, 0.00rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.035rgb]1.0, 0.0, 0.0Srgb]1.0, 0.0, 0.0$$
$$1S \ - 0\ .35S = \$1794$$
$$0.65S = \$1794$$
$$rgb]0.0, 0.0, 1.0S = rgb]0.0, 0.0, 1.0\$rgb]0.0, 0.0, 1.02rgb]0.0, 0.0, 1.0,rgb]0.0, 0.0, 1.0760rgb]0.0, 0.0, 1.0.rgb]0.0, 0.0, 1.000$$

---

b. Profit or Loss

Markup

$$M = 0.35S$$
$$M = 0.35(\$2760.00)$$
$$rgb]1.0, 0.0, 0.0M = rgb]1.0, 0.0, 0.0\$rgb]1.0, 0.0, 0.0966$$

Profit

$$P = 0.20S$$
$$P = 0.20\ (\$2760)$$
$$rgb]0.5, 0.0, 0.5P = rgb]0.5, 0.0, 0.5\$rgb]0.5, 0.0, 0.5552$$

Expenses

$$E = M - P$$
$$E = \$966 – \$552$$
$$rgb]0.0, 0.0, 1.0E = rgb]0.0, 0.0, 1.0\$rgb]0.0, 0.0, 1.0414$$

Discount

$$D = 0.175(\$2760)$$
$$rgb]1.0, 0.0, 1.0D = rgb]1.0, 0.0, 1.0\$rgb]1.0, 0.0, 1.0483$$

Selling Price Reduced

$$S_{red} = S – D$$
$$S_{red} = \$2760 - \$483$$
$$rgb]0.68, 0.46, 0.12S_{rgb]0.68,0.46,0.12red} = rgb]0.68, 0.46, 0.12\$rgb]0.68, 0.46, 0.122277$$

Reduced Profit

$$P_{red} = rgb]0.68, 0.46, 0.12S_{rgb]0.68,0.46,0.12red} - rgb]0.0, 0.5, 0.0C - rgb]0.0, 0.0, 1.0\ rgb]0.0, 0.0, 1.0E$$
$$P_{red} = rgb]0.68, 0.46, 0.12\$rgb]0.68, 0.46, 0.122277 - rgb]0.0, 0.5, 0.0\$rgb]0.0, 0.5, 0.01794 - rgb]0.0, 0.0, 1.0\$rgb]0.0, 0.0, 1.0414$$
$$P_{red} = +\$69.00$$

---

c. Rate of Markdown at Break-Even Price

Method 1

Break even is when C + E

$$BE = C + E$$
$$BE = \$1794 \; + \; \$414$$
$$BE = \$2208.00$$

$$\text{Markdown } (D) = \text{Selling Price} - BE$$
$$\text{Markdown } (D) = \$2760 \; - \; \$2208$$
$$\text{Markdown } (D) = \$552$$

$$\text{Markdown Rate} = \frac{\$552}{\$2760} \times 100\%$$
$$\text{Markdown Rate} = 20\%$$

Method 2

Table 4.5.4

Merchandising Calculation Table			
	Amount	**Rate**	
		on C	On S
P	$552		20%
$+E$	$414		15%
M	$966		35%
$+C$	$1794		
S	$2760		
$-D$	**-$483**		17.5%
S_{red}	**$2277**		
**	-$1794		
**	-$414		
P_{red}	**-$69.00**		

**

$rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.0 B rgb]0.0, 0.0, 1.0 E$

***Step 5:* Write a Statement**

The selling price of the computer is $\$2760$ and with a 17.5% discount at the trade show, there is a profit of $\$69.00$. The rate of markdown if the computer is sold at the break-even price is 20%.

Example 4.5.4

Carlton Printers manufactures and sells printers for $\$108.50$ each. The operating profit is 30% on cost and markup is 75% on cost. During the back to school sale the printer is offered at a rate of markdown that will provide a profit of $\$5.58$ per printer.

a. How much does it cost Carlton Printers to manufacture each printer?
b. Calculate the rate of markdown offered during a sale if a profit of $\$5.58$ is made in profit on each printer.
c. What should be the rate of markdown offered if the printer is selling at its cost price?

Solution

***Step 1:* List what you know.**

$$S = \$108.50$$
$$P = 0.30C$$
$$M = 0.75C$$

***Step 2:* Where can you find the information in known formulas?**

Using Formula 4.3a, $S = C + E + P$, will give us the selling price.

Using Formula 4.3b, $M = E + P$, will give us the Markup.

Using Formula 4.4c, $S_{red} = S - D$, will give us the reduced sale price.

Using Formula 4.5a, $P_{red} = S_{red} - C - E$, will give us the reduced profit.

Using Formula 4.5b, $P_{red} = P - D$, will give us the reduced profit.

***Step 3:* Where do you find the unknown variables:**

Same as Step 2.

***Step 4:* Solve the questions.**

a. Cost of the Printer

$$\begin{aligned} S &= \$108.50 \\ P &= 0.30C \\ M &= 0.75C \end{aligned}$$

$$\begin{aligned} S &= C + M \\ \$108.50 &= C + 0.75C \\ \$108.50 &= 1.75C \\ C &= \$62.00 \text{ Each printer cost is } \$62.00 \end{aligned}$$

---

b. Rate of Markdown with $\$5.58$ in profit after discount.

$$\begin{aligned} M &= E + P \\ 0.75C &= E \ + \ .30C \\ E &= 0.45C \\ E &= 0.45(\$62.00) \\ E &= \$27.90 \end{aligned}$$

$$\begin{aligned} P_{red} &= S_{red} – C – E \\ \$5.58 &= S_{red} - \$62.00 - \$27.90 \\ S_{red} &= 95.48 \end{aligned}$$

$$\begin{aligned} S_{red} &= S – D \\ 95.48 &= \$108.50 – D \\ D &= \$13.02 \end{aligned}$$

$$\begin{aligned} \text{Rate of Markdown} &= \frac{\$13.02}{\$108.50} \times 100\% \\ \text{Rate of Markdown} &= 12\% \end{aligned}$$

---

c. Rate of Markdown with a price at cost.

Method 1

$$\begin{aligned} S_{red} &= C = \$62.00 \\ S_{red} &= S – D \\ D &= S - S_{red} \\ D &= \$108.50 – \$62.00 \\ D &= \$46.50 \end{aligned}$$

$$\text{Rate of Markdown} = \frac{\$46.50}{\$108.50} \times 100\%$$

$$\text{Rate of Markdown} = 42.86\%$$

Method 2

Table 4.5.5

Merchandising Calculation Table			
	Amount	**Rate**	
		on C	On S
P	$18.60	30%	
$+E$	$27.90		
M	$46.50	75%	
$+C$	$62.00		
S	$108.50		
$-D$	$13.02		12%
S_{red}	$95.48		
**	-$62.00		
**	-$27.90		
P_{red}	$5.58		

**

$rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.0 B rgb]0.0, 0.0, 1.0 E$

Step 5: **Write a Statement**

The cost for each printer for Carlton Printers is $\$62.00$. With a profit of $\$5.58$ per printer the rate of markdown would be 12%.

Example 4.5.5

A retailer of electronic goods, purchases 1000 TVs from a distributor after trade discounts of 10% and 8% on the list price of $\$600$ per TV. The retailer then marks up the TVs by 55% on selling price and sells 660 units at the regular selling price. During a sale, they offer a markdown of 10% and sell another 300 units at that sale price. They finally sell the remaining units at the break-even price. Overhead expenses for the retailer are 20% of the selling price.

a. What is the selling price of the 660 units sold?
b. What was the price of the 300 units sold?
c. What is the break-even price at which the final units were sold?
d. What is the total profit or loss from the sale of the 1000 units?

Solution

Step 1: **List what you know.**

$$L = \$600$$
$$M = 0.55S$$
$$D = 0.10S$$
$$E = 0.20S$$

Step 2: **Where can you find the information in known formulas?**

Using Formula 4.1d, $\$D = L \times (1-d)(1-d)\ldots$, will give us the discount amount.

Using Formula 4.3c, $S = C + M$, will give us the selling price, M and P.

Using Formula 4.4b, $\$M = S \times d$, will give us the Markdown amount.

Using Formula 4.4c, $S_{red} = S - D$, will give us the reduced sale price.

Using Formula 4.5a, $P_{red} = S_{red} - C - E$, will give us the reduced profit.

Using Formula 4.5b, $P_{red} = P - D$, will give us the reduced profit.

Step 3: **Where do you find the unknown variables?**

Same as Step 2.

Step 4: **Solve the questions.**

Method 1

a.

$$\begin{aligned} \text{Discount Amount} &= L\ x\ (1-d)(1-d)\ldots \\ \text{Discount Amount} &= \$600(1-0.10)(1-0.08) \\ \text{Discount Amount} &= \$600(0.9)(0.92) \\ \text{Discount Amount} &= \$496.80 \end{aligned}$$

$$\begin{aligned} S &= C + M \\ S &= \$496.80 + 0.55S \\ 0.45S &= \$496.80 \\ S &= \$1104.00 \\ \text{660 units sold at } \$1104.00 \end{aligned}$$

---

b.

$$\begin{aligned} \text{Markdown Amount} &= S \times d \\ \text{Markdown Amount} &= \$1104.00 \times (0.10)) \\ \text{Markdown Amount} &= \$110.40 \end{aligned}$$

$$\begin{aligned} S_{red} &= S - D \\ S &= \$1104.00 - \$110.40 \\ S &= \$993.60 \\ \text{300 Units sold at } \$993.00 \end{aligned}$$

---

c.

$$\begin{aligned}
BE \text{ price} &= C + E \\
BE \text{ price} &= \$496.80 + (0.20 \times \$1104.00) \\
BE \text{ price} &= \$496.80 + \$220.80 \\
BE \text{ price} &= \$717.60
\end{aligned}$$

The BE price at which the remaining TVs were sold is $\$717.60$.

---

d. Profits from all TVs.

$$\text{Total cost of 1000 Units} = 1000 \times 496.80 = \$496,800.00$$

$$\text{Total operating expenses} = 1000 \times 0.20 \times \$1104.00 = \$220,800.00$$

$$\text{Total Selling price} = (660 \times \$1104.00) + (300 \times \$993.00) + (40 \times \$717.60)$$
$$\text{Total Selling price} = \$1,055,424.00$$

$$\text{Total Profit} = rgb]1.0, 0.0, 0.0S - rgb]0.0, 0.5, 0.0C - rgb]0.0, 0.0, 1.0E$$
$$\text{Total Profit} = rgb]1.0, 0.0, 0.0\$rgb]1.0, 0.0, 0.01rgb]1.0, 0.0, 0.0,rgb]1.0, 0.0, 0.0055rgb]1.0, 0.0, 0.0,rgb]1.0, 0.0, 0.0424rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.000 - rgb]0.0, 0.5, 0.0\$rgb]0.0, 0.5, 0.0496rgb]0.0, 0.5, 0.0,rgb]0.0, 0.5, 0.0800rgb]0.0, 0.5, 0.0.rgb]0.0, 0.5, 0.000 - rgb]0.0, 0.0, 1.0\$rgb]0.0, 0.0, 1.0220rgb]0.0, 0.0, 1.0,rgb]0.0, 0.0, 1.0800rgb]0.0, 0.0, 1.0.rgb]0.0, 0.0, 1.000$$
$$\text{Total Profit} = \$337,824.00$$

Method 2

Table 4.5.6

Merchandising Calculation Table			
	Amount	**Rate**	
		on C	On S
P	$rgb]1.0, 0.0, 0.0\boxed{\$386.40}$		
$+E$	$220.80		20%
M	$607.20		55%
$+C$	$496.80		
S	$1104.00		
$-D$	$110.40		10%
S_{red}	$993.60		
**	-$496.80		
**	-$220.80		
P_{red}	$rgb]1.0, 0.0, 0.0\boxed{\$276.00}$		

**

$rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.0 B rgb]0.0, 0.0, 1.0 E$

*Boxed values indicate profit for each unit sold at regular price (660 units)

$$\text{Total Profit} = (600 \times \$386.40)\ +\ (300 \times \$276.00)\ +\ (40 \times 00.00)$$
$$\text{Total Profit} = \$337,824.00$$

Step 5: **Write a Statement**

The selling price of the 660 units sold at regular price was $\$1104.00$. The sale price of the 300 units sold was $\$993.60$. The break-even price of the remaining 40 units sold was $\$717.60$. Finally, the total profit made from the sale of all 1000 TVs was $\$337,824.00$.

Section 4.5 Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

1. Calculate the missing values for the table below:

	Cost	Amount of Markup	Selling Price	Operating Profit	Overhead Expenses	Rate of Markup on Cost
a.	$12.50	?	$18.50	$4.00	?	?
b.	?	$180.50	$750.50	?	$45.75	?
c.	?	?	$12,400	?	$1280.50	?
d.	?	?	?	$0.25	?	30%
e.	?	?	?	$40.50	$80.50	?

2. Ziggys skates purchases ice skates for $\$30$ each pair and sells them at a regular price of $\$42$ each pair.
   a. If the profit made is $\$5.25$ per pair of ice skates, calculate the overhead expense per pair.
   b. If the discount offered during a Holiday Sale is 20%, calculate the reduced selling price and the profits or loss made on the sale of each pair.
3. Belanger Acoustics purchased acoustic guitars for $\$80$ each and has marked them up by 20% on cost. The overhead expenses were 10% on cost.
   a. Calculate the regular selling price of each guitar and the profit made.
   b. If Belanger decides to offer a markdown of 5% what would be the reduced selling price and profit or loss they would make on the sale of each guitar?
4. Julia makes a 10% profit on the cost of T shirts, which she purchases at $\$15.00$ each. The overhead expenses are 25% on cost. During a slae, she maked the T shirts down by 10%.

a. What is the profit or loss on the sale of this item?
b. What is the amount of markup on the sale price of the item?

5. Guilherme was in the business of purchasing painting from Brazil and selling them in Toronto at his shop. On one shipment listed at $\$2400$, he received trade discounts of 10%, 8%, and 6%. The overhead expenses were 15% of his costs and he wanted to make an operating profit of 20% on cost.

   a. Calculate the regular selling price of the painting.
   b. Calculate the loss or profit he will make if he decides to markdown the selling prices by 15%.
   c. Calculate the maximum amount of markdown that he can offer so that he breaks even on the sale.

6. Juanita purchased designer purses for $\$242.88$ each, less 12% and 8%. The markup is 35% on selling price and the operating profit is 15% on cost. During a sale , the designer purses were marked down to $\$260.00$.

   a. What was the regular selling price?
   b. What was the rate of markdown?
   c. At the sale price, what was the profit of loss?

7. The regular selling price of cell phones at a store is $\$125$ each. During a sale, it was sold at a markdown of 45%. Calculate the profit or loss made on the sale of the cell phone if the break-even price is $\$75$.
8. A retailer purchased shirts for $\$50$ each, less 10%. The retailer has a markup of 20% on selling price and an operating profit of 10% on cost. During a sale, the shirts were marked down and sold at break-even price.

   a. What was the regular selling price of each shirt?
   b. What was the sale price?
   c. What was the rate of markdown offered during the sale?

Solutions

1. a. $\$6.00$, $\$2.00$, 48%, 32.43%, 28.65%, $\$13.20$;
   b. $\$570.00$, $\$134.75$, 31.67%, 24.05%, 5.33%, $\$94.75$;
   c. $\$9548$, $\$2852$, $\$1571.50$, 29.87%, $\$11,284$, $\$455.50$;

   d. $\$0.45$, $\$1.95$, $\$0.20$, 23.08%, $\$1.84$, $\$0.14$
2. a. $\$6.75$; b. $\$33.60$ and $-\$3.15$
3. a. $\$96.00$, $\$8.00$; b. $\$91.20$, $\$3.20$
4. a. $-\$0.53$; b. $\$3.22$
5. a. $\$2521.76$; b. 3.52%; c. 25.93%
6. a. $\$302.52$; b. 14.06%; c. $-\$13.02$
7. $-\$6.25$
8. a. $\$56.25$; b. $\$51.75$; c. 8%

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $\mathrm{M\$} = \mathrm{E} + \mathrm{P}$
$MoC\% = \frac{\mathrm{M\$}}{\mathrm{C}} \times 100$ $\mathrm{MoS\%} = \frac{\mathrm{M\$}}{\mathrm{S}} \times 100$

Attribution

4.6: COST-REVENUE-NET INCOME ANALYSIS

Formula & Symbols Hub

For this section you will need the following:

Symbols Used

- $\times 100$ = Percentage multiplier
- $\%C$ = Percent change
- CM = Unit contribution margin
- CR = Contribution rate
- n = Level of output
- NI = Net income
- $n(CM)$ = Total contribution margin
- $n(S)$ = Total revenue
- S = Unit selling price
- SAvg = Simple Average
- TR = Total revenue
- TFC = Total fixed costs
- $TFC + n(VC)$ = Total costs
- TVC = Total variable cost
- VC = Unit variable cost
- V_{i} = Initial value
- V_{f} = Final value

Formulas Used

- Formula 2.4a – **Simple Average**

$$\text{SAvg} = \frac{\sum x}{n}$$

- Formula 3.2a – **Percent Change**

$$\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$$

- Formula 4.6a – **Unit Variable Cost**:

$$VC = \frac{TVC}{n}$$

- Formula 4.6b – **Net Income Using a Total Revenue and Cost Approach**

$$NI = n(S) - (TFC + n(VC))$$

- Formula 4.6c – **Unit Contribution Formula**

$$CM = S - VC$$

- Formula 4.6d – **Net Income Using Total Contribution Margin Approach Formula**

$$NI = n(CM) - TFC$$

- Formula 4.6e – **Contribution Rates Formula**

$$CR = \frac{CM}{S} \times 100$$

- Formula 4.6f – **Contribution Rate if Aggregate Information Is Known**

$$CR = \frac{TR - TVC}{TR} \times 100$$

Introduction

The end of the month is approaching, and bills are coming due. As you sit at your kitchen table trying to figure out your budget for next month, you wonder whether you will be able to afford that concert you had been planning on attending. Some of your costs remain unchanged from month to month, such as your rent, Internet service, gym membership, and insurance. Other costs tend to fluctuate with your usage, such as your utilities, cellphone bill, vehicle fuel, and the amount of money spent on recreational activities. Together, these regular and irregular costs total to next month's costs.

Examining a few recent paycheque stubs, you calculate the average monthly net income you bring home from your hourly cashier position at Sobeys. The exact amount of each paycheque, of course, depends on how many hours you work. Besides your short-term costs, you need to start saving for next year's tuition. Therefore, your budget needs to include regular deposits into your savings account to meet that goal. Once you have put your bills, paycheques, and goals together, you hope that your budget will balance. If there is a shortfall, you will have to miss out on those concert tickets.

Budgeting at work is no different in principle from your home budget. Businesses also need to recognize the different types of costs they incur each month, some of which remain the same and some of which fluctuate. Businesses need to pay for these costs by generating revenues, which correspond to your paycheque. As with your education goals, businesses also require profits to grow. A business needs to understand all of these numbers so it can plan its activities realistically.

This section explores the various types of costs and establishes a model relating total costs to total revenues to determine total profitability levels. You will then apply this model to see how the sale of an individual product contributes to covering costs and how each product individually contributes to overall profitability.

Types of Costs

A **cost** is an outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services. Costs can come in three forms:

1. A **fixed cost** is a cost that does not change with the level of production or sales (call this "output" for short). In other words, whether the business outputs nothing or outputs $10,000$ units, these costs remain the same. Some examples include rent, insurance, property taxes, salaries unrelated to production (such as management), production equipment, office furniture, and much more. **Total fixed costs** are the sum of all fixed costs that a business incurs.
2. A **variable cost** is a cost that changes with the level of output. In other words, if the business outputs nothing there is no variable cost. However, if the business outputs just one unit (or more) then a cost appears. Some examples include material costs of products, production labor (hourly or piecework wages), sales commissions, repairs, maintenance, and more. **Total variable costs** are the sum of all

variable costs that a business incurs at a particular level of output.

3. A **blended cost** is a cost that comprises both fixed cost and variable cost components. In other words, a portion of the total cost remains unchanged while another portion depends on the output. For calculation purposes, you must separate a blended cost into its fixed and variable cost components. A few examples will illustrate the concept of blended costs:

    - Residential natural gas bills from Manitoba Hydro include a fixed charge per month of $\$14$ plus charges for cubic meters of actual consumption based on transportation, distribution, and primary and supplemental gas rates. In this situation, the $\$14$ is a fixed cost while the actual consumption of natural gas is a variable cost.
    - A cellphone bill includes a fixed charge for the phone service plus any additional charges for usage, such as long distance, texting, or data.
    - If employees are paid a salary plus commission, then their salaries represent fixed costs while their commissions are a variable cost.

4.6a Unit Variable Cost

$$rgb]0.1,0.1,0.1\text{Unit Variable Cost: }rgb]1.0,0.0,0.0V rgb]1.0,0.0,0.0C = \frac{rgb]0.0,0.0,1.0T rgb]0.0,0.0,1.0V rgb]0.0,0.0,1.0C}{rgb]0.0,0.5,0.0n}$$

In calculating business costs, fixed costs are commonly calculated on a total basis only since the business incurs these costs regardless of any production. However, variable costs are commonly calculated both on a total and per-unit basis to reveal the overall cost along with the cost associated with any particular unit of output. When these variable costs are assigned on an individual basis it is called a **unit variable cost**. The calculation of unit variable cost has a further benefit because it allows managers to explore how the total business costs vary at different levels of output.

$rgb]1.0,0.0,0.0V rgb]1.0,0.0,0.0C rgb]0.1,0.1,0.1$ $rgb]0.1,0.1,0.1$is Unit Variable Cost: The unit variable cost is an adaptation of **Formula 2.4a** $\text{SAvg} = \frac{\sum x}{n}$ (Simple Average) with specific definitions for the data and the quantity. The end result of the calculation is the typical or average variable cost associated with an individual unit of output. Being a dollar cost, the unit variable cost is rounded to two decimals.

$rgb]0.0,0.0,1.0T rgb]0.0,0.0,1.0V rgb]0.0,0.0,1.0C rgb]0.1,0.1,0.1$ $rgb]0.1,0.1,0.1$is Total Variable Cost: The total variable costs in dollars that were incurred at a particular level of output. In the simple average formula, this is represented by the symbol $\sum_x$, which stands for the total of all pieces of data.

{\color[rgb]{0.0, 0.5, 0.0}n}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Level of Output:}}

In the simple average formula, n represented the number of pieces of data. For this chapter, the definition is further specified to represent the total number of units produced or sold or the total output that incurred the total variable costs.

HOW TO

Calculate the unit variable cost

Step 1: Identify all fixed, variable, and blended costs, along with the level of output. For variable costs, understand any important elements of how the cost is structured. For blended costs, separate the costs into variable and fixed components.

Step 2: Calculate the total variable cost (TVC) by totalling all variable costs at the indicated level of output. This involves taking any known unit variable costs and multiplying each by the level of output.

Step 3: Divide the total variable cost by the total level of output by applying **Formula 4.6a** $VC = \dfrac{TVC}{n}$.

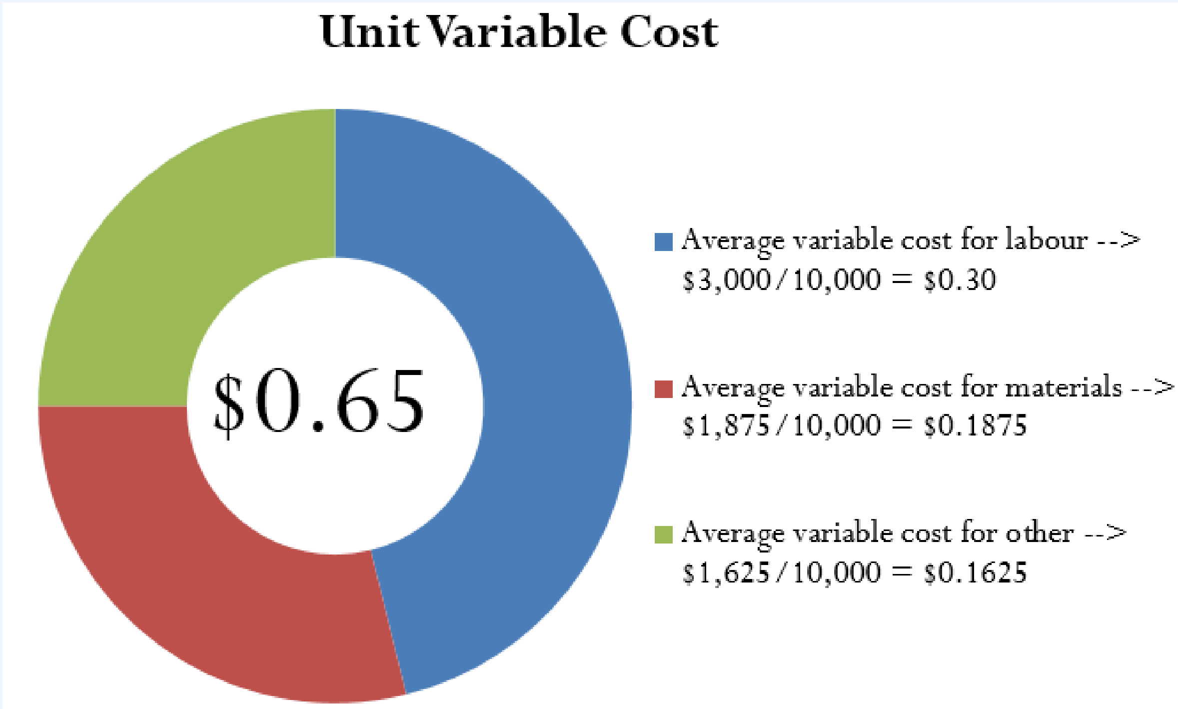


Figure 4.6.1

Assume a company produces $10,000$ units and wants to know its unit variable cost. It incurs production labour costs of $\$3,000$, material costs of $\$1,875$, and other variable costs totaling $\$1,625$.

Step 1: In this case, all costs are variable costs (production labor and material costs are always variable). The level of output is:

$$n = 10,000 \text{ units.}$$

Step 2: Total all variable costs together to get:

$$TVC = \$3,000 + \$1,875 + \$1,625$$
$$TVC = \$6,500$$

Step 3: Apply **Formula 4.6a** $VC = \dfrac{TVC}{n}$ to arrive at:

$$VC = \$6,500 \div 10,000$$
$$VC = \$0.65$$

This means that, on average, the variable cost associated with one unit of production is $\$0.65$

.

Key Takeaways

The definitions of variable, fixed, and blended costs along with their typical associated examples represent a simplified view of how the real world operates. The complexities involved in real-world business costs complicate the fundamentals of managing a business. Therefore, here is how this textbook addresses the complexities of atypical operations, changing fixed costs, and decreasing unit variable costs:

- **Atypical Operations.** Although there are "normal" ways that businesses operate, there are also businesses that have atypical operations. What is a fixed cost to one business may be a variable cost to another.
  - For example, rent is usually a fixed cost. However, some rental agreements include a fixed base cost plus a commission on the operational output. These rental agreements form blended costs. This textbook does not venture into any of these atypical costs and instead focuses on common cost categorizations.
- **Changing Fixed Costs**. In real-world applications, fixed costs do not remain flat at all levels of output. As output increases, fixed costs tend to move upwards in steps.
  - For example, at a low level of output only one manager (on salary) may be needed. As output increases, eventually another manager needs to be hired, perhaps one for every $20,000$ units produced. In other words, up to $20,000$ units the fixed costs would be constant, but at $20,001$ units the fixed costs take a step upwards as another manager is added. The model presented in this textbook does not address these upward steps and treats fixed costs as a constant at all levels of output.
- **Decreasing Unit Variable Costs**. Production tends to realize efficiencies as the level of output rises, resulting in the unit variable cost dropping. This is commonly known as achieving **economies of scale**. As a consumer, you often see a similar concept in your retail shopping. If you purchase one can of soup, it may cost $\$1$. However, if you purchase a bulk tray of 12 cans of soup it may cost only $\$9$, which works out to $75¢$ per can. This price is lower partly because the retailer incurs lower costs, such as fewer cashiers to sell 12 cans to one person than to sell one can each to 12 different people. Now apply this analogy to

production. Producing one can of soup costs $75¢$. However, a larger production run of 12 soup cans may incur a cost of only $\$6$ instead of $\$9$ because workers and machines can multitask. This means the unit variable cost would decrease by $25¢$ per can. However, the model in this textbook assumes that unit variable costs always remain constant at any given level of output.

Example 4.6.1

You are considering starting your own home-based Internet business. After a lot of research, you have gathered the following financial information:

Table 4.6.1

Dell computer	$\$214.48$ monthly lease payments
Office furniture (desk and chair)	$\$186.67$ monthly rental
Shaw high-speed Internet connection	$\$166.88$ per month
Your wages	$\$30$ per hour
Utilities	$\$13$per month plus $\$0.20$ per hour usage
Software (and ongoing upgrades)	$\$20.00$ per month
Business licences and permits	$\$27.00$ per month
Google click-through rate	$\$10.00$ per month $+\$0.01$per click payable as total clicks per sale

Generating and fulfilling sales of 430 units involves 80 hours of work per month. Based on industry response rates, your research also shows that to achieve your sales you require a traffic volume of $34,890$ Google clicks.

On a monthly basis, calculate the total fixed cost, total variable cost, and unit variable cost.

Solution

Step 1: **Sort the costs into fixed and variable. Separate the components for blended costs. You can total the fixed costs to arrive at the total fixed costs, or TFC.**

Table 4.6.2

Fixed Costs		Variable	
Dell computer	$\$214.48$	Wages	$\$30.00$ p
Office furniture	$\$186.67$	Utilities (blended cost)	$\$0.20$ per
Shaw high-speed Internet	$\$166.88$	Google clicks (blended cost)	$\$0.01$ per
Utilities (blended cost)	$\$13.00$ only		
Software	$\$20.00$		
Business licences/permits	$\$27.00$		
Google clicks (blended cost)	$\$10.00$ only		
TOTAL FIXED COSTS	$TFC = \$638.03$	TOTAL VARIABLE COSTS	TVC cal

Step 2: **Calculate variable costs, then total them to arrive at the total variable cost, or TVC.**

$TVC = (\$30.00 \times \text{hours}) + (\$0.20 \times \text{hours}) + (\$0.01 \times \text{Google clicks})$
$TVC = (\$30.00 \times rgb]1.0, 0.0, 0.080) + (\$0.20 \times rgb]1.0, 0.0, 0.080) + (\$0.01 \times rgb]0.0, 0.0, 1.034rgb]0.0, 0.0, 1.0, rgb]0.0, 0.0, 1.0890)$
$TVC = \$2,764.90$

Step 3: **Apply Formula 4.6a $\mathrm{VC} = \dfrac{\mathrm{TVC}}{n}$ to calculate the unit variable cost.**

$$\text{Unit Variable Cost} = \frac{\text{Total Variable Cost}}{\text{Level of Output}}$$

$$VC = \frac{\$2,764.90}{430}$$

$$VC = \$6.43$$

Step 4: **Write as a statement.**

Seven components make up the fixed costs: the computer, furniture, Internet service, fixed utilities, software, licenses/permits, and the fixed Google cost, all totaling $\$638.03$.

Three components make up the variable costs: wages, hourly utilities, and Google clicks, totaling $\$2,764.90$.

The average unit variable cost is $\$6.43$ per unit sold.

For-Profit Business

Most businesses are "for-profit" businesses, meaning that they operate to make money. In Example 4.1.1, you figured out the total fixed costs, total variable costs, and the unit variable cost for your Internet business. However, you left unanswered one of the most important questions in business: If you sell the planned 430 units, are you profitable? Is there any money left after you pay for all of those fixed and variable costs? You must also remember that 430 units is just an estimate. What happens if you sell only 350 units? What happens if you sell 525 units? What happens if you decide to pay yourself a higher wage?

There are no guarantees in business, and the future is always uncertain. Successful business managers plan for the future and perform many "what-if" scenarios to answer questions such as those above. This section develops a model for calculating total net income based on total revenues and total costs. The model allows managers to analyze various scenarios and determine the impact on profitability.

4.6b Net Income Using a Total Revenue and Cost Approach Formula

$$rgb]0.1, 0.1, 0.1\text{Net Income Using a Total Revenue \& Cost Approach:}rgb]1.0, 0.0, 0.0$$

$$rgb]1.0, 0.0, 0.0Nrgb]1.0, 0.0, 0.0I = rgb]0.0, 0.0, 1.0n(rgb]0.0, 0.5, 0.0S) - (rgb]0.5, 0.0, 0.5Trgb]0.5, 0.0, 0.5Frgb]0.5, 0.0, 0.5C + rgb]0.0, 0.0, 1.0n(rgb]0.68, 0.46, 0.12Vrgb]0.68, 0.46, 0.12C))$$

{\color[rgb]{1.0, 0.0, 0.0}N}{\color[rgb]{1.0, 0.0, 0.0}I}\;\text{is Net Income:} Net income is the amount of money left over after all costs are deducted from all revenues. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss since the costs are exceeding the revenues.

Note that many companies use the terms net earnings or net profit instead of the term net income. Net income is based on a certain level of output. This model assumes that the number of units that are produced or purchased (for resale) by the company exactly matches the number of units that are output or sold by the company. Therefore, the model does not consider inventory and its associated costs.

$rgb]0.0, 0.0, 1.0n$ is Level of Output: The number of units produced or sold or the output that incurred all of the variable costs.

{\color[rgb]{0.0, 0.5, 0.0}S}\;\text{is Unit Selling Price:} The unit selling price of the product.

\underbrace{{\color[rgb]{0.0, 0.0, 1.0}n}({\color[rgb]{0.0, 0.5, 0.0}S})}\;\text{is Total Revenue:}
This term in the formula calculates how much money or gross income the sale of the product at a certain output level brings into the organization. Total revenue is the entire amount of money received by a company for selling its product, calculated by multiplying the quantity sold by the selling price.

\underbrace{{\color[rgb]{0.5, 0.0, 0.5}T}{\color[rgb]{0.5, 0.0, 0.5}F}{\color[rgb]{0.5, 0.0, 0.5}C}+{\color[rgb]{0.0, 0.0, 1.0}n}({\color[rgb]{0.68, 0.46, 0.12}V}{\color[rgb]{0.68, 0.46, 0.12}C})}\;\text{ is Total Costs:}

This term in the formula calculates how much money is spent to generate the revenue. Total cost is the sum of all costs for the company, including both the total fixed costs and total variable costs.

Two terms make up the costs: Total fixed costs (TFC) are a constant since these costs do not change with the level of output; total variable costs, represented mathematically by $n(VC)$, are the level of output multiplied by the unit variable cost.

{\color[rgb]{0.5, 0.0, 0.5}T}{\color[rgb]{0.5, 0.0, 0.5}F}{\color[rgb]{0.5, 0.0, 0.5}C}\;\text{is Total Fixed Costs:}

The total of all costs that are not affected by the level of output.

{\color[rgb]{0.68, 0.46, 0.12}V}{\color[rgb]{0.68, 0.46, 0.12}C}\;\;\text{is Unit Variable Cost:}

From **Formula 4.6a** $VC = \dfrac{TVC}{n}$, this is the average variable cost associated with an individual unit of output.

HOW TO

Calculate net income using a total revenue and total income approach

Step 1: Calculate the total revenue. This requires identifying the unit selling price of the product and multiplying it by the level of sales.

Step 2: Calculate your total costs. This requires identifying and separating costs into fixed and variable components. Fixed costs are totaled to arrive at the total fixed cost. Total variable costs are either known or can be calculated through multiplying the unit variable cost by the level of output.

Step 3: Calculate the net income by applying **Formula 4.6b** $NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$.

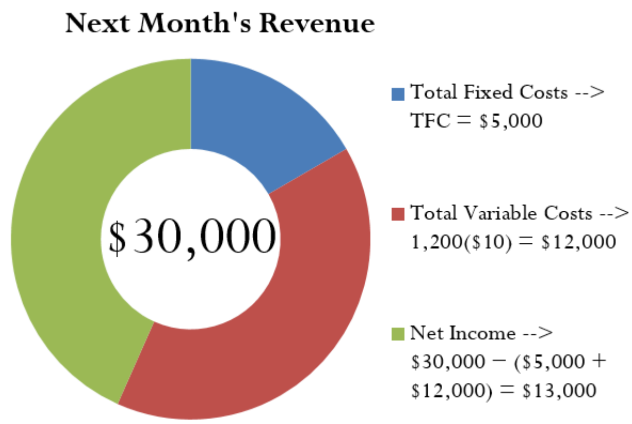


Figure 4.6.2

For example, assume that last month a company incurred total variable costs of $\$10,000$ in the course of producing $1,000$ units. For next month it forecasts total fixed costs of $\$5,000$ and all variable costs remaining unchanged. Projected production for next month is $1,200$ units selling for $\$25$ each. You want to estimate next month's net income.

Step 1: Using **Formula 4.6b**$NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$, you calculate total revenue from $n(S)$, or the total level of output multiplied by the price of the product. If you project sales of $1,200$ units (n) at $\$25$ each (S), then the total forecasted revenue is $1,200(\$25) = \$30,000.$

Step 2: Total fixed costs, or TFC, are $\$5,000$. To get the total variable costs, you must resolve

$n(VC)$. You calculate the unit variable cost, or VC, with **Formula 4.6a** $VC = \frac{TVC}{n}$. Using the current month figures, you see:

$$VC = \$10,000 \div 1,000$$
$$VC = \$10$$

If the projected level of output is $1,200$ units, then the total variable costs are $1,200(\$10) = \$12,000$.

Step 3: Applying **Formula 4.6b** $NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$ you have:

$$NI = \text{Total Revenue} - \text{Total Costs}$$
$$NI = \$30,000 - (\$5,000 + \$12,000)$$
$$NI = \$13,000$$

Based on the numbers, you forecast net income of $\$13,000$ for next month.

Key Takeaway

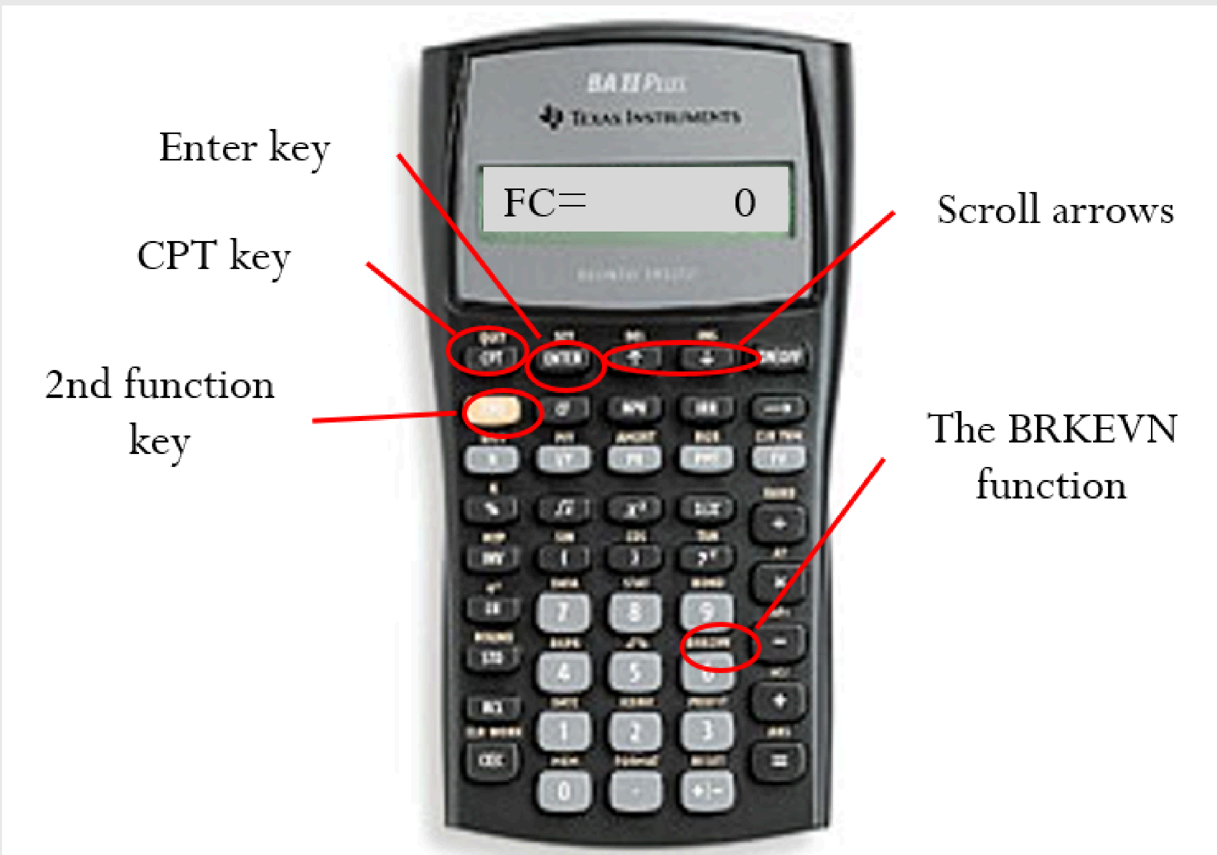


Figure 4.6.3

The Texas Instruments BAII Plus calculator is programmed with a version of **Formula 4.6b** $NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$. The function is called Brkevn, and you access it by pressing 2nd and then the number six key. The relationship between the formula symbols and the calculator symbols is displayed in the table below.

Table 4.6.3

Variable	Formula 4.6b Notation	Calculator Notation
Total fixed cost	TFC	FC
Unit variable cost	VC	VC
Price per unit	S	P (for price)
Net income	NI	PFT (for profit)
Level of output	*n*	Q (for quantity)

To solve **Formula 4.6b**$NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$ for the PFT or any other variable, enter data into all of the above variables except one. Keying in a variable requires inputting the value and pressing Enter. Use and to scroll through the display. When you are ready, scroll to the unknown variable and press CPT.

Paths To Success

An easy way to remember **Formula 4.6b**$NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$ is to understand what the formula represents. As explained, the calculation of $n(S)$ multiplies quantity by price to produce the total revenue. The $(TFC + n(VC))$ takes the total fixed costs and adds the total variable costs (which is a function of quantity multiplied by unit variable cost) to arrive at the total cost. Therefore, **Formula 4.6b** $NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$ expressed more simply is:

\text{Net Income = Total Revenue – Total Cost}

Try It

1) In the following situations, explain what would happen to net income and why.

a. The selling price is raised.
b. The hourly wages of production workers are increased to match the increase in the consumer price index.
c. The level of output decreases.
d. Your insurance company lowers your insurance premiums because your company has had no claims in the past year.

Solution

a. Net income increases because the total revenues increase with no similar increase in costs.
b. Net income decreases because the hourly wages of production workers are variable costs.If the costs go up, then less money is left over.
c. Net income decreases because both total revenueand total variable costs drop; however, the fixed costs remain the same, so there is less revenue to cover proportionally higher costs.
d. Net income increases because the insurance premium is a fixed cost that has become smaller. With lower costs, more money is left over.

Example 4.6.2

Recall from Example 4.6.1 that you are considering starting your own home-based Internet business. The following information is known:

FC = \$638.03 $VC = \$6.43$ forecasted $n = 430$

Based on these numbers, calculate:

- The forecasted net income if your price per unit is $\$10$.
- The dollar change in net income if you decide to pay yourself a higher wage of $\$35.38$ per hour instead of $\$30.00$ per hour while still working 80 hours. Note the total variable costs excluding wages were $\$364.90$.
- The dollar change in net income if sales are 30 lower than your initial forecast, ignoring part (b) calculations.

Solution

Step 1: **Write what you know from the question.**

The costs, level of output, and price are known:

$$\begin{aligned} FC &= \$638.03 \\ VC &= \$6.43 \\ \text{forecasted } n &= 430 \\ S &= \$10 \end{aligned}$$

For part b., the total variable costs excluding wages are known:

$$\begin{aligned} TVC\ (\text{excluding wages}) &= \$364.90 \\ \text{New wage} &= \$35.38 \\ \text{Hours} &= 80 \end{aligned}$$

For part c., the change in the level of output is known:

$$\%C = -30\%$$

a.

Step 2: **Calculate total revenue at the forecasted output.**

$$\begin{aligned} \text{Total revenue} \ &= \ n(S\,) \\ n(S) &= 430(\$10) \\ n(S) &= \$4,300 \end{aligned}$$

Step 3: **Calculate total costs at the forecasted output.**

$TFC = \$638.03$
$TVC = rgb]1.0, 0.0, 0.0n(rgb]0.0, 0.0, 1.0Vrgb]0.0, 0.0, 1.0C)$
$TVC = rgb]1.0, 0.0, 0.0430(rgb]0.0, 0.0, 1.0\$rgb]0.0, 0.0, 1.06rgb]0.0, 0.0, 1.0.rgb]0.0, 0.0, 1.043)$
$TVC = \$2,764.90$

Step 4: **Apply Formula 4.6b** $NI = n\,(S) - (TFC + n\,(VC))$

$$NI = \$4,300 - \ (\$638.03 \ + \ \$2,764.90)$$
$$NI = \$897.07$$

---

b.

Step 1: **Total revenue is unchanged since the output has not changed.**

$$\text{Total Revenue} = \$4,300$$

Step 2: **Fixed costs are unchanged. Recalculate the total variable costs by adding the new wages to the total variable costs excluding wages. By rearranging Formula 4.6a** $VC = \dfrac{TVC}{n}$ **, you can see** $TVC = n(VC)$**. Therefore, substitute** TVC **in place of** $n(VC)$ **in Formula 4.6b** $NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$

$$TFC = \$638.03$$
$$TVC = \$364.90 + (\$35.38 \ \times \ 80 \ hours)$$
$$TVC = \$3,195.30$$

Step 3: **Apply Formula 4.6b** $NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$ **to recalculate the new net income. Compare to net income in part a. and determine the change.**

$$NI = \$4,300 - (\$638.03 \ + \ \$3,195.30)$$
$$NI = \$466.67$$
$$\text{Change in } NI = \$466.67 - \$897.07 = -\$430.40$$

---

c.

