Calculus for the Sciences

Course Notes

Edition 1.0

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Contents

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A NOTE TO INSTRUCTORS

This is not your standard calculus textbook. It differs in four main ways:

1. Assumed Prior Knowledge: It assumes that students have had some previous exposure to calculus. In particular, it assumes students can evaluate limits graphically and find basic derivatives, including using derivative rules.

2. Order of Content: The content is presented in a unique order. For example, Chapter 1 covers derivatives, integrals, and differential equations. Introducing these topics early allows students to practice and master them throughout the term.

3. Problem Design: The end-of-section and end-of-chapter problems are designed with the spacing effect and interleaving principle in mind. Problems from previous sections, but on the same topic, can appear in the end-of-section/end-of-chapter problems. For example, the end-of-section problems for the integral test include problems that require a method other than the integral test. This helps students learn to recognize which test to use rather than relying solely on the section they are working on.

4. Practical Focus: This textbook prioritizes practical understanding over mathematical rigor, making it suitable for science students who do not require a deep theoretical understanding of calculus.

Content:

The following is a brief description of each chapter and its goals.

Chapters 1 - 5 are intended for a single-semester Calculus 1 course.

Chapter 1: An Overview of Calculus

This chapter provides a brief refresher on derivatives before introducing differential equations and integration. The aim is to get students started on these topics so they can practice them throughout the course and understand their interconnections. Notably, there is less emphasis on the area interpretation of the definite integrals, focusing instead on the fact that integrals are sums of smalls bit and on using definite integrals to calculate net change.

Chapter 2: Functions

This chapter covers a wide range of basic functions, including the absolute value function, inverse functions, exponential and logarithmic functions, and trigonometric functions. Since derivatives and integrals were discussed in Chapter 1, the derivatives and integrals of these functions are introduced here. Additionally, integral functions, the factorial function, and implicitly defined functions are also covered.

Chapter 3: Analyzing Functions

This chapter focuses on limits, intervals of increase/decrease, and intervals of concavity as tools for analyzing functions rather than as mere applications of derivatives. Two-sided limits are defined in terms of one-sided limits, and determinate forms are discussed before indeterminate forms.

Chapter 4: Optimization and Approximations

This chapter combines optimization and approximations. The optimization section is fairly standard. The approximations section, however, prepares students for Taylor polynomials and Taylor's Theorem by covering quadratic approximations and the error in linear approximations. Additionally, at the request of the physics department, the binomial approximation is included.

Chapter 5: Integration

This chapter covers Riemann sums, applications of integration, and integration by substitution. Emphasis is placed on using Riemann sums for approximations. Only the right Riemann sum is used for approximating definite integrals, and the error in the right Riemann sum is briefly discussed to further prepare students for Taylor's Theorem.

Chapters 6 - 12 are intended for a single semester Calculus 2 course.

Chapter 6: Sequences and Series

This chapter covers sequences and infinite series. It is fairly standard except that the standard comparison test and the root test are omitted. It precedes the techniques of integration chapter to introduce the Integral Test before improper integrals, helping students understand improper integrals as limits rather than as definite integrals with infinite bounds.

Chapter 7: Power Series

This chapter covers power series, Taylor series, Taylor polynomials, and Taylor's Theorem. Geometric series are introduced early to help students start thinking about power series before they are formally defined.

Chapter 8: Techniques of Integration

This chapter covers standard techniques such as trigonometric integrals, trigonometric substitutions, partial fractions, integration by parts, and improper integrals.

Chapter 9: Volume

This chapter introduces the calculation of volume using definite integrals and includes a standard treatment of volumes of revolution.

Chapter 10: Differential Equations

This chapter covers direction fields, Euler's method, solving first-order separable and linear differential equations, and rate in-rate out problems. Students will have a better understanding of differential equations by this chapter, having practiced solving simple differential equations since Chapter 1.

Chapter 11: Parametric Equations

This chapter covers the basics of parametric equations, including finding the slope (but not concavity), area, and arc length. There is an emphasis on parameterizing a curve, which is often needed in future calculus courses.

Chapter 12: Polar Coordinates

This chapter introduces polar coordinates, including graphing and area calculations. Calculus in polar coordinates is not covered.

A NOTE TO STUDENTS - READ THIS!

Best Selling Novel: "My Favourite Math," by Al G. Braw

Welcome to Calculus for the Sciences!

In this book, we will cover many of the core ideas in calculus: differentiation, approximations, integration, differential equations, power series, and more. This book uses two main strategies:

1. Scaffolding: Rather than overwhelming you with all the details of a concept all at once, we teach a little bit at a time. This allows you to learn the concept in manageable chunks which not only makes learning more efficient, but also prompts deeper learning. For example, integration is introduced in Chapter 1. But, rather than making you memorize a whole bunch of integration formulas all at once, we will slowly add more and more formulas throughout Chapter 2. You will then use some of what you have learned about integration in Chapter 3 and 4 so that when we have a more in-depth look in Chapter 5, you will already have a strong foundation.

2. Connections: Calculus is often taught as a collection of separate topics. A typical calculus textbook will have a chapter on limits, a chapter on derivatives, a chapter on integration, a chapter on differential equations, etc. However, this is misleading as all of these concepts are integrated (pun intended) together. So, in this book we try to stress how all of these topics are tied together. For example, in Chapter 1, we show how derivatives, integrals, and differential equations are intertwined.

This book gives students in science a good foundation in calculus, rather than a precise, theoretically treatment of calculus. That is, it does not always provide precise mathematical definitions of concepts (for example, it does not cover the epsilon-delta definition of a limit), and omits almost all proofs.

Prerequisites:

Teacher: Recall from your last course that...
Student: What?!? You mean we were supposed to remember that?

High school calculus is a prerequisite to this course. Therefore, we expect that you have some knowledge of

- Functions (domain, range, graphing, operations, etc.)
- Polynomials (especially factoring)
- Trigonometric functions
- Exponential and logarithmic functions
- Limits as $x \to a$
- Derivatives of power functions, basic trig functions, and exponential functions.

If you are unsure of any of these, we recommend that you do some self study to learn/remember these topics.

What is Calculus Used For?

Student: Will we ever use this in real life?

Professor: Not if you get a job flipping hamburgers.

Calculus is used in the social sciences, the natural sciences, engineering, business, computer science, statistics, and all branches of mathematics. A small sample of courses at the University of Waterloo that have calculus as a prerequisite include: BIOL 364, BIOL 382, CHEM 240, CHEM 254, CS 335, ECON 290, KIN 121, OPTOM 106, PHYS 121, PHYS 225, STAT 220.

As much as possible, this book demonstrates a couple of the ways that calculus is used in science. However, true applications of calculus in science require much more advanced knowledge of science then we can assume that first year students have, especially since this course is aimed at all branches of science. Most of the applications problems contained in the book are greatly simplified to make the problems accessible. This is another example of scaffolding... you start by learning how to solve relatively simple, but not realistic problems, before trying to tackle more complex problems.

How to Succeed in Math Calculus

Never drink pop while studying... it will dilute your concentration.

1. **Attend all the lectures.**

   Although this book has been written to help you learn calculus, the lectures are your primary resource to the material covered in the course. The lectures will provide additional examples and explanations to help you understand the course material. Moreover, during lectures you can ask when you do not understand something or need clarification... books do not provide this feature (yet).

2. **Read this book.**

   It is generally recommended that students read their course notes/textbooks before the content is covered in class. If you are able to teach yourself some of the material before it is covered in class, you will find the lectures considerably more helpful. The lecturers will then be there for you to clarify what you taught yourself and to help with the areas that you had difficulty with. Trust me, you will enjoy your classes much more if you understand most of what is being covered in the lectures.

3. **RED MEANS STOP!**

   The **mid-section exercises** are largely design to give you a way of testing your understanding of a concept before proceeding. You should always ensure that you understand the method used to solve an exercise before you continue reading as future concepts will often be easier to understand if you have a reasonable good understanding of the previous concepts.

4. **Study!**

   This might sound a little obvious, but most students do not do nearly enough of this. Moreover, you need to study properly. Staying up all night the night before a test is *not* studying properly! Learning math, and most other subjects, is like building a house. If you do not have a firm foundation, then it will be very difficult for you to build on top of it. It is extremely important that you know and do not forget the basics.

5. **Use learning resources.**

   There are many places you can get help outside of the lectures. They are there to assist you... make use of them!

Active Reading: How To Make The Most Of This Book

When reading science and mathematics textbooks it is important to ensure that you are reading for understanding as opposed to how you might read a novel for enjoyment. This is called **active reading**. Active reading is much, much slower than reading for enjoyment and involves critically evaluating what you are reading and constantly monitoring your understanding (the purpose of the mid-section exercises).

There are a variety of active reading techniques (for example, SQ3R). Here we present the MATH technique.

Make notes: Make notes on what you are reading. Your notes should include definitions, theorems, formula, and algorithms, along with any questions you have about the content. Write down any connections you see to other content. These connections could include similarities you notice, or prerequisites (for example, you might indicate to yourself that the properties of fractions are needed sometimes when doing integration).

Analyze: Carefully analyze what you are reading. For example, don't just read a definition or a theorem - take some time to try to understand why it is important and how it is going to be used. When reading examples or proofs, you should be checking the validity of every step. That is, justify to yourself that what is written is correct. Moreover, you should be trying to understand the method that is used. Whenever possible, write yourself an algorithm for solving the problem.

Test: Test your understanding. It is entirely too common for students to think they understood what they read when they actually didn't. The only way to know if you actually understood something is to test your knowledge. Some possible ways of doing this are trying the mid-section exercises, using flashcards, or having a conversation with someone else about it to see if their understanding is similar to your own.

Halt: Whenever you find that you do not understand a concept or a step, halt your reading and try to figure out what you don't understand. This could require reading some part of the book (it could even been an earlier chapter), reviewing a formula you are not sure of, or getting help (from a peer, an instructor, etc). Math builds upon itself. If you read past something that you don't understand, then it will very likely make it even more difficult to understand future concepts.

Yes, you are going to find that this makes the reading go much slower for the first couple of chapters, but students who use this technique consistently report that they feel that they end up spending a lot less time studying for the course as they learn the material so much better at the beginning (which makes future concepts much easier to learn).

How to Learn

What is Learning?

Learning is typically defined as the acquisition of knowledge, behaviours, and skills through instruction, study, experience, and/or practice. However, in education, it can sometimes be helpful to think of learning as the art of not forgetting. For example, many students sit in class, watch a video, or read a textbook and think they have learned it. However, if they cannot remember it for a test, did they really learn it?

The question becomes: how do you put something into memory so that you can access it when you need it? To answer this question, let's look at how the brain stores memory.

Memory

We will look at four types of memory: Working Memory, Early Long-Term Memory, Transitional Long-Term Memory, and Long Lasting Memory.

Working Memory: This is where your brain processes information. If you don't focus on information inside your working memory, then it disappears quickly (within a minute). Any distraction (like checking social media) can be enough for you to lose what was in your working memory. The way to get your brain to move information from working memory to early long-term memory is to deeply focus on it. For example, taking thoughtful notes (not just blindly copying), asking questions, trying to answer the instructor's questions, or comparing it to something you already known are all great ways of focusing on new information.

Example: Think about cases where you remember a dream when you first woke up. If you immediately focus on the details of the dream, you can remember them (I remember parts of dreams I had when I was a child). But, if you think about anything else, you will very quickly forget most or all of the dream.

Early Long-Term Memory: When you deeply focus on information in working memory it gets moved to early long-term memory. Items in early long-term memory are relatively easy to access. However, early long-term memory is temporary storage, up to about 12 hours, and does not have a huge capacity.

Cramming: This is why cramming is so bad... because it works, but not nearly as well as people think it does. While cramming, you are getting some information into early long-term memory. Since it is reliable memory, it may help you do better on a test later that day. However, since it has limited space, the amount that cramming can help you is actually minimal. Moreover, you will like forget most of the content after the test making you spend more time in the future having to relearn it.

Transitional Long-Term Memory: During sleep, memories in Early Long-Term Memory start moving to Transitional Long-Term Memory. It is theorized that even taking a nap can start this process for some memories (so, it is much, much better to take a nap after class rather than during class). Memories in Transitional Long-Term Memory can degrade quite quickly if they are not reinforced. In fact, they can degrade almost completely within just 7 days! Essentially, your brain only has so much 'material' to strengthen memories, so it prioritizes memories that are used more often. Note that strengthening memories not only means that they are more reliable to access, but also quicker to access!

Long Lasting Memory: Memories in transitional long-term memory that have been prioritized are moved to Long Lasting Memory. Memories in Long Lasting Memory degrade very, very slowly although their initial strength is dependent on how strong they were made in Transitional Long-Term Memory. Although memories here can last a long time, if a memory isn't accessed in a long time, then it becomes more and more difficult to access that memory (if you don't use it, you lose it).

Size Matters: In addition to the strength of a memory in Long Lasting Memory is, the size of it perhaps matters even more. That is, memories can be attached together in chunks called schemas. The larger the chunk is, the easier it is to remember because your brain only needs to remember one part for it to remember the whole thing.

Example: If you know the Pythagorean theorem really well, but don't know the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$ very well, then if you can connect these into a single chunk (i.e. memorize the relationship between them), then when trying to remember the identity, your brain just needs to remember the Pythagorean theorem. So, all of a sudden, you know the identity very well.

Learning

Our current understanding of how memory works in the brain tells us the following about learning:

1. The more you know about a topic or can connect it to something you already know, the easier it is to learn more about that topic.

2. Learning, especially initial learning, takes focus and effort.

3. You should at least briefly review any material you learned in lectures/videos/books on the same day that you initially learned it. This will help transition as much as possible from Early Long-Term Memory into Transitional Long-Term Memory.

4. You should do a second review of any material you learned within 7 days of initially learning it. The more times you review it, the more robust the memory will be. This is further improved if you do the reviewing over several days rather than all on one day.

5. The more you review a topic, the more likely you will be able recall that information, and the faster you will be able to recall the information.

6. The more you can find and understand connections in and between topics, the better you will be able to remember and use all of those topics.

Chapter 1: Introduction to Calculus

Section 1.1: Introduction to Differential Calculus

LEARNING OUTCOMES

1. Know how to calculate derivatives and rates of change.
2. Know how to use a tangent line to approximate the value of a function.
3. Know how to calculate differentials.
4. Know how to recognize and interpret a differential equation.
5. Know how to verify a function is a solution of a differential equation.

1.1.1 Derivatives

In high school you were introduced to one of the two main branches of calculus, differential calculus. You saw that the derivative of a function $y = f(x)$ at $x = a$ is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or equivalently by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1.1}$$

When the limit exists, we say that f is **differentiable** at a.

You were told that the derivative represents the instantaneous rate of change of f at $x = a$ and $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$.

We can rewrite the second formula using the notation Δ (read as 'delta') to represent the change of a quantity. In particular,

$$\Delta y = f(x) - f(a)$$

represents the change in the y value and

$$\Delta x = x - a$$

represents the change in the x value. Substituting these into equation (1.1) and using the fact that as x approaches a, Δx will approach 0 gives

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

This leads us to the other common notation for the derivative: $\dfrac{dy}{dx}$.

Two notations for the first and higher order derivatives are summarized below.

function	1st derivative	2nd derivative	n-th derivative	1st derivative at a
f	f'	f''	$f^{(n)}$	$f'(a)$
y	$\dfrac{d}{dx}y = \dfrac{dy}{dx}$	$\dfrac{d}{dx}\left(\dfrac{d}{dx}y\right) = \dfrac{d^2y}{dx^2}$	$\dfrac{d^ny}{dx^n}$	$\left.\dfrac{dy}{dx}\right\rvert_a$

We expect that you know the following formulas:

function	**derivative**
$f(x) = c$, c is a constant	$f'(x) = 0$
$f(x) = x^n$, $n \neq 0$	$f'(x) = nx^{n-1}$
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = \sin(x)$	$f'(x) = \cos(x)$
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$

And we expect that you know the following derivative rules:

- If functions f and g are both differentiable, then
  - If $h(x) = cf(x)$, then $h'(x) = cf'(x)$ for any constant c **(Constant Rule)**
  - If $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$ **(Sum Rule)**
  - If $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$ **(Product Rule)**
  - If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ **(Quotient Rule)**
- If $h(x) = (f \circ g)(x) = f(g(x))$ where g is differentiable at x and f is differentiable at $g(x)$, then
$$h'(x) = f'(g(x))g'(x) \qquad \textbf{(Chain Rule)}$$

EXAMPLE 1

Determine the derivative of $f(x) = 3x^2 + e^x + 2$.

Solution: Using the Sum Rule, we get

$$f'(x) = 6x + e^x + 0 = 6x + e^x$$

EXAMPLE 2

Determine the second derivative of $g(x) = (x^3 + x)\cos(x)$.

Solution: Using the Product Rule, we get

$$\begin{aligned} g'(x) &= (3x^2 + 1)\cos(x) - (x^3 + x)\sin(x) \\ g''(x) &= \Big[(6x + 0)\cos(x) - (3x^2 + 1)\sin(x)\Big] - \Big[(3x^2 + 1)\sin(x) + (x^3 + x)\cos(x)\Big] \\ &= (-x^3 + 5x)\cos(x) - (6x^2 + 2)\sin(x) \end{aligned}$$

EXAMPLE 3

Determine the derivative of $h(x) = \dfrac{3x^2 + x}{x - 1}$.

Solution: Using the Quotient Rule, we get

$$h'(x) = \frac{(6x + 1)(x - 1) - (1 - 0)(3x^2 + x)}{(x - 1)^2} = \frac{3x^2 - 6x - 1}{(x - 1)^2}$$

EXAMPLE 4

Determine the derivative of $j(s) = (s^2 + \sin(s))^{1/3}$.

Solution: Using the Chain Rule, we get

$$j'(s) = \frac{1}{3}(s^2 + \sin(s))^{-2/3} \cdot (2s + \cos(s))$$

EXERCISE 1

Determine the derivative of each function.

(a) $f(x) = x^3 + \dfrac{7}{x}$

(b) $f(x) = 5\sqrt[3]{x+1}$

(c) $f(x) = e^x \cos(x)$

(d) $g(r) = \sin^2(r)$

(e) $h(w) = e^{\sqrt{w}}$

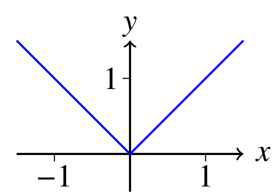
(f) $j(t) = \dfrac{e^t \sin(t)}{t^2}$

When the Derivative Does Not Exist

We briefly look at three conditions in which the derivative of a function won't exist at a point $x = a$ in the domain of f.

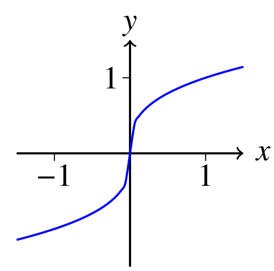
1. f has a corner at $x = a$.

The derivative does not exist in this case because the slope of the tangent line depends on the side from which we approach the corner.



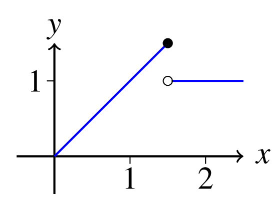
2. f has a vertical tangent line at $x = a$.

The derivative does not exist in this case because the slope of the tangent line is infinity, and infinity is not a number.



3. f is not continuous at $x = a$.

The derivative does not exist in this case because we can't calculate an instantaneous rate of change at a point where there is a break in the function.



The Meaning of Mathematics

On its own, mathematics has no physical meaning until one gives meaning to the functions, variables, etc. For example, the derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

is simply a mathematical construction. In this form, the derivative, if it exists, is just a number. However, as you saw in your previous mathematics, if we define the function f to represent a physical quantity (a population of birds, the heat generated by a chemical reaction, the gravitational force between two objects) and give meaning to the independent variable of f (time, concentration of a reactant, distance), then the derivative represents the instantaneous rate of change. Alternatively, we can interpret the derivative geometrically as the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

The fact that mathematics initially has no meaning is exactly what makes it a difficult subject for many students. In particular, students can find it difficult to understand why they are learning it. However, the fact that mathematics has no inherent meaning is what makes it such a powerful tool. Once a mathematical concept is defined, it can be interpreted in many different ways leading to a wide variety of applications across many different fields. So, learning how to solve a mathematical problem is actually learning how to solve many different applied problems.

1.1.2 Rates of Change

Algebraically, the derivative gives the rate of change of one quantity with respect to another. For example, if $Q(t)$ is a quantity changing with respect to time t, then $Q'(a)$ is the instantaneous rate of change of the quantity Q with respect to time at the moment that $t = a$.

EXAMPLE 5 Assume the mass M of an object in kg given by its radius r in cm is given by

$$M(r) = \frac{7r^3 + 2}{r^2}$$

Find the rate of change of the mass with respect to the radius and indicate the units.

Solution: We have

$$\begin{aligned}\frac{dM}{dr} &= \frac{(21r^2 + 0)(r^2) - 2r(7r^3 + 2)}{(r^2)^2} \\ &= \frac{21r^4 - 14r^4 - 4r}{r^4} \\ &= \frac{7r^3 - 4}{r^3}\end{aligned}$$

The notation $\frac{dM}{dr}$ actually tells us what the units are. Since M is in kg and r is in cm, we have units of kg/cm.

EXAMPLE 6

The velocity of an object in m/s is given by

$$v(t) = 3 + \sin(t)$$

where t is measured in seconds. Find the acceleration of the object at any time t.

Solution: The acceleration is

$$a(t) = v'(t) = \cos(t) \text{ m/s}^2$$

EXAMPLE 7

If the number of individuals in a population is given by $P(t) = 100 + \frac{1}{2}e^t$ where t is measured in days, then find the rate of change of the population with respect to time and indicate the units.

Solution: We have

$$P'(t) = 0 + \frac{1}{2}e^t = \frac{1}{2}e^t \quad \text{individuals/day}$$

EXAMPLE 8

Suppose that a company determines that the revenue in dollars for producing and selling x thingamajiggers is

$$R(x) = -0.1x^2 + 10x$$

(a) Calculate $R'(10)$and include its units.

(b) Explain the physical meaning of $R'(10)$.

Solution: (a) We have

$$R'(x) = -0.2x + 10 \quad \text{dollars/thingamajigger}$$

Thus,

$$R'(10) = -0.2(10) + 10 = 8 \quad \text{dollars/thingamajigger}$$

(b) As indicated by the units, $R'(10)$ is the approximate change in profit for producing and selling an eleventh thingamajigger. That is, it shows the company's profit will increase by approximately \$8 for producing and selling eleven thingamajiggers instead of ten.

NOTE: We can easily calculate the actual change in profit for producing and selling an eleventh thingamajigger. It is

$$R(11) - R(10) = 7.9 \text{ dollars}$$

So, what is the point of calculating R'? Although, R', called the marginal revenue, doesn't give us the exact change in profit (because it gives the instantaneous rate of change); it does give us a function that we can analyze the behaviour of. For example, economists can use the marginal revenue to calculate how to maximize profits.

EXERCISE 2 The position of a particle for time $t \geq 0$ in seconds is given by

$$s(t) = t^3 - 3t + 1 \text{ m}$$

(a) Calculate $s'(t)$, including its units.

(b) Explain the physical meaning of $s'(1)$.

EXERCISE 3 Microtubule tips are the dynamic, or moving, ends of microtubules, which are long, tubular structures made up of protein subunits called tubulin. If L_0 is the length, in μm, of the tip at time $t = 0$ and the length of the tip at time t, in seconds, is given by

$$L(t) = \frac{11}{14}t + L_0 \, \mu\text{m}$$

then what is the growth velocity of the microtubule tip? Indicate the units.

1.1.3 Tangent Line Approximations

Geometrically, the derivative of f evaluated at $x = a$, $f'(a)$, is the slope of the tangent line to the curve $y = f(x)$ where the point of tangency is $(a, f(a))$. We will use this fact for

(i) constructing approximations to functions,

(ii) investigating the behavior of functions, and

(iii) for finding maximum or minimum values of functions (i.e. optimization).

For now, we will focus on using the tangent line for approximations. We will look at the other applications later.

Point-Slope Form of a Line

In high school, you likely had considerable practice with the slope-intercept form of a line

$$y = mx + b$$

The problem with this equation for calculus is that it is focused on the location of the y-intercept of the line. In calculus, we want to be able to focus our attention at any point on the line we choose. So, in calculus, we almost always use the point-slope form of a line

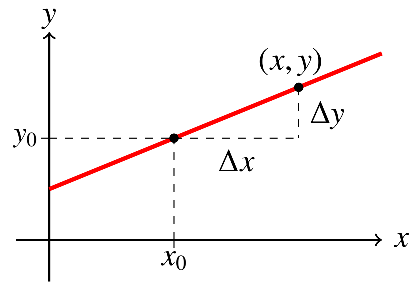
The **point-slope form** of a line with slope m passing through the point (x_0, y_0) is

$$y = y_0 + m(x - x_0)$$

Let's see where this equation comes from. Say we have a line with slope m that passes through the point (x_0, y_0). To get an equation for the line, we pick any other point (x, y) on the line.

By definition of slope, we get that

$$m = \frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0}$$



Rearranging this gives

$$m(x - x_0) = y - y_0$$
$$y_0 + m(x - x_0) = y$$

as required.

EXAMPLE 9 Find the point-slope equation of the line with slope $m = 5$ that passes through the point $(3, -2)$.

Solution: Substituting $m = 5$, $x_0 = 3$, and $y_0 = -2$ into the equation gives

$$y = -2 + 5(x - 3)$$

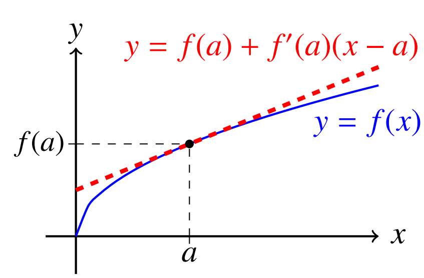
REMARK

There is a strong temptation to 'simplify' the point-slope form of a line. However, doing so would defeat the purpose of having the focus on the point (x_0, y_0).

EXERCISE 4 Find the point-slope equation of the line with slope $m = -3$ that passes through the point $(-2, 1)$.

Equation of the Tangent Line

To get the equation of the tangent line to a function $y = f(x)$ at a point $(a, f(a))$, we use the point-slope form of a line and the fact that the slope at $x = a$ is $f'(a)$. We get



The **equation of the tangent line** to $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)(x - a)$$

EXAMPLE 10 Let $f(x) = x^2$. Find the equation of the tangent line to $y = f(x)$ at $a = 1$.

Solution: We have $f'(x) = 2x$. So, the slope of the tangent line at $a = 1$ is

$$m = f'(1) = 2(1) = 2$$

Thus, the equation of the tangent line is

$$\begin{aligned} y &= f(1) + f'(1)(x-1) \\ y &= 1^2 + 2(x-1) \\ y &= 1 + 2(x-1) \end{aligned}$$

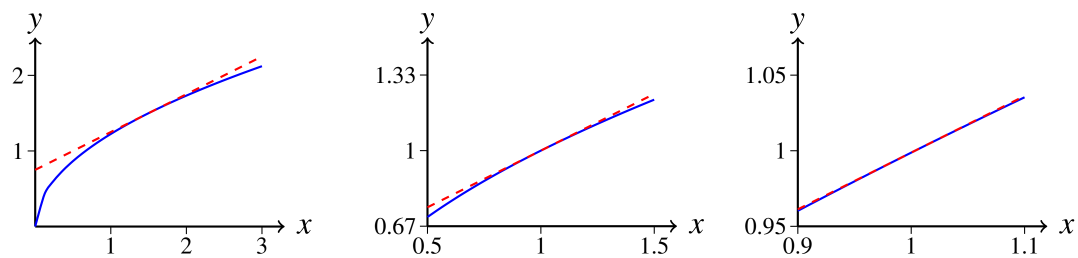
EXERCISE 5 Let $f(x) = x^3 + x$. Find the equation of the tangent line to $y = f(x)$ at $a = 2$.

EXERCISE 6 Let $h(x) = e^x$. Find the equation of the tangent line to $y = h(x)$ at $a = 0$.

EXERCISE 7 Let $g(\theta) = \sin(\theta)$. Find the equation of the tangent line to $y = g(\theta)$ at $a = \frac{\pi}{6}$.

We can use values on the tangent line of $y = f(x)$ at a to approximate values of f is because, on a small enough interval, the graph of a differentiable function will appear linear.

As an example, the figures below show $f(x) = \sqrt{x}$ (the solid curve) along with its tangent line $y = 1 + \frac{1}{2}(x-1)$ (the dashed line) on smaller and smaller intervals centered at $x = 1$.



We see that for values of x very close a the tangent line is almost indistinguishable from the graph of the function. We get:

The **tangent line (linear) approximation**

$$f(x) \approx f(a) + f'(a)(x-a)$$

for values of x near a.

EXAMPLE 11

Find the equation of the tangent line to $f(x) = \sqrt{x}$ at $a = 9$. Use the tangent line to approximate $\sqrt{10}$.

Solution: We have $f'(x) = \frac{1}{2}x^{-1/2}$. So, the slope of the tangent line at $a = 9$ is

$$f'(9) = \frac{1}{2}(9)^{-1/2} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

We get the equation of the tangent line is

$$\begin{aligned} y &= f(9) + f'(9)(x-9) \\ y &= \sqrt{9} + \frac{1}{6}(x-9) \\ y &= 3 + \frac{1}{6}(x-9) \end{aligned}$$

The tangent line approximation says

$$f(x) \approx 3 + \frac{1}{6}(x-9)$$

Taking $x = 10$ in this equation gives

$$\begin{aligned} \sqrt{10} &\approx 3 + \frac{1}{6}(10-9) \\ &= \frac{19}{6} \approx 3.1667 \end{aligned}$$

NOTE: Given that $\sqrt{10} = 3.162\ldots$, we see that we got a fairly accurate approximation for such little effort.

EXERCISE 8

Use the tangent line $y = 2 + \frac{1}{4}(x-4)$ of $f(x) = \sqrt{x}$ at $a = 4$ to approximate $\sqrt{2}$. Explain why the tangent line gives a relatively poor approximation in this case.

EXERCISE 9

Find the equation of the tangent line for $f(x) = \cos(x)$ at $a = \pi$ and use it to approximate $\cos\left(\frac{7\pi}{8}\right)$.

EXERCISE 10

Find the equation of the tangent line for $f(x) = \sqrt[3]{x+2}$ at $a = 6$ and use it to approximate $\sqrt[3]{9}$.

EXERCISE 11 Assume the position of an object is given by $s(t) = 17 + 3t + 2t^2$ where s is in meters and t is in seconds. Find the equation of the tangent line for $s(t)$ at $t = 1$ and use it to approximate $s(1.1)$.

1.1.4 Differentials

Say we want to launch a rocket to dock with the International Space Station. To meet up with the International Space Station, the rocket will need to follow a particular flight path given by some function. The rocket will have some capability of making small corrections to its flight path, but it is always possible that a very strong gust of wind during launch will change its trajectory by more than the rocket's thrusters can deal with. Before launching, we need to be able to calculate by how much the trajectory could change given the maximum gust strength. If on the day of launch the wind is too strong, we will know that we need to cancel the launch.

We can modify the tangent line approximation to estimate this kind of change.

If the tangent line of $y = f(x)$ at a is $y = f(a) + f'(a)(x - a)$, then we can write the tangent line approximation as

$$f(x) \approx f(a) + f'(a)(x - a)$$

In particular, as we saw above, we can approximate $f(b)$ by

$$f(b) \approx f(a) + f'(a)(b - a)$$

If we are interested in how much the value of $f(x)$ will change if we move from $x = a$ to $x = b$, then we can subtract $f(a)$ from both sides of this equation to arrive at

$$f(b) - f(a) \approx f'(a)(b - a) \tag{1.2}$$

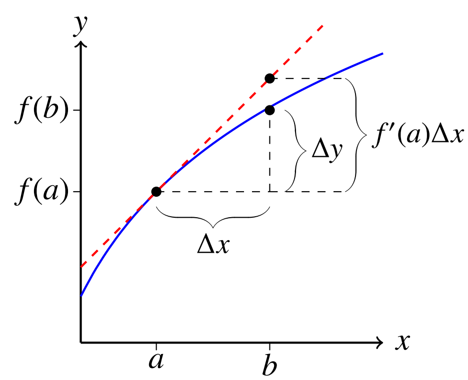
Recall that we use the notation Δ to represent change. So, we have $\Delta y = f(b) - f(a)$ is the change in the value of $f(x)$ (i.e. the change in the y-value) and $\Delta x = b - a$ is the change in the x value.

Substituting these into equation (1.2) gives

The **differential approximation**

$$\Delta y \approx f'(a)\Delta x$$

for values of x near a.



Observe in the figure that the tangent line (the dashed line) gets further and further away from the graph of $y = f(x)$ (the solid curve) as the x values move away from $x = a$. Therefore, we can only rely on the tangent line approximation being fairly accurate when Δx is very small. That is, the larger the value of Δx, the less confident we can be about the accuracy of the approximation. We will look more closely at the size of the error when using the tangent line approximation in Section 4.2.4.

EXAMPLE 12 Approximate the change in the surface area of a spherical balloon when the radius decreases from 4 m to 3.9 m.

Solution: We have $S(r) = 4\pi r^2$ m^2. Thus, by the differential approximation, the change in surface area with respect to radius is

$$\begin{aligned}\Delta S &\approx S'(r)\,\frac{\text{m}^2}{m}\cdot\Delta r\,\text{m}\\ &\approx (8\pi r)\Delta r\,\text{m}^2\end{aligned}$$

The initial radius is $r = 4$ m. The change in the radius is $\Delta r = -0.1$ m. So, the change is surface area will be approximately

$$\Delta S \approx (8\pi\cdot 4)(-0.1) = -3.2\pi\ \text{m}^2$$

EXAMPLE 13 Let $f(x) = 2x^3 + 9x^2 - 24x + 6$. Approximate the change in $f(x)$ as x changes from $x = 1$ to $x = 1.5$.

Solution: By the differential approximation, the change in $f(x)$ is given by

$$\begin{aligned}\Delta y &\approx f'(x)\Delta x\\ &\approx (6x^2 + 18x - 24)\Delta x\end{aligned}$$

The initial x value is $x = 1$. The change in the x value is $\Delta x = 0.5$. So, the approximate change in the y value will be

$$\Delta y \approx (6 + 18 - 24)(0.5) = 0$$

Wait!!! That doesn't make sense! The y value is definitely going to change. In fact, the actual value change is

$$\Delta y = f(1.5) - f(1) = 4$$

So, there was actually a huge change in y! What happened?

Part of the problem is the fact that the slope of the tangent line is 0 at $x = 1$. This results in the prediction that there will be no change in y. However, the main problem is that value of Δx, 0.5, is too large! For much smaller values of Δx, we would still have a relatively accurate approximation.

EXERCISE 12 Suppose we have a chemical reaction that is occurring with reaction rate r that is described by a polynomial function $r(c) = kc^2 - 2kc + \frac{k}{2}$ where c is the concentration of the reactant and k is a constant.

Approximate the change in the reaction rate as the concentration of the reactant changes from $c = 2$ to $c = 2.1$.

We now introduce some notation that will be extremely useful throughout calculus.

DEFINITION
Differential

Assume $y = f(x)$ is differentiable. We define the **differential** dx to be an infinitely small change in x. We define the **differential** dy by

$$dy = f'(x)\,dx$$

Essentially, we are defining dx to be Δx when Δx is infinitely small and we are defining dy to match with the equation

$$\frac{dy}{dx} = f'(x)$$

That is, dx and dy are defined so that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

In fact, this is exactly where the notation for the derivative $\frac{dy}{dx}$ comes from.

Since dx, and hence dy, are infinitely small, we cannot calculate them explicitly. In fact, the purpose of this notation is so that we can talk about and manipulate quantities that we cannot calculate. For example, in the next section, we will see that we can calculate the exact area under the graph of a function by adding up infinitely many infinitely small quantities. Differential notation gives us an effective way of representing this.

EXAMPLE 14

Find dy given that $y = f(t) = t^2 + t - 2$. Determine the units of dy given that the units of f are meters and the units of t are seconds.

Solution: We get

$$\begin{aligned} dy &= f'(t)\,\frac{\text{m}}{\text{s}} \cdot \ dt\,\text{s} \\ &= (2t+1)\,dt\,\text{m} \end{aligned}$$

Observe that the units make sense as dy is representing an infinitely small change in f which has units of meters.

EXAMPLE 15

Find du given that $u = f(x) = \cos(x^2)$.

Solution: We get

$$\begin{aligned} du &= f'(x)\,dx \\ du &= \left(-\sin(x^2)\cdot 2x\right)\,dx \end{aligned}$$

EXERCISE 13

Find dy given that $y = f(t) = 5t^3 - 3t^2$. If the units of $f(t)$ are kilograms and the units of t are minutes, then what are the units of dy?

EXERCISE 14

Find du given that $u = f(x) = \sin(x^2 + x)$.

1.1.5 Introduction to Differential Equations

Many scientific principles found in science and engineering relate quantities to their rates of change (e.g. Newton's 2nd Law of Motion, the Law of Mass Action for chemical kinetics, population growth laws). This often leads to problems where we need to find a function satisfying an equation involving one or more of its derivatives.

DEFINITION

Differential Equation

A **differential equation** (DE) is an equation involving an unknown function and one or more of its derivatives.

Differential equations are used frequently in the real world. Here are some examples modelling changes of quantities over time t.

1. If the rate of change of a population $P(t)$ is proportional to the current population with proportionality constant k, then the population satisfies the differential equation
$$\frac{dP}{dt} = kP$$
2. Denote the position of a 1 kg object attached to a spring by $x(t)$. If the spring exerts a force of 9 Newtons on the mass, then Hooke's Law says that
$$x'' = -9x$$
3. Newton's Law of Cooling says that if an object with temperature $T(t)$ is placed into an environment with constant temperature T_0, then the rate of change of the temperature of the object is proportional to the difference in the temperature of the object and the temperature of the environment. That is,
$$\frac{dT}{dt} = -k(T - T_0)$$
for some $k > 0$.
4. Let $B(t)$ represent the amount of money in a bank account at time t. If the bank account pays 2% interest annually (compounded continuously), then the amount of money in the account satisfies
$$B' = 0.02B$$

The Meaning of Mathematics

We discussed at the top of page 4 how an abstract mathematical equation can be assigned many different meanings. Differential equations provide many good examples of this. Consider the simple differential equation $\frac{dy}{dt} = ky$ (the derivative of the function is proportional to the function). Knowing how to solve this differential equation can help us solve problems related to population growth, continually compounded interest, carbon dating, feedback loops, and much, much more by giving an appropriate meaning to the function y.

As with many things in mathematics, we categorize differential equations according to some of their characteristics. For now, we will look at one broad categorization. In Chapter 10, we will see some additional categories.

DEFINITION
Order of a DE

The **order** of a differential equation is the order of the highest derivative of the unknown function in the equation.

Therefore, by definition, the differential equations

$$\frac{dP}{dt} = kP, \qquad \frac{dT}{dt} = -k(T - T_0), \qquad B' = 0.02B$$

are all first order differential equations. The differential equations

$$x'' = -9x, \quad \frac{d^2y}{dx^2} + (y')^3 = x^5$$

are second order differential equations.

EXERCISE 15

State the order of the following differential equations.

(a) $(y'')^3 + x^2y' = x$, (b) $\left(\frac{dy}{dx}\right)^2 = y^4$, (c) $\frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}$

In general, given a differential equation our goal is to find all unknown functions that satisfy the equation. We will look at how to solve some simple differential equations in the next section. In Chapter 10, we will look at how to solve some special types of differential equations.

DEFINITION
Solution

A function that satisfies a differential equation is called a **solution** of the differential equation.

When we say 'satisfies', we mean that it makes both sides of the differential equation equal. This is best demonstrated with some examples.

EXAMPLE 16

Show that $y(t) = 100e^{0.02t}$ is a solution of the first order differential equation

$$y' = 0.02y$$

Solution: Taking the derivative of y gives

$$\begin{aligned} y'(t) &= 100e^{0.02t} \cdot (0.02) \\ &= 0.02 \cdot 100e^{0.02t} \\ &= 0.02y(t) \end{aligned}$$

Thus, with this choice of y, the left hand side of the differential equation, y', is equal to the right hand side, $0.02y$, so $y(t) = 100e^{0.02t}$ is a solution of the differential equation.

EXAMPLE 17

Determine whether $y(x) = 3x^2 + 4$ is a solution of the second order differential equation

$$y'' = x$$

Solution: Taking derivatives of y gives

$$\begin{aligned} y'(x) &= 6x \\ y''(x) &= 6 \end{aligned}$$

Since the left hand side is 6 and the right hand side is x, they are not equal and hence $y(x) = 3x^2 + 4$ is not a solution of the differential equation.

EXERCISE 16

Determine which of the following is a solution of the second order differential equation $\frac{d^2f}{dx^2} = 1$.

(a) $f(x) = 3x^2 + 2x + 1$

(b) $f(x) = e^x + \cos(x)$

(c) $f(x) = \frac{1}{2}x^2 + 5x + 3$

(d) $f(x) = \frac{1}{2}x^2 - \sqrt{2}x + 101$

A differential equation may have infinitely many solutions. For example, because the derivative of a constant is 0, the first order differential equation

$$\frac{dy}{dx} = 2x$$

has solutions $y = x^2 + C$ for any real number C. We call C a **parameter** and we call $y = x^2 + C$ a **family of solutions**.

EXAMPLE 18

Show that $y(x) = \frac{1}{3}x^3 + C$ is a family of solutions of the first order differential equation

$$y'(x) = x^2$$

Solution: Taking the derivative of y gives

$$y'(x) = \frac{1}{3}(3x^2) + 0 = x^2$$

Since the left hand side y' equals the right hand side is x^2, all functions in the family $y = \frac{1}{3}x^3 + C$ are solutions of the differential equation.

EXAMPLE 19

Show that $y(x) = c_1 e^{2x} + c_2$ is a family of solutions of the second order differential equation

$$y'' = 2y'$$

Solution: We have

$$y'(x) = 2c_1 e^{2x}$$
$$y''(x) = 4c_1 e^{2x}$$

Therefore, the left hand side is $y'' = 4c_1 e^{2x}$.

On the right hand side, we have

$$2y' = 2(2c_1 e^{2x}) = 4c_1 e^{2x}$$

Since the left hand side equals the right hand side, all functions in the family $y(x) = c_1 e^{2x} + c_2$ are solutions of the differential equation.

EXERCISE 17

Show that $y(x) = -\cos(x) + C$ is a family of solutions of $y' = \sin(x)$.

EXERCISE 18

Show that $y(x) = \frac{1}{2}x^2 + C$ is a family of solutions of $y' = x$.

Section 1.1 Problems

1. Determine the derivative of each function.
   (a) $f(x) = 5x^3 - 2x + 1$
   (b) $f(x) = \sqrt{5x - 2}$
   (c) $f(x) = \dfrac{1}{(3 - x)^2}$
   (d) $f(x) = e^x \cos(x)$
   (e) $f(x) = \dfrac{\cos(x)}{\sin(x)}$
   (f) $f(x) = \sqrt{1 - x^2}$
   (g) $f(x) = \cos^3(x)$
   (h) $f(x) = (x^3 + x + 1)^{2/3}$
   (i) $f(x) = e^{\sqrt{x^2+x}}$
   (j) $f(x) = \dfrac{1}{\sqrt{x^2 + 1}}$
   (k) $f(x) = \dfrac{e^x - 1}{e^x + 1}$

2. Determine the second derivative of each function.
   (a) $f(x) = x^3 - 2x + 3$
   (b) $f(x) = x\sqrt{x}$
   (c) $f(x) = e^x$
   (d) $f(x) = \dfrac{\sin(x)}{\cos(x)}$
   (e) $f(x) = \dfrac{1}{x^3}$
   (f) $f(x) = \dfrac{1}{\sin(x)}$
   (g) $f(x) = e^{-x^2}$
   (h) $f(x) = \cos(x) + x^2 + 2$
   (i) $f(x) = e^x \sin(x)$
   (j) $f(x) = (x - 1)e^x$
   (k) $f(x) = \dfrac{1}{x} + \sqrt{x}$

3. Find the equation of the tangent line of the following functions at the given value of a.
   (a) $f(x) = \cos(x)$ at $a = 0$
   (b) $f(x) = e^x$ at $a = 0$
   (c) $f(x) = (1 + x)^{1/5}$ at $a = 0$
   (d) $f(x) = x^2 + 1$ at $a = -1$
   (e) $f(x) = x^3 - x$ at $a = 1$
   (f) $f(x) = \dfrac{e^x}{x}$ at $a = 2$

4. Use the tangent line of f at a to approximate the given value.
   (a) $f(x) = \sqrt{x}$ at $a = 9$; $\sqrt{10}$
   (b) $f(x) = x^{2/3}$ at $a = 8$; $7^{2/3}$
   (c) $f(x) = \sin(x)$ at $a = 0$; $\sin(0.1)$
   (d) $f(x) = (1 + x)^{1/4}$ at $a = 0$; $(1.001)^{1/4}$
   (e) $f(x) = (1 + x)^{1/3}$ at $a = 0$; $(0.99)^{1/3}$

5. Find the differential du.
   (a) $u = f(x) = x^2 + x$
   (b) $u = f(x) = \cos(3x + 1)$
   (c) $u = f(x) = \sqrt{2x + 3}$
   (d) $u = f(x) = e^{x^2}$
   (e) $u = f(x) = x^2 2^x$

6. Let $f(x) = x^2 + 3x - 1$. Approximate the change in f as x changes from $x = 1$ to $x = 1.1$.

7. The surface area of a spherical cell is related to its volume by the equation $S = (36\pi V^2)^{1/3}$. Approximate the change in surface area when the volume increases from 10 μm^3 to 10.5 μm^3.

8. Suppose the size of a bacterial population grows according to the model $\dfrac{dN}{dt} = 0.02N$, where t is in days and N is in millions. If we know that $N(10) = 250$, approximate the change in population when the time changes from 10 days to 10.5 days.

9. The population in millions of arctic flounder in the Atlantic Ocean can be modelled by the function $P(t) = \dfrac{8t + 1}{t^2 + 1}$, where t is measured in years since the beginning of 1980.
   (a) Calculate $P'(t)$ and state its units.
   (b) Find $P'(1)$ and briefly interpret the result.
   (c) Calculate $P(1)$ and $P(2)$. Explain why the difference between $P(1)$ and $P(2)$ does not contradict the result from part (b).

10. The head circumference H (in mm) of a fetus depends on the age t in weeks according to the model $H(t) = -30 + 1.5t^2$. Approximate the change in head circumference between $t = 8$ and $t = 9$ weeks.

11. Assume that $m(t)$ is the mass of an object in kg at any time t in hours.
    (a) What are the units for the functions $m'(t)$ and $m''(t)$?
    (b) What does it mean physically for the object if $m'(2) = 2$?

12. For a certain dosage of a drug, d (in mg), the resulting temperature change (in degrees Fahrenheit) of the person taking the drug is $T(d) = 0.05d - 0.3d^2$. Find the rate of change of temperature with respect to dosage and indicate the units.

13. State the order of the differential equation.
    (a) $y' = 2y(x - y)$
    (b) $y'' + 10y' - 5 = 2x + 3$
    (c) $\dfrac{dy}{dx} = y + x$
    (d) $x^2\dfrac{dy}{dx} = xy^3 + 1$
    (e) $y'' + y' + y = 0$
    (f) $\left(\dfrac{d^3y}{dx^3}\right)^2 = y$
    (g) $xy' = y$

14. Determine whether the given function is a family of solutions of the given differential equation.
    (a) $y = x + C$ for $y' = x + 2$
    (b) $y = Cx^2$ for $\dfrac{dy}{dx} = y^2$
    (c) $y = Ce^x$ for $\dfrac{dy}{dx} = y$
    (d) $y = Cx$ for $y' = \dfrac{y}{x}$
    (e) $y = -\dfrac{1}{e^x + C}$, for $y' = e^x y^2$
    (f) $y = -3x + Cx^2$, for $\dfrac{dy}{dx} - \dfrac{2}{x}y = 3$
    (g) $y = a\cos(t) + b\sin(t)$ for $\dfrac{d^2y}{dt^2} + y = 0$
    (h) $y = Ce^{3x} + 1$ for $y'' - y' - 6y = 0$

Section 1.2: Introduction to Integral Calculus

LEARNING OUTCOMES

1. Know how to use a definite integral to calculate a net change.
2. Know how to use a definite integral to calculate the area under a curve.
3. Know how to evaluate the indefinite integral of some basic functions.
4. Know how to use the indefinite integral to solve simple differential equations.

We now briefly look at the other main branch of calculus, integral calculus. We will first show how integral calculus is related to differential calculus through the net change problem. We will also show how integral calculus is used to help us solve simple differential equations.

1.2.1 The Net Change Problem

In many real world situations, we can find the rate of change $Q'(x)$ of a quantity $Q(x)$ and we want to use Q' to determine by how much the quantity Q will change due to a change in its independent variable x.

For example, let $s(t)$ denote the position of the object at any time t. Say that we are given that the velocity of the object is $s'(t) = v(t) = t^2$ m/s^2 from time $t = 1$ to $t = 2$, and we want to determine how far the object travelled during that time. If the velocity were constant, then we would simply use the formula

$$\text{distance} = \text{velocity} \times \text{time}$$

But, since the velocity isn't constant, this formula does not apply. In mathematics and science, if we cannot calculate something exactly, then the we try to approximate it. So, let's use the differential approximation

$$\Delta y \approx f'(x)\Delta x$$

that we saw in Section 1.1.4. In this case, the function is s and the variable is t, so the equation becomes

$$\Delta s \approx s'(a)\Delta t$$

which has units of m since $s'(a)$ has units m/s and Δt has units s. We also have that $s'(t) = v(t)$, so we get

$$\Delta s \approx v(a)\Delta t \text{ m}$$

As we saw in Section 1.1.4, this formula is inaccurate for large values of Δt. So, rather than using a large value like $\Delta t = 1$, let's approximate the change in s by approximating the change over four intervals of length $\Delta t = \frac{1}{4}$ s.

That is, the intervals will be $\left[1, \frac{5}{4}\right]$, $\left[\frac{5}{4}, \frac{6}{4}\right]$, $\left[\frac{6}{4}, \frac{7}{4}\right]$, and $\left[\frac{7}{4}, 2\right]$.

Over the first interval $\left[1, \frac{5}{4}\right]$, we have $a = 1$ and get

$$\Delta s_1 \approx v(1)\Delta t = (1)^2 \cdot \frac{1}{4} = \frac{1}{4} \text{ m}$$

Over the second interval $\left[\frac{5}{4}, \frac{3}{2}\right]$, we take $a = \frac{5}{4}$ and get

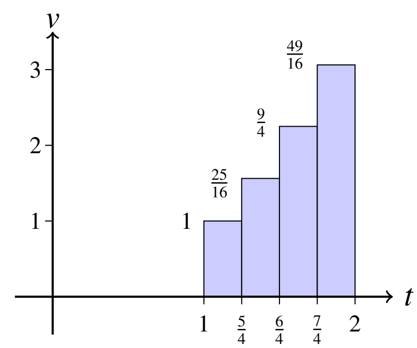
$$\Delta s_2 \approx v\left(\frac{5}{4}\right)\Delta t = \left(\frac{5}{4}\right)^2 \cdot \frac{1}{4} = \frac{25}{64} \text{ m}$$

Similarly, we find over the third interval $\left[\frac{3}{2}, \frac{7}{4}\right]$ that $\Delta s_3 \approx \frac{9}{16}$ m, and over the fourth interval $\left[\frac{7}{4}, 2\right]$ that $\Delta s_4 \approx \frac{49}{64}$ m.

So, over the entire interval $[1, 2]$, we have the approximate change in position is

$$\Delta s \approx \frac{1}{4} + \frac{25}{64} + \frac{9}{16} + \frac{49}{64} = \frac{63}{32} \text{ m}$$

Let's look at what we have done graphically. Over each interval we draw a rectangle with height equal to the value of $v(a)$ where a is the left end point of the interval. Observe the area of each rectangle corresponds to the approximate change in distance for that interval.



This approximation is not very accurate as $\Delta t = \frac{1}{4}$ is still quite large.

To get a better approximation, we pick a smaller value of Δt.

Let's try taking $\Delta t = \frac{1}{8}$. We will now have 8 sub-intervals. To use the approximation formula, we need the left end point of each sub-interval.

The left end point of the first sub-interval is 1.

The left end point of the second sub-interval is $1 + \frac{1}{8}$

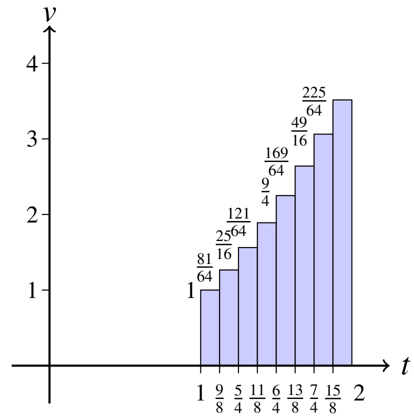
The left end point of the third sub-interval is $1 + \frac{1}{8} + \frac{1}{8}$.

We get the left end point of the i-th sub-interval is

$$t_i = 1 + (i-1)\frac{1}{8}, \qquad \text{for } 1 \le i \le 8$$

Making a rectangle over each sub-interval with height $v(t_i)$ gives the figure on the right.

Finding the area of each rectangle and adding them all together gives

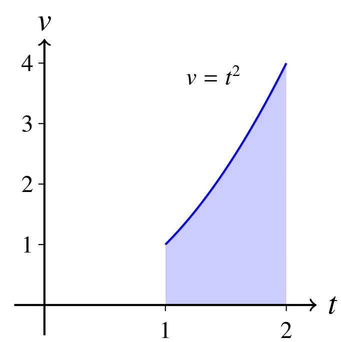


$$\begin{aligned}\Delta s &\approx (1)^2\frac{1}{8} + \left(\frac{9}{8}\right)^2\frac{1}{8} + \left(\frac{5}{4}\right)^2\frac{1}{8} + \left(\frac{11}{8}\right)^2\frac{1}{8} + \left(\frac{6}{4}\right)^2\frac{1}{8} + \left(\frac{13}{8}\right)^2\frac{1}{8} + \left(\frac{7}{4}\right)^2\frac{1}{8} + \left(\frac{15}{8}\right)^2\frac{1}{8}\\ &= \frac{275}{128}\text{ m}\end{aligned}$$

Of course, we can improve the approximation further by taking making Δt smaller and smaller (more and more sub-intervals).

If we were to make Δt infinitely small (turning Δt into dt and making the number of sub-intervals infinitely large), we would get the exact distance. Graphically, this would give the figure on the right.

The key things to observe are that we end up with the graph of $v(t) = t^2$ and that the net change in position is exactly equal to the area between the graph and the t-axis.



In general, if F is a quantity with rate of change $F'(x) = f(x)$, then the net change ΔF of quantity F as x changes from $x = a$ to $x = b$ is equal to the area under the graph of $f(x)$ from $x = a$ to $x = b$.

Since this is amazingly useful, we invent some notation for it.

DEFINITION

Definite Integral

Integrand

If a quantity F has rate of change $F'(x) = f(x)$, then we denote the net change of F from $x = a$ to $x = b$ by

$$\int_a^b f(x)\,dx$$

We call this a **definite integral**. The function f is called the **integrand** of the definite integral.

The Meaning of Mathematics

We again see how versatile mathematics is. By learning how to solve definite integrals, you can solve for the net change of any quantity that you know the rate of change of. Moreover, as we will see in Chapter 5, a definite integral can calculate a lot more than just net change or area.