Step 1: **The revised level of output is a percent change calculation where** $\%C = -30\%$ **and** $V_{\text{i}} = 430$**. Solve Formula 3.2a** $\%C = \dfrac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$ **(Percent Change) for** V_{f}**. Once** V_{f} **is known, recalculate total revenue.**

$$
\begin{aligned}
-30\% &= \frac{V_{\mathrm{f}} - 430}{430} \times 100 \\
-129 &= V_{\mathrm{f}} - 430 \\
301 &= V_{\mathrm{f}} \\
\text{Total revenue} &= 301(\$10) \\
\text{Total revenue} &= \$3,010
\end{aligned}
$$

Step 2: **Fixed costs are unchanged. Recalculate total variables costs using the revised level of output.**

$$TFC \ = \ \$638.03$$

$$
\begin{aligned}
TVC &= 301(\$6.43) \\
TVC &= \$1,935.43
\end{aligned}
$$

Step 3: **Apply Formula 4.6b** $NI = n\,(S) - (TFC + n\,(VC))$ **to recalculate the new net income. Compare to net income in part a. and determine the change.**

$$
\begin{aligned}
NI &= \$3,010 – (\$638.03 \ + \ \$1,935.43) \\
NI &= \$436.54 \\
\text{Change in } NI &= \$436.54 - \$897.07 = -\$460.53
\end{aligned}
$$

	FC	VC	P	PFT	Q
a.	638.03	6.43	10	Answer: 897.07	430
b.	✓	3195.3÷430=	✓	Answer: 466.67	✓
c.	✓	6.43	✓	Answer: 436.54	301*

*First use Δ% function to calculate this number:

	OLD	NEW	%CH	#PD
c.	430	Answer: 301	−30	1

Figure 4.6.4

Step 4: **Write as a statement.**

Under the initial scenario, a net income of $\$897.07$ on sales of 430 units is forecast. If you pay yourself a higher wage of $\$35.38$ per hour, your net income decreases by $\$430.40$ to $\$466.67$. If your forecast is inaccurate and is lower by 30%, then your net income is reduced by $\$460.53$, resulting in a lower net income of $\$436.54$.

Net Income Using a Total Contribution Margin Approach

In Example 4.6.2, you learned that if you sell the projected 430 units of product for your Internet business, the total net income is $\$897.07$. What if you sold 431 units of the product? How would your net income change? Clearly it should rise, but by how much? One approach to answering this question is to rerun the numbers through **Formula 4.6b**$NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$, revising the total revenues and total variable costs to calculate a new net income. This new net income can then be compared against the existing net income to determine how it changed. This is a multistep approach and involves a lot of work. An alternative approach explored in this section involves using a unit contribution margin to calculate the net income. The benefit of this approach is that with minimal calculations you can easily assess the impact of any change in the level of output.

In accounting and marketing, the **contribution margin** is the amount that each unit sold adds to the net income of the business. This approach allows you to understand the impact on net income of each unit sold. The contribution margin determines on a per-unit basis how much money is left over after unit variable costs are removed from the price of the product. This leftover money is then available to pay for the fixed costs. Ultimately, when all fixed costs have been paid for, the leftover money becomes the profits of the business. If not enough money is left over to pay for the fixed costs, then the business has a negative net income and loses money.

4.6c Unit Contribution Formula

The difference between the increase in the total revenue and the increase in the total variable costs is how much the sale of an individual product contributes toward the change in your net income. Formula 4.6c expresses this relationship.

Unit Contribution: $rgb]1.0, 0.0, 0.0C rgb]1.0, 0.0, 0.0M = rgb]0.0, 0.0, 1.0S - rgb]0.0, 0.5, 0.0V rgb]0.0, 0.5, 0.0C$

{\color[rgb]{1.0, 0.0, 0.0}C}{\color[rgb]{1.0, 0.0, 0.0}M}{\color[rgb]{0.0, 0.5, 0.0}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Unit Contribution Margin:}}

This is the amount of money that remains available to pay for fixed costs once the unit variable cost is removed from the selling price of the product.

{\color[rgb]{0.0, 0.0, 1.0}S}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Unit Selling Price:}}

The unit selling price of the product.

{\color[rgb]{0.0, 0.5, 0.0}V}{\color[rgb]{0.0, 0.5, 0.0}C}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Unit Variable Cost:}}

The typical or average variable cost associated with an individual unit of output, as determined from **Formula 4.6a** $VC = \dfrac{TVC}{n}$.

If you have no units sold, your net income is negative and equal to the total fixed costs associated with your business, since there is no offsetting revenue to pay for those costs. With each unit sold, the contribution margin of each product is available to pay off the fixed costs. Formula 4.6d expresses this relationship when calculating net income.

4.6d Net Income Using Total Contribution Margin Approach Formula

$rgb]0.1, 0.1, 0.1\text{Net Income Using Total Contribution Margin Approach:}$

$rgb]0.1, 0.1, 0.1\ rgb]1.0, 0.0, 0.0N rgb]1.0, 0.0, 0.0I rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1 = rgb]0.1, 0.1, 0.1\ rgb]0.0, 0.0, 1.0n rgb]0.1, 0.1, 0.1(rgb]0.0, 0.5, 0.0C rgb]0.0, 0.5, 0.0M rgb]0.1, 0.1, 0.1) rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1 - rgb]0.1, 0.1, 0.1\ rgb]0.5, 0.0, 0.5T rgb]0.5, 0.0, 0.5F rgb]0.5, 0.0, 0.5C$

{\color[rgb]{1.0, 0.0, 0.0}N}{\color[rgb]{1.0, 0.0, 0.0}I}{\color[rgb]{0.5, 0.0, 0.5}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Net Income:}}

The amount of money left over after all costs have been paid is the net income. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss.

{\color[rgb]{0.0, 0.0, 1.0}n}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Level of Output:}}

The number of units produced or sold or the output that incurred all of the variable costs.

{\color[rgb]{0.0, 0.5, 0.0}C}{\color[rgb]{0.0, 0.5, 0.0}M}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Unit Contribution Margin:}}

The amount of money per unit remaining after variable costs have been paid. It is available to cover fixed costs. Calculate the unit contribution margin by taking the unit selling price and subtracting the unit variable cost as per **Formula 4.6c** $CM = S - VC$.

{\color[rgb]{0.1, 0.1, 0.1}\underbrace{{\color[rgb]{0.0, 0.0, 1.0}n}({\color[rgb]{0.0, 0.5, 0.0}C}{\color[rgb]{0.0, 0.5, 0.0}M})}}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Total Contribution Margin:}}

This term in the formula calculates how much money is left over to pay the total fixed costs. Use the contribution margin calculated in **Formula 4.6c**$CM = S - VC$ and multiply it by the level of output to determine the total monies remaining after all variable costs are paid.

{\color[rgb]{0.5, 0.0, 0.5}T}{\color[rgb]{0.5, 0.0, 0.5}F}{\color[rgb]{0.5, 0.0, 0.5}C}{\color[rgb]{0.1, 0.1, 0.1}\;}{\color[rgb]{0.1, 0.1, 0.1}\text{is Total Fixed Cost:}}

The total of all costs that are not affected by the level of output.

HOW TO

Calculate the net income using a contribution margin approach

Step 1: If unit information is known, apply **Formula 4.6c**$CM = S - VC$ and calculate the unit contribution margin by subtracting the unit variable cost from the selling price. This may or may not require you to use **Formula 4.6a** $VC = \frac{TVC}{n}$ to calculate the unit variable cost.

Step 2: Calculate the total contribution margin by multiplying the contribution margin with the associated level of output.

Step 3: Calculate the total fixed costs by adding all fixed costs.

Step 4: Based on the level of output, calculate the net income by applying **Formula 4.6d**$NI = n\,(CM) - TFC$.

For example, using the contribution margin approach, calculate the net income for a product that sells for $\$75$, has unit variable costs of $\$31$, total fixed costs of $\$23,000$, and total sales of 800 units.

Step 1: The unit contribution margin is calculated from **Formula 4.6c**$CM = S - VC$. If the product sells for $\$75$ and has unit variable costs of $\$31$, then:

rgb]0.1, 0.1, 0.1Crgb]0.1, 0.1, 0.1Mrgb]0.1, 0.1, 0.1 = rgb]0.1, 0.1, 0.1\$rgb]0.1, 0.1, 0.175rgb]0.1, 0.1, 0.1−rgb]0.1, 0.1, 0.1\$rgb]0.1, 0.1, 0.131rgb]0.1, 0.1, 0.1
rgb]0.1, 0.1, 0.1Crgb]0.1, 0.1, 0.1Mrgb]0.1, 0.1, 0.1 = rgb]0.1, 0.1, 0.1\$rgb]0.1, 0.1, 0.144

This means that every unit sold has $\$44$ left over to contribute toward fixed costs.

Step 2: Now convert that into a total contribution margin. The first part of **Formula 4.6d** $NI = n\,(CM) - TFC$ calculates total contribution margin through $n(CM)$. With sales of 800 units, the total contribution margin is:

rgb]0.1, 0.1, 0.1800rgb]0.1, 0.1, 0.1(rgb]0.1, 0.1, 0.1\$rgb]0.1, 0.1, 0.144rgb]0.1, 0.1, 0.1)rgb]0.1, 0.1, 0.1 =rgb]0.1, 0.1, 0.1\$rgb]0.1, 0.1, 0.135rgb]0.1, 0.1, 0.1,rgb]0.1, 0.1, 0.1200

Step 3: Total fixed costs are known: $TFC = \$23,000$.

Step 4: Apply **Formula 4.6d**$NI = n\,(CM) - TFC$, which translates to:

$$\text{Net Income} = \text{Total Contribution Margin} - \text{Total Fixed Costs}$$
$$\text{Net Income} = \$35,200 - \$23,000$$
$$\text{Net Income} = \$12,200$$

Paths To Success

When you work with **Formula 4.6d**$NI = n\left(CM\right) - TFC$, sometimes unit information may not be known. Instead, you might just have a single aggregate number representing the total contribution margin for which somebody has already taken the total revenue and subtracted the total variable costs. In this case, skip step 1 and take the provided total contribution margin as the answer for Step 2 with no calculations necessary.

Example 4.6.3

Continuing with Example 4.6.1, calculate the unit contribution margin and net income using the contribution margin approach. From the previous examples, recall the unit variable cost is $\$6.43$, unit selling price is $\$10$, total fixed costs are $\$638.03$, and the projected sales are 430 units.

Solution

***Step 1:* Write what you know from the question.**

The cost, price, and sales information are known:

$$VC = \$6.43$$
$$S = \$10$$
$$TFC = \$638.03$$
$$n = 430$$

***Step 2:* Calculate the unit contribution margin using Formula 4.6c.**$CM = S - VC$

$$CM = \$10.00 - \$6.43$$
$$CM = \$3.57$$

***Step 3:* Calculate the total contribution margin by multiplying the unit contribution margin times the level of output.**

$$n(CM) = 430(\$3.57)$$
$$n(CM) = \$1,535.10$$

Step 4: **Determine the total fixed costs.**

$$TFC = \$638.03$$

Step 5: **To calculate net income, apply Formula 4.6d.**$NI = n\left(CM\right) - TFC$

$$NI = \$1,535.10 - \$638.03$$
$$NI = \$897.07$$

FC	VC	P	PFT	Q
638.03	6.43	10	Answer: 897.07	430

Figure 4.6.5

Step 6: **Write as a statement.**

> When a product sells for $\$10$, $\$6.43$ goes toward paying for the variable costs of your business, leaving $\$3.57$ as your unit contribution margin. This means that for every unit increase in sales, your net income rises by $\$3.57$. Thus, if you sell 430 units you have a total contribution margin of $\$1,535.10$, which results in a net income of $\$897.07$ after removing fixed costs of $\$638.03$.

Contribution Rates

It is difficult to compare different products and their respective dollar amount contribution margins if their selling prices and costs vary widely. For example, how do you compare a unit contribution margin of $\$1,390$ (selling price of $\$2,599.99$) on a big screen television to a unit contribution margin of $\$0.33$ on a chocolate bar (selling price of $\$0.67$)? On a per-unit basis, which contributes relatively more to fixed costs? To facilitate these comparisons, the products must be placed on equal terms, requiring you to convert all dollar amount contribution margins into percentages. A **contribution rate** is a contribution margin expressed as a percentage of the selling price.

4.6e Contribution Rate if Unit Information Is Known

Your choice between two formulas for calculating a contribution rate depends on whether you have unit information or only aggregate information. Both formulas are adaptations of the formula dealing with Rate, Portion, Base.

If unit information is known, including the unit variable cost and unit selling price, then calculate the contribution rate using unit information as expressed in Formula 4.6e.

Contribution Rate if Unit Information Is Known:

$$rgb]1.0, 0.0, 0.0Crgb]1.0, 0.0, 0.0R = \frac{rgb]0.0, 0.0, 1.0Crgb]0.0, 0.0, 1.0M}{rgb]0.0, 0.5, 0.0S} rgb]0.5, 0.0, 0.5 \times rgb]0.5, 0.0, 0.5100$$

$rgb]1.0, 0.0, 0.0Crgb]1.0, 0.0, 0.0R$ **is Contribution Rate:** This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business. When all fixed costs are paid for, this percentage is the portion of the selling price that will remain as profit. In portions, this is the rate.

$rgb]0.0, 0.0, 1.0Crgb]0.0, 0.0, 1.0M$ **is Unit Contribution:** This is the unit dollar amount that is left over after you subtract the unit variable costs from the unit selling price of the product. In portions formula this is the portion.

{\color[rgb]{0.0, 0.5, 0.0}S}\;\text{is Unit Selling Price:} The unit selling price of the product. In the portions formula, this is the base.

{\color[rgb]{0.5, 0.0, 0.5}\times}{\color[rgb]{0.5, 0.0, 0.5}100}\;\text{is Percent Conversion:} The contribution rate is always expressed as a percentage.

If any unit information, including the unit variable cost or unit selling price, is unknown or unavailable, then you cannot apply Formula 4.6e. Sometimes only aggregate information is known. When total revenue and total variable costs for any product are known or can at least be calculated, then you must calculate the contribution rate from the aggregate information as expressed in Formula 4.6f

$\boxed{4.6f}$ Contribution Rate if Aggregate Information Is Known

$$\text{Contribution Rate if Aggregate Information Is Known:}$$

$$rgb]1.0, 0.0, 0.0Crgb]1.0, 0.0, 0.0R = \frac{rgb]0.0, 0.0, 1.0Trgb]0.0, 0.0, 1.0R - rgb]0.68, 0.46, 0.12Trgb]0.68, 0.46, 0.12Vrgb]0.68, 0.46, 0.12C}{rgb]0.0, 0.0, 1.0Trgb]0.0, 0.0, 1.0R} rgb]0.5, 0.0, 0.5\times rgb]0.5, 0.0, 0.5100$$

$rgb]1.0, 0.0, 0.0Crgb]1.0, 0.0, 0.0R$ **is Contribution Rate:** This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business.

{\color[rgb]{0.0, 0.0, 1.0}T}{\color[rgb]{0.0, 0.0, 1.0}R}\;\text{is Total Revenue:} This is the total amount of money that the company has received from the sale of the product.

{\color[rgb]{0.68, 0.46, 0.12}T}{\color[rgb]{0.68, 0.46, 0.12}V}{\color[rgb]{0.68, 0.46, 0.12}C}\;\text{is Total Variable Costs:}

This is the total cost associated with the level of output. When total variable costs are subtracted from the total revenue, the remainder represents the portion of money left over to pay the fixed costs.

$rgb]0.5, 0.0, 0.5\times rgb]0.5, 0.0, 0.5100$ **is Percent Conversion:** The contribution rate is always expressed as a percentage.

HOW TO

Calculate a contribution rate

Step 1: Identify the required variables and calculate the margin, if needed.

- If unit information is known, this requires you to calculate the unit contribution margin. Otherwise, calculate the unit contribution margin by applying **Formula 4.6c** $CM = S - VC$. You must identify the unit selling price.
- If aggregate information is known, you need to identify total revenue and total variable costs.

Step 2: Calculate the contribution rate.

- If unit information is known, apply **Formula 4.6e** $CR = \frac{CM}{S} \times 100$.
- If only aggregate information is known, apply **Formula 4.6f** $CR = \frac{TR - TVC}{TR} \times 100$.

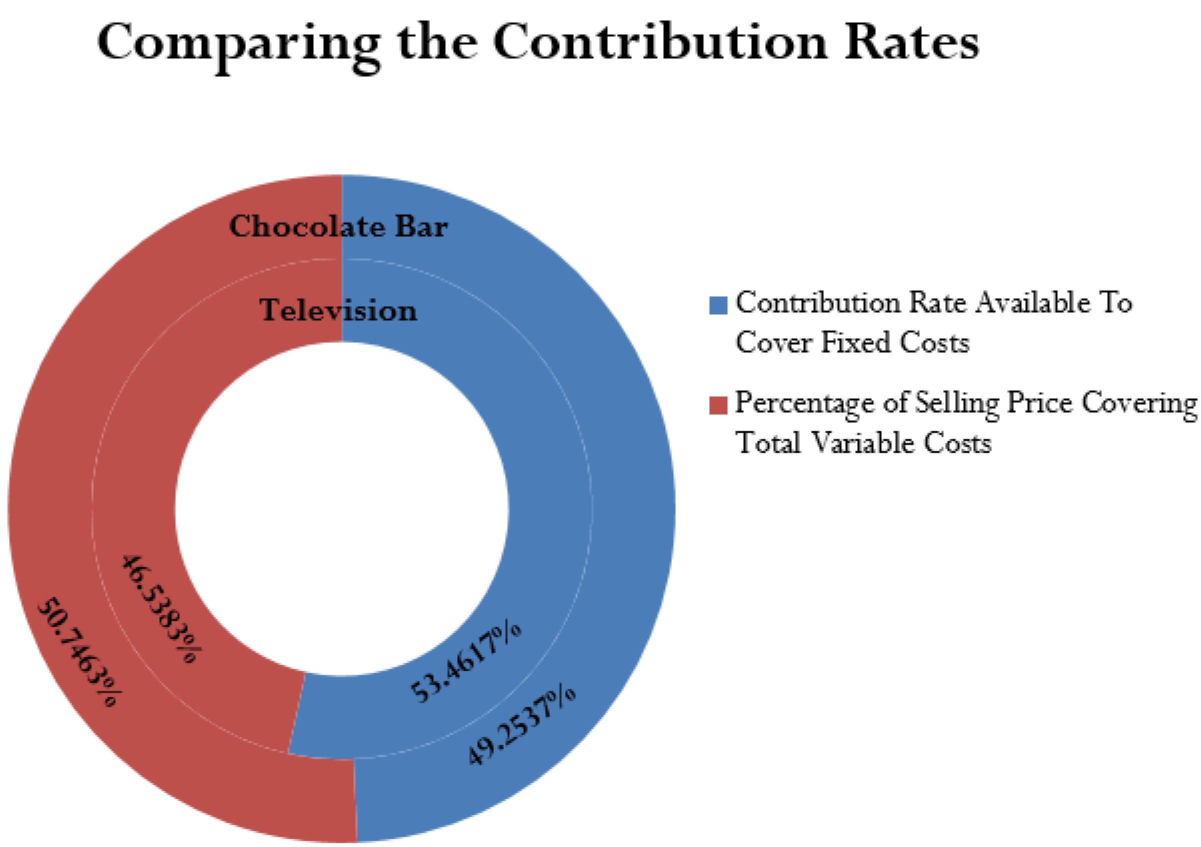


Figure 4.6.6

As an example of these steps, recall earlier that you wanted to compare the relative contributions of the big screen television and the chocolate bar. The television sells for $\$2,599.99$ and has a unit contribution margin of $\$1,390$. The chocolate bar sells for $\$0.67$ and has a unit contribution margin of $\$0.33$. Notice that the information being provided is on a per-unit basis. Calculate the contribution rate of each product.

Step 1: The contribution margins are known. For the television, $CM = \$1,390$, and for the chocolate bar, $CM = \$0.33$.

Step 2: Applying **Formula 4.6e** $CR = \frac{CM}{S} \times 100$, the television:

$$CR = \$1,390.00 \div \$2,599.99 \times 100 = 53.4617\%$$

while the chocolate bar

$$CR = \$0.33 \div \$0.67 \times 100 = 49.2537\%$$

It is now evident from the contribution rate that 4.208% more of the television's selling price is available to pay for fixed costs as compared to the chocolate bar's price.

Now change the facts. This time, assume there is no unit information. Instead, the television's total revenue is $\$129,999.50$ and associated total variable costs are $\$60,499.50$. The chocolate bar has total revenue of $\$3,886.00$ with total variable costs of $\$1,972.00$. Based on this information, you are to determine the product with the higher contribution rate.

Step 1: The aggregate numbers are known for both products. For the television, $TR = \$129,999.50$ and $TVC = \$60,499.50$. For the chocolate bar, $TR = \$3,886.00$ and $TVC = \$1,972.00$.

Step 2: Applying **Formula 4.6f** $CR = \frac{TR - TVC}{TR} \times 100$, the television:

$$CR = \frac{\$129,999.50 - \$60,499.50}{\$129,999.50} \times 100$$

$$CR = 53.4617\%$$

while the chocolate bar

$$CR = \frac{\$3,886 - \$1,972}{\$3,886} \times 100$$

$$CR = 49.2537\%$$

You have reached the same conclusion as above.

Try It

2) In each of the following situations, what would happen to the contribution rate and why?

a. The selling price is raised.
b. The hourly wages of production workers are increased to match the increase in the consumer price index.

c. The level of output decreases.
d. Your insurance company lowers your insurance premiums because your company has had no claims in the past year.

Solution

a. The contribution rate increases, since raising the price increases the unit contribution margin.
b. The contribution rate decreases, since total variable costs rise, eroding some of the unit contribution margin.
c. There is no effect on the contribution rate.The level of output is not a factor in calculating the contribution rate.
d. There is no effect on the contribution rate since the insurance premiums are a fixed cost.Fixed costs are not a factor in calculating contribution rates.

Example 4.6.4

Continuing with your ongoing Internet business from the three previous examples, calculate the contribution rate using both the unit and aggregate methods and show how they arrive at the same number. Remember that you are selling products for $\$10$ each, your unit contribution margin is $\$3.57$, total revenue is projected at $\$4,300$, and total variable costs are $\$2,764.90$.

Solution
***Step 1:* Write what you know from the question.**

Prices, margins, revenue, and costs are known:

$$\begin{aligned} S &= \$10 \\ CM &= \$3.57 \\ TR &= \$4,300 \\ TVC &= \$2,764.90 \end{aligned}$$

***Step 2:* Using the unit method, apply Formula 4.6e** $CR = \dfrac{CM}{S} \times 100$

$$CR = \frac{\$3.57}{\$10.00} \times 100$$
$$CR = 35.7\%$$

Step 3: **Using the aggregate method, apply Formula 4.6f** $CR = \frac{TR - TVC}{TR} \times 100$

$$CR = \frac{\$4,300.00 - \$2,764.90}{\$4,300.00} \times 100$$
$$CR = 35.7\%$$

Step 4: **Write a statement.**

Under both the unit and aggregate method, your contribution rate equals 35.7%. This means that every time you sell a $\$10$ product, 35.7% of the revenue remains after recovering the cost of the product.

Putting It All Together

You have studied costs, volume, and net income in this section. So far, you have considered each of these concepts separately while you worked through the various applications. It is time to put the types of costs, unit variable cost, net income, sales, contribution margin, and contribution rate together so that you can see all of these concepts in a single scenario. Look at the following example.

Example 4.6.5

In the commercial section of the newspaper you come across an ad for a pizza delivery business for sale. Upon inquiry, you discover that the owner, who wants to sell the business and then retire, has four salaried employees and owns two delivery vehicles. He invites you to look through his books, where you acquire the following information:

Table 4.6.4

Employees	
Owner's salary	$5,000 per month
Employee salaries and premiums	$2,000 per month each
Business	
Phone	$85 per month
Business insurance	$3,600 per year
Building lease	$2,000 per month
Delivery Vehicles	
Car insurance	$1,200 per year per vehicle
Fuel	$1,125 per month per vehicle
Oil changes	$225 per month per vehicle
Vehicle maintenance and repairs	$562.50 per month per vehicle
Operations	
Pizza ingredients, materials, and packaging	$19,125 per month
Selling price of pizza	$9 per pizza
Average number of pizzas sold	4,500 per month

Assume that this is all of the key information. You need to understand this business and therefore want to determine:

a. A unit variable cost
b. A typical monthly net income for the business
c. The contribution margin per pizza
d. The contribution rate

Solution

Step 1: **Write what you know from the question.**

You have unit information on costs, volume, and revenue, as listed above.

---

a. Unit variable cost:

Step 1: **Sort the costs into fixed and variable categories. Total the monthly fixed costs.**

Table 4.6.5

Fixed Costs per Month		**Variable Costs per Month**	
Owner's salary	$5,000	Fuel	$1,125 × 2 vehicles = $2,2
Employee salaries and premiums	$2,000 × 4 = $8,000	Oil changes	$225 × 2 vehicles =$450
Car insurance	$1,200 × 2 cars ÷ 12 months = $200	Vehicle maintenance and repairs	$562.50 × 2 vehicles = $1
Phone	$85	Pizza ingredients, materials, and packaging	$19,125
Business insurance	$3,600 ÷ 12 months = $300	TOTAL VARIABLE COSTS	TVC = $22,950 per mon
Building lease	$2,000		
TOTAL FIXED COSTS	TFC = $15,585 per month		

***Step 2:* Calculate the unit variable cost by applying Formula 4.6a.** $VC=\dfrac{TVC}{n}$

$$VC=\frac{\$2,250+\$450+\$1,125+\$19,125}{4,500\text{ pizzas}}$$

$$VC=\$5.10\text{ per pizza}$$

b. Net income:

***Step 1:* Calculate total revenue at the level of output.**

$$\begin{aligned}\text{Total Revenue}\ &=\ 4,500(\$9\ )\\ \text{Total Revenue}&=\ \$40,500\end{aligned}$$

Step 2: **Calculate total costs at the level of output.**

$$TFC = \$15,585$$

$$TVC = 4,500(\$5.10)$$
$$TVC = \$22,950$$

Calculator Instructions

FC	VC	P	PFT	Q
15585	5.1	9	Answer: 1,965	4500

Figure 4.6.7

Step 3: **Calculate monthly net income by applying Formula 4.6b.**

$NI = n(S) - (TFC + n(VC))$

$$NI = \$40,500 - (\$15,585 + \$22,950)$$
$$NI = \$40,500 - \$38,535$$
$$NI = \$1,965$$

---

c. and d. Contribution margin and rate:

Step 1: **Calculate the contribution margin by applying Formula 4.6c** $CM = S - VC$**.**

$$CM = \$9.00 - \$5.10$$
$$CM = \$3.90$$

Step 2: **Calculate the contribution rate by applying Formula 4.6e** $CR = \frac{CM}{S} \times 100$**.**

$$CR = \frac{\$3.90}{\$9.00} \times 100$$

$$CR = 43.\overline{3}\%$$

Step 2: **Write as a statement.**

This is a profitable business. For an average month, the business incurs $\$15,585$ in fixed costs along with $\$22,950$ in variable costs, which works out to a unit variable cost of $\$5.10$ for each of the $4,500$ pizzas sold. With total revenue of $\$40,500$, after removing both fixed and variable costs, net income is $\$1,965$ per month. Every pizza, after paying for the unit variable costs, has $\$3.90$, or $43.\overline{3}\%$, left over.

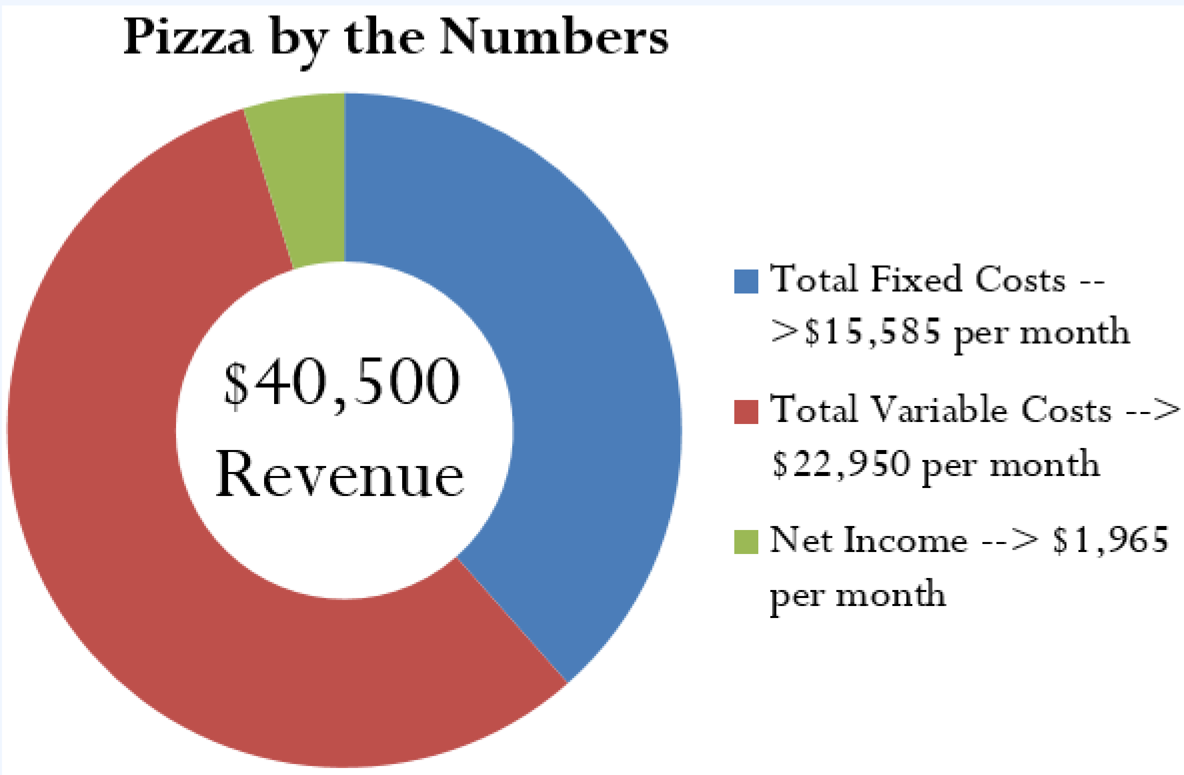


Figure 4.6.8

Section 4.6 Exercises

In each of the following questions, round all of the money and percentages to two decimals.

Mechanics

1. Classify each of the following costs as fixed costs, variable costs, or blended costs. If a cost is blended, separate it into its fixed and variable components.

   a. Natural gas bill for $\$15$ per month plus $\$0.33/m^2$.
   b. A chief executive officer salary of $\$240,000$ per year.
   c. An author earning a royalty of 5% of sales.
   d. Placing a commercial on television for $\$300,000$.
   e. A cellphone bill for $\$40$ per month plus \\$0.25/\text{minute for long distance}.
   f. Hourly production worker wages of $\$18/hr$.
   g. Sales staff who are compensated at a salary of $\$1,000$ per month plus 15% of sales.

For questions 2–7, solve for the unknown variables (identified with a ?) based on the information provided.

Table 4.6.6

	Total Fixed Costs	Total Variable Costs	Unit Variable Cost	Selling Price	Total Revenue	Level of Output	Net Income	Contribution Rate	Unit Contribution Margin
2.	$5,000	$6,600	?	$13	?	?	$4,000	?	$7.50
3.	$2,000	?	$5	$10	$10,000	?	?	?	?
4.	?	?	?	$75	$60,000	?	$14,500	35%	?
5.	$18,000	$45,000	?	?	$84,600	1,800	?	?	?
6.	?	?	?	?	$78,000	3,000	$18,000	?	$13
7.	?	$94,050	$75.24	?	?	?	–$19,500	38%	?

Solutions

1. a. Blended; $FC = \$15$, $VC = \$0.33$;
   b. Fixed cost;
   c. Variable cost;
   d. Fixed cost;
   e. Blended; $FC = \$40$; $VC = \$0.25$;
   f. Variable cost;
   g. Blended, $FC = \$1,000$; $VC = 15\%$ of sales
2. $TR = \$16,000$; $VC = \$5.50$; $n = 1,200$; $CR = 57.69\%$
3. $n = 1,000$; $TVC = \$5,000$; $NI = \$3,000$; $CR = 50\%$; $CM = \$5$
4. $n = 800$; $CM = \$26.25$; $VC = \$48.75$; $TVC = \$39,000$; $TFC = \$6,500$
5. $VC = \$25$; $S = \$47$; $NI = \$21,600$; $CR = 46.81\%$; $CM = \$22$
6. $S = \$26$; $CR = 50\%$; $VC = \$13$; $TVC = \$39,000$; $TFC = \$21,000$
7. $n = 1,250$; $S = \$121.35$; $TR = \$151,687.50$; $CM = \$46.11$; $TFC = \$77,137.50$

Applications

8. In the current period, Blue Mountain Packers in Salmon Arm, British Columbia, had fixed costs of $\$228,000$ and a total cost of $\$900,000$ while maintaining a level of output of $6,720$ units. Next period sales are projected to rise by 20%. What total cost should Blue Mountain Packers project?
9. Fred runs a designer candle-making business out of his basement. He sells the candles for $\$15$ each, and every candle costs him $\$6$ to manufacture. If his fixed costs are $\$2,300$ per month, what is his projected net income or loss next month, for which he forecasts sales of 225 units?
10. A college print shop leases an industrial Xerox photo copier for $\$1,500$ per month plus 1.5¢ for every page. Additional printing costs are estimated at 2¢ per page, which covers toner, paper, labour, and all other incurred costs. If copies are made for students at 10¢ each, determine the following:

    a. How does net income change with every 100 copies sold?
    b. What is the monthly net income if, on average, the shop makes $25,000$ copies for students each month?

11. Gayle is thinking of starting her own business. Total fixed costs are $\$19,000$ per month and unit variable costs are estimated at $\$37.50$. From some preliminary studies that she completed, she forecasts sales of $1,400$ units at $\$50$ each, $1,850$ units at $\$48$ each, $2,500$ units at $\$46$ each, and $2,750$ units at $\$44$ each. What price would you recommend Gayle set for her products?
12. Last year, A Child's Place franchise had total sales of $\$743,000$. If its total fixed costs were $\$322,000$ and net income was $\$81,000$, what was its contribution rate?
13. What level of output would generate a net income of $\$15,000$ if a product sells for $\$24.99$, has unit variable costs of $\$9.99$, and total fixed costs of $\$55,005$?
14. In the current year, a small Holiday Inn franchise had sales of $\$1,800,000$, fixed costs of $\$550,000$, and total variable costs of $\$750,000$. Next year, sales are forecast to increase by 25% but costs will remain the same. How much will net income change (in dollars)?

Solutions

8. $\$1,034,400$
9. $-\$275$
10. a. $\$6.50$ increase; b. $\$125$
11. $S = \$46$ (maximizes NI)
12. $CR = 54.24\%$
13. $4,667$
14. $\$262,000$ increase

Challenge, Critical Thinking, & Other Applications

15. Monsanto Canada reported the following on its income statement for one of its divisions:

$$\begin{aligned} \text{Sales} &= \$6,000,000 \\ \text{Total Fixed Costs} &= \$2,000,000 \\ \text{Total Variable Costs} &= \$3,200,000 \\ \text{Total Costs} &= \$5,200,000 \\ \text{Net Income} &= \$800,000 \end{aligned}$$

Calculate the total contribution margin in dollars and the contribution rate for this division.

16. Procter and Gamble is budgeting for next year. For one of its brands, P&G projects it will operate at 80% production capacity next year and forecasts the following:

$$\begin{aligned} \text{Sales} &= \$80,000,000 \\ \text{Total Fixed Costs} &= \$20,000,000 \\ \text{Total Variable Costs} &= \$50,000,000 \\ \text{Total Costs} &= \$70,000,000 \\ \text{Net Income} &= \$10,000,000 \end{aligned}$$

    Determine the net income if sales are higher than expected and P&G realizes 90% production capacity.

17. Through market research, a marketing manager determines that consumers are willing to pay 5% more for the company's product. However, some of their customers would not like this price increase, so the level of output would drop by 5%. Should the marketing manager leave things as they are or increase the price by 5%? Justify your solution.
18. Francesca is a departmental manager in the women's wear department for The Bay. She believes that if she places her line of $100 dresses on sale at 20% off, she would see her sales rise by 75%. The contribution rate on her dresses is 50%. On a strictly financial basis, should she place the dresses on sale?

Use the following information for questions 19 and 20:

$$\begin{aligned} TFC &= \$3,200,000.00 \\ S &= \$99.97 \\ TVC &= \$5,009,440.00 \\ n &= 131,000 \end{aligned}$$

19. Calculate the following information: unit variable cost, total revenue, net income, unit contribution margin, total contribution margin, and contribution rate.
20. Determine a new value for net income if the following situations occur:
    a. Fixed costs rise by 10%.
    b. The selling price is lowered by 25% during a sale, resulting in 50% more volume.
    c. Fixed costs are lowered by 5%, total variable costs rise by 3%, the price is lowered by 5%, and the level of output rises 10%.

Solutions

15. $TotalCM = \$2,800,000$; $CR = 46.67\%$
16. $\$13,750,000$
17. Increase price since NI rises
18. Put dresses on sale since NI rises
19. $VC = \$38.24$; $TR = \$13,096,070$; $NI = \$4,886,630$; $CM = \$61.73$; $TotalCM = \$8,086,630$; $CR = 61.75\%$
20. a. $\$4,556,630$; b. $\$4,019,410$; c. $\$4,969,078$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$ $\text{SAvg} = \dfrac{\sum x}{n}$ $NI = n\left(CM\right) - TFC$ $\%C = \dfrac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$ $VC = \dfrac{TVC}{n}$ $\text{VC} = \dfrac{\text{TVC}}{n}$ $CM = S - VC$ $CR = \dfrac{CM}{S} \times 100$ $CR = \dfrac{TR - TVC}{TR} \times 100$

Attribution

4.7: BREAK-EVEN ANALYSIS

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- CM = Unit contribution margin
- n = Breakeven level of output
- TFC = Total fixed cost
- TR = Total revenue at Break-even

Formulas Used

- Formula 4.7a – **Unit Break-Even**

$$n = \frac{TFC}{CM}$$

- Formula 4.7b – **Dollar Break-Even**

$$TR = \frac{TFC}{CR}$$

Introduction

Should you start up the Internet business described in the last section? Right now, all you have are some projected costs and a forecasted level of sales. You imagine you are going to sell 400 units. Is that possible? Is it reasonable to forecast this many sales?

Now you may say to yourself, "400 units a month . . . that's about 13 per day. What's the big deal?" But let's gather some more information. What if you looked up your industry in Statistics Canada data and learned that the product in question sells just $1,000$ units per month in total? Statistics Canada also indicates that there are eight existing companies selling these products. How does that volume of 400 units per month sound now? Unless you are revolutionizing your industry, it is unlikely you will receive a 40% market share in your first month of operations. With so few unit sales in the industry and too many competitors, you might be lucky to sell 100 units. If this is the case, are you still profitable?

Simply looking at the fixed costs, variable costs, potential revenues, contribution margins, and typical net income is not enough. Ultimately, all costs in a business need to be recovered through sales. Do you know how many units have to be sold to pay your bills? The answer to this question helps assess the feasibility of your business idea.

What Is Break-Even Analysis?

If you are starting your own business and head to the bank to initiate a start-up loan, one of the first questions the banker will ask you is your break-even point. You calculate this number through **break-even analysis**, which is the analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost. This **break-even point** is the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero. To determine the break-even point, you can calculate a break-even analysis in two different ways, involving either the number of units sold or the total revenue in dollars. Each of these two methods is discussed in this section.

Method 1: Break-Even Analysis in Units

In this method, your goal is to determine the level of output that produces a net income equal to zero. This method requires unit information, including the unit selling price and unit variable cost.

It is helpful to see the relationship of total cost and total revenue on a graph. Assume that a company has the following information:

$$\begin{aligned} TFC &= \$400 \\ S &= \$100 \\ VC &= \$60 \end{aligned}$$

The graph shows dollar information on the y-axis and the level of output on the x-axis. Here is how you construct such a graph:

1. Plot the total costs:
    a. At zero output you incur the total fixed costs of $\$400$. Denote this as Point 1 $(0, \$400)$.

b. As you add one level of output, the total cost rises in the amount of the unit variable cost. Therefore, total cost is:

$$TFC + (VC) = \$400 + 1(\$60) = \$460$$

Denote this as Point 2 $(1, \$460)$

c. As you add another level of output (2 units total), the total cost rises once again in the amount of the unit variable cost, producing $\$400 + 2(\$60) = \$520$. Denote this as Point 3 $(2, \$520)$.

d. Repeat this process for each subsequent level of output and plot it onto the figure. The red line plots these total costs at all levels of output.

2. Plot the total revenue:

a. At zero output, there is no revenue. Denote this as Point 4 $(0, \$0)$.

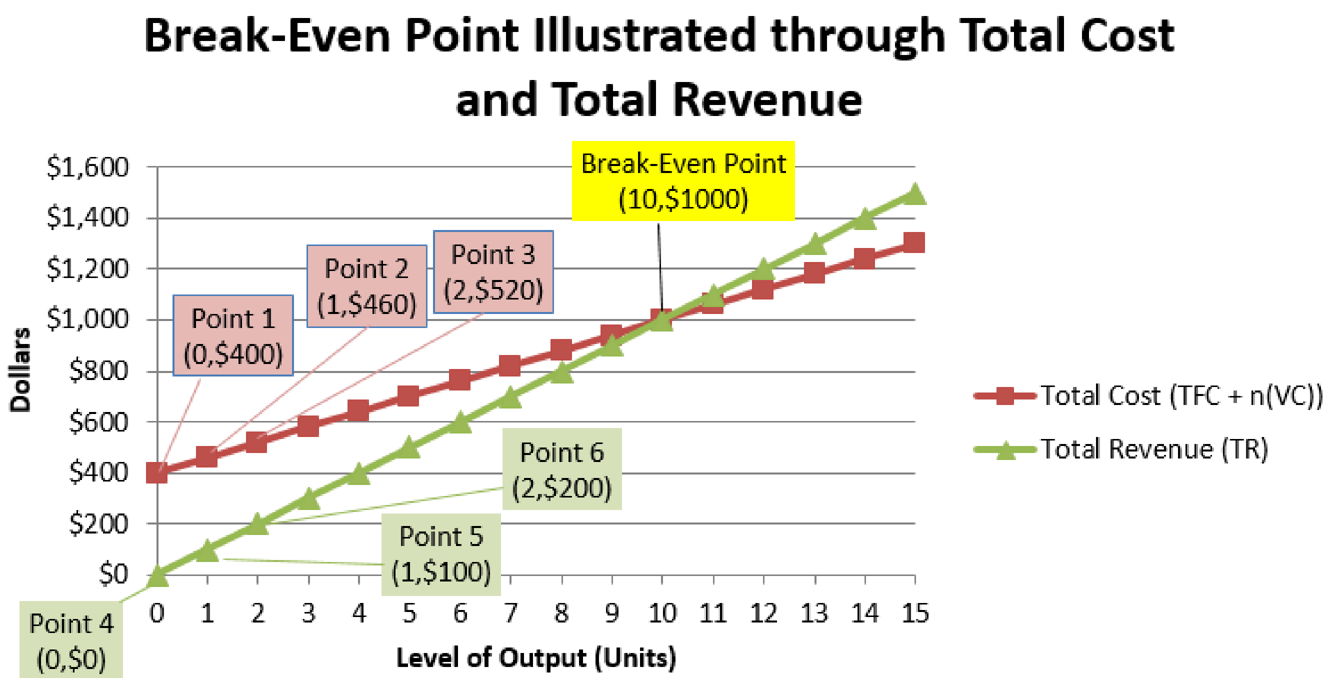


Figure 4.7.1

b. As you add one level of output, total revenue rises by the selling price of the product. Therefore, total revenue is $n(S) = 1(\$100) = \$100$. Denote this as Point 5 $(1, \$100)$.

c. As you add another level of output (2 units total), the total revenue rises once again in the amount of the selling price, producing $2(\$100) = \$200$. Denote this as Point 6 $(2, \$200)$.

d. Repeat this process for each subsequent level of output and plot it onto the figure. The green line plots the total revenue at all levels of output.

The purpose of break-even analysis is to determine the point at which total cost equals total revenue. The graph illustrates that the break-even point occurs at an output of 10 units. At this point, the total cost is

$\$400 + 10(\$60) = \$1,000$, and the total revenue is $10(\$100) = \$1,000$. Therefore, the net income is $\$1,000 - \$1,000 = \$0$; no money is lost or gained at this point.

Unit Break-Even Formula

Recall that **Formula 4.6b**$NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$ states that the net income equals total revenue minus total costs. In break-even analysis, net income is set to zero, resulting in:

$$0 = n(S) - (TFC + n(VC))$$

Rearranging and solving this formula for n gives the following:

$$0 = n(S) - TFC - n(VC)$$
$$TFC = n(S) - n(VC)$$
$$TFC = n(S - VC)$$
$$\frac{TFC}{(S - VC)} = n$$

Formula 4.6c$CM = S - VC$ states that $CM = S - VC$; therefore, the denominator becomes just CM. The calculation of the break-even point using this method is thus summarized in Formula 4.7a.