Our derivation of the definite integral shows that it calculates the net change of a function by summing up infinitely many infinitely small numbers (an infinite number of rectangles which all have infinitely small area). The notation $\int_c^d f(x)\,dx$ is designed to reflect this.

First, the symbol $\int$ is called a 'long S' (which used to be a letter in the English alphabet) and represents the fact that we are **s**umming up all infinitely many of the infinitely small quantities.

Next, if we define $F(x)$ so that $F'(x) = f(x)$, the notation becomes $\int_c^d F'(x)\,dx$. Recalling our work with differentials in Section 1.1.4, we have

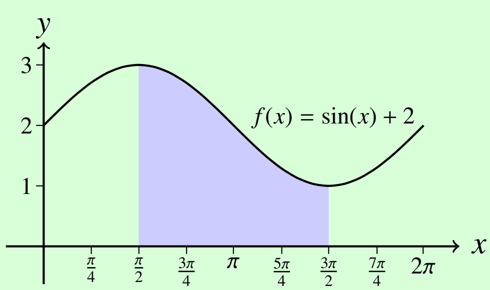
$$F'(x)\,dx = dy$$

That is, the infinitely small quantities that we are summing up are differentials dy.

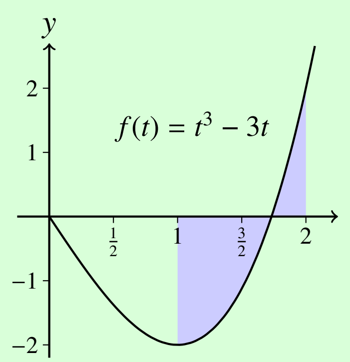
So, the notation $\int_c^d F'(x)\,dx$ represents "the sum of infinitely many differentials'.

EXAMPLE 1 Write the shaded area in the figure as a definite integral.

(a)



(b)



Solution: (a) The shaded area is the area under $f(x) = \sin(x) + 2$ where x runs from $x = \dfrac{\pi}{2}$ to $x = \dfrac{3\pi}{2}$. Thus, the area is represented by $\displaystyle\int_{\pi/2}^{3\pi/2} (\sin(x) + 2)\,dx$.

(b) The shaded area is the area under $f(t) = t^3 - 3t$ where t runs from $t = 1$ to $t = 2$. Thus, the area is represented by $\displaystyle\int_1^2 (t^3 - 3t)\,dt$.

EXAMPLE 2

If $v(t) = 70(1 + \sin(2\pi t))$ mL/s is the rate of blood flowing from the heart, what does $\int_0^5 v(t)\,dt$ represent?

Solution: $\int_0^5 v(t)\,dt$ represents the amount of blood that the heart has pumped out in 5 seconds.

EXAMPLE 3

If $b(t)$ is the yearly birth rate of a population and $d(t)$ is the yearly death rate of the population, then what does $\int_0^2 (b(t) - d(t))\,dt$ represent?

Solution: $\int_0^2 (b(t) - d(t))\,dt$ represents the net change in the population over a 2 year period.

We now look at some examples of how to evaluate a definite integral.

EXAMPLE 4

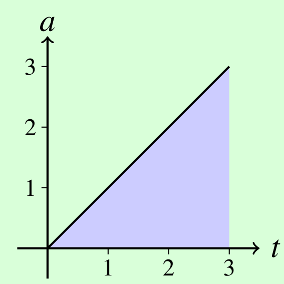
If an object accelerates at a rate of $a(t) = t$ m/s^2 from time $t = 0$ s to $t = 3$ s, what is the change in the object's velocity?

Solution: The net change in velocity is equal to the area under the acceleration graph.

Graphing $a(t) = t$ for $0 \leq t \leq 3$, we see that this is the area of a triangle.

Therefore, the net change in velocity is

$$\Delta v = \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2} \text{ m/s}$$



EXERCISE 1

If an object accelerates at a rate of $a(t) = -t + 1$ m/s^2 from time $t = 0$ s to $t = 3$ s, what is the net change in the object's velocity?

EXERCISE 2

The rate of change of an animal's weight from $t = 0$ years to $t = \sqrt{2}$ years is given by $w'(t) = \sqrt{2 - t^2}$. Given that the graph of $w'(t)$ for $0 \leq t \leq \sqrt{2}$ is one quarter of a circle of radius $\sqrt{2}$, find the net change is the animal's weight.

We can also do this process in reverse. That is, we can find the area under the graph by determining the net change. To do this, we would need to be able to figure out the function F from its rate of change F'.

EXAMPLE 5

What is the area under the graph $F'(x) = 3x^2$ for $x = 0$ to $x = 2$?

Solution: We are given the rate of change F'. We want to find a function F which has this rate of change. We observe that if we take $F(x) = x^3$, then we have $F'(x) = 3x^2$.

Therefore, we have that $F(x) = x^3$ is the amount of quantity for each value x. Hence, the net change in quantity from $x = 0$ to $x = 2$ will be the amount of quantity at $x = 2$, $F(2)$, minus the amount of quantity at $x = 0$, $F(0)$. That is, we get the net change is

$$\Delta F = \int_0^2 F'(x)\,dx = F(2) - F(0) = 2^3 - 0^3 = 8$$

Since the area under the graph is equal to the net change, we get that the area under the graph of $y = F'(x) = 3x^2$ for $x = 0$ to $x = 2$ is 8.

EXERCISE 3

In Example 1.2.5, observe that $F(x) = x^3 + 1$ also satisfies $F'(x) = 3x^2$. What would be the net change in the quantity from $x = 0$ to $x = 2$ if we use this $F(x)$ instead?

Generalizing what we did in Example 1.2.5 gives us the following result.

THEOREM 1

Fundamental Theorem of Calculus (FTC) - Part 2

If $F(x)$ is any function such that $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

NOTATION

When applying the Fundamental Theorem of Calculus, we often use the notation

$$F(x)\Big|_a^b$$

to represent $F(b) - F(a)$. The symbol $\Big|_a^b$ is read as "evaluated from a to b".

REMARK

Don't worry, you didn't miss Part 1 of the Fundamental Theorem of Calculus. We will look at that in Section 2.2.

We demonstrate how to use the Fundamental Theorem of Calculus and the new notation with a couple of examples.

EXAMPLE 6

Evaluate $\int_1^4 2x\,dx$.

Solution: The integrand is $f(x) = 2x$. We need to find a function $F(x)$ such that $F'(x) = 2x$. One choice is $F(x) = x^2$. Thus, we get

$$\int_1^4 2x\,dx = x^2\Big|_1^4 = 4^2 - 1^2 = 15$$

EXAMPLE 7

Evaluate $\int_0^{\pi/4} \cos(x)\,dx$.

Solution: The integrand is $f(x) = \cos(x)$. We need to find a function $F(x)$ such that $F'(x) = \cos(x)$. One choice is $F(x) = \sin(x)$. Thus, we get

$$\int \cos(x)\,dx = \sin(x)\Big|_0^{\pi/4} = \sin\left(\frac{\pi}{4}\right) - \sin(0) = \frac{1}{\sqrt{2}}$$

EXAMPLE 8

If an object has velocity $v(t) = t^2$ m/s^2, what is the net change in position from $t = 1$ s to $t = 2$ s.

Solution: Observe that the function $s(t) = \frac{1}{3}t^3$ satisfies $s'(t) = v(t)$.

Therefore, the net change in the object's position is

$$\Delta s = \int_1^2 v(t)\,dt = \frac{1}{3}t^3\Big|_1^2 = \frac{1}{3}2^3 - \frac{1}{3}1^3 = \frac{7}{3}\text{ m}$$

EXERCISE 4

Evaluate the following definite integrals.

(a) $\int_{-1}^3 5\,dx$ (b) $\int_0^2 e^x\,dx$ (c) $\int_0^{\pi/2} -\sin(x)\,dx$

EXERCISE 5

What is the area under the graph $y = f(x) = x$ from $x = c$ to $x = d$?

EXERCISE 6

If the rate of change of a quantity Q is $Q'(t) = 5t^4$, what is the net change in the quantity from $t = 0$ to $t = 2$?

1.2.2 Antiderivatives

To solve the net change problem we were doing the reverse operation of taking the derivative.

DEFINITION
Antiderivative

For a given function f, any function F such that $F'(x) = f(x)$ is called an **antiderivative** of f.

EXAMPLE 9

Find an antiderivative of $f(x) = x^4$.

Solution: An antiderivative of $f(x) = x^4$ is $F(x) = \frac{1}{5}x^5$ since $F'(x) = f(x)$.

EXAMPLE 10

Find an antiderivative of $g(x) = \sin(x)$.

Solution: An antiderivative of $g(x) = \sin(x)$ is $G(x) = -\cos(x)$ since $G'(x) = g(x)$.

EXAMPLE 11

Find an antiderivative of $h(x) = e^x$.

Solution: An antiderivative of $h(x) = e^x$ is $H(x) = e^x$ since $H'(x) = h(x)$.

REMARK

Observe that you can always check your answer when finding an antiderivative by taking the derivative of your F and making sure that you get f.

It is important to observe that other solutions to the examples above are available. For example, some other correct answers for Example 1.2.9 are

$$F(x) = \frac{1}{5}x^5 + 2$$
$$F(x) = \frac{1}{5}x^5 - \sqrt{2}$$
$$F(x) = \frac{1}{5}x^5 + \frac{\pi}{7}$$

In fact, there are infinitely many correct answers since we know that taking the derivative of a constant gives 0. That is, we could take any function of the form $F(x) = \frac{1}{5}x^5 + C$ for any real number C.

This motivates the following definition.

DEFINITION
Indefinite Integral

If F is any antiderivative of a function f, then the collection of all antiderivatives of f is called the **indefinite integral** of f and is denoted by

$$\int f(x)\,dx = F(x) + C$$

where C, called the **constant of integration**, is an arbitrary real number. The function f is called the **integrand** of the indefinite integral.

The indefinite integral of f gives a formula for *all* antiderivatives of f. Therefore, the '$+C$' is absolutely necessary. In particular, omitting '$+C$' makes the antiderivative look unique, which it is not. Note also that there's nothing special about our choice of the symbol C here; there will also be circumstances in which we will need to define multiple integration constants and hence will need to use different symbols for each.

EXAMPLE 12

Evaluate the indefinite integral $\int 3x^2\,dx$.

Solution: An antiderivative of $f(x) = 3x^2$ is $F(x) = x^3$ since $F'(x) = f(x)$. Therefore, the indefinite integral is

$$\int 3x^2\,dx = x^3 + C$$

EXAMPLE 13

Evaluate the indefinite integral $\int 3\,dt$.

Solution: The dt indicates that this time we have independent variable t. Since an antiderivative of $f(t) = 3$ is $F(t) = 3t$, the indefinite integral is

$$\int 3\,dt = 3t + C$$

EXERCISE 7

Find the indefinite integral. In each case, check your answer.

(a) $\int 2x^3\,dx$

(b) $\int \cos(x)\,dx$

(c) $\int -2\,dw$

(d) $\int (t+1)\,dt$

We can use our table of derivative formulas to make a table of indefinite integrals.

function	**indefinite integral**
$f(x) = 0$	$\int 0\,dx = C$
$f(x) = x^n,\ n \neq -1$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
$f(x) = e^x$	$\int e^x\,dx = e^x + C$
$f(x) = \cos(x)$	$\int \cos(x)\,dx = \sin(x) + C$
$f(x) = \sin(x)$	$\int \sin(x)\,dx = -\cos(x) + C$

Since finding an indefinite integral is essentially the opposite of taking a derivative, indefinite integrals inherit the basic properties of the derivatives.

THEOREM 2 If F is an antiderivative of f and G is an antiderivative of g, then for all real numbers k

$$\int kf(x)\,dx = kF(x) + C$$

$$\int \left(f(x) + g(x)\right)\,dx = F(x) + G(x) + C$$

In cases where the integrand is more complicated, we can try to use algebraic manipulations to convert the integrand into pieces that we know the antiderivative of. Here are two strategies to keep in mind.

1. See if you can apply a property to simplify the integrand.

EXAMPLE 14 Evaluate $\int (x^2+3)^2\,dx$.

Solution: We have

$$\begin{aligned}\int (x^2+3)^2\,dx &= \int (x^4 + 6x^2 + 9)\,dx \\ &= \frac{1}{5}x^5 + 2x^3 + 9x + C\end{aligned}$$

EXERCISE 8 Evaluate $\int (\sin^2(x) + \cos^2(x))\,dx$.

2. If you see a fraction in the integrand, see if you can use the property

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

EXAMPLE 15

Evaluate $\displaystyle\int \frac{t^2 - 2t^4}{t^4}\, dt$.

Solution: We have

$$\begin{aligned}\int \frac{t^2 - 2t^4}{t^4}\, dt &= \int \left(\frac{t^2}{t^4} - \frac{2t^4}{t^4}\right) dt \\ &= \int \left(\frac{1}{t^2} - 2\right) dt \\ &= -\frac{1}{t} - 2t + C\end{aligned}$$

EXERCISE 9

Evaluate $\displaystyle\int \frac{x + \sqrt{x}}{\sqrt[3]{x}}\, dx$

1.2.3 Solving Simple Differential Equations

In Section 1.1.5, we started looking at differential equations. We defined the order of a differential equation and what a solution of a differential equation is. We can now use antiderivatives to solve simple differential equations.

EXAMPLE 16

Find all solutions of the first order differential equation $\dfrac{dy}{dx} = \cos(x)$.

Solution: We want to find all functions y whose derivative is $y' = \cos(x)$. We have

$$y = \int \cos(x)\, dx = \sin(x) + C$$

EXAMPLE 17

Find all solutions of the first order differential equation $\dfrac{dy}{dt} = 8$.

Solution: We want to find all functions y whose derivative is $\dfrac{dy}{dt} = 8$. Observe that the notation tells us that the independent variable is t. Thus, we have

$$y = \int 8\, dt = 8t + C$$

EXAMPLE 18

Find all solutions of the first order differential equation $\dfrac{dy}{dx} = \dfrac{x^2 + \sqrt{x}}{x}$.

Solution: We want to find all functions y whose derivative is $y' = \dfrac{x^2 + \sqrt{x}}{x}$. Since we don't know an antiderivative of this, we simplify it using the property of fractions on page 29. This gives

$$\begin{aligned} y &= \int \frac{x^2 + \sqrt{x}}{x}\, dx \\ &= \int \left(\frac{x^2}{x} + \frac{\sqrt{x}}{x}\right) dx \\ &= \int (x + x^{-1/2})\, dx \\ &= \frac{1}{2}x^2 + 2x^{1/2} + C \end{aligned}$$

EXERCISE 10

Find all solutions of the first order differential equation.

(a) $\dfrac{dy}{dx} = x^2 + x + \dfrac{1}{x^2}$.

(b) $\dfrac{dy}{dt} = -2$.

(c) $\dfrac{dy}{dx} = e^x + 3$.

Finding all solutions to a differential equation can help us to understand a general scientific principle. For example, in physics, one model of the velocity v of a falling object of mass m says that the velocity satisfies the differential equation

$$m\frac{dv}{dt} = mg - kv$$

where g is the acceleration due to gravity and k is a constant corresponding to the force due to friction. The solutions to this differential equation can help us understand the behaviour of falling objects under this model.

However, if you are going skydiving, then you will likely be more interested in what your particular velocity will be than in understanding the general scientific principle. That is, you will want one particular solution to the differential equation that will match with what you will experience.

A differential equation that has initial conditions attached to it is called an **initial value problem.** The initial conditions allow us to find one particular solution that matches the particular situation.

EXAMPLE 19

Solve the initial value problem $y' = \sin(x)$, $y(0) = 5$.

Solution: The solutions to the differential equation $y' = \sin(x)$ are

$$y = \int \sin(x)\,dx$$
$$y = -\cos(x) + C$$

The initial condition says that $y = 5$ when $x = 0$. Hence,

$$5 = -\cos(0) + C$$
$$5 = -1 + C$$
$$6 = C$$

Hence, the solution of the initial value problem is $y(x) = -\cos(x) + 6$.

EXAMPLE 20

Solve the initial value problem $y' = e^t$, $y(0) = \pi$.

Solution: The solutions to the differential equation $y' = e^t$ are

$$y = \int e^t\,dt$$
$$y = e^t + C$$

The initial condition says that $y = \pi$ when $t = 0$. Hence,

$$\pi = e^0 + C$$
$$\pi = 1 + C$$
$$\pi - 1 = C$$

Hence, the solution of the initial value problem is $y(t) = e^t + \pi - 1$.

REMARK

Observe that if you forget the constant of integration (the $+C$), then you won't be able to finish solving initial value problems.

EXERCISE 11

Solve the initial value problem.

(a) $y' = x$, $y(1) = -3$.

(b) $y' = e^x$, $y(3) = 2$.

Section 1.2 Problems

1. Evaluate the indefinite integral.
   (a) $\int (3x - 1)\, dx$
   (b) $\int 3x^2\, dx$
   (c) $\int 4t^2\, dt$
   (d) $\int (y^5 + 1)\, dy$
   (e) $\int \frac{1}{r^2}\, dr$
   (f) $\int 2\cos(s)\, ds$
   (g) $\int e^t\, dt$
   (h) $\int \frac{1}{\sqrt{x}}\, dx$
   (i) $\int (5\sin(x) + 2)\, dx$
   (j) $\int e^{kt}\, dt$
   (k) $\int 4e^x\, dx$
   (l) $\int 0\, dw$
   (m) $\int -2\sin(3x)\, dx$
   (n) $\int e^{x/2}\, dx$

2. Find all solutions of the given differential equation.
   (a) $\frac{dy}{dx} = x$
   (b) $\frac{dy}{dx} = x^3 + 5$
   (c) $\frac{dy}{dx} = 4\sqrt{x}$
   (d) $\frac{dy}{dx} = x^2 + e^x$
   (e) $\frac{dy}{dx} = 3\sin(x) - 1$
   (f) $\frac{dy}{dx} = -x^{-3}$

3. Solving the following initial value problems.
   (a) $\frac{dy}{dx} = x,\ y(1) = 3$
   (b) $\frac{dy}{dx} = x,\ y(1) = -1$
   (c) $\frac{dy}{dx} = 1,\ y(\pi) = \pi$
   (d) $\frac{dy}{dx} = x^{-1/3},\ y(1) = 5$
   (e) $\frac{dy}{dx} = \cos(x),\ y(\pi) = 2$

4. Write the definite integral which represents the area under the graph of the given function over the given interval.
   (a) $f(x) = x + 5$ for $x = -1$ to $x = 3$
   (b) $f(x) = \cos(x)$ for $x = 0$ to $x = \pi$
   (c) $f(x) = e^x$ for $x = -2$ to $x = -1$

5. Evaluate the definite integral.
   (a) $\int_0^2 1\, dx$
   (b) $\int_0^3 x\, dx$
   (c) $\int_0^6 t\, dt$
   (d) $\int_0^x t\, dt$
   (e) $\int_1^2 3x^2\, dx$
   (f) $\int_0^\pi \sin(\theta)\, d\theta$

6. Given $f'(x)$, use a definite integral to find the net change in f from $x = c$ to $x = d$.
   (a) $f'(x) = x^2$ from $x = 1$ to $x = 2$.
   (b) $f'(x) = \sqrt{x}$ from $x = 1$ to $x = 4$.
   (c) $f'(x) = \cos(x)$ from $x = 0$ to $x = \frac{\pi}{2}$.
   (d) $f'(x) = \cos(x)$ from $x = \frac{\pi}{2}$ to $x = \pi$.
   (e) $f'(x) = \cos(x)$ from $x = 0$ to $x = \pi$.

7. Assume a balloon's volume is changing at a rate of 3 cm^3/s from time $t = 1$ s to $t = 3$ s. Use a definite integral to show that the net change in the balloon's volume is 6 cm^3.

8. The rate of rainfall in mm/h can be modelled by the periodic function

$$r(t) = 3\left(1 - \cos\left(\frac{2\pi t}{24}\right)\right)$$

where t is the number of hours after midnight. What does $\int_0^{24} r(t)\,dt$ represent?

9. If an object accelerates at $3t^2$ m/s^2 from time $t = 0$ s to $t = 2$ s, what is the net change in the object's velocity?

10. The velocity of a particle is given by $v(t) = 3t^2 - 6t$ in m/s. Determine the change in the particle's position from $t = 2$ to $t = 3$.

11. If an object has initial position $s(0) = s_0$, initial velocity $s'(0) = v_0$, and constant acceleration $s''(t) = a$, then show its position is given by

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0$$

End of Chapter Problems

1. Determine the second derivative of each function.
   (a) $f(x) = 3x^2 + 2x - 1$
   (b) $g(x) = \sin(x)$
   (c) $h(t) = e^{2t}$
   (d) $g(t) = (1 + t)^{1/4}$
   (e) $m(w) = \dfrac{w}{2w + 1}$
   (f) $f(x) = \cos(x^2)$
   (g) $f(x) = \dfrac{1}{x^2 + 1}$

2. Find the equation of the tangent line of the function at the given value of a.
   (a) $f(x) = \sin(x)$ at $a = 0$
   (b) $f(x) = x^2 - \dfrac{3}{5}x + \dfrac{1}{7}$ at $a = 0$
   (c) $f(x) = (1 + x)^{2/3}$ at $a = 0$
   (d) $f(x) = (1 + x)^{-3/4}$ at $a = 0$
   (e) $f(x) = \dfrac{1}{x^2}$ at $a = -1$
   (f) $f(x) = e^{x^2}$ at $a = 1$

3. Use the tangent line of the function at a to approximate the given value.
   (a) $f(x) = \sqrt{x}$ at $a = 1$; $\sqrt{3}$
   (b) $f(x) = \sqrt{x}$ at $a = 4$; $\sqrt{3}$
   (c) $f(x) = \cos(x)$ at $a = 0$; $\cos(0.1)$
   (d) $f(x) = (1 + x)^{1/3}$ at $a = 0$; $(1.001)^{1/3}$
   (e) $f(x) = (1 + x)^{-1/4}$ at $a = 0$; $(1.01)^{-1/4}$
   (f) $f(x) = (1 + x)^{2/5}$ at $a = 0$; $(0.99)^{2/5}$
   (g) $f(x) = \dfrac{1}{x^2}$ at $a = 1$; $\dfrac{1}{(1.01)^2}$
   (h) $f(x) = e^x$ at $a = 0$; e^2

4. Determine the differential du.
   (a) $u = f(x) = 3x^3 + 4x + 1$
   (b) $u = f(x) = \dfrac{3}{x}$
   (c) $u = f(x) = \sqrt{x}$
   (d) $u = f(x) = e^{3x}$
   (e) $u = f(x) = x^2 + 1$
   (f) $u = f(x) = \sqrt{x^2 + 1}$
   (g) $u = f(x) = \sin(x^2 + x)$

5. Evaluate the indefinite integral.
   (a) $\int 3\,dt$
   (b) $\int (2x+5)\,dx$
   (c) $\int (x^2+1)\,dx$
   (d) $\int \frac{3}{x^2}\,dx$
   (e) $\int 4\sin(x)\,dx$
   (f) $\int \frac{1}{\sqrt{x}}\,dx$
   (g) $\int \cos(2x)\,dx$
   (h) $\int \frac{2}{x^3}\,dx$
   (i) $\int x^{-1/3}\,dx$
   (j) $\int \sin\left(\frac{x}{3}\right)\,dx$
   (k) $\int e^{2x}\,dx$
   (l) $\int x^{2/3}\,dx$

6. Evaluate the definite integral.
   (a) $\int_{-1}^{3} 2\,dx$
   (b) $\int_{0}^{2} x\,dx$
   (c) $\int_{-1}^{3} t\,dt$
   (d) $\int_{1}^{4} \sqrt{x}\,dx$
   (e) $\int_{0}^{x} e^t\,dt$
   (f) $\int_{1}^{2} (x^2+1)\,dx$
   (g) $\int_{0}^{\pi/2} \cos(\theta)\,d\theta$
   (h) $\int_{0}^{\pi} \sin(2\theta)\,d\theta$
   (i) $\int_{1}^{3} \left(\frac{1}{x^2}+2x\right)\,dx$

7. State the order of the differential equation.
   (a) $y' = 10 - 3y$
   (b) $y'' + 10y' + y^3 = x^3$
   (c) $y' + x^2y = x^{-3}$
   (d) $y'' + (y')^3 = 0$
   (e) $y''' = (xy)^2 + 1$

8. Determine whether the given function is a family of solutions of the given differential equation.
   (a) $y = x^2 + C$ for $y' = 2x$
   (b) $y = -2x + C$ for $y' = 2x + y$
   (c) $y = -2x - 2 + Ce^x$ for $y' = 2x + y$
   (d) $y = x^3 + Cx^2$ for $y' = \frac{2}{x}y + x^2$
   (e) $y = \frac{5}{2} + Ce^{-2x}$ for $y' = 5 - 2y$

9. Find all solutions of the given differential equation.
   (a) $y' = 3x^2 + x + 1$
   (b) $y' = x^3 - 5$
   (c) $y' = x^{1/3}$
   (d) $y' = x + e^x$
   (e) $y' = \cos(2x+1)$
   (f) $y' = 5\sin(x)$

10. Solving the following initial value problems.
    (a) $y' = x + 1$, $y(0) = 0$
    (b) $y' = 2x^2$, $y(1) = 2$
    (c) $y' = 5$, $y(1) = 2$
    (d) $y' = x^{-1/4}$, $y(1) = 1$
    (e) $y' = \sin(x)$, $y(\pi) = 1$
    (f) $y' = 1 - 3e^x$, $y(0) = 5$

11. Given $f'(t)$, use a definite integral to find the net change in f from $t = c$ to $t = d$.
    (a) $f'(t) = 2t + 1$ from $t = 1$ to $t = 2$.
    (b) $f'(t) = \frac{1}{\sqrt{t}}$ from $t = 4$ to $t = 9$.
    (c) $f'(t) = \sin(2t)$ from $t = 0$ to $t = \frac{\pi}{4}$.
    (d) $f'(t) = \frac{1}{t^2}$ from $t = 1$ to $t = 2$.
    (e) $f'(t) = e^{3t}$ from $t = 0$ to $t = 1$.

Chapter 2: Important Functions

Section 2.1: Piecewise Defined Functions

LEARNING OUTCOMES

1. Know how to evaluate and graph piecewise defined functions.
2. Know how to convert a function with absolute values into a piecewise defined function.
3. Know how to find the interval defined by an absolute value function.
4. Know how to find the derivative of the absolute value function.

2.1.1 Piecewise Defined Functions

In many real world situations, the behavior of a quantity can change dramatically depending on the input. For example, the changing of a state (a light switch being turned on, a car changing gears), or some threshold being reached (dropping below critical mass in a nuclear reaction, a bridge exceeding its load strength). Such situations often require functions where the domain is divided into parts and each part is defined by a different formula. Such functions are called **piecewise defined functions**. We will call the values where a piecewise defined function changes from one definition to another **transition points**.

EXAMPLE 1

When a light switch is off, no electricity flows to the light bulb. Suppose that at time $t = a$, a light switch is turned on and 110 V of electricity flows to the light bulb. Write a mathematical equation for the voltage supplied to the light bulb.

Solution: Before the transition point at $t = a$, there is no electricity flowing to the light bulb. After $t = a$, there is 110 V flowing to the light bulb. Thus, we get the piecewise defined function

$$p(t) = \begin{cases} 0 & \text{if } t < a \\ 110 & \text{if } t > a \end{cases}$$

EXAMPLE 2

In Yellowstone National Park, the number of beaver colonies from 1994 to 1998 was constant at 1. In 1998, wolves were reintroduced to Yellowstone National Park, resulting in a roughly linear increase in the number of beaver colonies to 19 in 2015. Write a mathematical equation for the number of beaver colonies from 1994 to 2015.

Solution: Let $B(t)$ denote the number of beaver colonies where t is the number of years after 1994 (that is, $t = 0$ corresponds to 1994).

We are given that $B(t) = 1$ for $0 \leq t \leq 4$, and that we we have a transition point at $t = 4$ (corresponding to 1998).

We are told $B(t)$ is increasing linearly for $4 < t \leq 21$ $B(t)$. So, we model it using the point-slope form of a line. We get that the slope is

$$m = \frac{B(21) - B(4)}{21 - 4} = \frac{19 - 1}{17} = \frac{18}{17}$$

Hence,

$$\begin{aligned} B(t) &= y_0 + m(t - t_0) \\ &= 1 + \frac{18}{17}(t - 4) \end{aligned}$$

Thus, the number of beaver colonies is

$$B(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 4 \\ 1 + \frac{18}{17}(t - 4) & \text{if } 4 < t \leq 21 \end{cases}$$

EXERCISE 1

Suppose the temperature of a room is 20°C. At time $t = 0$, the heating system is turned on and the temperature increases by 2°C per hour until it reaches 24°C. At that point, the cooling system turns on and the temperature decreases by 1°C per hour until it goes back to 20°C. Represent this situation with a piecewise defined function.

EXERCISE 2

Let f be the function defined by $f(x) = \begin{cases} x & \text{if } x < 0 \\ -x + 2 & \text{if } 0 \leq x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$.

Sketch the graph of $y = f(x)$ over the interval $[-2, 3]$.

EXERCISE 3

Let g be the function defined by $g(x) = \begin{cases} 3 & \text{if } x \leq -2 \\ -x & \text{if } -2 < x < 2 \\ \sqrt{x} & \text{if } x \geq 2 \end{cases}$

(a) What is the domain of g?

(b) What is the value of $g(-2)$, $g(-1)$, $g(1)$, $g(2)$, and $g(3)$?

(c) Sketch a graph of $y = g(x)$.

EXERCISE 4

Let $A(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$.

Evaluate $A(3)$ and $A(-1)$.

2.1.2 Absolute Value Function

One very important piecewise defined functions is the absolute value function. This function will occur regularly throughout this text.

DEFINITION

Absolute Value Function

We define the absolute value function $|\cdot|$ by

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

EXAMPLE 3

Evaluate $|-3|$ using the definition.

Solution: Since $-3 < 0$, we use the first part of the definition to get

$$|-3| = -(-3) = 3$$

You may be wondering what the '$-x$' part is all about in the first place... many students are taught that the absolute value signs just make whatever is inside them positive. While this is essentially true, thinking about it this way leads to difficulties when the inside contains a variable.

For example, $|-x^3|$ does not simplify to x^3. In particular, if $x = -1$, then $|-x^3|$ evaluates to

$$|(-1)^3| = |-1| = -(-1) = 1$$

but x^3 evaluates to

$$(-1)^3 = -1$$

So, $|-x^3|$ does not equal x^3.

Taking the time to understand the above definition of the absolute value function will make the rest of this section much easier.

EXERCISE 5

Use the definition to evaluate $|-5|$, $|3|$, and $|-\sqrt{2}|$.

We will need the following properties of the absolute value function.

Properties of Absolute Values:

(i) $|x| = \sqrt{x^2}$

(ii) $|xy| = |x|\,|y|$

(iii) $|x| = a$ if and only if $x = \pm a$, for any $a \geq 0$

(iv) $|x| < a$ if and only if $-a < x < a$, for any $a > 0$

(v) $|x| > a$ if and only if $x < -a$ or $x > a$, for any $a > 0$

For now, we just look at a couple examples of property (i). We will look at the other properties soon.

EXAMPLE 4

Verify that $|x| = \sqrt{x^2}$ for $x = -2$.

Solution: On the left hand side, we use the definition of the absolute value function. Since $-2 < 0$, we get

$$|-2| = -(-2) = 2$$

On the right hand side, we get

$$\sqrt{(-2)^2} = \sqrt{4} = 2$$

Thus, $|-2| = \sqrt{(-2)^2}$.

EXAMPLE 5

Simplify $\sqrt{t^{2/3}}$ where $t < 0$.

Solution: By properties of exponents, we get

$$\sqrt{t^{2/3}} = \sqrt{(t^{1/3})^2}$$

The temptation would be to just cancel the squared with the square root. However, this would give the wrong answer (try it with $t = -1$). Instead, we must use property (i) which gives

$$\sqrt{t^{2/3}} = \sqrt{(t^{1/3})^2} = \left|t^{1/3}\right|$$

Since $t^{1/3} < 0$ when $t < 0$, by definition of the absolute value function, we get

$$\sqrt{t^{2/3}} = -t^{1/3}$$

EXERCISE 6

Simplify $\sqrt{x^4 + x^2}$ where $x < 0$.

EXERCISE 7

Explain why you do not need to use absolute value signs when simplifying $\left(\sqrt{x}\right)^2$.

Converting Functions Involving Absolute Values

What can make the absolute value function tricky to use in some situations is that it is a piecewise defined function in disguise. In these situations, it can be helpful to rewrite functions involving absolute values as piecewise defined functions.

The method for converting a function f containing absolute values into a piecewise defined function is to find the transition point(s) of any absolute value function contained in f and then apply the definition of absolute values.

The transition point(s) of an absolute value function are the values of the independent variable that make the inside of the absolute value function equal to 0 (as the transition point of $|x|$ is $x = 0$).

EXAMPLE 6 Rewrite $f(x) = |x + 2|$ without an absolute value sign.

Solution: The inside of $|x + 2|$ equals 0 when $x + 2 = 0$, so when $x = -2$. Thus, the transition point is $x = -2$. Therefore, we have two cases, $x < -2$ and $x \geq -2$.

Case 1: $x < -2$

When $x < -2$, the inside of $|x + 2|$ is negative, so, by definition of the absolute value function, we get

$$f(x) = -(x + 2)$$

Case 2: $x >= -2$

When $x >= -2$, the inside of $|x + 2|$ is non-negative, so by definition of the absolute value function we get

$$f(x) = x + 2$$

Consequently,

$$|x + 2| = \begin{cases} -(x + 2) & \text{if } x < -2 \\ x + 2 & \text{if } x \geq -2 \end{cases}$$

REMARK

When first learning these, it is highly recommended that you check several values of x to try to ensure your answer is correct. If you have made a mistake, finding a value of x for which your answer doesn't work can help you find and fix your mistake.

EXAMPLE 7 Rewrite $f(x) = \left| -x^3 \right|$ without an absolute value sign.

Solution: The transition point is when $-x^3 = 0$, so when $x = 0$.

Case 1: $x \leq 0$

When, $x \leq 0$, we get $-x^3 \geq 0$. So,

$$\left| -x^3 \right| = -x^3$$

Case 2: $x > 0$

When, $x > 0$, we get $-x^3 < 0$. So,

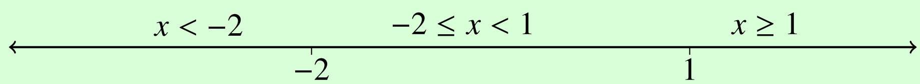
$$\left| -x^3 \right| = -(-x^3) = x^3$$

Therefore,

$$\left| -x^3 \right| = \begin{cases} -x^3 & \text{if } x \leq 0 \\ x^3 & \text{if } x > 0 \end{cases}$$

EXAMPLE 8 Rewrite $f(x) = 3|x+2| - 2|x-1|$ without absolute value signs.

Solution: This function has two different absolute value functions. So, we need to find the transition points for each one. The transition point for $|x+2|$ is $x = -2$ and the transition point of $|x-1|$ is $x = 1$. Since there is more than one transition point, to determine the cases, we label the transition points on a number line.



The number line indicates the cases we will have for the piecewise defined function. Then we just need to evaluate f in each case using the definition of the absolute value function.

Case 1: $x < -2$.

When $x < -2$, the inside of both $|x+2|$ and $|x-1|$ are negative. So, we get

$$|x+2| = -(x+2) \quad \text{and} \quad |x-1| = -(x-1)$$

Thus, we have

$$f(x) = 3|x+2| - 2|x-1| = 3(-(x+2)) - 2(-(x-1)) = -3x - 6 - (-2x+2) = -x - 8$$

Case 2: $-2 \leq x < 1$.

When $x \geq -2$, the inside of $|x+2|$ is non-negative, so $|x+2| = x+2$.

When $x < 1$, the inside of $|x-1|$ is negative. So, $|x-1| = -(x-1)$.

Thus, in this case we have

$$f(x) = 3|x+2| - 2|x-1| = 3(x+2) - 2(-(x-1)) = 3x + 6 - (-2x+2) = 5x + 4$$

Case 3: $x \geq 1$.

When $x \geq 1$, the inside of both $|x+2|$ and $|x-1|$ will be non-negative, so $|x+2| = x+2$ and $|x-1| = x-1$. Thus, in this case we have

$$f(x) = 3|x+2| - 2|x-1| = 3(x+2) - 2(x-1) = 3x + 6 - (2x-2) = x + 8$$

Therefore, we get

$$f(x) = \begin{cases} -x - 8 & \text{if } x < -2 \\ 5x + 4 & \text{if } -2 \leq x < 1 \\ x + 8 & \text{if } x \geq 1 \end{cases}$$

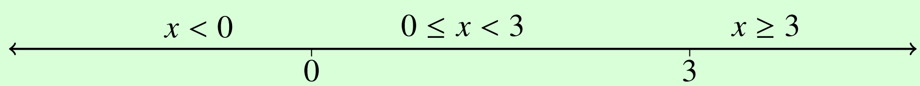
EXAMPLE 9 Rewrite $f(x) = |x^2 - 3x|$ without an absolute value sign.

Solution: We first rewrite this using property (ii) $|xy| = |x||y|$. In particular, we get

$$f(x) = |x^2 - 3x| = |x(x-3)| = |x|\,|x-3|$$

The transition point for $|x|$ is $x = 0$ and the transition point for $|x-3|$ is $x = 3$.

We draw a number line and label the two transition points to determine the cases.



Case 1: $x < 0$.

When $x < 0$, the inside of both $|x|$ and $|x-3|$ will be negative. So, we get that

$$|x| = -x \quad \text{and} \quad |x-3| = -(x-3)$$

Thus, we have

$$f(x) = |x|\,|x-3| = (-x)(-(x-3)) = x^2 - 3x$$

Case 2: $0 \le x < 3$.

When $x \ge 0$, the inside of $|x|$ is non-negative, so $|x| = x$.

When $x < 3$, the inside of $|x-3|$ is negative. So, $|x-3| = -(x-3)$.

Thus, we have

$$f(x) = |x|\,|x-3| = (x)(-(x-3)) = -x^2 + 3x$$

Case 3: $x \ge 3$.

When $x \ge 3$, the inside of both $|x|$ and $|x-3|$ will be non-negative, so $|x| = x$ and $|x-3| = x-3$. Thus, we have

$$f(x) = |x|\,|x-3| = (x)(x-3) = x^2 - 3x$$

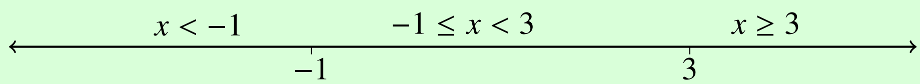
Therefore, we get

$$f(x) = \begin{cases} x^2 - 3x & \text{if } x < 0 \\ -x^2 + 3x & \text{if } 0 \le x < 3 \\ x^2 - 3x & \text{if } x \ge 3 \end{cases}$$

EXAMPLE 10 Rewrite $f(x) = x|x+1| + 5|x-3|$ without absolute value signs.

Solution: The transition point for $|x+1|$ is $x = -1$ and the transition point for $|x-3|$ is $x = 3$.

We draw a number line and label the two transition points to determine the cases.



Case 1: $x < -1$.

Since $x < -1$, then the inside of both $|x+1|$ and $|x-3|$ will be negative. So,

$$|x+1| = -(x+1) \quad \text{and} \quad |x-3| = -(x-3)$$

Thus, we have

$$f(x) = x|x+1|+5|x-3| = x(-(x+1))+5(-(x-3)) = -x^2-x+(-5x+15) = -x^2-6x+15$$

Case 2: $-1 \le x < 3$.

Since $x \ge -1$, then the inside of $|x+1|$ is non-negative, so $|x+1| = x+1$.

Since $x < 3$, the inside of $|x-3|$ is negative. So, $|x-3| = -(x-3)$.

Thus, in this case we have

$$f(x) = x|x+1|+5|x-3| = x(x+1)+5(-(x-3)) = x^2+x+(-5x+15) = x^2-4x+15$$

Case 3: $x \ge 3$.

Since $x \ge 3$, then the inside of both $|x+1|$ and $|x-3|$ will be non-negative, so $|x+1| = x+1$ and $|x-3| = x-3$. Thus, in this case we have

$$f(x) = x|x+1| + 5|x-3| = x(x+1) + 5(x-3) = x^2 + x + 5x - 15 = x^2 + 6x - 15$$

Thus, we get

$$f(x) = \begin{cases} -x^2 - 6x + 15 & \text{if } x < -1 \\ x^2 - 4x + 15 & \text{if } -1 \le x < 3 \\ x^2 + 5x - 15 & \text{if } x \ge 3 \end{cases}$$

EXERCISE 8

Rewrite the function without absolute value signs.

(a) $f(x) = |x - 3|$

(b) $f(x) = |(x - 2)(x + 1)|$

(c) $f(x) = x^2|x + 3| - x|x - 1|$

Absolute Value and Error

One of the main uses of calculus in science is to help us approximate values that would be otherwise very difficult or even impossible to calculate. Whenever we do an approximation, we would like to have an upper bound for the size of the error in the approximation. For this, we often use the absolute value function in the form

$$|a - b| < r$$

to say that an upper bound for the error when a is approximated by b is less than r. To work with such an equation we use property (iv) of the absolute value function:

$$|x| < a \text{ if and only if } -a < x < a.$$

For example, the notation

$$\left|\sqrt{2} - 1.4\right| < 0.02$$

means that "the error in approximating the number $\sqrt{2}$ by 1.4 is less than 0.02". Using property (iv) gives

$$-0.02 < \sqrt{2} - 1.4 < 0.02$$

If we now add 1.4 to all three sides, we get bounds on the value of $\sqrt{2}$.

$$\begin{aligned} 1.4 - 0.02 < \sqrt{2} > 1.14 + 0.02 \\ 1.38 < \sqrt{2} < 1.42 \end{aligned}$$

Let's look at a few more examples.

EXAMPLE 11

Find all values of x such that $|x - 3| < 2$.

Solution: Property (iv) gives

$$\begin{aligned} -2 < x - 3 < 2 \\ -2 + 3 < x < 2 + 3 \\ 1 < x < 5 \end{aligned}$$

Thus, all values of x are $-1 < x < 5$.

EXAMPLE 12

Find all values of x such that $|2x + 1| < 1$.

Solution: Property (iv) gives

$$\begin{aligned} -1 &< 2x + 1 < 1 \\ -2 &< 2x < 0 \\ -1 &< x < 0 \end{aligned}$$

Thus, all values of x are $-1 < x < 0$.

EXAMPLE 13

Find all values of x such that $x^2 > 2$.

Solution: Taking the square root of both sides gives

$$\begin{aligned} \sqrt{x^2} &> \sqrt{2} \\ |x| &> \sqrt{2} \end{aligned}$$

Thus, by property (v), we get $x < -\sqrt{2}$ or $x > \sqrt{2}$.

EXERCISE 9

Find all values of x satisfying the inequality.

(a) $|x + 2| < 4$

(b) $|3x + 2| > 3$

(c) $\left|2x^3\right| < 1$

Derivative of the Absolute Value Function

We can use property (i), $|x| = \sqrt{x^2}$, and the Chain Rule to get the derivative of the absolute value function. In particular, we get

$$\frac{d}{dx}|x| = \frac{d}{dx}(x^2)^{1/2} = \frac{1}{2}(x^2)^{-1/2} \cdot (2x) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$$

We add this to our list of derivatives to memorize.

function	**derivative**
$f(x) = \|x\|$	$f'(x) = \frac{x}{\|x\|}$

EXAMPLE 14

Determine the derivative of $f(x) = |\sin(x)|$.

Solution: We get

$$f'(x) = \frac{\sin(x)}{|\sin(x)|} \cdot \cos(x)$$

EXAMPLE 15

Determine the derivative of $f(x) = x|x|$.

Solution: We get

$$\begin{aligned} f'(x) &= 1 \cdot |x| + x \cdot \frac{x}{|x|} \\ &= |x| + \frac{x^2}{\sqrt{x^2}} \\ &= |x| + \sqrt{x^2} \\ &= |x| + |x| \\ &= 2|x| \end{aligned}$$

EXAMPLE 16

Determine the derivative of $f(x) = \dfrac{x}{|x|}$.

Solution: We get

$$\begin{aligned} f'(x) &= \frac{1 \cdot |x| - x \cdot \frac{x}{|x|}}{|x|^2} \\ &= \frac{|x| - \frac{x^2}{\sqrt{x^2}}}{|x|^2} \\ &= \frac{\sqrt{x^2} - \sqrt{x^2}}{|x|^2} \\ &= 0 \end{aligned}$$

EXERCISE 10

Determine the derivative of each function.

(a) $f(x) = |x^2 + \cos(x)|$

(b) $g(x) = |e^x - 1|$

Section 2.1 Problems

1. Let $f(x) = \begin{cases} 2 & \text{if } -2 \le x < 0 \\ \dfrac{1}{x} & \text{if } 1 < x \le 2 \\ x-3 & \text{if } 2 < x \le 3 \end{cases}$
   (a) Find $f(-1)$, $f(2)$, and $f(2.5)$.
   (b) Determine the domain of f.
   (c) Sketch the graph of f.
   (d) Use the graph to find the range of f.

2. State the domain and sketch the graph of the following function.
   (a) $f(x) = \begin{cases} 1-x & \text{if } -2 < x < 0 \\ x^3 & \text{if } 0 \le x \le 1 \\ 2^{-x} & \text{if } 1 < x \le 2 \end{cases}$
   (b) $f(x) = \begin{cases} (x+1)^2 & \text{if } -2 \le x \le 0 \\ x+1 & \text{if } 0 < x < 2 \\ \sqrt{11-x} & \text{if } 2 < x \le 4 \end{cases}$
   (c) $f(x) = \begin{cases} \dfrac{1}{x+1} & \text{if } -2 \le x < -1 \\ \dfrac{1}{x^2-1} & \text{if } -1 < x < 1 \\ \dfrac{1}{x+1} & \text{if } 1 < x \le 2 \end{cases}$

3. Rewrite each of the following without absolute value signs.
   (a) $f(x) = |1-x|$
   (b) $f(x) = |x^2-1|$
   (c) $f(x) = |(x-3)(x+2)|$
   (d) $f(x) = |(x+1)(x+4)|$
   (e) $f(x) = \sqrt{|x+3|}$
   (f) $f(x) = \sin(x)|x+1|$
   (g) $f(x) = e^{|x|}|x-2|$
   (h) $f(x) = 2|x+3| - 5|x-2|$
   (i) $f(x) = 3|x-1| + x|x+2|$
   (j) $f(x) = x^2|x| + (x+1)|x+1|$

4. Simplify $\sqrt{(x-1)^2}$ where $x > 1$.

5. Simplify $\sqrt{(x-1)^2}$ where $x < 1$.

6. Find all values x satisfying the equation.
   (a) $|x| = 3$
   (b) $|x| = \sqrt{2}$
   (c) $|x| < \dfrac{1}{5}$
   (d) $|x| > 1$
   (e) $|x| \le 2$
   (f) $|x-2| < 0.1$
   (g) $|x-1| > 5$
   (h) $|x+3| < 1$
   (i) $|x+2| \ge \dfrac{3}{2}$
   (j) $|x+1| \le \dfrac{1}{2}$
   (k) $|3-x| < 0.2$
   (l) $|x^2-1| < \dfrac{1}{2}$
   (m) $|x| < -2$

7. Consider $x\sqrt{1+\dfrac{1}{x^2}}$ where $x < 0$.
   (a) Simplify the expression by factoring out a $\dfrac{1}{x^2}$ from inside the square root.
   (b) Simplify the expression by moving the x inside the square root.

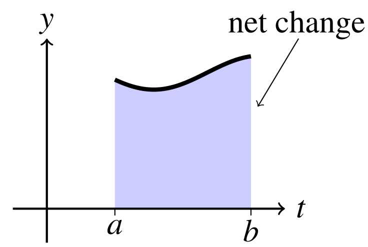
8. Determine the derivative of the following functions.
   (a) $f(x) = |3x+7|$
   (b) $f(x) = |1-x|$
   (c) $f(x) = |x^2-1|$
   (d) $f(x) = |1-\sqrt{x}|$
   (e) $f(x) = |\cos(x)|$
   (f) $f(x) = x\left|\dfrac{1}{x^2+1}\right|$
   (g) $f(x) = \dfrac{|x+1|}{x}$

Section 2.2: Integral Functions

LEARNING OUTCOMES

1. Know how to approximate values of an integral function.
2. Know how to compose a function with an integral function.
3. Know how to find the derivative of an integral function.

2.2.1 Introduction to Integral Functions



We saw in Section 1.2 that if we know the rate of change of a quantity $G(t)$ is $G'(t) = g(t)$, then the net change of G as t changes from $t = a$ to $t = b$ is equal to the area under the graph of $g(t)$.

The notation for this is $\Delta G = \int_a^b g(t)\,dt$.

Now, rather than just calculating the net change over a fixed interval, we look at how to turn this into a function that will calculate the net change from a fixed starting point to any ending point. In particular, if we define

$$f(x) = \int_a^x G'(t)\,dt$$

then the value of $f(x)$ is the net change in the quantity G from $t = a$ to $t = x$.

Since such functions occur regularly, we give them a name.

DEFINITION
Integral Function

A function of the form $f(x) = \int_a^x g(t)\,dt$ is called an **integral function**.

When working with integral functions $f(x) = \int_a^x g(t)\,dt$, it is very important to distinguish between the independent variable of the function x and the variable of integration t.

EXAMPLE 1

If $v(t)$ is the velocity of an object in m/s at time t in seconds, then the definite integral

$$\int_a^b v(t)\,dt$$

is the net change in position from time $t = a$ to $t = b$. The units, as expected, are meters since the units of $v(t) \cdot dt$ are $\frac{\text{m}}{\text{s}} \cdot \text{s} = \text{m}$.

Thus, the integral function

$$d(x) = \int_a^x v(t)\,dt$$

is a net distance-travelled function. It represents the net distance travelled in meters by the object from time $t = a$ to any time $t = x$.

EXAMPLE 2 Let $r(t)$ be the rate in mmol/hr at which urea is produced at time t in hours in a patient who kidneys are failing. What does the definite integral function $U(x) = \int_0^x r(t)\,dt$ represent?

Solution: Since $r(t) \geq 0$, $U(x)$ represents the total amount of urea produced in the x hours since time $t = 0$.

EXERCISE 1 Suppose $\frac{dL}{dt}$ represents the growth rate during month t of an organism measured in cm/month. Let

$H(x) = \int_3^x \frac{dL}{dt}dt$. What does $H(8)$ represent?

EXAMPLE 3 Let $f(x) = \int_1^x t\,dt$. Evaluate $f(2)$, $f(4)$, and $f(x)$.

Solution: An antiderivative of $g(t) = t$ is $G(t) = \frac{1}{2}t^2$. Thus,

$$f(2) = \int_1^2 t\,dt = \frac{1}{2}t^2\Big|_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = \frac{3}{2}$$
$$f(4) = \int_1^4 t\,dt = \frac{1}{2}t^2\Big|_1^4 = \frac{1}{2}(4)^2 - \frac{1}{2}(1)^2 = \frac{15}{2}$$
$$f(x) = \int_1^x t\,dt = \frac{1}{2}t^2\Big|_1^x = \frac{1}{2}(x)^2 - \frac{1}{2}(1)^2 = \frac{1}{2}x^2 - \frac{1}{2}$$

EXERCISE 2 The rate of change of a quantity Q is given by

$$Q'(r) = 3r^2$$

Write an integral function that represents the net change in Q from $r = 1$ to $r = x$. Evaluate the integral function to get an equation for Q in terms of x.

EXERCISE 3 The growth rate of a population in thousands at time t years is given by $n(t) = e^{-t}$.

(a) Let $P(x)$ be the net change in population size between time $t = 0$ and time $t = x$. Write $P(x)$ as an integral function.

(b) Evaluate $P(1)$, $P(5)$ and $P(x)$.

You may be wondering why we would use an integral function if we can just evaluate the definite integral to get an explicit formula for $f(x)$ like we did in the example and exercises above. The reason is that there are several very important integral functions that cannot be evaluated explicitly. For example, the error function

$$\text{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2}\, dt$$

and the sine integral function

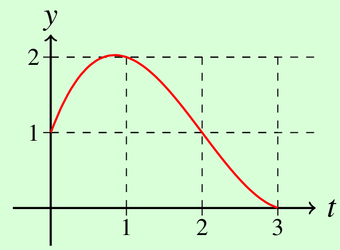
$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt$$

Since these cannot be evaluated explicitly, we can only approximate their values. One way of doing this is to use the method we saw in Section 1.2.1.

EXAMPLE 4 The graph of a function g is given. Let

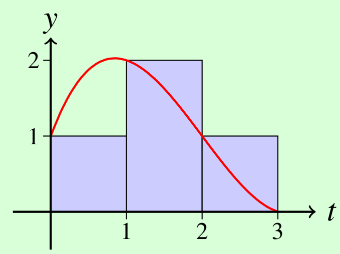
$$f(x) = \int_0^x g(t)\, dt$$

Use 3 subdivisions to approximate $f(3)$.



Solution: We subdivide the interval $[0, 3]$ into 3 equal pieces. So, the length of each subdivision is

$$\Delta x = \frac{3-0}{3} = 1$$



Over each interval, we draw a rectangle with height equal to the value of $g(a)$ where a is the left end point of the interval.

From the given graph of g, we see that the first rectangle has height $g(0) = 1$, the second rectangle has height $g(1) = 2$, and the third rectangle has height $g(2) = 1$.

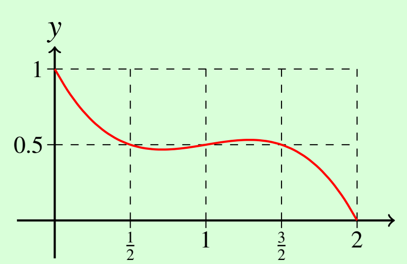
Thus,

$$\begin{aligned} f(3) &\approx g(0)\Delta t + g(1)\Delta t + g(2)\Delta t \\ &= 1 \cdot 1 + 2 \cdot 1 + 1 \cdot 1 \\ &= 4 \end{aligned}$$

EXAMPLE 5 The graph of a function g is given. Let

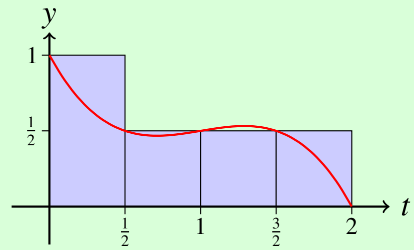
$$f(x) = \int_0^x g(t)\, dt$$

Use 4 subdivisions to approximate $f(2)$.



Solution: We subdivide the interval $[0, 2]$ into 4 equal pieces. So, the length of each subdivision is $\Delta x = \dfrac{2-0}{4} = \dfrac{1}{2}$.

Over each interval, we draw a rectangle with height equal to the value of $g(a)$ where a is the left end point of the interval.



From the given graph of g, we see that the first rectangle has height $g(0) = 1$,

the second rectangle has height $g\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$,

and the third rectangle has height $g(1) = \dfrac{1}{2}$.

and the fourth rectangle has height $g\left(\dfrac{3}{2}\right) = \dfrac{1}{2}$.