$\boxed{4.7\text{a}}$ Unit Break-Even

Unit Break-Even: $rgb]0.0, 0.0, 1.0n = \frac{rgb]0.5, 0.0, 0.5Trgb]0.5, 0.0, 0.5Frgb]0.5, 0.0, 0.5C}{rgb]1.0, 0.0, 0.0Crgb]1.0, 0.0, 0.0M}$

$rgb]0.0, 0.0, 1.0n$ is Breakeven Level of Output (Units): This is the level of output in units that produces a net income equal to zero.

{\color[rgb]{0.5, 0.0, 0.5}T}{\color[rgb]{0.5, 0.0, 0.5}F}{\color[rgb]{0.5, 0.0, 0.5}C}\;\text{is Total Fixed Costs:}

The total of all costs that are not affected by the level of output.

{\color[rgb]{1.0, 0.0, 0.0}C}{\color[rgb]{1.0, 0.0, 0.0}M}\;\text{is Unit Contribution Margin:}

The amount of money left over per unit after you have recovered your variable costs. Calculate this by taking the unit selling price and subtracting the unit variable cost (**Formula 4.6c**$CM = S - VC$). You use this money to pay off your fixed costs.

HOW TO

Calculate the break-even point in units

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the unit contribution margin (CM) by applying any needed techniques or formulas.

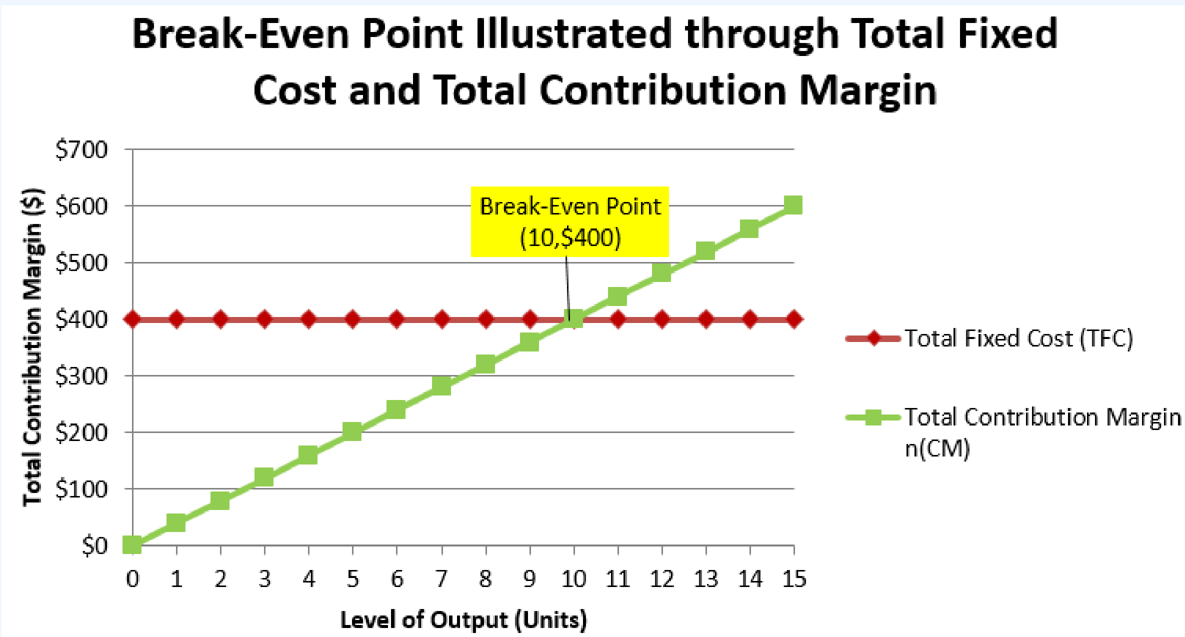


Figure 4.7.2

Step 3: Apply **Formula 4.7a** $n = \dfrac{TFC}{CM}$.

Continuing with the example that created the graph on the previous page:

Step 1: Total fixed costs are known, $TFC = \$400$.

Step 2: The unit contribution margin is $\$100 - \$60 = \$40$. For each unit sold this is the amount left over that can be applied against total fixed costs.

Step 3: Applying **Formula 4.7a** $n = \dfrac{TFC}{CM}$ results in $n = \$400 \div \$40 = 10 \text{ units}$.

Key Takeaways

When you calculate the break-even units, the formula may produce a number with decimals. For example, a break-even point might be 324.39 units. How should you handle the decimal? A partial unit cannot be sold, so the rule is always to round the level of output up to the next integer, regardless of the decimal. Why? The main point of a break-even analysis is to show the point at which you have recovered all of your costs. If you round the level of output down, you are 0.39 units short of recovering all of your costs. In the long-run, you always operate at a loss, which ultimately puts you out of business. If you round the level of output up to 325, all costs are covered and a tiny dollar amount, as close to zero as possible, is left over as profit. At least at this level of output you can stay in business.

Example 4.7.1

Recall the Internet business explored throughout Examples 4.6.1 to 4.6.4. Now let's determine the break-even point in units. As previously calculated, the total fixed costs are $\$638.03$ and the unit contribution margin is $\$3.57$.

Solution

***Step 1:* Write what you know from the question.**

The total fixed costs are known: $TFC = \$638.03$.

The contribution rate is known: $CM = \$3.57$.

Step 2: **Apply Formula 4.7a.** $n = \dfrac{TFC}{CM}$

$$n = \frac{\$638.03}{\$3.57}$$

$$n = 178.719888$$

Round this up to 179.

Calculator instructions:

FC	VC	P	PFT	Q
638.03	6.43	10	0	Answer: 178.719888

Figure 4.7.3

Step 3: **Write as a statement.**

In order for your Internet business to break even, you must sell 179 units. At a price of $\$10$ per unit, that requires a total revenue of $\$1,790$. At this level of output your business realizes a net income of $\$1$ because of the rounding.

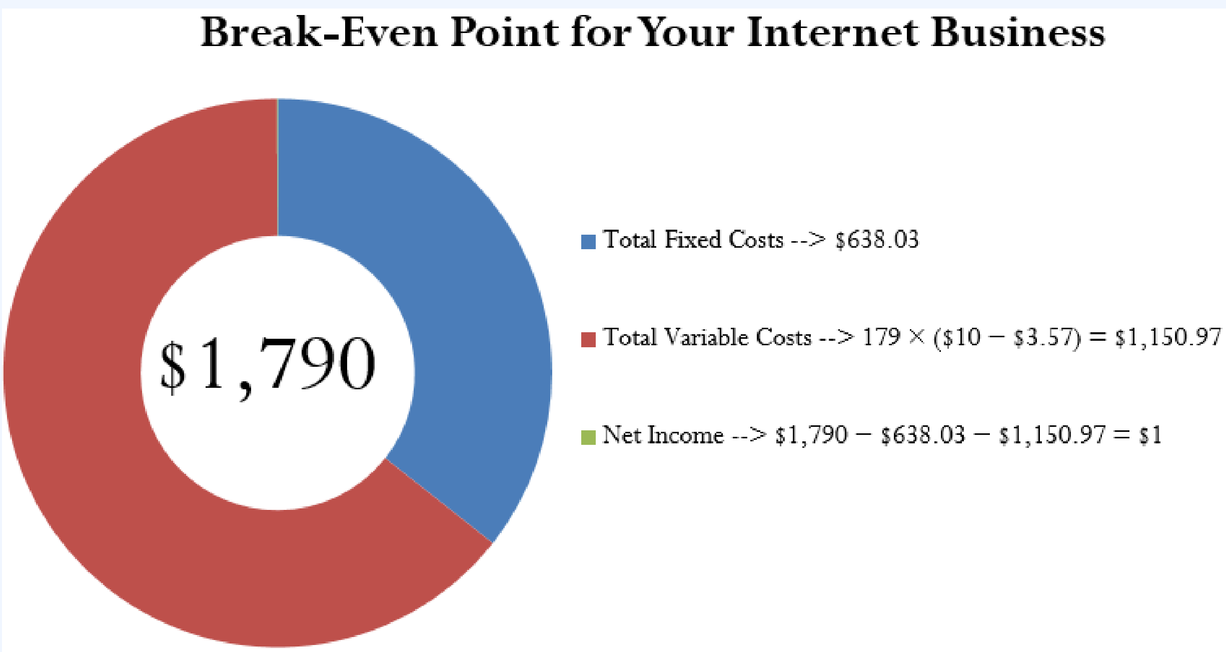


Figure 4.7.4

Method 2: Break-Even Analysis in Dollars

The income statement of a company does not display unit information. All information is aggregate, including total revenue, total fixed costs, and total variable costs. Typically, no information is listed about unit selling price, unit variable costs, or the level of output. Without this unit information, it is impossible to apply **Formula 4.7a** $n = \frac{TFC}{CM}$.

The second method for calculating the break-even point relies strictly on aggregate information. As a result, you cannot calculate the break-even point in units. Instead, you calculate the break-even point in terms of aggregate dollars expressed as total revenue.

Dollar Break-Even Formula

To derive the break-even point in dollars, once again start with **Formula 4.6b** $NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$, where total revenue at break-even less total fixed costs and total variable costs must equal a net income of zero:

$$NI = TR - (TFC + TVC)$$
$$0 = TR - (TFC + TVC)$$

Rearranging this formula for total revenue gives:

$$0 = TR - TFC - TVC$$
$$TR = TFC + TVC$$

Thus, at the break-even point the total revenue must equal the total cost. Substituting this value into the numerator of Formula 4.6f gives you:

$$CR = \frac{(TR - TVC)}{TR} \times 100$$

$$CR = \frac{((TFC \cancel{+TVC}) \cancel{-TVC})}{TR} \times 100$$

$$CR = \frac{TFC}{TR} \times 100$$

A final rearrangement results in Formula 4.7b, which expresses the break-even point in terms of total revenue dollars.

4.7b Dollar Break-Even

Dollar Break-Even: $rgb]0.0, 0.5, 0.5T rgb]0.0, 0.5, 0.5R = \frac{rgb]0.5, 0.0, 0.5T rgb]0.5, 0.0, 0.5F rgb]0.5, 0.0, 0.5C}{rgb]0.5, 0.0, 0.0C rgb]0.5, 0.0, 0.0R}$

{\color[rgb]{0.0, 0.5, 0.5}T}{\color[rgb]{0.0, 0.5, 0.5}R}\;\text{is Total Revenue at Break-even:}
The total amount of dollars that the company must earn as revenue to pay for all of the fixed and variable costs. At this level of revenue, the net income equals zero.

{\color[rgb]{0.5, 0.0, 0.5}T}{\color[rgb]{0.5, 0.0, 0.5}F}{\color[rgb]{0.5, 0.0, 0.5}C}\;\text{is Total Fixed Costs:}
The total of all costs that are not affected by the level of output.

{\color[rgb]{0.5, 0.0, 0.0}C}{\color[rgb]{0.5, 0.0, 0.0}R}\;\text{is Contribution Rate:} The percentage of the selling price, expressed in decimal format, available to pay for fixed costs.

HOW TO

Calculate the break-even point in total revenue dollars

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the contribution rate (CR), by applying any needed techniques or formulas. If not provided, typically the CR is calculated using **Formula 4.6f** $CR = \frac{TR - TVC}{TR} \times 100$, which requires aggregate information only.

Step 3: Apply **Formula 4.7b** $TR = \frac{TFC}{CR}$ to calculate the break-even point in dollars.

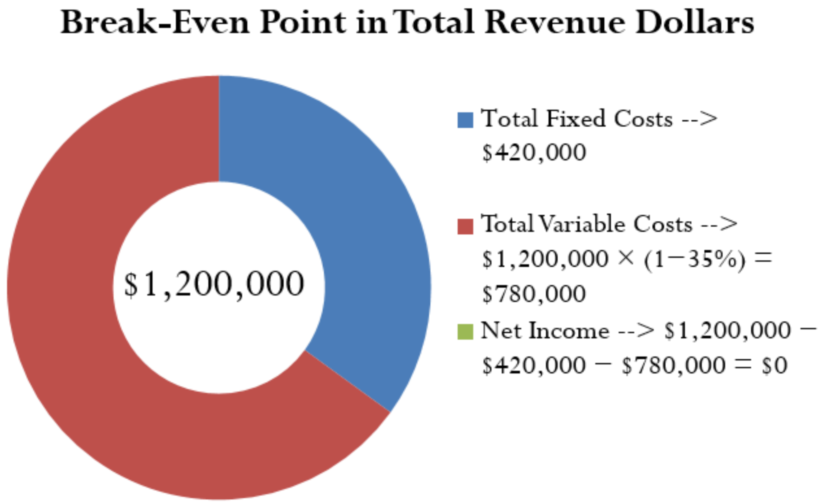


Figure 4.7.5

Assume that you are looking at starting your own business. The fixed costs are generally easier to calculate than the variable costs. After running through the numbers, you determine that your total fixed costs are $\$420,000$, or $TFC = \$420,000$. You are not sure of your variable costs but need to gauge your break-even point. Many of your competitors are publicly traded companies, so you go online and pull up their annual financial reports. After analyzing their financial statements, you see that your competitors have a contribution rate of 35%, or $CR = 0.35$, on average. What is your estimate of your break-even point in dollars?

Step 1: Total fixed costs are $TFC = \$420,000$.

Step 2: The estimated contribution rate is $CR = 0.35$.

Step 3: Applying **Formula 4.7b** $TR = \dfrac{TFC}{CR}$ results in:

$$TR = \$420,000 \div 0.35 = \$1,200,000$$

If you average a similar contribution rate, you require total revenue of $\$1,200,000$ to cover all costs, which is your break-even point in dollars.

Key Takeaways

You need to be very careful with the interpretation and application of a break-even number. In particular, the break-even must have a point of comparison, and it does not provide information about the viability of the business.

Break-Even Points Need to Be Compared. The break-even number by itself, whether in units or dollars, is meaningless. You need to compare it against some other quantity (or quantities) to determine the feasibility of the number you have produced. The other number needs to be some baseline that allows you to grasp the scope of what you are planning. This baseline could include but is not limited to the following:

- Industry sales (in units or dollars)
- Number of competitors fighting for market share in your industry
- Production capacity of your business

For example, in your Internet business the break-even point is 179 units per month. Is that good? In the section opener, you explored a possibility where your industry had total monthly sales of $1,000$ units and you faced eight competitors. A basic analysis shows that if you enter the industry and if everyone split the market evenly, you would have sales of $1,000$ divided by nine companies, equal to 111 units each. To just pay your bills, you would have to sell almost 61% higher than the even split and achieve a 17.9% market share. This doesn't seem very likely, as these other companies are already established and probably have satisfied customers of their own that would not switch to your business.

Break-Even Points Are Not Green Lights. A break-even point alone cannot tell you to do something, but it can tell you *not* to do something. In other words, break-even points can put up red lights, but at no point does it give you the green light. In the above scenario, your break-even of 179 units put up a whole lot of red lights since it does not seem feasible to obtain. However, what if your industry sold $10,000$ units instead of $1,000$ units? Your break-even would now be a 1.79% market share (179 units out of $10,000$ units), which certainly seems realistic and attainable. This does not mean "Go for it," however. It just means that from a strictly financial point of view breaking even seems possible. Whether you can actually sell that many units depends on a whole range of factors beyond just a break-even number. For instance, if your Google ad is written poorly you might not be able to generate that many sales. The break-even analysis cannot factor in this non-quantitative variable, and for that reason it cannot offer a "go ahead."

Try It

1) What would happen to the break-even point in each of the following situations? Would it increase, decrease, or remain the same?

a. The unit contribution margin increases.
b. The total fixed costs increase.
c. The contribution rate decreases.

Solution

a. Decrease. In **Formula 4.7a** $n = \frac{TFC}{CM}$, the denominator is larger, producing a lower break-even.

b. Increase. In both **Formula 4.7a** $n = \frac{TFC}{CM}$ and **Formula 4.7b** $TR = \frac{TFC}{CR}$, the numerator is larger, producing a higher break-even.

c. Increase. In **Formula 4.7b** $TR = \frac{TFC}{CR}$, the denominator is smaller, producing a higher break-even.

Example 4.7.2

In the annual report to shareholders, Borland Manufacturing reported total gross sales of $\$7,200,000$, total variable costs of $\$4,320,000$, and total fixed costs of $\$2,500,000$. Determine Borland's break-even point in dollars.

Solution

Step 1: **Write what you know from the question.**

The total fixed costs are known: $TFC = \$2,500,000$. Other known information includes the following:

$$TR = \$7,200,000$$
$$TVC = \$4,320,000$$

Step 2: **Calculate the contribution rate by applying Formula 4.6f.**

$$CR = \frac{TR - TVC}{TR} \times 100$$

$$CR = \frac{\$7,200,000 - \$4,320,000}{\$7,200,000} \times 100$$

$$CR = \frac{\$2,880,000}{\$7,200,000} \times 100$$

$$CR = 40\%$$

Step 3: **Apply Formula 4.7b** $TR = \dfrac{TFC}{CR}$ **and solve.**

$$TR = \frac{\$2,500,000}{40\%}$$

$$TR = \frac{\$2,500,000}{0.4}$$

$$TR = \$6,250,000$$

Step 4: **Write as a statement.**

Borland Manufacturing achieves its break-even point at $\$6,250,000$ in total revenue. At this point, total fixed costs are $\$2,500,000$ and total variable costs are $\$3,750,000$, producing a net income of zero.

Section 4.7 Exercises

In each of the following questions, round all of the money and percentages to two decimals unless otherwise specified.

Mechanics

1. Franklin has started an ink-jet print cartridge refill business. He has invested $\$2,500$ in equipment and machinery. The cost of refilling a cartridge including labor, ink, and all other materials is $\$4$. He charges $\$14.95$ for his services. How many cartridges does he need to refill to break even?
2. Hasbro manufactures a line of children's pet toys. If it sells the toy to distributors for $\$2.30$ each while variable costs are 75¢ toy, how many toys does it need to sell to recover the fixed cost investment in these toys of $\$510,000$? What total revenue would this represent?
3. You are thinking of starting your own business and want to get some measure of feasibility. You have determined that your total fixed costs would be $\$79,300$. From annual business reports and competitive studies, you estimate your contribution rate to be 65%. What is your break-even in dollars?
4. If a business has total revenue of $\$100,000$, total variable costs of $\$60,000$, and total fixed costs of $\$20,000$, determine its break-even point in dollars.
5. If the break-even point is $15,000$ units, the selling price is $\$95$, and the unit variable cost is $\$75$, what are the company's total fixed costs?
6. Louisa runs a secretarial business part time in the evenings. She takes dictation or

handwritten minutes and converts them into printed word-processed documents. She charges $\$5$ per page for her services. Including labor, paper, toner, and all other supplies, her unit variable cost is $\$2.50$ per page. She invested $\$3,000$ worth of software and equipment to start her business. How many pages will she need to output to break even?

7. If your organization has a contribution rate of 45% and knows the break-even point is $\$202,500$, what are your organization's total fixed costs?
8. What is the unit contribution margin on a product line that has fixed costs of $\$1,800,000$ with a break-even point of $360,000$ units?

Solutions

1. 229
2. $329,033$ toys; $TR = \$756,775.90$
3. $\$122,000$
4. $\$50,000$
5. $\$300,000$
6. $1,200$ pages
7. $\$91,125$
8. $\$5$

Applications

9. Ashley rebuilds old laptops as a home hobby business. Her variable costs are $\$125$ per laptop and she sells them for $\$200$. She has determined that her break-even point is 50 units per month. Determine her net income for a month in which she sells 60 units.
10. Burton Snowboards reported the following figures last week for its Custom V Rocker Snowboard:

$$\begin{aligned} \text{Sales} &= \$70,000 \\ \text{Total Fixed Costs} &= \$19,000 \\ \text{Total Variable Costs} &= \$35,000 \\ \text{Total Costs} &= \$54,000 \\ \text{Net Income} &= \$16,000 \end{aligned}$$

If the above numbers represent 70% operational capacity, express the weekly break-even point in dollars as a percentage of maximum capacity.

11. Shardae is starting a deluxe candy apple business. The cost of producing one candy apple is $\$4.50$. She has total fixed costs of $\$5,000$. She is thinking of selling her deluxe apples for $\$9.95$ each.

    a. Determine her unit break-even point at her selling price of $\$9.95$.
    b. Shardae thinks her price might be set too high and lowers her price to $\$8.95$. Determine her new break-even point.
    c. An advertising agency approaches Shardae and says people would be willing to pay the $\$9.95$ if she ran some "upscale" local ads. They would charge her $\$1,000$. Determine her break-even point.
    d. If she wanted to maintain the same break-even units as determined in a., what would the price have to be to pay for the advertising?

12. Robert is planning a wedding social for one of his close friends. Costs involve $\$865$ for the hall rental, $\$135$ for a liquor license, $\$500$ for the band, and refreshments and food from the caterer cost $\$10$ per person. If he needs to raise $\$3,000$ to help his friend with the costs of his wedding, what price should he charge per ticket if he thinks he can fill the social hall to its capacity of 300 people?
13. Boston Beer Company, the brewer of Samuel Adams, reported the following financial information to its shareholders:

$$\begin{aligned} \textbf{Total Revenue} &= \$388,600,000 \\ \textbf{Total Variable Costs} &= \$203,080,000 \\ \textbf{Total Fixed Costs} &= \$182,372,000 \\ \textbf{Total Costs} &= \$385,452,000 \\ \textbf{Net Income} &= \$3,148,000 \end{aligned}$$

    If this represented sales of $2,341,000$ barrels of beer, determine its break-even point in units and dollars.

14. In the beverage industry, PepsiCo and The Coca-Cola Company are the two big players. The following financial information, in millions of dollars, was reported to its shareholders:

$$\begin{aligned} &\text{PepsiCo} \\ \text{Total Revenue} &= \$14.296 \\ \text{Total Variable Costs} &= \$7.683 \\ \text{Total Fixed Costs} &= \$6.218 \\ \text{Total Costs} &= \$13.901 \\ \text{Net Income} &= \$0.395 \end{aligned}$$

$$\begin{aligned} &\text{The Coca-Cola Company} \\ \text{Total Revenue} &= \$21.807 \\ \text{Total Variable Costs} &= \$12.663 \\ \text{Total Fixed Costs} &= \$8.838 \\ \text{Total Costs} &= \$21.501 \\ \text{Net Income} &= \$0.306 \end{aligned}$$

Compare the break-even points in total dollars between the two companies based on these reports.

Solutions

9. $\$750$
10. 38%
11. a. 918; b. $1,124$; c. $1,101$; d. \$11.04
12. $\$25$
13. $2,301,277$ barrels; $TR = \$382,006,032.77$
14. PepsiCo = $\$13,442,088,008.47$; The Coca-Cola Company = $\$21,077,238,188.98$; PepsiCo is 36.22% lower

Challenge, Critical Thinking, & Other Applications

15. Calculate the following:
    a. By what percentage does the unrounded unit break-even point change if the unit contribution margin increases by 1% while all other numbers remain the same?
    b. By what percentage does the unrounded unit break-even point change if total fixed costs are reduced by 1% while all other numbers remain the same?
    c. What do the solutions to the above questions illustrate?

16. École Van Belleghem is trying to raise funds to replace its old playground equipment with a modern, child-safe structure. The Blue Imp playground equipment company has quoted the school a cost of $\$49,833$ for its $20m \times 15m$ megastructure. To raise the funds, the school wants to sell Show 'n' Save books. These books retail for $\$15.00$ each and cost $\$8.50$ to purchase. How many books must the school sell to raise funds for the new playground?
17. The Puzzle Company had total revenue of $\$4,750,000$, total fixed costs of $\$1,500,000$, and total variable costs of $\$2,750,000$. If the company desires to earn a net income of $\$1,000,000$, what total sales volume is needed to achieve the goal?
18. Whirlpool Corporation had annual sales of $\$18.907$ billion with a net income of $\$0.549$ billion. Total variable costs amounted to $\$16.383$ billion.
    a. Determine Whirlpool Corporation's break-even in dollars.
    b. If Whirlpool Corporation managed to increase revenues by 5% the following year while implementing cost-cutting measures that trimmed variable costs by 2%, determine the percent change in the break-even dollars.

Use the following information for questions 19 and 20:

$$S = \$100$$
$$VC = \$60$$
$$TFC = \$250,000$$

19. Calculate the current break-even point in both units and dollars.
20. A production manager is trying to control costs but is faced with the following tradeoffs under three different situations:
    a. Total fixed costs are reduced by 15%, but unit variable costs will rise by 5%.
    b. Unit variable costs are reduced by 10%, but fixed costs will rise by 5%
    c. Total fixed costs are reduced by 20%, but unit variable costs will rise by 10%.
    d. Unit variable costs are reduced by 15%, but fixed costs will rise by 15%.
    e. Based strictly on break-even calculations, which course of action would you recommend she pursue?

Solutions

15. a. $-0.\overline{9900}\%$; b. -1%; c. lowering TFC always better

16. $7,667$ books
17. $\$5,937,500$
18. a. $\$14.795$ billion; b. -30.2%
19. $n = 6,250$; $TR = \$625,000$
20. a. $n = 5,744$; $TR = \$574,000$; b. $n = 5,707$; $TR = \$570,700$; c. $n = 5,883$; $TR = \$588,300$; d. $n = 5,868$; $TR = \$586,800$; e. Option B

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER.$NI = n\left(S\right) - \left(TFC + n\left(VC\right)\right)$ $CM = S - VC$ $n = \dfrac{TFC}{CM}$ $CR = \dfrac{TR - TVC}{TR} \times 100$ $TR = \dfrac{TFC}{CR}$

Attribution

4.8: CHAPTER 4 SUMMARY

Formula & Symbol Hub Summary

For this chapter you used the following:

Symbols Used

- $\times 100$ = Percentage multiplier
- $\%C$ = Percent change
- C = Cost
- CM = Unit contribution margin
- CR = Contribution rate
- d = Discount rate or Cash discount rate
- $D\$$ = Discount amount
- d_1 = First discount
- d_2 = Second discount
- d_{equiv} = Single equivalent discount rate
- d_n = nth discount
- E = Expenses
- L = List price or Invoice amount
- $M\$$ = Markup amount
- $MoC\%$ = Markup on cost percentage
- $MoS\%$ = Markup on selling price Percentage
- n = Level of output / break even output level
- N = Net price or Net payment amount
- $n(CM)$ = Total contribution margin
- NI = Net income
- $n(S)$ = Total revenue

- P = Profit
- P_{onsale} = Planned profit amount
- P_{red} = Reduced profit
- S = Selling price
- SAvg = Simple Average
- S_{onsale} = Sale price
- S_{red} = Reduced sale price
- TR = Total revenue
- TFC = Total fixed costs
- $TFC + n(VC)$ = Total costs
- TR = Total revenue at Break-even
- TVC = Total variable cost
- VC = Unit variable cost
- V_{i} = Initial value
- V_{f} = Final value

Formulas Used

- Formula 2.4a – **Simple Average**

$$\text{SAvg} = \frac{\sum x}{n}$$

- Formula 3.2a – **Percent Change**

$$\%C = \frac{V_{\text{f}} - V_{\text{i}}}{V_{\text{i}}} \times 100$$

- Formula 4.1a – **Single Discount**

$$N = L \times (1 - d)$$

- Formula 4.1b – **Discount Amount**

$$D\$ = L \times d$$

- Formula 4.1c – **Discount Amount**

$$D\$ = L - N$$

- Formula 4.1d – **Multiple Discounts**

$$N = L \times (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$$

- Formula 4.1e – **Single Equivalent Discount**

$$d_{equiv} = 1 - (1 - d_1) \times (1 - d_2) \times \ldots \times (1 - d_n)$$

- Formula 4.2a – **Single Discount Rearranged**

$$L = \frac{N}{(1 - d)}$$

- Formula 4.2b – **Single Discount**

$$N = L \times (1 - d)$$

- Formula 4.3a – **The Selling Price of a Product**

$$S = C + E + P$$

- Formula 4.3b – **Markup Amount**

$$M\$ = E + P$$

- Formula 4.3c – **Selling Price Using Markup**

$$S = C + M\$$$

- Formula 4.3d – **Markup on Cost Percentage**

$$MoC\% = \frac{M\$}{C} \times 100$$

- Formula 4.3e – **Markup on Selling Price Percentage**

$$MoS\% = \frac{M\$}{S} \times 100$$

- Formula 4.4a – **The Sale Price of a Product**

$$S_{onsale} = S \times (1 - d)$$

- Formula 4.4b – **Markdown Amount**

$$D\$ = S \times d$$

- Formula 4.4c – **Markdown Amount**

$$D\$ = S - S_{onsale}$$

- Formula 4.4d – **Markdown Percentage**

$$d = \frac{D\$}{S} \times 100$$

- Formula 4.4e – **The Selling Price of a Product Adapted**

$$S_{onsale} = C + E + P_{onsale}$$

- Formula 4.5a – **Reduced Profit**

$$P_{red} = S_{red} - C - E$$

- Formula 4.5b – **Reduced Profit**

$$P_{red} = P - E$$

- Formula 4.6a – **Unit Variable Cost**:

$$VC = \frac{TVC}{n}$$

- Formula 4.6b – **Net Income Using a Total Revenue and Cost Approach**

$$NI = n(S) - (TFC + n(VC))$$

- Formula 4.6c – **Unit Contribution Formula**

$$CM = S - VC$$

- Formula 4.6d – **Net Income Using Total Contribution Margin Approach Formula**

$$NI = n(CM) - TFC$$

- Formula 4.6e – **Contribution Rates Formula**

$$\text{Contribution Rate if Unit Information Is Known:}$$
$$CR = \frac{CM}{S} \times 100$$

- Formula 4.6f – **Contribution Rate if Aggregate Information Is Known**

$$CR = \frac{TR - TVC}{TR} \times 100$$

- Formula 4.7a – **Unit Break-Even**

$$n = \frac{TFC}{CM}$$

- Formula 4.7b – **Dollar Break-Even**

$$TR = \frac{TFC}{CR}$$

Key Concepts Summary

Section 4.1: Figuring Out the Cost: Discounts (How Much?)

- The relationship between distribution and pricing

- Some of the types of discounts available to businesses and consumers
- How to calculate the net price when only one discount is involved
- How to calculate the net price when multiple discounts are involved
- Converting multiple discounts into single discounts

Section 4.2: Invoicing: Terms of Payment and Cash Discounts (Make Sure You Bill Them)

- How invoicing works: understanding invoice terms and invoice dating
- The calculations involved when a full payment amount is made
- The calculations involved when a partial payment amount is made
- The calculations involved when a late payment amount is made

Section 4.3: Markup: Setting the Regular Price (Need to Stay in Business)

- The three components that compose a selling price
- Calculating the markup in dollars and the relationship to pricing components
- Calculating the markup as a percentage under two different approaches
- Determining the price point where all costs are paid but no profits are earned—the break-even point

Section 4.4: Markdown: Setting the Sale Price (Everybody Loves a Sale)

- How to take a regular selling price and make it into a sale price by applying markdowns
- How to plan the merchandising of products that are always on sale

Section 4.5: Merchandising (How Does It All Come Together?)

- Combining discounts, markup, and markdown into complete merchandising situations
- Calculating the maintained markup when a product was sold at both a regular selling price and a sale price
- How coupons and mail-in rebates impact product pricing

Section 4.6: Cost-Revenue-Net Income Analysis (Need to Be in the Know)

- The three different types of costs
- How to determine the unit variable cost
- Calculating net income when you know total revenue and total cost
- Calculating net income when you know the contribution margin
- Comparing varying contribution margins by calculating a contribution rate
- Integrating all of the concepts together

Section 4.7: Break-Even Analysis (Sink or Swim)

- Explanation of break-even analysis
- How to calculate the break-even point expressed in the level of output
- How to calculate the break-even point expressed in total revenue dollars

The Language of Business Mathematics

cost An outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services.

coupon A promotion that entitles a consumer to receive certain benefits, usually in the form of a reduction of the selling price for a product.

coupon handling expense A handling charge that is paid to channel members for redeeming a coupon.

coupon marketing expense The expense associated with creating, distributing, and redeeming a coupon.

coupon redemption expense The face value price reduction offered by a coupon.

credit period The number of interest-free days from the date of commencement before full payment of the invoice is required.

date of commencement The first day from which the invoice terms extend forward in time and from which all due dates are established.

discount A reduction in the price of a product.

discount period The number of days from the date of commencement for which a cash discount is offered.

end-of-month invoice dating A term of payment where the date of commencement is the last day of the same month as indicated by the invoice date.

expenses A business's financial outlays incurred in the selling of a product.

list price A price for a product that has been published or advertised in some way.

loyalty discount A discount given from a seller to a purchaser for repeat business.

mail-in rebate A refund that occurs after a product has been purchased.

maintained markup The average level of markup that is maintained across all units sold at various price levels including the selling price and the sale price(s).

manufacturer's suggested retail price (MSRP) A recommended product retail price that a manufacturer sets for a retailer based on market research.

markdown A reduction from the regular selling price of a product resulting in a new lower sale price.

marketing price adjustment Any marketing activity executed by a member of the distribution channel for the purposes of altering a product's price.

markup The process of taking a product's cost and increasing it by a certain amount to arrive at a selling price.

markup amount The dollar amount of the expenses and profit combined together into a single number; it represents the difference between the price and cost in dollars.

markup on cost percentage The markup dollars expressed as a rate using cost as the base.

markup on selling price percentage The markup dollars expressed as a rate using the regular selling price as the base.

net price The price of the product after a discount is removed from the list price.

ordinary invoice dating A term of payment where the date of commencement is the same date as the invoice date.

profit The amount of money that remains after a business pays all of its costs and expenses.

quantity discount A discount for purchasing larger quantities of a certain product.

rebate marketing expense The expense associated with creating, distributing, and redeeming a rebate.

rebate redemption expense The face value amount that a consumer will receive as a refund if a submitted rebate fulfills all rebate conditions.

receipt-of-goods invoice dating A term of payment where the date of commencement is the day on which the customer physically receives the goods.

sale discount A temporary discount lowering the price from a product's regular selling price.

sale price A price for a product after a markdown that is lower than its regular selling price.

seasonal discount A discount offered to consumers and businesses for purchasing products out of season.

single equivalent discount A single discount rate that is equal to a series of multiple rate discounts.

trade discount A discount offered to businesses only based on the type of business and its location in the distribution system.

Technology

Calculator

The following calculator functions were introduced in this chapter:

Markup on Selling Price Percentage

2^{nd} Profit to access this feature.

Enter two of the three variables by pressing Enter after each input and using and to scroll through the display. The variables are:

CST = The cost of the item

SEL = The selling price of the item

MAR = The markup on selling price percentage (in $\%$ format)

Press CPT on the unknown (when it is on the screen display) to compute.

Attribution

CHAPTER 5: SIMPLE INTEREST: WORKING WITH SINGLE PAYMENTS AND APPLICATIONS

Outline of Chapter Topics

Learning Objectives

- Demonstrate the concept of simple interest.
- Determine the number of days between two calendar days using the pre-programmed financial calculator method.
- Calculate the amount of interest, principal, time, interest rate, and maturity value of investments and loans.
- Calculate equivalent payments that replace another payment or a series of payments.
- Use simple interest in solving problems involving business applications such as savings accounts, short term guaranteed investment certificates (GICs), treasury bills, and commercial paper.

Attribution

5.1: PRINCIPAL, RATE, TIME

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $I =$ Simple Interest
- $P =$ Present Value or Principal
- $r =$ Interest rate
- $t =$ Time period over which interest is charged

Formulas Used

- Formula 5.1 – **Simple Interest**

$$I = Prt$$

- Formula 3.1b – **Rate, Portion, and Base**

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

Introduction

Your investments may be at risk if stock and bond markets slump, as a story in the *Globe and Mail* predicts. You wonder if you should shift your money into relatively secure short-term investments until the market booms again. You consider your high-interest savings account, but realize that only the first $\$60,000$ of your savings account is insured. Perhaps you should put some of that money into treasury bills instead.

Looking ahead, what income will you live on once you are no longer working? As your career develops, you need to save money to fund your lifestyle in retirement. Some day you will have $\$100,000$ or more that you must invest and reinvest to reach your financial retirement goals.

To make such decisions, you must first understand how to calculate simple interest. Second, you need to understand the characteristics of the various financial options that use simple interest. Armed with this knowledge, you can make smart financial decisions!

The world of finance calculates interest in two different ways:

1. **Simple Interest:** A *simple interest* system primarily applies to short-term financial transactions, with a time frame of less than one year. In this system, which is explored in this chapter, interest accrues but does not compound.
2. **Compound Interest:** A *compound interest* system primarily applies to long-term financial transactions, with a time frame of one year or more. In this system, which the next chapter explores, interest accrues and compounds upon previously earned interest.

Simple Interest

In a simple interest environment, you calculate interest solely on the amount of money at the beginning of the transaction. When the term of the transaction ends, you add the amount of the simple interest to the initial amount. Therefore, throughout the entire transaction the amount of money placed into the account remains unchanged until the term expires. It is only on this date that the amount of money increases. Thus, an investor has more money or a borrower owes more money at the end.

The figure illustrates the concept of simple interest. In this example, assume $\$1,000$ is placed into an account with 12% simple interest for a period of 12 months. For the entire term of this transaction, the amount of money in the account always equals $\$1,000$. During this period, interest accrues at a rate of 12%, but the interest is never placed into the account. When the transaction ends after 12 months, the $\$120$ of interest and the initial $\$1,000$ are then combined to total $\$1,120$.

A loan or investment always involves two parties—one giving and one receiving. No matter which party you are in the transaction, the amount of interest remains unchanged. The only difference lies in whether you are earning or paying the interest.

- If you take out a personal loan from a bank, the bank gives you the money and you receive the money. In this situation, the bank earns the simple interest and you are being charged simple interest on your loan. In the figure, this means you must pay back not only the $\$1,000$ you borrowed initially but an additional $\$120$ in interest.

- If you place your money into an investment account at the bank, you have given the money and the bank has received the money. In this situation, you earn the simple interest on your money and the bank pays simple interest to your investment account. In the figure, this means the bank must give you back your initial $\$1,000$ at the end plus an additional $\$120$ of interest earned.

The best way to understand how simple interest is calculated is to think of the following relationship:

\text{Amount of Interest}\;=\;\underbrace{\text{How Much}}\text{ at }\underbrace{\text{What Simple Interest Rate}}\text{ for }\underbrace{\text{How Long}}

Notice that the key variables are the amount, the simple interest rate, and time. Formula 5.1 combines these elements into a formula for simple interest.

5.1 Simple Interest

$$I = Prt$$

Formula does not parse The **interest amount** is the dollar amount of interest that is paid or earned. To interpret the interest amount properly, remember who you are in the transaction. If you are borrowing the money, this is the interest amount charged to you. If you are investing the money, this is the interest amount you earn.

Formula does not parse The amount of money at the beginning of the time period being analyzed is known as the **present value**, or P. If this is in fact the amount at the start of the financial transaction, it is also called the **principal**, which is the original amount of money that is borrowed or invested. Or it can simply be the amount at some earlier time before the future value was known. In any case, the amount excludes the interest.

t is Time: The **time period** or term is the length of the financial transaction for which interest is charged or earned.

$rgb]0.5, 0.0, 0.5r\ rgb]0.0, 0.0, 0.0$is Interest Rate: The **interest rate** is the rate of interest that is charged or earned during a specified time period. It is usually expressed in percent format. Unless noted otherwise, interest rates are expressed on an annual basis. An interest rate is the result of **Formula 3.1b** $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$, where:

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

In this case, you calculate an annual interest rate in its decimal format as follows:

$$\text{Annual Interest Rate} = \frac{\text{Annual Interest Amount}}{\text{Principal}}$$

Thus, if you are earning $\$3$ of interest annually on a $\$100$ principal, then your annual interest rate is $\frac{3}{100} = 0.03$ or 3%.

HOW TO

Calculate Simple Interest

Follow these steps when you calculate the amount of simple interest:

Step 1: Formula 5.1 $I = Prt$ has four variables, and you need to identify three for any calculation involving simple interest. If necessary, draw a timeline to illustrate how the money is being moved over time.

Step 2: Ensure that the simple interest rate and the time period are expressed with a common unit. If they are not already, you need to convert one of the two variables to the same units as the other.

Step 3: Apply **Formula 5.1** $I = Prt$ and solve for the unknown variable. Use algebra to manipulate the formula if necessary.

Assume you have $\$500$ earning 3% simple interest for a period of nine months. How much interest do you earn?

Step 1: Note that your principal is $\$500$, or $P = \$500$. The interest rate is assumed to be annual, so $r = 3\%$ per year. The time period is nine months.

Step 2: Convert the time period from months to years: $t = \frac{9}{12}$.

Step 3: According to Formula 5.1:

$$I = Prt$$

$$I = \$500 \times 3\% \times \frac{9}{12}$$

$$I = \$11.25$$

Therefore, the amount of interest you earn on the $\$500$ investment over the course of nine months is $\$11.25$.

Key Takeaways

Recall that algebraic equations require all terms to be expressed with a common unit. This principle remains true for **Formula 5.1** $I = Prt$, particularly with regard to the interest rate and the time period. For example, if you have a 3% annual interest rate for nine months, then either

- The time needs to be expressed annually as $\frac{9}{12}$ of a year to match the yearly interest rate, or
- The interest rate needs to be expressed monthly as $\frac{3\%}{12} = 0.25\%$ per month to match the number of months.

It does not matter which you do so long as you express both interest rate and time in the same unit. If one of these two variables is your algebraic unknown, the unit of the known variable determines the unit of the unknown variable. For example, assume that you are solving **Formula 5.1** $I = Prt$ for the time period. If the interest rate used in the formula is annual, then the time period is expressed in number of years.

Paths To Success

Four variables are involved in the simple interest formula, which means that any three can be known, requiring you to solve for the fourth missing variable. To reduce formula clutter, the triangle technique illustrated in the video helps you remember how to rearrange the formula as needed.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=168#h5p-7

Concept Check

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=168#h5p-8

Try It

1) Answer the following:

a. If you have money in your savings account (or any other investment) and it earns simple interest over a period of time, would you have more or less money in your account in the future?
b. If you have a debt for which you haven't made any payments, yet it is being charged simple interest on the principal, would you owe more or less money in the future?

Solution

a. More, because the interest is earned and therefore is added to your savings account.
b. More, because you owe the principal and you owe the interest, which increases your total

amount owing

Example 5.1.1

Julio borrowed $\$1,100$ from Maria five months ago. When he first borrowed the money, they agreed that he would pay Maria 5% simple interest. If Julio pays her back today, how much interest does he owe her?

Solution

Step 1: **What are we looking for?**

You need to calculate the amount of interest that Julio owes Maria.

Step 2: **What do we already know?**

The terms of their agreement are as follows:

$$P = \$1,100 \qquad r = 5\% \qquad t = 5 \text{ months}$$

Step 3: **Make substitutions using the information known above.**

The rate is annual, and the time is in months. Convert the time to an annual number. Since five months out of 12 months in a year is $\frac{5}{12}$ of a year, $t = \frac{5}{12}$.

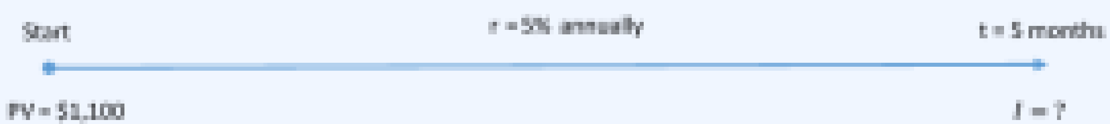


Figure 5.1.1

Apply **Formula 5.1** $I = Prt$.

$$I = \$1,100 \times 5\% \times \frac{5}{12}$$
$$I = \$1,100 \times 0.05 \times \frac{5}{12}$$
$$I = \$22.92$$

Step 4: **Provide the information in a worded statement.**

For Julio to pay back Maria, he must reimburse her for the $\$1,100$ principal borrowed plus an additional $\$22.92$ of simple interest as per their agreement.

Example 5.1.2

A $\$3,500$ investment earned $\$70$ of interest over the course of six months. What annual rate of simple interest did the investment earn?

Solution

Step 1: **What are we looking for?**

You need to calculate the annual interest rate (r).

Step 2: **What do we already know?**

The principal, interest amount, and time are known:

$$P = \$3,500 \qquad I = \$70 \qquad t = 6 \text{ months}$$

Step 3: **Make substitutions using the information known above.**

The computed interest rate needs to be annual, so you must express the time period annually as well. Since six months out of 12 months in a year is $\frac{6}{12}$ of a year, $t = \frac{6}{12}$.

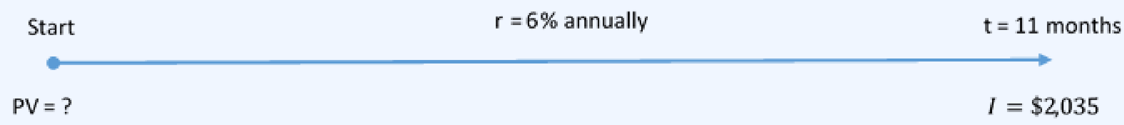


Figure 5.1.2

Apply **Formula 5.1** $I = Prt$, rearranging for r.

$$\$70 = \$3,500 \times r \times \frac{6}{12}$$

$$r = \frac{\$70}{\$3,500 \times \frac{6}{12}}$$

$$r = \frac{\$70}{\$1,750}$$

$$r = 0.04 \text{ or } 4\%$$

***Step 4:* Provide the information in a worded statement.**

For $\$3,500$ to earn $\$70$ simple interest over the course of six months, the annual simple interest rate must be 4%.

Example 5.1.3

What amount of money invested at 6% annual simple interest for 11 months earns $\$2,035$ of interest?

Solution

***Step 1:* What are we looking for?**

You need to calculate the amount of money originally invested, which is known as the present value or principal, symbolized by P.