Thus,

$$\begin{aligned} f(2) &\approx g(0)\Delta t + g\left(\frac{1}{2}\right)\Delta t + g(1)\Delta t + g\left(\frac{3}{2}\right)\Delta t \\ &= 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \\ &= \frac{5}{4} \end{aligned}$$

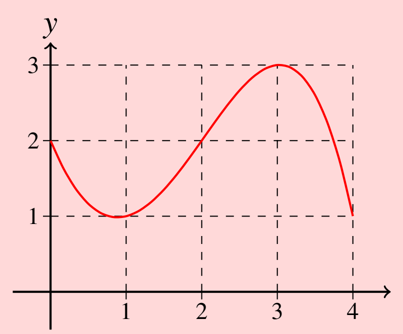
EXERCISE 4

The graph of a function g is given.

Let $f(x) = \displaystyle\int_1^x g(t)\,dt$.

(a) Use 3 subdivisions to approximate $f(4)$.

(b) Use 2 subdivisions to approximate $f(3)$.



EXERCISE 5

Some values of a function g are given in the table.

Let $f(x) = \displaystyle\int_1^x g(t)\,dt$.

Use 4 subdivisions to approximation $f(3)$.

t	1	1.5	2.0	2.5
$g(t)$	3	3.2	3.4	3.2

EXERCISE 6

Some values of a function g are given in the table.

Let $f(x) = \displaystyle\int_0^x g(t)\,dt$.

Use 5 subdivisions to approximation $f(0.5)$.

t	0	0.1	0.2	0.3	0.4
$g(t)$	1	1.01	1.04	1.09	1.17

Composition with Integral Functions

Since integral functions are just functions, we can perform operations on them just like any other function. However, composing a function into an integral function looks a little weird, so let's look at a couple of examples.

EXAMPLE 6

Let $f(x) = \int_1^x t\sin(t)\,dt$ and $g(x) = \cos(x)$. Determine $(f \circ g)(x)$.

Solution: When we do a composition of functions, we replace all the x's in the outside function with the definition of the inside function. In the integral function, the only x is the top bound of the definite integral. Therefore, we get

$$f(g(x)) = \int_1^{\cos(x)} t\sin(t)\,dt$$

EXAMPLE 7

Let $g(x) = \int_{-2}^x e^{t^2}\,dt$ and $h(x) = x + 1$. Determine $(g \circ h)(x)$.

Solution: We get

$$g(h(x)) = \int_{-2}^{x+1} e^{t^2}\,dt$$

EXERCISE 7

Determine $f(x) = (g \circ h)(x)$.

(a) $g(x) = \int_0^x te^t\,dt$, $h(x) = x^2 + x$.

(b) $g(x) = \int_3^x \frac{1}{t^2+1}\,dt$, $h(x) = e^x$.

EXERCISE 8

Given the integral function $g(x)$, what function $h(x)$ is required so that $f(x) = (g \circ h)(x)$?

(a) $g(x) = \int_{-2}^x \sin(t^2)\,dt$; $f(x) = \int_{-2}^{x^2} \sin(t^2)\,dt$

(b) $g(x) = \int_1^x \frac{e^t}{t}\,dt$; $f(x) = \int_1^{x^2+x} \frac{e^t}{t}\,dt$

2.2.2 Derivative of an Integral Function

To figure out how to take the derivative of an integral function, we look at some examples.

EXAMPLE 8 Assume at any time t the position of an object is given by $s(t)$ and the velocity of the object is given by $v(t)$. Recall that the area under the velocity curve over $[a, x]$ is the net distance travelled from time $t = a$ to time $t = x$. That is,

$$\Delta s(x) = \int_a^x v(t)\, dt$$

Taking the derivative with respect to x of both sides gives

$$\Delta s'(x) = \frac{d}{dx} \int_a^x v(t)\, dt$$

The instantaneous rate of change of the net distance travelled function $\Delta s(x)$ is the object's instantaneous velocity at time $t = x$, $v(x)$. So, we have

$$v(x) = \frac{d}{dx} \int_a^x v(t)\, dt$$

EXAMPLE 9 Let $r(t)$ be the rate at which urea is produced in a patient without functional kidneys.

(a) What does $G(x) = \int_0^x r(t)\, dt$ represent?

(b) What does $G'(x)$ represent?

Solution: (a) $G(x) = \int_0^x r(t)\, dt$ represents the total amount of urea in the body x hours since the last dialysis.

(b) $G'(x)$ represents the rate of change of the total amount of urea. That is, it is the rate at which urea is produced. Hence, $G'(x) = r(x)$.

We see that, not surprisingly, the derivative essentially 'cancels out' the definite integral. Importantly, we notice that the independent variable of the answer must be the same as the independent variable of the function we took the derivative of.

What we have derived is the other part of the Fundamental Theorem of Calculus (recall we saw Part 2 in section 1.2.1).

THEOREM 1 **Fundamental Theorem of Calculus (FTC) - Part 1**

Assume f is continuous on $[a, x]$ and $F(x) = \int_a^x f(t)\, dt$. Then,

$$F'(x) = f(x)$$

EXAMPLE 10

Evaluate $\dfrac{d}{dx}\displaystyle\int_0^x (t^3 + t + 1)\,dt$.

Solution: $\dfrac{d}{dx}\displaystyle\int_0^x (t^3 + t + 1)\,dt = x^3 + x + 1$

EXAMPLE 11

Evaluate $\dfrac{d}{dx}\displaystyle\int_3^x (t^3 + t + 1)\,dt$.

Solution: $\dfrac{d}{dx}\displaystyle\int_3^x (t^3 + t + 1)\,dt = x^3 + x + 1$

What was the different about the last two examples? The bottom bound of the integral was different. However, despite this, the answer was the same. You can think about this in two ways:

First, the rate of change at any point x is not affected by the starting point (for example, the velocity of a car is not affected by where the car started moving from).

Second, if we take the derivative of both sides of $\displaystyle\int_a^x f(t)\,dt = F(x) - F(a)$, the $F(a)$ on the right hand side will become 0 since it is a constant.

The next two examples stress that we must be careful about variable names.

EXAMPLE 12

If $f(r) = \displaystyle\int_{-3}^r \frac{\sin(x)}{x^2+1}\,dx$, find $f'(r)$.

Solution: $f'(r) = \dfrac{\sin(r)}{r^2+1}$

EXAMPLE 13

Evaluate $\dfrac{d}{dw}\displaystyle\int_{-2}^w \sin(\theta)\,d\theta$.

Solution: $\dfrac{d}{dw}\displaystyle\int_{-2}^w \sin(\theta)\,d\theta = \sin(w)$.

EXERCISE 9

Determine the derivative of the following functions.

(a) $f(x) = \displaystyle\int_{-2}^x (s^3 + s + 1)\,ds$

(b) $g(x) = \displaystyle\int_1^x t\sqrt{t^2+1}\,dt$

(c) $h(w) = \displaystyle\int_3^w (5x^2 + e^{\cos(x)})\,dx$

As usual, if we have a composition of functions, then we need to apply the Chain Rule.

EXAMPLE 14 Let $f(x) = \int_2^{x^2} \left(\sqrt{t} + t\right)\, dt$. Find $f'(x)$.

Solution: Observe that if we let $g(x) = \int_2^{x} \left(\sqrt{t} + t\right)\, dt$ and let $h(x) = x^2$, then

$$f(x) = g(h(x))$$

By the Chain Rule

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= \left(\sqrt{x^2} + x^2\right) \cdot (2x) \end{aligned}$$

EXAMPLE 15 Let $f(x) = \int_{-1}^{e^x} \cos(t^2)\, dt$. Find $f'(x)$.

Solution: Observe that if we let $g(x) = \int_{-1}^{x} \cos(t^2)\, dt$ and let $h(x) = e^x$, then

$$f(x) = g(h(x))$$

By the Chain Rule

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= \left(\cos\left((e^x)^2\right)\right) \cdot (e^x) \\ &= \cos\left(e^{2x}\right) \cdot e^x \end{aligned}$$

EXAMPLE 16 Let $f(x) = \int_{-1}^{\sin(x)} w^2\, dw$. Find $f'(x)$.

Solution: We let $g(x) = \int_{-1}^{x} w^2\, dw$, let $h(x) = \sin(x)$. Then, we have

$$f(x) = g(h(x))$$

Thus, by the Chain Rule

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= (\sin(x))^2 \cdot \cos(x) \end{aligned}$$

EXERCISE 10 Find the derivative of the integral function.

(a) $f(x) = \int_0^{\sqrt{x}} t^2\,dt$

(b) $f(x) = \int_3^{x^2+x} \sqrt{2t^2+t}\,dt$

(c) $f(x) = \int_{-2}^{e^{x^2}} t\sin(t)\,dt$

Section 2.2 Problems

1. Consider the integral function $f(x) = \int_0^x 1\,dt$ for $x \geq 0$.
   (a) Evaluate $f(0)$, $f(2)$, $f(3)$, and $f(4)$.
   (b) Sketch the graph of $y = f(x)$.

2. Consider the integral function $f(x) = \int_3^x 1\,dt$ for $x \geq 0$.
   (a) Evaluate $f(0)$, $f(2)$, $f(3)$, and $f(4)$.
   (b) Sketch the graph of $y = f(x)$.

3. Consider the integral function $f(x) = \int_{-2}^x t\,dt$ for $x \geq -2$.
   (a) Evaluate $f(-2)$, $f(-1)$, $f(0)$, and $f(1)$.
   (b) Sketch the graph of $y = f(x)$.

4. Determine $f(x) = (g \circ h)(x)$.
   (a) $g(x) = \int_1^x \sin^3(t)\,dt$, $h(x) = 2x + 1$
   (b) $g(x) = \int_0^x e^{-t^2}\,dt$, $h(x) = -x$
   (c) $g(x) = \int_2^x \frac{\sin(t)}{t}\,dt$, $h(x) = x^2$
   (d) $g(x) = \int_{-1}^x |t + \cos(t)|\,dt$, $h(x) = e^x$
   (e) $g(x) = \int_0^x \sqrt{t^2+1}\,dt$, $h(x) = \sin(x)$

5. Determine the derivative of the following functions.
   (a) $f(x) = \int_0^x (t^2 + \sqrt{t})\,dt$
   (b) $f(x) = \int_{-7}^x \ln(t^2+1)\,dt$
   (c) $f(x) = \int_{17}^x (w^3 - w)\,dw$
   (d) $g(w) = \int_3^w te^t\,dt$
   (e) $h(r) = \int_{-3}^r 1\,dx$
   (f) $j(x) = \int_\pi^x \tan(t^2+t)\,dt$
   (g) $f(x) = \int_0^x (r^2+r)\,dr$
   (h) $f(x) = \int_{-5}^{3x} (s^2+s)\,ds$
   (i) $f(x) = \int_0^{\sin(x)} \frac{1}{r}\,dr$
   (j) $f(x) = \int_3^{2x} (w^2+w)^{1/3}\,dw$
   (k) $f(x) = \int_0^{x^2} e^t\,dt$
   (l) $f(x) = \int_4^{3x} \sqrt{w+1}\,dw$
   (m) $f(x) = \int_0^{\cos(x)} \frac{1}{1+t^2}\,dt$

6. Let $f(x) = \int_0^x g(t)\,dt$. Use the table below to approximate $f(2)$ using 4 subdivisions.

t	0	0.5	1	1.5
$g(t)$	1.0	1.2	0.8	1.4

7. Let $f(x) = \int_0^x g(t)\,dt$. Use the table below to approximate $f(4)$ using 4 subdivisions.

t	0	1	2	3
$g(t)$	0.40	0.35	0.33	0.32

8. Let $f(x) = \int_1^x g(t)\,dt$. Use the table below to approximate $f(2)$ using 3 subdivisions.

t	1	$\frac{4}{3}$	$\frac{5}{3}$
$g(t)$	0.2	−0.1	−0.3

9. Let $f(x) = \int_{-1}^x g(t)\,dt$. Use the table below to approximate $f(1)$ using 4 subdivisions.

t	−1	−0.5	0	0.5
$g(t)$	12	23	32	42

10. Let $f(x) = \int_3^x g(t)\,dt$. Use the table below to approximate $f(4)$ using 4 subdivisions.

t	3	3.25	3.5	3.75
$g(t)$	−2.2	−2.0	−1.6	−1.0

11. Let $f(x) = \int_1^x g(t)\,dt$. Use the table below to approximate $f(2)$ using 5 subdivisions.

t	1	1.2	1.4	1.6	1.8
$g(t)$	0.2	0.4	0.4	0.4	0.6

12. The size of a mosquito population is changing at a rate of $432t^2 - 5t^4$ per month where t is in months. What does $f(t) = \int_0^t (432x^2 - 5x^4)\,dx$ represent?

13. The end product of a chemical reaction is produced at a rate of $\dfrac{\sqrt{t}-1}{t}$ mg/min. What does $f(x) = \int_0^x \dfrac{\sqrt{t}-1}{t}\,dt$ represent?

14. An object has velocity $v(t) = t + 1$ m/s. Determine the net change in the object's position from time $t = 2$ to the following times.
   (a) $t = 3$
   (b) $t = 4$
   (c) $t = x$

15. The rate of change of the mass of an object is $f(t) = \sin(t)$ kg/h. Determine the net change in the object's mass from time $t = 0$ to the following times.
   (a) $t = \dfrac{\pi}{4}$
   (b) $t = \pi$
   (c) $t = x$

16. The rate of change of air pressure at altitude h is given by $-3e^{-0.21h}$ kPa/km. Determine the net change in the air pressure from height $h = 0$ to the following heights.
   (a) $h = 1$
   (b) $h = 2$
   (c) $h = x$

Section 2.3: Inverse Functions

LEARNING OUTCOMES

1. Know how to find the inverse of a function on an interval.
2. Know how to determine whether a function is invertible on a given interval.
3. Know how to evaluate f^{-1} using the definition of the inverse.
4. Know how to find the domain and range of an inverse function.
5. Know how to use the cancellation properties.

2.3.1 Inverse Functions

Sometimes we require a function that does the opposite of a known function. For example, say we have a function $p = f(h)$ which outputs the air pressure p as a function of the height h above sea level. We may find ourselves in a situation where we need a function that instead returns the height above sea level given the air pressure.

DEFINITION

Inverse

Invertible

Suppose f is a function. If there exists a function g such that

$$g(y) = x \quad \text{if and only if} \quad f(x) = y$$

for all x in an interval I, then we call g the **inverse of f on the interval I**. We denote it by $g = f^{-1}$. If f has an inverse, then we say that f is **invertible**.

REMARK

Note that the notation f^{-1} means 'the inverse of f'... it does *not* mean $\dfrac{1}{f}$.

We have defined the inverse of a function to be on a specified interval. As you will soon see, for some functions it is necessary to specify the interval of the inverse. For example, a function may have different inverses on different intervals, or a function may have an inverse on one interval but not another. Note that in the case where a function has a unique inverse on its domain, we typically do not specify the interval. That is, if the interval is not specified, then we assume that the interval is the domain of the original function.

EXAMPLE 1

Find the inverse of $f(x) = 3x + 5$.

Solution: We set $y = 3x + 5$ and solve for x.

$$\begin{aligned} y &= 3x + 5 \\ y - 5 &= 3x \\ \frac{y-5}{3} &= x \end{aligned}$$

Therefore, $f^{-1}(y) = \dfrac{y-5}{3}$.

EXAMPLE 2 Find the inverse of $g(x) = x^2$ on $[0, \infty)$ and on $(-\infty, 0)$.

Solution: We set $y = x^2$ and solve for x.

For $0 \le x < \infty$, we have

$$\begin{aligned} y &= x^2 \\ \sqrt{y} &= \sqrt{x^2} \\ \sqrt{y} &= |x| \\ \sqrt{y} &= x \end{aligned}$$

since $x \ge 0$. Thus, on $[0, \infty)$, $g^{-1}(y) = \sqrt{y}$.

For $-\infty < x < 0$, we have

$$\begin{aligned} y &= x^2 \\ \sqrt{y} &= \sqrt{x^2} \\ \sqrt{y} &= |x| \\ \sqrt{y} &= -x \qquad \text{since } x < 0 \\ -\sqrt{y} &= x \end{aligned}$$

Thus, on $(-\infty, 0)$, $g^{-1}(y) = -\sqrt{y}$.

You were likely told in high-school to swap y and x when finding the inverse. The purpose of that was to make x the independent variable of the inverse function. However, this is generally not a good idea in science where variable names often mean something.

EXAMPLE 3 Let $p(t) = t^3 + 3$ denote the population of bacteria in a culture at any time $t \ge 0$ where the population is measured in millions and time is measured in hours. Find p^{-1} and use it to find the time t when the population of bacteria reaches 100 million.

Solution: We set $p(t) = P$ and solve for t.

$$\begin{aligned} P &= t^3 + 3 \\ P - 3 &= t^3 \\ \sqrt[3]{P - 3} &= t \end{aligned}$$

So, $p^{-1}(P) = \sqrt[3]{P - 3}$. This is a function that inputs the population and output time.

The time when the population reaches 100 million is

$$t = p^{-1}(100) = \sqrt[3]{100 - 3} = \sqrt[3]{97} \text{ hours}$$

EXERCISE 1

If $H(a) = \dfrac{100a}{100+a}$ represents the height of a tree in meters given the age a of the tree in years, find H^{-1} and use it to find the age of a tree that is 50 meters high.

REMARK

Observe from the example and the exercise if we had swapped variables, then we would be using P to represent time and t to represent population, or H to represent age and a to represent height. That would be confusing!

Existence of an Inverse

Recall that for a relation to be considered a function, it can only have one output for each input. Therefore, for us to be able to define an inverse for a function f on an interval I, f can only have one input in I that gives a particular output.

That is, if there are two different x-values, x_1, x_2, in an interval I such that $f(x_1) = f(x_2)$, then f is not invertible on I.

THEOREM 1

Horizontal Line Test/Invertiblity Test

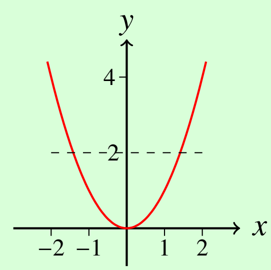
A function is invertible on an interval I if every horizontal line crosses the function at most once.

EXAMPLE 4

Show that $f(x) = x^2$ is not invertible on the interval $[-2, 2]$.

Solution: Graphing $y = x^2$ we see that, for example, the horizontal line $y = 2$ intersects the graph in two places.

Thus, by the Horizontal Line Test, $f(x) = x^2$ is not invertible on the interval $[-2, 2]$.



EXERCISE 2

Show that $f(x) = x^2 - 2x + 1$ is not invertible on $[0, 2]$.

EXERCISE 3

Explain/show why $\sin(x)$ is invertible on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, but not on $[0, \pi]$. Give another interval on which $\sin(x)$ is invertible.

EXERCISE 4

Give an example of a function f that is defined for all real numbers, but does not have an inverse on any interval (a, b).

2.3.2 Properties of Inverse Functions

Evaluating f^{-1}

Because of the definition of f^{-1}, namely the property that

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

sometimes we do not even need to find f^{-1} to be able to evaluate $f^{-1}(y)$. This is best demonstrated with some examples.

EXAMPLE 5

Given that $f(x) = x^3 + 2x + 1$ is invertible, find $f^{-1}(1)$ and $f^{-1}(3)$.

Solution: By definition, to find $f^{-1}(1)$ we ask ourselves "What value of x gives $f(x) = 1$?". By inspection, we see that $f(0) = 1$. Thus,

$$f^{-1}(1) = 0$$

Similarly, by inspection, we see that $f(1) = 3$. Thus,

$$f^{-1}(3) = 1$$

EXAMPLE 6

Given that $f(x) = \sin(x)$ is invertible on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, evaluate $f^{-1}\left(\frac{1}{2}\right)$.

Solution: Evaluating $f^{-1}\left(\frac{1}{2}\right)$ is equivalent to asking ourselves "What angle θ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ gives $\sin(\theta) = \frac{1}{2}$?"

We know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Hence,

$$f^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

EXERCISE 5

Given that $f(x) = x^3 + x^2 + x + 1$ is invertible, find $f^{-1}(1)$ and $f^{-1}(0)$.

EXERCISE 6

Given that $f(x) = \cos(x)$ is invertible on $[0, \pi]$, evaluate $f^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

EXERCISE 7

Given that $f(x) = x^2 + 2x$ is invertible on $(-\infty, -1)$, find $f^{-1}(0)$.

Domain and Range

Since the inverse function f^{-1} does the opposite of f, it means that the outputs of f become the inputs for f^{-1}, and the inputs of f become the outputs for f^{-1}. That is:

> If f^{-1} is the inverse of a function f on an interval I, then
> the domain of f^{-1} is the range of f restricted to values of x in I, and
> the range of f^{-1} is the interval I.

EXAMPLE 7

Let $f(x) = \sqrt{x+2}$. What is the domain and range of f^{-1}?

Solution: Since f is invertible on its domain, we take the interval I to be the domain of f. That is $I = [-2, \infty)$.

The domain of f^{-1} is the set of all values we get from $f(x)$ by plugging in values of x from I. So, the domain of f^{-1} is $[0, \infty)$.

The range of f^{-1} is $I = [-2, \infty)$.

EXAMPLE 8

Let $f(x) = x^2$. We saw that we can define at inverse, f^{-1}, on $I = (-\infty, 0)$. What is the domain and range of f^{-1}?

Solution: The domain of f^{-1} is the set of all values that we get from f by plugging in values of x from I. So, the domain of f^{-1} is $(0, \infty)$.

The range of f^{-1} is $I = (-\infty, 0)$.

EXAMPLE 9

The function $f(x) = \cos(x)$ is invertible on $I = [0, \pi]$. We denote this inverse by arccos. What is the domain and range of arccos?

Solution: Since $\cos(x)$ takes all values from -1 to 1 over the interval $0 \le x \le \pi$, the domain of arccos is $[-1, 1]$.

The range of arccos is $I = [0, \pi]$.

EXERCISE 8

Given that $f(x) = x^2 + 2x$ is invertible on $I = [-1, \infty)$, find the domain and range of f^{-1}.

EXERCISE 9

The function $f(x) = \sin(x)$ is invertible on $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We denote this inverse by arcsin. What is the domain and range of arcsin?

EXERCISE 10

The function $f(x) = \tan(x)$ is invertible on $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We denote this inverse by arctan. What is the domain and range of arctan?

Cancellation Properties

If f is invertible on the interval I with inverse f^{-1}, then

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \text{ for all } x \text{ in } I$$
$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = y \text{ for all } y \text{ in the domain of } f^{-1}$$

The cancellation properties give us an easy way to verify that two functions are inverses of each other.

EXAMPLE 10 Verify that if $f(x) = 3x + 5$, then $f^{-1}(x) = \dfrac{y-5}{3}$.

Solution: Since the domain of both f and f^{-1} are $(-\infty, \infty)$, we get

$$f^{-1}(f(x)) = f^{-1}(3x+5) = \frac{(3x+5)-5}{3} = x$$

and

$$f(f^{-1}(y)) = f\left(\frac{y-5}{3}\right) = 3\left(\frac{y-5}{3}\right) + 5 = y$$

Thus, f and f^{-1} are inverses of each other.

EXAMPLE 11 Verify that if $f(x) = x^2$, then on $I = (-\infty, 0)$, we have $f^{-1}(x) = -\sqrt{y}$.

Solution: We are restricting the domain of f to $I = (-\infty, 0)$. We see that the domain of $f^{-1} = -\sqrt{y}$ is $(0, \infty)$.

For any $x \in (-\infty, 0)$ we get

$$f^{-1}(f(x)) = f^{-1}(x^2) = -\sqrt{x^2} = -|x| = -(-x) = x$$

For any $y \in (0, \infty)$, we get

$$f(f^{-1}(y)) = f(-\sqrt{y}) = (-\sqrt{y})^2 = y$$

Thus, f and f^{-1} are inverses of each other.

It is important to note that we cannot use the Cancellation Properties when evaluating $f^{-1}(f(x))$ when x is not in the interval I.

EXAMPLE 12

Consider $f(x) = x^2$ on $I = (-\infty, 0)$. What is $f(f^{-1}(7))$? What is $f^{-1}(f(3))$?

Solution: The domain of f^{-1} is the range of f restricted to I. The range of $f(x) = x^2$ on I is $(0, \infty)$. Thus, the domain of f^{-1} is $(0, \infty)$. Since 7 is in the domain of f^{-1}, the Cancellation Properties give

$$f(f^{-1}(7)) = 7$$

For, $f^{-1}(f(3))$ we may be tempted to think that the answer is 3. However, 3 is not in the interval I. Thus, we *cannot* use the Cancellation Properties. Instead, we have to calculate this manually. We get

$$f^{-1}(f(3)) = f^{-1}(3^2) = f^{-1}(9) = -\sqrt{9} = -3$$

EXAMPLE 13

Given that $f(x) = \tan(x)$ is invertible on $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, evaluate the following.

(a) $f^{-1}\left(f\left(\frac{\pi}{4}\right)\right)$

(b) $f\left(f^{-1}\left(\frac{1}{2}\right)\right)$

(c) $f^{-1}\left(f\left(\frac{5\pi}{4}\right)\right)$

Solution: (a) Since $\frac{\pi}{4} \in I$, we get by the Cancellation Properties that

$$f^{-1}\left(f\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$$

(b) The domain of f^{-1} is the range of f restricted to values of x in I. On I, the range of $\tan(x)$ is $(-\infty, \infty)$. Thus, domain of f^{-1} is $(-\infty, \infty)$, so $y = \frac{1}{2}$ is in the domain of f^{-1}. Hence, we get by the Cancellation Properties that

$$f\left(f^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{2}$$

(c) Since $\frac{5\pi}{4}$ is not in I, we cannot use the Cancellation Properties. So, we must evaluate this manually. We have

$$f^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right) = f^{-1}(-1)$$

To evaluate $f^{-1}(1)$, we ask "what angle θ in I gives $\tan(\theta) = 1$?". We get $\theta = \frac{\pi}{4}$. Hence,

$$f^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right) = f^{-1}(1) = \frac{\pi}{4}$$

EXERCISE 11 Given that $f(x) = \sin(x)$ is invertible on $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, evaluate the following.

(a) $f^{-1}\left(f\left(\frac{\pi}{4}\right)\right)$

(b) $f\left(f^{-1}\left(\frac{1}{2}\right)\right)$

(c) $f^{-1}\left(f\left(\frac{3\pi}{4}\right)\right)$

Section 2.3 Problems

1. Find the inverse of each function.
   (a) $f(x) = 3x - 2$
   (b) $g(t) = 2\sqrt{t-3} + 1$
   (c) $h(r) = r^3 + 5$
   (d) $f(y) = -\sqrt[3]{y+1}$
   (e) $f(x) = \dfrac{x+1}{x-2}$
2. Determine graphically whether the function has an inverse on its domain. If it does, find the inverse.
   (a) $f(x) = \sqrt{x+1}$
   (b) $f(x) = x^2 + 2x + 1$
   (c) $f(x) = \dfrac{1}{x}$
   (d) $f(x) = \tan(x)$
   (e) $f(x) = |x| + 2$
3. Find the inverse of each function on the given interval. State the domain and the range of the inverse.
   (a) $f(x) = x^2 - 1$, $[0, \infty)$
   (b) $f(x) = x^2 - 1$, $(-\infty, 0]$
   (c) $f(x) = x + |x|$, $[0, \infty)$
   (d) $f(x) = x^2 - 4x + 4$, $[3, \infty)$.
   (e) $f(x) = \dfrac{1}{x^4}$, $(-\infty, -1]$.
4. Split the domain of f into two intervals such that f has an inverse function on each interval. Find the inverse function for each interval.
   (a) $f(x) = x^4$
   (b) $f(x) = x^2 + 2x + 3$
   (c) $f(x) = |x^5 - 10|$
5. Assuming the function is invertible on the given interval, find the specified quantities.
   (a) $f(x) = x^3 + x$ on $(-\infty, \infty)$; $f^{-1}(0)$, $f^{-1}(10)$
   (b) $g(x) = 3 + x + e^x$ on $(-\infty, \infty)$; $g^{-1}(4)$; $g^{-1}(4+e)$
   (c) $f(x) = \sqrt{x+1}$ on $[-1, \infty)$; $f^{-1}(2)$, $f^{-1}(9)$
   (d) $h(x) = \frac{1}{3}x^3 + x - 1$ on $(-\infty, \infty)$; $h^{-1}(-1)$, $h^{-1}\left(-\frac{7}{3}\right)$
   (e) $f(x) = x^2$ on $[0, \infty)$; $f^{-1}(1)$, $f^{-1}(3)$
   (f) $f(x) = x^2$ on $(-\infty, 0]$; $f^{-1}(1)$, $f^{-1}(3)$
   (g) $f(x) = \sin(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; $f^{-1}(0)$, $f^{-1}\left(\frac{1}{2}\right)$
   (h) $f(x) = \sin(x)$ on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$; $f^{-1}(0)$, $f^{-1}\left(\frac{1}{2}\right)$
   (i) $f(x) = \cos(x)$ on $[0, \pi]$; $f^{-1}(-1)$, $f^{-1}\left(\frac{1}{2}\right)$
   (j) $f(x) = \cos(x)$ on $[\pi, 2\pi]$; $f^{-1}(-1)$, $f^{-1}\left(\frac{1}{2}\right)$
6. Let $q = f(p)$ where p is the price of an item in dollars and q is the number of items sold at that price. What is the physical meaning of each of the following.
   (a) $f(10)$
   (b) $f^{-1}(5)$
7. Let $C = C(A)$ be the cost in thousands of dollars of building a house of area A square feet. What is the physical meaning of each of the following.
   (a) $C(1000)$
   (b) $C^{-1}(1000)$

8. fMRI is a method for determining brain activity through changes in blood flow to different parts of the brain. Data shows that blood flow F is linearly related to brain activity x.
   (a) Given the table below, determine a linear function $F(x)$ for blood flow in terms of brain activity.

x	F
0.16	0.52
1.0	1.0

   (b) Find $F^{-1}(x)$ and state what is represents.

9. Bacteria are growing in a culture. The time t (in hours) for the number of bacteria to double is a function of the temperature (in degree Celsius) of the culture according to the model

$$t = f(T) = \begin{cases} \frac{1}{24}T + \frac{11}{4}, 30 \leq T \leq 36 \\ \frac{4}{3}T - \frac{175}{4}, 36 < T \leq 39 \end{cases}$$

   (a) Determine the number of hours it takes for the bacteria to double at a temperature of 31° C.
   (b) Find the piecewise inverse function $T = f^{-1}(t)$.
   (c) Use your answer to (b) to determine the temperature that allows a bacteria to double in 8 hours.

10. Given that $f(x) = x^2$ is invertible on $I = [0, \infty)$, what is
   (a) $f(f^{-1}(2))$
   (b) $f^{-1}(f(5))$
   (c) $f^{-1}(f(-3))$

11. Given that $f(x) = x^2$ is invertible on $I = (-\infty, 0]$, what is
   (a) $f(f^{-1}(-2))$
   (b) $f^{-1}(f(5))$
   (c) $f^{-1}(f(-3))$

12. Given that $f(x) = \cos(x)$ is invertible on $I = [0, \pi]$, what is
   (a) $f(f^{-1}(1))$
   (b) $f\left(f^{-1}\left(\frac{1}{2}\right)\right)$
   (c) $f^{-1}(f(0))$
   (d) $f^{-1}\left(f\left(\frac{\pi}{4}\right)\right)$
   (e) $f^{-1}\left(f\left(-\pi\right)\right)$

Section 2.4: Exponential and Logarithmic Functions

LEARNING OUTCOMES

1. Know how to use properties of exponential and logarthmic functions.
2. Know how to integrate and differentiate exponential and logarithmic functions.
3. Know how to solve exponential growth and decay problems.
4. Know how to use and interpret a logarithmic scale.

2.4.1 Exponential Functions

We will first do a quick review of the basics of exponential functions that you covered in high-school.

DEFINITION
Exponential Function

For $a > 0$, we call a function of the form $f(x) = a^x$ an **exponential function**.

When working with exponential functions, it's useful to remember the following properties of exponents.

Exponent Properties: For $a > 0$ and $b > 0$, we have

$$\begin{aligned} a^x a^y &= a^{x+y} \\ \frac{a^x}{a^y} &= a^{x-y} \\ (a^x)^y &= a^{xy} \\ (ab)^x &= a^x b^x \\ a^0 &= 1 \end{aligned}$$

EXAMPLE 1

Let $f(x) = 2^x$. Evaluate $f(3)$, $f(1)$, $f(0)$, and $f(-4)$.

Solution: We have

$$\begin{aligned} f(3) &= 2^3 = 8 \\ f(1) &= 2^1 = 2 \\ f(0) &= 2^0 = 1 \\ f(-4) &= 2^{-4} = \frac{1}{16} \end{aligned}$$

EXERCISE 1

Let $f(x) = \left(\frac{1}{3}\right)^x$. Find $f(2)$, $f(0)$, and $f(-3)$.

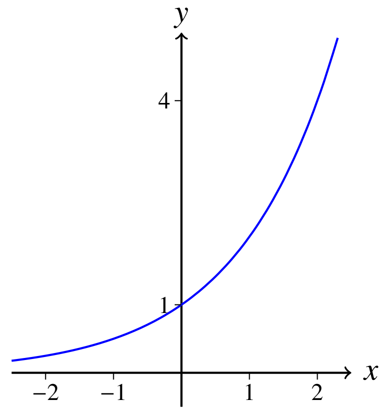
Graphs of Exponential Functions

It is also very important that you know and understand the behavior of exponential functions. One of the best ways of doing this is via their graphs.

For $a > 1$, the key features of the graph of $y = a^x$ are:

The key features we notice are:

- It is always increasing.
- It is always concave up.
- It grows very slowly at first ($x < 0$) and grows very fast later ($x > 0$).
- As $x \to \infty$, $a^x \to \infty$
- As $x \to -\infty$, $a^x \to 0$



Graph of $y = 2^x$

In calculus, to make calculations work out much nicer, we create a special base for the exponential function such that the slope of the graph at $x = 0$ is $m = 1$. We call this base e where $e = 2.71828\ldots$ (called Euler's number).

DEFINITION
e^x

The natural exponential function is $\exp(x) = e^x$.

2.4.2 Calculus with Exponential Functions

We now add the derivative of a^x to our list of derivatives to memorize.

function	**derivative**
$f(x) = a^x$	$f'(x) = a^x \ln(a)$

EXAMPLE 2

Determine the derivative of $f(x) = 2^x$.

Solution: We get $f'(x) = 2^x \ln(2)$.

EXERCISE 2

Determine the second derivative of $f(x) = x3^x$.

Since antiderivatives are the opposite of derivatives, we get the following:

function	**indefinite integral**
$f(x) = a^x$	$\int a^x \, dx = \dfrac{a^x}{\ln(a)}$

EXAMPLE 3

Determine $\int 5^x\, dx$.

Solution: We get $\int 5^x\, dx = \frac{5^x}{\ln(5)} + C$.

EXAMPLE 4

Find all solutions of the first order differential equation $y' = 3^x + x$.

Solution: All solutions are given by

$$y = \int (3^x + x)\, dx = \frac{3^x}{\ln(3)} + \frac{1}{2}x^2 + C$$

EXERCISE 3

Determine $\int 4^x\, dx$.

EXERCISE 4

Solve the initial value problem $\frac{dy}{dx} = 2^x$, $y(1) = 2$.

2.4.3 Logarithmic Functions

The inverse of the exponential function is called the logarithmic function. Although we use logarithms to solve problems involving the exponential function, perhaps the main uses of the logarithm function comes from its properties. In particular, its ability to make very large numbers small, the ability to turn multiplication/division into addition/subtraction, and the ability to turn exponentiation into multiplication.

DEFINITION

Logarithmic Function

For any $a > 1$, we define the **base** $\boldsymbol{a}$ **logarithmic function**, $\log_a$, to be the inverse of the exponential function with base a. In particular,

$$\log_a(x) = y \quad \text{if and only if} \quad x = a^y$$

The base e logarithm is called the **natural logarithmic function** and is denoted by ln. That is, $\ln(x) = \log_e(x)$.

We can use the fact that $\log_a$ is the inverse of a^x to get the domain and range of $\log_a$. In particular, since the range of a^x is $(0, \infty)$, the domain of $\log_a$ is $(0, \infty)$. Since the domain of a^x is $(-\infty, \infty)$, the range of $\log_a$ is $(-\infty, \infty)$.

Logarithms have the following properties:

Logarithm Properties:

$$
\begin{aligned}
a^{\log_a(x)} &= x \\
\log_a(a^y) &= y \\
\log_a(xy) &= \log_a(x) + \log_a(y) \\
\log_a\left(\frac{x}{y}\right) &= \log_a(x) - \log_a(y) \\
\log_a(x^y) &= y\log_a(x) \\
\log_a(b) &= \frac{\ln(b)}{\ln(a)} = \frac{\log_{10}(b)}{\log_{10}(a)}
\end{aligned}
$$

EXAMPLE 5 Evaluate the following

(a) $\log_3(9)$

(b) $\log_{10}\left(\frac{1}{100}\right)$

(c) $\log_3(729)$

(d) $\log_2(4^{2/3})$

(e) $e^{2\ln(5)}$

Solution: (a) Determining $y = \log_3(9)$ is the same as asking "For what value of y does $9 = 3^y$?". Since $3^2 = 9$, we get $\log_3(9) = 2$.

(b) Determining $\log_{10}\left(\frac{1}{100}\right)$ is the same as asking "For what value of y does $\frac{1}{100} = 10^y$?". Since $10^{-2} = \frac{1}{100}$, we get $\log_{10}\left(\frac{1}{100}\right) = -2$.

(c) Determining $\log_3(729)$ is the same as asking "For what value of y does $729 = 3^y$?". However, what if we don't know what power of 3 gives 729? We use the logarithm properties! We first observe that 9 divides evenly into 729. So,

$$
\begin{aligned}
\log_3(729) &= \log_3(9\cdot 81) \\
\log_3(729) &= \log_3((3\cdot 3)\cdot 3^4) \\
&= \log_3(3) + \log_3(3) + \log_3(3^4) \\
&= 1 + 1 + 4 \\
&= 6
\end{aligned}
$$

(d) Using logarithm properties gives

$$\log_2(4^{2/3}) = \log_2((2^2)^{2/3}) = \log_2(2^{4/3}) = \frac{4}{3}\log_2(2) = \frac{4}{3}\cdot 1 = \frac{4}{3}$$

(e) The property $e^{\log_e(x)} = x$ does not have a coefficient in front of $\log_e$. So, we must move the coefficient 2 before we can use this property. We get

$$e^{2\ln(5)} = e^{\ln(5^2)} = 5^2 = 25$$

EXERCISE 5 Evaluate the following

(a) $\log_4(64)$
(b) $\log_{10}(1000)$
(c) $\log_{10}\left(\dfrac{1}{1000000}\right)$
(d) $3^{\log_3(5)}$
(e) $2^{3\log_2(7)}$
(f) $\ln(e^2 + e^3)$

EXAMPLE 6 Let $f(x) = \dfrac{(x-1)(x+2)^2}{(x+3)^3(x+1)}$. Use the properties of logarithms to write $\ln(f(x))$ as a sum and difference of functions.

Solution: Taking ln of both sides gives

$$\begin{aligned}
\ln(f(x)) &= \ln\left(\frac{(x-1)(x+2)^2}{(x+3)^3(x+1)}\right) \\
&= \ln\Big((x-1)(x+2)^2\Big) - \ln\Big((x+3)^3(x+1)\Big) \\
&= \ln(x-1) + \ln(x+2)^2 - \Big[\ln(x+3)^3 + \ln(x+1)\Big] \\
&= \ln(x-1) + 2\ln(x+2) - 3\ln(x+3) - \ln(x+1)
\end{aligned}$$

EXERCISE 6 Express $\ln(a+b) + \ln(a-b) - 2\ln(c)$ as a single logarithm.

EXERCISE 7 Let $f(x) = \dfrac{(x-2)^5}{x^2(x+1)^3}$. Use the properties of logarithms to write $\ln(f(x))$ as a sum and difference of functions.

EXERCISE 8 Suppose that $f(x) = \dfrac{1}{1+e^{-(b+mx)}}$ where b and m are constants. This function is called a logistic function and it is used to describe the growth of populations with limited resources. Show that $\ln\left(\dfrac{f(x)}{1-f(x)}\right) = b + mx$. This is called the logistic transformation.

As previously mentioned, the properties of logarithms are often needed when solving problems involving exponential functions.

EXAMPLE 7 A dead body is found at 12 pm in a room that is maintained at 20°C. The body is 28°C when it is found and has cooled to 27°C at 1 pm. Given that the temperature of the body from time of death satisfies

$$T(t) = 20 + Ae^{kt}$$

estimate the time of death.

Solution: We are given that $T(0) = 28$ and $T(1) = 27$. We can use these conditions to solve for A and k.

Using $T(0) = 28$ gives

$$\begin{aligned} 28 &= 20 + Ae^0 \\ 8 &= A \end{aligned}$$

Using $T(1) = 27$ gives

$$\begin{aligned} 27 &= 20 + 8e^k \\ 7 &= 8e^k \\ \frac{7}{8} &= e^k \\ \ln\left(\frac{7}{8}\right) &= \ln(e^k) \\ \ln\left(\frac{7}{8}\right) &= k \end{aligned}$$

Thus, the temperature of the body at any time t is

$$T(t) = 20 + 8e^{\ln(7/8)t}$$

Thus, the body temperature was 37°C when

$$\begin{aligned} 37 &= 20 + 8e^{\ln(7/8)t} \\ 17 &= 8e^{\ln(7/8)t} \\ \frac{17}{8} &= e^{\ln(7/8)t} \\ \ln\left(\frac{17}{8}\right) &= \ln\left(e^{\ln(7/8)t}\right) \\ \ln\left(\frac{17}{8}\right) &= \ln\left(\frac{7}{8}\right)t \\ \frac{\ln(17/8)}{\ln(7/8)} &= t \end{aligned}$$

A calculator tells us that $\dfrac{\ln(17/8)}{\ln(7/8)} \approx -6$, so the estimated time of death is 6:00 pm.

The use of a calculator in the last example brings up an interesting question. How do we know values of $\ln(x)$? If you plug $\ln(2)$ into your calculator it will tell you

$$\ln(2) \approx 0.6931471806\ldots$$

But, how is this figured out? How did scientists determine the value of $\ln(2)$ before the invention of calculators? Of course, one way is to use the fact that $\ln(x)$ is the inverse of e^x and to approximate a value of x such that $e^x = 2$.

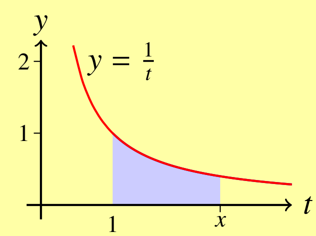
Another way we can do this is to define the natural logarithm function as an integral function. This can be proven to be equivalent to our previous definition of the natural logarithm function.

DEFINITION

Natural Logarithm Function

For any $x > 0$, we define

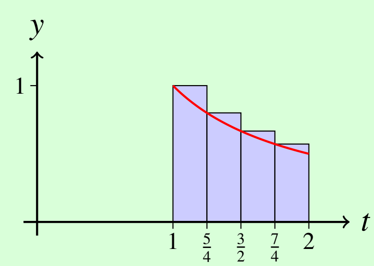
$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$



Since the natural logarithm is an integral function, we can approximate its values use the method we saw in Section 2.2.

EXAMPLE 8

Approximate $\ln(2)$ using 4 subdivisions.



Solution: By definition, $\ln(2) = \displaystyle\int_1^2 \frac{1}{t}\,dt$.
We subdivide the interval $[1, 2]$ into 4 equal pieces of length $\Delta t = \dfrac{2-1}{4} = \dfrac{1}{4}$.

Over each interval, we draw a rectangle with height equal to the value of $\dfrac{1}{t}$ where t is the left end point of the interval. We find the left end points are $1, \dfrac{5}{4}, \dfrac{3}{2}, \dfrac{7}{4}$. Thus,

$$\begin{aligned}\ln(2) &\approx \frac{1}{1}\cdot\Delta t + \frac{1}{\frac{5}{4}}\cdot\Delta t + \frac{1}{\frac{3}{2}}\Delta t + \frac{1}{\frac{7}{4}}\Delta t \\ &= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \\ &= \frac{319}{420}\end{aligned}$$

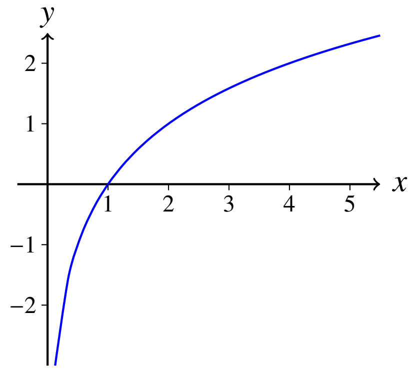
EXERCISE 9

Approximate $\ln(3)$ using 4 subdivisions.

Graph of Logarithmic Functions

For $a > 1$, the key features of the graph of $y = \log_a(x)$ are:

- It is always increasing.
- It is always concave down.
- It grows very quickly at first ($x < 1$) and grows very slowly later ($x > 1$).
- As $x \to \infty$, $\log_a(x) \to \infty$
- As $x \to 0^+$, $\log_a(x) \to -\infty$



Graph of $y = \log_2(x)$

2.4.4 Calculus with Logarithmic Functions

To calculate the derivative of the logarithmic function $\log_a(x)$ with base a, we can use the fact that it is the inverse function of a^x. In particular, we know that

$$a^{\log_a(x)} = x$$

Taking the derivative of both sides with respect to x (using the Chain Rule) gives

$$a^{\log_a(x)} \ln(a) \cdot \frac{d}{dx} \log_a(x) = 1$$

Dividing both sides by $a^{\log_a(x)} \ln(a)$ gives $\dfrac{d}{dx} \log_a(x) = \dfrac{1}{a^{\log_a(x)} \ln(a)} = \dfrac{1}{x \ln(a)}$.

In the case of the natural logarithm, this is $\dfrac{d}{dx} \ln(x) = \dfrac{1}{x \ln(e)} = \dfrac{1}{x}$.

function	**derivative**
$f(x) = \log_a(x)$	$f'(x) = \dfrac{1}{x \ln(a)}$
$f(x) = \ln(x)$	$f'(x) = \dfrac{1}{x}$

REMARK

Observe that the derivative also matches with the integral function definition of ln. In particular, by the Fundamental Theorem of Calculus

$$\frac{d}{dx} \int_1^x \frac{1}{t}\, dt = \frac{1}{x}$$

EXAMPLE 9

Determine the derivative of $f(x) = x\ln(x) - x$.

Solution: We get

$$\begin{aligned} f'(x) &= 1 \cdot \ln(x) + x \cdot \frac{1}{x} - 1 \\ &= \ln(x) \end{aligned}$$

EXAMPLE 10

Determine the derivative of $f(x) = x^2 \ln(2x)$.

Solution: We get

$$\begin{aligned} f'(x) &= 2x \cdot \ln(2x) + x^2 \cdot \frac{1}{2x} \cdot 2 \\ &= 2x\ln(2x) + x \end{aligned}$$

EXERCISE 10

Determine the derivative of $f(x) = x^2 \ln(2x)$ by first using the property of logarithms

$$\ln(2x) = \ln(2) + \ln(x)$$

and then taking the derivative.

The formulas for derivatives of logarithms gives us the following formulas for the antiderivatives.

function	**indefinite integral**
$f(x) = \dfrac{1}{x\ln(a)}$	$\displaystyle\int \frac{1}{x\ln(a)}\,dx = \log_a(\lvert x\rvert) + C$
$f(x) = \dfrac{1}{x}$	$\displaystyle\int \frac{1}{x}\,dx = \ln(\lvert x\rvert) + C$

REMARK

We need the absolute value signs in the antiderivatives so that the domain of an antiderivative is the same as the domain of the original function.

EXERCISE 11

Verify that $\dfrac{d}{dx}\ln(\lvert x\rvert) = \dfrac{1}{x}$.

EXAMPLE 11 Evaluate $\int \left(x + 1 + \frac{1}{x} + \frac{1}{x^2}\right) dx$.

Solution: We have

$$\int \left(x + 1 + \frac{1}{x} + \frac{1}{x^2}\right) dx = \frac{1}{2}x^2 + x + \ln(|x|) - \frac{1}{x} + C$$

EXAMPLE 12 Solve the initial value problem $y' = \frac{1}{x}$, $y(1) = 2$.

Solution: We have

$$\begin{aligned} y &= \int \frac{1}{x}\, dx \\ &= \ln(|x|) + C \end{aligned}$$

The initial condition says that $y = 2$ when $x = 1$. Hence,

$$\begin{aligned} 2 &= \ln(|1|) + C \\ 2 &= 0 + C \end{aligned}$$

Hence, the solution is $y(x) = \ln(|x|) + 2$.

EXERCISE 12 Evaluate $\int \left(e^x + \frac{1}{x}\right) dx$

EXERCISE 13 Solve the initial value problem $y' = \frac{1}{x \ln(2)}$, $y(4) = 3$.

2.4.5 Exponential Growth and Decay

When provided with an abundance of resources and space, most populations will grow at a rate proportional to the number of members in the population. That is, if $P(t)$ is the population after t units of time, then the rate of change of the population satisfies the differential equation

$$\frac{dP}{dt} = kP$$

To solve this differential equation, we ask ourselves what function would we take the derivative of to return a multiple of itself. We get

$$P(t) = Ae^{kt}$$

A quantity satisfying this differential equation is experiencing exponential growth when $k > 0$ and exponential decay when $k < 0$.

EXERCISE 14

Verify that $P(t) = Ae^{kt}$ is a solution of the first order differential equation $\frac{dP}{dt} = kP$ for any real numbers A and k.

DEFINITION

Exponential Growth

Exponential Decay

A quantity which has the form $Q(t) = Ae^{kt}$ for $k > 0$ is said to be experiencing **exponential growth**.

A quantity which has the form $Q(t) = Ae^{kt}$ for $k < 0$ is said to be experiencing **exponential decay**.

If we know a quantity Q is experiencing exponential growth/decay, then we know it satisfies $Q(t) = Ae^{kt}$ for some real numbers A and k. If we are given some conditions on Q, then we are able to solve for A and k.

EXAMPLE 13

Suppose a cell has an initial volume 1350 μm^3 and 2 hours later has a volume of 4050 μm^3. Given that the cell is growing exponentially, find a formula for the volume V of the cell at any time t in hours.

Solution: An exponential model is of the form $V(t) = ae^{kt}$. We are given that

$$V(0) = 1350$$
$$V(2) = 4050$$

Thus,

$$1350 = V(0) = ae^{k(0)} = a$$

Next, we solve for k.

$$\begin{aligned} 4050 &= V(2) = 1350e^{2k} \\ 3 &= e^{2k} \\ \ln(3) &= \ln(e^{2k}) \\ \ln(3) &= 2k \\ \frac{\ln(3)}{2} &= k \end{aligned}$$

Thus, the volume at any time t is

$$V(t) = 1350e^{\ln(3)t/2}\ \mu\text{m}^3$$

EXAMPLE 14

A doctor prescribes a drug to be administered intravenously to a patient. Immediately after the drug is injected, its concentration in the bloodstream is 1.5 mg/ml. After four hours the concentration has dropped to 0.25 mg/ml. Given that the concentration is decaying exponentially, find a formula for the concentration $C(t)$ of the drug at any time t.

Solution: An exponential model is of the form $C(t) = ae^{kt}$. We are given that

$$C(0) = \frac{3}{2}$$
$$C(4) = \frac{1}{4}$$

Thus,

$$\frac{3}{2} = C(0) = ae^{k(0)} = a$$

Next, we use the information that $C(4) = 0.25$ to solve for k.

$$\begin{aligned}
\frac{1}{4} &= C(4) = \frac{3}{2}e^{4k} \\
\frac{1}{6} &= e^{4k} \\
\ln\left(\frac{1}{6}\right) &= \ln(e^{4k}) \\
\ln\left(\frac{1}{6}\right) &= 4k \\
\frac{1}{4}\ln\left(\frac{1}{6}\right) &= k
\end{aligned}$$

Thus, the concentration at any time t is

$$C(t) = \frac{3}{2}e^{\ln(1/6)t/4} \text{ mg/ml}$$

WAIT! The concentration of the drug in is experiencing exponential decay, don't we need k to be negative?

It is negative! Remember that if $0 < x < 1$, then $\ln(x) < 0$. So, $k = \frac{1}{4}\ln\left(\frac{1}{6}\right)$ is negative.

EXERCISE 15

Suppose a population of a species is initially counted to be 1000 members and 2 years later is has a population of 1200. Given that the population is growing exponentially, find a formula for the population after t years.

Doubling Time/Half-Life

In some important applications of exponential functions, it is helpful to know the length of time in which it takes a quantity experiencing exponential growth to double in size or, similarly, the length of time it takes a quantity experiencing exponential decay to halve in size.

DEFINITION
Doubling Time
Half-Life

The amount of time it takes the quantity to double in size is called the **doubling time**.

The amount of time it takes the quantity to halve in size is called the **half-life**.

EXAMPLE 15

While living, Carbon-14, a radioactive isotope of carbon, is absorbed by plants through photosynthesis. When a plant dies, its amount of Carbon-14 begins to decay exponentially. This property, along with the fact that the half-life of Carbon-14 is 5730 years, can be used to approximate how long ago a plant died.

(a) If a sample of the plant now has 10 g of Carbon-14, and it is calculated that it contained 40 g of Carbon-14 when it died. How long ago did it die?

(b) If a different sample initially had 25 g of Carbon-14 and now has 5 g of Carbon-14, how long ago did it die?

Solution: (a) We know that the 40 g would have decayed to 20 g over a period of 5730 years. Then the 20 g would have decayed to 10 g over another 5370 years. So, the plant died 5730 + 5730 = 11460 years ago.

(b) The amount of Carbon-14 satisfies $C(t) = ae^{kt}$. We are given that $C(0) = 25$ g, and we want to find the time t such that $C(t) = 5$ g.

Using $C(0) = 25$ g, we get

$$25 = C(0) = ae^0 = a$$

Since the half-life is 5730 years, we also have that $C(5730) = \frac{25}{2}$ g. Thus,

$$\begin{aligned}
\frac{25}{2} &= 25e^{5730k} \\
\frac{1}{2} &= e^{5730k} \\
\ln\left(2^{-1}\right) &= \ln(e^{5730k}) \\
-\ln(2) &= 5730k \\
-\frac{\ln(2)}{5730} &= k
\end{aligned}$$

Thus, $C(t) = 25e^{-\ln(2)t/5730}$.

We now set $C(t) = 5$ and solve for t. We get

$$\begin{aligned}
25e^{-\ln(2)t/5730} &= 5 \\
e^{-\ln(2)t/5730} &= \frac{1}{5} \\
\ln\left(e^{-\ln(2)t/5730}\right) &= \ln\left(5^{-1}\right) \\
\frac{-\ln(2)}{5730}t &= -\ln(5) \\
t &= \frac{5730\ln(5)}{\ln(2)}
\end{aligned}$$

So, the plant died $\frac{5730\ln(5)}{\ln(2)}$ years ago.

2.4.6 Logarithmic Scales

There are many phenomena that take on a very large range of values. For example, we measure the concentration of hydrogen ions in moles per litre in a solution to determine how acidic or alkaline the solution is. Stomach acid has a hydrogen ion concentration of about $[H^+] = 1 \times 10^{-1}$ moles per litre while bleach has a hydrogen ion concentration of about about $[H^+] = 1 \times 10^{-13}$ moles per litre. Imagine trying to plot these values on a graph together! If you plot the 1×10^{-13} one unit above the x-axis, then you would have to plot the 1×10^{-1} one trillion units above the x-axis... you're going to need a big sheet of paper! So, rather than having to write out numbers like this each time, the pH scale was invented. It is defined as

$$pH = -\log_{10}([H^+])$$

Thus, we can simply say that the pH of stomach acid is 1 and the pH of bleach is 13.

When we use logarithms to adjust the scale of a quantity, we call it a **logarithmic scale**. Some other common examples of logarithmic scales include decibels, the Richter scale, and radiant energy flux (for measuring the brightness of a star).

REMARK

This is a great example of how scientists, like mathematicians, use notation to simplify writing. Like learning a new language, it takes time to learn all of the notations and understand what they mean. But, if you put the effort into learning them, you will see that it really does make things easier in the long run.

EXAMPLE 16

The Richter scale is a base-10 logarithmic scale that measures the amplitude (height) of the largest recorded wave at a specific distance from the source of an earthquake.

How much larger is the amplitude of a magnitude 6 earthquake compared to a magnitude 3 earthquake?

Solution: Since it is a base-10 logarithm, every increase of 1 increases the amplitude by a factor of 10. So, the magnitude 6 earthquake would be 1000 times larger than the magnitude 3 earthquake.

Let's work this out mathematically. Let A_1 and A_2 represent the two amplitudes of the two earthquakes. We are given

$$\log_{10}(A_1) = 6$$
$$\log_{10}(A_2) = 3$$

By definition of logarithms, this means

$$A_1 = 10^6$$
$$A_2 = 10^3$$

So, the amplitude of the magnitude 6 earthquake would be $\dfrac{A_1}{A_2} = \dfrac{10^6}{10^3} = 1000$ times larger.

EXAMPLE 17

How much larger is the amplitude of a magnitude 4.5 earthquake compared to a magnitude 2 earthquake?

Solution: Let A_1 and A_2 represent the two amplitudes of the two earthquakes. We are given the

$$\log_{10}(A_1) = 4.5$$
$$\log_{10}(A_2) = 2$$

By definition of logarithms, this means

$$A_1 = 10^{4.5}$$
$$A_2 = 10^2$$

So, $\dfrac{A_1}{A_2} = \dfrac{10^{4.5}}{10^2} = 10^{2.5}$. Since $\sqrt{10} \approx 3.17$, we get that the amplitude of the magnitude 4.5 earthquake would be approximately 317 times larger.

EXAMPLE 18

The decibel scale is used to measure the intensity of sounds. In particular, if I is the intensity of sounds waves in watts per square meter, then the loudness of the sound is defined to be

$$D = 10\log_{10}\left(\frac{I}{10^{-12}}\right) \text{ decibels}$$

How many times more intense is a sound that measures $D_1 = 110$ decibels versus a sound that measures $D_2 = 20$ decibels?

Solution: We first find the intensity of each sound. We have

$$\begin{aligned} D_1 &= 10\log_{10}\left(\frac{I_1}{10^{-12}}\right) \\ 110 &= 10\log_{10}\left(\frac{I_1}{10^{-12}}\right) \\ 11 &= \log_{10}\left(\frac{I_1}{10^{-12}}\right) \\ 10^{11} &= \frac{I_1}{10^{-12}} \\ 10^{-1} &= I_1 \end{aligned}$$

and

$$\begin{aligned} D_2 &= 10\log_{10}\left(\frac{I_2}{10^{-12}}\right) \\ 20 &= 10\log_{10}\left(\frac{I_2}{10^{-12}}\right) \\ 2 &= \log_{10}\left(\frac{I_2}{10^{-12}}\right) \\ 10^{2} &= \frac{I_2}{10^{-12}} \\ 10^{-10} &= I_2 \end{aligned}$$

Thus, the first sound is

$$\frac{I_1}{I_2} = \frac{10^{-1}}{10^{-10}} = 10^9$$

times louder.

EXERCISE 16

What is the pH of a substance that has a concentration of $[H^+] = 1 \times 10^{-6}$ moles per litre?

EXERCISE 17

How many times louder is a sound that measures $D_1 = 50$ decibels versus a sound that measures $D_2 = 35$ decibels?

Section 2.4 Problems

1. Evaluate the following.
   (a) $\log_5(25)$
   (b) $\log_{10}\left(\frac{1}{1000}\right)$
   (c) $e^{\ln(7)}$
   (d) $\log_2(2^5)$
   (e) $\ln\left(\frac{1}{e}\right)$
   (f) $\log_5(\sqrt{5})$
   (g) $e^{3\ln(4)}$
   (h) $e^{\ln(7)/2}$
2. Simplify the following expressions.
   (a) $x \cdot (x^2)^3$
   (b) $2^3 \cdot \sqrt{2}$
   (c) $\ln(6) - \ln(2)$
   (d) $\log_2(6) - \log_2(15) + \log_2(20)$
   (e) $\log_{a^2}(a^3)$
   (f) $(x^n)^3$ where n is a positive integer
   (g) $((-1)^n x^n)^2$ where n is a positive integer
   (h) $\dfrac{2^n}{3^{2n}} \cdot \dfrac{9^n}{4^n}$ where n is a positive integer
   (i) $\ln(|x|) - \ln(|x-1|)$
3. Find the inverse of f.
   (a) $f(x) = e^{2x+1}$
   (b) $f(x) = \ln(x^2)$
   (c) $f(x) = \ln(7-2x) + 1$
   (d) $f(x) = \dfrac{e^{2x}}{2}$
   (e) $f(x) = \ln(2 + \ln(x))$
   (f) $f(t) = 14 \cdot 2^t$
   (g) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
4. Use properties of logarithms to write $\ln(f(x))$ as a sum and difference of functions.
   (a) $f(x) = \dfrac{x+2}{x^2(x-1)^3}$
   (b) $f(x) = \dfrac{(x-5)^3}{x(x+3)}$
   (c) $f(x) = \dfrac{\sqrt{x}(x-1)(x+1)}{x+2}$
5. Determine the derivative of the function.
   (a) $f(x) = \ln(x^2)$
   (b) $f(x) = e^{\ln(x)}$
   (c) $f(x) = x\ln(x+1)$
   (d) $f(x) = e^{\sin(x)}$
   (e) $f(x) = \log_2(3x+5)$
   (f) $f(x) = \log_3(e^{x^2})$
   (g) $f(x) = \dfrac{\cos(x) - \ln(x)}{x}$
   (h) $f(x) = \ln(\ln(x))$
6. Verify the given function y is a solution of the differential equation.
   (a) $y = 2e^{3t}$ for $\dfrac{dy}{dt} = 3y$
   (b) $y = -1 + e^{x^2/2}$ for $\dfrac{dy}{dx} = x(y+1)$
   (c) $y = -\dfrac{1}{\ln(x)+1}$ for $\dfrac{dy}{dx} = \dfrac{y^2}{x}$
   (d) $y = |x|$ for $y' = \dfrac{y}{x}$
   (e) $y = xe^x + e^x$ for $y' = y + e^x$
7. Solve the initial value problem.
   (a) $y' = \dfrac{1}{x}$, $y(-3) = 1$
   (b) $y' = e^x + x$, $y(1) = 2$
   (c) $y' = \dfrac{1}{x\ln(3)}$, $y(3) = 3$
   (d) $y' = \dfrac{3}{x}$, $y(-2) = 5$
8. Sketch the graph of $y = f(x)$ and state the domain and range of f.
   (a) $f(x) = e^{-x+1}$
   (b) $f(x) = e^{x+1} - 1$
   (c) $f(x) = \ln(x-1)$
   (d) $f(x) = |\ln(x+1)|$
   (e) $f(x) = 2e^{x-2} + 2$
9. Use the definition $\ln(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$ to approximate the following values using 4 subdivisions.
   (a) $\ln(3)$
   (b) $\ln(5)$
   (c) $\ln(7)$

10. Find the pH of a substance that has the given concentration of $[H^+]$.
    (a) $[H^+] = 1 \times 10^{-2}$ moles per litre
    (b) $[H^+] = 1 \times 10^{-7}$ moles per litre
    (c) $[H^+] = 1 \times 10^{-3.5}$ moles per litre
    (d) $[H^+] = 1 \times 10^{-4.1}$ moles per litre

11. Show mathematically (as in Example 2.4.16) that the amplitude of a magnitude 8 earthquake is 10000 times larger than a magnitude 4 earthquake.

12. Show mathematically (as in Example 2.4.16) that the amplitude of a magnitude 2 earthquake is $\frac{1}{100}$ times the size of a magnitude 4 earthquake.

13. Find the intensity of each sound (as in Example 2.4.18) and use the intensity to determine how many times louder D_1 is than D_2.
    (a) $D_1 = 40$ decibels, $D_2 = 30$ decibels.
    (b) $D_1 = 60$ decibels, $D_2 = 20$ decibels.
    (c) $D_1 = 95$ decibels, $D_2 = 35$ decibels.
    (d) $D_1 = 50$ decibels, $D_2 = 45$ decibels.

14. If a culture of bacteria has an initial population of 100 and has a doubling time of 3 hours. What will be the population after 9 hours?

15. A biodome is initially home to a population of 500 birds. After 1 year, the population doubles to 1000 birds.
    (a) Write a function $P(t)$ for the population using the natural exponential function.
    (b) Write a function $P(t)$ for the population using an exponential function with base 2.
    (c) Assuming exponential growth, how long will it take for the population to reach 4000?
    (d) Assuming exponential growth, how long will it take for the population to reach 2500?

16. A culture of a bacterium Salmonela enteritidis initially contains 50 cells. When introduced into a nutrient broth, the culture grows at a rate proportional to its size. After 2 hours the population has increased to 1000. Find the number of bacteria after 3 hours. After how many hours will the population reach 250 000?

17. The half-life of ^{14}C is 5730 years. Carbon 14 is an unstable element in the atmosphere that is ingested by plants and animals. When an organism dies, ^{14}C starts to decay to ^{12}C. Suppose a piece of parchment is found and has 90% the ^{14}C content compared with paper today. What is the age of the artifact?

18. If a substance has a half-life of 16 years and there is initially 24 mg, determine the appropriate model for the amount of substance at any given time t (in years). How many years until there is 2 mg remaining?

19. Suppose that the half-life of morphine in the bloodstream is 3 hours and that it decays at a rate proportional to the amount in the bloodstream.
    (a) Write a differential equation for the amount of morphine in the bloodsteam.
    (b) If there's initially 0.4 mg of morphine in the system, how long does it take until there's only 0.01 mg of morphine remaining in the bloodstream?

20. A bacteria population is 2500 at time $t = 0$ and its growth rate is $1200e^{2t}$ bacteria per hour after t hours.
    (a) What is the population after 2 hours?
    (b) What is the population after x hours?

21. A colored dye is injected into the heart at time $t = 0$ at a constant positive rate R and mixes with blood. As fresh blood flows into the heart, the diluted mixture of blood and dye flows out at a constant positive rate r. Assuming the heart has constant volume $V > 0$, the rate of change of concentration of dye in the heart at time t is given by $\frac{dC}{dt} = \frac{R}{V} - \frac{r}{V}C(t)$.
    (a) Verify that the solution to the given differential equation is $C(t) = \frac{R}{r}(1 - e^{-(r/V)t})$.
    (b) Find C^{-1} and describe what it calculates.

Section 2.5: Trigonometric Functions

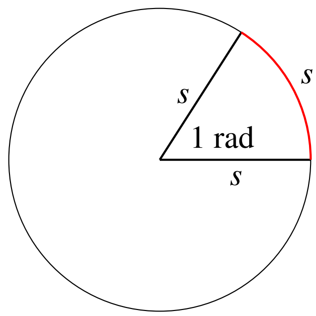
LEARNING OUTCOMES

1. Know how to create and solve mathematical models using trigonometric functions.
2. Know how to use and evaluate inverse trigonometric functions.
3. Know how to integrate and differentiate trigonometric functions.

2.5.1 Radians and Trigonometric Functions

Radians

In calculus, angles are *always* measured in **radians** where one radian is defined to be the angle required so that the arc length s subtended by the angle is equal to the radius of the circle.



Since the circumference of a circle of radius r is $2\pi r$, we get that there are 2π radians in a circle. That is

$$360° = 2\pi \text{ rad}$$

or

$$\frac{180°}{\pi} = 1 \text{ rad}$$

EXERCISE 1 Fill in the following table converting between radians and degrees.

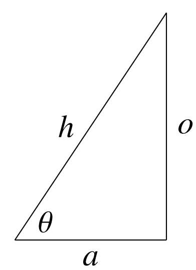
degrees	0	30	45	60		120	135		180	270	360
rads					$\frac{\pi}{2}$			$\frac{5\pi}{6}$			

Basic Trigonometric Functions

For more than 1000 years, trigonometry was mainly used by Greek, Arab, and Hindu scholars for their work in astronomy. Nowadays, trigonometry is a vital part of many branches of science.

In some sense, there is only one basic trigonometric function, **sine**, which is defined to be the ratio between the opposite side and the hypotenuse of an acute angle θ of a right triangle. We write

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$



All of the other trigonometric functions that we use can be defined in terms of sine. For example, the sine of the complementary angle of θ, that is the angle $\frac{\pi}{2} - \theta$, is called the *complementary sine* which we abbreviate as **cosine**. We write

$$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

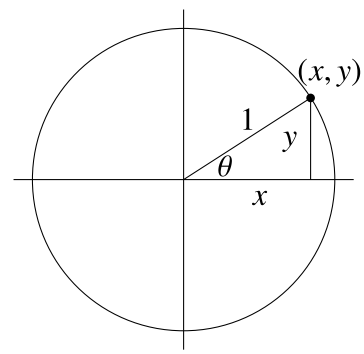
From here, we can define the four other main trigonometric functions.

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{h}{a} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{o}{a}$$
$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{h}{o} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{a}{o}$$

It is also important to note that sine and cosine can be defined using the unit circle $x^2 + y^2 = 1$. In particular, we define

$$\sin(\theta) = y$$
$$\cos(\theta) = x$$

For this reason, sine and cosine are often referred to as **circular functions**.



For an acute angle θ, these definitions match with our definition above. These definitions extend the domain of sine and cosine, and hence the other four trigonometric functions, to any real number θ.

You are required to know the value of all the basic trigonometric functions at all of the basic angles. There are many different ways of remembering the values for the angels $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ (for example, remembering the special triangles) and for relating these values to other quadrants (for example, the CAST rule or using a graph). Different students will have learned different ways of calculating these. As long as it is correct and you remember it, it is fine.

EXERCISE 2 Fill in the following table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$\sin(\theta)$												
$\cos(\theta)$												
$\tan(\theta)$												

EXERCISE 3

Write all angles θ with $0 \leq \theta \leq 4\pi$ such that

(a) $\sin(\theta) = \dfrac{1}{2}$

(b) $\cos(\theta) = \dfrac{\sqrt{3}}{2}$

(c) $\tan(\theta) = \dfrac{1}{\sqrt{3}}$

(d) $\sec(\theta) = 2$

(e) $\cot(\theta) = 1$

(f) $\sin(\theta) = 0$

(g) $\cos(\theta) = 0$

(h) $\cot(\theta) = 0$

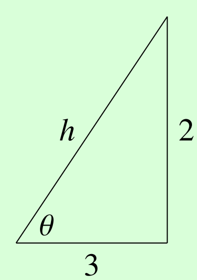
Trigonometric Values for Non-Standard Angles

In addition to being able to evaluate the trigonometric functions at the standard angles, you will need to know how to determine the value of a trigonometric function given the value of a different trigonometric function. The method we will use here, drawing a triangle, will be very useful throughout this text.

EXAMPLE 1

Given that $\tan(\theta) = \dfrac{2}{3}$ and $0 < \theta < \dfrac{\pi}{2}$, find $\cos(\theta)$.

Solution: We are given that $\tan(\theta) = \dfrac{2}{3}$. So, we can draw a right triangle where the opposite side has length 2 and the adjacent side has length 3. We can then use the Pythagorean Theorem to calculate the hypotenuse.



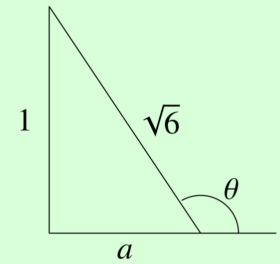
$$
\begin{aligned}
h^2 &= o^2 + a^2 \\
h^2 &= 2^2 + 3^2 \\
h^2 &= 13 \\
|h| &= \sqrt{13} \\
h &= \sqrt{13}
\end{aligned}
$$

since the hypotenuse is always positive

Thus, $\cos(\theta) = \dfrac{a}{h} = \dfrac{3}{\sqrt{13}}$.

EXAMPLE 2 Given that $\sin(\theta) = \dfrac{1}{\sqrt{6}}$ and $\dfrac{\pi}{2} < \theta < \pi$, find $\tan(\theta)$.

Solution: Drawing the right triangle corresponding to $\sin(\theta) = \dfrac{1}{\sqrt{6}}$ and using the Pythagorean theorem gives



$$\begin{aligned} h^2 &= o^2 + a^2 \\ (\sqrt{6})^2 &= 1^2 + a^2 \\ 6 &= 1 + a^2 \\ 5 &= a^2 \\ \sqrt{5} &= |a| \end{aligned}$$

Since we are in the second quadrant, a will be negative. So, $a = -\sqrt{5}$.

Thus, $\tan(\theta) = \dfrac{o}{a} = \dfrac{1}{-\sqrt{5}}$.

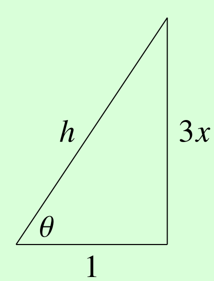
EXERCISE 4 Given that $\cos(\theta) = \dfrac{1}{3}$ and $\dfrac{3\pi}{2} < \theta < 2\pi$, find $\tan(\theta)$.

EXERCISE 5 Given that $\tan(\theta) = 4$ and $\pi < \theta < \dfrac{3\pi}{2}$, find $\sin(\theta)$.

The next couple of examples are similar to what we will need in Chapter 8.

EXAMPLE 3 Given that $\tan(\theta) = 3x$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find $\sec(\theta)$.

Solution: Drawing the right triangle (in the first quadrant) corresponding to $\tan(\theta) = \dfrac{3x}{1}$ and using the Pythagorean theorem gives



$$\begin{aligned} h^2 &= o^2 + a^2 \\ h^2 &= (3x)^2 + 1^2 \\ h^2 &= 9x^2 + 1 \\ |h| &= \sqrt{9x^2+1} \\ h &= \sqrt{9x^2+1} \end{aligned}$$

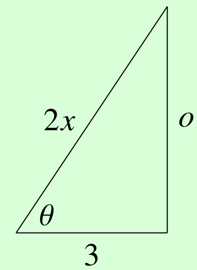
since the hypotenuse is always positive.

Thus, $\sec(\theta) = \dfrac{h}{a} = \dfrac{\sqrt{9x^2+1}}{1} = \sqrt{9x^2+1}$.

EXAMPLE 4

Given that $\sec(\theta) = \dfrac{2x}{3}$ and $0 < \theta < \dfrac{\pi}{2}$, find $\tan(\theta)$.

Solution: Drawing the right triangle corresponding to $\sec(\theta) = \dfrac{2x}{3}$ and using the Pythagorean theorem gives



$$\begin{aligned} h^2 &= o^2 + a^2 \\ (2x)^2 &= o^2 + 3^2 \\ 4x^2 - 9 &= o^2 \\ \sqrt{4x^2 - 9} &= |o| \\ \sqrt{4x^2 - 9} &= o \end{aligned}$$

since we are in the first quadrant.

Thus, $\tan(\theta) = \dfrac{o}{a} = \dfrac{\sqrt{4x^2 - 9}}{3}$.

EXERCISE 6

Given that $\sin(\theta) = \dfrac{2x}{3}$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find $\cos(\theta)$.

EXERCISE 7

Given that $\sec(\theta) = 2x$ and $0 < \theta < \dfrac{\pi}{2}$, find $\tan(\theta)$ and $\sin(\theta)$.

Trigonometric Identities

You may have seen in high-school that there are many, many trigonometric identities. The good news is that if you to know the following six identities, then you can easily derive all the trigonometric identities needed in this book.

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \sin(-x) &= -\sin(x), && \text{sin is an odd function} \\ \cos(-x) &= \cos(x), && \text{cos is an even function} \\ \sin(2x) &= 2\sin(x)\cos(x) \\ \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \end{aligned}$$

EXAMPLE 5

Show that $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

Solution: We have

$$\begin{aligned} \cos(2\theta) &= \cos(\theta + \theta) \\ &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) \\ &= \cos^2(\theta) - \sin^2(\theta) \end{aligned}$$

EXERCISE 8

Show that $\sec^2(x) = \tan^2(x) + 1$.

EXAMPLE 6

Find all values of x in the interval $[0, 2\pi]$ that satisfy $2\cos(x) + \sin(2x) = 0$.

Solution: Using the identity $\sin(2x) = 2\sin(x)\cos(x)$ gives

$$\begin{aligned} 2\cos(x) + 2\sin(x)\cos(x) &= 0 \\ 2\cos(x)(1 + \sin(x)) &= 0 \end{aligned}$$

For this to equal 0, we must have either $\cos(x) = 0$ or $\sin(x) = -1$.

We have $\cos(x) = 0$ when $x = \dfrac{\pi}{2}$ or $x = \dfrac{3\pi}{2}$, and $\sin(x) = -1$ when $x = \dfrac{3\pi}{2}$.

Thus, all x are $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$.

EXERCISE 9

Find all values of x in the interval $[0, 2\pi]$ that satisfy $\sin(x)\cos(x) = \dfrac{\sqrt{3}}{4}$.

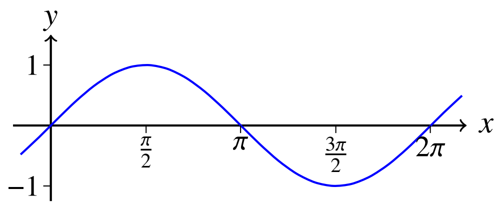
EXERCISE 10

Find all values of x in the interval $[0, 2\pi]$ that satisfy $2\sin^2(x) + \cos(x) = 1$.

Graphs of Trigonometric Functions

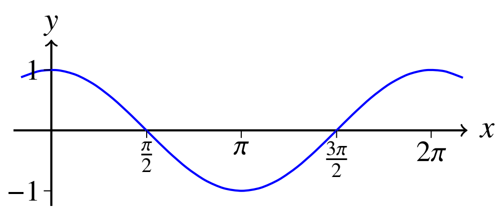
We expect that you know the graphs of the three basic trigonometric functions: sine, cosine, and tangent.

Key features of $y = \sin(x)$**:**



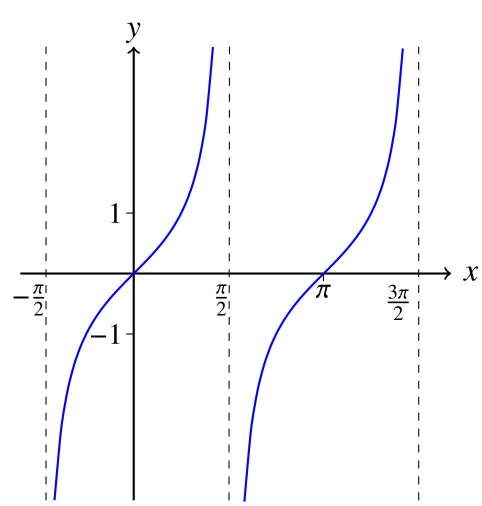
- Periodic with period 2π.
- Range $[-1, 1]$.
- Increasing on $\left[-\dfrac{\pi}{2} + 2k\pi, \dfrac{\pi}{2} + 2k\pi\right]$, for any integer k.
- Decreasing on $\left[\dfrac{\pi}{2} + 2k\pi, \dfrac{3\pi}{2} + 2k\pi\right]$, for any integer k.
- Concave down on $[2k\pi, \pi + 2k\pi]$, for any integer k.
- Concave up on $[\pi + 2k\pi, 2\pi + 2k\pi]$, for any integer k.

Key features of $y = \cos(x)$**:**



- Periodic with period 2π.
- Range $[-1, 1]$.
- Increasing on $[\pi + 2k\pi, 2\pi + 2k\pi]$, for any integer k.
- Decreasing on $[2k\pi, \pi + 2k\pi]$, for any integer k.
- Concave down on $\left[-\dfrac{\pi}{2} + 2k\pi, \dfrac{\pi}{2} + 2k\pi\right]$, for any integer k.
- Concave up on $\left[\dfrac{\pi}{2} + 2k\pi, \dfrac{3\pi}{2} + 2k\pi\right]$, for any integer k.

Key features of $y = \tan(x)$**:**



- Periodic with period π.
- Range $(-\infty, \infty)$.
- Increasing on $\left(-\dfrac{\pi}{2} + k\pi, \dfrac{\pi}{2} + k\pi\right)$ for any integer k.
- Concave down on $\left(-\dfrac{\pi}{2} + k\pi, k\pi\right)$ for any integer k.
- Concave up on $\left(k\pi, \dfrac{\pi}{2} + k\pi\right)$ for any integer k.
- Vertical asymptotes at $x = -\dfrac{\pi}{2} + k\pi$.

It is also expected that you can use transformations on graphs to sketch functions of the form

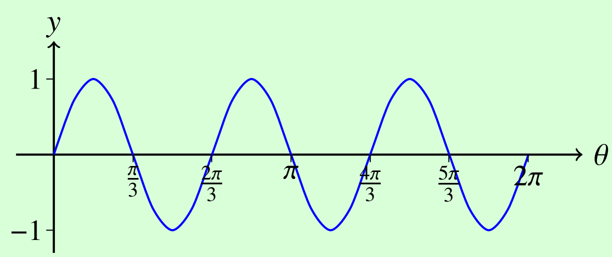
$$y = a\sin(b(x-c)) + d$$
$$y = a\cos(b(x-c)) + d$$
$$y = a\tan(b(x-c)) + d$$

In particular, recall that:

- a corresponds to a scaling in y, and, if $a < 0$, a reflection over the x-axis.
- b corresponds to a scaling in x, and, if $b < 0$, a reflection over the y-axis.
- c corresponds to a horizontal translation.
- d corresponds to a vertical translation.

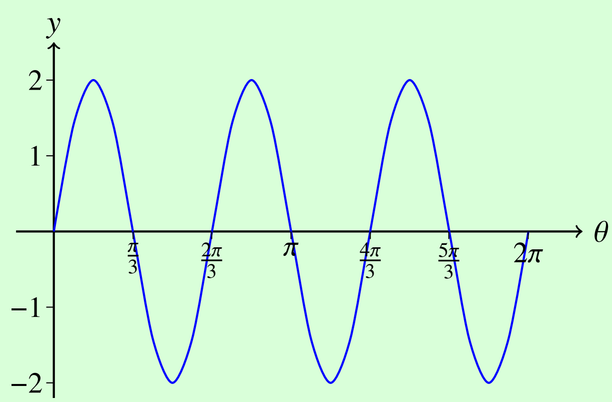
EXAMPLE 7 Sketch the graph of $f(\theta) = 2\sin(3\theta) + 1$ for $0 \leq \theta \leq 2\pi$.

Solution: The argument 3θ is a horizontal compression by a factor of 3.



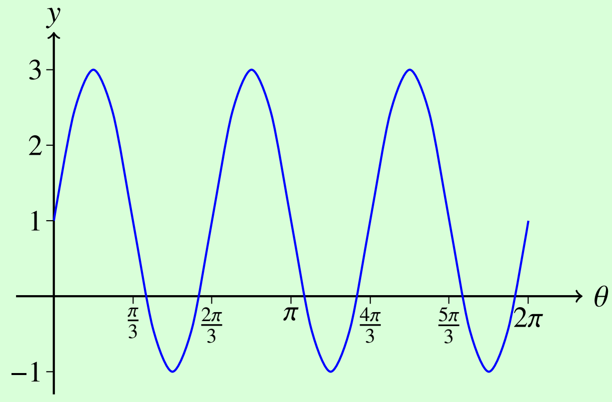
Graph of $y = \sin(3\theta)$.

The coefficient 2 of $\sin(3\theta)$ is a vertical stretch by a factor of 2.



Graph of $y = 2\sin(3\theta)$.

The +1 is a vertical translation by 1 unit.

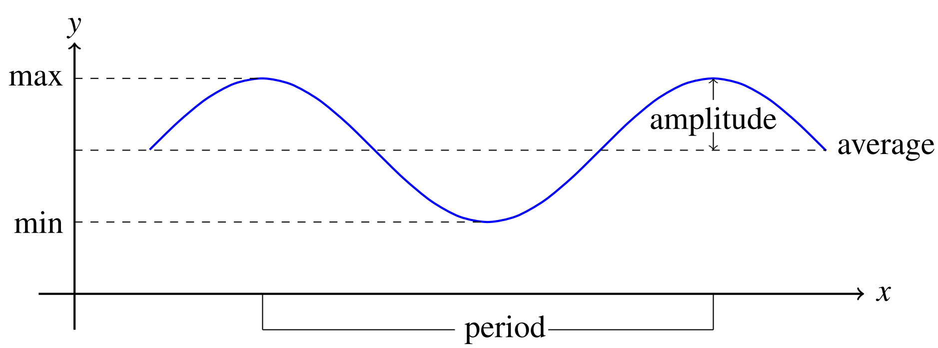


Graph of $y = 2\sin(3\theta) + 1$.

EXERCISE 11 Sketch one period of the graph of $f(\theta) = 3\cos\left(\frac{\theta}{2}\right) - 1$.

EXERCISE 12 Sketch one period of the graph of $f(\theta) = -\tan(2\theta)$.

For modelling, it is important to be able to determine the following values directly from an equation of the form $f(x) = a\sin(b(x-c)) + d$ or $f(x) = a\cos(b(x-c)) + d$:



Maximum: The highest y-value.

Minimum: The lowest y-value.

Average: The middle value on the curve (half-way between the maximum and minimum).

Amplitude: The difference between the maximum and the average.

Period: The distance between successive peaks.

With a deep understanding of how the a, b, c, and d modify the graph of the function, one can read all of the values directly from the equation. To develop the level of understanding, it is recommend to practice by graphing these equations step-by-step as we did in Example 2.5.7.

EXAMPLE 8 Determine the max, min, average, amplitude, and period of $f(x) = 4\cos\left(\frac{x}{2}\right) + 3$.

Solution: The coefficient of 4 corresponds to a vertical stretch by a factor of 4. So, the range of $4\cos\left(\frac{x}{2}\right)$ will be $[-4, 4]$. But, we then do a vertical translation by 3, so the range of $4\cos\left(\frac{x}{2}\right) + 3$ will be $[-1, 7]$. Therefore, the maximum value is 7 and the minimum value is -1.

Since the graph has been translated up 3 units, the average value is 3.

The value of a tells us the amplitude. So, the amplitude is 4.

The argument $\frac{x}{2}$ is a horizontal stretch by a factor of 2. Thus, the period has been stretched to be 4π.

EXERCISE 13 Determine the max, min, average, amplitude, and period of $f(\theta) = 3\sin(2\theta) + 1$.

EXERCISE 14 Determine the max, min, average, amplitude, and period of $f(\theta) = -2\cos(\theta/3) + \pi$.

2.5.2 Modelling with Trigonometric Functions

There are two main ways in which trigonometric functions are used in mathematical modelling: when dealing with triangles, and when dealing with wave functions. We will demonstrate this with some examples.

EXAMPLE 9 One's ultradian rhythm is a cosine wave with a period of approximately 4 hours, an average value of 0, and an amplitude of 0.4. Write an equation for one's ultradian rhythm U at any time t.

Solution: We want $U(t) = a\cos(bt)$. The period of cos is normally 2π, but we know want it to be 4 hours. So, we need to do a horizontal compression. In particular, we want to choose b so that when $t = 4$, the inside of $\cos(bt)$ will equal 2π. That is, we want

$$b \cdot 4 = 2\pi \Rightarrow b = \frac{2\pi}{4} = \frac{\pi}{2}$$

Next, we want the 'height' of the wave to be 0.4. Since the maximum of cos is 1, we need to scale the height by a factor of 0.4. So, $a = 0.4$.

Hence, $U(t) = 0.4\cos\left(\frac{\pi}{2}t\right)$.

EXAMPLE 10

Assume that a city's high tide occurs at midnight and the height of the water h in feet at t hours after midnight is given by

$$h(t) = 5 + 4.1\cos\left(\frac{\pi}{6}t\right)$$

(a) What is the average height of the water?

(b) What is the maximum height?

(c) What is the period of this function, and what does it represent in terms of tides?

(d) When was low tide and what was the water level at that time?

Solution: (a) The vertical translation of the cosine graph is 5, so the average height of the water is 5ft.

(b) The maximum value of a cosine graph is the average value plus the amplitude. Thus, the maximum height of the water is $5 + 4.1 = 9.1$ ft.

(c) The period of cosine is normally 2π. So, it will now be when

$$\begin{aligned} 2\pi &= \frac{\pi}{6}t \\ 12\pi &= \pi t \\ 12 &= t \end{aligned}$$

So, the period is 12 hours. This means that there will be a 12 hour period between high-tides.

(d) Low tide occurs when cos reaches its minimum value of -1. We know that occurs at $\theta = \pi + 2\pi k$ for any integer k. So, when

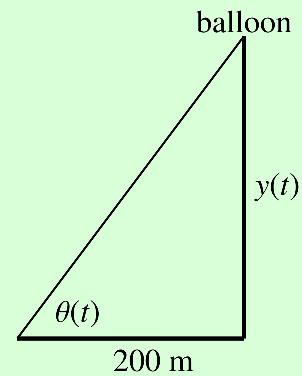
$$\pi = \frac{\pi}{6}t \Rightarrow 6 = t$$

But, since the period is 12 hours, we know that it will also occur 12 hours later. So, low tide will be at 6 am and 6 pm. The height will be 0.9 feet.

EXAMPLE 11

An observer stands 200 m from the launch site of a hot air balloon. The balloon rises vertically at a constant rate of 4 m/s. As the balloon rises, its distance from the ground y and its angle of elevation θ change simultaneously.

Find an equation relating the angle of elevation to the distance above the ground. Use the equation to determine the height of the balloon when the angle is $\frac{\pi}{3}$ radians.



Solution: From the picture we see that we have a triangle where we are given the adjacent side and want to relate the angle to the opposite side. This tells us that we want to use the tangent function. We get

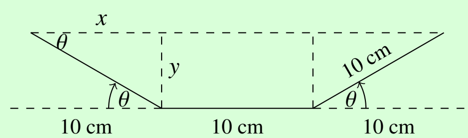
$$\tan(\theta) = \frac{y}{200}$$

Therefore, the height y of the balloon when $\theta = \frac{\pi}{3}$ is

$$y = 200\tan\left(\frac{\pi}{3}\right) = 200\sqrt{3}\text{ m}$$

EXAMPLE 12

A rain gutter is constructed from a 30 cm wide metal sheet by bending up one third of the width on each side by an angle of θ. Find the cross-sectional area of the gutter as a function of θ.



Solution: We let y denote the height of the gutter and x denote the width of the diagonal piece. The cross-sectional area of the gutter is then

$$\text{Area} = \frac{1}{2}xy + 10y + \frac{1}{2}xy = xy + 10y$$

Using the triangle, we get that $\frac{x}{10} = \cos(\theta)$ and $\frac{y}{10} = \sin(\theta)$. Hence, $x = 10\cos(\theta)$ and $y = 10\sin(\theta)$. Therefore, the cross-sectional area of the gutter is

$$\begin{aligned}\text{Area} &= 10\cos(\theta)\cdot 10\sin(\theta) + 10\cdot 10\sin(\theta)\\ &= 50\sin(2\theta) + 100\sin(\theta)\end{aligned}$$

EXERCISE 15

Suppose high tide is at midnight and the water level at high tide is 8.7 feet while the water level at low tide is 0.3 feet. Assuming the next high tide occurs 12 hours later at noon, find a formula for the water level as a function of t where t is the time in hours (suppose midnight is given by $t = 0$).

EXERCISE 16

Suppose we have data that can be modelled by a trigonometric function of the form $y = A\cos(B(t - C)) + D$, where A, B, C, D are constants. A maximum point occurs at $(7, 23)$ and the preceding minimum point occurs at $(2, 7)$. Determine a cosine function that fits this data.

EXERCISE 17

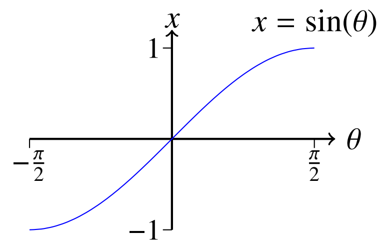
A 3 m long ladder rests against a vertical wall. Create a function which represents the distance that the bottom of the ladder is from the wall in terms of the angle between the ladder and the floor.

2.5.3 Inverse Trigonometric Functions

We now define inverses of the sine, cosine, and tangent functions. To do so, we need to ensure we pick an interval on which the function not only passes the horizontal line test, but also contains all values in the range of the function. The tricky part here is that there are infinitely many different intervals we can choose. Here we will indicate the standard choice for each function, called the **principal inverse**. Note that it is very, very important that you remember which intervals we picked as this determines the ranges of the inverse trigonometric function.

Arcsine

Observe that sin is invertible on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We use this interval to define the principal inverse.



DEFINITION
Arcsine

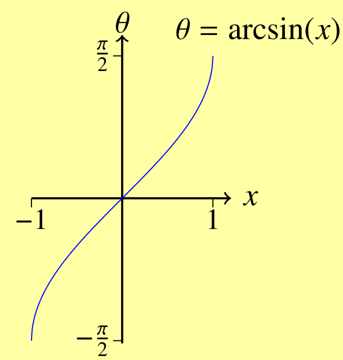
We define arcsin to be the inverse function of sin on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Te domain of arcsin is $[-1, 1]$.

The range of arcsin is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

That is, if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $\sin(\theta) = x$, then

$$\arcsin(x) = \theta$$



Observe that since sin inputs an angle θ in radians and outputs a ratio, arcsin inputs a ratio and outputs an angle in radians.

REMARK

It is common to use the notation $\sin^{-1}$ for arcsin. Do *not* confuse this with

$$\frac{1}{\sin(x)} = \csc(x)$$

EXAMPLE 13

Evaluate the following:

(a) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

(b) $\arcsin\left(-\frac{1}{2}\right)$

Solution: (a) The question is essentially asking for what angle θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is $\sin(\theta) = \frac{\sqrt{3}}{2}$.

Since we know that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, we get that $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

(b) We want to find the angle θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(\theta) = -\frac{1}{2}$.

Since, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$, we get that $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

EXERCISE 18

Evaluate the following:

(a) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

(b) $\arcsin(1)$

(c) $\arcsin(2)$

EXAMPLE 14

Simplify $\sin\left(\arcsin\left(\frac{1}{3}\right)\right)$.

Solution: The key to solving these types of problems is to use let statements.

Let $\theta = \arcsin\left(\frac{1}{3}\right)$. Then, we are trying to evaluate $\sin(\theta)$. By definition of inverse functions we have that

$$\theta = \arcsin\left(\frac{1}{3}\right) \quad \text{if and only if} \quad \sin(\theta) = \frac{1}{3}$$

So,

$$\sin\left(\arcsin\left(\frac{1}{3}\right)\right) = \sin(\theta) = \frac{1}{3}$$

EXAMPLE 15

Evaluate $\arcsin\left(\sin\left(\dfrac{11\pi}{5}\right)\right)$.

Solution: Let $y = \sin\left(\dfrac{11\pi}{5}\right)$. We want to figure out $\arcsin(y)$. By definition of inverses:

$$y = \sin\left(\frac{11\pi}{5}\right) \quad \text{if and only if} \quad \arcsin(y) = \ldots$$

WAIT! The range of arcsin is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, so arcsin cannot output $\dfrac{11\pi}{5}$.

To fix this, we use the fact that sin is periodic with period 2π. In particular, we know that

$$\sin\left(\frac{11}{5}\pi\right) = \sin\left(2\pi + \frac{1}{5}\pi\right) = \sin\left(\frac{1}{5}\pi\right)$$

So,

$$\arcsin\left(\sin\left(\frac{11}{5}\pi\right)\right) = \arcsin\left(\sin\left(\frac{1}{5}\pi\right)\right)$$

Now, we can let $y = \sin\left(\dfrac{\pi}{5}\right)$ and we are trying to figure out $\arcsin(y)$.

By definition of inverses we have

$$y = \sin\left(\frac{\pi}{5}\right) \quad \text{if and only if} \quad \arcsin(y) = \frac{\pi}{5}$$

Thus,

$$\begin{aligned}\arcsin\left(\sin\left(\frac{11}{5}\pi\right)\right) &= \arcsin\left(\sin\left(\frac{1}{5}\pi\right)\right)\\ &= \frac{\pi}{5}\end{aligned}$$

EXERCISE 19

What is the domain and range of $\arcsin(\sin(x))$?

Sketch the graph of $f(x) = \arcsin(\sin(x))$.

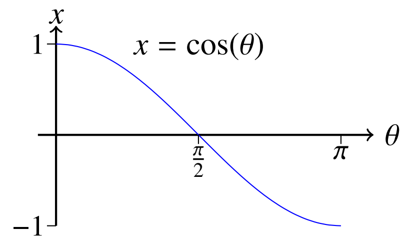
EXERCISE 20

What is the domain and range of $\sin(\arcsin(x))$?

Sketch the graph of $f(x) = \sin(\arcsin(x))$.

Arccosine

Observe that cos is invertible on $[0, \pi]$. We use this interval to define the principal inverse.



DEFINITION
Arccosine

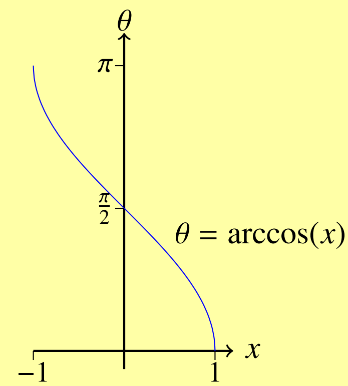
We define arccos to be the inverse function of cos on $[0, \pi]$.

The domain of arccos is $[-1, 1]$.

The range of arccos is $[0, \pi]$.

That is, if $0 \leq \theta \leq \pi$ and $\cos(\theta) = x$, then

$$\arccos(x) = \theta$$



Of course, just like arcsine, arccosine inputs a ratio and outputs an angle in radians.

REMARK

It is common to use the notation $\cos^{-1}$ for arccos. Do *n*ot confuse this with

$$\frac{1}{\cos(x)} = \sec(x)$$

EXAMPLE 16

Evaluate the following:

(a) $\arccos\left(-\dfrac{\sqrt{3}}{2}\right)$

(b) $\arccos(1)$

Solution: (a) We want to find the angle θ in $[0, \pi]$ such that $\cos(\theta) = -\dfrac{\sqrt{3}}{2}$.

We know that $\cos\left(\dfrac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$. Hence, $\arccos\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}$.

(b) We want to find the angle θ in $[0, \pi]$ such that $\cos(\theta) = 1$.

Since $\cos(0) = 1$, we get that $\arccos(1) = 0$.

EXERCISE 21

Evaluate the following:

(a) $\arccos\left(\frac{1}{2}\right)$

(b) $\arccos(-1)$

EXAMPLE 17

Evaluate $\arccos\left(\cos\left(-\frac{\pi}{7}\right)\right)$.

Solution: We observe that $-\frac{\pi}{7}$ is not in the range of arccos. Therefore, just like in Example 2.5.15 we need to shift this angle into the principle interval.

Using the trigonometric identity $\cos(-x) = \cos(x)$ gives

$$\cos\left(-\frac{\pi}{7}\right) = \cos\left(\frac{\pi}{7}\right)$$

Thus,

$$\arccos\left(\cos\left(-\frac{\pi}{7}\right)\right) = \arccos\left(\cos\left(\frac{\pi}{7}\right)\right)$$

Let $y = \cos\left(\frac{\pi}{7}\right)$. By definition of inverses we have

$$y = \cos\left(\frac{\pi}{7}\right) \quad \text{if and only if} \quad \arccos(y) = \frac{\pi}{7}$$

Thus,

$$\arccos\left(\cos\left(-\frac{\pi}{7}\right)\right) = \frac{\pi}{7}$$

EXERCISE 22

What is the domain and range of $\arccos(\cos(x))$?

Sketch the graph of $f(x) = \arccos(\cos(x))$.

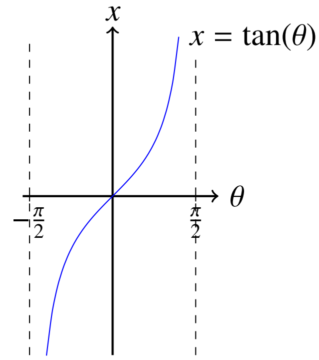
EXERCISE 23

What is the domain and range of $\cos(\arccos(x))$?

Sketch the graph of $f(x) = \cos(\arccos(x))$.

Arctangent

Observe that tan is invertible on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We use this interval to define the principle inverse.



DEFINITION
Arctangent

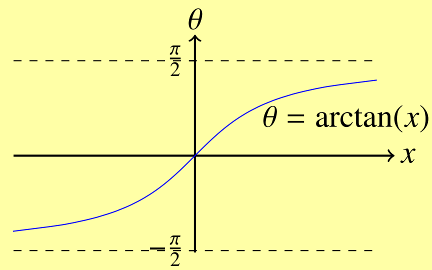
We define arctan to be the inverse function of tan on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The domain of arctan is (∞, ∞).

The range of arctan is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

That is, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan(\theta) = x$, then

$$\arctan(x) = \theta$$



As with arcsine and arccosine, we note that arctan inputs a ratio and outputs an angle in radians.

REMARK

It is also common to use the notation $\tan^{-1}$ for arctan. Again, do *not* confuse this with

$$\frac{1}{\tan(x)} = \cot(x)$$

EXAMPLE 18

Evaluate the following:

(a) $\arctan(1)$

(b) $\arctan\left(-\sqrt{3}\right)$

Solution: (a) We want to find the angle θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(\theta) = 1$.

We know that $\tan\left(\frac{\pi}{4}\right) = 1$. Hence, $\arctan(1) = \frac{\pi}{4}$.

(b) We want to find the angle θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(\theta) = -\sqrt{3}$.

Since $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$, we get that $\arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$.

EXERCISE 24 Evaluate the following:

(a) $\arctan(0)$

(b) $\arctan\left(-\frac{1}{\sqrt{3}}\right)$

EXERCISE 25 What is the domain and range of $\arctan(\tan(x))$?

EXERCISE 26 What is the domain and range of $\tan(\arctan(x))$?

For the next type of problem, there are two main methods one can use to solve them. The first method, which we will typically use, is to draw a triangle. The other method is to use trigonometric identities.

EXAMPLE 19 Evaluate $\sin(\arctan(7))$.

Solution: As we did with the similar problems above, we start with a let statement.

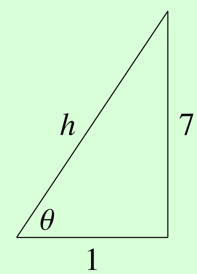
Let $\theta = \arctan(7)$. We want to find $\sin(\theta)$. By definition of inverses

$$\theta = \arctan(7) \quad \text{if and only if} \quad \tan(\theta) = 7$$

Now, the trick to evaluate $\sin(\theta)$ is to draw a triangle using the fact that we know that

$$\tan(\theta) = \frac{7}{1} = \frac{o}{a}$$

From the triangle, we can calculate the hypotenuse:



$$\begin{aligned} h^2 &= o^2 + a^2 \\ h^2 &= 7^2 + 1^2 \\ h^2 &= 50 \\ |h| &= \sqrt{50} \\ h &= \sqrt{50} \quad \text{since } h > 0 \end{aligned}$$

Thus, we get that $\sin(\arctan(7)) = \sin(\theta) = \frac{o}{h} = \frac{7}{\sqrt{50}}$.

EXAMPLE 20

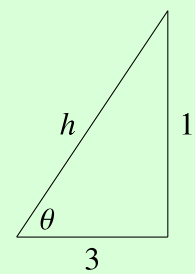
Evaluate $\cos\left(\arctan\left(\frac{1}{3}\right)\right)$.

Solution: Let $\theta = \arctan\left(\frac{1}{3}\right)$. We want to find $\cos(\theta)$.

By definition of inverses, we have that $\tan(\theta) = \frac{1}{3} = \frac{\text{o}}{\text{a}}$.

We use this to find the hypotenuse

$$\begin{aligned} h^2 &= 3^2 + 1^2 \\ h^2 &= 10 \\ \sqrt{h^2} &= \sqrt{10} \\ |h| &= \sqrt{10} \\ h &= \sqrt{10} \quad \text{since } h > 0 \end{aligned}$$



Thus,

$$\cos(\arctan(1/3)) = \cos(\theta) = \frac{\text{a}}{\text{h}} = \frac{3}{\sqrt{10}}$$

EXAMPLE 21

Evaluate $\sin\left(\arccos\left(-\frac{1}{4}\right)\right)$.

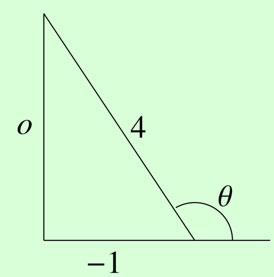
Solution: Let $\theta = \arccos\left(-\frac{1}{4}\right)$. We want to find $\sin(\theta)$.

By definition of inverses, we have that $\cos(\theta) = -\frac{1}{4} = \frac{\text{a}}{\text{h}}$.

Since the hypotenuse is always positive, the adjacent side has length -1.

We use this to find the length of the opposite side

$$\begin{aligned} 4^2 &= (-1)^2 + o^2 \\ 15 &= o^2 \\ \sqrt{15} &= \sqrt{o^2} \\ \sqrt{15} &= |o| \end{aligned}$$



We need to determine whether o is positive or negative. Observe that the range of arccos is $[0, \pi]$. So, $0 < \theta < \pi$, and hence $\sin(\theta) > 0$. Thus, o must be positive. Hence, $o = \sqrt{15}$.

Thus,

$$\sin\left(\arccos\left(-\frac{1}{4}\right)\right) = \sin(\theta) = \frac{\text{o}}{\text{h}} = \frac{\sqrt{15}}{4}$$

EXERCISE 27 Evaluate $\tan\left(\arcsin\left(\frac{1}{2}\right)\right)$.

EXERCISE 28 Evaluate $\cos\left(\arcsin\left(\frac{1}{\sqrt{3}}\right)\right)$.

2.5.4 Calculus of Trigonometric Functions

Derivatives of Trigonometric Functions

You know the derivative of sine and cosine from high-school. From these we can find the derivatives of the other basic trigonometric functions.

function	**derivative**
$f(x) = \sin(x)$	$f'(x) = \cos(x)$
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$
$f(x) = \tan(x)$	$f'(x) = \sec^2(x)$
$f(x) = \sec(x)$	$f'(x) = \sec(x)\tan(x)$
$f(x) = \csc(x)$	$f'(x) = -\csc(x)\cot(x)$
$f(x) = \cot(x)$	$f'(x) = -\csc^2(x)$

EXAMPLE 22 Determine the derivative of the following functions.

(a) $f(x) = \tan(x)\sec(x)$
(b) $g(t) = \csc(e^t)$
(c) $h(y) = \ln(|\sec(y)|)$

Solution: (a) We get

$$\begin{aligned} f'(x) &= \sec^2(x)\cdot\sec(x) + \tan(x)\cdot\sec(x)\tan(x) \\ &= \sec^3(x) + \tan^2(x)\sec(x) \end{aligned}$$

(b) We get $g'(t) = -\csc(e^t)\cot(e^t)\cdot e^t$.

(c) We get

$$\begin{aligned} h'(y) &= \frac{1}{|\sec(y)|}\cdot\frac{\sec(y)}{|\sec(y)|}\cdot\sec(y)\tan(y) \\ &= \frac{\sec^2(y)}{|\sec^2(y)|}\cdot\tan(y) \\ &= \tan(y) \end{aligned}$$

EXERCISE 29

Determine the derivative of the following functions.

(a) $f(x) = \cot(3x) + 2^x$

(b) $g(x) = \tan(\sqrt{t+1})$

(c) $h(x) = \ln(|\sec(x) + \tan(x)|)$

For the derivative of inverse trigonometric functions, we can use a method similar to how we found the derivative of $\log_a(x)$. We get:

function	**derivative**
$f(x) = \arcsin(x)$	$f'(x) = \dfrac{1}{\sqrt{1-x^2}}$
$f(x) = \arccos(x)$	$f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$
$f(x) = \arctan(x)$	$f'(x) = \dfrac{1}{1+x^2}$

EXAMPLE 23

Determine the derivative of the following functions.

(a) $f(x) = x\arcsin(x+1)$

(b) $g(x) = \arctan(\ln(x))$

(c) $h(x) = x\arcsin(x) + \sqrt{1-x^2}$

Solution: (a) We get

$$\begin{aligned} f'(x) &= 1 \cdot \arcsin(x+1) + \frac{x}{\sqrt{1-(x+1)^2}} \\ &= \arcsin(x+1) + \frac{x}{\sqrt{-x^2-2x}} \end{aligned}$$

(b) We get $g'(x) = \dfrac{1}{1+(\ln(x))^2} \cdot \dfrac{1}{x}$.

(c) We get

$$\begin{aligned} h'(x) &= \left(1 \cdot \arcsin(x) + \frac{x}{\sqrt{1-x^2}}\right) + \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} \cdot (-2x) \\ &= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \arcsin(x) \end{aligned}$$

EXERCISE 30 Determine the derivative of the following functions.

(a) $f(x) = e^x \arctan(x)$

(b) $g(x) = \arcsin(x^2 + 1)$

(c) $h(x) = x \arccos(x) - \sqrt{1 - x^2}$

Antiderivatives of Trigonometric Functions

We can use our known derivatives (and the results of some of the examples/exercises) to make a list of antiderivatives of some trigonometric functions.

$\int \sin(x)\,dx = -\cos(x) + C$	$\int \cos(x)\,dx = \sin(x) + C$
$\int \tan(x)\,dx = \ln(\lvert\sec(x)\rvert) + C$	$\int \sec(x)\,dx = \ln(\lvert\sec(x) + \tan(x)\rvert) + C$
$\int \sec^2(x)\,dx = \tan(x) + C$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$\int \csc(x)\,dx = -\ln(\lvert\csc(x) + \cot(x)\rvert) + C$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$
$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C$	$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

EXAMPLE 24 Find all solutions of the differential equation $y' = \sec^2(x)$.

Solution: We have

$$y = \int \sec^2(x)\,dx$$
$$y = \tan(x) + C$$

EXERCISE 31 Solve the initial value problem $y' = \sec(x)\tan(x)$, $y(0) = \frac{2\pi}{3}$.