Step 2: **What do we already know?**

The interest rate, time, and interest earned are known:

$$r = 6\% \qquad t = 11 \text{ months} \qquad I = \$2,035$$

Step 3: **Make substitutions using the information known above.**

Convert the time from months to an annual basis to match the interest rate. Since eleven months out of 12 months in a year is $\frac{11}{12}$ of a year, $t = \frac{11}{12}$.

Apply **Formula 5.1** $I = Prt$, rearranging for P.

$$\$2,035 = P \times 6\% \times \frac{11}{12}$$

$$P = \frac{\$2,035}{6\% \times \frac{11}{12}}$$

$$P = \frac{\$2,035}{0.06 \times 0.916}$$

$$P = \$37,000$$

Step 4: **Provide the information in a worded statement.**

To generate $\$2,035$ of simple interest at 6% over a time frame of 11 months, $\$37,000$ must be invested.

Example 5.1.4

For how many months must $\$95,000$ be invested to earn $\$1,187.50$ of simple interest at an interest rate of 5%?

Solution

Step 1: **What are we looking for?**

You need to calculate the length of time in months (t) that it takes the money to acquire the interest.

Step 2: **What do we already know?**

The amount of money invested, interest earned, and interest rate are known:

$$P = \$95,000.00 \qquad I = \$1,187.50 \qquad r = 5\%$$

Step 3: **Make substitutions using the information known above.**

Express the time in months. Convert the interest rate to a "per month" format. 5% per year converted into a monthly rate is $r = \frac{0.05}{12}$.

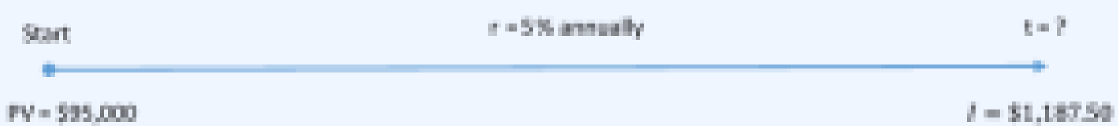


Figure 5.1.3

Apply **Formula 5.1** $I = Prt$, rearranging for t.

$$\$1,187.50 = \$95,000 \times \frac{0.05}{12} \times t$$

$$t = \frac{\$1,187.50}{\$95,000 \times \frac{0.05}{12}}$$

$$t = \frac{\$1,187.50}{\$95,000 \times 0.00416}$$

$$t = 3 \text{ months}$$

Step 4: **Provide the information in a worded statement.**

For $\$95,000$ to earn $\$1,187.50$ at 5% simple interest, it must be invested for a three-month period.

Try It

2) If you want to earn $\$1,000$ of simple interest at a rate of 7% in a span of five months, how much money must you invest?

Solution

***Step 1:* Given information:**

$$I = \$1,000; \qquad r = 7\% \text{ annually}; \qquad t = 5 \text{ months}$$

***Step 2:* Convert monthly t to match annual r:**

$$t = \frac{5}{12}$$

***Step 3:* Solve for P.**

$$P = \frac{I}{rt}$$

$$P = \frac{\$1,000}{0.07\left(\frac{5}{12}\right)}$$

$$P = \$34,285,71$$

***Step 4:* Write as a statement.**

I must invest $\$34,285.71$.

Try It

3) If you placed $\$2,000$ into an investment account earning 3% simple interest, how many months does it take for you to have $\$2,025$ in your account?

Solution

Step 1: **Given information:**

$$I = \$2,025 - \$2,000 = \$25; \qquad P = \$2,000; \qquad r = 3\% \text{ annually}$$

Step 2: **Convert annual r to match monthly t:**

$$r = \frac{3\%}{12}$$

Step 3: **Solve for t.**

$$t = \frac{I}{Pr}$$

$$t = \frac{\$25}{\$2,000\left(\frac{0.03}{12}\right)}$$

$$t = 5 \text{ months}$$

Step 4: **Write as a statement.**

It takes 5 months to have $\$2,025$ in the account.

Try It

4) A $\$3,500$ investment earned $\$70$ of interest over the course of six months. What annual rate of simple interest did the investment earn?

Solution

***Step 1*: Given information:**

$$P = \$3,500; \quad I = \$70; \quad t = 6 \text{ months}$$

***Step 2*: Convert the time period from months to years:**

$$t = \frac{6}{12}$$

***Step 3*: Solve for r.**

$$r = \frac{I}{Pt}$$

$$r = \frac{\$70}{\$3,500\left(\frac{6}{12}\right)}$$

$$r = 0.04 \text{ or } 4\%$$

***Step 4:* Write as a statement.**

The investment earned 4% simple interest.

Time and Dates

In the examples of simple interest so far, the time period was given in months. While this is convenient in many situations, financial institutions and organizations calculate interest based on the exact number of days in the transaction, which changes the interest amount.

To illustrate this, assume you had money saved for the entire months of July and August, where $t = \dfrac{2}{12}$ or $t = 0.16666... = 0.1\overline{6}$ of a year. However, if you use the exact number of days, the 31 days in July and

31 days in August total 62 days. In a 365-day year that is $t = \dfrac{62}{365}$ or $t = 0.169863$ of a year. Notice a difference of 0.003196 $(0.169863 - 0.16)$ occurs. Therefore, to be precise in performing simple interest calculations, you must calculate the **exact number of days** involved in the transaction.

Using The BA 2+ Plus Date Function to Calculate the Exact Number of Days

In the video below we'll demonstrate how to use the BA2+ Date Function:

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=168#h5p-9

Key Takeaways

When solving for t, decimals may appear in your solution. For example, if calculating t in days, the answer may show up as 45.9978 or 46.0023 days; however, interest is calculated only on complete days. This occurs because the interest amount (I) used in the calculation has been rounded off to two decimals. Since the interest amount is imprecise, the calculation of t is imprecise. When this occurs, round t off to the nearest integer.

Example 5.1.5

On September 13, 2011, Aladdin decided to pay back the Genie on his loan of $\$15,000$ at 9% simple interest. If he paid the Genie the principal plus $\$1,283.42$ of interest, on what day did he borrow the money from the Genie?

Solution

Step 1: **Given variables:**

$$P = \$15,000; \quad I = \$1,283.42; \quad r = 9\% \text{ per year;}$$
$$\text{End Date} = \text{September 13, 2011}$$

Step 2: **The time is in days, but the rate is annual. Convert the rate to a daily rate:**

$$r = \frac{9\%}{365}$$

Step 3: **Solve for the time,** t**.**

$$t = \frac{I}{Pr}$$
$$t = \frac{\$1{,}283.42}{\$15{,}000 \times \frac{0.09}{365}}$$
$$t = 346.998741 = 347 \text{ days}$$

Step 4: **Use the DATE function to calculate the start date (DT1). Use the time in days.**

Calculator Instructions:

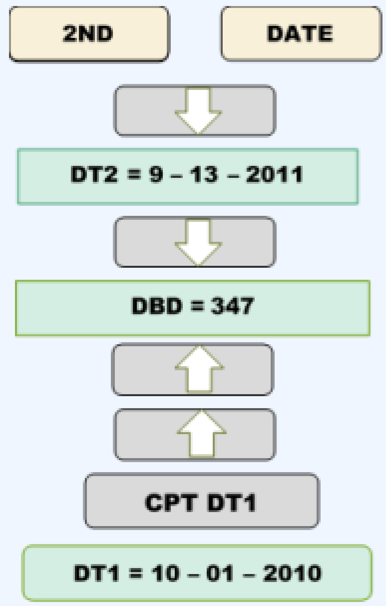


Figure 5.1.4

Step 5: **Write as a statement.**

If Aladdin owed the Genie $\$1,283.42$ of simple interest at 9% on a principal of $\$15,000$, he must have borrowed the money 347 days earlier, which is October 1, 2010.

Section 5.1 Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Brynn borrowed $\$25,000$ at 1% per month from a family friend to start her

entrepreneurial venture on December 2, 2011. If she paid back the loan on June 16, 2012, how much simple interest did she pay?

Solution

Step 1: **Given information:**

$$P = \$25,000; \quad r = 1\% \text{ monthly};$$
$$t = \text{December 2, 2011 to June16, 2012}$$

Use DATE function on calculator to get the number of days. Total days for $t = 197$

Step 2: **Convert both the monthly r and the daily t annual numbers:**

$$r = 1\% \times 12 = 12\% \text{ annually}; \quad t = 197365$$

Step 3: **Solve for I.**

$$I = Prt$$
$$I = \$25,000\,(0.12)\left(\frac{197}{365}\right)$$
$$I = \$1,619.18$$

Step 4: **Write as a statement.**

She payed $\$1,619.18$ simple interest.

2. If $\$6,000$ principal plus $\$132.90$ of simple interest was withdrawn on August 14, 2011, from an investment earning 5.5% interest, on what day was the money invested?

Solution

Need to calculate t in days first:

Step 1: **Given information:**

$$I = \$132.90; \quad P = \$6,000; \quad r = 5.5\% \text{ annually}$$

Step 2: **Convert annual r to match daily t:**

$$r = \frac{5.5\%}{365}$$

Step 3: **Solve for t.**

$$t = \frac{I}{Pr}$$

$$t = \frac{\$132.90}{\$6,000\,(0.055365)}$$

$$t = 146.995454 \text{ days} \rightarrow 147 \text{ days}$$

Use the DATE function on the calculator to find the date when the money was invested.

Step 4: **Write as a statement.**

The money was invested on March 20, 2011.

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$ $I = Prt$

Attribution

5.2: MOVING MONEY INVOLVING SIMPLE INTEREST

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $I =$ Simple interest
- $P =$ Present or principal value
- $r =$ Interest rate
- $S =$ Maturity value or future value
- $t =$ Time period of transaction

Formulas Used

- Formula 5.1 – **Simple Interest**

$$I = Prt$$

- Formula 5.2a – **Simple Interest Future Value**

$$S = P(1 + rt)$$

- Formula 5.2b – **Simple Interest Amount**

$$I = S - P$$

Maturity Value (or Future Value)

The maturity value of a transaction is the amount of money resulting at the end of a transaction, an amount that includes both the interest and the principal together. It is called a maturity value because in the financial world the termination of a financial transaction is known as the "maturing" of the transaction. The amount of principal with interest at some point in the future, but not necessarily the end of the transaction, is known as the future value.

For any financial transaction involving simple interest, the following is true:

$$\text{Amount of money at the end} = \text{Amount of money at the beginning} + \text{Interest}$$

Applying algebra, you can summarize this expression by the following equation, where the future value or maturity value is commonly denoted by the symbol S.

$$S = P + I$$

Substituting in **Formula 5.1** $I = Prt$ (Simple Interest; see Section 5.1) yields the equation

$$S = P + Prt$$

or

$$S = P(1 + rt)$$

5.2a Simple Interest Future Value

$$S = P(1 + rt)$$

Formula does not parse The maturity value is the amount of money resulting at the end of a transaction, including both the interest and the principal.

Formula does not parse The present value is the amount borrowed or invested at the beginning of a period.

Formula does not parse The interest rate is the rate of interest that is charged or earned during a specified time period. It is expressed as a percent.

$rgb]0.5, 0.0, 0.5t$ $rgb]0.1, 0.1, 0.1$is Time Period: The time period or term is the length of the financial transaction for which interest is charged or earned.

—

From the future value formula $S = P(1 + rt)$ you can derive the present value formula (P):

$$P = \frac{S}{1 + rt}$$

Sometimes you will be required to calculate the simple interest dollar amount (I). the formula is given below.

5.2b Simple Interest Amount

$$I = S - P$$

$rgb]1.0, 0.0, 0.0I\ rgb]0.1, 0.1, 0.1$is Interest Amount: The interest amount is the dollar amount of interest that is paid or received.

Formula does not parse The maturity value is the amount of money resulting at the end of a transaction, including both the interest and the principal.

Formula does not parse The present value is the amount borrowed or invested at the beginning of a period.

Example 5.2.1

Assume that today you have $\$10,000$ that you are going to invest at 7% simple interest for 11 months. How much money will you have in total at the end of the 11 months? How much interest do you earn?

Solution

***Step 1:* Given variables:**

$$P = \$10,000; \qquad r = 7\%; \qquad t = 11 \text{ months}$$

***Step 2:* Express the time in years to match the annual rate.**

$$t = \frac{11}{12}$$

***Step 3:* Solve for the future value, S.**

$$S = P \times (1 + rt)$$

$$S = \$10,000 \times \left(1 + 0.07 \times \frac{11}{12}\right)$$

$$S = \$10,641.67$$

This is the total amount after 11 months.

Step 4: **Solve for the interest amount, I.**

$$I = \$10,641.67 - \$10,000.00$$
$$I = \$641.67$$

Step 5: **Write as a statement.**

The $\$10,000$ earns $\$641.67$ in simple interest over the next 11 months, resulting in $\$10,641.67$ altogether.

Concept Check

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=180#h5p-10

Example 5.2.2

You just inherited $\$35,000$ from your uncle's estate and plan to purchase a house four months from today. If you use your inheritance as your down payment on the house, how much will you be able to put down if your money earns $4\frac{1}{4}\%$ simple interest? How much interest will you have earned?

Solution

Calculate the amount of money four months from now including both the principal and interest earned. This is the maturity value (S). Also calculate the interest earned (I).

Step 1: **Given variables:**

$$P = \$35,000; \qquad t = 4 \text{ months}; \qquad r = 4\frac{1}{4}\% \text{ per year}$$

Step 2: **Express the time in years to match the annual rate.**

$$t = \frac{4}{12}$$

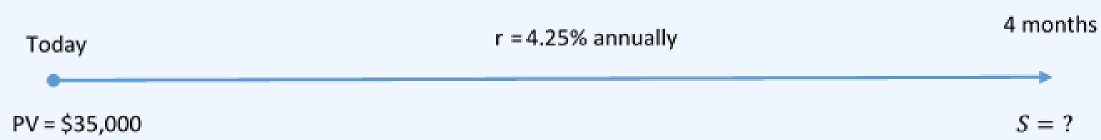


Figure 5.2.1

Step 3: **Solve for the future value, S.**

$$S = P \times (1 + rt)$$

$$S = \$35,000 \times \left(1 + 4\frac{1}{4}\% \times \frac{4}{12}\right)$$

$$S = \$35{,}000 \times (1 + 0.0425 \times 0.\overline{3})$$
$$S = \$35{,}495.83$$

***Step 4:*Solve for the amount of interest, I.**

$$I = \$35,495.83 - \$35,000.00$$
$$I = \$495.83$$

Step 5: **Write as a statement.**

Four months from now you will have $\$35,495.83$ as a down payment toward your house, which includes $\$35,000$ in principal and $\$495.83$ of interest.

Example 5.2.3

Recall the section opener, where you needed $\$8,000$ for tuition in the fall and the best simple interest rate you could find was 4.5%. Assume you have eight months before you need to pay your tuition. How much money do you need to invest today?

Solution

Calculate the principal amount of money today P that you must invest such that it will earn interest and end up at the $\$8,000$ required for the tuition.

Step 1: **Given variables:**

$$S=\$8,000; \qquad r=4.5\% \text{ per year}; \qquad t=8 \text{ months}$$

Step 2: **Express the time in years to match the annual rate.**

$$t=\frac{8}{12}$$

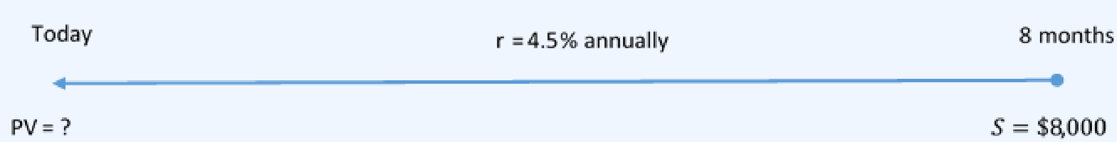


Figure 5.2.2

Step 3: **Solve for the present value,** P**.**

$$P = \frac{S}{1+rt}$$

$$P = \frac{\$8,000}{(1+4.5\% \times \frac{8}{12})}$$

$$P = \frac{\$8,000}{(1+0.045 \times 0.\overline{6})}$$

$$P = \$7,766.99$$

Step 4: **Write as a statement.**

If you place $\$7,766.99$ into the investment, it will grow to $\$8,000$ in the eight months.

Example 5.2.4

You are sitting in an office at your local financial institution on August 4. The bank officer says to you, "We will make you a great deal. If we advance that line of credit and you borrow $\$20,000$ today, when you want to repay that balance on September 1 you will only have to pay us $\$20,168.77$, which is not much more!" Before answering, you decide to evaluate the statement. Calculate the simple interest rate that the bank officer used in her calculations.

Solution

Determine the rate of interest that you would be charged on your line of credit.

Step 1: **Given variables:**

$$P = \$20,000; \quad S = \$20,168.77; \quad t = \text{August 4 to September 1}$$

Step 2: **Calculate the number of days in the transaction.**

Calculator Instructions: Assume the year 2011 and use the DATE function to find the exact number of days.

Table 5.2.1

DT1	DT2	DBD	Mode
8.0411	9.0111	Answer: 28	ACT

Step 3: **Since interest rates are usually expressed annually, convert the time from days to an annual number.**

$$t = \frac{28}{365}$$

Step 4: **Calculate the amount of interest, I.**

$$I = \$20{,}168.77 - \$20{,}000$$
$$I = \$168.77$$

Step 5: **Solve for r.**

$$r = \frac{I}{Pt}$$

$$r = \frac{\$168.77}{\$20,000.00 \times \frac{28}{365}}$$

$$r = 0.110002 \text{ or } 11.0002\%$$

Step 6: **Write as a statement.**

The interest rate on the offered line of credit is 11.0002% (note that it is probably exactly 11%; the extra 0.0002% is most likely due to the rounded amount of interest used in the calculation).

Equivalent Payments

Late Payments: If a debt is paid late, then a financial penalty that is fair to both parties involved should be imposed. That penalty should reflect a current rate of interest and be added to the original payment. Assume you owe $\$100$ to your friend and that a fair current rate of simple interest is 10%. If you pay this debt one year late, then a 10% late interest penalty of $\$10$ should be added, making your debt payment $\$110$. This is no different from your friend receiving the $\$100$ today and investing it himself at 10% interest so that it accumulates to $\$110$ in one year.

Early Payments: If a debt is paid early, there should be some financial incentive (otherwise, why bother?).

Therefore, an interest benefit, one reflecting a current rate of interest on the early payment, should be deducted from the original payment. Assume you owe your friend $\$110$ one year from now and that a fair current rate of simple interest is 10%. If you pay this debt today, then a 10% early interest benefit of $\$10$ should be deducted, making your debt payment today $\$100$. If your friend then invests this money at 10% simple interest, one year from now he will have the $\$110$, which is what you were supposed to pay.

Notice in these examples that a simple interest rate of 10% means $\$100$ today is the same thing as having $\$110$ one year from now. This illustrates the concept that two payments are equivalent payments if, once a fair rate of interest is factored in, they have the same value on the same day. Thus, in general you are finding two amounts at different points in time that have the same value, as illustrated in the figure below.

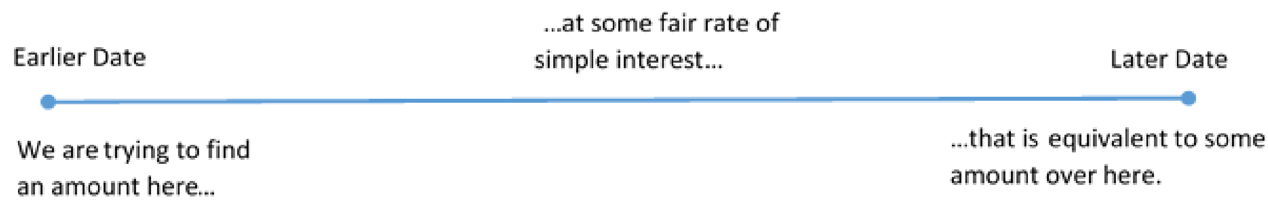


Figure 5.2.3

HOW TO

Calculate an Equivalent Payment

The steps required to calculate an equivalent payment are no different from those for single payments. If an early payment is being made, then you know the future value, so you solve for the present value (which removes the interest). If a late payment is being made, then you know the present value, so you solve for the future value (which adds the interest penalty).

Example 5.2.5

Erin owes Charlotte $\$1,500$ today. Unfortunately, Erin had some unexpected expenses and is unable to make her debt payment. After discussing the issue, they agree that Erin can make the payment nine months late and that a fair simple interest rate on the late payment is 5%. Use 9 months from now as your focal date and calculate how much Erin needs to pay. What is the amount of her late penalty?

Solution

A late payment is a future value amount (S). The late penalty is equal to the interest (I).

Step 1: **Given variables:**

$$P=\$1,500; \quad r=5\% \text{ annually}; \quad t=9 \text{ months}$$

Step 2: **Express the time in years to match the annual rate.**

$$t=\frac{9}{12}$$

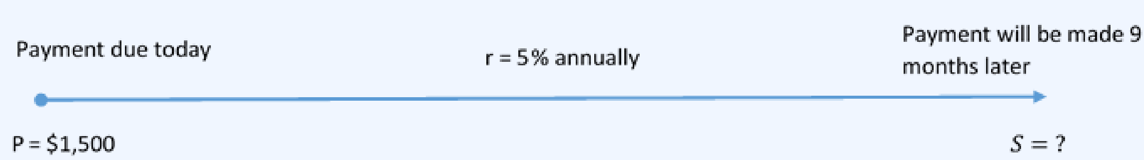


Figure 5.2.4

Step 3:Calculate the future value, S.

$$S=P\times(1+rt)$$

$$S=\$1,500\times\left(1+5\%\times\frac{9}{12}\right)$$

$$S=\$1,500\times(1+0.05\times0.75)$$
$$S=\$1,556.25$$

Step 4:**Solve for the amount of interest, I.**

$$I = \$1,556.25 - \$1,500.00$$
$$I = \$56.25$$

***Step 5:* Write as a statement.**

Erin's late payment is for $\$1,556.25$, which includes a $\$56.25$ interest penalty for making the payment nine months late.

Example 5.2.6

Rupert owes Aminata two debt payments: $\$600$ four months from now and $\$475$ eleven months from now. Rupert came into some money today and would like to pay off both of the debts immediately. Aminata has agreed that a fair interest rate is 7%. Using today as a focal date, what amount should Rupert pay? What is the total amount of his early payment benefit?

An early payment is a present value amount (P). Both payments will be moved to today and summed. The early payment benefit will be the total amount of interest removed (I).

Solution

***Step 1:* Given variables:**

$$r = 7\% \text{ annually}$$

The two payments and payment due dates are known.

Payment #1: $S_1 = \$600; t = 4 \text{ months from now}$

Payment #2: $S_2 = \$475; t = 11 \text{ months from now}$

Replacement payment is being made today (the focal date).

Payment #1:

***Step 1:* Express the time in years to match the annual rate.**

$$t = \frac{4}{12}$$

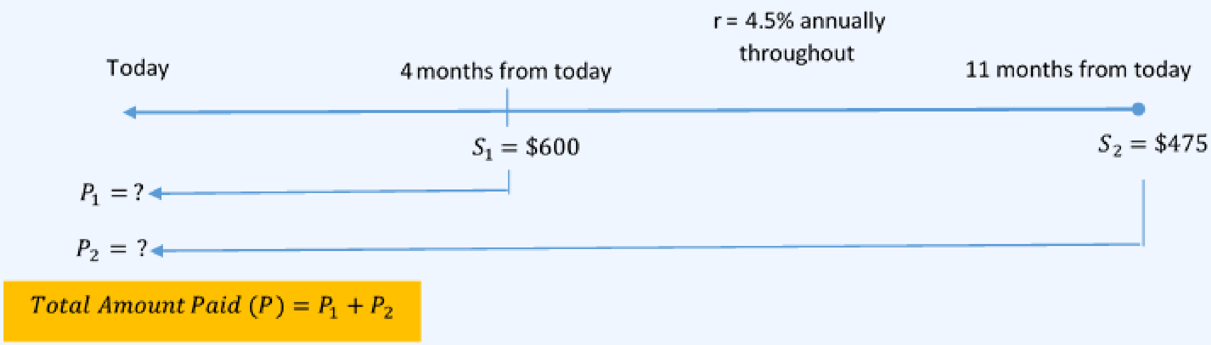


Figure 5.2.5

Step 2: **Solve for P_1.**

$$P_1 = \frac{S_1}{1+rt}$$

$$P_1 = \frac{\$600}{\left(1+0.07\times 0.\overline{3}\right)}$$

$$P_1 = \$586.32$$

Payment #2:

Step 1:**While the rate is annual, the time is in months. Convert the time to an annual number.**

$$t = \frac{11}{12}$$

Step 2: **Solve for P_2.**

$$P_2 = \frac{S_2}{1+rt}$$

$$P_2 = \frac{\$475}{(1+0.07 \times 0.91\overline{6})}$$

$$P_2 = \$446.36$$

Step 2: **Calculate the total amount of interest, I.**

Payment #1:

$$I_1 = S_1 - P_1$$
$$I_1 = \$600 - \$586.32$$
$$I_1 = \$13.68$$

Payment #2:

$$I_2 = S_2 - P_2$$
$$I_2 = \$475 - \$446.36$$
$$I_2 = \$28.64$$

Total Interest:

$$I = \$13.68 + \$28.64$$
$$I = \$42.32$$

Total Payment:

$$P_{\text{total paid today}} = P_1 + P_2$$
$$P_{\text{total paid today}} = \$586.32 + \$446.36$$
$$P_{\text{total paid today}} = \$1,032.68$$

Step 3: **Write as a statement.**

To clear both debts today, Rupert pays $\$1,032.68$, which reflects a $\$42.32$ interest benefit reduction for the early payment.

Section 5.2 Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. An accountant needs to allocate the principal and simple interest on a loan payment into the appropriate ledgers. If the amount received was $\$10,267.21$ for a loan that spanned April 14 to July 31 at 9.1%, how much was the principal and how much was the interest?

Solution

Step 1: **Given information:**

$$S = \$10,267.21; \quad r = 9.1\% \text{ annually}; \\ t = \text{April 14 to July 31} = 108 \text{ days}$$

Step 2: **Convert daily t to match annual r:**

$$t = \frac{108}{365}$$

Step 3: **Solve for P.**

$$P = \frac{S}{1+rt}$$
$$P = \frac{\$10,267.21}{1+0.091 \times \frac{108}{365}}$$
$$P = \$9,998$$

Step 4: **Solve for I.**

$$I = S - P$$
$$I = \$10,267.21 - \$9,998$$
$$I = \$269.21$$

Step 5: **Write as a statement.**

The principal was $\$9,998$ and the simple interest on the loan was $\$269.21$.

2. Suppose Robin borrowed $\$3,600$ on October 21 and repaid the loan on February 21 of the following year. What simple interest rate was charged if Robin repaid $\$3,694.63$?

Solution

***Step 1:* Given information:**

$$P = \$3,600; \qquad S = \$3,694.63;$$
$$t = \text{October 21 to February 21} = 123 \text{ days}$$

***Step 2:* Compute I.**

$$I = S - P$$
$$I = \$3,694.63 - \$3,600$$
$$I = \$94.63$$

***Step 3:* Convert daily t to match annual r:**

$$t = \frac{123}{365}$$

***Step 4:* Solve for r:**

$$r = \frac{I}{Pt}$$
$$r = \frac{\$94.63}{\$3,600\left(\frac{123}{365}\right)}$$
$$r = 0.078004 \text{ or } 7.8004\%$$

***Step 5:* Write as a statement.**

The simple interest charged was 7.80%.

3. Jayne needs to make three payments to Jade requiring $\$2,000$ each 5 months, 10 months, and 15 months from to day. She proposes instead making a single payment eight months from today. If Jade agrees to a simple interest rate of 9.5%, what amount should Jayne pay?

Solution

***Step 1:* Given information:**

$$r = 9.5\% \text{ annually}$$

Payment #1: $P = \$2,000; t = 8 \text{ months} - 5 \text{ months} = 3 \text{ months}$

Payment #2: $S = \$2,000; t = 10 \text{ months} - 8 \text{ months} = 2 \text{ months}$

Payment #3: $S = \$2,000$; $t = 15 \text{ months} - 8 \text{ months} = 7 \text{ months}$

Payment #1:

Step 1: **Convert monthly t to match annual r:**

$$t = \frac{3}{12}$$

Step 2: **Solve for S:**

$$S = P\left(1 + rt\right)$$
$$S = \$2,000 \times \left(1 + 0.095 \times \frac{3}{12}\right)$$
$$S = \$2,000 \times 1.02375$$
$$S = \$2,047.50$$

Payment #2:

Step 2: **Convert monthly t to match annual r:**

$$t = \frac{2}{12}$$

Step 3: **Solve for P:**

$$P = \frac{S}{1 + rt}$$
$$P = \frac{\$2,000}{1 + 0.095 \times \frac{2}{12}}$$
$$P = \frac{\$2,000}{1.0158\overline{3}}$$
$$P = \$1,968.83$$

Payment #3:

Step 2: **Convert monthly t to match annual r:**

$$t = \frac{7}{12}$$

Step 3: **Solve for P:**

$$P = \frac{S}{1+rt}$$
$$P = \frac{\$2,000}{1+0.095\times\frac{7}{12}}$$
$$P = \frac{\$2,000}{1.05541\overline{6}}$$
$$P = \$1,894.99$$

$$\text{Replacement payment eight months from today} = \$2,047.50 + \$1,968.83 + \$1,894.99$$
$$\text{Replacement payment eight months from today} = \$5,911.32$$

Step 2: **Write as a statement.**

Jayne should pay $\$5,911.32$.

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $I = Prt$

Attribution

5.3: SAVINGS ACCOUNTS AND SHORT-TERM GICS

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $I =$ Simple interest
- $P =$ Present value or principal
- $r =$ Interest rate
- $S =$ Maturity or future value
- $t =$ time period of transaction

Formulas Used

- Formula 5.1 – **Simple Interest**

$$I = Prt$$

- Formula 5.2a – **Simple Interest Future Value**

$$S = P(1 + rt)$$

Savings Accounts

A savings account is a deposit account that bears interest and has no stated maturity date. These accounts are found at most financial institutions, such as commercial banks (Royal Bank of Canada, TD Canada Trust, etc.), trusts (Royal Trust, Laurentian Trust, etc.), and credit unions (FirstOntario, Steinbach, Assiniboine, Servus, etc.). Owners of such accounts make deposits to and withdrawals from these accounts at any time, usually accessing the account at an automatic teller machine (ATM), at a bank teller, or through online banking.

A wide variety of types of savings accounts are available. This textbook focuses on the most common features of most savings accounts, including how interest is calculated, when interest is deposited, insurance against loss, and the interest rate amounts available.

- **How Interest Is Calculated:** There are two common methods for calculating simple interest:

    1. Accounts earn simple interest that is calculated based on the daily closing balance of the account. The closing balance is the amount of money in the account at the end of the day. Therefore, any balances in the account throughout a single day do not matter. For example, if you start the day with $\$500$ in the account and deposit $\$3,000$ at 9:00 a.m., then withdraw the $\$3,000$ at 4:00 p.m., your closing balance is $\$500$. That is the principal on which interest is calculated, not the $\$3,500$ in the account throughout the day.
    2. Accounts earn simple interest based on a minimum monthly balance in the account. For example, if in a single month you had a balance in the account of $\$900$ except for one day, when the balance was $\$500$, then only the $\$500$ is used in calculating the entire month's worth of interest.

- **When Interest Is Deposited:** Interest is accumulated and deposited (paid) to the account once monthly, usually on the first day of the month. Thus, the interest earned on your account for the month of January appears as a deposit on February 1.
- **Insurance against Loss:** Canadian savings accounts at commercial banks are insured by the national Canada Deposit Insurance Corporation (CDIC), which guarantees up to $\$100,000$ in savings. At credit unions, this insurance is usually provided provincially by institutions such as the Deposit Insurance Corporation of Ontario (DICO), which also guarantees up to $\$100,000$. This means that if your bank were to fold, you could not lose your money (so long as your deposit was within the maximum limit). Therefore, savings accounts carry almost no risk.
- **Interest Rate Amounts:** Interest rates are higher for investments that are riskier. Savings accounts carry virtually no risk, which means the interest rates on savings accounts tend to be among the lowest you can earn. At the time of writing, interest rates on savings accounts ranged from a low of 0.05% to a high of 1.95%. Though this is not much, it is better than nothing and certainly better than losing money!

While a wide range of savings accounts are available, these accounts generally follow one of two common structures when it comes to calculating interest. These structures are flat rate savings accounts and tiered savings accounts. Each of these is discussed separately.

HOW TO

Calculate Interest for Flat-Rate Savings Accounts

A flat-rate savings account has a single interest rate that applies to the entire balance. The interest rate may fluctuate in sync with short-term interest rates in the financial markets.

Step 1: Identify the interest rate, opening balance, and the monthly transactions in the savings account.

Step 2: Set up a flat-rate table as illustrated here. Create a number of rows equaling the number of monthly transactions (deposits or withdrawals) in the account plus one.

Table 5.3.1

Date	Closing Balance in Account	# of Days	Simple Interest Earned
			$I = Prt$
Total Interest earned			

Step 3: For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

Step 4: Calculate the number of days that the closing balance is maintained for each row.

Step 5: Apply **Formula 5.1** $I=Prt$ (Simple Interest) to each row in the table. Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts (I).

Step 6: Sum the Simple Interest Earned column and round off to two decimals.

When you are calculating interest on any type of savings account, pay careful attention to the details on how interest is calculated and any restrictions or conditions on the balance that is eligible to earn the interest.

Example 5.3.1

The RBC High Interest Savings Account pays 0.75% simple interest on the daily closing balance in the account and the interest is paid on the first day of the following month. On March 1, the opening balance in the account was $\$2,400$. On March 12, a deposit of $\$1,600$ was made. On March 21, a withdrawal of $\$2,000$ was made. Calculate the total simple interest earned for the month of March.

Solution

Calculate the total interest amount (I) for the month.

***Step 1:* Given variables:**

The following transactions dates are known:

$$\text{March 1 opening balance} = \$2,400$$

$$\text{March 12 deposit} = \$1,600$$

$$\text{March 21 withdrawal} = \$2,000$$

Step 2: **Set up a flat-rate table (see table below).**

Step 3: **Determine the date ranges for each balance throughout the month and calculate the closing balances (see table below).**

Step 4: **For each row of the table, calculate the number of days involved.**

Step 5: **Apply simple interest formula $I = Prt$ to calculate simple interest on each row.**

Step 6: **Sum the Simple Interest Earned.**

Table 5.3.2

Dates (Step 2)	Closing Balance in Account (Step 3)	# of Days (Step 4)	Simple Interest Earned ($I = Prt$) (Step 5)
March 1 to March 12	$\$2,400$	$12-1=11$	$I=\$2,400(0.0075)\left(\frac{11}{365}\right)$ $I=\$0.542465$
March 12 to March 21	$\$2,400+\$1,600=\$4,000$	$21-12=9$	$I=\$4,000(0.0075)\left(\frac{9}{365}\right)$ $I=\$0.739726$
March 21 to April 1	$\$4,000-\$2,000=\$2,000$	$31+1-21=11$	$I=\$2,000(0.0075)\left(\frac{11}{365}\right)$ $I=\$0.452054$
Step 6: Total Monthly Interest Earned.			$I=\$0.542465+\$0.739726+\$0.452054$ $I=\$1.73$

Step 7: **Write as a statement.**

For the month of March, the savings account earned a total simple interest of $\$1.73$, which was deposited to the account on April 1.

Try It

1) Canadian Western Bank offers a Summit Savings Account with posted interest rates as indicated in the table below. Only each tier is subject to the posted rate, and interest is calculated daily based on the closing balance.

Table 5.3.3

Balance	Interest Rate
$\$0$ – $\$5,000.00$	0%
$\$5,000.01$ – $\$1,000,000.00$	1.05%
$\$1,000,000.01$ and up	0.80%

December's opening balance was $\$550,000$. Two deposits in the amount of $\$600,000$ each were made on December 3 and December 21. Two withdrawals in the amount of $\$400,000$ and $\$300,000$ were made on December 13 and December 24, respectively. What interest for the month of December will be deposited to the account on January 1?

Solution

Step 1: **Interest rates as per table in question.**

$$\text{December opening balance} = \$550,000$$

$$\text{December 3 Deposit} = \$600,000$$

$$\text{December 13 Withdrawal} = \$400,000$$

$$\text{December 21 Deposit} = \$600,000$$

$$\text{December 24 Withdrawal} = \$300,000$$

Step 2: **See the information below as per the question.**

Putting this information into a table can be helpful for visualizing what values to use.

Date: Dec 3 to Dec 13

Closing Balance In Account: $\$550,000$

# of Days: $3 - 1 = 2$

0% $0 to $5,000 (This portion only): $$P = \$5,000$$ $$I = \$0$$

1.05% $5,000.01 to $1,000,000 (This portion only): $$P = \$545,000$$ $$I = \$545,000\,(0.0105)\left(\frac{2}{365}\right)$$ $$I = \$31.356164$$

Date: Dec 1 to Dec 3

Closing Balance In Account: $\$550,000 + \$600,000 = \$1,150,000$

# of Days: $13 - 3 = 10$

0% $0 to $5,000 (This portion only): $$P = \$5,000$$ $$I = \$0$$

1.05% $5,000.01 to $1,000,000 (This portion only): $$P = \$995,000$$ $$I = \$955,000\,(0.0105)\left(\frac{10}{365}\right)$$ $$I = \$286.232876$$

0.8% $1,000,000.01 and up (This portion only): $$P = \$150,000$$ $$I = \$150,000\,(0.008)\left(\frac{10}{365}\right)$$ $$I = \$32.876712$$

Date: Dec 13 to Dec 21

Closing Balance In Account: $\$1,150,000 - \$400,000 = \$750,000$

# of Days: $21 - 13 = 8$

0% $0 to $5,000 (This portion only): $$\begin{aligned} P &= \$5,000 \\ I &= \$0 \end{aligned}$$

1.05% $5,000.01 to $1,000,000 (This portion only): $$\begin{aligned} P &= \$745,000 \\ I &= \$745,000\,(0.0105)\left(\frac{8}{365}\right) \\ I &= \$171.452054 \end{aligned}$$

Date: Dec 21 to Dec 24

Closing Balance In Account: $\$750,000 + \$600,000 = \$1,350,000$

# of Days: $24 - 21 = 3$

0% $0 to $5,000 (This portion only): $$\begin{aligned} P &= \$5,000 \\ I &= \$0 \end{aligned}$$

1.05% $5,000.01 to $1,000,000 (This portion only): $$\begin{aligned} P &= \$995,000 \\ I &= \$995,000\,(0.0105)\left(\frac{3}{365}\right) \\ I &= \$85.869863 \end{aligned}$$

0.8% $1,000,000.01 and up (This portion only): $$\begin{aligned} P &= \$350,000 \\ I &= \$350,000\,(0.008)\left(\frac{3}{365}\right) \\ I &= \$23.013698 \end{aligned}$$

Date: Dec 24 to Jan 1

Closing Balance In Account: $\$1,350,000 - \$300,000 = \$1,050,000$

# of Days: $31 + 1 - 24 = 8$

0% $0 to $5,000 (This portion only): $$\begin{aligned} P &= \$5,000 \\ I &= \$0 \end{aligned}$$

1.05% $5,000.01 to $1,000,000 (This portion only):
$$P = \$995,000$$
$$I = \$995,000\,(0.0105)\left(\frac{8}{365}\right)$$
$$I = \$228.986301$$

0.8% $1,000,000.01 and up (This portion only):
$$P = \$50,000$$
$$I = \$50,000\,(0.008)\left(\frac{8}{365}\right)$$
$$I = \$8.767123$$

Step 3: **Total Monthly Interest Earned I:**

$$I = \$31.356164 + \$286.232876 + \$32.876712 + \$171.452054 + \$85.869863 + \$23.013698 + \$228.986301 + \$8.767123$$
$$I = \$868.55$$

Step 4: **Write as a statement.**

The total monthly interest earned is $\$868.55$.

Tiered Savings Accounts

A tiered savings account pays higher rates of interest on higher balances in the account. This is very much like a graduated commission on gross earnings. For example, you might earn 0.25% interest on the first $\$1,000$ in your account and 0.35% for balances over $\$1,000$. Most of these tiered savings accounts use a portioning system. This means that if the account has $\$2,500$, the first $\$1,000$ earns the 0.25% interest rate and it is only the portion above the first $\$1,000$ (hence, $\$1,500$) that earns the higher interest rate.

HOW TO

Calculate Monthly Interest for a Tiered Savings Account

Follow these steps to calculate the monthly interest for a tiered savings account:

Step 1: Identify the interest rate, opening balance, and the monthly transactions in the savings account.

Step 2: Set up a tiered interest rate table as illustrated below. Create a number of rows equaling the number of monthly transactions (deposits or withdrawals) in the account plus one. Adjust the number of columns to suit the number of tiered rates. Fill in the headers for each tiered rate with the balance requirements and interest rate for which the balance is eligible.

Table 5.3.4

Dates	Closing Balance in Account	# of Days	Tier Rate #1 Balance Requirements and Interest Rate	Tier Rate #2 Balance Requirements and Interest Rate	Tier Rate #3 Balance Requirements and Interest Rate
			• Eligible P • $I=Prt$	• Eligible P • $I=Prt$	• Eligible P • $I=Prt$
Total Monthly Interest Earned					

Step 3: For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

Step 4: For each row, calculate the number of days that the closing balance is maintained.

Step 5: Assign the closing balance to the different tiers, paying attention to whether portioning is being used. In each cell with a balance, apply simple interest formula $I=Prt$. Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts (I).

Step 6: To calculate the Total Monthly Interest Earned, sum all interest earned amounts from all tier columns and round off to two decimals.

Example 5.3.2

The Rate Builder savings account at your local credit union pays simple interest on the daily closing balance as indicated in the table below:

Table 5.3.5

Balance	Interest Rate
$\$0.00$ to $\$500.00$	0% on entire balance
$\$500.01$ to $\$2,500.00$	0.5% on entire balance
$\$2,500.01$ to $\$5,000.00$	0.95% on this portion of balance only
$\$5,000.01$ and up	1.35% on this portion of balance only

In the month of August, the opening balance on an account was $\$2,150.00$. Deposits were made to the account on August 5 and August 15 in the amounts of $\$3,850.00$ and $\$3,500.00$. Withdrawals were made from the account on August 12 and August 29 in the amounts of $\$5,750.00$ and $\$3,000.00$. Calculate the simple interest earned for the month of August.

Solution

Calculate the total interest amount (I) for the month of August.

***Step 1:* The interest rate structure is in the table above.**

The transactions and dates are also known:

$$\begin{aligned}
&\text{August 1 opening balance} = \$2,150.00\\
&\text{August 5 deposit} = \$3,850.00\\
&\text{August 12 withdrawal} = \$5,750.00\\
&\text{August 15 deposit} = \$3,500.00\\
&\text{August 29 withdrawal} = \$3,000.00
\end{aligned}$$

***Step 2:* Set up a tiered interest rate table with four columns for the tiered rates (see table below).**

***Step 3:* Determine the date ranges for each balance throughout the month and calculate the closing balances (see table below).**

Step 4: **Calculate the number of days involved on each row of the table.**

Step 5: **Assign the closing balance to each tier accordingly. Apply Formula 5.1** $I = Prt$ **to any cell containing a balance (see table below).**

Step 6: **Total up all of the interest from all cells of the table.**

Table 5.3.6

Calculations of Interest Based on Date with Total Interest Earned at the Bottom
Dates: *Aug 1 to Aug 5* ***Closing Balance in Account:*** $$\$2,150$$ ***# of Days:***

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$5-1=4$$

0.5% $500.01 to $2,500 (Entire balance):

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$2,150$$
$$I = \$2,150(0.005)\left(\frac{4}{365}\right)$$
$$I = \$0.117808$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

Dates: *Aug 5 to Aug 12*
Closing Balance in Account:

$$\$2,150 + \$3,850 = \$6,000$$

# of Days:

$$12 - 5 = 7$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom
0.5% $500.01 to $2,500 (Entire balance):

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$2,500$$

$$I = \$2,500(0.005)\left(\frac{7}{365}\right)$$

$$I = \$0.239726$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

0.95% $2,500.01 to $5,000 (This portion only):

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$2,500$$

$$I = \$2,500(0.0095)\left(\frac{7}{365}\right)$$

$$I = \$0.455479$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom
1.35% $5,000.01 and up (This portion only):

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$1,000$$

$$I = \$1,000(0.0135)\left(\frac{7}{365}\right)$$

$$I = \$0.258904$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

Dates: *Aug 12 to Aug 15*
Closing Balance in Account:

$$\$6,000 - \$5,750 = \$250$$

# of Days:

$$15 - 12 = 3$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

0% $0 to $500 (Entire balance):

$$P = \$250.00$$
$$I = \$0.00$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

Dates: *Aug 15 to Aug 29*

Closing Balance in Account:

$$\$250 + \$3,500 = \$3,750$$

# of Days:

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$29 - 15 = 14$$

0.5% $500.01 to $2,500 (Entire balance):

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$2,500$$

$$I = \$2,500(0.005)\left(\frac{14}{365}\right)$$

$$I = \$0.479452$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom
0.95% $2,500.01 to $5,000 (This portion only):

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$1,250$$

$$I = \$1,250(0.005)\left(\frac{14}{365}\right)$$

$$I = \$0.455479$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

Dates: *Aug 29 to Sep 1*

Closing Balance in Account:

$$\$3,750 - \$3,000 = \$750$$

# of Days:

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$31 - 29 + 1 = 3$$

0.5% $500.01 to $2,500 (Entire balance):

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$750$$

$$I = \$750(0.005)\left(\frac{3}{365}\right)$$

$$I = \$0.030821$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

Total Interest Earned:

$I = \$0.117808 + \$0.239726 + \$0.455479 + \$0.258904 + \$0.00 + \$0.479452 + \$0.455479 + \$0.030821$

$I = \$2.04$

Step7: **Write as a statement.**

For the month of August, the tiered savings account earned a total simple interest of $\$2.04$, which was deposited to the account on September 1.