Section 2.5 Problems

1. Write all angles θ with $-2\pi \le \theta \le 2\pi$ such that
   (a) $\cos(\theta) = \dfrac{1}{\sqrt{2}}$
   (b) $\sin(\theta) = \dfrac{\sqrt{3}}{2}$
   (c) $\tan(\theta) = \sqrt{3}$
   (d) $\sec(\theta) = \dfrac{2}{\sqrt{3}}$
   (e) $\cot(\theta) = \sqrt{3}$
   (f) $\sin(\theta) = 1$
   (g) $\cos(\theta) = -1$
   (h) $\cot(\theta) = -1$
   (i) $\csc(\theta) = \sqrt{2}$
   (j) $\sec(\theta) = 0$
2. Given $\cos(\theta) = \dfrac{2}{3}$ and $0 < \theta < \dfrac{\pi}{2}$, find $\sin(\theta)$.
3. Given $\sin(\theta) = \dfrac{1}{5}$ and $0 < \theta < \dfrac{\pi}{2}$, find $\tan(\theta)$.
4. Given $\tan(\theta) = -4$ and $\dfrac{\pi}{2} < \theta < \pi$, find $\cos(\theta)$.
5. Given $\sin(\theta) = -\dfrac{1}{4}$ and $\pi < \theta < \dfrac{3\pi}{2}$, find $\tan(\theta)$.
6. Given $\cos(\theta) = \dfrac{3}{5}$ and $\dfrac{3\pi}{2} < \theta < 2\pi$, find $\sin(\theta)$.
7. Given $\cos(\theta) = \dfrac{x}{\sqrt{x^2+1}}$ and $0 < \theta < \pi$, find $\tan(\theta)$.
8. Given $\sin(\theta) = \dfrac{3}{x}$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find $\cos(\theta)$.
9. Given $\tan(\theta) = \dfrac{x}{2}$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find $\sec(\theta)$.
10. Given $\sec(\theta) = x$ and $0 < \theta < \dfrac{\pi}{2}$, find $\tan(\theta)$.
11. Given $\sin(\theta) = \dfrac{2x}{3}$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find $\sec(\theta)$ and $\tan(\theta)$.
12. Find all values of x in the interval $[0, 2\pi]$ that satisfy $\sin(x)\cos(x) = 0$.
13. Find all values of x in the interval $[0, 2\pi]$ that satisfy $2x\cos(x) + x = 0$.
14. Find all values of x in the interval $[0, 2\pi]$ that satisfy $\sin^2(x) = \dfrac{1}{4}$.
15. Find all values of x in the interval $[0, 2\pi]$ that satisfy $2\cos(x) + \sin(2x) = 0$.
16. Sketch one period of the graph of each function. Show each step as in Example 2.5.7.
   (a) $f(\theta) = 2\cos(\theta) + 1$
   (b) $f(\theta) = \sin(2\theta) - 1$
   (c) $f(\theta) = \tan\left(\dfrac{\theta}{2}\right)$
   (d) $f(\theta) = -2\cos(2\theta)$
   (e) $f(\theta) = \arcsin(2\theta)$
   (f) $f(\theta) = -\arccos\left(\dfrac{\theta}{2}\right)$
   (g) $f(\theta) = 2\arctan(\theta)$
17. Evaluate the following
   (a) $\arcsin\left(\dfrac{1}{\sqrt{2}}\right)$
   (b) $\arccos\left(\dfrac{\sqrt{3}}{2}\right)$
   (c) $\arccos(0)$
   (d) $\arctan(1)$
   (e) $\arcsin(-1)$
   (f) $\arctan(-\sqrt{3})$
   (g) $\sin\left(\arcsin\left(\dfrac{1}{4}\right)\right)$
   (h) $\arcsin\left(\sin\left(\dfrac{3\pi}{5}\right)\right)$
   (i) $\arcsin\left(\sin\left(\dfrac{7\pi}{3}\right)\right)$
   (j) $\arctan\left(\tan\left(\dfrac{2\pi}{3}\right)\right)$
18. Evaluate the following
   (a) $\sin(\arctan(3))$
   (b) $\cos\left(\arcsin\left(\dfrac{1}{4}\right)\right)$
   (c) $\tan\left(\arccos\left(\dfrac{1}{2}\right)\right)$
   (d) $\sin\left(\arccos\left(-\dfrac{2}{3}\right)\right)$
   (e) $\cos(\arctan(-2))$

19. Determine the derivative of the following functions.
    (a) $f(x) = \sin(2x+1)$
    (b) $f(x) = x\cos(x^2)$
    (c) $f(x) = \dfrac{1}{2x+3}$
    (d) $f(x) = |e^x - 1|$
    (e) $f(x) = \sec(x)\ln(x)$
    (f) $f(x) = \dfrac{\tan(e^x)}{x}$
    (g) $f(x) = \csc(2x^2+3x)$
    (h) $f(x) = \arcsin(3x) + x$
    (i) $f(x) = \arccos(xe^x)$
    (j) $f(x) = \arctan(\sqrt{x})$
    (k) $f(x) = \dfrac{\csc(2x)}{\sqrt{x}}$
    (l) $f(x) = \arctan(x+x^2)$
    (m) $f(x) = \arcsin(x)\tan(2x)$
    (n) $f(x) = \ln(|\csc(x)|)$

20. Verify the given function y is a solution of the differential equation.
    (a) $y = x\ln(x) - x$ for $y' = \ln(x)$
    (b) $y = \dfrac{x}{2} - \dfrac{\sin(2x)}{4}$ for $y' = \dfrac{1}{2} - \dfrac{1}{2}\cos(2x)$
    (c) $y = e^{\sin(x)}$ for $y' = \cos(x)y$
    (d) $y = \sec(x)$ for $y' = \tan(x)y$

21. Solve the initial value problem.
    (a) $y' = \dfrac{1}{\sqrt{1-x^2}}$, $y(0) = 2$
    (b) $y' = \sec^2(x)$, $y\left(\dfrac{3\pi}{4}\right) = 1$
    (c) $y' = \dfrac{-3}{\sqrt{1-x^2}}$, $y(1) = 1$
    (d) $y' = \dfrac{1}{1+x^2}$, $y(-1) = 2$
    (e) $y' = e^x + x$, $y(0) = 3$
    (f) $y' = \sin(x)$, $y\left(\dfrac{11\pi}{6}\right) = 1$
    (g) $y' = \sec(x)\tan(x)$, $y(0) = 2$
    (h) $y' = \dfrac{1}{1+(2x)^2}$, $y(1) = 0$
    (i) $y' = \dfrac{1}{\sqrt{1-\frac{x^2}{4}}}$, $y(1) = 1$

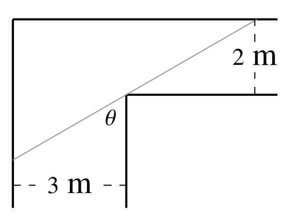
22. Determine the maximum, minimum, average, amplitude, and period of each function.
    (a) $f(\theta) = 4\cos(\theta) - 4$
    (b) $f(\theta) = 2\cos(3\theta) + 1$
    (c) $f(\theta) = -\sin\left(\dfrac{\theta}{2}\right)$
    (d) $f(\theta) = 3\sin(\pi\theta) - 2$
    (e) $f(\theta) = \sqrt{2}\cos\left(\dfrac{\theta}{3}\right) + \sqrt{2}$
    (f) $f(\theta) = 2\sin\left(\dfrac{\theta}{4}\right) + 2$
    (g) $f(\theta) = -2\cos(2\theta) + 4$

23. Let $f(x) = \sin(\arccos(x))$. State the domain and range of f and sketch its graph.

24. Let $f(x) = \tan(\arcsin(x))$. State the domain and range of f and simplify the equation.

25. A boat sails towards a lighthouse. Let θ be the angle from the base of the boat to the top of the lighthouse. Given that the top of the lighthouse is 20 m above sea level, determine the distance between the boat and the top of the lighthouse as a function of θ.

26. A steel pipe is being carried down a hallway 3 m wide. At the end of the hall, there is a right turn into a narrower hallway 2 m wide. Find a relationship between the length of the pipe and the angle of the pipe (see the figure) when the pipe simultaneously touches both walls.



27. A 15 ft long ladder leans against a vertical wall. The top of the ladder starts sliding down the wall. Find a relationship between the distance the top of the ladder is from the floor and the angle the ladder makes with the floor.

28. Show that the following equations are true using the identities on page 88
    (a) $1 + \cot^2(\theta) = \csc^2(\theta)$
    (b) $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
    (c) $\cos(2\theta) = 2\cos^2(\theta) - 1$
    (d) $\sin(x)\sin(2x) + \cos(x)\cos(2x) = \cos(x)$

Section 2.6: The Factorial Function

LEARNING OUTCOMES

1. Know how to evaluate the factorial function.
2. Know how to simplify expressions with factorial functions.
3. Know how to evaluate the binomial coefficient.

2.6.1 The Factorial Function

We now briefly look at the factorial function. This function will be used a lot in Chapter 7.

DEFINITION
Factorial

For any integer $n \geq 1$ we define $n!$ (read as 'n factorial') by

$$n! = 1 \cdot 2 \cdot 3 \cdots n$$

That is, it is the product of all integers between 1 and n inclusive.

For $n = 0$, we make a special definition of

$$0! = 1$$

REMARK

It is natural to wonder why 0! is defined to equal 1. The short answer is that we need $0! = 1$ to make formulas used in permutations and combinations work.

EXAMPLE 1

Evaluate 3! and 5!.

Solution: We have

$$\begin{aligned} 3! &= 1 \cdot 2 \cdot 3 = 6 \\ 5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120 \end{aligned}$$

EXAMPLE 2

Simplify $\dfrac{n!}{(n+2)!}$.

Solution: By definition, $n!$ is the product of all numbers from 1 to n and $(n+2)!$ is the product of all numbers from 1 to $n+2$. That is,

$$\frac{n!}{(n+2)!} = \frac{1 \cdot 2 \cdots (n-1) \cdot n}{1 \cdot 2 \cdots (n-1) \cdot n \cdot (n+1) \cdot (n+2)}$$

Cancelling off the common terms gives

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}$$

EXERCISE 1 Evaluate 4!.

EXERCISE 2 Simplify $\frac{n!}{(n-1)!}$.

Since this is a calculus course, the next natural question to ask is "What is the derivative and antiderivative of the factorial function?". The answer is also really easy... it doesn't have a derivative or antiderivative. Why not? Because it is only defined for integer values, so the limit which defines the derivative cannot exist.

REMARK

Of course, there are real world situations in which we need to take the derivative of the factorial function. In these cases we use a variation of the factorial function, called the Gamma function, which is defined for all real numbers.

2.6.2 Binomial Coefficients

In high school, you learned how to expand $(a+b)^2$ to get

$$(a+b)^2 = a^2 + 2ab + b^2$$

Perhaps, you also saw that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

But, how would you expand $(a+b)^{10}$?

It turns out that there is a nice formula for the coefficients when expanding $(a+b)^n$.

DEFINITION

Binomial Coefficient

For any non-negative numbers n and k, the **binomial coefficient** is defined by

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

EXAMPLE 3 Calculate $\binom{5}{0}$, $\binom{5}{1}$, $\binom{5}{2}$, and $\binom{5}{4}$.

Solution: We have

$$\binom{5}{0} = \frac{5!}{(5-0)! \cdot 0!} = \frac{5!}{5! \cdot 1} = 1$$
$$\binom{5}{1} = \frac{5!}{(5-1)! \cdot 1!} = \frac{5!}{4! \cdot 1} = 5$$
$$\binom{5}{2} = \frac{5!}{(5-2)! \cdot 2!} = \frac{5!}{3! \cdot 2!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2} = 10$$
$$\binom{5}{4} = \frac{5!}{(5-4)! \cdot 4!} = \frac{5!}{1 \cdot 4!} = 5$$

EXERCISE 3 Calculate $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$.

Using the binomial coefficients, we get

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}b^n$$

This is best demonstrated with some examples.

EXAMPLE 4 Expand $(a+b)^3$ using binomial coefficients.

Solution: We have

$$\binom{3}{0} = \frac{3!}{(3-0)! \cdot 0!} = 1$$
$$\binom{3}{1} = \frac{3!}{(3-1)! \cdot 1!} = \frac{3!}{2! \cdot 1} = 3$$
$$\binom{3}{2} = \frac{3!}{(3-2)! \cdot 2!} = \frac{3!}{1! \cdot 2!} = 3$$
$$\binom{3}{3} = \frac{3!}{(3-3)! \cdot 3!} = \frac{3!}{1 \cdot 3!} = 1$$

Thus,

$$\begin{aligned}(a+b)^3 &= \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}a^1b^2 + \binom{3}{3}b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

EXAMPLE 5 Expand $(x+3)^2$ using binomial coefficients.

Solution: We have

$$\binom{2}{0} = \frac{2!}{(2-0)! \cdot 0!} = 1$$
$$\binom{2}{1} = \frac{2!}{(2-1)! \cdot 1!} = \frac{2!}{1 \cdot 1} = 2$$
$$\binom{2}{2} = \frac{2!}{(2-2)! \cdot 2!} = \frac{2!}{1! \cdot 2!} = 1$$

Thus,

$$\begin{aligned}(x+3)^2 &= \binom{2}{0}x^2 + \binom{2}{1}x^1(3)^1 + \binom{2}{2}(3)^2 \\ &= x^2 + 6x + 9\end{aligned}$$

EXAMPLE 6 Expand $(x+2)^4$ using binomial coefficients.

Solution: We have

$$\binom{4}{0} = \frac{4!}{(4-0)! \cdot 0!} = 1$$
$$\binom{4}{1} = \frac{4!}{(4-1)! \cdot 1!} = \frac{4!}{3! \cdot 1} = 4$$
$$\binom{4}{2} = \frac{4!}{(4-2)! \cdot 2!} = \frac{4!}{2! \cdot 2!} = \frac{24}{2 \cdot 2} = 6$$
$$\binom{4}{3} = \frac{4!}{(4-3)! \cdot 3!} = \frac{4!}{1 \cdot 3!} = 4$$
$$\binom{4}{4} = \frac{4!}{(4-4)! \cdot 4!} = \frac{4!}{1 \cdot 4!} = 1$$

Thus,

$$\begin{aligned}(x+2)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3(2)^1 + \binom{4}{2}x^2(2)^2 + \binom{4}{3}x^1(2^3) + \binom{4}{4}(2)^4 \\ &= x^4 + 8x^3 + 24x^2 + 32x + 16\end{aligned}$$

EXERCISE 4 Expand the following using binomial coefficients.

(a) $(a+b)^2$
(b) $(x+1)^3$
(c) $(2x+3)^3$

Section 2.6 Problems

1. Simplify:
   (a) $\dfrac{5!}{3! \cdot 2!}$
   (b) $\dfrac{17!}{15!}$
   (c) $\dfrac{23!}{24!}$
   (d) $\dfrac{3!4!}{2!0!}$
   (e) $\dfrac{15!10!}{13!11!}$
   (f) $\dfrac{12!8!}{(9!)^2}$
   (g) $\dfrac{n!}{(n+1)!}$
   (h) $\dfrac{n!}{(n+2)!}$
   (i) $\dfrac{(n+1)!}{(n-1)!}$
   (j) $\dfrac{(2n)!}{(2n+2)!}$
   (k) $\dfrac{(2n)!}{2n!}$

2. Evaluate the binomial coefficient.
   (a) $\dbinom{5}{3}$
   (b) $\dbinom{7}{6}$
   (c) $\dbinom{6}{4}$
   (d) $\dbinom{8}{2}$

3. Expand using binomial coefficients.
   (a) $(a+b)^4$
   (b) $(x+1)^4$
   (c) $(a+b)^5$
   (d) $(x+2)^5$
   (e) $(x-2)^3$
   (f) $(2x+3)^4$
   (g) $(2x-1)^4$

4. Find the value of n that makes the expression true.
   (a) $n! = 24$
   (b) $n! = 120$
   (c) $(n-1)! = 6$
   (d) $n! = 56(n-2)!$

5. If two brown-eyed parents have 3 children, the probability that there will be exactly r blue-eyed children is given by $P(r) = \dfrac{3!(\frac{1}{4})^r(\frac{3}{4})^{3-r}}{r!(3-r)!}$, where $r = 0, 1, 2, 3$. Find the probability that exactly two of the children will be blue-eyed.

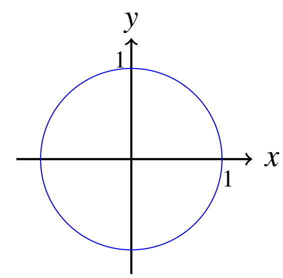
Section 2.7: Implicitly Defined Functions

LEARNING OUTCOMES

1. Understand the difference between explicitly and implicitly defined functions.
2. Know how to find the derivative of an implicitly defined function.
3. Know how to use logarithmic differentiation.

So far, we have just been looking at functions of the form $y = f(x)$. However, there are many situations where we have functions that cannot be written in this form.

For example, consider the circle $x^2 + y^2 = 1$. You are probably thinking "Wait! That is not a function as it does not pass the vertical line test!"



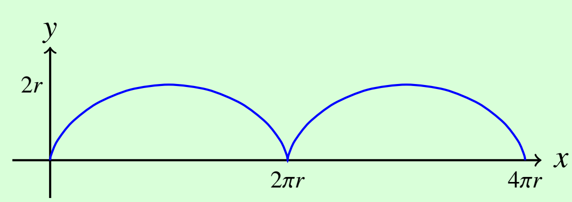
However, the vertical line test does not determine if a relation is a function or not. It only tests if the relation can be written in the form $y = f(x)$ (you weren't exactly lied to in high-school, but what you were taught wasn't the whole truth either). We will see in Chapter 11 that a circle is the graph of a function!

In cases where we cannot write y as a function of x, it is often helpful to treat instances of the y variable as if such a definition were possible. In doing so, we are treating the y in an expression like $x^2 + y^2 = 1$ as an "implicitly defined function" — we cannot provide an explicit (written) form for the function, but we know there's some relationship specified by the equation that guides y's behaviour relative to a second variable, x.

Importantly, treating y like a function will allow us to apply many calculus tools in a familiar way, allowing us, for example, to find tangent lines to points on curves that would not satisfy the vertical line test (like the circle above). More generally, treating y as an implicitly defined function will allow us to apply calculus to equations without first (or ever) "solving them for y".

EXAMPLE 1 A cycloid is the path that a point on a circle takes as the circle rolls along a straight line. For example, imagine a rock is stuck in a bicycle tire. The path that the rocks takes as the bicycle moves is a cycloid.

The graph of a cycloid is given.



A cycloid can be represented as an implicitly defined function

$$r\cos\left(\frac{x + \sqrt{y(2r-y)}}{r}\right) + y = r$$

where r is the radius of the circle.

2.7.1 Implicit Differentiation

To differentiate an implicitly defined function, we use the chain rule. In these cases, it is very important to be able to identify which variables are functions of the variable we are taking the derivative with respect to. This method is called **implicit differentiation**.

EXAMPLE 2

Find y' where $x^2 + \ln(xy) = \cos(y)$.

Solution: We take the derivative implicitly with respect to x of both sides.

Everywhere we see y, we think of it as a function of x. So, for example, we think of as xy as $x \cdot y(x)$. Hence, xy is now a product of functions, so we need to use the product rule. Similarly, $\cos(y)$ is $\cos(y(x))$ is now a composition of functions, so we need to use the chain rule. The implicit derivative of $x^2 + \ln(xy) = \cos(y)$ is

$$2x + \frac{1}{xy}(y + xy') = -\sin(y)y'$$

Then, we just solve for y'.

$$\begin{aligned}
\frac{1}{xy}(y + xy') &= -2x - \sin(y)y' \\
y + xy' &= -2x^2y - xy\sin(y)y' \\
xy' + xy\sin(y)y' &= -2x^2y - y \\
y'(x + xy\sin(y)) &= -2x^2y - y \\
y' &= \frac{-2x^2y - y}{x + xy\sin(y)}
\end{aligned}$$

EXAMPLE 3

Find y' if $x^3y^5 = 8y^3$.

Solution: We take the derivative implicitly with respect to x of both sides to get

$$\begin{aligned}
3x^2y^5 + 5x^3y^4y' &= 24y^2y' \\
5x^3y^4y' - 24y^2y' &= -3x^2y^5 \\
y' &= \frac{-3x^2y^5}{5x^3y^4 - 24y^2}
\end{aligned}$$

EXERCISE 1

Find y' if $xy^2 = \arctan(y)$.

EXAMPLE 4

Find the tangent line to $x^2 + y^2 = 4$ at the point $(1, \sqrt{3})$.

Solution: We take the derivative implicitly with respect to x of both sides to get

$$2x + 2yy' = 0$$

Substituting in $x = 1$ and $y = \sqrt{3}$ gives

$$\begin{aligned} 2 + 2 \cdot \sqrt{3} \cdot y' &= 0 \\ 2\sqrt{3}y' &= -2 \\ y' &= -\frac{1}{\sqrt{3}} \end{aligned}$$

The equation of the tangent line is $y = f(a) + f'(a)(x - a)$. What is a and $f(a)$ in this case?

a is the x-value of the point of tangency, and $f(a)$ is the y-value of the point of tangency. So, $a = 1$ (the x-coordinate) and $f(a) = \sqrt{3}$ (the y-coordinate).

$$\begin{aligned} y &= f(a) + f'(a)(x - a) \\ y &= \sqrt{3} - \frac{1}{\sqrt{3}}(x - 1) \end{aligned}$$

Implicitly defined functions also regularly occur as solutions to differential equations.

EXAMPLE 5

The general solution to the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y + \cos(y)}$ is the implicitly defined function

$$y^2 + \sin(y) = 2x^3 + C$$

EXERCISE 2

Find the tangent line to $\sin(x + y) = y^2 \cos(x)$ at the point $(0, 0)$.

EXERCISE 3

Kepler's equation describes the mean anomaly M in terms of the eccentric anomaly E and the eccentricity of the orbit e by $M = E - e\sin(E)$. Solving for E in terms of M cannot be done without approximation techniques. Use implicit differentiation to find $\frac{dE}{dM}$.

EXERCISE 4

Elliptic curve cryptography is the foundation behind the cryptocurrencies Bitcoin and Ethereum. The implicitly defined function $y^2 = x^3 - x + 1$ is an example of an elliptic curve. Find the equation of the tangent line to this curve at $(1, -1)$.

Derivatives of Inverse Functions

We can use implicit differentiation to find the derivative of inverse functions. This is best demonstrated with some examples.

EXAMPLE 6 Use implicit differentiation and the fact that $\ln(x)$ is the inverse of e^x to show that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$.

Solution: We have that

$$e^{\ln(x)} = x$$

Taking the derivative of both sides gives

$$\begin{aligned} e^{\ln(x)} \cdot \frac{d}{dx}(\ln(x)) &= 1 \\ x \cdot \frac{d}{dx}(\ln(x)) &= 1 \\ \frac{d}{dx}(\ln(x)) &= \frac{1}{x} \end{aligned}$$

as required.

EXAMPLE 7 Use implicit differentiation and the fact that $\arcsin(x)$ is the inverse of $\sin(x)$ to show the $\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$.

Solution: For any x with $-1 < x < 1$ we have

$$\sin(\arcsin(x)) = x$$

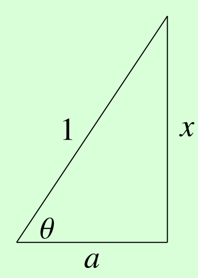
Taking the derivative of both sides gives

$$\cos(\arcsin(x)) \cdot \frac{d}{dx} \arcsin(x) = 1 \tag{2.1}$$

Let $\theta = \arcsin(x)$. Since $-1 < x < 1$, we get $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We want to find $\cos(\theta)$.

By definition of inverses, we have that $\sin(\theta) = x$.

Drawing the right triangle corresponding to $\sin(\theta) = \frac{x}{1}$ and then using the Pythagorean Theorem gives



$$\begin{aligned} 1^2 &= a^2 + x^2 \\ 1 - x^2 &= a^2 \\ \sqrt{1-x^2} &= |a| \\ \sqrt{1-x^2} &= a \text{ since } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \end{aligned}$$

Thus, $\cos(\arcsin(x)) = \frac{a}{h} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.

Substituting this into equation (2.1) we get

$$\begin{aligned}\sqrt{1-x^2}\cdot\frac{d}{dx}\arcsin(x) &= 1\\ \frac{d}{dx}\arcsin(x) &= \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

as required.

EXERCISE 5

Use implicit differentiation and the fact that $\arctan(x)$ is the inverse of $\tan(x)$ to show the $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$.

2.7.2 Logarithmic Differentiation

Consider the three functions $f(x) = x^n$, $g(x) = a^x$, $a > 0$, and $h(x) = x^x$, $x > 0$. These all look similar, but are actually very different. In particular:

- f is a power function and to take its derivative we use the rule $f'(x) = nx^{n-1}$.
- g is an exponential function and to take its derivative we use the rule $g'(x) = a^x\ln(a)$.
- h is not a power function, since the exponent is not a constant. It is also not an exponential function because its base is not a constant.

So, we need to find a new method for taking the derivative of a function like h... a function which has the form of a non-constant function to the power of another non-constant function.

The method we will use is called **logarithmic differentiation**. It is just a combination of implicit differentiation with logarithms.

We demonstrate logarithmic differentiation with a couple of examples.

EXAMPLE 8

Determine the derivative of $y = (x+1)^x$, $x > -1$.

Solution: Taking ln of both sides gives

$$\ln(y) = \ln((x+1)^x) = x\ln(x+1)$$

Take the derivative implicitly with respect to x of both sides to get

$$\begin{aligned}\frac{1}{y}y' &= \ln(x+1) + \frac{x}{x+1}\\ y' &= (x+1)^x\left(\ln(x+1) + \frac{x}{x+1}\right)\end{aligned}$$

EXAMPLE 9

Determine the derivative of $f(x) = (\ln(x))^{e^x}$, $x > 1$.

Solution: To make everything look nicer, we let $y = (\ln(x))^{e^x}$. Taking ln of both sides gives

$$\ln(y) = \ln((\ln x)^{e^x}) = e^x \ln(\ln(x))$$

Take the derivative implicitly with respect to x of both sides to get

$$\begin{aligned} \frac{1}{y}y' &= e^x \ln(\ln(x)) + e^x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\ y' &= (\ln(x))^{e^x} \left[e^x \ln(\ln(x)) + \frac{e^x}{x\ln(x)} \right] \end{aligned}$$

EXERCISE 6

Determine the derivative of $y = x^{\arctan(x)}$, $x > 0$.

We can also combine logarithmic differentiation with the properties of logarithms to simplify derivatives involving products, quotients, and powers.

EXAMPLE 10

Determine the derivative of $y = \dfrac{(x-2)^2(x-1)^3}{(x+2)^5(x-6)^4}$, $x > 6$.

Solution: Taking ln of both sides gives

$$\begin{aligned} \ln(y) &= \ln\left(\frac{(x-2)^2(x-1)^3}{(x+2)^5(x-6)^4}\right) \\ &= 2\ln(x-2) + 3\ln(x-1) - 5\ln(x+2) - 4\ln(x-6) \end{aligned}$$

by the properties of logarithms.

Next, take the derivative implicitly with respect to x of both sides to get

$$\begin{aligned} \frac{1}{y}y' &= \frac{2}{x-2} + \frac{3}{x-1} - \frac{5}{x+2} - \frac{4}{x-6} \\ y' &= \frac{(x-2)^2(x-1)^3}{(x+2)^5(x-6)^4}\left[\frac{2}{x-2} + \frac{3}{x-1} - \frac{5}{x+2} - \frac{4}{x-6}\right] \end{aligned}$$

EXERCISE 7

Use logarithmic differentiation to calculate the derivative of $f(x) = \dfrac{x^3\sqrt{x+1}}{(x+2)^2(x-1)}$, $x > 1$.

We conclude this section by remarking that when we have a function of the form $y = f(x)^{g(x)}$, it is often helpful to use logarithms to simplify it. We will see this again in the next Chapter.

Section 2.7 Problems

1. Find $\frac{dy}{dx}$ by implicit differentiation.
   (a) $x = y^2$
   (b) $xy^2 = 1$
   (c) $\frac{x}{y} + y = 1$
   (d) $3x + y^2 = x^2$
   (e) $x^3 + y^3 = 1$
   (f) $x^3 + y^3 = 6xy$
   (g) $y^2 = x^2 - x^4$
   (h) $y + y^2 = \sin(x)$
   (i) $x^2 + xy - y^2 = 4$
   (j) $4\cos(x)\sin(y) = 1$
   (k) $y\sin(x^2) = x\sin(y^2)$
   (l) $\ln(y) = \arccos(x)$
   (m) $x\sec(y) = 5$
   (n) $\frac{x}{x+y} - \frac{y}{x} = 4$

2. Use implicit differentiation to find the equation of the tangent line to the curve at the given point.
   (a) $x^2 = y^2 + y$ at $(\sqrt{2}, 1)$.
   (b) $x^2 + y^2 = 1$ at $(0, -1)$.
   (c) $x^2 + xy + y^2 = 3$ at $(1, 1)$.
   (d) $x^2 + 2xy - y^3 + x = -2$ at $(1, 2)$.
   (e) $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at $(0, 1/2)$.

3. Find y'.
   (a) $y = x^x$
   (b) $y = x^{\sin(x)}$
   (c) $y = 2^x$
   (d) $y = (\arcsin(x))^3$
   (e) $y = x^{\ln(x)}$
   (f) $y = (\cos(x))^x$
   (g) $y = (\sin(x))^{\ln(x)}$

4. Verify the implicitly defined function is a solution of given the differential equation.
   (a) $y' = -\frac{x}{y}$, $x^2 + y^2 = c^2$
   (b) $y' = x^2 y$, $\ln(y) = \frac{x^3}{3} + C$
   (c) $y' = \frac{6x^2}{2y + \cos(y)}$, $y^2 + \sin(y) = 2x^3 + C$

5. Use implicit differentiation and the fact that $\arccos(x)$ is the inverse of $\cos(x)$ to show the $\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$.

6. Given that $\sec(x)$ is invertible on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, use implicit differentiation and the fact that $\operatorname{arcsec}(x)$ is the inverse of $\sec(x)$ to show the $\frac{d}{dx}(\operatorname{arcsec}(x)) = \frac{1}{x\sqrt{x^2-1}}$.

End of Chapter Problems

1. Determine the derivative of the following functions.

   (a) $f(x) = x^2 + \sqrt{x}$
   (b) $f(x) = \cos(2x)$
   (c) $f(x) = |\tan(x)|$
   (d) $f(x) = e^{x^2}$
   (e) $f(x) = \dfrac{1}{3 - 2x}$
   (f) $f(x) = \ln(2x + 5)$
   (g) $f(x) = \displaystyle\int_0^x e^{-t^2}\, dt$
   (h) $f(x) = \arcsin(x^2 + 1)$
   (i) $f(x) = \displaystyle\int_0^x \sqrt{t^3 + t}\, dt$
   (j) $f(x) = \sec(3 - x)$
   (k) $f(x) = \displaystyle\int_{-1}^{3x+2} \ln(t^2 + 1)\, dt$
   (l) $f(x) = x\ln(2x + 1)$
   (m) $f(x) = \dfrac{x + 2}{|x|}$
   (n) $f(x) = xe^{2x}$
   (o) $f(x) = \dfrac{\sin(x)}{x^2}$
   (p) $f(x) = \displaystyle\int_0^{\tan(x)} (t^2 + t)\, dt$
   (q) $f(x) = x^{2/3}\csc(x)$
   (r) $f(x) = \ln(|\csc(x)|)$
   (s) $f(x) = \arccos(\sqrt{x})$
   (t) $f(x) = \cot(1 - 3x^2)$
   (u) $f(x) = \displaystyle\int_{-2}^{x+1} \arctan(t)\, dt$
   (v) $f(x) = \arctan(\ln(x))$
   (w) $f(x) = \dfrac{\arcsin(x)}{x^{1/3}}$
   (x) $f(x) = \displaystyle\int_0^x t^2\sin(t)\, dt$
   (y) $f(x) = \dfrac{1}{(2 - 5x)^3}$
   (z) $f(x) = \tan(2x + 5)$

2. Evaluate the following integrals.

   (a) $\displaystyle\int (x^2 + \sqrt{x})\, dx$
   (b) $\displaystyle\int \frac{1}{x}\, dx$
   (c) $\displaystyle\int \frac{1}{x^2}\, dx$
   (d) $\displaystyle\int 2\sin(x)\, dx$
   (e) $\displaystyle\int 5\cos(x)\, dx$
   (f) $\displaystyle\int x^{-2/3}\, dx$
   (g) $\displaystyle\int 4e^{-x}\, dx$
   (h) $\displaystyle\int_{-2}^{-1} \frac{2}{x}\, dx$
   (i) $\displaystyle\int_0^{\pi/4} \sec^2(x)\, dx$
   (j) $\displaystyle\int_{-\pi/6}^{\pi/6} \tan(x)\, dx$
   (k) $\displaystyle\int \frac{1}{\sqrt{1 - x^2}}\, dx$
   (l) $\displaystyle\int_0^1 \frac{1}{1 + x^2}\, dx$
   (m) $\displaystyle\int \sec(x)\tan(x)\, dx$
   (n) $\displaystyle\int_{2\pi/3}^{5\pi/6} \cos(x)\, dx$
   (o) $\displaystyle\int \csc^2(x)\, dx$
   (p) $\displaystyle\int_{-1}^{1} -x^3\, dx$
   (q) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\, dx$
   (r) $\displaystyle\int \frac{-1}{\sqrt{1 - x^2}}\, dx$
   (s) $\displaystyle\int \sin(3x)\, dx$
   (t) $\displaystyle\int 3\cos(2x)\, dx$
   (u) $\displaystyle\int \csc(2x)\cot(2x)\, dx$

3. Sketch the graph of the following function.
   (a) $f(x) = 3\sin(2x) + 1$
   (b) $f(x) = -2\cos\left(\frac{x}{3}\right)$
   (c) $f(x) = \frac{1}{2}\sin(\pi x) - 1$
   (d) $f(x) = \arccos(x)$
   (e) $f(x) = \arctan(x)$
   (f) $f(x) = \begin{cases} x+1 & \text{if } -2 < x < 0 \\ 3 & \text{if } x = 0 \\ 3-x & \text{if } 0 < x < 3 \end{cases}$
   (g) $f(x) = \begin{cases} \cos(x) & \text{if } -\pi \le x < 0 \\ \sin(x) & \text{if } 0 \le x \le \pi \end{cases}$
   (h) $f(x) = \begin{cases} e^x & \text{if } -2 < x \le 0 \\ \ln(x) & \text{if } 0 < x < 1 \\ \sqrt{x+1} & \text{if } 1 < x \le 3 \end{cases}$

4. If $x > 0$, show that $\dfrac{\sqrt{x^2+1}}{x} = \sqrt{1 + \dfrac{1}{x^2}}$.

5. If $x < 0$, show that $\dfrac{\sqrt{x^2+1}}{x} = -\sqrt{1 + \dfrac{1}{x^2}}$.

6. Show that $\dfrac{\sqrt{x^2+5x}}{x^2} = \sqrt{\dfrac{1}{x^2} + \dfrac{5}{x^3}}$.

7. Let $f(x) = \int_0^x g(t)\,dt$. Use the table below to approximate $f(2)$ using 3 subdivisions.

t	0	$\frac{2}{3}$	$\frac{4}{3}$
$g(t)$	$\frac{1}{3}$	1	$\frac{1}{2}$

8. Let $f(x) = \int_0^x g(t)\,dt$. Use the table below to approximate $f(2)$ using 4 subdivisions.

t	0	0.5	1	1.5
$g(t)$	0.4	0.6	1.0	1.4

9. Let $f(x) = \int_2^x g(t)\,dt$. Use the table below to approximate $f(4)$ using 4 subdivisions.

t	2	2.5	3	3.5
$g(t)$	-3	-1.5	0.5	0

10. Find all values x satisfying the equation.
    (a) $|x| = 0$
    (b) $|x| = 5$
    (c) $|x| < \frac{2}{3}$
    (d) $|x| > 2$
    (e) $|x+1| < \frac{1}{2}$
    (f) $|x-1| > 1$
    (g) $2|x-2| < 1$
    (h) $3|x+2| \le 1$
    (i) $4|x-3| \ge 2$

11. Find the inverse of each function on the given interval. State the domain and the range of the inverse.
    (a) $f(x) = x^2 + 2$, $[0, \infty)$
    (b) $f(x) = x^2 + 2$, $(-\infty, 0)$
    (c) $f(x) = e^{2x+1}$, $(-\infty, \infty)$
    (d) $f(x) = \ln(x-1)$, $(1, \infty)$.
    (e) $f(x) = 2\tan(3x)$, $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$.

12. Use implicit differentiation to find the equation of the tangent line at the given point.
    (a) $x^2 + 2y^2 = 9$ at $(-1, 2)$.
    (b) $x = \sin(y)$ at $(0, \pi)$.
    (c) $2x + xy = y$ at $(2, -4)$.
    (d) $y = x^{2x+1}$ at $(2, 32)$.
    (e) $\frac{x}{y} + 1 = y^2$ at $(0, 1)$.
    (f) $y^2 e^x = 1$ at $(0, -1)$.
    (g) $y = x^{\sin(x)}$ at $(\pi, 1)$.
    (h) $y = (\ln(x))^x$ at $(2, (\ln(2))^2)$.
    (i) $y = \cos(x)^{x^2}$ at $(0, 1)$.

13. Simplify the following expressions.
    (a) $\dfrac{(n-2)!}{n!}$
    (b) $\ln(x) + \ln(x+1)$
    (c) $e^{2\ln(|x|)}$
    (d) $\ln(2x^2) - \ln(x)$
    (e) $\dfrac{(n+1)!}{n!}$
    (f) $(\sqrt{x+1})^2$
    (g) $e^{-\ln(1/x)}$

14. Evaluate the following.
    (a) $e^{\ln(3)}$
    (b) $5!$
    (c) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
    (d) $\arctan(\sqrt{3})$
    (e) $\arccos(1)$
    (f) $\arcsin(-1)$
    (g) $\arccos\left(-\frac{1}{\sqrt{2}}\right)$
    (h) $\ln(e^3)$
    (i) $e^{4\ln(2)}$
    (j) $\frac{5!}{3!}$
    (k) $\frac{6!}{4!\cdot 3!}$
    (l) $e^{-\ln(1/3)}$
    (m) $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$
    (n) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$
    (o) $\sin(\arctan(2))$
    (p) $\arcsin\left(\cos\left(\frac{2\pi}{3}\right)\right)$
    (q) $\arctan\left(\tan\left(\frac{7\pi}{6}\right)\right)$
    (r) $\cos\left(\arcsin\left(\frac{1}{3}\right)\right)$

15. Rewrite each of the following without absolute value signs.
    (a) $f(x) = |x+3|$
    (b) $f(x) = |3-2x|$
    (c) $f(x) = |(x+1)(x-3)|$
    (d) $f(x) = |(x-2)(x-5)|$
    (e) $f(x) = 3|x| - 2x|x+1|$
    (f) $f(x) = x|x-1| + x^2|x+3|$

16. Use the definition $\ln(x) = \int_1^x \frac{1}{t}\,dt$ to approximate the following values using 3 subdivisions.
    (a) $\ln(2)$
    (b) $\ln(4)$
    (c) $\ln(7)$

17. Find the intensity of each sound (as in Example 2.4.18) and use the intensity to determine how many times louder D_1 is than D_2.
    (a) $D_1 = 30$ decibels, $D_2 = 10$ decibels.
    (b) $D_1 = 58$ decibels, $D_2 = 33$ decibels.

18. How much larger is the amplitude of a magnitude 4.5 earthquake compared to a magnitude 2.5 earthquake?

19. Given $\cos(\theta) = \frac{1}{3}$ and $0 < \theta < \frac{\pi}{2}$, find $\sin(\theta)$.

20. Given $\tan(\theta) = \frac{3}{2}$ and $0 < \theta < \frac{\pi}{2}$, find $\cos(\theta)$.

21. Given $\sin(\theta) = \frac{x}{3}$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find $\tan(\theta)$.

22. Given $\sec(\theta) = 2x$ and $0 < \theta < \frac{\pi}{2}$, find $\tan(\theta)$.

23. Given $\tan(\theta) = \frac{x}{2}$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find $\sec(\theta)$.

24. Determine the maximum, minimum, average, amplitude, and period of each function.
    (a) $f(\theta) = 2\sin(\theta) + 1$
    (b) $f(\theta) = \cos\left(\frac{\theta}{2}\right) - 1$
    (c) $f(\theta) = -\frac{1}{2}\sin\left(\frac{\theta}{\pi}\right)$
    (d) $f(\theta) = 2\sin(2\theta) - 3$

25. Evaluate the binomial coefficient.
    (a) $\binom{3}{0}$
    (b) $\binom{4}{2}$
    (c) $\binom{5}{1}$
    (d) $\binom{6}{3}$

26. Verify the given function y is a solution of the differential equation.
    (a) $y = \cos(x) + x\sin(x)$ for $y' = x\cos(x)$
    (b) $y = 5e^{x^2/2}$ for $y' = xy$
    (c) $y = -\frac{x}{2} + \frac{1}{4}$ for $y' - 2y = x - 1$
    (d) $y = -\frac{x}{2} + \frac{1}{4} + e^{2x}$ for $y' - 2y = x - 1$

27. Solve the initial value problem.
    (a) $y' = 2x + 3$, $y(-1) = -3$
    (b) $y' = \cos(x)$, $y\left(\frac{2\pi}{3}\right) = 1$
    (c) $y' = \frac{1}{x}$, $y(-2) = 4$
    (d) $y' = x^2$, $y(-1) = 2$
    (e) $y' = \sin(x)$, $y\left(\frac{3\pi}{2}\right) = -1$
    (f) $y' = \frac{x}{|x|}$, $y(1) = 3$
    (g) $y' = 2^x$, $y(0) = \sqrt{2}$
    (h) $y' = \sec(x)\tan(x)$, $y\left(\frac{\pi}{4}\right) = 1$
    (i) $y' = \frac{1}{\sqrt{1-x^2}}$, $y(1) = 0$
    (j) $y' = \frac{2}{x}$, $y(1) = -6$
    (k) $y' = \tan(x)$, $y\left(\frac{5\pi}{6}\right) = 0$
    (l) $y' = \sec^2(x)$, $y\left(\frac{5\pi}{6}\right) = 0$
    (m) $y' = \frac{1}{1+x^2}$, $y(3) = 1$
    (n) $y' = \frac{1}{x\ln(3)}$, $y(1) = -2$
    (o) $y' = \frac{3}{x}$, $y(-1) = \pi$
    (p) $y' = -\cot(x)$, $y\left(\frac{\pi}{2}\right) = 1$

28. Expand using binomial coefficients.
    (a) $(x+3)^3$
    (b) $(x-2)^3$
    (c) $(x+2)^4$
    (d) $(3x+1)^4$
    (e) $(2x-2)^5$
    (f) $(2x+1)^5$

29. A \$40 monthly cellphone plan includes 200 anytime minutes, and for each additional minute it costs \$0.25. Write a piecewise defined function for the cost per minute used.

30. In Mathanada, federal income tax is calculated using a graduate tax system as follows: there is a 15% tax on income up to \$49 200. The next \$49 200 of income is taxed at a rate of 20.5% (on the portion of income over \$49 200 up to \$98 400). The next \$53 939 of taxable income (over \$98 400 and up to \$151 978) is taxed at 26%. For example, an income of \$100 000 would have the first \$49 200 taxed at a rate of 15%, the next 49 200 taxed at a rate of 20.5% and the remaining \$1600 taxed at a rate of 26
    Write a piecewise function for the tax rate R (as a percent) as a function of taxable income I (in dollars).

Chapter 3: Analyzing the Behaviour of Functions

In Chapter 2, we explored a variety of functions along with their derivatives and antiderivatives. However, the functions we often encounter in the sciences tend to be more complicated than these foundational functions. Moreover, as scientists strive to make ever more accurate mathematical models, the functions continually become more and more complicated. This is where calculus comes in! As we will see in this chapter, calculus gives us the ability to analyze local behaviour (behaviour around a point) and long-term behaviour (behaviour as the independent variable gets infinitely large) of functions which helps us use mathematical models to make predictions.

Section 3.1: Limits and Continuity

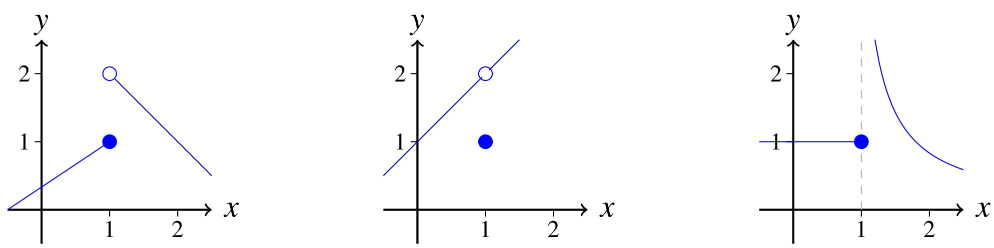
LEARNING OUTCOMES

1. Know how to evaluate limits graphically and algebraically.
2. Know how to recognize and evaluate determinate forms.
3. Know how to determine whether a function is continuous at a point.
4. Know how to find the intervals where a function is continuous.
5. Know how to apply the Intermediate Value Theorem.
6. Know how to find and classify discontinuities.

3.1.1 Limits as $x \to a$

One-Sided Limits

We begin by looking at the behaviour of a function around a point. In Section 2.1, we saw that functions can have different behaviours on either side of a point. For example, consider the functions graphed below.



All of these functions contain the point $(1, 1)$, but their behaviour around this point (i.e. for values of x close to but not equal to 1) are very different.

To be able to determine the behaviour of a function on either side of a point, we introduce left-hand and right-hand limits. We begin with an intuitive definition of these.

DEFINITION

Left-Hand Limits

If the values of $f(x)$ get infinitely close to L as x gets infinitely close to a where $x < a$, then we say the **limit of $f(x)$ as x approaches a from the left** is equal to L and write

$$\lim_{x \to a^-} f(x) = L$$

If the values of $f(x)$ are not getting infinitely close to a single number, then we say that the limit does not exist.

DEFINITION

Right-Hand Limits

If the values of $f(x)$ get infinitely close to L as x gets infinitely close to a where $x > a$, then we say the **limit of $f(x)$ as x approaches a from the right** is equal to L and write

$$\lim_{x \to a^+} f(x) = L$$

If the values of $f(x)$ are not getting infinitely close to a single number, then we say that the limit does not exist.

REMARK

These are not mathematical precise definitions. It would be more accurate (but still not fully precise), to say $\lim\limits_{x \to a^+} f(x) = L$ if the values of $f(x)$ can be made arbitrarily close to L and only L, by making the values of x sufficiently close to a with $x > a$.

We can use these intuitive definitions to evaluate one-sided limits graphically.

EXAMPLE 1

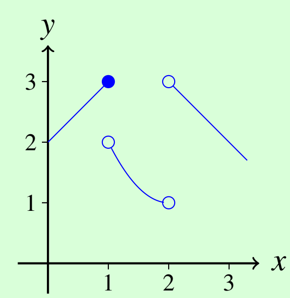
Use the graph of the function f to evaluate the limit.

(a) $\lim\limits_{x \to 1^-} f(x)$

(b) $\lim\limits_{x \to 1^+} f(x)$

(c) $\lim\limits_{x \to 2^-} f(x)$

(d) $\lim\limits_{x \to 2^+} f(x)$



Solution: As x approaches 1 from the left, the values of $f(x)$ are getting infinitely close to 3. Thus, $\lim\limits_{x \to 1^-} f(x) = 3$.

As x approaches 1 from the right, the values of $f(x)$ are getting infinitely close to 2. Thus, $\lim\limits_{x \to 1^+} f(x) = 2$.

As x approaches 2 from the left, the values of $f(x)$ are getting infinitely close to 1. Thus, $\lim\limits_{x \to 2^-} f(x) = 1$.

As x approaches 2 from the right, the values of f are getting infinitely close to 3. Thus, $\lim\limits_{x \to 2^+} f(x) = 3$.

Observe in the example above that the fact that $f(1) = 3$ did not affect the value of $\lim_{x\to 1^+} f(x)$. Similarly, the fact that $x = 2$ is not in the domain of f did not affect the values of either $\lim_{x\to 2^-} f(x)$ or $\lim_{x\to 2^+} f(x)$.

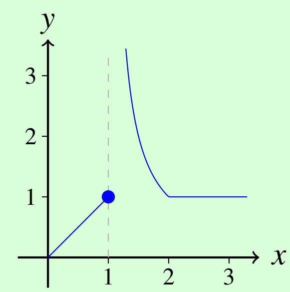
EXAMPLE 2 Use the graph of the function f to evaluate the limit.

(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 2^-} f(x)$

(d) $\lim_{x\to 2^+} f(x)$



Solution: As x approaches 1 from the left, the values of $f(x)$ are getting infinitely close to 1. Thus, $\lim_{x\to 1^-} f(x) = 1$.

As x approaches 1 from the right, the values of $f(x)$ become infinitely large. Since the values are not approaching a single number, $\lim_{x\to 1^+} f(x)$ does not exist.

As x approaches 2 from the left, the values of $f(x)$ are getting infinitely close to 1. Thus, $\lim_{x\to 2^-} f(x) = 1$.

As x approaches 2 from the right, the values of f are always equal to 1. Thus, $\lim_{x\to 2^+} f(x) = 1$.

In the example above, stating that $\lim_{x\to 1^+} f(x)$ did not exist was correct, but unsatisfactory. In particular, since our goal is to describe the local behaviour of a function, it would be better to state why the limit does not exist. To do this, we use the following notation.

NOTATION If the left-hand limit of a function f as x approaches a does not exist because the values of $f(x)$ will exceed any positive number as x gets infinitely close to a, then we write

$$\lim_{x\to a^-} f(x) = \infty$$

If the left-hand limit does not exist because the values of $f(x)$ will be below any negative number as x gets infinitely close to a, then we write

$$\lim_{x\to a^-} f(x) = -\infty$$

We define $\lim_{x\to a^+} f(x) = \infty$ and $\lim_{x\to a^+} f(x) = -\infty$ similarly.

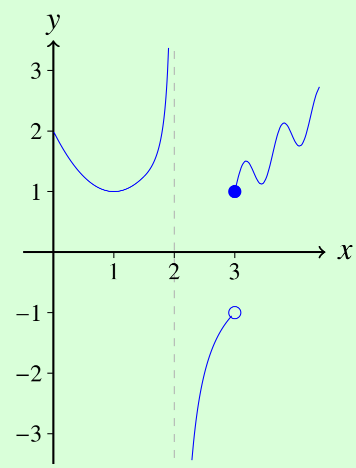
EXAMPLE 3 In Chapter 2, we saw that

$$\lim_{x\to 0^+} \ln(x) = -\infty$$

and

$$\lim_{x\to \frac{\pi}{2}^-} \tan(x) = \infty$$

EXAMPLE 4 Use the graph of the function g to evaluate the limit.



(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 2^-} f(x)$

(d) $\lim_{x\to 2^+} f(x)$

(e) $\lim_{x\to 3^-} f(x)$

(f) $\lim_{x\to 3^+} f(x)$

Solution: We have

(a) $\lim_{x\to 1^-} f(x) = 1$

(b) $\lim_{x\to 1^+} f(x) = 1$

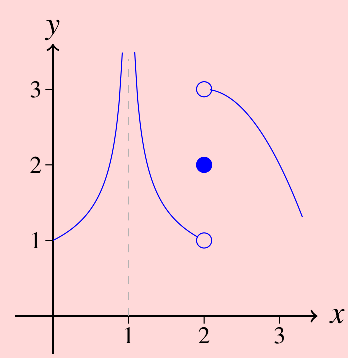
(c) $\lim_{x\to 2^-} f(x) = \infty$

(d) $\lim_{x\to 2^+} f(x) = -\infty$

(e) $\lim_{x\to 3^-} f(x) = -1$

(f) $\lim_{x\to 3^+} f(x) = 1$

EXERCISE 1 Use the graph of the function f to evaluate the limit.



(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 2^-} f(x)$

(d) $\lim_{x\to 2^+} f(x)$

Evaluating Limits Algebraically

The purpose of the examples above was to give you an intuitive understanding of one-sided limits. Realistically, there isn't much value in evaluating these limits if we already have the graph of the function. Therefore, we now look at how to evaluate limits when we do not already have the graph.

To evaluate limits algebraically, we can use the following limit laws.

Limits Laws: Suppose c is a constant and the limits $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^+} g(x)$ exist. Then

1. $\lim\limits_{x\to a^+} c = c$
2. $\lim\limits_{x\to a^+} x = a$
3. $\lim\limits_{x\to a^+} [f(x) \pm g(x)] = \lim\limits_{x\to a^+} f(x) \pm \lim\limits_{x\to a^+} g(x)$
4. $\lim\limits_{x\to a^+} [cf(x)] = c \lim\limits_{x\to a^+} f(x)$
5. $\lim\limits_{x\to a^+} [f(x)g(x)] = \lim\limits_{x\to a^+} f(x) \lim\limits_{x\to a^+} g(x)$
6. $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a^+} f(x)}{\lim\limits_{x\to a^+} g(x)}$, if $\lim\limits_{x\to a^+} g(x) \neq 0$.
7. $\lim\limits_{x\to a^+} [f(x)]^n = [\lim\limits_{x\to a^+} f(x)]^n$, for any n, if $\lim\limits_{x\to a^+} f(x)$ is in the domain of $(\cdot)^n$.

We have stated the limit laws for right-hand limits. Of course, they also apply for left-hand limits.

EXAMPLE 5

Evaluate $\lim\limits_{x\to 1^+} (2x^2 - 3x - 5)$ using the limit laws.

Solution: We can use the limit laws to write the given limit as

$$\begin{aligned}
\lim_{x\to 1^+} (2x^2 - 3x - 5) &= \lim_{x\to 1^+} (2x^2) - \lim_{x\to 1^+} (3x) - \lim_{x\to 1^+} 5 && \text{(law 3)}\\
&=2 \lim_{x\to 1^+} x^2 - 3 \lim_{x\to 1^+} x - \lim_{x\to 1^+} 5 && \text{(law 4)}\\
&=2(\lim_{x\to 1^+} x)^2 - 3 \lim_{x\to 1^+} x - \lim_{x\to 1^+} 5 && \text{(law 7)}\\
&=2(1)^2 - 3(1) - 5 && \text{(laws 1 and 2)}\\
&= -6
\end{aligned}$$

If a is in the domain of a function f and f is any combination of constant functions, power functions, polynomials, rational functions, and the absolute value function, then

$$\lim_{x\to a^+} f(x) = f(a)$$

and

$$\lim_{x\to a^-} f(x) = f(a)$$

This arises from fact that any combination of these functions is simply a combination of the operations described in the above limit laws.

EXAMPLE 6

Determine the following limits for $f(x) = \begin{cases} x^3 + 4x & x \leq -1 \\ \sqrt{x+3} & -1 < x < 1 \\ 3 - x^2 & 1 < x \leq 2 \end{cases}$.

(a) $\lim_{x\to -1^-} f(x)$

(b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to 1^-} f(x)$

(d) $\lim_{x\to 1^+} f(x)$

Solution: We have

(a) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^3 + 4x) = (-1)^3 + 4(-1) = -5.$

(b) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \sqrt{x+3} = \sqrt{-1+3} = \sqrt{2}.$

(c) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x+3} = \sqrt{1+3} = 2.$

(d) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3 - x^2) = 3 - 1^2 = 2.$

EXERCISE 2

Determine the following limits for $f(x) = \begin{cases} 2 - x & x \leq 1 \\ 1 & 1 < x < 2 \\ 5 & x = 2 \\ 4 - \frac{1}{2}x^2 & 2 < x \leq 3 \end{cases}$.

(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 2^-} f(x)$

(d) $\lim_{x\to 2^+} f(x)$

Determinate Forms

We frequently encounter limits where we cannot apply the limit laws. For example, we cannot apply limit law 6 to

$$\lim_{x\to 1^-} \frac{x+1}{x-1}$$

since the limit of the denominator is 0. Similarly, we cannot apply the limit laws to

$$\lim_{x\to 0^+} \frac{x}{\ln(x)}$$

since the limit of the denominator does not exist.

For some of these limits, we can deduce the value of the limit by considering the behaviour of the numerator and the behaviour of the denominator.

For example, for

$$\lim_{x\to 1^-} \frac{x+1}{x-1}$$

we see that as x approaches 1 from the left, the numerator is approaching 2 while the denominator is approaching 0 from the left. Thus, we have numbers getting closer and closer to 2 being divided by smaller and smaller negative numbers. This results in larger and larger negative numbers. Hence,

$$\lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty$$

This limit is said to have the form $\frac{L}{0^-}$ (where L is implied to be a non-zero number). With any limit that has this form, we can use the same reasoning as above to determine that the limit is approaching $-\infty$ (in the case where L is positive) or ∞ (in the case where L is negative).

Forms representing undefined expressions where we can immediately deduce the value of the expression or that the value of the expression is $\pm\infty$ are called **determinate forms**.

Some determinate forms are (remembering L indicates a non-zero number)

$$\frac{L}{0^+},\quad \frac{L}{0^-},\quad \frac{L}{\infty},\quad \frac{0}{\infty},\quad \frac{\infty}{L},\quad \frac{\infty}{0},\quad \infty+\infty,\quad L\cdot\infty,\quad 0^\infty,\quad \infty^\infty$$

When we get a limit in a determinate form, we reason out the value of the limit. We demonstrate this with some examples.

EXAMPLE 7 Evaluate $\displaystyle\lim_{x\to 1^+} \frac{5}{x-1}$.

Solution: As x approaches 1 from the right, the numerator is 5 while the denominator is approaching 0 from the right. So, the limit has the determinate form $\frac{L}{0^+}$. Thus, we have 5 being divided by smaller and smaller positive numbers. This results in larger and larger positive numbers. Hence,

$$\lim_{x\to 1^+} \frac{5}{x-1} = \infty$$

EXAMPLE 8

Evaluate $\lim_{x \to -3^+} \frac{x-5}{x+3}$.

Solution: As x approaches -3 from the right, the numerator is approaching -8 while the denominator is approaching 0 from the right. So, the limit has the determinate form $\frac{L}{0^+}$. In this case, we have numbers getting closer and closer to -8 being divided by smaller and smaller positive numbers. This results in larger and larger negative numbers. Hence,

$$\lim_{x \to -3^+} \frac{x-5}{x+3} = -\infty$$

EXAMPLE 9

Evaluate $\lim_{x \to 0^+} x^{1/x}$.

Solution: As x approaches 0 from the right, the base is approaching 0 and the power is getting larger and larger. So, this limit has the determinate form 0^{∞}. Taking large and larger powers of smaller and smaller numbers results in smaller and smaller numbers. Thus,

$$\lim_{x \to 0^+} x^{1/x} = 0$$

EXAMPLE 10

Evaluate $\lim_{x \to 0^+} \frac{x}{\ln(x)}$.

Solution: As x approaches 0 from the right, the numerator is approaching 0 and the denominator is approaching $-\infty$. So, this limit has the determinate form $\frac{0}{-\infty}$. Dividing smaller and smaller numbers by larger and larger number results in smaller and smaller numbers. Thus,

$$\lim_{x \to 0^+} x^{1/x} = 0$$

EXERCISE 3

Evaluate the following limits.

(a) $\lim_{x \to 1^-} \frac{1}{x-1}$

(b) $\lim_{x \to -2^-} \frac{x+1}{x+2}$

(c) $\lim_{x \to 1^+} \frac{(x+3)^2}{(x+1)^2}$

(d) $\lim_{x \to \frac{\pi}{2}^-} (\tan(x) + \sec(x))$

(e) $\lim_{x \to 0^-} e^{1/x}$

(f) $\lim_{x \to 0^+} e^{1/x}$

Two-sided Limits

We have seen several examples where $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$. We invent notation for this.

DEFINITION

Limit of f as $x \to a$

If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, then we say the **limit of $f(x)$ as x approaches a** is equal to L and write

$$\lim_{x\to a} f(x) = L$$

If either $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ do not exist, or if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then we say $\lim_{x\to a} f(x)$ does not exist.

All of the limit laws, and hence our method for evaluating limits, apply to two-sided limits as well (i.e. as $x \to a$). The only times we have to consider left-hand and right-hand limits are when the limit laws do not apply or at a transition point for a piecewise defined function.

EXAMPLE 11

Evaluate the following limits.

(a) $\lim_{x\to 3} (x^2 - x + 5)$

(b) $\lim_{x\to 0} \frac{1}{x}$

Solution: We have

(a) $\lim_{x\to 3} (x^2 - x + 5) = 3^2 - 3 + 5 = 11$.

(b) Since the denominator approaches 0 as x as approaches 0, the limit laws do not apply. So, we use left-hand and right-hand limits. We see that $\lim_{x\to 0^+} \frac{1}{x}$ has the form $\frac{L}{0^+}$. Thus,

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

Since this limit does not exist, $\lim_{x\to 0} \frac{1}{x}$ does not exist.

EXERCISE 4

Evaluate the following limits.

(a) $\lim_{x\to 1} (x^3 + 2x^2)$

(b) $\lim_{x\to \frac{1}{2}} \frac{x}{x+1}$

(c) $\lim_{x\to -3} \sqrt{|2 + x|}$

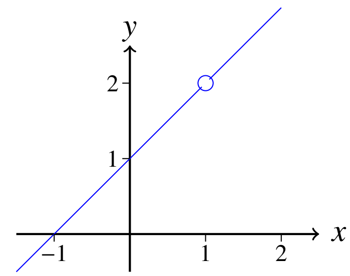
(d) $\lim_{x\to 2} \frac{|x-3|}{x-2}$

3.1.2 Continuity

Intuitively, a function is continuous at a point if its graph can be drawn through the point without lifting the pen from the page. To create a more precise definition, it is helpful to look at some examples.

Example 1: A hole in the graph.

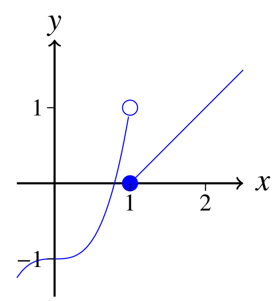
Consider the function $f(x) = \dfrac{x^2-1}{x-1}$. Since $x = 1$ is not in the domain of f, we cannot sketch the graph without lifting the pen from the page. So, f is discontinuous at $x = 1$.



From this, we see that for a function to be continuous at a point $x = a$, a must be in the domain of the function.

Example 2: A jump in the graph.

Consider the function $f(x) = \begin{cases} 2x^3 - 1 & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 1 \end{cases}$.
In this case, 1 is in the domain of f, but f is still discontinuous at $x = 1$ because the graph jumps at $x = 1$. In particular, $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$.



Therefore, to be continuous at $x = a$, the graph of the function must 'meet up' at a. That is, not only do we need both $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$ to exist and equal each other (so that $\lim\limits_{x\to a} f(x)$ exists), but we need the value of the limits to match the value of the function at $x = a$. That is, we require

$$\lim_{x\to a} f(x) = f(a)$$

We get the following definition.

DEFINITION
Continuous

A function f is said to be **continuous at the point $x = a$** if

$$\lim_{x\to a} f(x) = f(a)$$

Otherwise, f is said to be **discontinuous at the point $x = a$**.

If f is continuous at every point in an interval I, then f is said to be continuous on I.

A function that is continuous at every point in its domain is called a **continuous function**.

Let's look at some examples.

EXAMPLE 12

Determine whether $f(x) = \begin{cases} 3x+4 & \text{if } x < 2 \\ 5 & \text{if } x = 2 \\ \dfrac{5x^3}{(x-4)^2} & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$.

Solution: We have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3x+4) = 3(2)+4 = 10$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{5x^3}{(x-4)^2} = \frac{5(2)^3}{(2-4)^2} = 10$$

Thus,

$$\lim_{x\to 2} f(x) = 10$$

But, $f(2) = 5$. Therefore, f is discontinuous at $x = 2$.

EXAMPLE 13

Determine whether $f(x) = \begin{cases} x^2-2 & \text{if } x \le -3 \\ 2x+13 & \text{if } x > -3 \end{cases}$ is continuous at $x = -3$.

Solution: We have

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} (x^2-2) = (-3)^2 - 2 = 7$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} (2x+13) = 2(-3)+13 = 7$$

Thus,

$$\lim_{x\to -3} f(x) = 7$$

Since we also have $f(-3) = (-3)^2 - 2 = 7$, f is continuous at $x = -3$.

EXAMPLE 14

Determine whether $f(x) = \begin{cases} \dfrac{3}{(x-1)^2} & \text{if } x < 1 \\ \sqrt{x+1} & \text{if } x \ge 1 \end{cases}$ is continuous at $x = 1$.

Solution: Observe that $\displaystyle\lim_{x\to 1^-} \frac{3}{(x-1)^2}$ has the form $\dfrac{L}{0^+}$. We get

$$\lim_{x\to 1^-} \frac{3}{(x-1)^2} = \infty$$

Since the limit does not exist, f is discontinuous at $x = 1$.

EXAMPLE 15 For what values of a and b is $f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ b & \text{if } x = 1 \\ ax^2+1 & \text{if } x > 1 \end{cases}$ continuous at $x = 1$?

Solution: For f to be continuous at $x = 1$, we require

$$\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$$

So, we need

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= f(1) \\ \lim_{x\to 1^-}(2x-1) &= b \\ 1 &= b \end{aligned}$$

Using $b = 1$, we get

$$\begin{aligned} \lim_{x\to 1^+} f(x) &= f(1) \\ \lim_{x\to 1^+}(ax^2+1) &= 1 \\ a+1 &= 1 \\ a &= 0 \end{aligned}$$

Taking both $a = 0$ and $b = 1$, we get

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= 2(1)-1 = 1 \\ \lim_{x\to 1^+} f(x) &= 1 \\ f(1) &= 1 \end{aligned}$$

Therefore, the only values of a and b that make f continuous at $x = 1$ are $a = 0$ and $b = 1$.

EXERCISE 5 Determine whether $f(x) = \begin{cases} \sqrt{2x+1} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ 2x^2+x+1 & \text{if } x > 0 \end{cases}$ is continuous at $x = 0$.

EXERCISE 6 Determine whether $f(x) = \begin{cases} x^2-x+1 & \text{if } x < 1 \\ \dfrac{3x-2}{x} & \text{if } x \geq 1 \end{cases}$ is continuous at $x = 1$.

EXERCISE 7

For what values of a is $f(x) = \begin{cases} ax+2 & \text{if } x<2 \\ x^2+1 & \text{if } x \geq 2 \end{cases}$ continuous at $x = 2$?

Why do we care so much about whether or not a function is continuous at a point? If a function is continuous at $x = a$, then it means that a small change in x from a will result in a relatively small change in $f(x)$ from $f(a)$. That is, the function is 'predictable' near $x = a$. This is best demonstrated with an example.

EXAMPLE 16

The function

$$f(x) = \begin{cases} 2x+1 & \text{if } x \leq 1 \\ 3x^2 & \text{if } x > 1 \end{cases}$$

is continuous at $x = 1$. So, moving from $x = 1$ to $x = 1.001$ will result in a small change in the value of $f(x)$. Indeed, we find that

$$f(1) = 3$$
$$f(1.001) = 3.006003$$

On the other hand, the function

$$g(x) = \begin{cases} 3x+4 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

is not continuous at $x = 1$. For this function, we have

$$g(1) = 3$$
$$g(1.001) = 7.003$$

So, the small change from $x = 1$ to $x = 1.001$ resulted in a relatively large change in the value of $g(x)$.

Continuity Theorems

Some more good news. We only have to apply the definition of continuity at the transition points of piecewise defined functions as it can be proven that all of our basic functions (constant functions, power functions, polynomials, rational functions, exponential functions, logarithmic functions, and the absolute value function) are continuous at every point in their respective domains.

REMARK

At first glance, it can seem strange to call $\tan(x)$ a continuous function when it has infinitely many discontinuities. However, the discontinuities are not in the domain of $\tan(x)$. So, it is continuous at every point in its domain.

Moreover, the following theorems, known collectively as the **continuity theorems**, tell us that combinations of continuous functions are continuous.

THEOREM 1

If f and g are continuous at $x = a$, then so are the function $f + g$, $f - g$, cf (for any real number c), fg, and $\frac{f}{g}$ (provided $g(a) \neq 0$).

THEOREM 2

If g is continuous at $x = a$ and f is continuous at $g(a)$, then $f \circ g$ is continuous at $x = a$.

EXAMPLE 17

Show that $f(x) = 3(x^2 - \sin(x))e^x$ is continuous at $x = 0$.

Solution: The given function can be broken down into simpler functions $g(x) = x^2$, $h(x) = \sin(x)$ and $k(x) = e^x$. We know that $g(x) = x^2$, $h(x) = \sin(x)$, and $k(x) = e^x$ are all continuous at $x = 0$. Since $f(x)$ can be written as the product of $g(x) - h(x)$ and $k(x)$, by the continuity theorems, f is also continuous at $x = 0$.

In practice, we often want to find the intervals where a function is continuous. We can use the above theorems to break a function down into simpler pieces.

EXAMPLE 18

Determine all intervals on which $h(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 25}$ is continuous.

Solution: In the numerator, we have $f(x) = \sqrt{x^2 - 9}$. This function is a composition of the continuous functions $x^2 - 9$ and $\sqrt{x}$ and hence is continuous on its domain. To find its domain, we need to determine where $x^2 - 9 \geq 0$. We get

$$\begin{aligned} x^2 - 9 &\geq 0 \\ x^2 &\geq 9 \\ |x| &\geq 3 \end{aligned}$$

So, by property (v) of absolute values, we get $x \leq -3$ or $x \geq 3$.

In the denominator, we have $g(x) = x^2 - 25$. This function is continuous everywhere, but since it is in the denominator oh h, we need to remove any places where $g(x)$ is equal to zero. We get

$$\begin{aligned} x^2 - 25 &= 0 \\ x^2 &= 25 \\ |x| &= 5 \end{aligned}$$

So, h is not continuous at $x = \pm 5$.

Therefore, $h(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 25}$ is continuous on the intervals $(-\infty, -5)$, $(-5, 3]$, $[3, 5)$, and $(5, \infty)$.

EXAMPLE 19

Determine all intervals on which $f(x) = \dfrac{x}{x}$ is continuous.

Solution: Since $x = 0$ is not the domain of f, f cannot be continuous there. Hence, f is continuous on the intervals $(-\infty, 0), (0, \infty)$.

The moral of the last example is that, just like when finding the domain of a function, you must determine the intervals where the function is continuous before simplifying.

EXERCISE 8

Determine all intervals on which the following functions are continuous.

(a) $f(x) = \ln(x^2 - 1)$

(b) $g(x) = \sqrt{x^2 - 5x + 6}$

(c) $h(x) = \dfrac{x^2 - 5x + 6}{x^2 + x - 12}$

(d) $j(x) = \tan(\arccos(x))$

For piecewise defined functions, we can use the continuity theorems at all points in the domain except for transition points.

EXAMPLE 20

Determine all intervals on which $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < 0 \\ x^2 + 2 & \text{if } 0 \le x \le 2 \\ 6\cos(\pi x) & \text{if } x > 2 \end{cases}$ is continuous.

Solution: We first determine whether f is continuous at its transition points.

For $x = 0$, we see that

$$\lim_{x \to 0^-} f(x) = -\infty$$

Since the limit does not exist, f is discontinuous at $x = 0$.

For $x = 2$, we see that

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (x^2 + 2) = 6 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} 6\cos(\pi x) = 6\cos(2\pi) = 6 \end{aligned}$$

So, $\lim\limits_{x \to 2} f(x) = 6$ and $f(2) = 6\cos(\pi 2) = 6$. Therefore, f is continuous at $x = 2$.

By the continuity theorems, f is continuous on the intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$.

Since, f is also continuous at $x = 2$, we can combine the last two intervals into a single interval $(0, \infty)$.

EXAMPLE 21

The gravitational force exerted by the planet Earth on a unit mass at a distance $r > 0$ from the centre of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3}, & 0 < r < R \\ \\ \dfrac{GM}{r^2}, & r \geq R \end{cases}$$

where M is the mass of the Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

Solution: By the continuity theorems, $F(r)$ is continuous for all $r > 0$ except possible at $r = R$. At $r = R$. We get

$$\lim_{r \to R^-} F(r) = \lim_{r \to R^-} \frac{GMr}{R^3} = \frac{GMR}{R^3} = \frac{GM}{R^2}$$
$$\lim_{r \to R^+} F(r) = \lim_{r \to R^+} \frac{GM}{r^2} = \frac{GM}{R^2}$$

Thus, $\lim\limits_{r \to R} F(r) = \dfrac{GM}{R^2}$. We also have that $F(R) = \dfrac{GM}{R^2}$. Therefore, F is continuous at $r = R$ and so is continuous for all $r > 0$.

EXERCISE 9

Let $f(x) = \begin{cases} \sin(x) & \text{if } x < 0 \\ x^2 + 2 & \text{if } 0 \leq x \leq 2 \\ 4\cos(\pi x) & \text{if } 2 < x \leq 5 \\ -\dfrac{4}{5}x & \text{if } x > 5 \end{cases}$.

List all intervals where f is continuous.

3.1.3 Intermediate Value Theorem

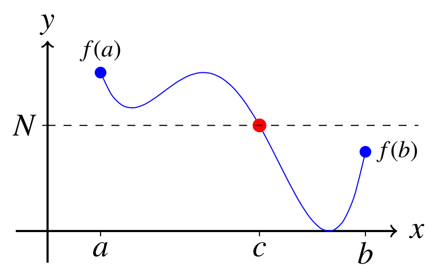
We have already discussed the benefit of knowing a function is continuous at a point (a small change in the independent variables leads to a relatively small change in the dependent variable). We now look at one of the many benefits of knowing a function is continuous on an interval.

THEOREM 3

Intermediate Value Theorem (IVT)

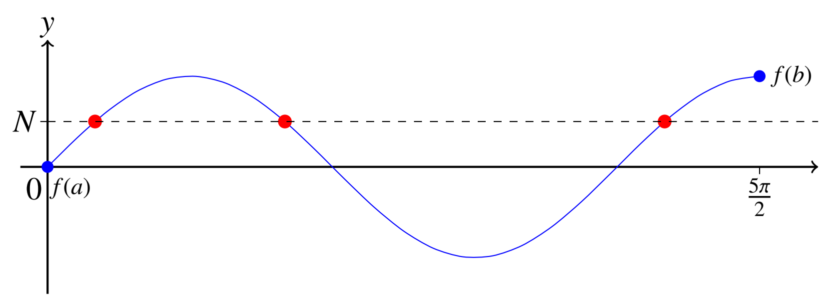
Let f be continuous on the closed interval $[a, b]$ such that $f(a) \neq f(b)$. If N is any number between $f(a)$ and $f(b)$, then there exists a number c in the open interval (a, b) such that $f(c) = N$.

Graphically, the Intermediate Value Theorem is clear. It says that if f is continuous (the graph can be drawn without lifting a pencil off the page) and either $f(a)$ or $f(b)$ is beneath the line $y = N$ and the other is above the line, then at some point, which we call c, between a and b the graph of f must cross the line.



It is important to notice that while the Intermediate Value Theorem guarantees the existence of a point c, it does not state that the point c is unique.

For example, consider $f(x) = \sin(x)$ on the interval $\left[0, \frac{5\pi}{2}\right]$. If we take $N = \frac{1}{2}$, then we actually have three choices for the point c: $\frac{\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{13\pi}{6}$.



One of the main uses of the Intermediate Value Theorem is to show that a function has a root (a value of x such that $f(x) = 0$) or that an equation has a solution.

EXAMPLE 22 Prove that $f(x) = x^3 + x + 1$ has a real root.

Solution: We want to show that the equation $x^3 + x + 1 = 0$ has a solution.

We observe that f is continuous for all x and that

$$f(0) = 1$$
$$f(-1) = -1$$

Since we have $f(-1) < 0 < f(0)$, by the Intermediate Value Theorem, there exists a point c in the interval $(-1, 0)$ such that $f(c) = 0$. That is, f has a root between $x = -1$ and $x = 0$.

EXAMPLE 23

Use the Intermediate Value Theorem to show that the equation $2\sqrt{x^2+1} = x^2 - 1$ has a solution in the interval $(\sqrt{3}, \sqrt{8})$.

Solution: We first need to create a function f to apply the Intermediate Value Theorem to. To do this, we rearrange the equation to get it into the form $f(x) = N$. We get

$$2\sqrt{x^2+1} = x^2 - 1$$
$$2\sqrt{x^2+1} - x^2 = -1$$

Thus, we take $f(x) = 2\sqrt{x^2+1} - x^2$ and $N = -1$. By the continuity theorems, f is continuous for all x.

Now, we see that

$$f(\sqrt{3}) = 2\sqrt{3+1} - 3 = 2(2) - 3 = 4 - 3 = 1$$
$$f(\sqrt{8}) = 2\sqrt{8+1} - 8 = 2(3) - 8 = -2$$

Therefore, $f(\sqrt{8}) < -1 < f(\sqrt{3})$. Hence, by the Intermediate Value Theorem, there exists a number c between $\sqrt{3}$ and $\sqrt{8}$ such that $f(c) = -1$. That is, $x = c$ is a solution of the equation $2\sqrt{x^2+1} = x^2 - 1$.

EXERCISE 10

Prove that the equation $x^3 + x^2 - 2x = 1$ has a solution in the interval $[-1, 1]$.

3.1.4 Classifying Discontinuities

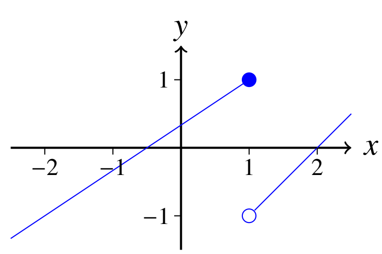
We now look at three of the different ways in which a function can be discontinuous at a point.

DEFINITION

Jump Discontinuity

A function f has a **jump discontinuity** at $x = a$ when both one-sided limits exist at a, but have different values. That is, f has a jump discontinuity at $x = a$ if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$



A jump discontinuity.

EXAMPLE 24

Use the definition to show that the Heaviside function $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ has a jump discontinuity at $x = 0$.

Solution: We can see that $\lim_{x\to 0^-} H(x) = 0$ and $\lim_{x\to 0^+} H(x) = 1$. Therefore, $H(x)$ has a jump discontinuity at $x = 0$.

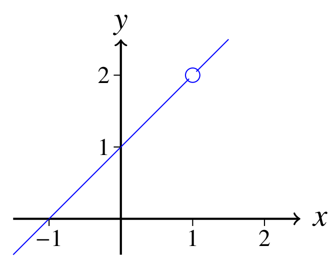
DEFINITION

Removable Discontinuity

A function f has a **removable discontinuity** at $x = a$ when the limit exists at a but is different from the function's value. That is, f has a removable discontinuity at $x = a$ when $\lim_{x\to a} f(x)$ exists but

$$\lim_{x\to a} f(x) \neq f(a)$$

or $f(a)$ does not exist.



A removable discontinuity.

EXAMPLE 25

Suppose $g(x) = \begin{cases} x, & \text{if } x < 1 \\ 3, & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$.

Determine the types of discontinuities at $x = 1$ and $x = 2$.

Solution: Since

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x = 1$$
$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (2 - x^2) = 2 - 1 = 1$$

we get that $\lim_{x\to 1} g(x) = 1$.

However, $g(1) = 3 \neq \lim_{x\to 1} g(x) = 1$. Thus, g has a removable discontinuity at $x = 1$.

Since

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (2 - x^2) = 2 - 4 = -2$$
$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} x - 3 = -1$$

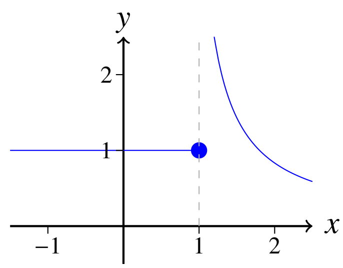
we get that g has a jump discontinuity at $x = 2$.

DEFINITION

Infinite Discontinuity

A function has an **infinite discontinuity** at $x = a$ if one of the one-sided limits at a is infinite. That is, f has an infinite discontinuity at $x = a$ when either

$$\lim_{x\to a^-} f(x) = \pm\infty \text{ or } \lim_{x\to a^+} f(x) = \pm\infty$$



An infinite discontinuity.

EXAMPLE 26

Show that $f(x) = \dfrac{x^2+1}{3x-2x^2}$ has an infinite discontinuity at $x = 0$ and at $x = \dfrac{3}{2}$.

Solution: The limit $\displaystyle\lim_{x\to 0^+} \frac{x^2+1}{x(3-2x)}$ has the form $\dfrac{L}{0^+}$. We find that

$$\lim_{x\to 0^+} \frac{x^2+1}{x(3-2x)} = \infty$$

Therefore, there is an infinite discontinuity at $x = 0$.

The limit $\displaystyle\lim_{x\to \frac{3}{2}^-} \frac{x^2+1}{3x-2x^2}$ has the form $\dfrac{L}{0^-}$. We find that

$$\lim_{x\to \frac{3}{2}^-} \frac{x^2+1}{3x-2x^2} = \infty$$

Hence, there is also an infinite discontinuity at $x = \dfrac{3}{2}$.

EXERCISE 11

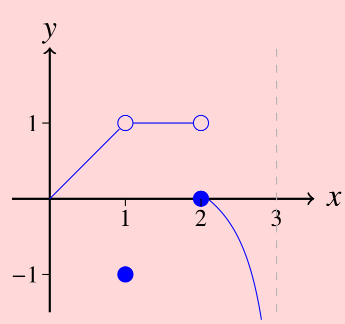
The graph of a function f is given.

Classify the discontinuities at

(a) $x = 1$

(b) $x = 2$

(c) $x = 3$



3.1.5 Limits at Infinity

So far in this section we have been analyzing the behaviour of functions around a point $x = a$. However, in many real world situations, we are interested in the long term behaviour of a function. That is, the behaviour of the function as x gets infinitely large (either positive or negative). To capture this behaviour, we define limits at infinity.

DEFINITION

Limits at Infinity

If the values of $f(x)$ can be made infinitely close to L by taking x infinitely large, then we write

$$\lim_{x\to\infty} f(x) = L$$

If the values of $f(x)$ are not approaching a single number, then we say the limit does not exist.

Moreover, if the limit does exist because the values of $f(x)$ are positive and growing larger and larger, then we write

$$\lim_{x\to\infty} f(x) = \infty$$

The notations

$$\lim_{x\to\infty} f(x) = -\infty$$
$$\lim_{x\to-\infty} f(x) = L$$
$$\lim_{x\to-\infty} f(x) = \infty$$
$$\lim_{x\to-\infty} f(x) = -\infty$$

are defined similarly.

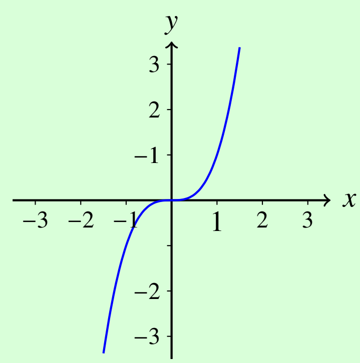
It is VERY important to remember that infinity is NOT a number. You can never substitute ∞ directly into a function. Consequently, to evaluate limits at infinity, we need to know the end behaviour of the basic functions. This is why these were specified in Sections 2.4 and 2.5.

EXAMPLE 27

Evaluate the following limits.

(a) $\lim_{x\to\infty} x^3$

(b) $\lim_{x\to-\infty} x^3$



Solution: From the graph, we see that

(a) $\lim_{x\to\infty} x^3 = \infty$

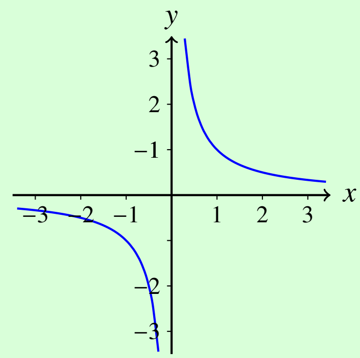
(b) $\lim_{x\to-\infty} x^3 = -\infty$

EXAMPLE 28

Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{1}{x}$

(b) $\lim_{x\to-\infty} \frac{1}{x}$



Solution: From the graph, we see that

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$

(b) $\lim_{x\to-\infty} \frac{1}{x} = 0$

EXERCISE 12

Evaluate the following limits. Assume that r is a positive rational number such that x^r is defined for all x.

(a) $\lim_{x\to\infty} x^r$

(b) $\lim_{x\to\infty} \frac{1}{x^r}$

(c) $\lim_{x\to\infty} e^x$

(d) $\lim_{x\to-\infty} e^x$

(e) $\lim_{x\to\infty} e^{-x}$

(f) $\lim_{x\to-\infty} e^{-x}$

(g) $\lim_{x\to\infty} \ln(x)$

(h) $\lim_{x\to\infty} \cos(x)$

(i) $\lim_{x\to\infty} \arctan(x)$

(j) $\lim_{x\to-\infty} \arctan(x)$

We also encounter determinate forms when evaluating limits at infinity.

EXAMPLE 29

Evaluate $\lim_{x\to\infty} \frac{e^{1/x}}{x}$.

Solution: As x gets larger and larger, the value of $\frac{1}{x}$ will get smaller and smaller. Thus, the values of $e^{1/x}$ will be approaching $e^0 = 1$. So, the limit has the determinate form $\frac{L}{\infty}$. Numbers getting closer and closer to 1 divided by larger and larger numbers results in smaller and smaller numbers. Thus,

$$\lim_{x\to\infty} \frac{e^{1/x}}{x} = 0$$

One strategy to evaluate limits at $\pm\infty$ of more complicated functions will be to use algebraic manipulations and properties of limits to reduce the problem to limits involving these basic functions ... remembering that all limits must exist for us to use the limit laws.

For now, our main step will be to first divide the numerator and denominator by the fastest growing term in the fraction.

EXAMPLE 30 Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$.

Solution: Since the fastest growing term (as $x \to \infty$) is x^2, we divide the numerator and denominator by x^2 to get

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} &= \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} \\ &= \frac{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty} \dfrac{1}{x} - \lim\limits_{x\to\infty} \dfrac{2}{x^2}}{\lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty} \dfrac{4}{x} + \lim\limits_{x\to\infty} \dfrac{1}{x^2}}\end{aligned}$$

since all six of these limits exist. Evaluating each limit gives

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} &= \frac{3 - 0 - 0}{5 + 0 + 0} \\ &= \frac{3}{5}\end{aligned}$$

In general, we skip writing the step where we applied the limit laws. Again, we stress that it is important that you ensure all of the limits will exist before using the limits laws.

EXAMPLE 31 Evaluate $\lim\limits_{x\to\infty} \dfrac{2x^2 + 3}{2x + 1 - 5x^3}$.

Solution: The fastest growing term is x^3. We divide the numerator and denominator by x^3 to get

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^2 + 3}{2x + 1 - 5x^3} &= \lim_{x\to\infty} \frac{\frac{2}{x} + \frac{3}{x^3}}{\frac{2}{x^2} + \frac{1}{x^3} - 5} \\ &= \frac{0 + 0}{0 + 0 - 5} \\ &= 0\end{aligned}$$

EXAMPLE 32 Evaluate $\lim_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

Solution: The fastest growing term as $x \to \infty$ is e^x. So, we divide the numerator and the denominator by e^x. We get

$$\begin{aligned}\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} &= \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} \\ &= \frac{1-0}{1+0} \\ &= 1\end{aligned}$$

EXAMPLE 33 Evaluate $\lim_{x\to\infty} \dfrac{4x^3 - 2x + 1}{5x + 1}$.

Solution: The fastest growing term is x^3. We divide the numerator and denominator by x^3 to get

$$\lim_{x\to\infty} \frac{4x^3 - 2x + 1}{5x + 1} = \lim_{x\to\infty} \frac{4 - \frac{2}{x^2} + \frac{1}{x^3}}{\frac{5}{x^2} + \frac{1}{x^3}}$$

This new limit has the determinate form $\dfrac{L}{0^+}$. Consequently,

$$\lim_{x\to\infty} \frac{4x^3 - 2x + 1}{5x + 1} = \lim_{x\to\infty} \frac{4 - \frac{2}{x^2} + \frac{1}{x^3}}{\frac{5}{x^2} + \frac{1}{x^3}} = \infty$$

EXERCISE 13 Evaluate the following limits.

(a) $\lim_{x\to-\infty} \dfrac{5x^4 + 2x^3}{3x^4 + 3x}$

(b) $\lim_{x\to-\infty} \dfrac{\sqrt{x^2 + 16} - 4}{x}$

(c) $\lim_{x\to\infty} \dfrac{3e^{2x} + 5}{4e^{3x} + 7e^{2x}}$

(d) $\lim_{x\to-\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(e) $\lim_{x\to\infty} \dfrac{x^2 + 2x - 1}{x^3 - 3x + 1}$

Applications

As previously mentioned, we use infinite limits to look at the long term behavior of real world phenomena.

EXAMPLE 34

Suppose the velocity of a falling skydiver can be modelled by $v(t) = \frac{mg}{\alpha}\left(1 - e^{-\alpha t/m}\right)$ where m is the mass of the skydiver, g is the acceleration due to gravity, α is the air drag coefficient, and t is the time in seconds. Evaluate $\lim\limits_{t\to\infty} v(t)$ and describe what it calculates.

Solution: We get

$$\lim_{t\to\infty} \frac{mg}{\alpha}(1 - e^{-\alpha t/m}) = \frac{mg}{\alpha}(1-0) = \frac{mg}{\alpha} \text{ m/s}$$

This is the maximum velocity the skydiver will approach. It is often called the **terminal velocity**.

EXAMPLE 35

A tank contains 5000 L of pure water. Brine that contains 30 g of salt per litre of water is pumped into the tank at a rate of 25 litres per minute and drained at the same rate. Given that the concentration of salt after t minutes is $C(t) = \frac{30t}{200+t}$ g/L, what happens to the concentration in the long run (as $t \to \infty$)?

Solution: We take the limit as $\to \infty$ of $C(t)$:

$$\begin{aligned}
\lim_{t\to\infty} C(t) &= \lim_{t\to\infty} \frac{30t}{200+t} \\
&= \lim_{t\to\infty} \frac{\frac{30t}{t}}{\frac{200+t}{t}} \\
&= \lim_{t\to\infty} \frac{30}{\frac{200}{t}+1} \\
&= 30 \text{ g/L}
\end{aligned}$$

EXERCISE 14

Suppose that the size of a population is given by $N(t) = \frac{100}{1+3e^{-t}}$. Determine the size of the population in the long run.

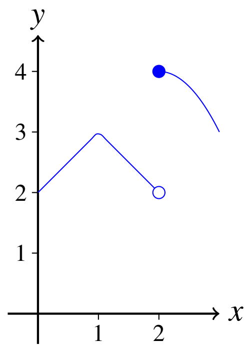
EXERCISE 15

The Monod model is a mathematical model for the growth rate of microorganisms in terms of the abundance of available nutrients. A Monod model for the reproduction rate r of an *E. coli* culture is $r(C) = \frac{1.35C}{C + 0.22 \times 10^{-4}}$, where C is the concentration of glucose in millimolar. How does the reproduction rate behave as the glucose concentration C gets very high?

Section 3.1 Problems

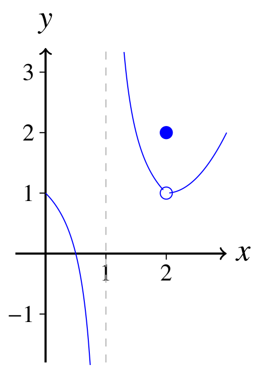
1. Use the graph of the function f to evaluate the limit.

   (a) $\lim_{x\to 1^-} f(x)$
   (b) $\lim_{x\to 1^+} f(x)$
   (c) $\lim_{x\to 1} f(x)$
   (d) $\lim_{x\to 2^-} f(x)$
   (e) $\lim_{x\to 2^+} f(x)$
   (f) $\lim_{x\to 2} f(x)$



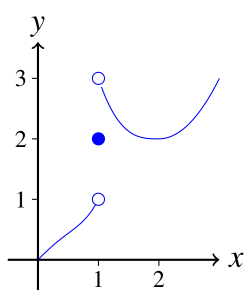
2. Use the graph of the function f to evaluate the limit.

   (a) $\lim_{x\to 1^-} f(x)$
   (b) $\lim_{x\to 1^+} f(x)$
   (c) $\lim_{x\to 1} f(x)$
   (d) $\lim_{x\to 2^-} f(x)$
   (e) $\lim_{x\to 2^+} f(x)$
   (f) $\lim_{x\to 2} f(x)$



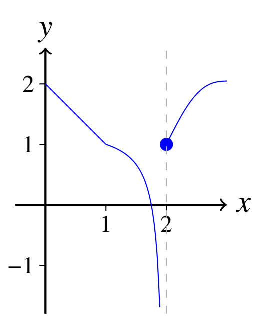
3. Use the graph of the function f to evaluate the limit.

   (a) $\lim_{x\to 1^-} f(x)$
   (b) $\lim_{x\to 1^+} f(x)$
   (c) $\lim_{x\to 1} f(x)$
   (d) $\lim_{x\to 2^-} f(x)$
   (e) $\lim_{x\to 2^+} f(x)$
   (f) $\lim_{x\to 2} f(x)$



4. Use the graph of the function f to evaluate the limit.

   (a) $\lim_{x\to 1^-} f(x)$
   (b) $\lim_{x\to 1^+} f(x)$
   (c) $\lim_{x\to 1} f(x)$
   (d) $\lim_{x\to 2^-} f(x)$
   (e) $\lim_{x\to 2^+} f(x)$
   (f) $\lim_{x\to 2} f(x)$



5. Evaluate the following limits.

   (a) $\lim_{x\to 1} \dfrac{x^3+1}{x+1}$
   (b) $\lim_{x\to -1} (x^2+3x)$
   (c) $\lim_{x\to 0^+} \dfrac{-3}{x}$
   (d) $\lim_{x\to 0} \dfrac{-3}{x}$
   (e) $\lim_{x\to \infty} \dfrac{x^2-3}{x^2+5x+2}$
   (f) $\lim_{x\to -\infty} \dfrac{4-x}{3-2x-x^2}$
   (g) $\lim_{x\to \frac{\pi}{2}^-} \dfrac{5}{\tan(x)}$
   (h) $\lim_{x\to \infty} \dfrac{x^3+5x-1}{3-4x^2}$
   (i) $\lim_{x\to \infty} \dfrac{\arctan(x)}{x}$
   (j) $\lim_{x\to 5} \dfrac{x}{x-5}$
   (k) $\lim_{x\to -\infty} \dfrac{-3}{e^x}$
   (l) $\lim_{x\to \infty} 2e^{-x}$
   (m) $\lim_{x\to 3} \dfrac{x}{|2-x|}$
   (n) $\lim_{x\to 0^+} \ln(x)$
   (o) $\lim_{x\to 1^-} \dfrac{2}{(1-x)^2}$
   (p) $\lim_{x\to 3} \dfrac{x^2+5x+6}{x^2+x-6}$.
   (q) $\lim_{x\to 2} \dfrac{x^2+\cos\left(\frac{\pi}{6}x\right)+1}{x^2-9}$.
   (r) $\lim_{x\to -\infty} \dfrac{x^3-3x}{2x^3+5x+234}$
   (s) $\lim_{x\to -\infty} \dfrac{2}{\arctan(x)}$
   (t) $\lim_{x\to \infty} \dfrac{\sqrt{x^3+x}}{x\sqrt{x}+x}$
   (u) $\lim_{x\to 0^+} \left(-\dfrac{7}{x}+\ln(x)\right)$
   (v) $\lim_{x\to -\infty} \dfrac{\sqrt{x^4+x^2}}{x^2+1}$

6. Let $f(x) = \begin{cases} 3x^2 - 5 & \text{if } x \leq 1 \\ 2x - 3 & \text{if } 1 < x \leq 2 \\ \dfrac{x^2 - x - 2}{x - 2} & \text{if } x > 2 \end{cases}$.

   Evaluate the following limits:
   (a) $\lim_{x \to 1^-} f(x)$
   (b) $\lim_{x \to 1^+} f(x)$
   (c) $\lim_{x \to 1} f(x)$
   (d) $\lim_{x \to 2^-} f(x)$
   (e) $\lim_{x \to 2^+} f(x)$
   (f) $\lim_{x \to 2} f(x)$

7. Let $f(x) = \begin{cases} \dfrac{x^2 + 2x + 1}{x - 1} & \text{if } x \leq 1 \\ \ln(x - 1) & \text{if } 1 < x \leq 2 \\ \dfrac{x^2 - 4x + 4}{x - 2} & \text{if } x > 2 \end{cases}$.

   Evaluate the following limits:
   (a) $\lim_{x \to 1^-} f(x)$
   (b) $\lim_{x \to 1^+} f(x)$
   (c) $\lim_{x \to 1} f(x)$
   (d) $\lim_{x \to 2^-} f(x)$
   (e) $\lim_{x \to 2^+} f(x)$
   (f) $\lim_{x \to 2} f(x)$

8. Is the function continuous on the interval?
   (a) $\dfrac{1}{x - 2}$, $[-1, 2)$
   (b) $\dfrac{1}{x + 3}$, $[-4, 4)$
   (c) $\tan(x)$, $(0, \pi)$
   (d) $\arcsin(x)$, $[-1, 1]$
   (e) $\dfrac{1}{\cos(x)}$, $[0, \pi]$
   (f) $x \ln(|x|)$, $[-1, 1]$
   (g) $\dfrac{1}{e^x - 1}$, $[-2, 0)$

9. Find all values of a and b that make the following function continuous

$$f(x) = \begin{cases} x + ax^2 + 1 & \text{if } x < 1 \\ bx^2 + 3ax & \text{if } 1 \leq x \leq 2 \\ bx + a + x & \text{if } x > 2 \end{cases}$$

10. Determine the value of a so that f is continuous on its domain.
    (a) $f(x) = \begin{cases} ax^2 & \text{if } x \leq 1 \\ 3 & \text{if } x > 1 \end{cases}$
    (b) $f(x) = \begin{cases} 2\ln(x) + 1 & \text{if } 0 < x < e \\ \dfrac{3}{a} & \text{if } x \geq e \end{cases}$
    (c) $f(x) = \begin{cases} \dfrac{x^2 + 3x + 2}{x + 1} & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$

11. Sketch the graph of a function f such that:
    (a) f is discontinuous at $x = 1$ and the limit as $x \to 1$ of f exists.
    (b) $x = 1$ is in the domain of f and f is discontinuous at $x = 1$.

12. Classify the discontinuities of each function.
    (a) $f(x) = \dfrac{|x - 2|}{x - 2}$
    (b) $f(x) = \begin{cases} -\dfrac{1}{3} & x \leq -1 \\ \dfrac{1}{x^2 - 4} & -1 < x < 2 \\ 5 & x \geq 2 \end{cases}$
    (c) $f(x) = \dfrac{x^3 - 4x^2 + 4x}{x(x - 1)}$
    (d) $f(x) = \dfrac{x^2 - 7x + 10}{x - 2}$

13. Using the Intermediate Value Theorem show that there exists x with $0 < x < 1$ such that:
    (a) $x^3 + x^2 - 1 = 0$
    (b) $2^x = 3x$
    (c) $\sqrt{x^4 + 25x^3 + 10} = 5$

14. Using the Intermediate Value Theorem show that there is at least one negative solution to $e^x = -x$.

15. The height h of a tree as a function of the tree's age t in years is given by $h = 121e^{-17/t}$ ft, for $t > 0$. Determine the limit of the height of the tree.

16. Ducks pant to control their internal temperature. If $H(T)$ is the rate of heat loss in watts per kg of body mass as a function of the duck's core temperature T in degrees Celsius, then it has been found that

$$H(T) = \begin{cases} 0.6 & T \leq T_c \\ 4.3T - 183 & T > T_c \end{cases}$$

Find the value of T_c that makes $H(T)$ continuous for all T.

17. The magnitude of the electric field for a uniformly charged solid sphere of radius $R > 0$ with total charge Q is given by

$$F(d) = \begin{cases} \dfrac{Q}{d^2}, & 0 < d \leq R \\ \dfrac{Q}{R^3}d, & d > R \end{cases}$$

where d is distance from the centre of the charged sphere. Determine if this function is continuous at $d = R$.

18. On a cold day, you measure the temperature inside your home to be 22°C and the temperature outside to be -9°C. The temperature in your walls, however, varies so that temperature as a function of position, $T(x)$, is a continuous function. Based on the composition of your 5 cm thick walls, you are able to construct the following piecewise model for $T(x)$:

$$T(x) = \begin{cases} 22, & x \leq 0 \\ a - bx^2, & 0 < x \leq 5 \\ -9, & x > 5 \end{cases}$$

(a) Find a and b so that $T(x)$ is continuous.
(b) What is the temperature in the middle of the wall (i.e., 2.5 cm in)?

Section 3.2: Indeterminate Forms

LEARNING OUTCOMES

1. Know how to identify indeterminate forms.
2. Know how to evaluate indeterminate forms.

3.2.1 Indeterminate Forms

In the last section, we encountered limits that took on determinate forms; undefined expressions where we could deduce the value of the expression from the form. We now look at **indeterminate forms**; undefined expressions where we *cannot* immediately deduce the value of the expression.

For example, consider the three limits

$$\lim_{x\to 0^+} \frac{\sin(x)}{2x}, \quad \lim_{x\to 0^+} \frac{\ln(x+1)}{x^2}, \quad \lim_{x\to 0^+} \frac{\cos(x)-1}{x}$$

In all of these limits both the numerator and denominator approach 0 as x approaches 0 from the right. That is, these all have the form $\frac{0}{0}$. If we graph these functions for positive values of x near 0, we find that

$$\begin{aligned} \lim_{x\to 0^+} \frac{\sin(x)}{2x} &= \frac{1}{2} \\ \lim_{x\to 0^+} \frac{\ln(x+1)}{x^2} &= \infty \qquad \text{(i.e. does not exist)} \\ \lim_{x\to 0^+} \frac{\cos(x)-1}{x} &= 0 \end{aligned}$$

So, $\frac{0}{0}$ is an indeterminate form because, because we cannot determine its value without doing further work.

The other most common indeterminate form is $\frac{\infty}{\infty}$. We can verify this is an indeterminate form by observing that the limits

$$\lim_{x\to\infty} \frac{e^x}{x}, \qquad \lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}, \qquad \lim_{x\to 0^+} \frac{x^2+1}{-3x^2+2}$$

all have the form $\frac{\infty}{\infty}$, but graphing these shows that

$$\begin{aligned} \lim_{x\to\infty} \frac{e^x}{x} &= \infty \\ \lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} &= 0 \\ \lim_{x\to 0^+} \frac{x^2+1}{-3x^2+2} &= -\frac{1}{3} \end{aligned}$$

EXAMPLE 1

Decide whether the limit can be determined or has an indeterminate form.

(a) $\lim_{x\to\infty} \frac{x^2+1}{e^x}$

(b) $\lim_{x\to\infty} \frac{e^{-x}}{e^x}$

(c) $\lim_{x\to 0} \frac{\tan(x)}{x}$

(d) $\lim_{x\to 0^+} \frac{\ln(x)}{x^5}$

Solution: (a) Since $\lim_{x\to\infty}(x^2+1) = \infty$ and $\lim_{x\to\infty} e^x = \infty$, the limit has the indeterminate form $\frac{\infty}{\infty}$.

(b) Since $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to\infty} e^x = \infty$, the limit has the form $\frac{0}{\infty}$. This is a determinate form. We find that

$$\lim_{x\to\infty} \frac{e^{-x}}{e^x} = 0$$

(c) Since $\lim_{x\to 0} \tan(x) = 0$ and $\lim_{x\to 0} x = 0$, the limit has the indeterminate form $\frac{0}{0}$.

(d) Since $\lim_{x\to 0^+} \ln(x) = -\infty$ and $\lim_{x\to 0^+} x^5 = 0$, the limit has the form $\frac{-\infty}{0^+}$. This is a determinate form. We find that

$$\lim_{x\to 0^+} \frac{\ln(x)}{x^5} = -\infty$$

EXERCISE 1

Decide whether the limit has an indeterminate form.

(a) $\lim_{x\to\infty} \frac{\ln(x)}{x}$

(b) $\lim_{x\to 2} \frac{x^2-4}{|x-2|}$

(c) $\lim_{x\to\infty} \frac{\arctan(x)}{x^2}$

(d) $\lim_{x\to 1} \frac{\ln(x)}{x-1}$

(e) $\lim_{x\to -\infty} \frac{e^{-x}}{\sqrt{|x|}}$

(f) $\lim_{x\to \frac{\pi}{2}^-} \tan(x)$

3.2.2 Algebraic Methods for Evaluating Indeterminate Forms

Simplifying

We can evaluate some limits which have an indeterminate form by using algebraic manipulations. This often includes factoring and canceling off a common factor, but can also involve dividing by the largest term as we saw in Section 3.1.5.

EXAMPLE 2 Evaluate $\lim\limits_{h\to 0} \dfrac{(2+h)^2-4}{h}$.

Solution: Since both the numerator and denominator approach 0 as h approaches 0, this has the indeterminate form $\dfrac{0}{0}$.

We get

$$\begin{aligned}\lim_{h\to 0} \frac{(2+h)^2-4}{h} &= \lim_{h\to 0} \frac{4+4h+h^2-4}{h}\\ &= \lim_{h\to 0} \frac{4h+h^2}{h}\\ &= \lim_{h\to 0}(4+h)\\ &= 4\end{aligned}$$

EXAMPLE 3 Evaluate $\lim\limits_{x\to 1^-} \dfrac{x^2-1}{(x-1)^2}$.

Solution: Since both the numerator and denominator approach 0 as x approaches 1 from the left, this has the indeterminate form $\dfrac{0}{0}$.

$$\begin{aligned}\lim_{x\to 1^-} \frac{x^2-1}{(x-1)^2} &= \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)^2}\\ &= \lim_{x\to 1^-} \frac{x+1}{x-1}\end{aligned}$$

The new limit has the determinate form $\dfrac{L}{0^-}$. We find that

$$\lim_{x\to 1^-} \frac{x^2-1}{(x-1)^2} = \lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty$$

EXAMPLE 4

Evaluate $\lim\limits_{x\to 1}\dfrac{|x-1|}{x-1}$.

Solution: Since both the numerator and denominator approach 0 as x approaches 1, this has the indeterminate form $\dfrac{0}{0}$.

Since $|x-1|$ is technically a piecewise defined function, we need to use left-hand and right-hand limits. We get

$$\begin{aligned}\lim_{x\to 1^+}\frac{|x-1|}{x-1} &= \lim_{x\to 1^+}\frac{x-1}{x-1} \qquad \text{since } x-1>0 \text{ when } x>1\\ &= \lim_{x\to 1^+} 1\\ &= 1\\ \lim_{x\to 1^-}\frac{|x-1|}{x-1} &= \lim_{x\to 1^-}\frac{-(x-1)}{x-1} \qquad \text{since } x-1<0 \text{ when } x<1\\ &= \lim_{x\to 1^-} -1\\ &= -1\end{aligned}$$

Since $\lim\limits_{x\to 1^+}\dfrac{|x-1|}{x-1} \neq \lim\limits_{x\to 1^-}\dfrac{|x-1|}{x-1}$, the limit $\lim\limits_{x\to 1}\dfrac{|x-1|}{x-1}$ does not exist.

EXAMPLE 5

Evaluate $\lim\limits_{x\to \frac{\pi}{2}^+}\dfrac{\tan(x)}{\sec(x)}$.

Solution: Since both the numerator and denominator approach ∞ as x approaches $\dfrac{\pi}{2}$ from the right, this has the indeterminate form $\dfrac{-\infty}{-\infty}$. We get

$$\begin{aligned}\lim_{x\to \frac{\pi}{2}^+}\frac{\tan(x)}{\sec(x)} &= \lim_{x\to \frac{\pi}{2}^+}\frac{\sin(x)}{\cos(x)}\cdot\frac{\cos(x)}{1}\\ &= \lim_{x\to \frac{\pi}{2}^+}\sin(x)\\ &= 1\end{aligned}$$

EXERCISE 2

Evaluate the following limits.

(a) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}$

(b) $\lim\limits_{x\to 2^+}\dfrac{x^2-2x-2}{x^2+x-6}$

(c) $\lim\limits_{x\to 1}\dfrac{|x^2-1|}{x-1}$

Multiplying by the Conjugate

When we have a square root term being added to or subtracted from a quantity (which sometimes could be another square root term), we can try multiplying the numerator and denominator by the conjugate of the expression with the square root term to get a difference of squares.

EXAMPLE 6 Evaluate $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$.

Solution: The limit has the indeterminate form $\dfrac{0}{0}$. The conjugate of the numerator is $\sqrt{x}+2$. Multiplying the numerator and denominator by this conjugate gives

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} &= \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} \\
&= \lim_{x\to 4} \frac{(\sqrt{x})^2 - 2^2}{(x-4)(\sqrt{x}+2)}, \qquad \text{by the difference of squares formula} \\
&= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} \\
&= \lim_{x\to 4} \frac{1}{\sqrt{x}+2} \\
&= \frac{1}{4}
\end{aligned}$$

EXAMPLE 7 Evaluate $\lim_{x\to 1^+} \dfrac{(x-1)^2}{\sqrt{x+2}-\sqrt{2x+1}}$.

Solution: The limit has the indeterminate form $\dfrac{0}{0}$. The conjugate of the denominator is $\sqrt{x+2}+\sqrt{2x+1}$. Multiplying the numerator and denominator by this conjugate gives

$$\begin{aligned}
\lim_{x\to 1^+} \frac{(x-1)^2}{\sqrt{x+2}-\sqrt{2x+1}} &= \lim_{x\to 1^+} \frac{(x-1)^2}{\sqrt{x+2}-\sqrt{2x+1}} \cdot \frac{\sqrt{x+2}+\sqrt{2x+1}}{\sqrt{x+2}+\sqrt{2x+1}} \\
&= \lim_{x\to 1^+} \frac{(x-1)^2(\sqrt{x+2}-\sqrt{2x+1})}{(\sqrt{x+2})^2-(\sqrt{2x+1})^2} \\
&= \lim_{x\to 1^+} \frac{(x-1)^2(\sqrt{x+2}-\sqrt{2x+1})}{x+2-(2x+1)} \\
&= \lim_{x\to 1^+} \frac{(x-1)^2(\sqrt{x+2}-\sqrt{2x+1})}{-(x-1)} \\
&= \lim_{x\to 1^+} -(x-1)(\sqrt{x+2}-\sqrt{2x+1}) \\
&= 0
\end{aligned}$$

EXAMPLE 8

Evaluate $\lim_{x\to 2} \frac{1-x}{\sqrt{x-1}+1}$.

Solution: The temptation here is to multiply top and bottom by the conjugate of the denominator. But, this doesn't have an indeterminate form! In particular, since the function is continuous at $x = 2$, we get

$$\lim_{x\to 2} \frac{1-x}{\sqrt{x-1}+1} = \frac{1-2}{\sqrt{2-1}+1} = \frac{-1}{\sqrt{2}}$$

The moral of the last example is that spending a little bit of timing thinking about a problem before you start solving it, can save you time and effort.

EXERCISE 3

Evaluate the following limits.

(a) $\lim_{x\to 2^+} \frac{\sqrt{x-1}-1}{(x-2)^2}$

(b) $\lim_{x\to 3} \frac{\sqrt{x^2-1}-\sqrt{8}}{x-3}$

(c) $\lim_{x\to\infty} \frac{\sqrt{x+1}-1}{x}$

3.2.3 L'Hospital's Rule

Some indeterminate forms can be very difficult to solve using algebraic manipulations. For these, we can try to apply the following powerful tool.

THEOREM 1

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a) and the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ has an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

as long as the new limit exists or equals $\pm\infty$.

Observe that applying L'Hospital's Rule gives a new limit to evaluate that has the same answer as the original. If this new limit also has an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we may apply L'Hospital's Rule again to get a third limit that is also equal to the original.