Short-Term Guaranteed Investment Certificates (GICs)

A guaranteed investment certificate (GIC) is an investment that offers a guaranteed rate of interest over a fixed period of time. GICs are found mostly at commercial banks, trust companies, and credit unions. In this section, you will deal only with short-term GICs, defined as those that have a time frame of less than one year.

The table below summarizes three factors that determine the interest rate on a short-term GIC: principal, time, and redemption privileges.

Table 5.3.7

Factors Determining Interest Rate	**Higher Interest Rates**	**Lower Interest Rates**
Principal Amount	Large	Small
Time	Longer	Shorter
Redemption Privileges	Nonredeemable	Redeemable

- **Amount of Principal:** Typically, a larger principal is able to realize a higher interest rate than a smaller principal.
- **Time:** The length of time that the principal is invested affects the interest rate. Short-term GICs range from 30 days to 364 days in length. A longer term usually realizes higher interest rates.
- **Redemption Privileges:** The two types of GICs are known as redeemable and nonredeemable. A redeemable GIC can be cashed in at any point before the maturity date, meaning that you can access your money any time you want it. A nonredeemable GIC "locks in" your money for the agreed-upon term. Accessing that money before the end of the term usually incurs a stiff financial penalty, either on the interest rate or in the form of a financial fee. Nonredeemable GICs carry a higher interest rate.

To summarize, if you want to receive the most interest it is best to invest a large sum for a long time in a nonredeemable short-term GIC.

HOW TO

Calculate Interest of a Short-Term GIC

Short-term GICs involve a lump sum of money (the principal) invested for a fixed term (the time) at a guaranteed interest rate (the rate). Most commonly the only items of concern are the amount of interest earned and the maturity value. Therefore, you need the same four steps as for single payments involving simple interest shown in Section 5.2.

Example 5.3.3

Your parents have $\$10,000$ to invest. They can either deposit the money into a 364-day nonredeemable GIC at Assiniboine Credit Union with a posted rate of 0.75%, or they could put their money into back-to-back 182-day nonredeemable GICs with a posted rate of 0.7%. At the end of the first 182 days, they will reinvest both the principal and interest into the second GIC. The interest rate remains unchanged on the second GIC. Which option should they choose?

Solution

For both options, calculate the future value (S), of the investment after 364 days. The one with the higher future value is your parents' better option.

Step 1: **Given variables:**

For the first GIC investment option:

$$P=\$10,000; \quad r=0.75\% \text{ per year}; \quad t=364 \text{ days}$$

For the second GIC investment option:

$$\text{Initial } P=\$10,000; \quad r=0.7\% \text{ per year}; \quad t=182 \text{ days each}$$

Step 2: **The rate is annual, the time is in days. Convert the time to an annual number. Transforming both time variables,** $t=\frac{364}{365}$ **and** $t=\frac{182}{365}$

Step 3: **(1st GIC option): Calculate the maturity value** S_1 **of the first GIC option after its** 364**-day term.**

$$S_1 = \$10,000\left(1+(0.0075)\left(\frac{364}{365}\right)\right)$$

$$S_1 = \$10,074.79$$

Step 4: **(2nd GIC option, 1st GIC): Calculate the maturity value** S_2 **after the first** 182**-day term.**

$$S_2 = \$10,000\left(1+(0.007)\left(\frac{182}{365}\right)\right)$$

$$S_2 = \$10{,}034.90$$

Step 5: **(2nd GIC option, 2nd GIC): Reinvest the first maturity value as principal for another term of** 182 **days and calculate the final future value** S_3**.**

$$S_3 = \$10,034.90\left(1+(0.007)\left(\frac{182}{365}\right)\right)$$

$$S_3 = \$10,069.93$$

Step 6: **Write as a statement.**

> The 364-day GIC results in a maturity value of $\$10,074.79$, while the two back-to-back 182-day GICs result in a maturity value of $\$10,069.93$. Clearly, the 364-day GIC is the better option as it will earn $\$4.86$ more in simple interest.

Section 5.3 Exercises

In the exercise that follow, try it on your own. Full solution is available should you get stuck.

1. If you place $\$25,500$ into an 80-day short-term GIC at TD Canada Trust earning 0.55% simple interest, how much will you receive when the investment matures?

Solution
***Step 1:* Given information:**

$$P = \$25,500; \qquad t = 80 \text{ days}; \qquad r = 0.55\% \text{ annually}$$

***Step 2:* Convert daily t to match annual r:**

$$t = \frac{80}{365}$$

Step 3: Solve for S.

$$S = P(1 + rt)$$
$$S = \$25,5000 \times \left(1 + 0.0055 \times \frac{80}{365}\right)$$
$$S = \$25,530.74$$

***Step 4:* Write as a statement.**

When the investment matures I will receive $\$25,530.74$.

2. Interest rates in the GIC markets are always fluctuating be cause of changes in the short-term financial markets. If you have $\$50,000$ to invest today, you could place the money into a 180-day GIC at Canada Life earning a fixed rate of 0.4%, or you could take two consecutive 90-day GICs. The current posted fixed rate on 90-day GICs at Canada Life is 0.3%. Trends in the short-term financial markets suggest that within the next 90 days short-term GIC rates will be rising. What does the short-term 90-day rate need to be 90 days from now to arrive at the same maturity value as the 180-day GIC? Assume that the entire maturity value of the first 90-day GIC would be reinvested.

Solution
***Step 1:* Given information:**

For the first GIC investment option:

$$P = \$50,000; \qquad r = 0.4\% \text{ per year}; \qquad t = 180 \text{ days}$$

For the second GIC investment option:

$$\text{Initial } P = \$50,000; \qquad r = 0.3\% \text{ per year}; \qquad t = 90 \text{ days}$$

then,

$$P = S \text{ of first 90-day GIC}; \qquad S = \text{maturity value of 180-day GIC}; \\ t = 90 \text{ days}$$

***Step 2:* Transforming both time variables:**

$$t = \frac{180}{365} \text{ and } t = \frac{90}{365}$$

Step 3: (1st GIC option):

$$S_1 = \$50,000\left(1 + 0.004 \times \frac{180}{365}\right)$$

$$S_1 = \$50,098.63$$

Step 3: (2nd GIC option, 1st GIC):

$$S_2 = \$50,000\left(1 + 0.003 \times \frac{90}{365}\right)$$

$$S_2 = \$50,036.99$$

Step 3: (2nd GIC option, 2nd GIC):

$$\begin{aligned} I &= (S \text{ of 180-day GIC}) - (S \text{ of 90-day GIC}) \\ I &= \$50,098.63 - \$50,036.99 \\ I &= \$61.64 \end{aligned}$$

$\$61.64$ is what the second 90-day GIC must earn in interest.

$$r = \frac{I}{Pt}$$

$$r = \frac{\$61.64}{\$50,036.99\left(\frac{90}{365}\right)}$$

$$r = 0.004995 \text{ or } 0.4995\%$$

***Step 4:* Write as a statement.**

The short-term 90-day rate needs to be 0.50% in 90 days from now to arrive at the same maturity value as the 180-day GIC.

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER.$I = Prt$

Attribution

5.4: APPLICATION: TREASURY BILLS AND COMMERCIAL PAPER

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $I =$ Simple interest
- $P =$ Present value or principal
- $r =$ Interest rate
- $t =$ time period of transaction
- $S =$ Maturity or future value

Formulas Used

- Formula 5.1 – **Simple Interest**

$$I = Prt$$

- Formula 5.2a – **Simple Interest Future Value**

$$S = P(1 + rt)$$

Treasury Bills: The Basics

Treasury bills, also known as T-bills, are short-term financial instruments that both federal and provincial governments issue with maturities no longer than one year. Approximately 27% of the national debt is borrowed through T-bills.

Here are some of the basics about T-bills:

- The Government of Canada regularly places T-bills up for auction every second Tuesday. Provincial governments issue them at irregular intervals.
- The most common terms for federal and provincial T-bills are 30 days, 60 days, 90 days, 182 days, and 364 days.
- T-bills do not earn interest. Instead, they are sold at a discount and redeemed at full value. This follows the principle of "buy low, sell high." The percentage by which the value of the T-bill grows from sale to redemption is called the **yield** or rate of return. From a mathematical perspective, the yield is calculated in the exact same way as an interest rate is calculated, and therefore the yield is mathematically substituted as the discount rate in all simple interest formulas. Up-to-date yields on T-bills can be found at https://www.bankofcanada.ca/rates/interest-rates/t-bill-yields/selected-treasury-bill-yields-10-year-lookup/.
- The **face value of a T-bill** (also called *par value*) is the maturity value, payable at the end of the term. It includes both the principal and yield together.
- T-bills do not have to be retained by the initial investor throughout their entire term. At any point during a T-bill's term, an investor is able to sell it to another investor through secondary financial markets. Prevailing yields on T-bills at the time of sale are used to calculate the price.

Commercial Papers – The Basics

A **commercial paper** (or paper for short) is the same as a T-bill except that it is issued by a large corporation instead of a government. It is an alternative to short-term bank borrowing for large corporations. Most of these large companies have solid credit ratings, meaning that investors bear very little risk that the face value will not be repaid upon maturity.

Commercial papers carry the same properties as T-bills. The only fundamental differences lie in the term and the yield:

1. The terms are usually less than 270 days but can range from 30 days to 364 days. The most typical terms are 30 days, 60 days, and 90 days.
2. The yield on commercial papers tends to be slightly higher than on T-bills since corporations do carry a higher risk of default than governments.

HOW TO

Calculate the Price of a T-Bill

Mathematically, T-bills and commercial papers operate in the exact same way. The future value for both of these investment instruments is always known since it is the face value. Commonly, the two calculated variables are either the present value (price) or the yield (interest rate). The yield is explored later in this section. Follow these steps to calculate the price:

Step 1: The face value, yield, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify the following:

a. The face value (S).
b. The yield (r) on the date of the sale, which is always expressed annually. Remember that mathematically the yield is the same as the discount rate.
c. The number of days (t) remaining between the date of the sale and the maturity date. Count the first day but not the last day. Express the number of days annually to match the annual yield.

Figure 5.4.1

Step 2: Solve for the present value using $P = \dfrac{S}{1+rt}$, which is the price of the T-bill or commercial paper. This price is always less than the face value.

Concept Check

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=186#h5p-11

Example 5.4.1

A Government of Canada 182-day issue T-bill has a face value of $\$100,000$. Market yields on these T-bills are 1.5%. Calculate the price of the T-bill on its issue date.

Solution

***Step 1:* Given variables:**

$$S=\$100,000; \qquad r=1.5\%; \qquad t=\frac{182}{365}$$

***Step 2:* Solve for the present value, P.**

$$P=\frac{S}{1+rt}$$

$$P=\frac{\$100,000}{1+(0.015)\left(\frac{182}{365}\right)}$$

$$P=\$99,257.61$$

***Step 3:* Write as a statement.**

An investor will pay $\$99,257.61$ for the T-bill. If the investor holds onto the T-bill until maturity, the investor realizes a yield of 1.5% and receives $\$100,000$.

Example 5.4.2

Pfizer Inc. issued a 90-day, $\$250,000$ commercial paper on April 18 when the market rate of return was 3.1%. The paper was sold 49 days later when the market rate of return was 3.63%. Calculate the price of the commercial paper on its date of sale.

Solution

Note that the historical rate of return of 3.1% is irrelevant to the price of the commercial paper today. The number of days elapsed since the date of issue is also unimportant. The number of days before maturity is the key piece of information.

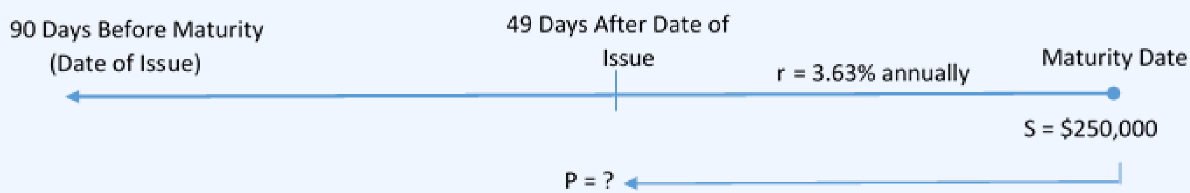


Figure 5.4.2

Step 1: **Given variables:**

$$S = \$250,000; \qquad r = 3.63\%; \qquad t = 90 - 49 = 41 \text{ days or } \frac{41}{365} \text{ years}$$

Step 2: Solve for the present value, P.

$$P = \frac{S}{1+rt}$$

$$P = \frac{\$250,000}{1 + (0.0363)\left(\frac{41}{365}\right)}$$

$$P = \$248,984.76$$

Step 3: **Write as a statement.**

An investor pays $\$248,984.76$ for the commercial paper on the date of sale. If the investor holds onto the commercial paper for 41 more days (until maturity), the investor realizes a yield of 3.63% and receives $\$250,000$.

HOW TO

Calculate a Rate of Return

Sometimes the unknown value when working with T-bills and commercial papers is the yield, or rate of return. In these cases, follow these steps to solve the problem:

Step 1: The face value, price, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify:

a. The face value (S).
b. The price on the date of the sale (P).
c. The number of days (t) remaining between the date of the sale and the maturity date. Count the first day but not the last day. Express the number of days annually so that the calculated yield will be annual.

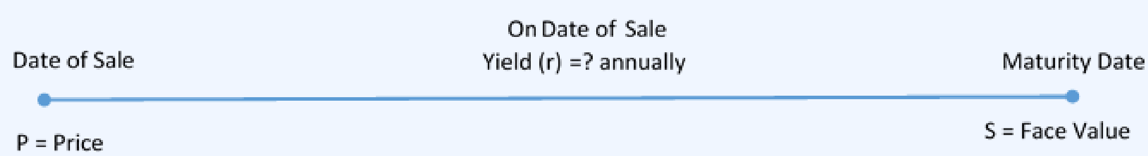


Figure 5.4.3

Step 2: Apply formula $I = S - P$, to calculate the interest earned during the investment.

Step 3: Apply simple interest formula, $I = Prt$, rearranging for r to solve for the interest rate (or yield or rate of return).

Example 5.4.3

Figuring Out Rates of Return for Multiple Investors

Marlie paid $\$489,027.04$ on the date of issue for a $\$500,000$ face value T-bill with a 364-day term. Marlie received $\$496,302.21$ when he sold it to Josephine 217 days after the date of issue. Josephine held the T-bill until maturity. Determine the following:

a. Marlie's actual rate of return.
b. Josephine's actual rate of return.
c. If Marlie held onto the T-bill for the entire 364 days instead of selling it to Josephine, what would his rate of return have been?

Comment on the answers to (a) and (c).

Solution

Calculate three yields or rates of return (r) involving Marlie and the sale to Josephine, Josephine herself, and Marlie without the sale to Josephine. Afterwards, comment on the rate of return for Marlie with and without the sale.

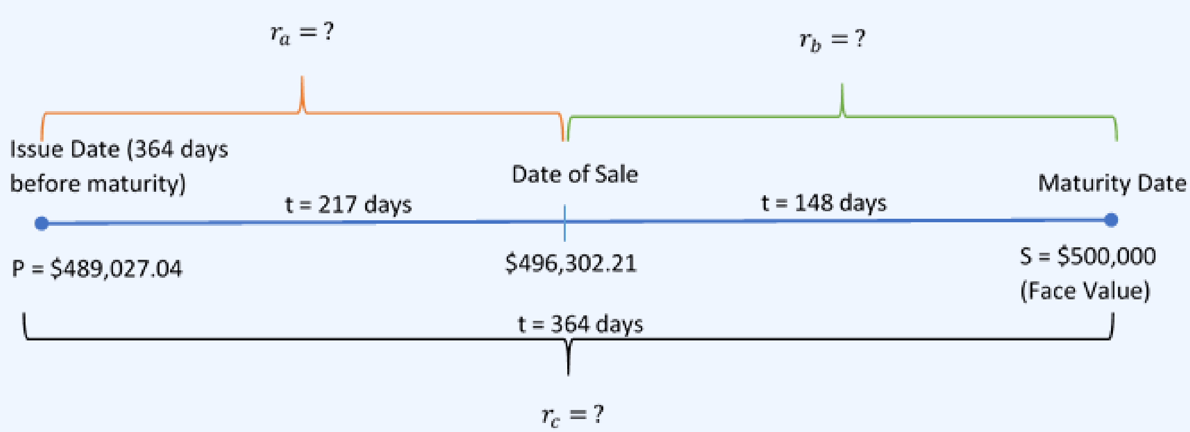


Figure 5.4.4

Step 1: **Given information:**

The present values, maturity value, and terms are known.

a. Marlie with sale:

$$P = \$489,027.04; \quad S = \$496,302.21; \quad t = \frac{217}{365}$$

b. Josephine:

$$P = \$496,302.21; \quad S = \$500,000; \quad 364 - 217 = 147 \text{ days remaining}; \quad t = \frac{147}{365}$$

c. Marlie without sale:

$$P = \$489,027.04; \quad S = \$500,000; \quad t = \frac{364}{365}$$

Step 2: **For each situation, calculate the interest amount (I).**

a. Marlie with sale to Josephine:

$$I = \$496,302.21 - \$489,027.04$$
$$I = \$7,275.17$$

b. Josephine by herself:

$$I = \$500,000 - \$496,302.21$$
$$I = \$3,697.79$$

c. Marlie without sale to Josephine:

$$I = \$500,000 - \$489,027.04$$
$$I = \$10,972.96$$

Step 3: **For each situation, apply Formula 5.1 $I = Prt$, rearranging for r.**

a. Marlie with sale to Josephine:

$$r = \frac{\$7,275.17}{(\$489,027.04)\left(\frac{217}{365}\right)}$$
$$r = 2.50\%$$

b. Josephine by herself:

$$r = \frac{\$3,697.79}{(\$496,302.21)\left(\frac{147}{365}\right)}$$
$$r = 1.85\%$$

c. Marlie without sale to Josephine:

$$r = \frac{\$10,972.96}{(\$489,027.04)\left(\frac{364}{365}\right)}$$
$$r = 2.25\%$$

When Marlie sold the T-bill after holding it for 217 days, he realized a 2.50% rate of return. Josephine then held the T-bill for another 148 days to maturity, realizing a 1.85% rate of return. If Marlie hadn't sold the note to Josephine and instead held it for the entire 364 days, he would have realized a 2.25% rate of return.

Step 4: **Compare the answers for (a) and (c) and comment.**

The yield on the date of issue was 2.25%. Marlie realized a higher rate of return because the interest rates in the market decreased during the 217 days he held it (to 1.85%, which is what Josephine is able to obtain by holding it until maturity). This raises the selling price of the T-bill. If his investment of $\$489,027.04$ grows by 2.25% for 217 days, he has $\$6,541.57$ in interest. The additional $\$733.60$ of interest (totaling $\$7,275.17$) is due to the lower yield in the market, increasing his rate of return to 2.50% instead of 2.25%.

Section 5.4 Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. A 60-day, $\$90,000$ face value commercial paper was issued when yields were 2.09%. What was its purchase price?

Solution

***Step 1:* Given information:**

$$t = \frac{60}{365}; \qquad r = 2.09\%; \qquad S = \$90,000$$

***Step 2:* Solve for P.**

$$P = \frac{S}{1+rt}$$

$$P = \frac{\$90,000}{1 + 0.209 \times \frac{60}{365}}$$

$$P = \$89,691.85$$

***Step 3:* Write as a statement.**

Its purchase price was $\$89,691.85$.

2. A 90-day Province of Ontario T-bill with a $\$35,000$ face value matures on December 11. Farrah works for Hearthplace Industries and notices that the company temporarily has some extra cash available. If she invests the money on October 28, when the yield is 4.94%, and sells the T-bill on November 25, when the yield is 4.83%, calculate how much money Farrah earned and the rate of return she realized.

Solution

Calculate the purchase price for the T-bill:

***Step 1:* Given information:**

$r = 4.94\%$ (only the rate on the day of purchase matters);

$S = \$35,000$

$$t = \text{October 28 to December 11}$$
$$t = 3 + 30 + 11$$
$$t = 44 \text{ days left on T-bill, or } \frac{44}{365}$$

(only the time remaining on the T-bill is important)

Step 2: **Solve for P.**

$$P = \frac{S}{1+rt}$$

$$P = \frac{\$35,000}{1+0.0494 \times \frac{44}{365}}$$

$$P = \$34,792.81$$

Calculate the price when sold for the T-bill:

Step 1: **Given information:**

$r = 4.83\%$ (only the rate on the day of sale matters)

$S = \$35,000$

$$t = \text{November 25 to December 11}$$
$$t = 5 + 11$$
$$t = 16 \text{ days left on T-bill, or } \frac{16}{365}$$

(Only the time remaining on the T-bill is important)

Step 2: **Solve for P.**

$$P = \frac{S}{1+rt}$$

$$P = \frac{\$35,000}{1+0.0483 \times \frac{16}{365}}$$

$$P = \$34,926.05$$

Calculate the amount of interest:

$$\begin{aligned}\text{Amount earned} &= \text{Sold Price} - \text{ Purchase Price}\\ \text{Amount earned} &= \$34,926,05 - \$34,792,81\\ \text{Amount earned} &= \$133.24\end{aligned}$$

Calculate Farrah's rate of return:

Step 1: **Given information:**

$$\begin{aligned}I &= \$133.24\\ P &= \$34,792.81\end{aligned}$$

$$\begin{aligned}t &= \text{October 28 to November 25 (the time held)}\\ t &= 3 + 25\\ t &= 28 \text{ days, or } \frac{28}{365}\end{aligned}$$

Step 2: **Solve for r.**

$$\begin{aligned}r &= \frac{I}{Pt}\\ r &= \frac{\$133.24}{\$34,792.81 \times \frac{28}{365}}\\ r &= 0.049920 \text{ or } 4.99\%\end{aligned}$$

Step 3: **Write as a statement.**

Farrah earned $\$133.24$ and the rate of return she realized was 4.99%.

3. Philippe purchased a $\$100,000$ Citicorp Financial 220-day commercial paper for $\$96,453.93$. He sold it 110 days later to Damien for $\$98,414.58$, who then held onto the commercial paper until its maturity date.
   a. What is Philippe's actual rate of return?
   b. What is Damien's actual rate of return?
   c. What is the rate of return Philippe would have realized if he had held onto the note instead of selling it to Damien?

Solution

a. Calculate Phillippe's actual rate of return.

***Step 1:* Given information:**

$$P = \$96,453.93; \quad t = \frac{110}{365}; \quad S = \$98,414.58$$

***Step 2:* Calculate I.**

$$I = S - P$$
$$I = \$98,414.58 - \$96,453.93$$
$$I = \$1,960.65$$

***Step 3:* Solve for r.**

$$r = \frac{I}{Pt}$$
$$r = \frac{\$1,960.65}{\$96,453.93 \times \frac{110}{365}}$$
$$r = 0.067449 \text{ or } 6.74\%$$

***Step 4:* Write as a statement.**

Philippe's actual rate of return is 6.74%.

---

b. Calculate Damien's actual rate of return.

***Step 1:* Given information:**

$$S = \$100,000; \quad t = \frac{110}{365}; \quad P = \$98,414.58$$

***Step 2:* Calculate I.**

$$I = S - P$$
$$I = \$100,000 - \$98,414.58$$
$$I = \$1,585.42$$

***Step 3:* Solve for r.**

$$r = \frac{I}{Pt}$$
$$r = \frac{\$1,585.42}{\$98,414.58 \times \frac{110}{365}}$$
$$r = 0.053454 \text{ or } 5.35\%$$

Step 4: **Write as a statement.**

Damien's actual rate of return is 5.35%.

---

c. Calculate rate of return Phillippe would have realized if he had held onto the note.

Step 1: **Given information:**

$$S = \$100,000; \qquad t = \frac{220}{365}; \qquad P = \$96,453.93$$

Step 2: **Calculate I.**

$$I = S - P$$
$$I = \$100,000 - \$96,453.93$$
$$I = \$3,546.07$$

Step 3: **Solve for r.**

$$r = \frac{I}{Pt}$$
$$r = \frac{\$3,546.07}{\$96,453.93 \times \frac{220}{365}}$$
$$r = 0.060995 \text{ or } 6.10\%$$

Step 4: **Write as a statement.**

Philippe would have realized 6.10% rate of return if he had held onto the note instead of selling it to Damien.

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $I = Prt$

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CHAPTER 5: SIMPLE INTEREST TERMINOLOGY (INTERACTIVE ACTIVITY)

Complete the following activity.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=187#h5p-4

Attribution

CHAPTER 5: SUMMARY

Formula & Symbol Hub Summary

For this chapter you used the following:

Symbols Used

- $I =$ Simple Interest
- $P =$ Present Value or Principal
- $r =$ Interest rate
- $t =$ Time period over which interest is charged
- $S =$ Maturity value or future value

Formulas Used

- Formula 3.1b – **Rate, Portion, and Base**

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

- Formula 5.1 – **Simple Interest**

$$I = Prt$$

- Formula 5.2a – **Simple Interest Future Value**

$$S = P(1 + rt)$$

- Formula 5.2b – **Simple Interest Amount**

$$I = S - P$$

Key Concepts Summary

5.1: Principal, Rate, Time

- Calculating the amount of simple interest either earned or charged in a simple interest environment
- Calculating the time period when specific dates or numbers of days are involved
- Calculating the simple interest amount when the interest rate is variable throughout the transaction

5.2: Moving Money Involving Simple Interest

- Putting the principal and interest together into a single calculation known as maturity value
- Altering a financial agreement and establishing equivalent payments

5.3: Application: Savings Accounts and Short-Term GICs

- How to calculate simple interest for flat-rate and tiered savings accounts
- How to calculate simple interest on a short-term GIC

5.4 Application: Treasury Bills and Commercial Papers

- The characteristics of treasury bills
- The characteristics of commercial papers
- Calculating the price of T-Bills and commercial papers
- Calculating the yield of T-Bills and commercial papers

The Language of Business Mathematics

accrued interest Any interest amount that has been calculated but not yet placed (charged or earned) into an account.

commercial paper A short-term financial instrument with maturity no longer than one year that is issued by large corporations.

compound interest A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more; interest is periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

current balance The balance in an account plus any accrued interest.

demand loan A short-term loan that generally has no specific maturity date, may be paid at any time without any interest penalty, and where the lender may demand repayment at any time.

discount rate An interest rate used to remove interest from a future value.

equivalent payments Two payments that have the same value on the same day factoring in a fair interest rate.

face value of a T-bill The maturity value of a T-bill, which ispayable at the end of the term. It includes both the principal and interest together.

fixed interest rate An interest rate that is unchanged for the duration of the transaction.
focal date A single date that is chosen to locate all values in a financial scenario so that equivalent amounts can be determined.

future value The amount of principal with interest at a future point of time for a financial transaction. If this future point is the same as the end date of the financial transaction, it is also called the maturity value.

guaranteed investment certificate (GIC) An investment that offers a guaranteed rate of interest over a fixed period of time.

interest amount The dollar amount of interest that is paid or earned.

interest rate The rate of interest that is charged or earned during a specified time period.

legal due date of a note Three days after the term specified in an interest-bearing promissory note is the date when a promissory note becomes legally due. This grace period allows the borrower to repay the note without penalty in the event that the due date falls on a statutory holiday or weekend.

maturity date The date upon which a transaction, such as a promissory note, comes to an end and needs to be repaid.

maturity value The amount of money at the end of a transaction, which includes both the interest and the principal together.

present value The amount of money at the beginning of a time period in a transaction. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some time earlier before the future value was known. In any case, the amount excludes the interest.

prime rate An interest rate set by the Bank of Canada that usually forms the lowest lending rate for the most secure loans.

principal The original amount of money that is borrowed or invested in a financial transaction.

proceeds The amount of money received from a sale.

promissory note An unconditional promise in writing made by one person to another person to pay a sum of money on demand or at a fixed or determinable future time.

repayment schedule A table that details the financial transactions in an account, including the balance, interest amounts, and payments.

savings account A deposit account that bears interest and has no stated maturity date.
secured loan Those loans that are guaranteed by an asset such as a building or a vehicle that can be seized to pay the debt in case of default.

simple interest A system for calculating interest that primarily applies to short-term financial transactions with a time frame of less than one year.

student loan A special type of loan designed to help students pay for the costs of tuition, books, and living expenses while pursuing postsecondary education.

time period The length of the financial transaction for which interest is charged or earned. It may also be called the term.

treasury bills Short-term financial instruments with maturities no longer than one year that are issued by both federal and provincial governments.

unsecured loan Those loans backed up by the general goodwill and nature of the borrower.

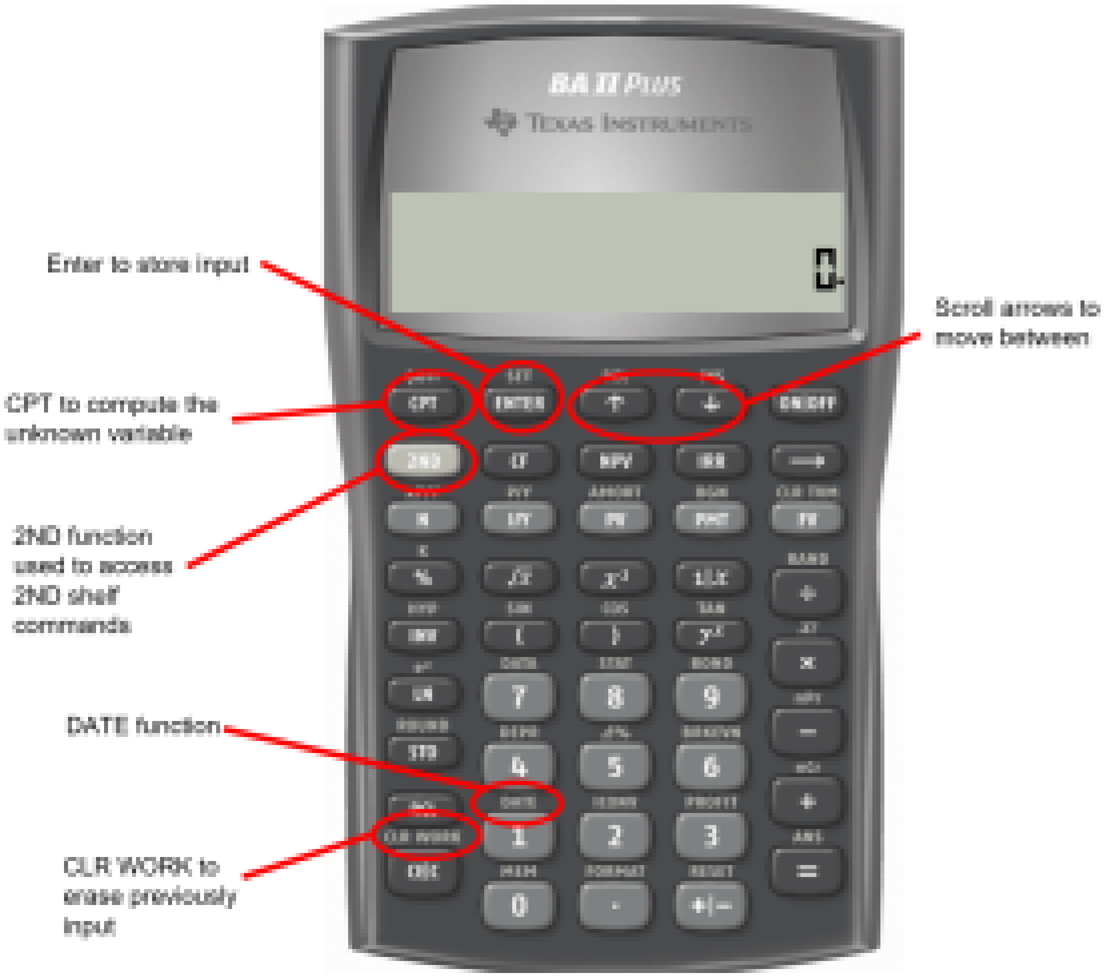
variable interest rate An interest rate that is open to fluctuations over the duration of a transaction.

yield The percentage increase between the sale price and redemption price on an investment such as a T-bill or commercial paper.

Technology Introduced

Calculator

The following calculator functions were introduced in this chapter:



BAII Plus Calculator

Date Function

- 2nd DATE to access.
- Enter two of the three variables $(\text{DT1}, \text{DT2}, \text{DBD})$ by pressing Enter after each input and using $\uparrow$ and $\downarrow$ to scroll through the display. The variables are:
    - DT1 $=$ The starting date of the transaction
    - DT2 $=$ The ending date of the transaction
    - DBD $=$ The days between the dates, counting the first day but not the last, which is the time period of the transaction.
    - $\text{ACT} / 360$ $=$ A setting for determining how the calculator determines the DBD. In Canada,

you should maintain this setting on ACT, which is the actual number of days. In other countries, such as the United States, they treat each year as having 360 days (the 360 setting) and each month as having 30 days. If you need to toggle this setting, press $\mathrm{2nd\ SET}$.

- Enter all dates in the format of MMDDYY, where MM is the numerical month, DD is the day, and YY is the last two digits of the year. DD and YY must always be entered with both digits.
- Press CPT on the unknown (when it is on the screen display) to compute the answer.

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CHAPTER 6: COMPOUND INTEREST

Outline of Chapter Topics

Learning Objectives

- Differentiate between the concept of compound interest and simple interest.
- Calculate the future value and present value of investments and loans in compound interest applications using both the algebraic and financial calculator methods.
- Calculate equivalent payments that replace another payment or a set of payments.
- Calculate the effective and equivalent interest rates for nominal interest rates.
- Calculate periodic and nominal interest rates.
- Calculate the number of compounding periods and time period of an investment or loan.

Attribution

6.1: COMPOUND INTEREST AND FUNDAMENTALS

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $\mathrm{C/Y} =$ Compounds per year
- $i =$ Periodic interest rate
- $\mathrm{I/Y} =$ Nominal interest rate per year

Formulas Used

- Formula 6.1 – **Periodic Interest Rate**

$$i = \frac{\mathrm{I/Y}}{\mathrm{C/Y}}$$

Compound Interest and Fundamentals

Compound interest is used for most transactions lasting one year or more. In simple interest, interest is converted to principal at the end of the transaction. Therefore, all interest is based solely on the original principal amount of the transaction. **Compound interest**, by contrast, involves interest being periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

Calculating the Periodic Interest Rate

The first step in learning about investing or borrowing under compound interest is to understand the interest rate used in converting interest to principal. You commonly need to convert the posted interest rate to find the exact rate of interest earned or charged in any given time period.

6.1 Periodic Interest Rate

Formula does not parse

Formula does not parse The percentage of interest earned or charged at the end of each compounding period is called the periodic interest rate. You calculate it by taking the nominal interest rate and dividing by the compounding frequency. For example, 12% compounded quarterly has a periodic interest rate of $12\% \div 4 = 3\%$. This means that at the end of every three months, you calculate 3% interest and convert it to principal.

Formula does not parse A compound interest rate consists of two elements: a nominal number for the annual interest rate, known as the nominal interest rate, and words that state the compounding frequency. For example, a 12% compounded quarterly interest rate is interpreted to mean that you accumulate 12% nominal interest per year but the interest is converted to principal every compounding period, or every three months. The word nominal is used because if compounding occurs more than once per year, the true amount of interest that you earn per year is greater than the nominal interest rate.

Formula does not parse The determination of $\mathrm{C/Y}$ involves two key concepts: Compounding Period and Compounding Frequency. The Compounding Period is the amount of time that elapses between the dates of successive conversions of interest to principal is known as the compounding period (for example, a quarterly compounded interest rate converts interest to principal every three months, therefore, the compounding period is three months). The Compounding Frequency is the number of compounding periods in a complete year (for example, a quarterly compounded interest rate compounds every three months, or $\mathrm{C/Y} = 4$ times in a single year).

Concept Check

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=194#h5p-12

Example 6.1.1

Calculate the periodic interest rate, i, for the following nominal interest rates:

a. 9% compounded monthly
b. 6% compounded quarterly

Solution

***Step 1:* Given information:**

a. $\mathrm{I/Y} = 9\%; \mathrm{C/Y} = \text{monthly} = 12 \text{ times per year}$
b. $\mathrm{I/Y} = 6\%; \mathrm{C/Y} = \text{quarterly} = 4 \text{ times per year}$

***Step 2:* For each question apply Formula 6.1.**

a.

$$I = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}}$$

$$I = \frac{9\%}{12}$$

$$I = 0.75\% \text{ per month}$$

Nine percent compounded monthly is equal to a periodic interest rate of 0.75% per month. This means that interest is converted to principal 12 times throughout the year at the rate of 0.75% each time.

---

b.

$$I = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}}$$

$$I = \frac{6\%}{4}$$

$$I = 1.5\% \text{ per quarter}$$

Step 3: **Write as a statement.**

Six percent compounded quarterly is equal to a periodic interest rate of 1.5% per quarter. This means that interest is converted to principal 4 times (every three months) throughout the year at the rate of 1.5% each time.

Example 6.1.2

Calculate the nominal interest rate, I/Y, for the following periodic interest rates:

a. $0.58\overline{3}\%$ per month
b. 0.05% per day

Solution

Step 1: **Given information:**

a. $i = 0.58\overline{3}\%$; $\mathrm{C/Y} = \text{monthly} = 12 \text{ times per year}$
b. $i = 0.05\%$; $\mathrm{C/Y} = \text{daily} = 365 \text{ times per year}$

Step 2: **For each question, apply Formula 6.1 and rearrange for the nominal rate, $\mathrm{I/Y}$.**

a.

$$\begin{aligned}\mathrm{I/Y} &= i \times \mathrm{C/Y}\\ \mathrm{I/Y} &= 0.58\overline{3} \times 12\\ \mathrm{I/Y} &= 7\%\end{aligned}$$

A periodic interest rate of $0.58\overline{3}$ per month is equal to a nominal interest rate of 7% compounded monthly.

---

b.

$$\begin{aligned}\mathrm{I/Y} &= i \times \mathrm{C/Y}\\ \mathrm{I/Y} &= 0.05 \times 365\\ \mathrm{I/Y} &= 18.25\%\end{aligned}$$

Step 3: **Write as a statement.**

A periodic interest rate of 0.05% per day is equal to a nominal interest rate of 18.25% compounded daily.

Example 6.1.3

Calculate the compounding frequency $(\mathrm{C/Y})$ for the following nominal and periodic interest rates:

a. $\text{nominal interest rate} = 6\%$, $\text{periodic interest rate} = 3\%$
b. $\text{nominal interest rate} = 9\%$, $\text{periodic interest rate} = 2.25\%$

Solution
***Step 1:* Given information:**

a. $\text{I/Y} = 6\%$; $i = 3\%$
b. $\text{I/Y} = 9\%$; $i = 2.25\%$

***Step 2:* For each question, apply Formula 6.1 $i = \dfrac{\text{I/Y}}{\text{C/Y}}$ and rearrange for the compounding frequency, C/Y.**

a.

$$\text{C/Y} = \frac{\text{I/Y}}{i}$$
$$\text{C/Y} = \frac{6\%}{3\%}$$
$$\text{C/Y} = 2 \text{ compounds per year}$$
$$\text{C/Y} = \text{semi-annually}$$

For the nominal interest rate of 6% to be equal to a periodic interest rate of 3%, the compounding frequency must be twice per year, which means a compounding period of every six months, or semi-annually.

---

b.

$$\text{C/Y} = \frac{\text{I/Y}}{i}$$
$$\text{C/Y} = \frac{9\%}{2.25\%}$$
$$\text{C/Y} = 4 \text{ compounds per year}$$
$$\text{C/Y} = \text{quarterly}$$

***Step 3:* Write as a statement.**

For the nominal interest rate of 9% to be equal to a periodic interest rate of 2.25%, the compounding frequency must be four times per year, which means a compounded period of every three months, or quarterly.

Section 6.1 Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Calculate the periodic interest rate if the nominal interest rate is 7.75% compounded monthly.
   Solution
   $$i=\frac{\text{I/Y}}{\text{C/Y}}$$
   $$i=\frac{7.75\%}{12}$$
   $$i=0.6458\%\text{ per month}$$
   The periodic interest rate is 0.65%.
2. Calculate the compounding frequency for a nominal interest rate of 9.6% if the periodic interest rate is 0.8%.
   Solution
   $$\text{C/Y}=\text{I/Y}\times i$$
   $$\text{C/Y}=9.6\%\times 0.8\%$$
   $$\text{C/Y}=12\text{ (monthly)}$$

The compounding frequency is 12 (monthly).

3. Calculate the nominal interest rate if the periodic interest rate is 2.0875% per quarter.
Solution

$$\begin{aligned} \text{I/Y} &= i \times \text{C/Y} \\ \text{I/Y} &= 2.0875\% \times 4 \\ \text{I/Y} &= 8.35\% \text{ compounded quarterly} \end{aligned}$$

The nominal interest rate is 8.35% compounded quarterly.

4. After a period of three months, Alese saw one interest deposit of $\$176.40$ for a principal of $\$9,800$. What nominal rate of interest is Alese earning?
Solution
Step 1: **First convert the interest amount into a periodic interest rate per quarter.**

$$\begin{aligned} \text{Portion} &= \text{Rate} \times \text{ Base} \\ I &= i \times PV \\ \$176.40 &= i \times \$9,800 \\ i &= \frac{\$176.40}{\$9,800} \\ i &= 0.018 \text{ or } 1.8\% \text{ per quarter} \end{aligned}$$

Step 2: **Now convert the result in Step 1 to a nominal rate.**

$$\begin{aligned} \text{I/Y} &= i \times \text{C/Y} \\ \text{I/Y} &= 1.8\% \times 4 \\ \text{I/Y} &= 7.2\% \text{ compounded quarterly} \end{aligned}$$

Alese is earning 7.2% compounded quarterly.

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $i = \frac{\text{I/Y}}{\text{C/Y}}$

Attribution

6.2: DETERMINING THE FUTURE (MATURITY) VALUE

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- C/Y = Compounds per year
- FV = Future value or maturity value
- i = Periodic interest rate
- I/Y = Nominal interest rate per year
- n = Total number of compounding periods
- PV = Present value or principal value

Formulas Used

- Formula 6.1 – **Periodic Interest Rate**

$$i=\frac{\text{I/Y}}{\text{C/Y}}$$

- Formula 6.2a – **Number of Compound Periods**

$$n=\text{C/Y}\times\text{Number of Years}$$

- Formula 6.2b – **Future (Maturity) Value**

$$FV=PV\times(1+i)^n$$

Determining the Future (Maturity) Value

The simplest future value scenario for compound interest is for all of the variables to remain unchanged throughout the entire transaction. To understand the derivation of the formula, continue with the following scenario. If $\$4,000$ was borrowed two years ago at 12% compounded semi-annually, then a borrower will owe two years of compound interest in addition to the original principal of $\$4,000$. That means $PV = \$4,000$. The compounding frequency is semi-annually, or twice per year, which makes the periodic interest rate $i = \frac{\text{I/Y}}{\text{C/Y}} = \frac{12\%}{2} = 6\%$ .Therefore, after the first six months, the borrower has 6% interest converted to principal. This a future value, or FV, calculated as follows:

Principal after one compounding period (six months) = Principal plus interest:

$$FV = PV + i(PV)$$
$$FV = \$4,000 + 0.06(\$4,000)$$
$$FV = \$4,000 + \$240$$
$$FV = \$4,240$$

Now proceed to the next six months. The future value after two compounding periods (one year) is calculated in the same way.

Note that the equation $FV = PV + i(PV)$ can be factored and rewritten as $FV = PV(1 + i)$.

$$FV_{\text{after two compounding periods}} = PV(1 + i)$$
$$FV_{\text{after two compounding periods}} = \$4,240(1 + 0.06)$$
$$FV_{\text{after two compounding periods}} = \$4,240(1.06)$$
$$FV_{\text{after two compounding periods}} = \$4,494.40$$

Since the $PV = \$4,240$ is the result of the previous calculation where $PV(1 + i) = \$4,240$, the following algebraic substitution is possible:

$$FV_{\text{after two compounding periods}} = PV(1 + i)(1 + i)$$
$$FV_{\text{after two compounding periods}} = \$4,000(1.06)(1.06)$$
$$FV_{\text{after two compounding periods}} = \$4,240(1.06)$$
$$FV_{\text{after two compounding periods}} = \$4,494.40$$

Simplifying algebraically, you get:

$$FV = PV(1 + i)(1 + i)$$
$$FV = PV(1 + i)^2$$

Do you notice a pattern? With one compounding period, the formula has only one $(1 + i)$. With two compounding periods involved, it has two factors of $(1 + i)$. Each successive compounding period multiplies a further $(1 + i)$ onto the equation. This makes the exponent on the $(1 + i)$ exactly equal to the number of times that interest is converted to principal during the transaction.

The Formula

First, you need to know how many times interest is converted to principal throughout the transaction. You can then calculate the future value. Use Formula 6.2a below to determine the number of compound periods involved in the transaction.

6.2a Number of Compound Periods

Formula does not parse

n is the total number of compounding periods.

Formula does not parse is the number of compounding periods per year.

Formula does not parse is the total number of years.

Once you know n, substitute it into Formula 6.2b, which finds the amount of principal and interest together at the end of the transaction, or the future (maturity) value, FV.

6.2b Future (Maturity) Value

$$FV = PV \times (1 + i)^n$$

FV is Future or Maturity Value.

PV is the Present Value or principal. This is the starting amount upon which compound interest is calculated.

i is the periodic interest rate from **Formula 6.1** $i = \frac{\text{I/Y}}{\text{C/Y}}$.

n is the number of compound periods from **Formula 6.2a**$n = \text{C/Y} \times \text{Number of Years}$.