Although L'Hospital's Rule can be very useful, it does not always produce a new limit that is easier than the original. In fact, there are limits that cannot be solved just with applications of L'Hospital's Rule (see Example 3.2.12 below). There are other limits where the algebraic methods discussed above will be much faster than using L'Hospital's Rule. In general, we recommend using L'Hospital's Rule as a last resort. That is, only use it when algebraic methods won't work.

REMARKS

1. L'Hospital's Rule also applies to one-sided limits and limits at infinity.
2. It is important to communicate whenever you are applying L'Hospital's Rule. In this book, we will do this by placing an H above the equals sign.
3. Only use L'Hospital's Rule when you have either the form $\frac{0}{0}$ or the form $\frac{\infty}{\infty}$. Trying to apply it in any other case will likely give you the wrong answer.

EXAMPLE 9

Evaluate $\lim\limits_{x\to 0} \frac{\sin(x)}{2x}$.

Solution: The limit has the form $\frac{0}{0}$. We get

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)}{2x} &\overset{H}{=} \lim_{x\to 0}\frac{\frac{d}{dx}(\sin(x))}{\frac{d}{dx}(2x)}\\ &= \lim_{x\to 0}\frac{\cos(x)}{2}\\ &= \frac{1}{2}\end{aligned}$$

EXAMPLE 10

Evaluate $\lim\limits_{x\to\infty} \frac{\ln(x)}{e^x}$.

Solution: The limit has the form $\frac{\infty}{\infty}$. We get

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln(x)}{e^x} &\overset{H}{=} \lim_{x\to\infty}\frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(e^x)}\\ &\overset{H}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{e^x}\\ &= \lim_{x\to\infty}\frac{1}{xe^x}\\ &= 0\end{aligned}$$

EXAMPLE 11 Evaluate $\lim_{x\to\infty} \frac{e^x}{x^3}$.

Solution: The limit has the form $\frac{\infty}{\infty}$. We get

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^3)} = \lim_{x\to\infty} \frac{e^x}{3x^2}$$

This new limit also has the form $\frac{\infty}{\infty}$. So,

$$\lim_{x\to\infty} \frac{e^x}{3x^2} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(3x^2)} = \lim_{x\to\infty} \frac{e^x}{6x}$$

This limit also has the form $\frac{\infty}{\infty}$. So, we apply L'Hospital's rule a third time to get

$$\lim_{x\to\infty} \frac{e^x}{6x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(6x)} = \lim_{x\to\infty} \frac{e^x}{6}$$

This limit has the form $\frac{\infty}{L}$. Since this determinate form evaluates to ∞, we have that

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \infty$$

Normally, we apply L'Hospital's Rule and take the derivatives in a single step as in the examples below.

EXAMPLE 12 Evaluate $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$.

Solution: The limit has the form $\frac{\infty}{\infty}$. We get

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} &\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x}{1} \\ &= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}\end{aligned}$$

This new limit also has the form $\frac{\infty}{\infty}$. So,

$$\begin{aligned}\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} &\overset{H}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x} \\ &= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}\end{aligned}$$

WAIT! This is the limit we started with! As promised, this is an example where L'Hospital's Rule does not work.

If we instead divided the numerator and denominator by the fastest growing term, x, (which we should have done in the first place), we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} &= \lim_{x\to\infty} \frac{\sqrt{x^2+1}\cdot\frac{1}{x}}{1} \\
&= \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}}\cdot\sqrt{x^2}\cdot\frac{1}{x} \\
&= \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}}\cdot|x|\cdot\frac{1}{x} \\
&= \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}}\cdot x\cdot\frac{1}{x}, \qquad \text{since } x>0 \\
&= \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} \\
&= \sqrt{1+0} \\
&= 1
\end{aligned}$$

EXERCISE 4

Evaluate the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{x^{1/3}}$

(b) $\displaystyle\lim_{x\to0} \frac{\ln(x+1)}{x^2}$

(c) $\displaystyle\lim_{x\to0} \frac{\tan(x)-x}{x^3}$

(d) $\displaystyle\lim_{x\to\infty} \frac{x^{11}-x^9-x^7-1}{x^{13}}$

(e) $\displaystyle\lim_{x\to0^+} \frac{\ln(x)}{x}$

(f) $\displaystyle\lim_{x\to0^-} \frac{\sqrt{x^2+x^4}-x}{x}$

EXERCISE 5

Use L'Hospital's Rule to show that $\displaystyle\lim_{x\to0} \frac{\cos(x)-1}{x} = 0$.

We now look at some other types of indeterminate forms.

3.2.4 Indeterminate Products

Our next indeterminate form is $0 \cdot \infty$. We can verify this is an indeterminate form by observing that the limits

$$\lim_{x\to 0^+} x\ln(x), \qquad \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right), \qquad \lim_{x\to\infty} xe^{1/x}$$

all have the form $0 \cdot \infty$, but if we evaluate them, we get

$$\lim_{x\to 0^+} x\ln(x) = 0$$
$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = 1$$
$$\lim_{x\to\infty} xe^{1/x} = \infty$$

EXAMPLE 13 Identify which of the following limits have an indeterminate form.

(a) $\lim_{x\to\infty}(x^2+1)e^x$

(b) $\lim_{x\to\infty} e^{-x}\ln(x)$

(c) $\lim_{x\to 0^+} \sqrt{x^3}\ln(x)$

(d) $\lim_{x\to\infty} 0 \cdot e^x$

Solution: (a) The limit has the determinate form $\infty \cdot \infty$. Since larger and larger numbers multiplied by larger and larger numbers are even larger and larger number, we get

$$\lim_{x\to\infty}(x^2+1)e^x = \infty$$

(b) Since $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to\infty}\ln(x) = \infty$, the limit has the indeterminate form $0 \cdot \infty$.

(c) Since $\lim_{x\to 0^+}\sqrt{x^3} = 0$ and $\lim_{x\to 0^+}\ln(x) = -\infty$, the limit has the indeterminate form $0 \cdot (-\infty)$.

(d) Since 0 times any number is 0, we have

$$\lim_{x\to\infty} 0 \cdot e^x = \lim_{x\to\infty} 0 = 0$$

EXERCISE 6 Identify which of the following limits have an indeterminate form.

(a) $\lim_{x\to\infty} xe^{-x}$

(b) $\lim_{x\to 0^+} e^{1/x}\ln(x)$

(c) $\lim_{x\to \frac{\pi}{2}^+}\left(x-\frac{\pi}{2}\right)\tan(x)$

We can deal with the indeterminate form $0 \cdot \infty$ by writing the product fg as a quotient

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ so that we can then use the methods above.

EXAMPLE 14

Evaluate $\displaystyle\lim_{x\to 0^+} \sqrt[3]{x}\ln(x)$.

Solution: The limit has the form $0 \cdot (-\infty)$.

We will bring one of the factors down to the denominator by writing it as its reciprocal. But, which one?

Let's see what happens if we bring the $\ln(x)$ down. We write $\sqrt[3]{x}$ as $x^{1/3}$ and we rewrite $\ln(x)$ as $\dfrac{1}{\frac{1}{\ln(x)}}$ to get

$$\lim_{x\to 0^+} \frac{x^{1/3}}{\frac{1}{\ln(x)}}$$

This has the indeterminate form $\dfrac{0}{0}$, so we apply L'Hospital's Rule.

$$\begin{aligned}\lim_{x\to 0^+} \frac{x^{1/3}}{\frac{1}{\ln(x)}} &\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{3}x^{-2/3}}{\frac{-1/x}{(\ln(x))^2}} \\ &= \lim_{x\to 0^+} -\frac{1}{3}\sqrt[3]{x}(\ln(x))^2\end{aligned}$$

This looks worse than what we started with...and is still an indeterminate form $0 \cdot \infty$!

Let's start over, this time keeping the $\ln(x)$ in the numerator and bringing the $\sqrt[3]{x}$ to the denominator as $x^{-1/3}$. Doing so, we end up with the indeterminate form $\dfrac{-\infty}{\infty}$. We can now apply L'Hospital's rule to get

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/3}} &\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{3}x^{-4/3}} \\ &= \lim_{x\to 0^+} -3x^{1/3} \\ &= 0\end{aligned}$$

REMARK

When deciding which factor to move to the denominator, think about what will be easy to differentiate as a reciprocal. There is no rule to memorize for doing this. It is through practice and experience that you will learn how to recognize which function to move into the denominator.

EXAMPLE 15

Evaluate $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$.

Solution: The limit has the form $\infty \cdot 0$. We get

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

This has the form $\frac{0}{0}$. We now get

$$\begin{aligned}\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} &\overset{H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) \\ &= \cos(0) \\ &= 1\end{aligned}$$

EXERCISE 7

Evaluate the limits.

(a) $\lim_{x\to\infty} xe^{-x}$

(b) $\lim_{x\to 0^+} x^2\ln(x)$

3.2.5 Indeterminate Powers

The final three indeterminate forms that we will consider are 0^0, ∞^0, and 1^∞.

Of all of our indeterminate forms, 1^∞ is the one that surprises most people. Most people would guess that this is a determinate form with a value of 1. However, we can verify that it is indeterminate by looking at some examples.

The limits $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ and $\lim_{x\to 0^+}(1-2x)^{1/x}$ both have the form 1^∞, but we can show that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to 0^+}(1-2x)^{1/x} = \frac{1}{e^2}$$

EXAMPLE 16 Identify which of the following have an indeterminate form.

(a) $\lim\limits_{x\to\infty} x^{1/x}$

(b) $\lim\limits_{x\to\infty} (\ln(x))^x$

(c) $\lim\limits_{x\to 0^+} \sqrt{x}^{x^2}$

(d) $\lim\limits_{x\to 0^+} x^{\ln(x)}$

Solution: (a) We have $\lim\limits_{x\to\infty} x = \infty$ and $\lim\limits_{x\to\infty} \frac{1}{x} = 0$. Thus, the limit has the indeterminate form ∞^0.

(b) We have $\lim\limits_{x\to\infty} \ln(x) = \infty$ and $\lim\limits_{x\to\infty} x = \infty$. This limit has the determinate form ∞^∞. Since larger and larger numbers to the power of larger and larger numbers will be even larger and larger numbers, we get

$$\lim_{x\to\infty} (\ln(x))^x = \infty$$

(c) We have $\lim\limits_{x\to 0^+} \sqrt{x} = 0$ and $\lim\limits_{x\to\infty} x^2 = 0$, so the limit has the indeterminate form 0^0.

(d) We have $\lim\limits_{x\to 0^+} x = 0$ and $\lim\limits_{x\to 0^+} \ln(x) = -\infty$. This limit has the determinate form $0^{-\infty}$. Since smaller and smaller numbers to larger and larger negative powers will be larger and larger numbers, we get

$$\lim_{x\to 0^+} x^{\ln(x)} = \infty$$

We deal with indeterminate powers using the amazing property of logarithms that

$$\ln(a^b) = b\ln(a)$$

This turns the indeterminate power into an indeterminate product.

Mathematically, it looks like this:

$$\begin{aligned} L &= \lim_{x\to a}[f(x)]^{g(x)} \\ \ln(L) &= \lim_{x\to a} \ln\left([f(x)]^{g(x)}\right) \\ \ln(L) &= \lim_{x\to a} g(x)\ln(f(x)) \end{aligned}$$

We then can convert this indeterminate product into an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as we learned above.

Note that we will end up finding the limit of the logarithm of the original expression. So, at the end, we need to undo the logarithm by applying the exponential function.

EXAMPLE 17

Evaluate $\lim_{x\to 0^+} x^x$.

Solution: The limit has the form 0^0. Let $L = \lim_{x\to 0^+} x^x$. Taking ln of both sides gives

$$\begin{aligned}
\ln(L) &= \lim_{x\to 0^+} \ln(x^x) \\
\ln(L) &= \lim_{x\to 0^+} x\ln(x) \qquad (\text{ form } 0\cdot\infty) \\
\ln(L) &= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \qquad \left(\text{form } \frac{\infty}{\infty}\right) \\
\ln(L) &\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
\ln(L) &= \lim_{x\to 0^+} -x \\
\ln(L) &= 0 \\
L &= e^0 = 1
\end{aligned}$$

Therefore, $\lim_{x\to 0^+} x^x = L = 1$.

EXAMPLE 18

Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution: The limit has the form 1^∞. Let $L = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. Taking ln of both sides gives

$$\begin{aligned}
\ln(L) &= \lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right) \\
\ln(L) &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \qquad (\text{ form } \infty\cdot 0) \\
\ln(L) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \left(\text{form } \frac{0}{0}\right) \\
\ln(L) &\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} \\
\ln(L) &= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} \\
\ln(L) &= \frac{1}{1+0} \\
\ln(L) &= 1 \\
L &= e^1
\end{aligned}$$

Therefore, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = L = e$.

EXAMPLE 19 Evaluate $\lim_{x\to 0}\left(\frac{1}{x}\right)^x$.

Solution: The limit has the form ∞^0. Let $L = \lim_{x\to 0}\left(\frac{1}{x}\right)^x$. Taking ln of both sides gives

$$\begin{aligned}
\ln(L) &= \lim_{x\to 0}\ln\left(\left(\frac{1}{x}\right)^x\right) \\
\ln(L) &= \lim_{x\to 0} x\ln\left(\frac{1}{x}\right) && (\text{ form } 0\cdot\infty) \\
\ln(L) &= \lim_{x\to 0}\frac{-\ln(x)}{\frac{1}{x}} && \left(\text{form } \frac{-\infty}{\infty}\right) \\
\ln(L) &\overset{H}{=} \lim_{x\to 0}\frac{-\frac{1}{x}}{\frac{-1}{x^2}} \\
\ln(L) &= \lim_{x\to 0} x \\
\ln(L) &= 0 \\
L &= e^0 = 1
\end{aligned}$$

Therefore, $\lim_{x\to 0}\left(\frac{1}{x}\right)^x = L = 1$.

EXAMPLE 20 Evaluate $\lim_{x\to 0^+}(\tan(2x))^x$.

Solution: The limit has the form 0^0. Let $L = (\tan(2x))^x$. Taking ln of both sides gives

$$\begin{aligned}
\ln(L) &= \lim_{x\to 0^+}\ln(\tan(2x)^x) \\
\ln(L) &= \lim_{x\to 0^+} x\ln(\tan(2x)) && (\text{ form } 0\cdot -\infty) \\
\ln(L) &= \lim_{x\to 0^+}\frac{\ln(\tan(2x))}{\frac{1}{x}} && \left(\text{form } \frac{-\infty}{\infty}\right) \\
\ln(L) &\overset{H}{=} \lim_{x\to 0^+}\frac{\frac{2\sec^2(2x)}{\tan(2x)}}{-x^{-2}} \\
\ln(L) &= \lim_{x\to 0^+}\frac{-2x^2\sec^2(2x)}{\tan(2x)} && \left(\text{form } \frac{0}{0}\right) \\
\ln(L) &\overset{H}{=} \lim_{x\to 0^+}\frac{-4x\sec^2(2x) - 2x^2\cdot 2\sec(2x)\cdot\sec(2x)\tan(2x)\cdot 2}{\sec^2(2x)\cdot 2} \\
\ln(L) &= \lim_{x\to 0^+}(-2x - 4x^2\tan(2x)) \\
\ln(L) &= 0 \\
L &= e^0 = 1
\end{aligned}$$

Therefore, $\lim_{x\to 0^+}(\tan(2x))^x = L = 1$.

EXERCISE 8 Evaluate the following limits.

(a) $\lim_{x\to\infty} x^{1/x}$

(b) $\lim_{x\to 0^+} (1-2x)^{1/x}$

(c) $\lim_{x\to 0} (\cos(x))^{1/x^2}$

Section 3.2 Problems

1. Is the form determinate or indeterminate? If it is determinate, state its value.
   (a) $\frac{0^+}{\infty}$
   (b) $\frac{\infty}{0^+}$
   (c) $\frac{0}{0}$
   (d) $\frac{\infty}{\infty}$
   (e) $\infty \cdot (-\infty)$
   (f) 1^∞
   (g) $(0^+)^\infty$
   (h) 0^0

2. Evaluate the limit.
   (a) $\lim_{x\to 3} \frac{x^2-9}{x^2-x-6}$
   (b) $\lim_{x\to -5} \frac{x^2+3x-10}{x^2+6x+5}$
   (c) $\lim_{x\to 2} \frac{x^2-1}{x-1}$
   (d) $\lim_{x\to 9} \frac{3-\sqrt{x}}{x-9}$
   (e) $\lim_{x\to\infty} \frac{x^2+5}{x+1}$
   (f) $\lim_{x\to 0^-} \frac{e^x}{x}$
   (g) $\lim_{x\to 1} \frac{|x^2+x-2|}{x^2-1}$
   (h) $\lim_{x\to 0^+} \left(\frac{1}{x}\right)^0$
   (i) $\lim_{x\to 0^+} (\tan(2x))^x$
   (j) $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{e^{-x}}$

3. Evaluate the following limits.
   (a) $\lim_{h\to 0} \frac{(1+h)^2-1}{h}$
   (b) $\lim_{x\to 2} \frac{x^2-4}{x-2}$
   (c) $\lim_{x\to\infty} \frac{x^2}{e^x}$
   (d) $\lim_{x\to -1} \frac{x^2+4x+3}{x^2-2x-3}$
   (e) $\lim_{x\to -1} \frac{1}{\sqrt{x^2+x}-x}$
   (f) $\lim_{x\to 0} \frac{e^{2x}-1}{x}$
   (g) $\lim_{x\to 0} \frac{x}{\arcsin(x)}$
   (h) $\lim_{x\to\infty} x^2 e^{-x^2}$
   (i) $\lim_{x\to 0} \frac{x-\sin(x)}{x-\tan(x)}$
   (j) $\lim_{x\to\infty} \frac{(\ln(x))^2}{x}$
   (k) $\lim_{x\to\infty} \frac{x^3}{e^{x/2}}$
   (l) $\lim_{x\to 0^+} x^{x^2}$
   (m) $\lim_{x\to\infty} \frac{3x-7}{e^x}$
   (n) $\lim_{x\to\infty} (x(e^{1/x}-1))$
   (o) $\lim_{x\to 0^+} (\csc(x))^{\cos(x)}$
   (p) $\lim_{x\to 0^+} \sin(x)\ln(x)$
   (q) $\lim_{x\to 0^+} (\cos(x))^{1/x^2}$
   (r) $\lim_{x\to\infty} \left(\frac{2x+5}{2x+3}\right)^{2x+1}$

Section 3.3: Analysis of Functions

LEARNING OUTCOMES

1. Know how to find the vertical and horizontal asymptotes of a curve.
2. Know how to find intervals of increase and decrease of a function.
3. Know how to find and classify critical points of a function.
4. Know how to find local maxima and minima points of a function.
5. Know how to find intervals of concavity of a curve.
6. Know how to find points of inflection of a curve.

We now look at how to use limits and derivatives to further analyze the behavior of a function.

3.3.1 Asymptotes

Horizontal Asymptotes

DEFINITION

Horizontal Asymptote

If

$$\lim_{x\to\infty} f(x) = L$$

then the line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$.

If

$$\lim_{x\to-\infty} f(x) = M$$

then the line $y = M$ is called a **horizontal asymptote** of the curve $y = f(x)$.

As we will see, it is possible for the graph of a function to have two different asymptotes (one as $x \to \infty$ and a different one as $x \to -\infty$), one asymptote (which can be just one in one of the directions, or the same one in both directions), or no horizontal asymptotes.

For this reason, when looking for horizontal asymptotes, we always check both the limit as $x \to \infty$ and as $x \to -\infty$ unless the domain of the function prevents this (see Example 3.3.3 below).

Note that if either of these limits is $\pm\infty$, then there is no horizontal asymptote in that direction. However, the fact that $f(x)$ is growing larger and larger as x gets larger and larger is still extremely valuable information about the behaviour of the function. We will see this when graphing functions in the next section.

EXAMPLE 1

Find the horizontal asymptotes of $y = f(x) = \dfrac{\arctan(x)}{\pi}$.

Solution: We have

$$\lim_{x\to-\infty} \frac{\arctan(x)}{\pi} = \frac{\frac{-\pi}{2}}{\pi} = -\frac{1}{2}$$

Therefore, $y = -\dfrac{1}{2}$ is a horizontal asymptote of $y = f(x)$.

We also have

$$\lim_{x\to\infty} \frac{\arctan(x)}{\pi} = \frac{\frac{\pi}{2}}{\pi} = \frac{1}{2}$$

Therefore, $y = \dfrac{1}{2}$ is also a horizontal asymptote of $y = f(x)$.

EXAMPLE 2

Find the horizontal asymptotes of $y = f(x) = \dfrac{x^3 + 3x}{x^2 + 1}$.

Solution: Dividing the numerator and denominator by the fastest growing term, x^3, gives

$$\lim_{x\to\infty} \frac{x^3 + 3x}{x^2 + 1} = \lim_{x\to\infty} \frac{1 + \frac{3}{x^2}}{\frac{1}{x} + \frac{1}{x^3}}$$

This new limit has the determinate form $\dfrac{L}{0^+}$. We find that

$$\lim_{x\to\infty} \frac{x^3 + 3x}{x^2 + 1} = \infty$$

Similarly, we get

$$\lim_{x\to-\infty} \frac{x^3 + 3x}{x^2 + 1} = \lim_{x\to-\infty} \frac{1 + \frac{3}{x^2}}{\frac{1}{x} + \frac{1}{x^3}}$$

This new limit is a determinate form $\dfrac{L}{0^-}$. Thus,

$$\lim_{x\to\infty} \frac{x^3 + 3x}{x^2 + 1} = -\infty$$

Therefore, the curve $y = f(x)$ does not have any horizontal asymptotes.

EXAMPLE 3

Find the horizontal asymptotes of $y = f(x) = \dfrac{3\ln(x) + 2(\ln(x))^2}{3(\ln(x))^2 - 2\ln(x)}$.

Solution: We first note that we do not need to check the limit as $x \to -\infty$ as the domain of f is $(0, \infty)$.

Dividing the numerator and the denominator by the fastest growing term, $(\ln(x))^2$, gives

$$\begin{aligned}\lim_{x\to\infty} \frac{3\ln(x) + 2(\ln(x))^2}{3(\ln(x))^2 - 2\ln(x)} &= \lim_{x\to\infty} \frac{\frac{3}{\ln(x)} + 2}{3 - \frac{2}{\ln(x)}} \\ &= \frac{0+2}{3-0} \\ &= \frac{2}{3}\end{aligned}$$

So, the only horizontal asymptote is $y = \dfrac{2}{3}$.

EXERCISE 1

Find the horizontal asymptotes for each curve.

(a) $y = f(x) = e^{-x}$

(b) $y = f(x) = \arctan(x)$

(c) $y = f(x) = \sin(x)$

(d) $y = f(x) = \dfrac{3x^2 + 3x - 2}{5 - x^2}$

(e) $y = f(x) = \dfrac{x^2 + 5}{x}$

Vertical Asymptotes

DEFINITION

Vertical Asymptotes

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a^+} f(x) = \pm\infty$$
$$\lim_{x\to a^-} f(x) = \pm\infty$$

Essentially, vertical asymptotes occur at infinity discontinuities. So, we look for vertical asymptotes at the function's discontinuities.

Note that when checking for a vertical asymptote, as soon as we have one of the limits above equaling $\pm\infty$, we do not need to check the other limit. However, when graphing functions, we will want to do the other limit anyway to determine its behaviour.

EXAMPLE 4 Show that $x = 0$ is a vertical asymptote of the curve $y = \dfrac{1}{\sqrt{x}}$.

Solution: We have

$$\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$$

Thus, $x = 0$ is a vertical asymptote.

EXAMPLE 5 Determine if the curve $y = f(x) = \dfrac{x^2 - 10}{3 - x}$ has any vertical asymptotes.

Solution: The only possible discontinuity of f is at $x = 3$. We find that

$$\lim_{x\to 3^+} \frac{x^2 - 10}{3 - x}$$

has the determinate form $\dfrac{L}{0^-}$. We get

$$\lim_{x\to 3^+} \frac{x^2 - 10}{3 - x} = \infty$$

Hence, $x = 3$ is a vertical asymptote.

EXAMPLE 6 Determine if the curve $y = f(x) = \begin{cases} x & \text{if } x \le 2 \\ \dfrac{1}{2 - x} & \text{if } x > 2 \end{cases}$ has a vertical asymptote.

Solution: The only possible discontinuity of f is at $x = 2$. We find that

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x = 2$$

and that

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1}{2 - x}$$

has the determinate form $\dfrac{L}{0^-}$. We get

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1}{2 - x} = -\infty$$

Thus, $x = 2$ is a vertical asymptote.

EXAMPLE 7 Find the vertical and horizontal asymptotes of $y = f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$.

Solution: The only possible discontinuity is at $x = \dfrac{5}{3}$. The limit

$$\lim_{x\to\frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5}$$

has the determinate form $\dfrac{L}{0^+}$. We get

$$\lim_{x\to\frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$$

Hence, $x = \dfrac{5}{3}$ is a vertical asymptote.

To find horizontal asymptotes, we will look at the limits as $x \to \infty$ and $x \to -\infty$.

First, we see that

$$\frac{\sqrt{2x^2+1}}{3x-5} = \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{3x-5} = \frac{|x|\sqrt{2+\frac{1}{x^2}}}{3x-5}$$

Hence,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to\infty} \frac{x\sqrt{2+\frac{1}{x^2}}}{3x-5} \qquad \text{since } x > 0 \\
&= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to-\infty} \frac{-x\sqrt{2+\frac{1}{x^2}}}{3x-5} \qquad \text{since } x < 0 \\
&= \lim_{x\to-\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} \\
&= -\frac{2}{3}
\end{aligned}$$

Thus, the horizontal asymptotes are $y = \dfrac{2}{3}$ and $y = -\dfrac{2}{3}$.

EXERCISE 2

Find the vertical asymptotes for each curve.

(a) $y = f(x) = \ln(x)$

(b) $y = f(x) = \dfrac{1}{x^2(x-3)}$

(c) $y = f(x) = \dfrac{x-2}{x^2-4}$

3.3.2 Intervals of Increase/Decrease

DEFINITION

Increasing

Decreasing

A function f is said to be increasing on an interval if $f(a) < f(b)$ for all a, b in the interval where $a < b$.

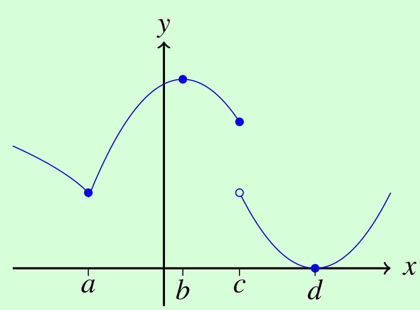
A function f is said to be decreasing on an interval if $f(a) > f(b)$ for all a, b in the interval where $a < b$.

One important thing to notice in the definition of increasing and decreasing is that they are defined over an interval. That is, it does *not* make sense to talk about whether a function is increasing or decreasing at a point.

One of the main purposes of the next example and the next exercise is to help you understand how this definition of increasing or decreasing affects whether or not we should include the end points of an interval in the interval of increase or decrease.

EXAMPLE 8

Consider the function f whose graph is depicted. Find the intervals of increase/decrease of f.



Solution: Observe that if we take any point two points $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, then we will have $f(x_2) > f(x_1)$. Thus, f is increasing on the closed interval $[a, b]$.

On the interval $[a, c]$, the function is neither increasing nor decreasing. f isn't decreasing on this interval since $a < b$ but $f(a) < f(b)$. f isn't increasing on this interval since $b < c$ but $f(b) > f(c)$.

Even though there is a jump discontinuity at c, if we take any two points $x_3, x_4 \in [b, d]$ with $x_3 < x_4$, then we will have $f(x_4) < f(x_3)$, so f is decreasing on $[b, d]$.

Using the same type of argument, we can also show that f is decreasing on $(-\infty, a]$ and increasing on $[d, \infty)$.

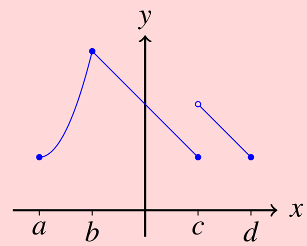
EXERCISE 3

Consider the function f whose graph is depicted. Determine whether f is increasing, decreasing, or neither on the given intervals.

(a) The interval $[a, b]$.

(b) The interval $[b, c]$.

(c) The interval $[c, d]$.



When we do not have the graph of the function, we can use the derivative to determine the intervals of increase and the intervals of decrease of the function. In particular, recall that the derivative tells us the slope of the tangent line to the curve. For the parts of the graph where this slope is positive, the curve is moving up and to the right (increasing). When the slope is negative, the curve is moving down and to the right (decreasing). This gives us the following test.

THEOREM 1

Increasing/Decreasing Test

1. If $f'(x) > 0$ for all x in an interval, then f is increasing on the interval.
2. If $f'(x) < 0$ for all x in an interval, then f is decreasing on the interval.

To find intervals of increase/decrease, we need to know what intervals to look at. The statement of the Increasing/Decreasing Test implies that the only places where a continuous function f can switch between an interval of increase and an interval of decreasing is where $f'(x) = 0$ or $f'(x)$ does not exist. This gives us a method for finding intervals of increase/decrease which is demonstrated in the next few examples.

EXAMPLE 9

Find the intervals of increase/decrease of $f(x) = 3x^4 + 8x^3 + 6x^2$.

Solution: First we find all x such that $f'(x) = 0$ or $f'(x)$ does not exist. We have

$$f'(x) = 12x^3 + 24x^2 + 12x = 12x(x^2 + 2x + 1) = 12x(x + 1)^2$$

Thus, $f'(x) = 0$ when $x = -1, 0$. There are no values of x where $f'(x)$ does not exist.

Next, we divide the domain of f into subintervals using these values. Finally, we determine whether the derivative is positive or negative on each of these intervals.

	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
$12x$	-	-	+
$(x + 1)^2$	+	+	+
$f'(x)$	-	-	+
graph	\	\	/

So, f is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$ (we can combine the intervals $(-\infty, -1]$ and $[-1, 0]$ since f is continuous at $x = -1$).

EXAMPLE 10 Find the intervals of increase/decrease of $f(x) = x^4 - 4x^3 - 12x^2 + 32x - 7$.

Solution: We have

$$f'(x) = 4x^3 - 12x^2 - 24x + 32 = 4(x^3 - 3x^2 - 6x + 8) = 4(x-1)(x-4)(x+2)$$

Therefore, $f'(x) = 0$ when $x = -2, 1, 4$. There are no values of x where $f'(x)$ does not exist. Next, we find that

	$(-\infty, -2)$	$(-2, 1)$	$(1, 4)$	$(4, \infty)$
$4(x-1)$	-	-	+	+
$(x-4)$	-	-	-	+
$(x+2)$	-	+	+	+
$f'(x)$	-	+	-	+
graph	\	/	\	/

So, f is decreasing on $(-\infty, -2]$ and $[1, 4]$, and is increasing on $[-2, -1]$ and $[4, \infty)$.

EXAMPLE 11 Find the intervals of increase/decrease of $f(x) = \dfrac{x}{\ln(x)}$.

Solution: We first observe that the domain of f is $x > 0$, $x \neq 1$. Next, we find that

$$f'(x) = \frac{1 \cdot \ln(x) - x \cdot \frac{1}{x}}{(\ln(x))^2} = \frac{\ln(x) - 1}{(\ln(x))^2}$$

Therefore, $f'(x)$ exists for all x in the domain of f, and $f'(x) = 0$ when

$$\begin{aligned} \ln(x) - 1 &= 0 \\ \ln(x) &= 1 \\ x &= e \end{aligned}$$

We divide the domain of f into subintervals using these values. We get

	$(0, 1)$	$(1, e)$	(e, ∞)
$\ln(x) - 1$	-	-	+
$(\ln(x))^2$	+	+	+
$f'(x)$	-	-	+
graph	\	\	/

So, f is decreasing on $(0, 1)$ and $(1, e]$, and is increasing on $[e, \infty)$.

NOTE: We cannot combine $(0, 1)$ and $(1, e]$ because $x = 1$ is not in the domain of f.

EXAMPLE 12

Find the intervals of increase/decrease of $f(x) = \cos^2(x)$.

Solution: We have $f'(x) = 2\cos(x)(-\sin(x))$. There are no values of x where $f'(x)$ does not exist.

There are infinitely places where $\sin(x)$ and $\cos(x)$ are 0, so a table is not possible. Instead we use a trigonometric identity to get

$$f'(x) = -2\cos(x)\sin(x) = -\sin(2x)$$

We know that $\sin(2x)$ is positive on $(0, \pi/2)$, $(\pi, 3\pi/2)$, ..., and negative on $(\pi/2, \pi)$, $(3\pi/2, 2\pi)$, ...

Thus, $f'(x) < 0$ when $\left(k\pi, k\pi + \frac{\pi}{2}\right)$ for all integers k. So, f is decreasing on $\left[k\pi, k\pi + \frac{\pi}{2}\right]$ for all integers k

Since $f'(x) > 0$ when $\left(k\pi - \frac{\pi}{2}, k\pi\right)$ for all integers k, f is increasing on $\left[k\pi - \frac{\pi}{2}, k\pi\right]$ for all integers k.

EXERCISE 4

Find the intervals of increase/decrease of $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$.

EXERCISE 5

Find the intervals of increase/decrease of $f(x) = x^{1/3}(x+1)$.

EXERCISE 6

Suppose y is related to x by the implicit equation

$$\ln(y) = 1 - \frac{y}{x}$$

where $x > 0$ and $y > 0$. Determine the intervals of increase/decrease for y as a function of x.

These methods can also help us analyze the behaviour of solutions to first order differential equations without needing to solve the differential equation.

EXAMPLE 13

Assuming a constant gravitational force $g = 9.8$ and air resistance proportional to the velocity v m/s with proportionality constant $k = \frac{1}{3}$ kg/s, then an object of mass $m = 1$ kg is dropped from rest will satisfy the differential equations

$$\frac{dv}{dt} = g - \frac{1}{3}v \text{m/s}^2$$

where k is the constant of proportionality. For what values of v will the object be speeding up?

Solution: The velocity will be increasing when its derivative $\frac{dv}{dt}$ is positive. So, when

$$\begin{aligned} \frac{dv}{dt} &> 0 \\ g - \frac{1}{3}v &> 0 \\ -\frac{1}{3}v &> -g \\ v &< 3g = 29.4 \text{ m/s} \end{aligned}$$

Thus, the velocity will increase whenever the velocity is less than 29.4 m/s.

EXERCISE 7

Suppose a quantity y satisfies the differential equation

$$\frac{dy}{dt} = y(10 - y)$$

For what values of y is the quantity y decreasing?

Critical Numbers

Since the numbers where a function's derivative equals 0 or does not exist seem to be important for analyzing the behaviour of a function, we give them a name.

DEFINITION
Critical Number

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

EXAMPLE 14

Find all the critical numbers of $f(x) = x^3 - 6x^2 - 36x + 2$.

Solution: We have

$$\begin{aligned} f'(x) &= 3x^2 - 12x - 36 \\ &= 3(x^2 - 4x - 12) \\ &= 3(x - 6)(x + 2) \end{aligned}$$

Thus, $f'(x) = 0$ when $x = 6$ and $x = -2$. Since both of these are in the domain of f, they are both critical numbers of f.

EXAMPLE 15

Find the critical numbers of $f(x) = \dfrac{\ln(x)}{x^2}$.

Solution: We have

$$\begin{aligned} f'(x) &= \frac{(\frac{1}{x})x^2 - (2x)\ln(x)}{x^4} \\ &= \frac{1 - 2\ln(x)}{x^3} \end{aligned}$$

Thus, $f'(x) = 0$ when

$$\begin{aligned} 1 - 2\ln(x) &= 0 \\ \ln(x) &= \frac{1}{2} \\ x &= e^{1/2} \end{aligned}$$

So, $x = e^{1/2}$ is a critical number.

We also see that f' does not exist when $x = 0$. But $x = 0$ is not in the domain of f, so it is not a critical number.

EXERCISE 8

Find the critical numbers of each function.

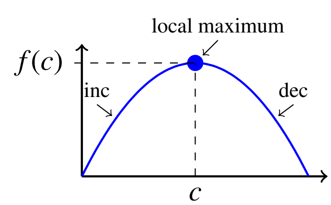
(a) $f(x) = \dfrac{1}{3}x^3 - \dfrac{5}{2}x^2 + 4x$

(b) $f(x) = \dfrac{2x}{1 + x^2}$

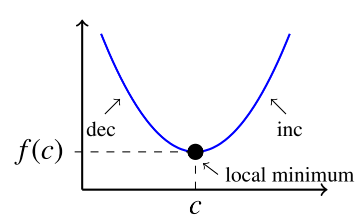
(c) $f(x) = x^{2/3}(x - 2)$

3.3.3 Local Maxima and Local Minima

If $x = c$ is a critical number of a continuous function f such that f is increasing for values of x just less than c and is decreasing for values of x just greater than c, then $f(c)$ will be the largest y-value f has around $x = c$.



Similarly, if f is decreasing up to $x = c$ and then switches to increasing at $x = c$, then $f(c)$ will be the smallest y-value f has around $x = c$.



Such points are not only useful for understanding the behaviour of a function, but as we will see in Chapter 4, they play an important role is solving optimization problems. We make the following definitions.

DEFINITION

Local Maximum

Local Minimum

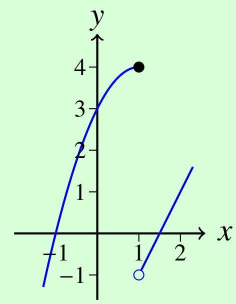
A function f has a **local maximum**, $f(M)$, at $x = M$ if $f(M) \geq f(x)$ for all x very close to M. f has a **local minimum**, $f(m)$, at $x = m$ if $f(m) \leq f(x)$ for all x very close to m.

The reason that we did not use increasing/decreasing in our definition of a local maximum or a local minimum is that we can have a local maximum at $x = c$ even if f is increasing on both sides of c. This is demonstrated in the next example.

EXAMPLE 16

For example, consider the function

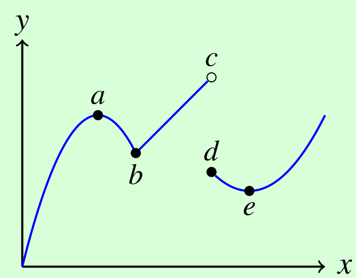
$$f(x) = \begin{cases} 4 - (x-1)^2 & \text{if } x \leq 0 \\ 2x - 3 & \text{if } x > 0 \end{cases}$$



Observe from its graph that $f(0) = 4$ is a local maximum even though f is increasing on $(-\infty, 0]$ and on $(0, \infty)$.

EXAMPLE 17

Consider a function f defined on a closed interval with the following graph. Determine whether a local maximum, a local minimum, or neither occurs at the points a, b, c, d, and e.



Solution: The y-values of all points immediately to the right or left of the point a are smaller than the y-value at a. Thus, a local maximum occurs at a.

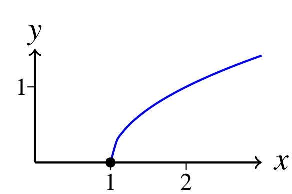
The y-values of all points immediately to the right or left of the point b are larger than the y-value at b. Thus, a local minimum occurs at b. For the same reason, we get that a local minimum occurs at e.

There is no local maximum at c since we never actually reach a largest value because the point is not included in the graph.

There is no local maximum at d since if we look at x-values immediately to the left of d, they correspond to y-values that are larger than the y-value at d. There is also no local minimum at d since the y-values on the graph immediately to the right of d that are lower than the y-value at d.

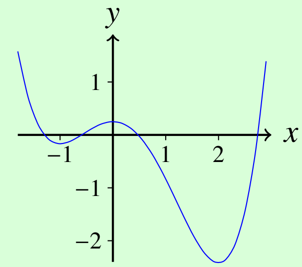
In the example above that we did not label the end points of the graph. In this text, an end point of the graph of a function will not be labelled as a local maximum or a local minimum.

For example, we will *not* call 0 a local minimum of the function $y = \sqrt{x-1}$ even though the y-value of 0 is the lowest point on the graph.



EXAMPLE 18

The graph of a function f is shown. Where are the local maximums and local minimums of f?



Solution: The y-values around $x = 0$ are lower than $f(0)$, thus there is a local maximum at $x = 0$.

The y-values around $x = -1$ are greater than $f(-1)$, thus there is a local minimum at $x = -1$.

Similarly, the y-values around $x = 2$ are greater than $f(2)$, thus there is a local minimum at $x = 2$.

In Examples 3.3.17 and 3.3.18 we see that the local maximums and local minimums occur at critical numbers. Indeed, it can be proven that this is always the case.

THEOREM 2

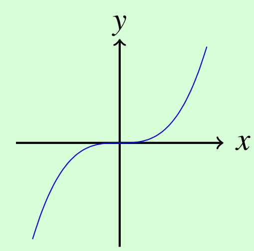
Fermat's Theorem

If f has a local maximum/minimum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist.

The converse of Fermat's Theorem is not true. That is, just because $f'(c) = 0$ or $f'(c)$ does not exist, does not mean that c is a local maximum or a local minimum.

EXAMPLE 19

Let $f(x) = x^3$. Show that $x = 0$ is a critical number of f, but is neither a local maximum nor a local minimum.



Solution: Observe that $f'(x) = 3x^2$, so $f'(0) = 0$. Thus, $x = 0$ is a critical number of f.

Observe that $x = 0$ is not a local minimum because for any $x < 0$ we have $f(x) < f(0)$. Similarly, $x = 0$ is not a local maximum since for any $x > 0$ we have $f(x) > f(0)$.

EXERCISE 9

Let $f(x) = x^{1/3}$. Show that $x = 0$ is a critical number of f, but is neither a local maximum nor a local minimum.

Since a critical number may not be a local maximum or a local minimum, we need a method for classifying critical numbers. To do this, we look at whether the function is switching between increasing and decreasing. To prevent the special case that we saw in Example 3.3.16, we require the additional condition that f is continuous at the critical number.

THEOREM 3

First Derivative Test

Suppose c is a critical number of a function f and f is continuous at c.

1. If f' changes from positive to negative at c, then $f(c)$ is a local maximum.
2. If f' changes from negative to positive at c, then $f(c)$ is a local minimum.

EXAMPLE 20 Find the local maxima and local minima of $f(x) = x^3 - 3x^2 - 9x + 2$.

Solution: We begin by finding the critical numbers of f. We have

$$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$$

Thus, the critical numbers are $x = -1, 3$. Next, we find the intervals of increase/decrease.

	$(-\infty, -1)$	$(-1, 3)$	$(3, \infty)$
$3(x-3)$	-	-	+
$x+1$	-	+	+
$f'(x)$	+	-	+
graph	/	\	/

Since f is continuous at $x = -1$ and f' changes positive to negative at $x = -1$, we get that $f(-1) = 4$ is a local maximum.

Since f is continuous at $x = 1$ and f' changes negative to positive at $x = 3$, we get that $f(3) = -28$ is a local minimum.

EXAMPLE 21 Find the local maxima and local minima of $f(x) = 5x^{2/3} - 2x^{5/3}$.

Solution: We begin by finding the critical numbers of f. We have

$$f'(x) = \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3} = \frac{10}{3}x^{-1/3}(1-x)$$

Thus, the critical numbers are $x = 0, 1$. Next, we find the intervals of increase/decrease.

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$-\frac{10}{3}$	-	-	-
$x^{-1/3}$	-	+	-
$1-x$	+	+	-
$f'(x)$	-	+	-
graph	\	/	\

Since f is continuous, we get from the table that $f(0) = 0$ is a local minimum and $f(1) = 3$ is a local maximum.

EXAMPLE 22 Find the local maxima and local minima of $f(x) = x^2 \ln(|x|)$.

Solution: First, we note the domain of f is all $x \neq 0$. We find that

$$f'(x) = 2x\ln(|x|) + x^2 \cdot \frac{1}{|x|} \cdot \frac{x}{|x|} = 2x\ln(|x|) + x \cdot \frac{x^2}{|x|^2} = x(2\ln(|x|) + 1)$$

Since $x = 0$ is not in the domain of f, we ignore it. $f'(x) = 0$ when

$$\begin{aligned} 2\ln(|x|) + 1 &= 0 \\ \ln(|x|) &= -\frac{1}{2} \\ |x| &= e^{-1/2} \\ x &= \pm e^{-1/2} \end{aligned}$$

So, the critical numbers of f are $x = \pm e^{-1/2}$. Next, we find that

	$(-\infty, -e^{-1/2})$	$(-e^{-1/2}, 0)$	$(0, e^{-1/2})$	$(e^{-1/2}, \infty)$
x	-	-	+	+
$2\ln(\lvert x\rvert) + 1$	+	-	-	+
$f'(x)$	-	+	-	+
graph	\	/	\	/

Since f is continuous, the table shows us that $f(-e^{-1/2})$ and $f(e^{-1/2})$ are both local minimums.

EXAMPLE 23 The sine integral function, $\mathrm{Si}(x) = \displaystyle\int_0^x \frac{\sin(t)}{t}dt$, occurs regularly in science. Given that $\mathrm{Si}(x)$ is continuous on $(-2\pi, \pi)$, find its local maxima and local minima on $(-2\pi, 2\pi)$.

Solution: We get $\mathrm{Si}'(x) = \dfrac{\sin(x)}{x}$. Thus, the critical numbers of Si are $x = 0$ (where S' does not exist) and $x = \pi, -\pi$ (where $\mathrm{Si}'(x) = 0$). Next, we find that

	$(-2\pi, -\pi)$	$(-\pi, 0)$	$(0, \pi)$	$(\pi, 2\pi)$
x	-	-	+	+
$\sin(x)$	+	-	+	-
$S'(x)$	-	+	+	-
graph	\	/	/	\

Since S is continuous, the table shows us that $f(-\pi)$ is a local minimum and $f(\pi)$ is a local maximum.

EXERCISE 10

Find the local maxima and local minima of each function.

(a) $f(x) = (x^2 - 1)^3$

(b) $f(x) = \dfrac{x^2}{x-1}$

(c) $f(x) = \displaystyle\int_0^x e^{t^2}\,dt$

3.3.4 Intervals of Concavity

We are also often interested in finding intervals where the rate of change is either increasing or decreasing.

DEFINITION
Concave Up

If f' is increasing on an interval I, then the graph of f is said to be **concave up** on I.

If f' is positive and increasing, then it means that the slope of f is getting steeper. In this case, the graph of f will have the shape:

If f' is negative and increasing, then it means that the slope of f is getting less steep. In this case, the graph of f will have the shape:

DEFINITION
Concave Down

If f' is decreasing on an interval I, then the graph of f is said to be **concave down** on I.

If f' is positive and decreasing, then it means that the slope of f is getting less steep. In this case, the graph of f will have the shape:

If f' is negative and decreasing, then it means that the slope of f is getting steeper. In this case, the graph of f will have the shape:

EXAMPLE 24

Show that the graph of $f(x) = x^2$ is concave up on $(-\infty, \infty)$.

Solution: We get that the derivative of $f'(x) = 2x$ is $f''(x) = 2$. Since $f''(x) > 0$ for all x, f' is always increasing. Thus, the graph of f is concave up on $(-\infty, \infty)$.

EXERCISE 11

Show that the graph of $f(x) = \sqrt{x}$ is concave down on $(0, \infty)$.

From the Increasing/Decreasing Test, we get the following test for concavity.

THEOREM 4

Concavity Test

1. If $f''(x) > 0$ for all x in an interval, then the graph of f is concave up on the interval.
2. If $f''(x) < 0$ for all x in an interval, then the graph of f is concave down on the interval.

We saw that when trying to find intervals of increase/decrease we needed to first subdivide the domain of f into subintervals using values where $f'(x)$ was zero or did not exist (the critical numbers). For determining intervals of concavity, we do the same except using values where $f''(x)$ is zero or does not exist.

EXAMPLE 25

Determine the intervals of concavity for the curve $y = f(x) = xe^{-x}$.

Solution: We first determine the second derivative of f.

$$\begin{aligned} f'(x) &= e^{-x} - xe^{-x} \\ f''(x) &= -e^{-x} - e^{-x} + xe^{-x} \\ &= e^{-x}(x-2) \end{aligned}$$

Thus, $f''(x) = 0$ when $x = 2$.

In this case, since $e^{-x} > 0$ for all x, we see that $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$. Thus, the graph of f is concave up on $[2, \infty)$ and is concave down on $(-\infty, 2]$.

EXAMPLE 26

Find the intervals of concavity of $y = f(x) = x^4 - 4x^3$.

Solution: We have

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 \\ f''(x) &= 12x^2 - 24x \\ &= 12x(x-2) \end{aligned}$$

Thus, $f''(x) = 0$ when $x = 0$ and $x = 2$. Next, we get the table on the right.

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$12x$	-	+	+
$x - 2$	-	-	+
$f''(x)$	+	-	+
graph	$\cup$	$\cap$	$\cup$

So, the graph of f is concave up on $(-\infty, 0]$ and $[2, \infty)$, and is concave down on $[0, 2]$.

EXAMPLE 27

Find the intervals of concavity of $y = f(x) = x^{1/3}(2 - x)^{2/3}$.

Solution: We have

$$f'(x) = \frac{-3x + 2}{3x^{2/3}(2-x)^{1/3}}$$

$$f''(x) = \frac{-8}{9x^{5/3}(2-x)^{4/3}}$$

We see that $f''(x)$ does not exist when $x = 0$ or $x = 2$. Next, we get the table on the right.

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
-8	-	-	-
$9x^{5/3}$	-	+	+
$(2-x)^{4/3}$	+	+	+
$f''(x)$	+	-	-
graph	$\cup$	$\cap$	$\cap$

Thus, graph of f is concave up on $(-\infty, 0)$, and is concave down on $(0, 2)$ and $(2, \infty)$.

EXAMPLE 28 The function

$$Y(N) = \frac{50N}{5+N}, \qquad N \geq 0$$

can be used to model the yield of a crop, Y, as a function of the amount of nitrogen in the soil. Show that Y is an increasing function of N and that its graph is concave down. Interpret these results.

Solution: We find that

$$Y'(N) = \frac{50(5+N) - 1(50N)}{(5+N)^2} = \frac{250}{(5+N)^2} > 0$$

for $N \geq 0$. So, Y is always increasing. Hence, more nitrogen in the soil will increase the crop yield.

Differentiating again gives

$$Y''(N) = \frac{-500}{(5+N)^3} < 0$$

for all $N \geq 0$. Thus, the graph of Y is always concave down.

This means that although increasing nitrogen levels will increase the crop yield, it will increase it by smaller and smaller amounts (the rate of increase is decreasing). This behaviour is known as *the law of diminishing returns*. In other words, the more nitrogen already in the soil, the lesser an effect additional nitrogen will have on the yield.

EXAMPLE 29 Consider the initial value problem

$$y' = x^2 + y^2 + 1, \qquad y(1) = -3$$

(a) Show that the solution $y(x)$ is an increasing function.

(b) Determine whether the graph of the solution is concave up or concave down around $x = 1$.

Solution: (a) We have that $y' = x^2 + y^2 + 1 > 0$, so the solution y is increasing.

(b) To find the second derivative of the solution $y(x)$, we take the derivative of y' implicitly with respect to x. We get

$$y'' = 2x + 2yy' + 0$$

Since $y(1) = -3$, we have $x = 1$, $y = -3$, and

$$y'(1) = (1)^2 + (-3)^2 + 1 = 11$$

Therefore,

$$y''(1) = 2(1) + 2(-3)(11) = -64$$

Therefore, around $x = 1$, the graph of the solution $y(x)$ is concave down.

EXERCISE 12

Find the intervals of concavity for each curve.

(a) $y = f(x) = \dfrac{1}{1 + e^x}$

(b) $y = g(x) = 3x^4 - 4x^3 - 6x^2 + 12x$

(c) $y = h(x) = x^{1/3}(x + 1)$

(d) $y = f(x) = \displaystyle\int_0^x \sqrt{1 + t^2}\, dt$

Points of Inflection

We have seen how to use the First Derivative Test to locate local maximums and minimums of a function f. In many situations, it is very helpful to know where the rate of change, f', is locally maximized or minimized.

By the First Derivative Test, f' will have a local maximum when its derivative, $f''(x)$, changes from positive to negative. We have defined the graph of a function to be concave up on an interval if $f''(x)$ is positive (f' is increasing) on that interval, and the graph to be concave down on an interval if $f''(x)$ is negative (f' is decreasing) on that interval. So, we can look for local maximums of f' at points where the graph of f switches from concave up to concave down. Similarly, we look for local minimums of f' at points where the graph of f where switches from concave down to concave up.

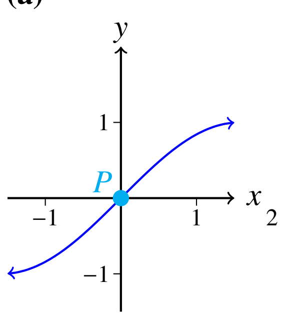
We make the following definition.

DEFINITION
Point of Inflection

A point $P(x, y)$ on a curve $y = f(x)$ is called a **point of inflection** if f is continuous at P and the graph changes from concave up to concave down or concave down to concave up at P.

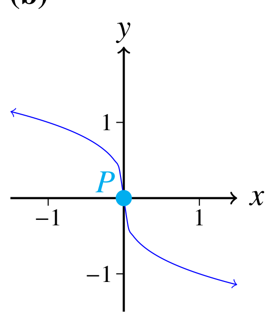
For example, each of the graphs below have a point of inflection at $x = 0$.

(a)



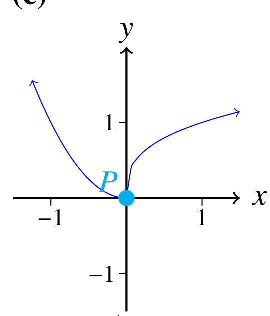
$f(x) = \sin(x)$

(b)



$g(x) = -\sqrt[3]{x}$

(c)



$h(x) = \begin{cases} x^2 & x < 0 \\ \sqrt[3]{x}, & x \geq 0 \end{cases}$

Since a point of inflection occurs where the second derivative changes sign, we look for possible points of inflection where $f''(x) = 0$ or $f''(x)$ does not exist.

REMARK

Since the definition specifies that f must be continuous at a point of inflection, if a point P is not in the domain of f, then it cannot be a point of inflection.

EXAMPLE 30

Show algebraically that the graph of $f(x) = \sin(x)$ has a point of inflection at $x = 0$.

Solution: We have

$$\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \end{aligned}$$

We see that $f''(x) > 0$ for $-1 < x < 0$, so the graph is concave up on $(-1, 0)$.

We also have $f''(x) < 0$ for $0 < x < 1$, so the graph is concave down on $(0, 1)$.

Therefore, since f is continuous at $x = 0$ and the graph switches from concave up to concave down there, there is a point of inflection at $x = 0$. We have $f(0) = 0$, so the point of inflection is $(0, 0)$.

EXAMPLE 31

Show algebraically that the graph of $g(x) = -\sqrt[3]{x}+1$ has a point of inflection at $x = 0$.

Solution: We have

$$\begin{aligned} g'(x) &= -\frac{1}{3}x^{-2/3} \\ g''(x) &= \frac{2}{9}x^{-5/3} = \frac{2}{9x^{5/3}} \end{aligned}$$

We see that $g''(x) < 0$ for $-1 < x < 0$, so the graph is concave down on $(-1, 0)$.

We also have $g''(x) > 0$ for $0 < x < 1$, so the graph is concave up on $(0, 1)$.

Therefore, since g is continuous at $x = 0$ and the graph of f switches from concave down to concave up there, there is a point of inflection at $x = 0$. We have $g(0) = 1$, so the point of inflection is $(0, 1)$.

EXERCISE 13

Show algebraically that $y = h(x) = \begin{cases} x^2 & x < 0 \\ \sqrt[3]{x}, & x \geq 0 \end{cases}$ has a point of inflection at $x = 0$.