Key Takeaways

Calculating the Interest Amount (I)

In any situation of lump-sum compound interest, you can isolate the interest amount using the formula

$$I = FV - PV$$

HOW TO

Calculate the Future Value of a Singe Payment

Follow these steps to calculate the future value of a single payment:

Step 1: Calculate the periodic interest rate using Formula 6.1:

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

Step 2: Calculate the total number of compound periods (n) using Formula 6.2a:

$$n = C/Y \times (\text{Number of years})$$

Step 3: Calculate the future value using Formula 6.2b:

$$FV = PV(1+i)^n$$

Note: You will first need to calculate i and n using Steps 1 and 2.

Your BAII Plus Calculator

We will be using the function keys that are presented in the third row of your calculator, known as the TVM row or (time value of money row). The five buttons located on the third row of the calculator are five of the seven variables required for time value of money calculations. This row's buttons are different in color from the rest of the buttons on the keypad.

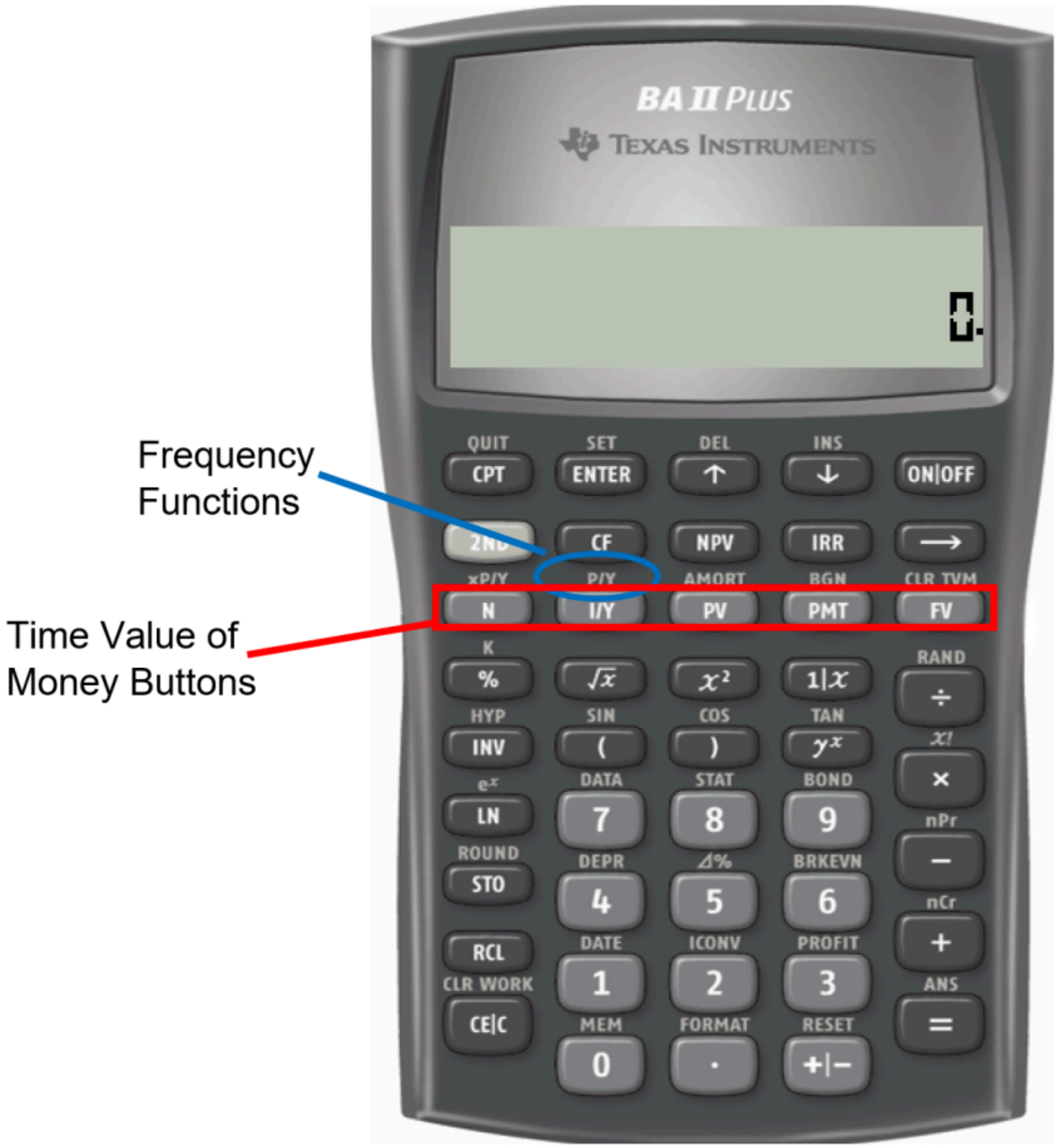


Figure 6.2.1

The table below relates each button (variable) to its meaning:

Table 6.2.1

Variable	Meaning
N	Number of compounding periods
I/Y	Interest rate per year (nominal interest rate). This is entered in percent form (without the % sign). For example, 5% is entered as 5.
PV	Present value or principal
PMT	Periodic annuity payment. For lump sum payments set this variable to zero.
FV	Future value or maturity value.
C/Y	Pressing 2ND key then I/Y will open the P/Y worksheet. P/Y stands for periodic payments per year and this will be covered in annuities. We only need to assign a value for C/Y as the calculation does not involve an annuity. We need to set payments per year (P/Y) to the same value as the number of compounding periods per year (C/Y) then press ENTER. When you scroll down (using the down arrow key), you will notice that C/Y will automatically be set to the same value. Pressing 2nd then CPT (Quit button) will close the worksheet.

To enter any information into any one of these buttons, key in the data first and then press the appropriate button. For example, if you want to enter $N = 34$, then key in 34 followed by pressing N.

Cash Flow Sign Convention

Calculating FV (PV is given)

For investments: When money is invested (paid-out), this amount is considered as a cash-outflow and this amount has to be entered as a negative number for PV.

For Loans: When money is received (loaned), this amount is considered as a cash-inflow and this amount has to be entered as a positive number for PV.

Calculating PV (FV is given)

For investments: When you receive your matured investment at the end of the term this is considered as a cash-inflow for you and the future value should be entered as a positive amount.

For Loans: When the loan is repaid at the end of the term this is considered as a cash-outflow for you and the future value should be entered as a negative amount.

Key Takeaways

When you compute solutions on the BAII Plus calculator, one of the most common error messages displayed is "Error 5." This error indicates that the cash flow sign convention has been used in a manner that is financially impossible. Some examples of these financial impossibilities include loans with no repayment or investments that never pay out. In these cases, the PV and FV have been incorrectly set to the same cash flow sign.

BAII Plus Memory

Your calculator has permanent memory. Once you enter data into any of the time value buttons it is permanently stored until

- You override it by entering another piece of data and pressing the button;
- You clear the memory of the time value buttons by pressing 2nd CLR TVM before proceeding with another question; or
- The reset button on the back of the calculator is pressed.

Example 6.2.1

If you invested $\$5,000$ for 10 years at 9% compounded quarterly, how much money would you have? What is the interest earned during the term?

Solution

The timeline for the investment is below.

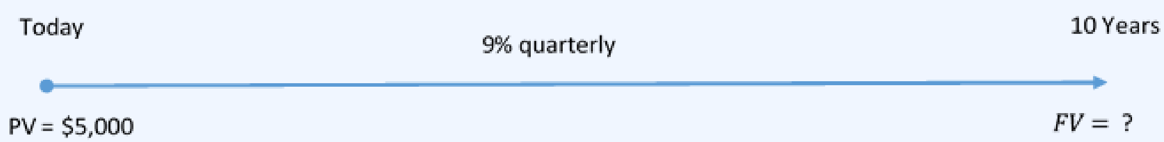


Figure 6.2.2

Step 1: **Given information:**

$$PV = 5,000; \quad \text{I/Y} = 9\%; \quad \text{C/Y} = 4$$

Step 2: **Calculate the periodic interest rate,** i**.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{9\%}{4}$$
$$i = 2.25\%$$
$$i = 0.0225$$

Step 3: **Calculate the total number of compoundings,** n**.**

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 4 \times 10$$
$$n = 40$$

Step 4: **Solve for the future value,** FV**.**

$$FV = \$5,000(1 + 0.0225)^{40}$$
$$FV = \$12,175.94$$

Step 5: **Find the interest earned.**

$$I = FV - PV$$
$$I = \$12,175.94 - \$5,000$$
$$I = \$7,175.94$$

Calculator instructions:

Table 6.2.2

N	I/Y	PV	PMT	FV	P/Y	C/Y
40	9	-5,000	0	?	12	12

Step 6: **Write as a statement.**

After 10 years, the principal grows to $\$12,175.94$, which includes your $\$5,000$ principal and $\$7,175.94$ of compound interest.

Future Value Calculations with Variable Changes

What happens if a variable such as the nominal interest rate, compounding frequency, or even the principal changes somewhere in the middle of the transaction? When any variable changes, you must break the timeline into separate time fragments at the point of the change. To arrive at the solution, you need to work from left to right one time segment at a time using **Formula 6.2b**$FV = PV \times (1+i)^n$.

HOW TO

Calculate Future Value From Lump-Sum Compound Interest

Follow these steps when variables change in calculations of future value based on lump-sum compound interest:

Step 1: Read and understand the problem. Identify the present value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the length of the time segment in years.

Step 2: For each time segment, calculate the periodic interest rate (i) using **Formula 6.1**

$$i = \frac{\mathrm{I/Y}}{\mathrm{C/Y}}$$

Step 3: For each time segment, calculate the total number of compound periods (n) using **Formula 6.2a** $n = \mathrm{C/Y} \times \text{Number of Years}$.

Step 4: Starting with the present value in the first time segment (starting on the left), solve for the future value using **Formula 6.2b** $FV = PV \times (1+i)^n$.

Step 5: Let the future value calculated in the previous step become the present value for the next step. If the principal changes, adjust the new present value accordingly.

Step 6: Using **Formula 6.2b** $FV = PV \times (1+i)^n$ calculate the future value of the next time segment.

Step 7: Repeat Steps 5 and 6 until you obtain the final future value from the final time segment.

Key Takeaways

The BAII Plus Calculator:

Transforming the future value from one time segment into the present value of the next time segment does not require re-entering the computed value. Instead, apply the following technique:

1. Load the calculator with all known compound interest variables for the first time segment.
2. Compute the future value at the end of the segment.
3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the ± key, and then store the unrounded number back into the present value button by pressing PV. Change the N, I/Y, and C/Y as required for the next segment.
4. Return to step 2 for each time segment until you have completed all time segments.

Concept Check

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=199#h5p-13

Example 6.2.2

Five years ago Coast Appliances was supposed to upgrade one of its facilities at a quoted cost of $\$48,000$. The upgrade was not completed, so Coast Appliances delayed the purchase until now. The construction company that provided the quote indicates that prices rose 6% compounded quarterly for the first $1\dfrac{1}{2}$ years, 7% compounded semi-annually for the following $2\dfrac{1}{2}$ years, and 7.5% compounded monthly for the final year. If Coast Appliances wants to perform the upgrade today, what amount of money does it need?

Solution

The timeline below shows the original quote from five years ago until today.

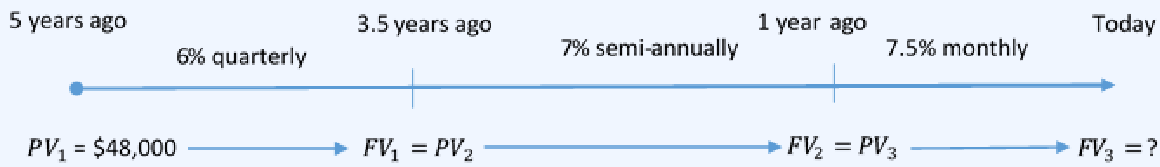


Figure 6.2.3

Step 1: **First time segment:**

$$PV_1 = \$48,000; \quad \text{I/Y} = 6\%; \quad \text{C/Y} = 4; \quad \text{Years} = 2$$

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{6\%}{4}$$
$$i = 1.5\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 4 \times 1.5$$
$$n = 6$$

Find FV_1

$$FV_1 = PV_1(1+i)^n$$
$$FV_1 = \$48,000(1+0.015)^6$$
$$FV_1 = \$24,500(1.015)^6$$
$$FV_1 = \$52,485.27667$$

This becomes PV_2 for the next calculation in Step 2.

Step 2: **Second line segment:**

$$PV_2 = FV_1 = \$52,485.27667; \quad \text{I/Y} = 7\%; \quad \text{C/Y} = 2; \quad \text{Years} = 2.5$$

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{7\%}{2}$$
$$i = 3.5\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 2 \times 2.5$$
$$n = 5$$

Find FV_2

$$FV_2 = PV_2(1+i)^n$$
$$FV_2 = \$52,485.27667(1+0.035)^5$$
$$FV_2 = \$62,336.04435$$

This becomes PV_3 for the next calculation in Step 3.

Step 3: **Third line segment:**

$$PV_3 = FV_2 = \$62,336.04435; \quad \text{I/Y} = 7.5\%; \quad \text{C/Y} = 12; \quad \text{Years} = 1$$

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{7.5\%}{12}$$
$$i = 0.625\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 12 \times 1$$
$$n = 12$$

Find FV_3

$$FV_3 = PV_3(1+i)^n$$
$$FV_3 = \$62,336.04435(1+0.00325)^{12}$$
$$FV_3 = \$67,175.35$$

The future value is $\$67,175.35$.

Calculator instruction:

Table 6.2.3

Step	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	6	6	48,500	0	?	4	4
2	5	7	52,485.27667	0	?	2	2
3	12	7.5	62,336.04435	0	?	12	12

***Step 4:* Write as a statement.**

Coast Appliances requires $\$67,175.35$ to perform the upgrade today. This consists of $\$48,000$ from the original quote plus $\$19,175.35$ in price increases.

Example 6.2.3

Two years ago Lorelei placed $\$2,000$ into an investment earning 6% compounded monthly. Today she makes a deposit to the investment in the amount of $\$1,500$. What is the maturity value of her investment three years from now?

Solution

The timeline for the investment is below.

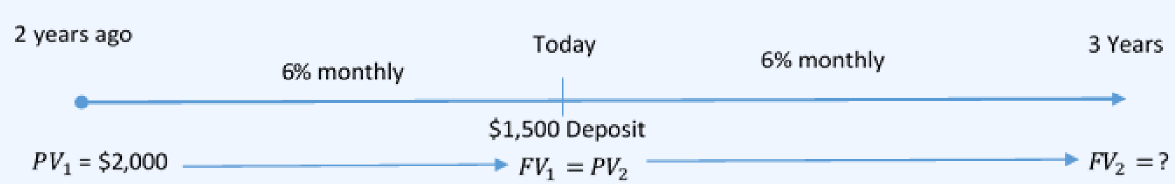


Figure 6.2.4

Step 1: **First time segment:**

$$PV_1 = \$2,000; \quad \text{I/Y} = 6\%; \quad \text{C/Y} = 12; \quad \text{Years=2}$$

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{6\%}{12}$$

$$i = 0.5\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$n = 12 \times 2$$

$$n = 24$$

Find FV_1

$$FV_1 = PV_1(1+i)^n$$

$$FV_1 = \$2,000(1+0.005)^24$$

$$FV_1 = \$2,000(1.005)^24$$

$$FV_1 = \$2,254.319552$$

$$\$2,254.319552 + \$1,500 = \$3,754.319552$$

This becomes PV_2 for the second line segment in Step 2.

Step 2: **Second line segment:**

$$PV_1 = FV_1 = 3,754.319552; \quad \text{I/Y} = 6\%; \quad \text{C/Y} = 12; \quad \text{Years} = 3$$

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{6\%}{12}$$

$$i = 0.5\%$$

$$n = \text{C/Y} \times \text{(Number of Years)}$$
$$n = 12 \times 3$$
$$n = 36$$

Find FV_2

$$FV_2 = PV_2(1+i)^n$$
$$FV_2 = \$3,754.319552(1+0.005)^{36}$$
$$FV_2 = \$4,492.72$$

The future value is $\$4,492.72$

Calculator instructions:

Table 6.2.4 Calculator Instructions for Example 6.2.3

Step	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	24	6	-2,000	0	?	12	12
2	36	6	-3,754.319552	0	?	12	12

Step 3: **Write as a statement.**

Three years from now Lorelei will have $\$4,492.72$. This represents $\$3,500$ of principal and $\$992.72$ of compound interest.

Section 6.2 Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Find the future value if $\$53,000$ is invested at 6% compounded monthly for 4 years and 3 months.
   Solution
   ***Step 1:* Given information:**
   $$PV = \$53,000; \quad \mathrm{I/Y} = 6\%; \quad \mathrm{C/Y} = 12; \quad t = 4\frac{3}{12} \text{ years}$$
   ***Step 2:* Find i.**
   $$\begin{aligned} i &= \frac{\mathrm{I/Y}}{\mathrm{C/Y}} \\ i &= \frac{6\%}{12} \\ i &= 0.5\% \end{aligned}$$
   ***Step 3:* Find n.**
   $$\begin{aligned} n &= \mathrm{C/Y} \times (\text{Number of Years}) \\ n &= 12 \times \left(4\frac{3}{12}\right) \\ n &= 4.25 \times 12 \\ n &= 51 \end{aligned}$$
   ***Step 4:* Solve for FV.**
   $$\begin{aligned} FV &= PV(1+i)^{51} \\ FV &= \$53,000(1+0.005)^{51} \\ FV &= \$53,000(1.005)^{51} \\ FV &= \$68,351.02 \end{aligned}$$
   ***Step 5:* Write as a statement.**

   The future value is $\$68,351.02$.

Calculator instructions:

$$\begin{aligned}&\text{N} = 51\\&\text{I/Y} = 6\\&\text{PV} = -53,000\\&\text{PMT} = 0\\&\text{P/Y} = 12\\&\text{C/Y} = 12\\&\text{CPT FV} = \$68,351.02\end{aligned}$$

2. Find the future value if $\$24,500$ is invested at 4.1% compounded annually for 4 years; then 5.15% compounded quarterly for 1 year, 9 months; then 5.35% compounded monthly for 1 year, 3 months.

Solution

Step 1: **Find** FV_1**.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{4.1\%}{1}$$

$$i = 4.1\%$$

$$\begin{aligned}n &= \text{C/Y} \times (\text{Number of Years})\\n &= 1 \times 4\\n &= 4\end{aligned}$$

$$\begin{aligned}FV_1 &= PV_1(1+i)^n\\FV_1 &= \$24,500(1+0.041)^4\\FV_1 &= \$28,771.03049 \text{ (This becomes } PV \text{ in Step 2)}\end{aligned}$$

Step 2: **Find** FV_2**.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{5.15\%}{4}$$

$$i = 1.2875\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$n = 4 \times \left(1\frac{9}{12}\right)$$

$$n = 1.75 \times 4$$

$$n = 7$$

$$FV_2 = PV_2(1+i)^n$$

$$FV_2 = \$28,771.93049(1.012875)^7$$

$$FV_2 = \$31,467.33516 \;\; (\text{This becomes } PV \text{ in Step 3})$$

Step 3: **Find** FV_3

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{5.35\%}{12}$$

$$i = 0.4458\overline{3}$$

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$n = 12 \times \left(1\frac{3}{12}\right)$$

$$n = 1.25 \times 12$$

$$n = 15$$

$$FV_3 = PV_3(1+i)^n$$

$$FV_3 = \$31,467.33516\left(1.004458\overline{3}\right)^{15}$$

$$FV_3 = \$33,638.67$$

Step 4: **Write as a statement.**

The future value is $\$33,638.67$.

Calculator Instructions:

$$\text{N} = 4$$
$$\text{I/Y} = 4.1$$
$$PV = -24,500$$
$$\text{PMT} = 0$$
$$\text{P/Y} = 1$$
$$\text{C/Y} = 1$$
$$\text{CPT FV} = \$28,771.93049 \text{ (This becomes negative PV in Step 2.)}$$

3. Nirdosh borrowed $\$9,300$ at 6.35% compounded semi-annually $4\frac{1}{4}$ years ago. The interest rate changed to 6.5% compounded quarterly $1\frac{3}{4}$ years ago. What amount of

money today is required to pay off this loan?

Solution

Step 1: **Find** FV_1**.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{6.35\%}{2}$$

$$i = 2 \times 2.5$$

$$i = 3.175\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$n = 2 \times 2.5$$

$$n = 5$$

$$FV_1 = PV(1+i)^n$$

$$FV_1 = \$9,300(1.03175)^5$$

$$FV_1 = \$10,873.14892 \text{ (This becomes } PV \text{ in Step 2)}$$

Step 2: **Find** FV_2**.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{6.5\%}{4}$$

$$i = 1.625\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$n = 4 \times 1.75$$

$$n = 7$$

$$FV_1 = PV(1+i)^n$$

$$FV_1 = \$10,873.14892(1.001625)^7$$

$$FV_1 = \$12,171.92 \text{ (Round at this step)}$$

***Step 3:* Write as a statement.**

It is required today $\$12,171.92$ to pay off the loan.

Calculator Instructions for Step 1 and Step 2:

Step 1:

$$\text{N} = 5$$
$$\text{I/Y} = 6.35$$
$$PV = 9,300$$
$$\text{PMT} = 0$$
$$\text{P/Y} = 2$$
$$\text{C/Y} = 2$$
$$\text{CPT FV} = \$10,873.14892 \ \text{(This becomes negative } PV \text{ in Step 2)}$$

Step 2:

$$\text{N} = 7$$
$$\text{I/Y} = 6.5$$
$$PV = -10,873.14892$$
$$\text{PMT} = 0$$
$$\text{P/Y} = 4$$
$$\text{C/Y} = 4$$
$$\text{CPT FV} = \$12,171.92$$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER.$n = \text{C/Y} \times \text{Number of Years}$

$FV = PV \times (1+i)^n$ $i = \dfrac{\text{I/Y}}{\text{C/Y}}$

Attribution

"9.2 Determining the Future (Maturity) Value" from Business Math: A Step-by-Step Handbook

6.3: DETERMINING THE PRESENT VALUE

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- FV = Future value or maturity value
- PV = Present value or principal value
- i = Periodic interest rate
- C/Y = Compounds per year
- I/Y = Nominal interest rate per year
- n = Total number of compounding periods

Formulas Used

- Formula 6.1 – **Periodic Interest Rate**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

- Formula 6.2a – **Number of Compound Periods**

$$n = \text{C/Y} \times \text{Number of Years}$$

- Formula 6.2b – **Future (Maturity) Value**

$$FV = PV \times (1+i)^n$$

- Formula 6.3 – **Present Value (Principal)**

$$PV = \frac{FV}{(1+i)^n}$$

Determining the Present Value

PV is the **Present Value** or **Principal**. This is the new unknown variable. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some earlier point in time than when the future value is known. In any case, the amount excludes the future interest. To calculate this variable, substitute the values for the other three variables into the formula and then algebraically rearrange to isolate PV.

Solving for present value requires you to use the future value formula we introduced in Section 6.2 (**Formula 6.2b** $FV = PV \times (1+i)^n$). We rearrange the future value formula to solve for PV.

6.3 Present Value (Principal)

Formula does not parse

PV is the Present Value or principal.
FV is Future or Maturity Value.
i is the periodic interest rate from **Formula 6.1** $i = \dfrac{\text{I/Y}}{\text{C/Y}}$.
n is the number of compound periods from **Formula 6.2a** $n = \text{C/Y} \times \text{Number of Years}$.

HOW TO

Calculate the Present Value of a Single Payment

Follow these steps to calculate the present value of a single payment:

Step 1: Calculate the periodic interest rate (i) using Formula 6.1:

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}}$$

Step 2: Calculate the total number of compound periods (n) using Formula 6.2a

$$n = \text{C/Y} \times (\text{Number of years})$$

Step 3: Calculate the present value using Formula 6.3:

$$PV = \frac{FV}{(1+i)^n}$$

Your BAII Plus Calculator

You use the financial calculator in the exact same manner as described in Section 6.2. The only difference is that the unknown variable is PV instead of FV. You must still load the other six variables into the calculator and apply the cash flow sign convention carefully.

Example 6.3.1

Castillo's Warehouse will need to purchase a new forklift for its warehouse operations three years from now, when its new warehouse facility becomes operational. If the price of the new forklift is $\$38,000$ and Castillo's can invest its money at 7.25% compounded monthly, how much money should it put aside today to achieve its goal?

Solution

***Step 1:* Given variables:**

$$FV = 38,000; \quad \text{I/Y} = 7.25\%; \quad \text{C/Y} = 12; \quad \text{Years} = 3$$

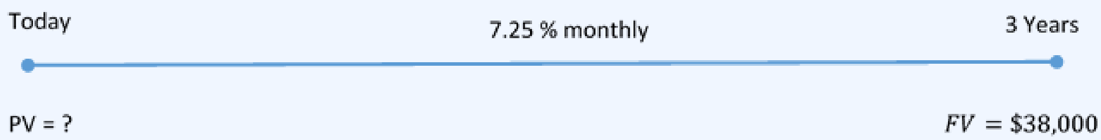


Figure 6.3.1

Step 2: **Calculate the periodic interest rate, i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{7.25\%}{12}$$

$$i = 0.6041\overline{6}\%$$

$$i = 0.006041\overline{6}$$

Step 3: **Calculate the number of compound periods,n.**

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$n = 3 \times 12$$

$$n = 36$$

Step 4: **Solve for the present value, PV.**

$$PV = \frac{FV}{(1+i)^n}$$

$$PV = \frac{38{,}000}{(1+0.006041\overline{6})^{36}}$$

$$PV = 30{,}592.06$$

Table 6.3.1 Calculator Instructions for Example 6.3.1

N	I/Y	PV	PMT	FV	P/Y	C/Y
36	7.25	?	0	38,000	12	12

Step 5: **Write as a statement.**

If Castillo's Warehouse places $\$30,592.06$ into the investment, it will earn enough interest to grow to $\$38,000$ three years from now to purchase the forklift.

Present Value Calculations with Variable Changes

Addressing variable changes in present value calculations follows the same techniques as future value calculations. You must break the timeline into separate time segments, each of which involves its own calculations.

Solving for the unknown PV at the left of the timeline means you must start at the right of the timeline. You must work from right to left, one time segment at a time using the formula for PV each time. Note that the present value for one time segment becomes the future value for the next time segment to the left.

HOW TO

Calculate Present Value Involving Variable Changes (Single Payment)

Follow these steps to calculate a present value involving variable changes in single payment compound interest:

Step 1: Read and understand the problem. Identify the future value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the segment's length in years.

Step 2: For each time segment, calculate the periodic interest rate, i.

Step 3: For each time segment, calculate the total number of compounding periods, n.

Step 4: Starting with the future value in the first time segment on the right, solve for the present value.

Step 5: Let the present value calculated in the previous step become the future value for the next time segment to the left. If the principal changes, adjust the new future value accordingly.

Step 6: Using the present value formula, calculate the present value of the next time segment.

Step 7: Repeat steps 5 and 6 until you obtain the present value from the leftmost time segment.

Your BAII Plus Calculator

To use your calculator efficiently in working through multiple time segments, follow a procedure similar to that for future value:

1. Load the calculator with all the known compound interest variables for the first time segment on the right.
2. Compute the present value at the beginning of the segment.
3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the $\pm$ key, then store the unrounded number back into the future value button by pressing FV. Change the N, I/Y, and C/Y as required for the next segment.

Return to Step 2 for each time segment until you have completed all time segments.

Example 6.3.2

Sebastien needs to have $\$9,200$ saved up three years from now. The investment he is considering pays 7% compounded semi-annually, 8% compounded quarterly, and 9% compounded monthly in successive years. To achieve his goal, how much money does he need to place into the investment today?

Solution

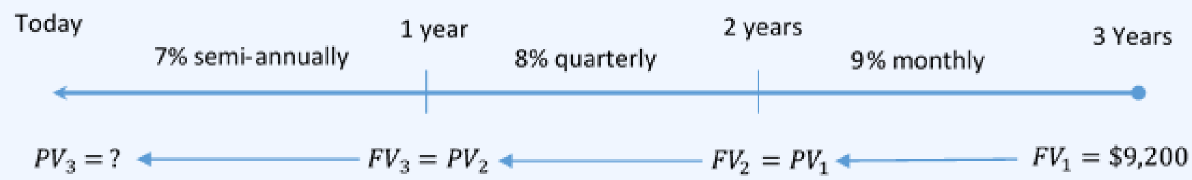


Figure 6.3.2 Timeline

Starting from the right end of the timeline and working backwards:

Step 1: **First time segment:**

$$FV_1 = \$9,200; \quad \text{I/Y} = 9\%; \quad \text{C/Y} = 12; \quad \text{Years} = 1$$

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{9\%}{12}$$
$$i = 0.75\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 1 \times 12$$
$$n = 12$$

Find PV_1

$$PV_1 = \frac{FV_1}{(1+i)^n}$$
$$PV_1 = \frac{9{,}200}{(1+0.0075)^{12}}$$
$$PV_1 = 8{,}410.991026$$

This becomes FV_2 in Step 2.

Step 2: **Second line segment:**

$$FV_2 = PV_1 = 8,410.991026; \quad \text{I/Y} = 8\%; \quad \text{C/Y} = 4; \quad \text{Years} = 1$$

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{8\%}{4}$$

$$i = 2\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 1 \times 4$$
$$n = 4$$

Find PV_2

$$PV_2 = \frac{FV_2}{(1+i)^n}$$

$$PV_2 = \frac{8{,}410.991026}{(1+0.02)^4}$$

$$PV_2 = 7{,}770.455587$$

This becomes FV_3 in Step 3.

Step 3: **Third line segment:**

$$FV_3 = PV_2 = 7,770.455587; \quad \text{I/Y} = 7\%; \quad \text{C/Y} = 2; \quad \text{Years} = 1$$

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{7.5\%}{12}$$

$$i = 0.625\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 1 \times 2 = 2$$

Find PV_3

$$PV_3 = \frac{FV_3}{(1+i)^n}$$

$$PV_3 = \frac{7{,}770.455587}{(1+0.035)^2}$$

$$PV_3 = 7{,}253.80$$

The present value is $\$7,253.80$.

Table 6.3.2 Calculator Instructions for Example 6.3.2

Step	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	12	9	?	0	9,200	12	12
2	4	8	?	0	±(PV from Step 1)	4	4
3	2	7	?	0	±(PV from Step 1)	2	2

***Step 4:* Write as a statement.**

Sebastien needs to place $\$7,253.80$ into the investment today to have $\$9,200$ three years from now.

When you calculate the present value of a single payment for which only the interest rate fluctuates, it is possible to find the principal amount in a single division:

$$PV = \frac{FV}{(1+i_1)^{n_1} \times (1+i_2)^{n_2} \times (1+i_3)^{n_3} \times \ldots (1+i_n)^{n_n}}$$

where n represents the time segment number.

In the previous example you can calculate the same principal as follows:

$$PV = \frac{\$9,200}{(1+0.0075)^{12} \times (1+0.02)^4 \times (1+0.035)^2}$$

$$PV = \$7{,}253.80$$

Section 6.3 Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. A debt of $\$37,000$ is owed 21 months from today. If prevailing interest rates are 6.55% compounded quarterly, what amount should the creditor be willing to accept today?

Solution

Step 1: **Given information:**

$$FV = \$37,000; \quad \text{I/Y} = 6.55\%; \quad t = \frac{21}{12} = 1.75 \text{ years};$$
$$\text{C/Y} = \text{quarterly} = 4$$

Step 2: **Find i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{6.55\%}{4}$$
$$i = 1.6375\%$$

Step 3: **Find n.**

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 4 \times \frac{21}{12}$$
$$n = 7$$

Step 4: **Solve for PV.**

$$PV = \frac{FV}{(1+i)^n}$$
$$PV = \frac{\$37,000}{(1.016375)^7}$$
$$PV = \$33,023.56$$

The creditor should be willing to accept $\$33,023.56$ today.

Calculator Instructions:

$\text{N} = 7$
$\text{I/Y} = 6.55$
$\text{PMT} = 0$
$\text{FV} = 37,000$
$\text{P/Y} = 4$
$\text{C/Y} = 4$
$\text{CPT PV} = \$33,023.5$

2. For the first $4\frac{1}{2}$ years, a loan was charged interest at 4.5% compounded semi-annually. For the next 4 years, the rate was 3.25% compounded annually. If the maturity value was $\$45,839.05$ at the end of the $8\frac{1}{2}$ years, what was the principal of the loan?

Solution
Step 1: **Find PV_1.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{3.25\%}{1}$$

$$i = 3.25\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$n = 1 \times 4$$

$$n = 4$$

$$PV_1 = \frac{FV}{(1+i)^n}$$

$$PV_1 = \frac{\$45,839.05}{(1.0325)^4}$$

$$PV_1 = \$40,334.37829 \ \ (\text{This becomes } FV \text{ in Step 2})$$

Step 2: **Find PV_2.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{4.5\%}{2}$$

$$i = 2.25\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$n = 2 \times 4.5$$

$$n = 9$$

$$PV_2 = \frac{FV}{(1+i)^n}$$

$$PV_2 = \frac{\$40,334.37829}{(1.0225)^9}$$

$$PV_2 = \$33,014.56 \ \ (\text{Round at this step})$$

The principal of the loan is $\$33,014.56$.

Calculator Instructions:

Step 1: **Find** PV_1**.**

$\text{N} = 4$
$\text{I/Y} = 3.25$
$\text{PMT} = 0$
$\text{FV} = -45,839.05$
$\text{P/Y} = 1$
$\text{C/Y} = 1$
$\text{CPT PV} = 40,334.37829$ (This becomes negative FV for Step 2)

Step 2: **Find** PV_2**.**

$\text{N} = 9$
$\text{I/Y} = 4.5$

$$\text{PMT} = 0$$
$$\text{FV} = -40,334.37829$$
$$\text{P/Y} = 2$$
$$\text{C/Y} = 2$$
$$\text{CPT PV} = \$33,014.56$$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $PV = \frac{FV}{(1+i)^n}$ $i = \frac{\text{I/Y}}{\text{C/Y}}$ $FV = PV \times (1+i)^n$ $n = \text{C/Y} \times \text{Number of Years}$

Attribution

6.4: EQUIVALENT PAYMENTS

Formula & Symbol Hub

For this chapter you used the following:

Symbols Used

- FV = Future value or maturity value
- PV = Present value or principal value
- i = Periodic interest rate
- C/Y = Compounds per year
- I/Y = Nominal interest rate per year
- n = Total number of compounding periods

Formulas Used

- Formula 6.1 – **Periodic Interest Rate**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

- Formula 6.2a – **Number of Compound Periods**

$$n = \text{C/Y} \times \text{Number of Years}$$

- Formula 6.2b – **Future (Maturity) Value**

$$FV = PV \times (1+i)^n$$

- Formula 6.3 – **Present Value (Principal)**

$$PV = \frac{FV}{(1+i)^n}$$

Equivalent Payments

This section explores the concept of equivalent payment streams. This involves equating two or more alternative financial streams to ensure that neither party is penalized by any choice. You then apply the concept of present value to loans and loan payments.

The Fundamental Concept of Equivalency

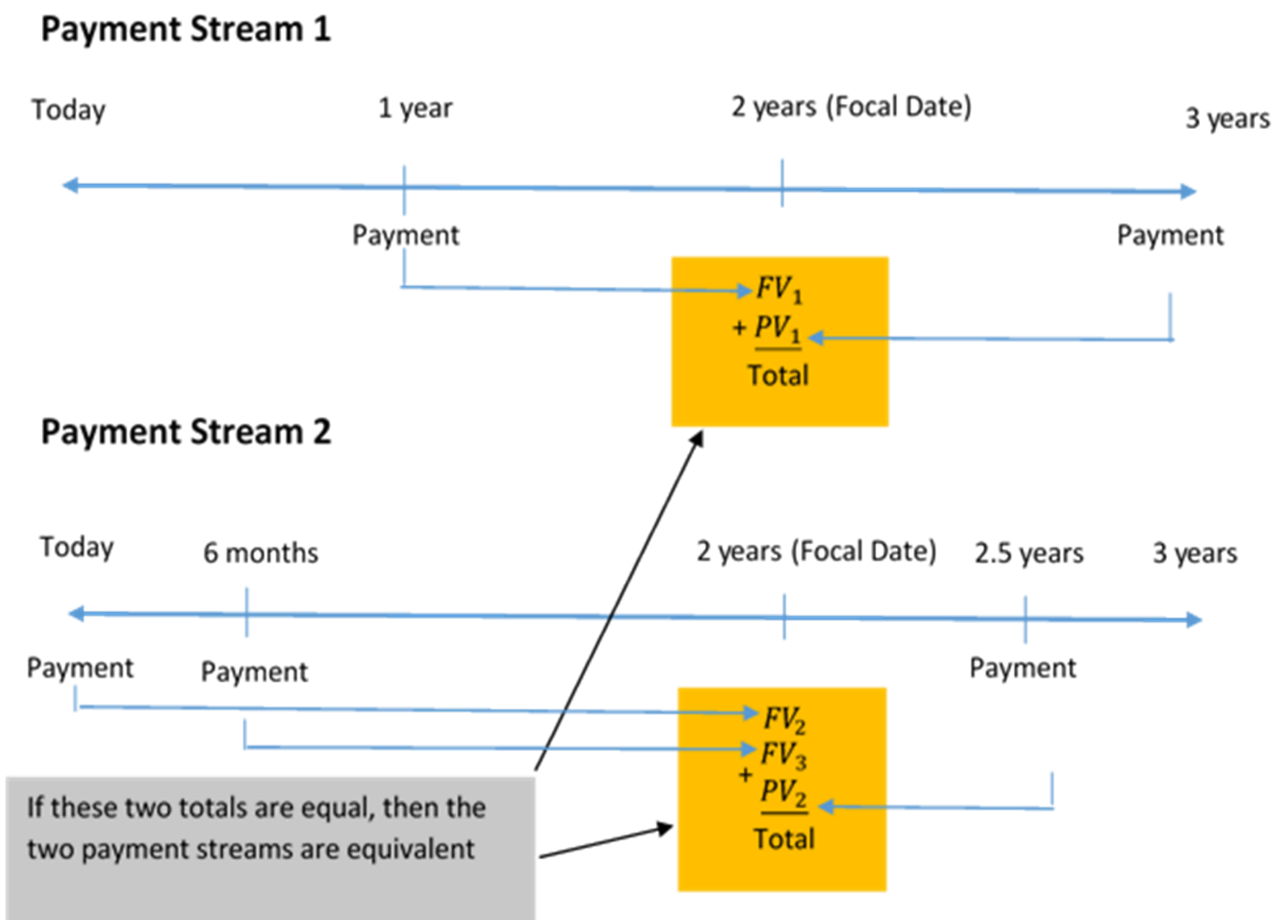


Figure 6.4.1

The fundamental concept of equivalency states that two or more payment streams are equal to each other if they have the same economic value on the same focal date. As illustrated in the figure, the two alternative financial streams are equivalent if the total of Payment Stream 1 is equal to the total of Payment Stream 2 on the same focal date. Note that the monies involved in each payment stream can be summed on the focal date because of the fundamental concept of time value of money.

HOW TO

Solve Questions Involving Equivalent Payments

Follow these steps to solve an equivalent payment question:

Step 1: Draw as many timelines as needed to illustrate each of the original and proposed agreements. Clearly indicate dates, payment amounts, and the interest rate(s). If you draw two or more timelines, align them vertically, ensuring that all corresponding dates are in the same columns. This allows you to see which payments need to be future valued and which need to be present valued to express them in terms of the chosen focal date.

Step 2: Choose a focal date to which all money will be moved. You should simplify your calculations by selecting a focal date corresponding to the date of an unknown variable.

Step 3: Calculate all needed periodic interest rates (i) using Formula 6.1:

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compound per Year (C/Y)}}$$

Step 4: Calculate the total number of compounds, (n) for each payment using Formula 6.2a:

$$n = (\text{C/Y}) \times (\text{Number of Years})$$

Step 5: Perform the appropriate time value calculation for each payment using either the formula for FV or PV.

Step 6: Equate the values of the original and proposed agreements on the focal date and solve for any unknowns.

Example 6.4.1

Assume you owe $\$1,000$ today and $\$1,000$ one year from now. You find yourself unable to make that payment today, so you indicate to your creditor that you want to make both payments six months from now instead. Prevailing interest rates are at 6% compounded semi-annually. What single payment six months from now (the proposed payment stream) is equivalent to the two payments (the original payment stream)?

Solution

Step 1: **The timeline illustrates the scenario.**

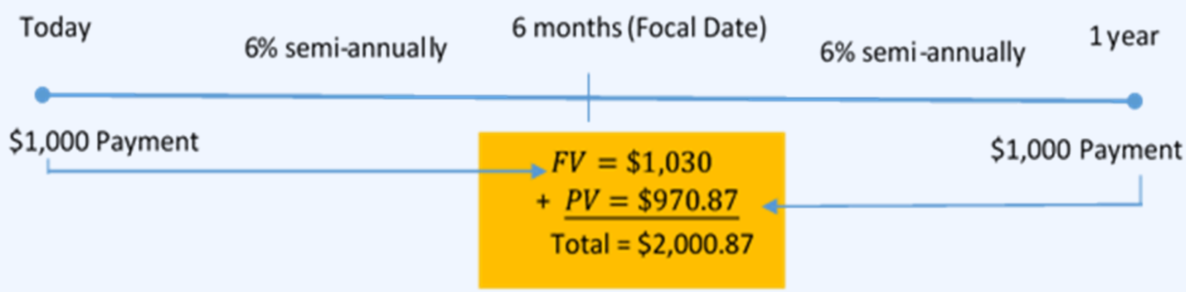


Figure 6.4.2

$$\mathrm{I/Y} = 6\%; \qquad \mathrm{C/Y} = \text{semi-annually} = 2$$

Step 2:**Apply the fundamental concept of time value of money, moving all of the money to the same date. Since the proposed payment is for six months from now, you choose a focal date of six months.**

Step 3: **Calculate the periodic interest rate, i.**

$$i = \frac{\mathrm{I/Y}}{\mathrm{C/Y}}$$

$$i = \frac{6\%}{2}$$

$$i = 3\%$$

Step 4: **Calculate the total number of compounds, n.**

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 2 \times \frac{1}{2}$$
$$n = 1$$

Step 5: **Perform the appropriate time value calculations.**

Moving today's payment of $\$1,000$ six months into the future, you have

$$FV = \$1{,}000(1 + 0.03)^1$$
$$FV = \$1{,}030$$

Moving the future payment of $\$1,000$ six months earlier, you have

$$PV = \frac{\$1{,}000}{(1 + 0.03)^1}$$
$$PV = \$970.87$$

Table 6.4.1

Payment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	1	6	$1,000$	0	Answer: $-\$1,030$	2	2
2	1	6	Answer: $-\$970.87$	0	$1,000$	2	2

Step 6: Now that money has been moved to the same focal date you can sum the two totals to determine the equivalent payment, which is $\$1,030 + \$970.87 = \$2,000.87$. Note that this is financially fair to both parties. For making your $\$1,000$ payment six months late, the creditor is charging you $\$30$ of interest. Also, for making your second $\$1,000$ payment six months early, you are receiving a benefit of $\$29.13$. This leaves both parties compensated equitably: Neither party is financially better or worse off because of the change in the deal.

Example 6.4.2

Johnson's Garden Centre has recently been unprofitable and concludes that it cannot make two debt payments of $\$4,500$ due today and another $\$6,300$ due in three months. After discussions between Johnson's Garden Centre and its creditor, the two parties agree that both payments could be made nine months from today, with interest at 8.5% compounded monthly. What total payment does Johnson's Garden Centre need nine months from now to clear its debt?

Solution

***Step 1:* The timeline illustrates the scenario.**

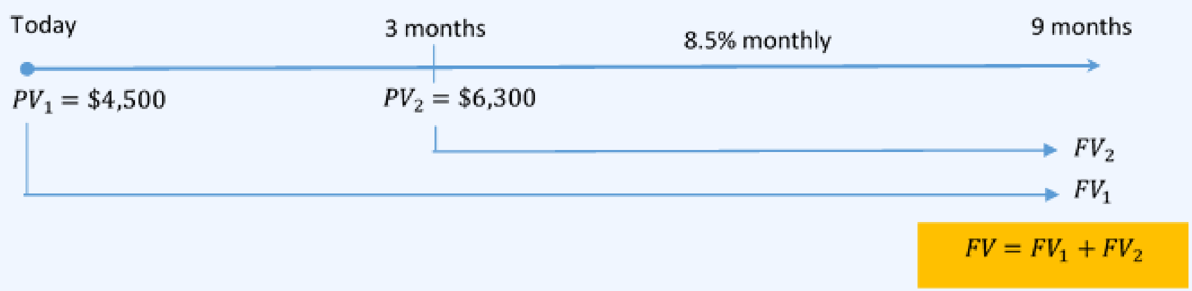


Figure 6.4.3

$$PV_1 \text{ (today)} = \$4,500; \quad PV_2 \text{ (3 months from now)} = \$6,300;$$
$$\text{I/Y} = 8.5\%; \quad \text{C/Y} = \text{monthly} = 12$$

***Step 2:*Due date for all payments $= 9$ months from today. This is your focal date.**

***Step 3:* Calculate the periodic interest rate,i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{8.5\%}{12}$$

$$i = 0.708\overline{3}\%$$

$$i = 0.00708\overline{3}$$

Step 4: **Calculate the total number of compounds, n.**

The first payment moves nine months into the future, or $\frac{9}{12}$ of a year.

$$n_1 = \text{C/Y} \times (\text{Number of Years})$$
$$n_1 = 12 \times \frac{9}{12}$$
$$n_1 = 9$$

The second payment moves six months into the future, or $\frac{6}{12}$ of a year.

$$n_2 = \text{C/Y} \times (\text{Number of Years})$$
$$n_2 = 12 \times \frac{6}{12}$$
$$n_2 = 6$$

Step 5: **Perform the appropriate time value calculations.**

Moving the first payment of $\$4,500$ nine months into the future, you have

$$FV_1 = \$4{,}500(1 + 0.00708\overline{3})^9$$
$$FV_1 = \$4{,}795.138902$$

Moving the second payment of $\$6,300$ six months into the future, you have

$$FV_2 = \$6{,}300(1 + 0.007083\overline{3})^6$$
$$FV_2 = \$6{,}572.536425$$

Table 6.4.2

Payment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	9	8.5	$4,500$	0	Answer: $-\$4,795.138902$	12	12
2	6	8.5	$6,300$	0	Answer: $-\$6,572.526425$	12	12

Step 6: **Write as a statement.**

With interest, the two payments total $\$11,367.68$. This is the $\$10,800$ of the original principal plus $\$567.68$ in interest for making the late payments.

Example 6.4.3

You have three debts to the same creditor: $\$3,000$ due today, $\$2,500$ due in $2\frac{1}{4}$ years, and $\$4,250$ due in 3 years 11 months. Unable to fulfill this obligation, you arrange with your creditor to make two alternative payments: $\$3,500$ in nine months and a second payment due in two years. You agree upon an interest rate of 9.84% compounded monthly. What is the amount of the second payment?

Solution

Determine the amount of the second payment that is due two years from today. Apply the fundamental concept of time value of money, moving all money from the original and proposed payment streams to a focal date positioned at the unknown payment. Once all money is moved to this focal date, apply the fundamental concept of equivalence, solving for the unknown payment, or x.

Step 1: **With two payment streams and multiple amounts all on different dates, visualize two timelines. The payment amounts, interest rate, and due dates for both payment streams are known.**

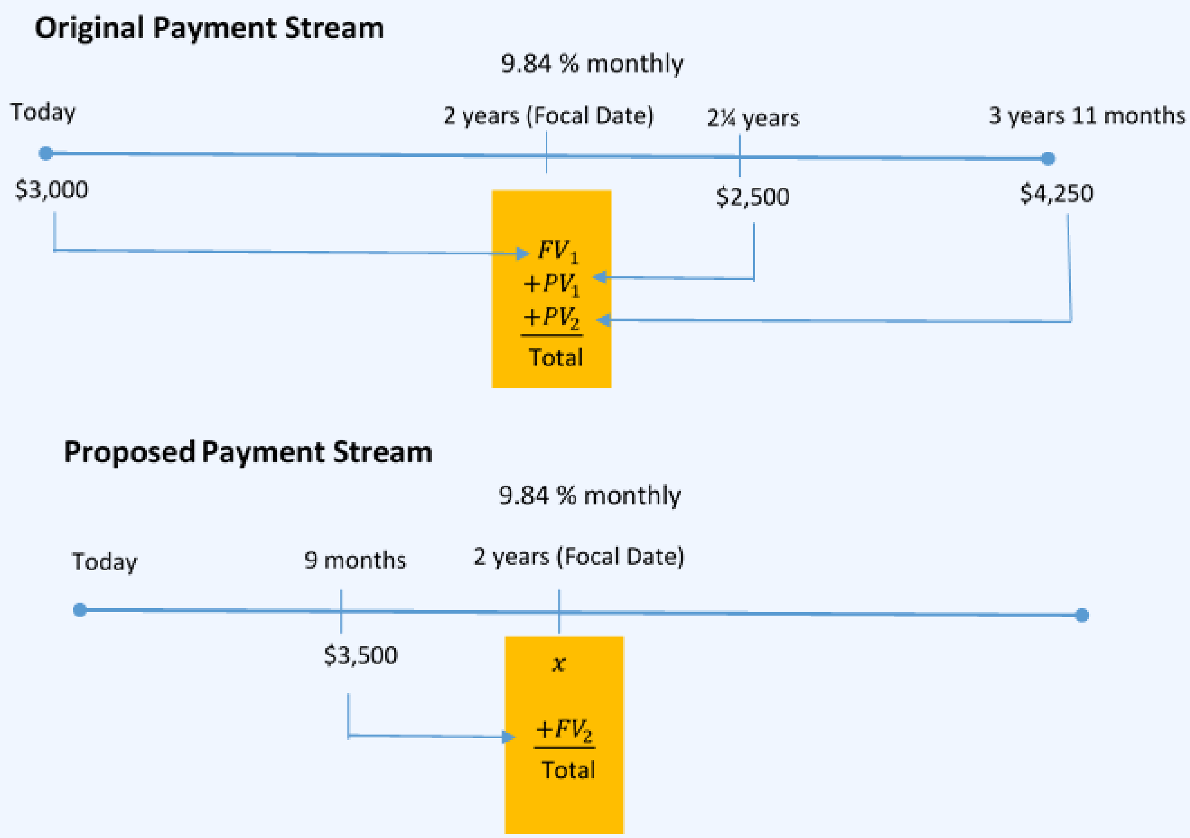


Figure 6.4.4

$$\mathrm{I/Y} = 9.84\%; \qquad \mathrm{C/Y} = \text{monthly} = 12$$

Step 2: **Focal Date** $= 2$ **years.**

Step 3: **Calculate the periodic interest rate,** i**.**

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}}$$

$$i = \frac{9.84\%}{12}$$

$$i = 0.82\% \text{ or } 0.0082$$

Step 4: For each payment, calculate the number of compoundings per year by applying Formula 6.2a:

$$n = C/Y \times (\text{Number of Years})$$

Step 5: For each payment, calculate the appropriate time value calculation. Note that all payments before the two year focal date require you to calculate future values, while all payments after the two-year focal date require you to calculate present values.

***Steps 4 and 5:* Using the circled number references from the timelines:**

$$①\ n = 12 \times 2 = 24;\ FV_1 = \$3{,}000(1 + 0.0082)^{24} = \$3{,}649.571607$$

$$②\ n = 12 \times \frac{1}{4} = 3;\ PV_1 = \frac{\$2{,}500}{1.0082^3} = \$2{,}439.494983$$

$$③\ n = 12 \times 1\frac{11}{12} = 23;\ PV_2 = \frac{\$4{,}250}{1.0082^{23}} = \$3{,}522.207915$$

$$④\ n = 12 \times 1\frac{1}{4} = 15;\ FV_2 = \$3{,}500(1 + 0.0082)^{15} = \$3{,}956.110749$$

Step 6:

$$\begin{aligned} \$3{,}649.571607 + \$2{,}439.494983 + \$3{,}522.207915 &= x + \$3{,}956.110749 \\ \$9{,}611.274505 &= x + \$3{,}956.110749 \\ \$5{,}655.16 &= x \end{aligned}$$

Table 6.4.3

Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y
①	24	9.84	3,000	0	Answer: $-\$3{,}649.571607$	12	12
②	3	9.84	Answer: $-\$2{,}439.494983$	0	2,500	12	12
③	23	9.84	Answer: $-\$3{,}522.207915$	0	4,250	12	12
④	15	9.84	3,500	0	Answer: $-\$3{,}956.110749$	12	12

Section 6.4 Exercises

1. A winning lottery ticket offers the following two options:

    a. A single payment of $\$1,000,000$ today or
    b. $\$250,000$ today followed by annual payments of $\$300,000$ for the next three years.

If money can earn 9% compounded annually, which option should the winner select? How much better is that option in current dollars?

Solution

a. The $\$1,000,000$ is already today.
b. To fairly compare the payment plan, move all money to today as well.

$$\text{Focal Date} = \text{Today}$$

Step 1: **Find i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{9\%}{1}$$

$$i = 9\%$$

Step 2: **Find n of the payments.**

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$\text{Payment \#1: } n = 1 \times 1 = 1$$

$$\text{Payment \#2: } n = 1 \times 2 = 2$$

$$\text{Payment \#3: } n = 1 \times 3 = 3$$

Step 3: **Find the present value of the payments.**

$$PV_1 = \frac{FV}{(1+i)^n}$$
$$PV_1 = \frac{\$300,000}{(1.09)^1}$$
$$PV_1 = \$275,229.3578$$

$$PV_2 = \frac{FV}{(1+i)^n}$$
$$PV_2 = \frac{\$300,000}{(1.09)^2}$$
$$PV_2 = \$252,503.998$$

$$PV_3 = \frac{FV}{(1+i)^n}$$
$$PV_3 = \frac{\$300,000}{(1.09)^3}$$
$$PV_3 = \$231,655.044$$

Total Present Value

$$\text{Today} = \&250,000 + \$275,229.3578 + \$252,503.998 + \$231,655.044$$
$$\text{Today} = \$1,009,388.40$$

Conclusion: The payment plan is better by $\$1,009,388.40 - \$1,000,000 = \$9,388.40$.

Calculator instructions:

Payment #1

$\text{N} = 1$
$\text{I/Y} = 9$
$\text{PMT} = 0$

$\mathrm{FV} = 300,000$
$\mathrm{P/Y} = 1 = \mathrm{C/Y}$
$\mathrm{PV} = ?$

Payment #2

$\mathrm{N} = 2$
$\mathrm{I/Y} = 9$
$\mathrm{PMT} = 0$
$\mathrm{FV} = 300,000$
$\mathrm{P/Y} = 1 = \mathrm{C/Y}$
$\mathrm{PV} = ?$

Payment #3

$\mathrm{N} = 3$
$\mathrm{I/Y} = 9$
$\mathrm{PMT} = 0$
$\mathrm{FV} = 300,000$
$\mathrm{P/Y} = 1 = \mathrm{C/Y}$
$\mathrm{PV} = ?$

2. James is a debt collector. One of his clients has asked him to collect an outstanding debt from one of its customers. The customer has failed to pay three amounts: $\$1,600$ eighteen months ago, $\$2,300$ nine months ago, and $\$5,100$ three months ago. In discussions with the customer, James finds she desires to clear up this situation and proposes a payment of $\$1,000$ today, $\$4,000$ nine months from now, and a final payment two years from now. The client normally charges 16.5% compounded quarterly on all outstanding debts. What is the amount of the third payment?

Solution

$$\text{Focal Date} = 2 \text{ years from today}$$

Step 1: **Find i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{16.5\%}{4}$$

$$i = 4.125\%$$

Step 2: **Find n of the payments.**

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$\text{Original Payment \#1: } n = 4 \times 3.5 = 14$$

$$\text{Original Payment \#2: } n = 4 \times 2.75 = 11$$

$$\text{Original Payment \#3: } n = 4 \times 2.25 = 9$$

$$\text{Proposed Payment \#1: } n = 4 \times 1.25 = 5$$

$$\text{Proposed Payment \#2: } n = 4 \times 2 = 8$$

Step 3: **Find the future value of the payments.**

$$\text{Original Payment \#1: } FV_1 = \frac{\$1,600}{(1+0.04125)^{14}}$$
$$FV_1 = \$2,817.670366$$

$$\text{Original Payment \#2: } FV_2 = \frac{\$2,300}{(1+0.04125)^{11}}$$
$$FV_2 = \$3,587.839398$$

$$\text{Original Payment \#3: } FV_3 = \frac{\$5,100}{(1+0.04125)^{9}}$$
$$FV_3 = \$7,337.790461$$

$$\text{Proposed Payment \#1: } FV_4 = \frac{\$4,000}{(1+0.04125)^{5}}$$
$$FV_4 = \$4,895.928462$$

$$\text{Proposed Payment \#2: } FV_5 = \frac{\$1,000}{(1+0.04125)^{8}}$$
$$FV_5 = \$1,381.783859$$

$$\begin{aligned}\text{Total Dated Debts} &= \text{Total Dated Payments}\\ FV_1+FV_2+FV_3 &= x+FV_4+FV_5\\ \$13,743.30023 &= x+\$6,277.712321\\ x &= \$7,465.59\end{aligned}$$

The amount of the third payment is $\$7,465.59$.

Calculator instructions:

Original Payment #1

$\mathrm{N} = 14$
$\mathrm{I/Y} = 16.5$
$\mathrm{PV} = 1,600$
$\mathrm{PMT} = 0$
$\mathrm{P/Y} = 4 = \mathrm{C/Y}$
$\mathrm{FV} = ?$

Original Payment #2

$\mathrm{N} = 11$
$\mathrm{I/Y} = 16.5$
$\mathrm{PV} = 2,300$
$\mathrm{PMT} = 0$
$\mathrm{P/Y} = 4 = \mathrm{C/Y}$
$\mathrm{FV} = ?$

Original Payment #3

$\mathrm{N} = 9$
$\mathrm{I/Y} = 16.5$
$\mathrm{PV} = 5,100$
$\mathrm{PMT} = 0$
$\mathrm{P/Y} = 4 = \mathrm{C/Y}$
$\mathrm{FV} = ?$

Proposed Payment #1

$\mathrm{N} = 8$
$\mathrm{I/Y} = 16.5$
$\mathrm{PV} = 1,000$
$\mathrm{PMT} = 0$
$\mathrm{P/Y} = 4 = \mathrm{C/Y}$
$\mathrm{CPT\ FV} = \$7,337.790461$

Proposed Payment #2

$$\begin{aligned}&\mathrm{N}=5\\&\mathrm{I/Y}=16.5\\&\mathrm{PV}=4,000\\&\mathrm{PMT}=0\\&\mathrm{P/Y}=4=\mathrm{C/Y}\\&\mathrm{CPT\ FV}=\$4,895.928462\end{aligned}$$

3. Four years ago, Aminata borrowed $\$5,000$ from Randal with interest at 8% compounded quarterly to be repaid one year from today. Two years ago, Aminata borrowed another $\$2,500$ from Randal at 6% compounded monthly to be repaid two years from today. Aminata would like to restructure the payments so that she can pay 15 months from today and $2\frac{1}{2}$ years from today. The first payment is to be twice the size of the second payment. Randal accepts an interest rate of 6.27% compounded monthly on the proposed agreement. Calculate the amounts of each payment assuming the focal date is 15 months from today.

Solution

First, calculate the amounts owing under Aminata's original loans.

Original Loan 1:

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{8\%}{4}$$

$$i = 2\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$

$$n = 4 \times 5$$

$$n = 20$$

$$FV_1 = \frac{PV}{(1+i)^n}$$

$$FV_1 = \frac{\$5,000}{(1.02)^{20}}$$

$$FV_1 = \$7,429.74 \text{ (Due in 1 year from today)}$$

Original Loan 2:

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{6\%}{12}$$
$$i = 0.5\%$$

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 12 \times 4$$
$$n = 48$$

$$FV_2 = \frac{PV}{(1+i)^n}$$
$$FV_2 = \frac{\$2,500}{(1.005)^{48}}$$
$$FV_2 = \$3,176.22 \text{ (Due in 2 years from today)}$$

Calculator instructions:

Original Loan 1:

$\text{N} = 20$
$\text{I/Y} = 8$
$\text{PV} = 5,000$
$\text{PMT} = 0$
$\text{P/Y} = 4 = \text{C/Y}$
$\text{FV} = ?$

Original Loan 2:

$\text{N} = 48$
$\text{I/Y} = 6$
$\text{PV} = 2,500$
$\text{PMT} = 0$

$\text{P/Y} = 12 = \text{C/Y}$
$\text{FV} = ?$

Now calculate the equivalent payments under the proposed arrangement:

$$\begin{aligned} 1 \text{ year} &= 12 \text{ months} \\ 2 \text{ years} &= 24 \text{ months} \\ 2.5 \text{ years} &= 30 \text{ months} \end{aligned}$$

$$\begin{aligned} i &= \frac{6.27\%}{12} \\ i &= 0.5225\% \end{aligned}$$

$$\begin{aligned} \text{Total Dated Debts} &= \text{Total Dated Payments} \\ FV_1 + PV_1 &= 2x + PV_2 \\ 7,429.74(1.005225)^3 + \frac{3,176.22}{(1.005225)^9} &= 2x + \frac{x}{(1.005225)^{15}} \\ 7,546.810744 + 3,030.686729 &= 2x + 0.924806x \\ \$10,577.49747 &= 2.924806x \\ x &= \$3,616.48 \text{ (second payment)} \\ \\ 2x &= 2(\$3,616.48) \\ 2x &= \$7,232.96 \text{ (first payment)} \end{aligned}$$

The amount of each payment is $\$7,232.96$.

Calculator Instructions:

Original Payment 1:

$\text{N} = 3$
$\text{I/Y} = 6.27$
$\text{PV} = 7,429.74$
$\text{PMT} = 0$

$\mathrm{P/Y} = 12 = \mathrm{C/Y}$
$\mathrm{FV} = ?$

Original Payment 2:

$\mathrm{N} = 9$
$\mathrm{I/Y} = 6.27$
$\mathrm{FV} = 3,176.22$
$\mathrm{PMT} = 0$
$\mathrm{P/Y} = 12 = \mathrm{C/Y}$
$\mathrm{PV} = ?$

Proposed Payment 1:

$\mathrm{N} = 15$
$\mathrm{I/Y} = 6.27$
$\mathrm{FV} = 1$
$\mathrm{PMT} = 0$
$\mathrm{P/Y} = 12 = \mathrm{C/Y}$
$\mathrm{PV} = ?$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER.$n = \mathrm{C/Y} \times \text{Number of Years}$

$i = \frac{\mathrm{I/Y}}{\mathrm{C/Y}}$

Attribution

6.5 DETERMINING THE INTEREST RATE

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $FV =$ Future value or maturity value
- $PV =$ Present value or principal value
- $i =$ Periodic interest rate
- $\text{C/Y} =$ Compounds per year
- $\text{I/Y} =$ Nominal interest rate per year
- $n =$ Total number of compounding periods

Formulas Used

- Formula 6.1 – **Periodic Interest Rate**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

- Formula 6.2a – **Number of Compound Periods**

$$n = \text{C/Y} \times \text{Number of Years}$$

- Formula 6.2b – **Future (Maturity) Value**

$$FV = PV \times (1 + i)^n$$

- Formula 6.3 – **Present Value (Principal)**

$$PV = \frac{FV}{(1+i)^n}$$

Determining the Interest Rate

This section shows how to calculate the nominal interest rate on single payments when you know both the future value and the present value.

HOW TO

Calculate Nominal Interest Rate on a Single Payment

Follow these steps to solve for the nominal interest rate on a single payment:

Step 1: Draw a timeline to help you visualize the question. Of utmost importance is identifying the values of PV and FV, the number of years involved, and the compounding for the interest rate.

Step 2: Calculate the number of compounds (n) using Formula 6.2a:

$$n = \text{C/Y} \times (\text{Number of Years}).$$

Step 3: Substitute known variables into **Formula 6.2b** $FV = PV \times (1+i)^n$, rearrange and solve for the periodic interest rate, i.

$$i = \left(\frac{FV}{PV}\right)^{\frac{1}{n}} - 1$$

Step 4: Substitute the periodic interest rate and the compounding frequency into the **Formula 6.1** $i = \frac{\text{I/Y}}{\text{C/Y}}$ and rearrange, and solve for the nominal interest rate, I/Y.

$$\text{I/Y} = i \times \text{C/Y}$$

Ensure that the solution is expressed with the appropriate compounding words.

Key Takeaways

Handling Decimals in Interest Rate Calculations

Rule 1: A Clear Marginal Effect

Use this rule when it is fairly obvious how to round the interest rate. The dollar amounts used in calculating the interest rate are rounded by no more than a half penny. Therefore, the calculated interest rate should be extremely close to its true value. For example, if you calculate an $\mathrm{I/Y}$ of 7.999884%, notice this value would have a marginal difference of only 0.000116% from a rounded value of 8%. Most likely the correct rate is 8% and not 7.9999%. However, if you calculate an $\mathrm{I/Y}$ of 7.920094%, rounding to 8% would produce a difference of 0.070006%, which is quite substantial. Applying marginal rounding, the most likely correct rate is 7.92% and not 7.9201%, since the marginal impact of the rounding is only 0.000094%.

Rule 2: An Unclear Marginal Effect

Use this rule when it is not fairly obvious how to round the interest rate. For example, if the calculated $\mathrm{I/Y} = 7.924863\%$, there is no clear choice of how to round the rate. In these cases or when in doubt, apply the standard rule established for this book of rounding to four decimals. Hence, $\mathrm{I/Y} = 7.9249\%$ in this example.

It is important to stress that the above recommendations for rounding apply to final solutions. If the calculated interest rate is to be used in further calculations, then you should carry forward the unrounded interest rate.

Your BAII Plus Calculator

Enter values for the known variables, PV, FV, N, and both of the values in the $\mathrm{P/Y}$ window ($\mathrm{P/Y}$ and $\mathrm{C/Y}$) following the procedures established in Section 6.2. Ensure proper application of the cash flow sign convention to PV and FV. One number must be negative while the other is positive, otherwise an ERROR message will appear on your calculator display.

Example 6.5.1

When Sandra borrowed $\$7,100$ from Sanchez, she agreed to reimburse him $\$8,615.19$ three years from now including interest compounded quarterly. What nominal quarterly compounded rate of interest is being charged?

Solution
***Step 1:* The present value, future value, term, and compounding are known, as illustrated in the timeline.**

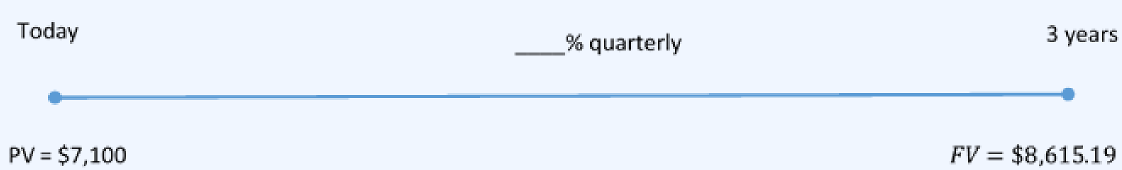


Figure 6.5.1

$$PV = \$7,100; \quad FV = \$8,615.19; \quad \mathrm{C/Y} = \text{quarterly} = 4; \quad \text{Term} = 3 \text{ years}$$

***Step 2:* Calculate the total number of compoundings,n.**

\begin{align*}n&=\text{C/Y}\times\text{(Number of Years)}\\n&=4\times 3\\n&=12\end{align*}

***Step 3:* Calculatethe periodic rate, i.**

$$\begin{aligned}FV &= PV(1+i)^n\\ \$8{,}615.19 &= \$7{,}100(1+i)^{12}\\ 1.213407 &= (1+i)^{12}\\ 1.213407^{\frac{1}{12}} &= 1+i\\ 1.01624996 &= 1+i\\ i &= 0.01624996\end{aligned}$$

***Step 4:* Calculate the nominal rate,$\mathrm{I/Y}$.**

$$\begin{aligned}i &= \frac{\mathrm{I/Y}}{\mathrm{C/Y}}\\ 0.01624996 &= \frac{\mathrm{I/Y}}{4}\\ \mathrm{I/Y} &= 0.06499985\\ \mathrm{I/Y} &= 0.065 \text{ or } 6.5\% \text{ (compounded quarterly)}\end{aligned}$$

Table 6.5.1

N	I/Y	PV	PMT	FV	P/Y	C/Y
12	Answer: 6.499985	$-7,100$	0	$8,615.19$	4	4

***Step 5:* Write as a statement.**

Sanchez is charging an interest rate of 6.5% compounded quarterly on the loan to Sandra.

Example 6.5.2

Five years ago, Taryn placed $\$15,000$ into an RRSP that earned $\$6,799.42$ of interest compounded monthly. What was the nominal interest rate for the investment?

Solution

***Step 1:* The present value, interest earned, term, and compounding are known, as illustrated in the timeline.**

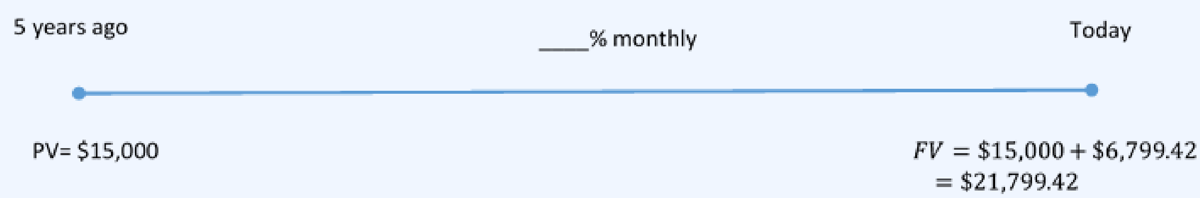


Figure 6.5.2

$$FV = \$15,000 + \$6,799.42 = \$21,799.42;$$
$$PV = \$15,000; \text{C/Y} = \text{monthly} = 12; \quad \text{Term} = 5 \text{ years}$$

***Step 2:* Calculate the total number of compoundings, n.**

\begin{align*}n&=\text{C/Y}\times\text{(Number of Years)}\\n&=12\times 5\\n&=60\end{align*}

***Step 3:*Calculate the periodic rate, i.**

$$\begin{aligned} FV &= PV(1+i)^n \\ \$21{,}799.42 &= \$15{,}000(1+i)^{60} \\ 1.453294 &= (1+i)^{60} \\ 1.453294^{\frac{1}{60}} &= 1+i \\ 1.00625 &= 1+i \\ i &= 0.00625 \end{aligned}$$

***Step 4:* Calculate the nominal rate,I/Y.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$0.00625 = \frac{\text{I/Y}}{12}$$

$$\text{I/Y} = 0.075 \text{ or } 7.5\% \text{ (compounded monthly)}$$

Table 6.5.2

N	I/Y	PV	PMT	FV	P/Y	C/Y
60	Answer: 7.500003	$-15,000$	0	$21,799.42$	12	12

Step 5: Write as a statement.

Taryn's investment in his RRSP earned 7.5% compounded monthly over the five years.

Converting Variable Interest Rates to a Fixed Interest Rate

When you deal with a series of variable interest rates it is extremely difficult to determine their overall effect. This also makes it hard to choose wisely between different series. For example, assume that you could place your money into an investment earning interest rates of 2%, 2.5%, 3%, 3.5%, and 4.5% over the course of five years, or alternatively you could invest in a plan earning 1%, 1.5%, 1.75%, 3.5%, and 7% (all rates compounded semi-annually). Which plan is better? The decision is unclear. But you can make it clear by converting the variable rates on each investment option into an equivalent fixed interest rate.

HOW TO

Convert Variable Interest Rates to Equivalent Fixed Interest Rates

Follow these steps to convert variable interest rates to their equivalent fixed interest rates:

Step 1: Draw a timeline for the variable interest rate. Identify key elements including any known PV or FV, interest rates, compounding, and terms.

Step 2: For each time segment, calculate the periodic interest rate (i) and the number of compoundings (n).

Step 3: One of three situations will occur, depending on what variables are known:

PV Is Known: Calculate the future value at the end of the transaction in each time segment, working left to right across the timeline.

FV Is Known: Calculate the present value at the beginning of the transaction in each time segment, working right to left across the timeline.

Neither PV nor FV Is Known: Pick an arbitrary number for PV ($\$10,000$ is recommended) and solve for the future value in each time segment at the end of the transaction, working left to right across the timeline.

Step 4: Determine the compounding required on the fixed interest rate ($\mathrm{C/Y}$) and calculate a new value for n to reflect the entire term of the transaction.

Step 5: Solve for i using the n from Step 4 along with the starting PV and ending FV for the entire timeline.

Step 6: Solve for $\mathrm{I/Y}$.

Example 6.5.3

Continue working with the two investment options mentioned previously. The choices are to place your money into a five year investment earning semi-annually compounded interest rates of either:

a. 2%, 2.5%, 3%, 3.5%, and 4.5%
b. 1%, 1.5%, 1.75%, 3.5%, and 7%

Calculate the equivalent semi-annual fixed interest rate for each plan and recommend an investment.

Solution

Step 1: **Draw a timeline for each investment option, as illustrated below.**

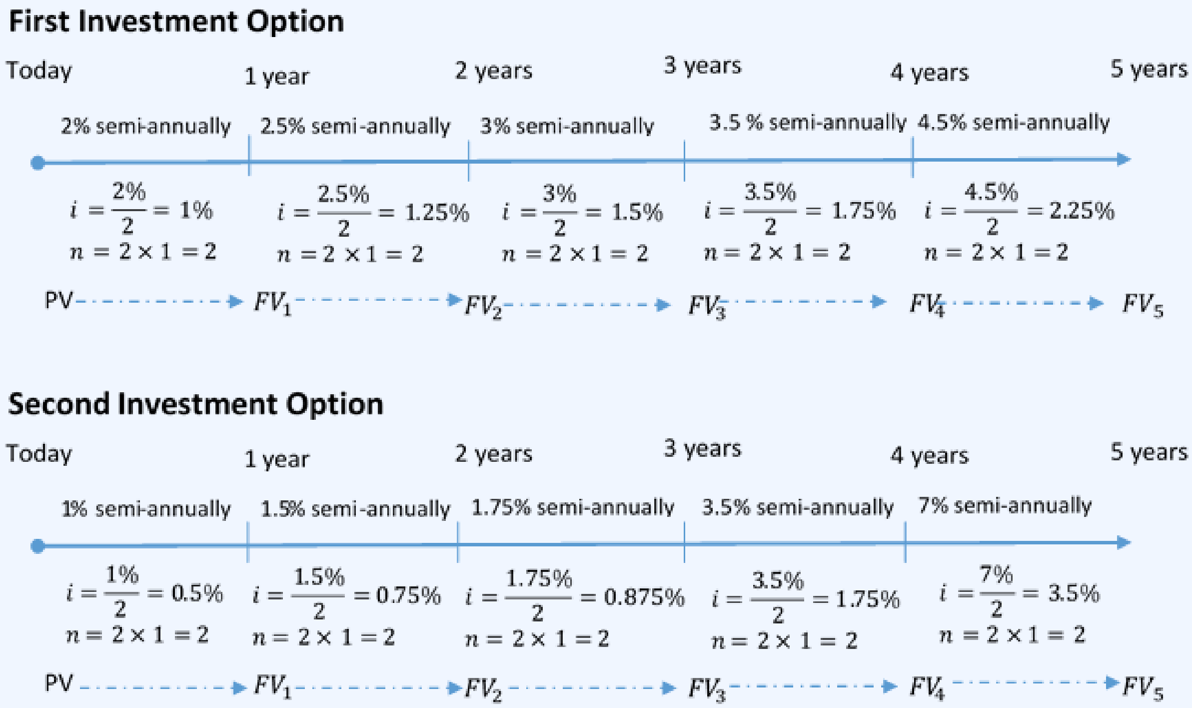


Figure 6.5.3

Step 2: **For each time segment calculate i and n using Formula 6.1** $i = \dfrac{\text{I/Y}}{\text{C/Y}}$ **and Formula 6.2a** $n = \text{C/Y} \times \text{Number of Years}$.

Calculations are found in the timeline figure above.

Step 3: **There is no value for PV or FV. Choose an arbitrary value of $PV = \$10,000$ and solve for FV. Since only the interest rate fluctuates, solve in one calculation.**

$$FV_5 = PV \times (1+i_1)^{n_1} \times (1+i_2)^{n_2} \times \cdots \times (1+i_5)^{n_5}$$

First Investment:

$$\begin{aligned} FV_5 &= 10{,}000(1+0.01)^2(1+0.0125)^2(1+0.015)^2(1+0.0175)^2(1+0.0225)^2 \\ FV_5 &= \$11{,}661.65972 \end{aligned}$$

Second Investment:

$$\begin{aligned} FV_5 &= 10{,}000(1+0.005)^2(1+0.0075)^2(1+0.00875)^2(1+0.0175)^2(1+0.035)^2 \\ FV_5 &= \$11{,}570.14666 \end{aligned}$$

Step 4:

First Investment: $n = 2 \times 5 = 10$
Second Investment: $n = 2 \times 5 = 10$

Step 5:

First Investment:

$$\begin{aligned} \$11{,}66165972 &= \$10{,}000(1+i)^{10} \\ 1.166165 &= (1+i)^{10} \\ 1.166165^{\frac{1}{10}} &= 1+i \\ 1.001549 &= 1+i \\ i &= 0.001549 \end{aligned}$$

Second Investment:

$$\begin{aligned} \$11{,}570.14666 &= \$10{,}000(1+i)^{10} \\ 1.157014 &= (1+i)^{10} \\ 1.157015^{\frac{1}{10}} &= 1+i \\ 1.014691 &= 1+i \\ i &= 0.014691 \end{aligned}$$

Step 6:

First Investment:

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$0.001549 = \frac{\text{I/Y}}{2}$$

$$\text{I/Y} = 0.030982 \text{ or } 3.0982\%$$

Second Investment:

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$0.014691 = \frac{\text{I/Y}}{2}$$

$$\text{I/Y} = 0.029382 \text{ or } 2.9382\%$$

First Investment:

Table 6.5.3

Time segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	2	2	$-10,000$	0	Answer: $10,201$	2	2
2	2	2.5	$-10,201$	0	Answer: $10,457.61891$	2	2
3	2	3	$-10,457.61891$	0	Answer: $10,773.70044$	2	2
4	2	3.5	$-10,773.70044$	0	Answer: $11,154.0794$	2	2
5	2	4.5	$-11,154.0794$	0	Answer: 11661.65972	2	2
All	2	Answer: 0.030982	-10000	$11,661.66$	2	2	

Second Investment:

Table 6.5.4

Time segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	60	1	$-10,000$	0	Answer: $10,100.25$	2	2
2	60	1.5	$-10,100.25$	0	Answer: $10,252.32189$	2	2
3	60	1.75	$-10,252.32189$	0	Answer: $10,432.52247$	2	2
4	60	3.5	$-10,432.52247$	0	Answer: $10,800.85571$	2	2
5	60	7	$-10,800.85571$	0	Answer: $11,570.14666$	2	2
All	10	Answer: 0.029382	$-10,000$	0	$11,570.15$	2	2

Step 7: **Write as a statement.**

The variable interest rates on the first investment option are equivalent to a fixed interest rate of 3.0982% compounded semi-annually. For the second option, the rates are equivalent to 2.9382% compounded semi-annually. Therefore, recommend the first investment since its rate is higher by $3.0982\% - 2.938$ compounded semi-annually.

Section 6.5 Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Your company paid an invoice five months late. If the original invoice was for $\$6,450$ and the amount paid was $\$6,948.48$, what monthly compounded interest rate is your supplier charging on late payments?
   Solution
   ***Step 1:* Given information:**

   $$PV = \$6,450; \qquad FV = \$6,948.48; \qquad \text{C/Y} = \text{monthly} = 12$$

   ***Step 2:* Find n.**

   $$n = \text{C/Y} \times (\text{Number of Years})$$
   $$n = 12 \times 512$$
   $$n = 5$$

   ***Step 3:* Using Formula 6.2b $FV = PV \times (1+i)^n$ for FV, rearrange for i.**

$$\begin{aligned} FV &= PV \times (1+i)^n \\ \$6,948.48 &= \$6,450(1+i)^5 \\ 1.077283 &= (1+i)^5 \\ 1.077283^{\frac{1}{5}} &= (1+i) \\ 1.014999 &= 1+i \\ i &= 0.014999 \end{aligned}$$

Step 4: **Solve for the nominal rate, $\mathrm{I/Y}$.**

$$\begin{aligned} &\mathrm{I/Y} = i \times 12 \\ &\mathrm{I/Y} = 0.179999 \\ &\mathrm{I/Y} = 18\% \text{ (compounded monthly)} \end{aligned}$$

The supplier is charging 18% compounded monthly on late payments?

Calculator instructions:

$$\begin{aligned} &\mathrm{N} = 5 \\ &\mathrm{PV} = -6,450 \\ &\mathrm{PMT} = 0 \\ &\mathrm{FV} = 6,948.48 \\ &\mathrm{P/Y} = 12 \\ &\mathrm{C/Y} = 12 \\ &\mathrm{I/Y} = ? \end{aligned}$$

2. At what monthly compounded interest rate does it take five years for an investment to double?

Solution

Step 1: **Pick any two values for PV and FV where FV is double the PV.**

$$PV = \$10,000; \qquad FV = \$20,000$$

Step 2: **Find n.**

$$\begin{aligned} &n = \mathrm{C/Y}(\text{Number of Years}) \\ &n = 5 \times 12 \\ &n = 60 \end{aligned}$$

Step 3: **Using the formula for FV solve for i.**

$$\begin{aligned} FV &= PV \times (1+i)^n \\ \$20,000 &= \$10,000 \times (1+i)^{60} \\ 2 &= (1+i)^{60} \\ 2^{\frac{1}{60}} &= (1+i) \\ 1.011619 &= 1+i \\ i &= 0.011619 \end{aligned}$$

Step 4: **Solve for the nominal rate, I/Y.**

$$\begin{aligned} \text{Nominal Rate} &= i \times 12 \\ \text{Nominal Rate} &= 0.139428 \\ \text{Nominal Rate} &= 13.94\% \text{ compounded monthly} \end{aligned}$$

Step 5: **Write as a statement.** The investment will double in five years at 13.94% compounded monthly.

Calculator instructions:

$$\begin{aligned} &\text{N} = 60 \\ &\text{PV} = -10,000 \\ &\text{PMT} = 0 \\ &\text{FV} = 20,000 \\ &\text{P/Y} = 12 \\ &\text{C/Y} = 12 \\ &\text{I/Y} = ? \end{aligned}$$

3. Indiana just received a maturity value of $\$30,320.12$ from a semi-annually compounded investment that paid 4%, 4.1%, 4.35%, 4.75%, and 5.5% in consecutive years. What amount of money did Indiana invest? What fixed quarterly compounded nominal interest rate is equivalent to the variable rate his investment earned?

   Solution

   Step 1: **Given information:**

$$\begin{aligned}
&\text{Year 1: I/Y} = 4\%; && \text{C/Y} = 2 \\
&\text{Year 2: I/Y} = 4.1\%; && \text{C/Y} = 2 \\
&\text{Year 3: I/Y} = 4.35\%; && \text{C/Y} = 2 \\
&\text{Year 4: I/Y} = 4.75\%; && \text{C/Y} = 2 \\
&\text{Year 5: I/Y} = 5.5\%; && \text{C/Y} = 2
\end{aligned}$$

Step 2: **Calculate n and i for all years:**

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 2 \times 1$$
$$n = 2$$

$$\text{Year 1: } i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{4\%}{2}$$
$$i = 2\%$$

$$\text{Year 2: } i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{4.1\%}{2}$$
$$i = 2.05\%$$

$$\text{Year 3: } i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{4.35\%}{2}$$
$$i = 2.175\%$$

$$\text{Year 4: } i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{4.75\%}{2}$$
$$i = 2.375\%$$

$$\text{Year 5: } i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{5.5\%}{2}$$

Step 3: **Solve for PV.**

$$\text{Year 5: } PV = \frac{\$30,320.12}{(1+0.0275)^2}$$
$$PV = \$28,718.86385$$

$$\text{Year 4: } PV = \frac{\$28,718.86385}{(1+0.02375)^2}$$
$$PV = \$27,401.82101$$

$$\text{Year 3: } PV = \frac{\$27,401.82101}{(1+0.02175)^2}$$
$$PV = \$26,247.63224$$

$$\text{Year 2: } PV = \frac{\$26,247.63224}{(1+0.0205)^2}$$
$$PV = \$25,203.68913$$

$$\text{Year 1: } PV = \frac{\$25,203.68913}{(1+0.02)^2}$$
$$PV = \$24,225$$

Step 4: **Solve for n.**

$$n = \text{C/Y} \times (\text{Number of Years})$$
$$n = 4 \times 5$$
$$n = 20$$

Step 5: **Use the formula for FV and rearrange for i.**

$$\begin{aligned} FV &= PV(1+i)^n \\ \$30,320.12 &= \$24,225(1+i)^{20} \\ 1.251604 &= (1+i)^{20} \\ 1.251604^{\frac{1}{20}} &= 1+i \\ 1.011284 &= 1+i \\ i &= 0.011284 \end{aligned}$$

Step 6: **Find the nominal rate,** I/Y**.**

$$\begin{aligned} &\text{I/Y} = i \times \text{C/Y} \\ &\text{I/Y} = 0.011284 \times 4 \\ &\text{I/Y} = 0.045138 \\ &\text{I/Y} = 4.51\% \text{ compounded quarterly} \end{aligned}$$

Step 7: **Write as a statement.** $\$24,225$ investment earned 4.51% compounded quarterly.

Calculator instructions:

Year 5:

$\text{N} = 2$
$\text{I/Y} = 5.5$
$\text{PMT} = 0$
$\text{FV} = 30,320.12$
$\text{P/Y} = 2$
$\text{C/Y} = 2$
$\text{PV} = ?$

Year 4:

$\text{N} = 2$
$\text{I/Y} = 4.75$
$\text{PMT} = 0$
$\text{FV} = \pm PV \text{ from above}$
$\text{P/Y} = 2$

$\text{C/Y} = 2$

$\text{PV} = ?$

Year 3:

$\text{N} = 2$
$\text{I/Y} = 4.35$

$\text{PMT} = 0$
$\text{FV} = \pm PV \text{ from above}$
$\text{P/Y} = 2$

$\text{C/Y} = 2$

$\text{PV} = ?$

Year 2:

$\text{N} = 2$
$\text{I/Y} = 4.1$

$\text{PMT} = 0$
$\text{FV} = \pm PV \text{ from above}$
$\text{P/Y} = 2$

$\text{C/Y} = 2$

$\text{PV} = ?$

Year 1:

$\text{N} = 2$
$\text{I/Y} = 4$

$\text{PMT} = 0$
$\text{FV} = \pm PV \text{ from above}$
$\text{P/Y} = 2$

$\text{C/Y} = 2$

$\text{PV} = ?$

Nominal Rate:

$\text{N} = 20$
$\text{PV} = -24,225$

$\text{PMT} = 0$
$\text{FV} = 30,320.12$

$$\mathrm{P/Y} = 4$$
$$\mathrm{C/Y} = 4$$
$$\mathrm{I/Y} = ?$$

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER.$n = \mathrm{C/Y} \times \text{Number of Years}$ $i = \frac{\mathrm{I/Y}}{\mathrm{C/Y}}$ $FV = PV \times (1+i)^n$

Attribution

6.6 EFFECTIVE AND EQUIVALENT INTEREST RATES

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- $FV =$ Future value or maturity value
- $PV =$ Present value or principal value
- $i =$ Periodic interest rate
- $\mathrm{C/Y} =$ Compounds per year
- $\mathrm{I/Y} =$ Nominal interest rate per year
- $n =$ Total number of compounding periods

Formulas Used

- Formula 6.1 – **Periodic Interest Rate**

$$i = \frac{\mathrm{I/Y}}{\mathrm{C/Y}}$$

- Formula 6.2a – **Number of Compound Periods**

$$n = \mathrm{C/Y} \times \text{Number of Years}$$

- Formula 6.2b – **Future (Maturity) Value**

$$FV = PV \times (1+i)^n$$

- Formula 6.3 – **Present Value (Principal)**

$$PV = \frac{FV}{(1+i)^n}$$

- Formula 6.6 – **Interest Rate Conversion**

$$i_{\text{new}} = (1+i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1$$

Effective and Equivalent Interest Rates

How can you compare interest rates posted with different compounding? For example, let's say you are considering the purchase of a new home, so for the past few weeks you have been shopping around for financing. You have spoken with many banks as well as onsite mortgage brokers in the show homes. With semi-annual compounding, the lowest rate you have come across is 6.6%. In visiting another show home, you encounter a mortgage broker offering a mortgage for 6.57%. In the fine print, it indicates the rate is compounded quarterly. You remember from your business math class that the compounding is an important component of an interest rate and wonder which one you should choose — 6.6% compounded semi-annually or 6.57% compounded quarterly.

When considering interest rates on loans, you clearly want the best rate. If all of your possible loans are compounded in the same manner, selecting the best interest rate is a matter of picking the lowest number. However, when interest rates are compounded differently the lowest number may in fact not be your best choice. For investments, on the other hand, you want to earn the most interest. However, the highest nominal rate may not be as good as it appears depending on the compounding.

To compare interest rates fairly and select the best, they all have to be expressed with equal compounding. This section explains the concept of an effective interest rate, and you will learn to convert interest rates from one compounding frequency to a different frequency.

$\boxed{6.6}$ Interest Rate Conversion

Formula does not parse

Formula does not parse The new periodic interest rate expressed in a compounding frequency equal to CY_{new}. If CY_{new} equals 1, then the new periodic interest rate is also the effective rate of interest (IY); that is, the rate compounded on an annual basis.

i_{old} **is Original Periodic Interest Rate:** This is the unrounded periodic rate for the original interest rate that is to be converted to its new compounding. This periodic rate results from **Formula 6.1** $i=\dfrac{\mathrm{I/Y}}{\mathrm{C/Y}}$, where the original nominal interest rate is your IY and the original compounding frequency is your CY.

Formula does not parse This is how many times in a single year the original nominal interest rate is compounded.

Formula does not parse This is how many times in a single year the newly converted interest rate compounds. If this variable is set to 1, then the result of the formula is the effective rate of interest.

HOW TO

Calculate Effective Interest Rate

Follow these steps to calculate effective interest rates:

Step 1: Identify the known variables including the original nominal interest rate $(\mathrm{I/Y})$ and original compounding frequency $(\mathrm{C/Y}_{\text{old}})$. Set the $\mathrm{C/Y}_{\text{new}}=1$.

Step 2: Calculate i_{old} using **Formula 6.1** $i=\dfrac{\mathrm{I/Y}}{\mathrm{C/Y}}$.

$$i_{\text{old}}=\frac{\mathrm{I/Y}}{\mathrm{C/Y}_{\text{old}}}$$

Step 3: Apply the formula for i_{new} to convert to the effective interest rate.

$$i_{\text{new}}=(1+i_{\text{old}})^{\frac{\mathrm{C/Y}_{\text{old}}}{\mathrm{C/Y}_{\text{new}}}}-1$$

Note: With a compounding frequency of 1, this makes $i_{\text{new}}=\mathrm{I/Y}$ compounded annually.

Comparing the interest rates of 6.6% compounded semi-annually and 6.57% compounded quarterly requires you to express both rates in the same units. Therefore, you could convert both nominal interest rates to effective rates.

Table 6.6.1

Steps	**6.6% compounded semi-annually**	**6.57% compounded quarterly**
Step 1	$\mathrm{I/Y} = 6.6$	$\mathrm{I/Y} = 6.57$
Step 2	$i_{\text{old}} = 6.6$	$i_{\text{old}} = 6.57$
Step 3	$i_{\text{new}} = (1 + 0.033)^{\frac{2}{1}} - 1 = 6.7089$	$i_{\text{new}} = (1 + 0.016425)^{\frac{4}{1}} - 1 = 6.7336$

The rate of 6.6% compounded semi-annually is effectively charging 6.7089%, while the rate of 6.57% compounded quarterly is effectively charging 6.7336%. The better mortgage rate is 6.6% compounded semi-annually, as it results in annually lower interest charges.

Your BAII Plus Calculator

The Texas Instruments BAII Plus calculator has a built-in effective interest rate converter called ICONV located on the second shelf above the number 2 key. To access it, press $2\mathrm{nd}\ \mathrm{ICONV}$. You access three input variables using your $\uparrow$ or $\downarrow$ scroll buttons. Use this function to solve for any of the three variables, not just the effective rate.

Table 6.6.2

Variable	**Description**	**Algebraic Symbol**
NOM	Nominal Interest Rate	$\mathrm{I/Y}$
EFF	Effective Interest Rate	i_{new} (annually compounded)
C/Y	Compound Frequency	$\mathrm{C/Y}_{\text{old}}$

To use this function, enter two of the three variables by keying in each piece of data and pressing ENTER to store it. When you are ready to solve for the unknown variable, scroll to bring it up on your display and press CPT. For example, use this sequence to find the effective rate equivalent to the nominal rate of 6.6% compounded semi-annually:

$$2\text{nd ICONV}, 6.6 \text{ Enter } \uparrow, 2 \text{ Enter } \downarrow, \text{CPT}$$

Answer: 6.7089

Concept Check

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=212#h5p-14

Example 6.6.1

If your investment earns 5.5% compounded monthly, what is the effective rate of interest?

Solution

***Step 1:* Given information:**

$$\text{I/Y} = 5.5; \quad \text{C/Y}_{\text{old}} = \text{monthly} = 12; \quad \text{C/Y}_{\text{new}} = 1;$$

***Step 2:*Calculate i_{old}.**

$$i_{\text{old}} = \frac{\text{I/Y}}{\text{C/Y}_{\text{old}}}$$

$$i_{\text{old}} = \frac{5.5\%}{12}$$

$$i_{\text{old}} = 0.458\overline{3}\% \text{ or } 0.00458\overline{3}$$

***Step 3:* Calculate i_{new}.**

$$i_{\text{new}} = (1 + i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1$$
$$i_{\text{new}} = (1 + 0.00458\overline{3})^{\frac{12}{1}} - 1$$
$$i_{\text{new}} = 0.056408 \text{ or } 5.6408\%$$

Calculator instructions:

2nd ICONV

Table 6.6.3

NOM	C/Y	EFF
5.5	12	Answer: 5.640786

Step 4: **Write as a statement.**

You are effectively earning 5.6408% interest per year.

Example 6.6.2

As you search for a car loan, all banks have quoted you monthly compounded rates (which are typical for car loans), with the lowest being 8.4%. At your last stop, the credit union agent says that by taking out a car loan with them, you would effectively be charged 8.65%. Should you go with the bank loan or the credit union loan?

Solution

Step 1: **Given information:**

$$i_{\text{new}} = 8.65\% \text{ effective rate}; \quad \text{C/Y}_{\text{old}} = \text{monthly} = 12; \quad \text{C/Y}_{\text{new}} = 1$$

(Note: In this case the i_{new} is known, so the process is reversed to arrive at the I/Y).

Step 2:**Using Formula 6.6 for i_{new}, solve and rearrange for i_{old}.**

$$\begin{aligned} i_{\text{new}} &= (1+i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1 \\ 0.0865 &= (1+i_{\text{old}})^{\frac{12}{1}} - 1 \\ 1.0865 &= (1+i_{\text{old}})^{12} \\ 1.0865^{\frac{1}{12}} &= 1+i_{\text{old}} \\ 1.006937 &= 1+i_{\text{old}} \\ i_{\text{old}} &= 0.006937 \end{aligned}$$

Step 3:**Solve for the nominal rate, I/Y.**

$$\begin{aligned} i_{\text{old}} &= \frac{\text{I/Y}}{\text{C/Y}_{\text{old}}} \\ 0.006937 &= \frac{\text{I/Y}}{12} \end{aligned}$$

$$\text{I/Y} = 0.083249\% \text{ or } 8.3249$$

Calculator instructions:

2nd ICONV

Table 6.6.4

NOM	C/Y	EFF
Answer: 8.324896	12	8.65

Step 4: **Write as a statement.**

The offer of 8.65% effectively from the credit union is equivalent to 8.3249% compounded monthly. If the lowest rate from the banks is 8.4% compounded monthly, the credit union offer is the better choice.

Equivalent Interest Rates

At times you must convert a nominal interest rate to another nominal interest rate that is not an effective rate. This brings up the concept of equivalent interest rates, which are interest rates with different compounding that produce the same effective rate and therefore are equal to each other. After one year, two equivalent rates have the same future value.

HOW TO

Convert Nominal Interest Rates

To convert nominal interest rates you need no new formula. Instead, you make minor changes to the effective interest rate procedure and add an extra step. Follow these steps to calculate any equivalent interest rate:

Step 1: Identify the given nominal interest rate ($\mathrm{I/Y}$) and compounding frequency ($\mathrm{C/Y}_{\text{old}}$). Also identify the new compounding frequency ($\mathrm{C/Y}_{\text{new}}$).

Step 2: Calculate the original periodic interest rate (i_{old}) using the formula

$$i_{\text{Old}} = \frac{\mathrm{I/Y}}{\mathrm{C/Y}_{\text{Old}}}$$

Step 3: Calculate the new periodic interest rate (i_{new}) using the formula.

$$i_{\text{new}} = (1 + i_{\text{old}})^{\frac{\mathrm{C/Y}_{\text{old}}}{\mathrm{C/Y}_{\text{new}}}} - 1$$

Step 4: Using the formula rearrange and solve for the new converted nominal rate $\mathrm{I/Y}$.

$$i_{\text{new}} = \frac{\mathrm{I/Y}}{\mathrm{C/Y}_{\text{new}}}$$

Your BAII Plus Calculator

Converting nominal rates on the BAII Plus calculator takes two steps:

Step 1: Convert the original nominal rate and compounding to an effective rate. Input NOM (this is the given nominal rate $\mathrm{I/Y}$) and the corresponding old $\mathrm{C/Y}$, then compute the EFF.

Step 2: Input the new $\mathrm{C/Y}$ and compute the new converted nominal rate NOM.

Example 6.6.3

Revisiting the mortgage rates from the section opener, compare the 6.6% compounded semi-annually rate to the 6.57% compounded quarterly rate by converting one compounding to another.

Solution

It is arbitrary which interest rate you convert. In this case, choose to convert the 6.57% compounded quarterly rate to the equivalent nominal rate compounded semi-annually.

Step 1: **Given information:**

$$\mathrm{I/Y} = 6.57\%; \quad \mathrm{C/Y_{old}} = \text{quarterly} = 4;$$
$$\text{Convert to } \mathrm{C/Y_{new}} = \text{semi-annually} = 2$$

Step 2: **Calculate i_{old}.**

$$i_{\text{old}} = \frac{\mathrm{I/Y}}{\mathrm{C/Y_{old}}}$$
$$i_{\text{old}} = \frac{6.57\%}{4}$$
$$i_{\text{old}} = 1.6425\%$$
$$i_{\text{old}} = 0.016425$$

Step 3: **Calculate i_{new}.**

$$i_{\text{new}} = (1 + i_{\text{old}})^{\frac{\mathrm{C/Y_{ld}}}{\mathrm{C/Y_{new}}}} - 1$$
$$i_{\text{new}} = (1 + 0.016425)^{\frac{4}{2}} - 1$$
$$i_{\text{new}} = 0.033119$$

Step 4: **Solve for the new converted nominal rate$\mathrm{I/Y}$.**

$$i_{\text{new}} = \frac{\text{I/Y}}{\text{C/Y}_{\text{new}}}$$

$$0.033229 = \frac{\text{I/Y}}{2}$$

$$\text{I/Y} = 0.06624 \text{ or } 6.624\%$$

Step 5: **Write as a statement.**

Thus, 6.57% compounded quarterly is equivalent to 6.624% compounded semi-annually. Pick the mortgage rate of 6.6% compounded semi-annually since it is the lowest rate available.

Calculator instructions:

2nd ICONV

Table 6.6.5

Step	NOM	C/Y	EFF
1	6.57	4	Answer: 6.733648
2	Answer: 6.623956	2	6.733648

Use this sequence:

2nd ICONV, 6.57 Enter ↑, 4 Enter ↑, CPT ↓, 2 Enter ↓, CPT

Answer: 6.623956

When converting interest rates, the most common source of error lies in confusing the two values of the compounding frequency, or C/Y. When working through the steps, clearly distinguish between the old compounding (C/Y_{old}) that you want to convert from and the new compounding (C/Y_{new}) that you want to convert to. A little extra time spent on double-checking these values helps avoid mistakes.

Example 6.6.4

You are looking at three different investments bearing interest rates of 7.75% compounded semi-annually, 7.7% compounded quarterly, and 7.76% compounded semi-annually. Which investment offers the highest interest rate?

Solution

Notice that two of the three interest rates are compounded semi-annually while only one is compounded quarterly. Although you could convert all three to effective rates (requiring three calculations), it is easier to convert the quarterly compounded rate to a semi-annually compounded rate. Then all rates are compounded semi-annually and are therefore comparable.

Step 1: **Given information:**

$$\mathrm{I/Y} = 7.7\%; \quad \mathrm{C/Y}_{\mathrm{old}} = \text{quarterly} = 4; \quad \mathrm{C/Y}_{\mathrm{new}} = \text{semi-annually} = 2$$

Step 2: **Calculate i_{old}.**

$$\begin{aligned} i_{\mathrm{old}} &= \frac{\mathrm{I/Y}}{\mathrm{C/Y}_{\mathrm{old}}} \\ i_{\mathrm{old}} &= \frac{7.7\%}{4} \\ i_{\mathrm{old}} &= 1.925\% \\ i_{\mathrm{old}} &= 0.01925 \end{aligned}$$

Step 3: **Calculate i_{new}.**

$$\mathrm{i}_{New} \quad = (1+\mathrm{i}_{Old})^{\frac{\mathrm{C/Y}_{Old}}{\mathrm{C/Y}_{New}}} - 1 \ \mathrm{i}_{New} \quad = (1+0.01925)^{\frac{4}{2}} - 1 \ \mathrm{i}_{New} \quad = \quad 0.038870$$

Step 4: **Solve for the new converted nominal rate$\mathrm{I/Y}$.**

$$i_{\text{new}} = \frac{\text{I/Y}}{\text{C/Y}_{\text{new}}}$$

$$0.038870 = \frac{\text{I/Y}}{2}$$

$$\text{I/Y} = 0.077741 \text{ or } 7.7741\%$$

Step 5: **Write as a statement.**

The quarterly compounded rate of 7.7% is equivalent to 7.7741% compounded semi-annually. In comparison to the semi-annually compounded rates of 7.75% and 7.76%, the 7.7% quarterly rate is the highest interest rate for the investment.

Calculator instructions:

Table 6.6.7

Step	NOM	C/Y	EFF
1	7.7	4	Answer: 7.925204
2	Answer: 7.774112	2	7.925204

Concept Check

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=212#h5p-15

Section 6.6 Exercises

1. The HBC credit card has a nominal interest rate of 26.44669% compounded monthly. What effective rate is being charged?

   Solution

   ***Step 1:* Given information:**

   $$\mathrm{I/Y}=26.44669\%;\quad \mathrm{C/Y}_{\mathrm{old}}=12;\quad \mathrm{C/Y}_{\mathrm{new}}=1$$

   Step 2:

   $$\begin{aligned} i_{\mathrm{old}} &= \frac{\mathrm{I/Y}}{\mathrm{C/Y}_{\mathrm{old}}} \\ i &= \frac{26.44669\%}{12} \\ i &= 2.203890\% \end{aligned}$$

   Step 3:

   $$\begin{aligned} i_{\mathrm{new}} &= (1+i_{\mathrm{old}})^{\frac{\mathrm{C/Y}_{\mathrm{old}}}{\mathrm{C/Y}_{\mathrm{new}}}}-1 \\ i_{\mathrm{new}} &= (1+0.02203890)^{\frac{12}{1}}-1 \\ i_{\mathrm{new}} &= (1.02203890)\,12-1 \\ i_{\mathrm{new}} &= 1.299-1 \\ i_{\mathrm{new}} &= 0.299 \end{aligned}$$

   ***Step 4:* Write as a statement.** A rate of 29.9% is effectively being charged.

   Calculator instructions:

   $$\begin{aligned} &\mathrm{NOM}=26.44669 \\ &\mathrm{C/Y}=12 \\ &\mathrm{EFF}=? \end{aligned}$$

2. Louisa is shopping around for a loan. TD Canada Trust has offered her 8.3% compounded monthly, Conexus Credit Union has offered 8.34% compounded quarterly, and ING Direct has offered 8.45% compounded semi-annually. Rank the three offers and show calculations to support your answer.

   Solution

   Convert all to effective rates to facilitate a fair comparison.

TD Canada Trust:

***Step 1:* Given information:**

$$\mathrm{I/Y} = 8.3\%; \qquad \mathrm{C/Y}_{\text{old}} = 12; \qquad \mathrm{C/Y}_{\text{new}} = 1$$

Step 2:

$$\begin{aligned} i_{\text{old}} &= \frac{\mathrm{I/Y}}{\mathrm{C/Y}_{\text{old}}} \\ i_{\text{old}} &= \frac{8.3\%}{12} \\ i_{\text{old}} &= 0.691\overline{6}\% \end{aligned}$$

Step 3:

$$\begin{aligned} i_{\text{new}} &= (1 + i_{\text{old}})^{\frac{\mathrm{C/Y}_{\text{old}}}{\mathrm{C/Y}_{\text{new}}}} - 1 \\ i_{\text{new}} &= \left(1 + 0.00691\overline{6}\right)^{\frac{12}{1}} - 1 \\ i_{\text{new}} &= \left(1.00691\overline{6}\right)^{12} - 1 \\ i_{\text{new}} &= 1.086231 - 1 \\ i_{\text{new}} &= 0.086231 \end{aligned}$$

***Step 4:* Write as a statement.** The TD Canada Trust rate is effectively 8.6231%.

Conexus Credit Union:

***Step 1:* Given information:**

$$\mathrm{I/Y} = 8.34\%; \qquad \mathrm{C/Y}_{\text{old}} = 4; \qquad \mathrm{C/Y}_{\text{new}} = 1$$

Step 2:

$$i_{\text{old}} = \frac{\text{I/Y}}{\text{C/Y}_{\text{old}}}$$
$$i_{\text{old}} = \frac{8.34\%}{4}$$
$$i_{\text{old}} = 2.085\%$$

Step 3:

$$i_{\text{new}} = (1 + i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1$$
$$i_{\text{new}} = (1 + 0.02085)^{\frac{4}{1}} - 1$$
$$i_{\text{new}} = (1.02085)^{4} - 1$$
$$i_{\text{new}} = 1.086044 - 1$$
$$i_{\text{new}} = 0.086045$$

***Step 4:* Write as a statement.** The Conexus Credit Union rate is effectively 8.6045%.

ING Direct:

***Step 1:* Given information:**

$$\text{I/Y} = 8.45\%; \quad \text{C/Y}_{\text{old}} = 2; \quad \text{C/Y}_{\text{new}} = 1$$

Step 2:

$$i_{\text{old}} = \frac{\text{I/Y}}{\text{C/Y}_{\text{old}}}$$
$$i_{\text{old}} = \frac{8.45\%}{2}$$
$$i_{\text{old}} = 4.225\%$$

Step 3:

$$i_{\text{new}} = (1 + i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1$$
$$i_{\text{new}} = (1 + 0.04225)^{\frac{2}{1}} - 1$$
$$i_{\text{new}} = (1.04225)^2 - 1$$
$$i_{\text{new}} = 1.086285 - 1$$
$$i_{\text{new}} = 0.086285$$

***Step 4:* Write as a statement.** The ING Direct rate is effectively 8.6285%.

Ranking:

Table 6.6.8

Rank	Company	Effective Rate
1	ING Direct	8.6285%
2	TD Canada Trust	8.6231%
3	CONEXUS Credit Union	8.6045%

Calculator instructions:

Table 6.6.9

	NOM	C/Y	EFF
TD	8.3	12	?
CONEXUS	8.34	4	?
ING	8.45	2	?

3. The TD Emerald Visa card wants to increase its effective rate by 1%. If its current interest rate is 19.067014% compounded daily, what new daily compounded rate should it advertise?

Solution

First calculate the effective rate.

***Step 1:* Given information:**

$$\text{I/Y} = 19.067014\%; \quad \text{C/Y}_{\text{old}} = 365; \quad \text{C/Y}_{\text{new}} = 1$$

Step 2:

$$\begin{aligned} i_{\text{old}} &= \frac{\text{I/Y}}{\text{C/Y}_{\text{old}}} \\ i_{\text{old}} &= \frac{19.067014\%}{365} \\ i_{\text{old}} &= 0.052238\% \end{aligned}$$

Step 3:

$$\begin{aligned} i_{\text{new}} &= (1+i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1 \\ i_{\text{new}} &= (1+0.00052238)^{\frac{365}{1}} - 1 \\ i_{\text{new}} &= (1.00052238)^3 65 - 1 \\ i_{\text{new}} &= 1.209999 - 1 \\ i_{\text{new}} &= 0.21 \end{aligned}$$

***Step 4:* Write as a statement.** The effective interest rate is 21%.

Now convert it back to a daily rate after making the adjustment: (reverse steps 2 & 3)

Step 1:

$$i_{\text{new}} = 21\% + 1\% = 22\%; \quad \text{C/Y}_{\text{old}} = 365; \quad \text{C/Y}_{\text{new}} = 1$$

Step 3:

$$\begin{aligned} i_{\text{new}} &= (1+i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1 \\ 0.22 &= (1+i_{\text{old}})^{\frac{365}{1}} - 1 \\ 1.22 &= (1+i_{\text{old}})^{365} \\ 1.22^{\frac{1}{365}} &= 1 + i_{\text{old}} \\ 1.000544 &= 1 + i_{\text{old}} \\ i_{\text{old}} &= 0.000544 \end{aligned}$$

Step 2:

$$i_{\text{old}} = \frac{\text{I/Y}}{\text{C/Y}_{\text{old}}}$$
$$0.000544 = \frac{\text{I/Y}}{365}$$
$$\text{I/Y} = 0.198905$$

Step 4: **Write as a statement.** The interest rate is 19.89% compounded daily.

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $i = \frac{\text{I/Y}}{\text{C/Y}}$

$i_{\text{new}} = (1 + i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1$

Attribution

6.7 DETERMINING THE NUMBER OF COMPOUNDS

Formula & Symbol Hub

For this section you will need the following:

Symbols Used

- FV = Future value or maturity value
- PV = Present value or principal value
- i = Periodic interest rate
- C/Y = Compounds per year
- I/Y =Nominal interest rate per year
- n =Total number of compounding periods
- $\ln$ = Natural logarithm

Formulas Used

- Formula 6.1 – **Periodic Interest Rate**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

- Formula 6.2a – **Number of Compound Periods**

$$n = \text{C/Y} \times \text{Number of Years}$$

- Formula 6.2b – **Future (Maturity) Value**

$$FV = PV \times (1+i)^n$$

- Formula 6.3 – **Present Value (Principal)**

$$PV = \frac{FV}{(1+i)^n}$$

- Formula 6.6 – **Interest Rate Conversion**

$$i_{\text{new}} = (1+i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1$$

Determining the Number of Compounds

How long will it take to reach a financial goal? At a casual get-together at your house, a close friend discusses saving for a 14-day vacation to the Blue Bay Grand Esmeralda Resort in the Mayan Riviera of Mexico upon graduation. The estimated cost from Travelocity.ca is $\$1,998.94$ including fares and taxes. He has already saved $\$1,775$ into a fund earning 8% compounded quarterly. Assuming the costs remain the same and he makes no further contributions, can you tell him how soon he will be basking in the sun on the beaches of Mexico?

This section shows you how to calculate the time frame for single payment compound interest transactions. You can apply this knowledge to any personal financial goal. Or in your career, if you work at a mid-size to large company, you might need to invest monies with the objective of using the funds upon maturity to pursue capital projects or even product development opportunities. So knowing the time frame for the investment to grow large enough will allow you to schedule the targeted projects.

The number of compounding periods could work out to be an integer. More challenging scenarios involve time frame computations with non-integer compounding periods.

HOW TO

Compute the Number of Compounding Periods

Follow these steps to compute the number of compounding periods, n.

Step 1: Draw a timeline to visualize the question. Most important at this step is to identify PV, FV, and the nominal interest rate (both $\mathrm{I/Y}$ and $\mathrm{C/Y}$).

Step 2: Solve for the periodic interest rate (i) using Formula 6.1

$$i=\frac{\mathrm{I/Y}}{\mathrm{C/Y}}$$

Step 3: Use the formula for the future value, rearrange, and solve for n. Note that the value of n represents the number of compounding periods. For example, if the compounding is quarterly, a value of $n=9$ is nine quarters.

Step 4: Take the value of n and convert it back to a more commonly expressed format such as years and months. When the number of compounding periods calculated in Step 3 works out to an integer, you an solve for the number of years using the formula

$$\mathrm{Years}=\frac{n}{\mathrm{C/Y}}$$

- If the Years is *an integer*, you are done.
- If the Years is a *non–integer*, the whole number portion (the part in front of the decimal) represents the number of years. As needed, take the decimal number portion (the part after the decimal point) and multiply it by 12 to convert it to months. For example, if you have $\mathrm{Years}=8.25$ then you have 8 years plus $0.25\times12=30.25\times12=3$ months, or 8 years and 3 months.

Concept Check

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=214#h5p-16

Your BAII Plus Calculator

Enter values for the known variables, PV, FV, $\mathrm{I/Y}$, and both of the values in the $\mathrm{P/Y}$ window ($\mathrm{P/Y}$ and $\mathrm{C/Y}$) following the procedures established in Section 6.2. Ensure proper application of the cash flow sign convention to PV and FV. One number must be negative while the other is positive, otherwise an ERROR message will appear on your calculator display.

Example 6.7.1

Jenning Holdings invested $\$43,000$ at 6.65% compounded quarterly. A report from the finance department shows the investment is currently valued at $\$67,113.46$. How long has the money been invested?

Solution
Determine the amount of time that the principal has been invested. This requires calculating the number of compounding periods (n).

***Step 1:* Given variables:**

$$PV = \$43,000; \quad \mathrm{I/Y} = 6.65\%; \quad \mathrm{C/Y} = \mathrm{quarterly} = 4; \\ FV = \$67,113.46$$

***Step 2:* Solve for the periodic interest rate, i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{6.65\%}{4}$$

$$i = 1.6625\%$$

Step 3: **Use the formula for the future value, rearrange, and solve for n.**

$$\begin{aligned} FV &= PV(1+i)^n \\ \$67{,}113.46 &= \$43{,}000(1+0.016625)^n \\ 1.560778 &= 1.016625^n \\ \ln(1.560778) &= \ln(1.016625)^n \text{ (by taking ln of both sides)} \\ \ln(1.560778) &= n\ln(1.016625) \text{ (by using the property } \ln(x)^n = n\ln(x)) \\ n &= \frac{\ln(1.560778)}{\ln(1.016625)} \\ n &= \frac{0.445184}{0.016488} \\ n &= 26.99996 \text{ or } 27 \text{ quarterly compounds} \end{aligned}$$

Step 4: **Convert n to years and months.**

$$\begin{aligned} \text{Years} &= \frac{n}{\text{C/Y}} \\ \text{Years} &= \frac{27}{4} \\ \text{Years} &= 6.75 \text{ years} \\ \text{Years} &= 6 \text{ years and } .75 \times 12 = 9 \text{ months} \end{aligned}$$

Calculator instructions:

Table 6.7.1

N	I/Y	PV	PMT	FV	P/Y	C/Y
Answer: 26.999996	6.65	$-43,000$	0	$67,113.46$	4	4

Step 5: **Write as a statement.**

Jenning Holdings has had the money invested for six years and nine months.

Non-integer Compounding Periods

When the number of compounding periods does not work out to an integer, the method of calculating n does not change.

Typically, the non-integer involves a number of years, months, and days.

As summarized in the table below, to convert the compounding period into the correct number of days you can make the following assumptions:

Table 6.7.2

Compounding Period	**# of Days in the Period**
Annual	365
Semi-annual	182*
Quarter	91*
Month	30*
Week	7
Daily	1

HOW TO

Calculate Number of Compounding Periods for Non-Integer n-values

You still use the same four steps to solve for the number of compounding periods when n works out to a non-integer as you did for integers.

1. Separate the integer from the decimal for your value of n.
2. With the integer portion, apply the same technique used with an integer n to calculate the number of years and months as we discussed before.
3. With the decimal portion, multiply by the number of days in the period to determine the number of days and round off the answer to the nearest day (treating any decimals as a result of a rounded interest amount included in the future value).

Example 6.7.2

Tabitha estimates that she will need $\$20,000$ for her daughter's postsecondary education when she turns 18. If Tabitha is able to save up $\$8,500$, how far in advance of her daughter's 18th birthday would she need to invest the money at 7.75% compounded semi-annually? Answer in years and days. Round to the nearest day.

Solution

Step 1: **Given information:**

The principal, future value, and interest rate are known, as illustrated in the timeline.

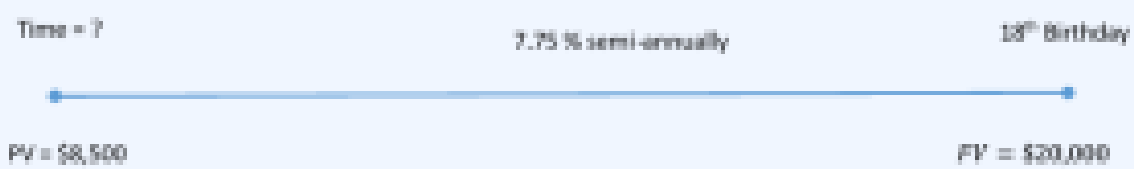


Figure 6.7.1

$$PV = \$8,500; \quad \text{I/Y} = 7.75\%; \quad \text{C/Y} = \text{semi-annually} = 2; \quad FV = \$20,000$$

Step 2: **Solve for the periodic interest rate, i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{7.75\%}{2}$$

$$i = 3.875\%$$

Step 3: **Use the formula for the future value, rearrange, and solve for n.**

$$\begin{aligned} FV &= PV(1+i)^n \\ \$20{,}000 &= \$8{,}500(1+0.03875)^n \\ 2.352941 &= 1.03875^n \\ \ln(2.352941) &= \ln(1.03875)^n \text{ (by taking ln of both sides)} \\ \ln(2.352941) &= n\ln(1.03875) \text{ (by using the property} \ln(x)^n = n\ln(x)) \\ n &= \frac{\ln(2.352941)}{\ln(1.03875)} \\ n &= \frac{0.855666}{0.038018} \\ n &= 22.506828 \text{ semi-annual compounds} \end{aligned}$$

Step 4: **Convert n to years and days.**

Take the integer:

$$\begin{aligned} \text{Years} &= \frac{n}{\text{C/Y}} \\ \text{Years} &= \frac{22}{2} \\ \text{Years} &= 11 \end{aligned}$$

Take the decimal:

$$\begin{aligned} \text{Days} &= 0.506828 \times 182 \\ \text{Days} &= 92 \end{aligned}$$

Calculator instructions:

Table 6.7.3 Calculator Instructions for Example 6.7.2

N	I/Y	PV	PMT	FV	P/Y	C/Y
Answer: 22.506828	7.75	$-8,500$	0	$20,000$	2	2

Step 5: **Write as a statement.**

If Tabitha invests the $\$8,500$ 11 years and 92 days before her daughter's 18th birthday, it will grow to $\$20,000$.

Section 6.7 Exercises

1. You just took over another financial adviser's account. The client invested $\$15,500$ at 6.92% compounded monthly and now has $\$24,980.58$. How long (in years and months) has this client had the money invested?

Solution

***Step 1:* Given information:**

$$PV = \$15,500; \quad \mathrm{I/Y} = 6.92\%; \quad FV = \$24,980.58$$

***Step 2:* Calculate i.**

$$i = \frac{\mathrm{I/Y}}{\mathrm{C/Y}}$$

$$i = \frac{6.92\%}{12}$$

$$i = 0.57\overline{6}\%$$

***Step 3:* Use Formula 6.2b for FV and rearrange for n.**

$$FV = PV(1+i)^n$$

$$\$24,980.58 = \$15,500\left(1 + 0.0057\overline{6}\right)^n$$

$$1.611650 = \left(1.0057\overline{6}\right)^n$$

$$\ln(1.611650) = n \times \ln\left(1.0057\overline{6}\right)$$

$$0.477258 = n \times 0.005750$$

$$n = 83 \text{ monthly compounds}$$

***Step 4:* Convert the time to years and months.**

$$\text{Years} = \frac{83}{12} = 6.91\overline{6}$$ which is 6 years plus $0.91\overline{6} \times 12 = 11$ months

***Step 5:* Write as a statement.** The client has had the money invested for 6 years and 11 months.

Calculator instructions:

Table 6.7.4

N	PV	I/Y	PMT	FV	P/Y	C/Y
?	$15,500$	6.92	0	$24,980.58$	12	12

2. Your organization has a debt of $\$30,000$ due in 13 months and $\$40,000$ due in 27 months. If a single payment of $\$67,993.20$ was made instead using an interest rate of 5.95% compounded monthly, when was the payment made? Use today as the focal date.

Solution

Step 1: **First figure out what the money is worth today.**

Original Agreement:

$$\text{Payment } \#1 = \$30,000 \text{ due in } 13 \text{ months}$$
$$\text{Payment } \#2 = \$40,000 \text{ due in } 27 \text{ months}$$
$$\text{I/Y} = 5.95\%; \quad \text{C/Y} = 12$$

Proposed Agreement:

$\$67,993.20$ due in x months

Step 2:

$$\text{Focal date} = \text{today}$$

Step 3: **Calculate i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$
$$i = \frac{5.95\%}{12}$$
$$i = 0.4958\overline{3}\%$$

Step 4: **Calculate n of the payments.**

Payment #1:

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$
$$n = 1\frac{1}{12} \times 12$$
$$n = 1.08\overline{3} \times 12$$
$$n = 13$$

Payment #2:

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$
$$n = 2\frac{3}{12} \times 12$$
$$n = 2.25 \times 12$$
$$n = 27$$

Step 5: **Use Formula 6.3** $PV = \dfrac{FV}{(1+i)^n}$ **to calculate** PV **of the payments.**

Payment #1:

$$PV = \frac{\$30,000}{(1.004958)^{13}}$$
$$PV = \$28,131.73574$$

Payment #2:

$$PV = \frac{\$40,000}{(1.004958)^{27}}$$
$$PV = \$34,999.55193$$

Step 6: **Find the total** PV **of the payments.**

$$\text{Total today} = \$28,131.73574 + \$34,999.55193$$
$$\text{Total today} = \$63,131.28768$$

Now figure out where the payment occurs:

Step 1:

$$PV = \$63,131.28768; \quad FV = \$67,993.20; \quad \text{I/Y} = 5.95\%; \quad \text{C/Y} = 12$$

Step 2: **Find i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{5.95\%}{12}$$

$$i = 0.4958\overline{3}\%$$

Step 3: **Use Formula 6.2b for FV and rearrange for n.**

$$\begin{aligned} FV &= PV(1+i)^n \\ \$67,993.20 &= \$63,131.28768(1+0.004958)^n \\ 1.121112 &= (1.004958)^n \\ \ln(1.077012) &= n \times \ln(1.004958) \\ 0.074191 &= n \times 0.004946 \\ n &= 15 \text{ monthly compounds} \end{aligned}$$

Step 4: **Convert the time to years and months.**

$$\text{Number of years} = \frac{15}{12}$$

$$\text{Number of years} = 1.25 \text{ which is 1 year plus } 0.25 \times 12 = 3 \text{ months}$$

Step 5: **Write as a statement.** Payment is made 15 months from today.

Calculator instructions:

Calculation	N	I/Y	PMT	PV	FV	P/Y	C/Y
Payment #1	13	5.95	0	?	30,000	12	12
Payment #2	27	5.95	0	?	40,000	12	12
	?	5.95	0	63,131.28768	67,993.2	12	12

3. A $\$9,500$ loan requires a payment of $\$5,000$ after $1\frac{1}{2}$ years and a final payment of

$\$6,000$. If the interest rate on the loan is 6.25% compounded monthly, when should the final payment be made? Use today as the focal date. Express your answer in years and months.

Solution

***Step 1:* Given information:**

$$P = \$9,500; \quad \text{I/Y} = 6.25\%; \quad \text{C/Y} = 12$$

$$\text{Payment \#1} = \$5,000 \text{ due in } 1\frac{1}{2} \text{ years}$$

$$\text{Payment \#2} = \$6,000 \text{ due in } x \text{ years}$$

Step 2:

$$\text{Focal date} = \text{today}$$

***Step 3:* Find i.**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

$$i = \frac{6.25\%}{12}$$

$$i = 0.52083\overline{3}\%$$

***Step 4:* Calculate n for the first payment.**

Payment #1:

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$

$$n = 1\frac{1}{2} \times 12$$

$$n = 1.5 \times 12$$

$$n = 18$$

Payment #2:

$$n = ?$$

***Step 5:* Use Formula 6.3 $PV = \dfrac{FV}{(1+i)^n}$ to calculate PV of the payments.**

Payment #1:

$$\$5,000 = PV\left(1 + 0.005208\overline{3}\right)^{18}$$

$$PV = \frac{\$5,000}{\left(1.005208\overline{3}\right)^{18}}$$

$$PV = \$4,553.65956$$

Payment #2:

$$\$6,000 = PV\left(1 + 0.005208\overline{3}\right)^{n}$$

$$PV = \frac{\$6,000}{\left(1.005208\overline{3}\right)^{n}}$$

Step 6: **Solve for n of the final payment.**

$$\$9,500 = \$4,553.65956 + \frac{\$6,000}{\left(1.005208\overline{3}\right)^{n}}$$

$$\$4,946.34044 = \frac{\$6,000}{\left(1.005208\overline{3}\right)^{n}}$$

$$\left(1.005208\overline{3}\right)^{n} = \frac{\$6,000}{\$4,946.34044}$$

$$\left(1.005208\overline{3}\right)^{n} = 1.213018$$

$$n \times \ln(1.005208) = \ln(1.213018)$$

$$n \times 0.005194 = 0.193111$$

$$n = 37.173874 \text{ monthly compounds (round up to 38 months)}$$

Step 7: **Convert the time to years and months.**

$$\text{Number of years} = \frac{38}{12}$$

$$\text{Number of years} = 3.1\overline{6} \text{ which is 3 years plus } 0.1\overline{6} \times 12$$

$$\text{Number of years} = 2 \text{ months}$$

Step 8: Write as a statement. The final payment should be made in 3 years and 2 months

THE FOLLOWING LATEX CODE IS FOR FORMULA TOOLTIP ACCESSIBILITY. NEITHER THE CODE NOR THIS MESSAGE WILL DISPLAY IN BROWSER. $FV = PV \times (1+i)^n$ $i = \frac{\text{I/Y}}{\text{C/Y}}$

$PV = \frac{FV}{(1+i)^n}$

Attribution

CHAPTER 6: COMPOUND INTEREST TERMINOLOGY (INTERACTIVE ACTIVITY)

Complete the following activity.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=215#h5p-5

Attribution

CHAPTER 6: SUMMARY

Formula & Symbol Hub Summary

For this chapter you used the following:

Symbols Used

- C/Y = Compounds per year
- FV = Future value or maturity value
- i = Periodic interest rate
- I/Y = Nominal interest rate per year
- $\ln$ = Natural logarithm
- n = Total number of compounding periods
- PV = Present value or principal value

Formulas Used

- Formula 6.1 – **Periodic Interest Rate**

$$i = \frac{\text{I/Y}}{\text{C/Y}}$$

- Formula 6.2a – **Number of Compound Periods**

$$n = \text{C/Y} \times \text{Number of Years}$$

- Formula 6.2b – **Future (Maturity) Value**

$$FV = PV \times (1+i)^n$$

- Formula 6.3 – **Present Value (Principal)**

$$PV = \frac{FV}{(1+i)^n}$$

- Formula 6.6 – **Interest Rate Conversion**

$$i_{\text{new}} = (1+i_{\text{old}})^{\frac{\text{C/Y}_{\text{old}}}{\text{C/Y}_{\text{new}}}} - 1$$

Key Concepts Summary

6.1: Compound Interest Fundamentals

- How compounding works
- How to calculate the periodic interest rate

6.2: Determining the Future Value

- The basics of taking a single payment and moving it to a future date
- Moving single payments to the future when variables change

6.3: Determining the Present Value

- The basics of taking a single payment and moving it to an earlier date
- Moving single payments to the past when variables change

6.4: Equivalent Payments

- The concept of equivalent payments
- The fundamental concept of time value of money
- The fundamental concept of equivalency

- Applying single payment concepts to loans and payments

6.5: Determining the Interest Rate

- Solving for the nominal interest rate
- How to convert a variable interest rate into its equivalent fixed interest rate

6.6: Equivalent and Effective Interest Rates

- The concept of effective rates
- Taking any nominal interest rate and finding its equivalent nominal interest rate

6.7: Determining the Number of Compounds

- Figuring out the term when n is an integer
- Figuring out the term when n is a non-integer

The Language of Business Mathematics

compound interest A system for calculating interest that primarily applies to long-term financial transactions with a timeframe of one year or more; interest is periodically converted to principal throughout a transaction, with the result that theinterest itself also accumulates interest.

compounding frequency The number of compounding periods in a complete year.

compounding period The amount of time that elapses between the dates of successive conversions of interest to principal.

discount rate An interest rate used to remove interest from a future value.

effective interest rate The true annually compounded interest rate that is equivalent to an interest rate compounded at some other (non-annual) frequency.

equivalent payment streams Equating two or more alternative financial streams such that neither party receives financial gain or harm by choosing either stream.

equivalent interest rates Interest rates with different compounding that produce the same effective rate and therefore areequal to each other.

focal date A point in time to which all monies involved in all payment streams will be moved using time value of money calculations.

fundamental concept of equivalency Two or more payment streams are equal to each other if they have the same economic value on the same focal date.

fundamental concept of time value of money All monies must be brought to the same focal date before any mathematical operations, decisions, or equivalencies can be determined.

nominal interest rate A nominal number for the annual interest rate, which is commonly followed by words that state the compounding frequency.

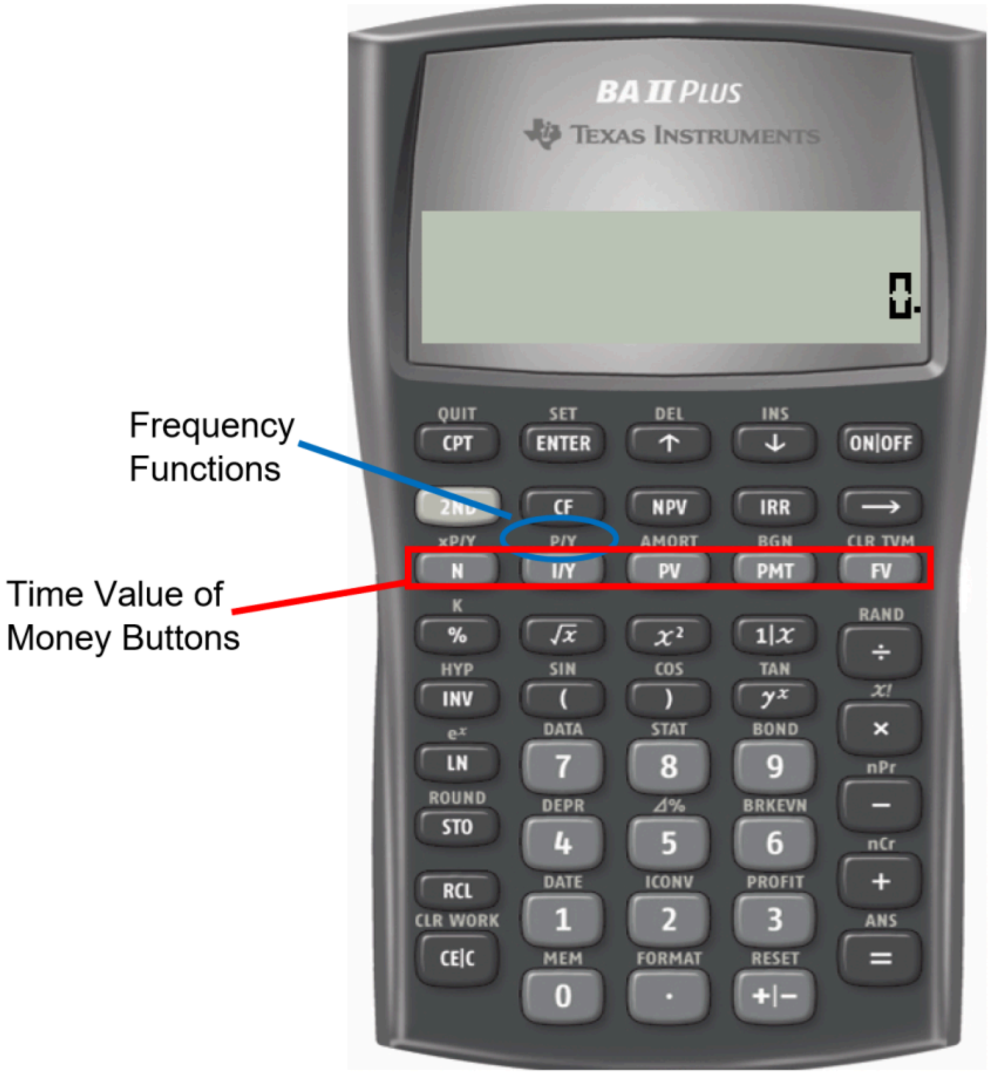
periodic interest rate The percentage of interest earned or charged at the end of each compounding period.

present value principle for loans The present value of all payments on a loan is equal to the principal that was borrowed.

Rule of 72 A rule of thumb stating that 72 divided by the annually compounded rate of return closely approximates the number of years required for money to double.

Technology Introduced

Calculator



Texas Instrument BAII Plus Calculator

Time Value of Money Buttons

1. The time value of money buttons are the five buttons located on the third row of your calculator.

Calculator Symbol	**Characteristic**	**Data Entry Requirements**
N	The number of compounding periods	An integer or decimal number; no negatives
$\mathrm{I/Y}$	The nominal interest rate per year	Percent format without the $\%$ sign (i.e., 7% is just 7)
PV	Present value or principal	An integer or decimal number
PMT	Used for annuity payment amounts (covered in Chapter 11) and is not applicable to lump-sum amounts; it needs to be set to zero for lump-sum calculations	An integer or decimal number
FV	Future value or maturity value	An integer or decimal number

To enter any information into any one of these buttons or variables, called loading the calculator, key in the information first and then press the appropriate button.

2. The frequency function is logically placed above the

I/<span id="MathJax-Span-237" class="mi">Y < /span >< /span >< /span >

button and is labelled $\mathrm{P/Y}$. This function addresses compound interest frequencies, such as the compounding frequency. Access the function by pressing $\mathrm{2nd\ P/Y}$ to find the following entry fields, through which you can scroll using your arrow buttons.

Calculator Symbol	**Characteristic**	**Data Entry Requirements**
$\mathrm{P/Y}$	Annuity payments per year (payment frequency is introduced in Chapter 11); when working with lump-sum payments and not annuities, the calculator requires this variable to be set to match the $\mathrm{C/Y}$	A positive, nonzero number only
$\mathrm{C/Y}$	Compounds per year (compounding frequency)	A positive, nonzero number only

- To enter any information into one of these fields, scroll to the field on your screen, key in the data, and press Enter.
- When you enter a value into the $\mathrm{P/Y}$ field, the calculator will automatically copy the value into the $\mathrm{C/Y}$ field for you. If in fact the $\mathrm{C/Y}$ is different, you can change the number manually.
- To exit the $\mathrm{P/Y}$ window, press $\mathrm{2nd\ Quit}$.

Keying in a Question

- You must load the calculator with six of the seven variables.
- To solve for the missing variable, press CPT followed by the variable.

Cash Flow Sign Convention

- The cash flow sign convention is used for the PV, PMT, and FV buttons.
- If money leaves you, you must enter it as a negative.
- If money comes at you, you must enter it as a positive.

Clearing the Memory

- Once you enter data into any of the time value buttons, it is permanently stored until one of the following happens:
  - You override it by entering another piece of data and pressing the button.
  - You clear the memory of the time value buttons by pressing 2nd CLR TVM (a step that is recommended before you proceed with a separate question).
  - You press the reset button on the back of the calculator.

Attribution

H5P EXAMPLES

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/introbusinessmath/?p=882#h5p-6

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VERSIONING HISTORY

This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.1. If the edits involve a number of changes, the version number increases to the next full number.

The files posted alongside this book always reflect the most recent version.

Version	Date	Change	Affected Web Page
1.0	April 2023	First Publication	N/A