EXERCISE 14

Explain why the curve $y = f(x) = \dfrac{1}{x}$ does not have a point of inflection at $x = 0$.

EXAMPLE 32

For the curve $y = f(x) = 5x^{2/3} - 2x^{5/3}$, find the intervals of concavity and the points of inflection.

Solution: Taking derivatives, we have

$$f'(x) = \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3}$$
$$f''(x) = \frac{-10}{9}x^{-4/3} - \frac{20}{9}x^{-1/3} = \frac{-10(1+2x)}{9x^{4/3}}$$

Observe that $f''(x)$ does not exist at $x = 0$ and is equal to 0 when $x = -\dfrac{1}{2}$.

	$\left(-\infty, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, 0\right)$	$(0, \infty)$
$-\frac{10}{9}x^{4/3}$	-	-	-
$1 + 2x$	-	+	+
$f''(x)$	+	-	-
graph	$\cup$	$\cap$	$\cap$

So, the curve is concave up on $\left(-\infty, -\dfrac{1}{2}\right]$, and is concave down on $\left[-\dfrac{1}{2}, 0\right)$ and $(0, \infty)$.

Since f is continuous at $x = -\dfrac{1}{2}$ and the graph of f changes from concave up to concave down at that point, it is a point of inflection. We find that

$$f\left(-\frac{1}{2}\right) = 6 \cdot 2^{-2/3}$$

Hence, the point of inflection is $\left(-\dfrac{1}{2}, 6 \cdot 2^{-2/3}\right)$.

EXERCISE 15

Find the intervals of concavity and points of inflection for each curve.

(a) $y = f(x) = x^4 - 2x^3$

(b) $y = g(x) = e^{1/x}$

Section 3.3 Problems

1. Find the horizontal and vertical asymptotes of the graph of each function.
   (a) $f(x) = \dfrac{x^2 - 3x + 2}{x - 1}$
   (b) $f(x) = \dfrac{x^2 + 5x + 6}{x^2}$
   (c) $f(x) = \dfrac{x^2 - 3}{x^3}$
   (d) $f(x) = \dfrac{x|x|}{1 - x^2}$
   (e) $f(x) = \dfrac{1}{\sqrt{x}\ln(x)}$

2. Find the intervals of increase/decrease of each function.
   (a) $f(x) = \dfrac{1}{3}x^3 - \dfrac{5}{2}x^2 + 4x$
   (b) $f(x) = x^4 - 4x^3 + 4x^2$
   (c) $f(x) = x\sqrt{x + 3}$
   (d) $f(x) = 3x^{2/3} - x$
   (e) $f(x) = x\ln(x^2) + 1$
   (f) $f(x) = \dfrac{e^x}{e^{2x} + 1}$
   (g) $f(x) = x + \cos(x)$ on $[0, 2\pi]$
   (h) $f(x) = -2\cos(x) - x$ on $[0, 2\pi]$
   (i) $f(x) = \sin^2(x)$ on $[-\pi, \pi]$
   (j) $f(x) = x^{2/3}(x^2 - 4)$
   (k) $f(x) = x^2\sqrt{9 - x^2}$

3. Find the local maxima and local minima of each function.
   (a) $f(x) = \dfrac{1}{3}x^3 - \dfrac{1}{2}x^2 - 2x$
   (b) $f(x) = x^2e^x$
   (c) $f(x) = 2\cos(2x)$ on $[0, \pi]$
   (d) $f(x) = x\sqrt{4 - x^2}$

4. Find the intervals of concavity and points of inflection of each curve.
   (a) $y = f(x) = x^3 + 9x^2 + 7x + 4$
   (b) $y = f(x) = 3x^{2/3} - x$
   (c) $y = f(x) = x^{1/3}(x + 4)$
   (d) $y = f(x) = x\sqrt{x + 3}$
   (e) $y = f(x) = \dfrac{x^2}{x^2 - 1}$
   (f) $y = f(x) = \sin^2(x)$ on $[0, \pi]$

5. The diversity of a population that contains two species can be measured according to the Gini-Simpson diversity index. If the fraction of animals of the first species is p, then the diversity is given by $D(p) = 2p(1 - p)$. Find $D'(p)$ and the value of p for which $D'(p) = 0$.

6. The reverberation time of a room is the time it takes for the intensity level of a sound to fall 60 decibels after the source of the sound has been turned off. The formula $R(V) = \dfrac{0.05V}{A + cV}$ gives the reverberation time of a room where V is the room volume, A is the total room absorption, and c is the air absorption coefficient. If A and c are positive constants, show that the rate of change of R with respect to V is always positive.

7. The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$, where t is measured in seconds and s in meters.
   (a) Find the velocity after 4 s.
   (b) When is the particle at rest?
   (c) When is the particle moving forward?
   (d) When is the particle speeding up?

8. For what values of the numbers a and b does the function $f(x) = axe^{bx^2}$ have a local maximum value $f(8) = 1$?

Section 3.4: Curve Sketching

LEARNING OUTCOMES

1. Know how to use limits and derivatives to sketch the graphs of functions.

ALGORITHM

To sketch the graph of a function f, we:

A. **Domain:** Determine the set of values x for which $f(x)$ is defined.

B. **Intercepts:** Find the x and y intercepts.

C. **Asymptotes:** Find the vertical and horizontal asymptotes.

D. **Intervals of Increase/Decrease:** Find the intervals of increase/decrease along with local maximums and local minimums.

E. **Concavity:** Find the intervals of concavity along with any points of inflection.

F. **Sketch the Curve:** Using the information found in steps A-E, sketch the curve. Label all local maximums, local minimums, and points of inflection.

EXAMPLE 1 Sketch the graph of $f(x) = 5x^{2/3} - 2x^{5/3}$. NOTE: $f\left(-\frac{1}{2}\right) \approx 3.8$.

Solution:

A. The domain of f is $(-\infty, \infty)$.

B. We set $x = 0$ to find the y-intercept is at $(0, 0)$. Solving $0 = f(x) = x^{2/3}(5 - 2x)$ gives $x = 0$ or $x = \frac{5}{2}$. So, the x-intercepts are $(0, 0)$ and $\left(\frac{5}{2}, 0\right)$.

C. There are no vertical asymptotes since the function is continuous for all x.

   We have

$$\lim_{x\to\infty} 5x^{2/3} - 2x^{5/3} = \lim_{x\to\infty} x^{2/3}(5 - 2x) = -\infty$$
$$\lim_{x\to-\infty} 5x^{2/3} - 2x^{5/3} = \lim_{x\to-\infty} x^{2/3}(5 - 2x) = \infty$$

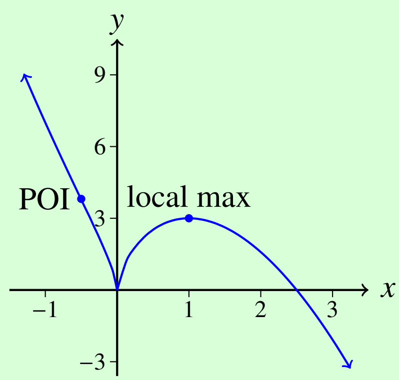
   Hence, there are no horizontal asymptotes.

D. In Example 3.3.21 we found that f is decreasing on $(-\infty, 0]$ and $[1, \infty)$, and increasing on $[0, 1]$, and that there is a local minimum at $(0, 0)$ and a local maximum at $(1, 3)$.

E. In Example 3.3.32 we found that the graph is concave up on $\left(-\infty, -\frac{1}{2}\right]$, and concave down on $\left[-\frac{1}{2}, 0\right)$ and $(0, \infty)$.

There is a point of inflection at $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right)$.

F. Finally, we sketch the curve.



EXAMPLE 2 Sketch the graph of $g(x) = \dfrac{x^2 + x + 1}{x^2}$.

Solution:

A. The domain is all $x \neq 0$.

B. There are no y-intercepts since we cannot set $x = 0$. There are no x-intercepts as $0 = g(x) = x^2 + x + 1$ has no real roots (the discriminant is $b^2 - 4ac = 1 - 4(1)(1) < 0$).

C. We look for a vertical asymptote at $x = 0$. Both limits have the form $\dfrac{L}{0^+}$, so

$$\lim_{x\to 0^-} \frac{x^2 + x + 1}{x^2} = \infty$$
$$\lim_{x\to 0^+} \frac{x^2 + x + 1}{x^2} = \infty$$

Thus, there is a vertical asymptote at $x = 0$.

Next, we consider the limits as $x \to \pm\infty$.

$$\lim_{x\to\infty} \frac{x^2 + x + 1}{x^2} = \lim_{x\to\infty} \left(\frac{x^2}{x^2} + \frac{x}{x^2} + \frac{1}{x^2}\right) = \lim_{x\to\infty} \left(1 + \frac{1}{x} + \frac{1}{x^2}\right) = 1$$
$$\lim_{x\to-\infty} \frac{x^2 + x + 1}{x^2} = \lim_{x\to-\infty} \left(\frac{x^2}{x^2} + \frac{x}{x^2} + \frac{1}{x^2}\right) = \lim_{x\to-\infty} \left(1 + \frac{1}{x} + \frac{1}{x^2}\right) = 1$$

Hence, there is a horizontal asymptote at $y = 1$ in both directions.

D. We have

$$g'(x) = \frac{x^2(2x+1) - (x^2+x+1)(2x)}{x^4} = \frac{-x-2}{x^3}$$

Since $x = 0$ is not in the domain of g, it is not a critical number. Therefore, the only critical number is $x = -2$.

	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
$-x-2$	+	-	-
x^3	-	-	+
$g'(x)$	-	+	-
graph	\	/	\

Thus, g is decreasing on $(-\infty, -2]$ and $(0, \infty)$, and increasing on $[-2, 0)$.

There is a local minimum at $(-2, g(-2)) = \left(-2, \frac{3}{4}\right)$.

E. We have

$$g''(x) = \frac{(-1)x^3 - (-x-2)3x^2}{x^6} = \frac{2x+6}{x^4}$$

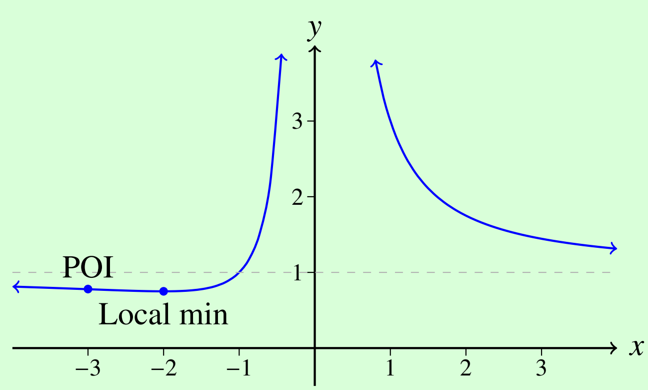
The second derivative is 0 when $x = -3$.

	$(-\infty, -3)$	$(-3, 0)$	$(0, \infty)$
$2x+6$	-	+	+
x^4	+	+	+
$g''(x)$	-	+	+
graph	$\cap$	$\cup$	$\cup$

Hence, the graph is concave down on the interval $(-\infty, -3]$, and is concave up on $[-3, 0)$ and $(0, \infty)$.

There is a point of inflection at $(-3, g(-3)) = \left(-3, \frac{(-3)^2 - 3 + 1}{(-3)^2}\right) = \left(-3, \frac{7}{9}\right)$.

F. Finally, we sketch the curve.



EXAMPLE 3 Sketch the graph of $f(x) = \dfrac{x^3}{x^2+1}$. NOTE: $f'(x) = \dfrac{x^2(x^2+3)}{(x^2+1)^2}$ and $f''(x) = \dfrac{2x(3-x^2)}{(x^2+1)^3}$.

Solution:

A. The domain of f is $(-\infty, \infty)$.

B. We set $x = 0$ to find the y-intercept is at $(0, 0)$. Solving $0 = f(x) = \dfrac{x^3}{x^2+1}$ gives $x = 0$. So, $(0, 0)$ is the only x-intercept.

C. There are no vertical asymptotes since f is continuous for all x.

We have

$$\lim_{x\to\infty} \frac{x^3}{x^2+1} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}+\frac{1}{x^3}} = \infty$$
$$\lim_{x\to-\infty} \frac{x^3}{x^2+1} = \lim_{x\to-\infty} \frac{1}{\frac{1}{x}+\frac{1}{x^3}} = -\infty$$

Thus, there are no horizontal asymptotes.

D. We are given that $f'(x) = \dfrac{x^2(x^2+3)}{(x^2+1)^2}$.

The only critical point is $x = 0$. Since $f'(x) > 0$ for $x \neq 0$, f is increasing for all x, and hence, there are no local maximums or local minimums.

E. We are given that $f''(x) = \dfrac{2x(3-x^2)}{(x^2+1)^3}$.

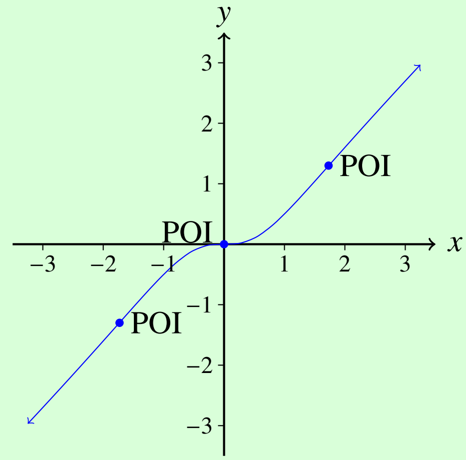
Thus, $f''(x) = 0$ when $x = 0$ and when $x = \pm\sqrt{3}$. We get

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$2x$	-	-	+	+
$3-x^2$	-	+	+	-
$(x^2+1)^3$	+	+	+	+
$f''(x)$	+	-	+	-
graph	$\cup$	$\cap$	$\cup$	$\cap$

Thus, the graph is concave up on $(-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$, and is concave down on $[-\sqrt{3}, 0]$ and $[\sqrt{3}, \infty)$.

We have points of inflection at $\left(-\sqrt{3}, \dfrac{-3\sqrt{3}}{4}\right)$, $(0, 0)$, and $\left(\sqrt{3}, \dfrac{3\sqrt{3}}{4}\right)$.

F. Finally, we sketch the curve.



EXAMPLE 4 Sketch the graph of $f(x) = \int_0^x e^{-t^2}\,dt$ for $x \geq 0$.

NOTE: There are no vertical asymptotes, and the graph has a horizontal asymptote at $y = \dfrac{\sqrt{\pi}}{2} \approx 0.9$ as $x \to \infty$.

Solution:

A. We are given the domain of $[0, \infty)$.

B. Setting $x = 0$ gives $f(0) = \int_0^0 e^{-t^2}\,dt = 0$. So, the y-intercept is at $(0, 0)$.

Since $e^{-t^2} > 0$ for all t, the area under the graph of e^{-t^2} will be positive over interval $[0, x]$ where $x > 0$. Thus, the only value of x such that $f(x) = 0$ is $x = 0$. So, $(0, 0)$ is the only x-intercept.

C. We are given that there are no vertical asymptotes.

We are given that the graph has a horizontal asymptote at $y = \dfrac{\sqrt{\pi}}{2} \approx 0.9$ as $x \to \infty$.

D. By the Fundamental Theorem of Calculus - Part 1, we have $f'(x) = e^{-x^2}$.

Since $f'(x) = e^{-x^2} > 0$ for all $x \geq 0$, f is always increasing.

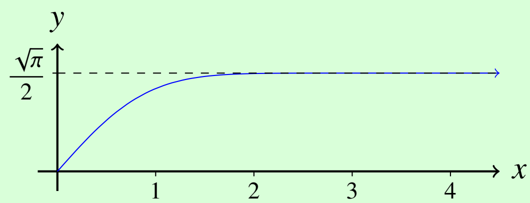
Since f is increasing, there are no local maximums or local minimums.

E. We get $f''(x) = -2xe^{-x^2}$.

Since $f''(x) = -2xe^{-x^2} < 0$ for all $x > 0$, the graph of f is always concave down.

There are no points of inflection.

F. Finally, we sketch the curve.



EXERCISE 1 Sketch the graph of the following functions.

(a) $f(x) = \dfrac{x^2}{x^2 - 4}$ (b) $g(x) = x^{1/3}(4 - x)$. Note: $2^{1/3} \approx 1.3$.

EXERCISE 2 The function $f(x) = e^{-x^2}$ is related to the standard normal-density function, and is very important in the study of probability. Sketch the graph of $f(x)$.

NOTE: $\sqrt{\dfrac{1}{2}} \approx 0.7$ and $e^{-0.5} \approx 0.6$.

EXERCISE 3 The logistic function is given by $N(t) = \dfrac{1}{1 + e^{-t}}$, $t \geq 0$ and represents the growth of a population. Sketch the graph of $N(t)$.

EXERCISE 4 Sketch the graph of $f(x) = \displaystyle\int_0^x \ln\left(\sin^2(t) + 1\right)\, dt$ on $[0, \pi]$. NOTE: $f\left(\dfrac{\pi}{2}\right) \approx 0.6$ and $f(\pi) \approx 1.2$.

Section 3.4 Problems

1. Sketch the graph of the following functions.
   (a) $f(x) = x(x+3)^2$
   (b) $f(x) = \dfrac{x+1}{x-2}$
   (c) $f(x) = x\sqrt{2 - x^2}$
   (d) $f(x) = \dfrac{x^2}{x^2 + 3}$
   (e) $f(x) = \dfrac{x}{x^2 + 4}$
   (f) $f(x) = \sqrt[3]{x^2 - 1}$

2. Sketch the graph of the following functions.
   (a) $f(x) = xe^{-1/x}$
   (b) $f(x) = \arctan(x^2)$. Note: $\left(\dfrac{1}{3}\right)^{1/4} \approx 0.8$.
   (c) $f(x) = 2x - 2\sqrt{x}$
   (d) $f(x) = x\ln(x)$
   (e) $f(x) = \dfrac{1}{(1 + e^x)^2}$. Note: $\ln\left(\dfrac{1}{2}\right) \approx -0.7$.
   (f) $f(x) = \sqrt{x^2 + 1} - x$
   (g) $f(x) = e^{-1/(x+1)}$

End of Chapter Problems

1. Use the graph of the function f to evaluate the limit.

   (a) $\lim_{x\to 1^-} f(x)$

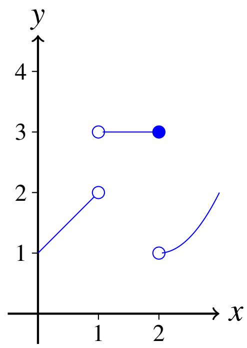
   (b) $\lim_{x\to 1^+} f(x)$

   (c) $\lim_{x\to 1} f(x)$

   (d) $\lim_{x\to 2^-} f(x)$

   (e) $\lim_{x\to 2^+} f(x)$

   (f) $\lim_{x\to 2} f(x)$



2. Use the graph of the function f to evaluate the limit.

   (a) $\lim_{x\to 1^-} f(x)$

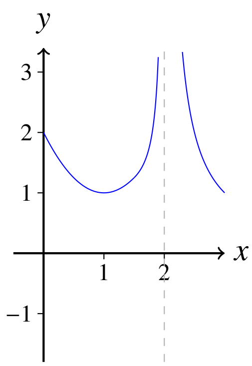
   (b) $\lim_{x\to 1^+} f(x)$

   (c) $\lim_{x\to 1} f(x)$

   (d) $\lim_{x\to 2^-} f(x)$

   (e) $\lim_{x\to 2^+} f(x)$

   (f) $\lim_{x\to 2} f(x)$



3. Use the graph of the function f to evaluate the limit.

   (a) $\lim_{x\to 1^-} f(x)$

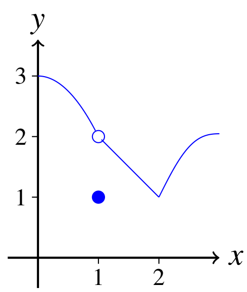
   (b) $\lim_{x\to 1^+} f(x)$

   (c) $\lim_{x\to 1} f(x)$

   (d) $\lim_{x\to 2^-} f(x)$

   (e) $\lim_{x\to 2^+} f(x)$

   (f) $\lim_{x\to 2} f(x)$



4. Use the graph of the function f to evaluate the limit.

   (a) $\lim_{x\to 1^-} f(x)$

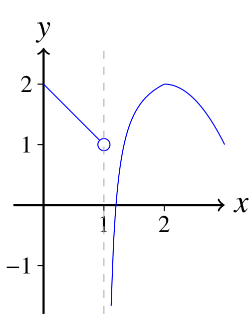
   (b) $\lim_{x\to 1^+} f(x)$

   (c) $\lim_{x\to 1} f(x)$

   (d) $\lim_{x\to 2^-} f(x)$

   (e) $\lim_{x\to 2^+} f(x)$

   (f) $\lim_{x\to 2} f(x)$



5. Is the form determinate or indeterminate? If it is determinate, state its value.

   (a) $\dfrac{\infty}{\infty}$

   (b) $\dfrac{L}{\infty}$

   (c) $\dfrac{0}{0}$

   (d) $\infty \cdot 0$

   (e) ∞^0

   (f) 1^0

   (g) 1^∞

   (h) 0^0

6. Evaluate the following limits.

   (a) $\lim_{x\to 2}(2x^2 - 1)$

   (b) $\lim_{x\to \frac{2\pi}{3}} \sin(x)$

   (c) $\lim_{x\to -\frac{5\pi}{6}} \cos(x)$

   (d) $\lim_{x\to\infty} \dfrac{x^3 + 3x}{2x^3 + x^2}$

   (e) $\lim_{x\to 2^+} \dfrac{x^2 - x - 2}{(x-2)^2}$

   (f) $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x} - 2}$

   (g) $\lim_{x\to -\infty} \arctan(x)$

   (h) $\lim_{x\to \frac{\pi}{2}^-} \tan(x)$

   (i) $\lim_{x\to 0} \dfrac{1}{x^2}$

   (j) $\lim_{x\to -\infty} x^2$

   (k) $\lim_{x\to -\infty} \dfrac{\sqrt{x^2+9}}{x}$

   (l) $\lim_{x\to 2} \dfrac{x^2 - 16}{x - 4}$

   (m) $\lim_{x\to\infty} \dfrac{5}{x^2+1}$

   (n) $\lim_{x\to\pi} \dfrac{\sin(x)}{\tan(x)}$

   (o) $\lim_{x\to 3} \dfrac{\sqrt{x-2} - 1}{x-3}$

   (p) $\lim_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x + e^{-x}}$

   (q) $\lim_{x\to 0} \dfrac{e^x - e^{-x}}{e^x - e^{-x}}$

7. Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$

(b) $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)}$

(c) $\lim_{x\to\infty} \frac{x^3}{e^x}$

(d) $\lim_{x\to\infty} \frac{2^x}{x^2}$

(e) $\lim_{x\to 0} \frac{\sin(x)}{x}$

(f) $\lim_{x\to \pi} \frac{\sin(x)}{\cos(x)-1}$

(g) $\lim_{x\to 0} \frac{\cos(x)-1}{x}$

(h) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

(i) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

(j) $\lim_{x\to\infty} xe^x$

(k) $\lim_{x\to \frac{\pi}{2}^+} \left(x-\frac{\pi}{2}\right)\tan(x)$

(l) $\lim_{x\to 0^+} x^{3x}$

(m) $\lim_{x\to 0^+} x^{\ln(x)}$

(n) $\lim_{x\to 0} \frac{\tan(x)-x}{x^2}$

(o) $\lim_{x\to 0^+} \frac{\ln(x)}{x}$

(p) $\lim_{x\to 0^+} x\ln(x^5)$

(q) $\lim_{x\to -2^-} \frac{x^2-3x+2}{x^2+2x-8}$

(r) $\lim_{x\to 0^+} \left(1-\frac{1}{x}\right)^x$

(s) $\lim_{x\to\infty} x^{1/x}$

(t) $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$

(u) $\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{\sqrt{3x^2+2}}$

(v) $\lim_{x\to 0^+} (1-2x)^{1/x}$

(w) $\lim_{x\to 0^+} (2x+1)^{\cot(x)}$

(x) $\lim_{x\to\infty} (x-x^2)$

(y) $\lim_{x\to\infty} (x^2-\ln(x))$

8. Classify the discontinuities of each function.

(a) $f(x) = \begin{cases} x^2-1 & \text{if } x<1 \\ 2 & \text{if } x=1 \\ \sqrt{x-1} & \text{if } x>1 \end{cases}$

(b) $f(x) = \begin{cases} \frac{2}{x} & \text{if } x<0 \\ \frac{x-2}{x^2-4} & \text{if } 0\le x<2 \\ \frac{x-1}{4} & \text{if } x\ge 2 \end{cases}$

(c) $g(x) = \begin{cases} \sin(x) & \text{if } x\le\pi \\ \cos(x) & \text{if } \pi<x<2\pi \\ \tan(x)+1 & \text{if } x>2\pi \end{cases}$

(d) $g(x) = \frac{x^2+5x+6}{(x+3)(x-2)^2}$

(e) $h(x) = \frac{|x+1|}{x+1}$

9. Find the vertical and horizontal asymptotes of each curve.

(a) $y = \arctan(x)$

(b) $y = e^{-x}$

(c) $y = \frac{e^x}{x}$

(d) $y = \frac{3x}{2x+5}$

(e) $y = \frac{|x+1|}{x+1}$

(f) $y = \frac{2x^2+1}{x-1}$

10. Find the intervals of increase/decrease of each function.

(a) $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + 1$

(b) $f(x) = 3x^4 - 8x^3$

(c) $f(x) = \int_0^x \sqrt{1+t^2}\,dt$

(d) $f(x) = \frac{x}{x^2+1}$

(e) $f(x) = x^x$ on $(0,\infty)$

11. Find the local maxima and local minima of each function.

(a) $f(x) = x^4 + 4x^3 + 4x^2$

(b) $f(x) = x\sqrt{1-x^2}$

(c) $f(x) = \sin(x) + \cos(x)$ on $[0, 2\pi]$

(d) $f(x) = xe^{-x}$

12. Find the intervals of concavity and points of inflection of each curve.
    (a) $y = f(x) = x^4 - 4x^3$
    (b) $y = f(x) = \ln(x^2 + 1)$
    (c) $y = f(x) = x^{1/3}(x + 8)$
    (d) $y = f(x) = \dfrac{x}{x^2 + 1}$
13. Is the function continuous on the interval?
    (a) $\dfrac{1}{x + 1}$, $[0, 4]$
    (b) $\arcsin(x)$, $[-1, 1]$
    (c) $\tan(x)$, $[0, \pi]$
    (d) $\dfrac{1}{\ln(x)}$, $(0, 2)$
    (e) $\dfrac{x - 3}{x^2 + x - 12}$, $[0, 4]$
14. Using the Intermediate Value Theorem show that there exists x such that:
    (a) $x^3 - 3x + 1 = 0$
    (b) $x^2 = 2^x$
    (c) $\sqrt{x^3 + 5x + 1} = 2$
15. Determine all values of a so that f is continuous on its domain.
    (a) $f(x) = \begin{cases} 2x + 3 & \text{if } x \leq 1 \\ ax^2 + 1 & \text{if } x > 1 \end{cases}$
    (b) $f(x) = \begin{cases} \sin(x) & \text{if } x < 0 \\ ax^3 & \text{if } x \geq 0 \end{cases}$
    (c) $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 2 \\ a & \text{if } x = 2 \\ 3 + 4(x - 2) & \text{if } x > 2 \end{cases}$
16. Sketch the graph of the following functions.
    (a) $f(x) = x^3 - 3x$
    (b) $f(x) = \sqrt{x^2 + 1}$
    (c) $f(x) = \dfrac{e^x}{x}$
    (d) $f(x) = \ln(4 - x^2)$. Note: $\ln(4) \approx 1.4$.
    (e) $f(x) = xe^{-x}$. Note: $e^{-1} \approx 0.4$.
    (f) $f(x) = \dfrac{x^2 - 4}{x + 1}$
    (g) $f(x) = (x^2 - 1)^{2/3}$. Note: $2^{2/3} \approx 1.6$.

Chapter 4: Applications of Derivatives

Section 4.1: Optimization

LEARNING OUTCOMES

1. Know how to find the absolute maximum and minimum of a function on a closed interval.
2. Know how to solve optimization problems.

In many real world situations, it is valuable to know the largest and/or smallest value of a function given some constraint. We first look at a method for finding the largest and small value of a function on an interval. We will then look at how to apply this method to real world situations (yay, word problems!).

4.1.1 Absolute Maxima and Absolute Minima

DEFINITION

Absolute Maximum

Absolute Minimum

Let f be a function defined on a interval I.

The value $f(M)$, is called the **absolute maximum of f on I** if $f(M) \geq f(x)$ for all x in the interval I.

The value $f(m)$ is called the **absolute minimum of f on I** if $f(m) \leq f(x)$ for all x in the interval I.

Observe that the absolute maximum and absolute minimum of a function are for a specified interval. So, the absolute maximum and/or absolute minimum of a function can change if we change the interval.

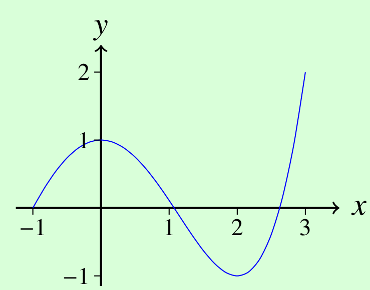
EXAMPLE 1

The graph of a function f is given. Find the absolute maximum and minimum of f on the given interval.

(a) $[-1, 3]$

(b) $[0, 2]$

(c) $[-1, 0]$



Solution: (a) The absolute maximum is 2 at $x = 3$.
The absolute minimum is -1 at $x = 2$.

(b) The absolute maximum is 1 at $x = 0$.
The absolute minimum is -1 at $x = 2$.

(c) The absolute maximum is 1 at $x = 0$.
The absolute minimum is 0 at $x = -1$.

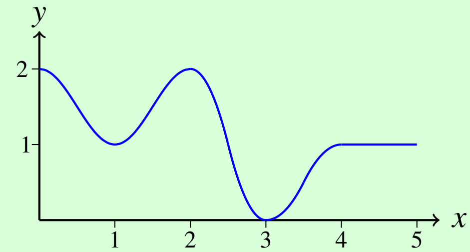
EXAMPLE 2 The graph of a function f is given.

Find the absolute maximum and minimum of f on the given interval.

(a) $[0,5]$

(b) $[1,2]$

(c) $[4,5]$



Solution: (a) The absolute maximum is 2 at $x = 0$ and $x = 2$.
The absolute minimum is 0 at $x = 3$.

(b) The absolute maximum is 2 at $x = 2$.
The absolute minimum is 1 at $x = 1$.

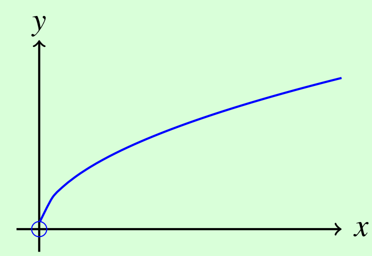
(c) The absolute maximum is 1 at every x with $4 \le x \le 5$.
The absolute minimum is 1 at every x with $4 \le x \le 5$.

In general, we don't have any guarantee that a function will have an absolute maximum or an absolute minimum on a given interval.

EXAMPLE 3 Explain why $f(x) = \sqrt{x}$ does not have an absolute maximum nor an absolute minimum on the *open* interval $(0, \infty)$.

Solution: It does not have an absolute minimum because $x = 0$ is not included. In particular, if we pick any small positive number, then we can find a value of $x > 0$ such that $\sqrt{x}$ is smaller than it.

It doesn't have an absolute maximum as the y-values just get larger and larger as $x \to \infty$.



EXERCISE 1 The graph of a function f is given.

Find the absolute maximum and minimum of f on the given interval.

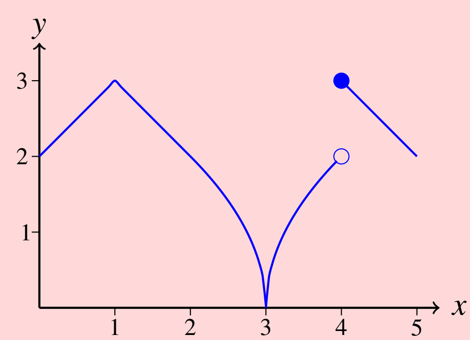
(a) $[0,5]$

(b) $[0,1]$

(c) $[1,3]$

(d) $[3,4]$

(e) $[4,5]$



EXERCISE 2 Find a function f that has an absolute maximum but no absolute minimum on $[0,2)$.

EXERCISE 3

Find a function f that has both an absolute maximum and an absolute minimum on $[0, 4]$, and the absolute minimum of f occurs at exactly two x values.

The next theorem gives us conditions under which we can guarantee the existence of an absolute maximum and an absolute minimum for a function on an interval.

THEOREM 1

Extreme Value Theorem

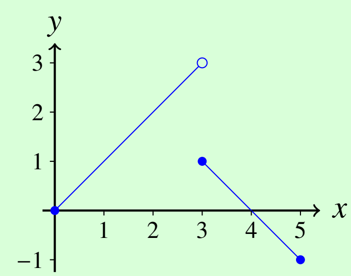
If f is continuous on a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum of f on $[a, b]$.

EXAMPLE 4

Show that the function f defined by $f(x) = \begin{cases} x & 0 \le x < 3 \\ 4 - x & 3 \le x \le 5 \end{cases}$

does not have an absolute maximum on $[0, 5]$. Does this contradict the Extreme Value Theorem?

Solution: We begin by graphing f.



Observe that the values of $f(x)$ gets infinitely close to 3, but never actually reaches 3. Therefore, f does not have an absolute maximum.

This does not violate the Extreme Value Theorem since f is not continuous on the interval $[0, 5]$.

Closed Interval Method

Looking at the examples above, we see that the absolute maximum/minimum of a continuous function on an interval $[a, b]$ always occur at a local maximum, at a local minimum, or at one of the end points of the interval. This gives us a nice algorithm for finding the absolute maximum and absolute minimum predicted by the Extreme Value Theorem.

ALGORITHM (Closed Interval Method)

To find the absolute maximum and absolute minimum of a continuous function f defined on the closed interval $[a, b]$:

Step 1. Find all critical numbers of f inside the interval.
Step 2. Compute the values of $f(x)$ at the critical numbers and at the endpoints of the interval.
Step 3. Take the largest of these values to be the absolute maximum and the smallest to be the absolute minimum.

EXAMPLE 5 Find the absolute maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on $[1, 4]$.

Solution: Step 1. We have $f'(x) = 3x^2 - 6x = 3x(x - 2)$.

Thus, the critical numbers are $x = 0, 2$. But, $x = 0$ is not inside the interval $[1, 4]$, so we ignore it.

Step 2. We have $f(2) = -3$, $f(1) = -1$, $f(4) = 17$.

Step 3. The absolute maximum is 17 and occurs at $x = 4$.
The absolute minimum is -3 and occurs at $x = 2$.

EXAMPLE 6 Find the absolute maximum and minimum of $f(x) = |x|$ on $[-3, 2]$.

Solution: Step 1. We have $f'(x) = \frac{x}{|x|}$.

We see that $f'(x)$ does not exist when $x = 0$. Since $x = 0$ is in the domain of f it is a critical number.

Step 2. We have $f(0) = 0$, $f(-3) = 3$, $f(2) = 2$.

Step 3. The absolute maximum is 3 and occurs at $x = -3$.
The absolute minimum is 0 and occurs at $x = 0$.

EXAMPLE 7 Find the absolute maximum and minimum for $f(x) = \sin^2(x) - \cos(2x)$ on $\left[0, \frac{3\pi}{4}\right]$.

Solution: Step 1. We have

$$\begin{aligned} f'(x) &= 2\sin(x)\cos(x) + 2\sin(2x) \\ &= 2\sin(x)\cos(x) + 2(2\sin(x)\cos(x)) \\ &= 6\sin(x)\cos(x) \\ &= 3\sin(2x) \end{aligned}$$

Thus, on the interval $\left[0, \frac{3\pi}{4}\right]$, the critical numbers are $x = 0, \frac{\pi}{2}$.

Step 2. We have

$$f(0) = \sin^2(0) - \cos(2(0)) = 0 - 1 = -1$$

$$f\left(\frac{3\pi}{4}\right) = \sin^2\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{2}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 - 0 = \frac{1}{2}$$

$$f\left(\frac{\pi}{2}\right) = \sin^2\left(\frac{\pi}{2}\right) - \cos(\pi) = 1 - (-1) = 2$$

Step 3. The absolute maximum is 2 and occurs at $x = \frac{\pi}{2}$.
The absolute minimum is 1 and occurs at $x = 0$.

EXERCISE 4 Find the absolute maximum and minimum for f on the given interval.

(a) $f(x) = -x^2 + 3x - 2$ on $[1, 3]$

(b) $f(x) = \dfrac{x}{x^2 + 1}$ on $[0, 2]$

(c) $f(x) = \sin(x) - \sin^2(x)$ on $[0, 2\pi]$

(d) $f(x) = x\sqrt{4 - x^2}$ on $[-2, 1]$

4.1.2 Optimization Problems

We now look at applying what we have been doing to word problems. As you will see, this is largely the same as finding the absolute maximum or absolute minimum. The only two possible differences are:

1. We will sometimes need to use the description of the problem to figure out the function and interval.

2. We won't always have a closed interval. In such cases, we use a limit to evaluate the function at an end of an interval that isn't closed. For example, if we were maximizing/minimizing $f(x)$ on the interval $1 < x < \infty$, then to evaluate the left end point, we would use

$$\lim_{x\to 1^+} f(x)$$

To evaluate the right end point, we would use

$$\lim_{x\to\infty} f(x)$$

Then, just like in the Closed Interval Method, we pick the largest/smallest of all the values found.

ALGORITHM

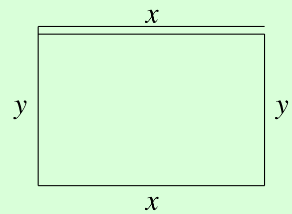
The basic steps for solving an optimization problem are:

Step 1: Read the Problem: Read the problem carefully and draw a picture. Give a name to the quantity that is to be optimized and give variable names to all the dependent and independent variables. Write down what information you are given and identify what you are trying to find.

Step 2. Mathematical Modelling: Use the information given to create a function for the quantity that is to be optimized. State any restrictions (mathematical and real-world) on the variables. If the function is dependent on more than one variable, use the information given to eliminate variables until you have a function of a single variable.

Step 3 Optimize: If you have a closed interval, use the Closed Interval Method. If you do not have a closed interval, use limits as necessary to evaluate the function at the ends of the interval.

EXAMPLE 8 A 12 cm length of wire is bent to form a rectangular loop such that the entire top of the rectangle is overlapped. What dimensions should the rectangle have to maximize the area of the loop?



Solution: Step 1. Let the top and bottom edge lengths equal x and left and right edge lengths equal y. We are given that the total length of the wire is 12 cm. We want to maximize the area.

Step 2. Since the set up requires the wire to include three segments of length x and two of length y and the wire is 12 cm long, then we have that

$$3x + 2y = 12 \tag{4.1}$$

Since length cannot be negative, this implies that we have the restrictions $0 \leq x \leq 4$ and $0 \leq y \leq 6$.

Next, let A be the area of the loop. Then,

$$A = xy \tag{4.2}$$

Since A is dependent on two variables, we use the information given to eliminate one of the variables. Solving equation (4.1) for y in terms of x gives

$$y = \frac{1}{2}(12 - 3x) = 6 - \frac{3}{2}x$$

We substitute this expression for y into equation (4.2) to get an expression for the area as a function of x.

$$A(x) = x\left(6 - \frac{3}{2}x\right) = 6x - \frac{3}{2}x^2$$

Step 3. We now have a function $A(x)$ and a closed interval $[0, 4]$. So, we can apply the Closed Interval Method. We get

$$A'(x) = 6 - 3x$$

Thus, the only critical number is $x = 2$. We see that

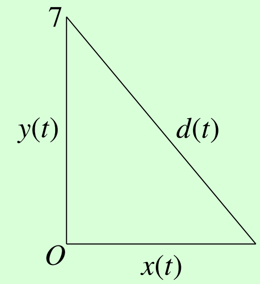
$$A(2) = 6$$
$$A(0) = 0$$
$$A(4) = 0$$

Thus, it is maximized at when $x = 2$. The corresponding value of y is

$$y = 6 - \frac{3}{2}(2) = 3$$

Therefore, dimensions of $x = 2$ cm and $y = 3$ cm maximize the area of the rectangular wire loop.

EXAMPLE 9 A mini train is travelling south at 8 km/h. At time $t = 0$ it is heading directly towards a deer that is 7 km away. At that time, the deer begins moving due east at a rate of 6 km/h. At what time during the first 30 minutes will the mini train be closest to the deer?



Solution: Step 1. Let O be the location of the deer at time $t = 0$. Let $y(t)$ be the distance the train is from O, let $x(t)$ be the distance the deer is from O, and let $d(t)$ be the distance between the deer and the train. We want to minimize $d(t)$.

Step 2. We are given that

$$\begin{aligned} x(t) &= 7 - 8t \\ y(t) &= 6t \end{aligned}$$

Thus,

$$\begin{aligned} d(t) &= \sqrt{(x(t))^2 + (y(t))^2} \\ &= \sqrt{(7-8t)^2 + (6t)^2} \\ &= \sqrt{49 - 112t + 100t^2} \end{aligned}$$

We are given the restriction on time that $0 \leq t \leq \frac{1}{2}$.

Step 3. We apply the Closed Interval Method.

We first look for critical numbers of d in the interval.

$$\begin{aligned} d'(t) &= \frac{1}{2}(49 - 112t + 100t^2)^{-1/2} \cdot (-112 + 200t) \\ 0 &= \frac{-56 + 100t}{\sqrt{49 - 112t + 100t^2}} \\ 0 &= -56 + 100t \\ 56 &= 100t \\ \frac{14}{25} &= t \end{aligned}$$

Since $\frac{14}{25} > \frac{1}{2}$, there are no critical numbers in the interval.

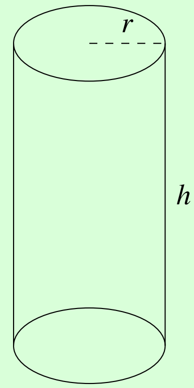
We have

$$\begin{aligned} d(0) &= 7 \\ d\left(\frac{1}{2}\right) &= \sqrt{3^2 + 3^2} = \sqrt{18} < 7 \end{aligned}$$

Thus, the train is closest to the deer at $t = \frac{1}{2}$ hours.

EXAMPLE 10 Pop cans with a volume of 300 cm^3 are made in the shape of right circular cylinders. Find the dimensions of the can that minimize its surface area.

Solution: Step 1. Let h be the height of the can, and let r be the radius of the top and bottom. We are told that the volume must be 300 cm^3. We want to minimize its surface area.



Step 2. We are given that

$$300 = \pi r^2 h \tag{4.3}$$

The surface area of the can is

$$\begin{aligned} S(r,h) &= \text{ area of top and bottom} + \text{ area of side} \\ &= 2\pi r^2 + 2\pi rh \end{aligned}$$

Solving equation (4.3) for h (since it is much easier than trying to solve for r) gives

$$h = \frac{300}{\pi r^2}$$

Since we divided by r, we have an added restriction that $r \neq 0$.

Substituting h into the surface area function, we get

$$S(r) = 2\pi r^2 + 2\pi r\left(\frac{300}{\pi r^2}\right) = 2\pi r^2 + 600r^{-1}$$

What is the real world domain? We already know that we must have $r > 0$. What is the maximum radius? Theoretically, there is some maximum size (it would not be optimal for the pop can to have a radius of 2 m!), but it is not worth trying to figure out what it is. So, we just say that domain is $0 < r < \infty$.

Step 3. Taking the derivative gives

$$S'(r) = 4\pi r - 600r^{-2}$$

Setting this equal to 0, we find

$$\begin{aligned} 0 &= 4\pi r - \frac{600}{r^2} \\ \frac{600}{r^2} &= 4\pi r \\ \frac{150}{\pi} &= r^3 \\ \left(\frac{150}{\pi}\right)^{1/3} &= r \end{aligned}$$

So, the only critical number is $r = \left(\frac{150}{\pi}\right)^{1/3}$.

Since we don't have a closed interval, we use limits to evaluate the end points. We see that

$$\lim_{r\to 0^+} S(r) = \lim_{r\to 0^+}(2\pi r^2 + 600r^{-1}) = \lim_{r\to 0^+}\frac{2\pi r^3 + 600}{r} = \frac{600}{0^+} = \infty$$
$$\lim_{r\to \infty} S(r) = \lim_{r\to \infty}(2\pi r^2 + 600r^{-1}) = \lim_{r\to \infty}\frac{2\pi r^3 + 600}{r} = \lim_{r\to\infty}\frac{2\pi + \frac{600}{r^3}}{\frac{1}{r^2}} = \frac{2\pi + 0}{0^+} = \infty$$

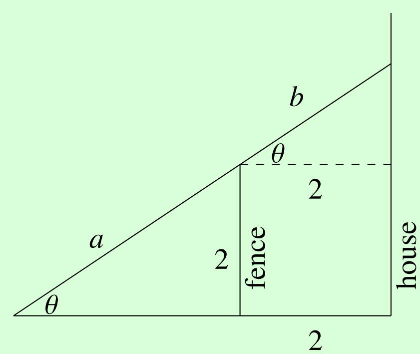
Since $S\left(\left(\frac{150}{\pi}\right)^{1/3}\right)$ is a finite number, this is the smallest, and so $r = \left(\frac{150}{\pi}\right)^{1/3}$ gives the minimum surface area.

The height that gives the minimum is then

$$h = \frac{300}{\pi\left(\frac{150}{\pi}\right)^{2/3}} = 2\left(\frac{150}{\pi}\right)^{1/3}$$

EXAMPLE 11 A 2 m tall fence is 2 m from a house. Find the length of the shortest ladder that will reach over the fence and reach the house.

Solution: Step 1. Let θ be the angle the ladder makes with the ground. Let L be the length of the ladder. Let a be the length of the ladder on the left size of the fence and b be the length of the ladder on the right side of the fence. Then, $L = a + b$. We want to minimize L.



Step 2. From the diagram, we get

$$\sin(\theta) = \frac{2}{a}$$
$$\cos(\theta) = \frac{2}{b}$$

Hence, $a = 2\csc(\theta)$ and $b = 2\sec(\theta)$. So,

$$L(\theta) = 2\csc(\theta) + 2\sec(\theta)$$

Physically, the angle must be between 0 and $\frac{\pi}{2}$. However, it cannot equal 0 nor $\frac{\pi}{2}$ since these are not in the domain of L. Thus, the restriction is

$$0 < \theta < \frac{\pi}{2}$$

Step 3. We have

$$\frac{dL}{d\theta} = -2\csc(\theta)\cot(\theta) + 2\sec(\theta)\tan(\theta)$$

Setting $\frac{dL}{d\theta} = 0$ gives

$$
\begin{aligned}
0 &= -2\frac{1}{\sin(\theta)}\frac{\cos(\theta)}{\sin(\theta)} + 2\frac{1}{\cos(\theta)}\frac{\sin(\theta)}{\cos(\theta)} \\
2\frac{\cos(\theta)}{\sin^2(\theta)} &= 2\frac{\sin(\theta)}{\cos^2(\theta)} \\
\frac{\cos^3(\theta)}{\sin^3(\theta)} &= 1 \\
\tan^3(\theta) &= 1 \\
\tan(\theta) &= 1
\end{aligned}
$$

Thus, the only critical number in $0 < \theta < \frac{\pi}{2}$ is $\theta = \frac{\pi}{4}$.

Evaluating the end points gives

$$
\begin{aligned}
\lim_{\theta\to 0^+} L(\theta) &= \lim_{\theta\to 0^+}(2\csc(\theta) + 2\sec(\theta)) = \infty \\
\lim_{\theta\to \frac{\pi}{2}^+} L(\theta) &= \lim_{\theta\to \frac{\pi}{2}^+}(2\csc(\theta) + 2\sec(\theta)) = \infty
\end{aligned}
$$

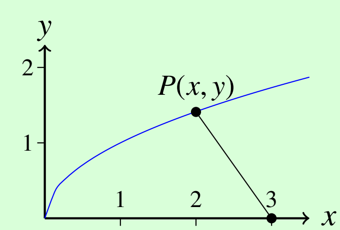
We also have

$$
L\left(\frac{\pi}{4}\right) = 2\csc\left(\frac{\pi}{4}\right) + 2\sec\left(\frac{\pi}{4}\right) = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}
$$

Thus, the length of the shortest ladder $4\sqrt{2}$ m.

EXAMPLE 12 Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

Solution: Step 1. Let $P(x, y)$ be the location of the closet point and d be the square of the distance between P and $(3, 0)$ (we choose the square of the distance so that the calculations are much easier!). We want to minimize d.



Step 2. We have that $y = \sqrt{x}$. Then, by the distance formula, we get

$$
d(x) = (x-3)^2 + (\sqrt{x} - 0)^2 = (x-3)^2 + x
$$

Since the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$, that is also the domain here.

Step 3. We have

$$
d'(x) = 2(x-3) + 1 = 2x - 5
$$

Hence, we have a critical number at $x = \frac{5}{2}$.

For the end points we use $x = 0$ and a limit as $x \to \infty$. We get

$$\begin{aligned} d(0) &= 9 \\ d(5/2) &= \frac{11}{4} \\ \lim_{x\to\infty} d(x) &= \infty \end{aligned}$$

Thus, the minimum occurs when $x = \frac{5}{2}$. We get that $y = \sqrt{\frac{5}{2}}$. So, the point $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$ is the point on the curve closest to $(3, 0)$.

EXERCISE 5 A shipping container is to be manufactured in the shape of a square based box with a volume of 1000 cm^3. Find the dimensions for such a container which minimizes the quantity of material required to build it (i.e., minimizes the surface area).

EXERCISE 6 You are in a dune buggy at a point P in the desert, 12 km due south of the nearest point A on a straight east-west road. You want to get to a town B on the road 18 km east of A. If your dune buggy can travel at an average speed of 15 km/h through the desert and 30 km/h along the road, towards what point Q on the road should you head to minimize your travel time from P to B?

Section 4.1 Problems

1. Find the absolute maximum and minimum of f on the given closed interval.
   (a) $f(x) = \sqrt{x-2}$ on $[2, 6]$
   (b) $f(x) = 3x^3 + \frac{3}{2}x^2$ on $[0, 1]$
   (c) $f(x) = (x+1)^{4/3}$ on $[-9, 7]$
   (d) $f(x) = e^{x^2}$ on $[1, 4]$
   (e) $f(x) = \sec(x)$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
   (f) $f(x) = \frac{1}{x^2+1}$ on $[0, 3]$
   (g) $f(x) = \frac{1}{3}x^3 + 3x^2 + 8x + 10$ on $[-3, -1]$

2. A square based closed box has a fixed surface area of 8 cm^2. Find the maximum volume of the box.

3. Find the minimum surface area of a square based closed box with a volume of 1000 cm^3.

4. Find the minimum distance from $(2, 0)$ to the curve $y = \sqrt{x^2+1}$.

5. A person is in a boat on a lake and is 4 km from the nearest point P on a straight shoreline. Their cabin is 9 km down the shoreline from P. Their plan is to row the boat to some point along the shore and then walk the rest of the way. Given that they can row 3 km/h and can walk 5 km/h, how far down the shoreline from the point P should they land their boat to minimize the time it takes to get to their cabin?

6. A person has 40 m of fencing and would like to create a large garden. One side of the garden will be along their house and hence requires no fencing. They will use three exterior fences and two interior fences to partition the garden into three rectangular regions. What are the dimensions of the garden that maximize the enclosed area?

7. A water trough is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side through an angle θ. Which θ will allow the trough to carry the maximum amount of water?

Section 4.2: Approximations

LEARNING OUTCOMES

1. Know how to use the linear approximation of a function.
2. Know how to use the binomial approximation.
3. Know how to use the quadratic approximation of a function.
4. Know how to find an upper bound for the error in the linear approximation.

4.2.1 Linear Approximation

Describing the precise behaviour of a system often involves working with a very complicated function. However, we often only need to worry about the behaviour of the system over a small part of its domain. In such a case, we can instead use an approximation of the function which will be much simpler to use.

For example, near the surface of the Earth we can calculate the gravitational potential energy, E_g, of a mass m as a function of height h above the surface of the Earth with the formula $E_g(h) = \dfrac{-mgR_E^2}{R_E + h}$ where R_E is the radius of Earth and $g = \dfrac{GM}{R_E^2}$ is the gravitational acceleration at the surface of the Earth. In contrast, if we don't need a very precise value, then we can replace the function by the much simpler (but approximate) function $E_g(h) = mgh$.

We can construct an approximate function like the one above through a process called *linearization*. This may sound fancy, but it just involves finding an equation for a tangent line to the curve described by the function we want to approximate.

DEFINITION
Linearization

The **linearization** of a function $f(x)$ at a is given by

$$L_a(x) = f(a) + f'(a)(x - a)$$

REMARK

The linearization can also be called the **first degree Taylor polynomial**. We will look at the second degree Taylor polynomial in Section 4.2.3.

EXAMPLE 1

Find the linearization, $L_0(x)$, of $f(x) = e^x$ at $a = 0$.

Solution: Since $f'(x) = e^x$, we get

$$L_0(x) = f(0) + f'(0)(x - 0) = e^0 + e^0(x - 0) = 1 + 1(x - 0) = x + 1$$

EXERCISE 1 Show that the linearization of $f(x) = x^{1/3}$ at $a = 1$ is given by $L_1(x) = 1 + \frac{1}{3}(x - 1)$.

Since the linearization $L_a(x)$ of a function f is just the tangent line to f at $x = a$, as we saw on page 8, we get

The **linear (tangent line) approximation**

$$f(x) \approx L_a(x) = f(a) + f'(a)(x - a)$$

for values of x near a.

EXAMPLE 2 Use the linear approximation $L_{16}(x)$ for $f(x) = \sqrt{x}$ to approximate $\sqrt{16.08}$.

Solution: First, the notation $L_{16}(x)$ tells us that $a = 16$. Next, we have $f'(x) = \dfrac{1}{2\sqrt{x}}$. Thus,

$$\begin{aligned} L_{16}(x) &= f(16) + f'(16)(x - 16) \\ &= 4 + \frac{1}{8}(x - 16) \end{aligned}$$

We now use this equation to approximate $\sqrt{16.08}$. We get

$$\sqrt{16.08} = f(16.08) \approx L_{16}(16.08) = 4 + \frac{1}{8}(16.08 - 16) = 4 + \frac{0.08}{8} = 4.01$$

For comparison, a calculator gives $\sqrt{16.08} \approx 4.009988$. So, using the linear approximation here yielded a very accurate approximation.

EXAMPLE 3 Use the linearization of $f(x) = e^x$ at $a = 0$ to approximate $e^{0.08}$.

Solution: In Example 4.2.1, we found that

$$L_0(x) = x + 1$$

Therefore, the linear approximation gives

$$e^{0.08} = f(0.08) \approx L_0(0.08) = 0.08 + 1 = 1.08$$

EXERCISE 2 Find the linearization of $f(x) = x^{2/5}$ at $a = 32$ and use it to approximate $30^{2/5}$.

EXAMPLE 4

The motion of a simple pendulum of length L is described by the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta)$$

where θ is the angle that the pendulum makes with a vertical and g is acceleration due to gravity. This differential equation does not have solutions which can be written down in terms of elementary functions. However, if we are only concerned with relatively small amplitude oscillations (i.e., θ is always near 0), then we can simplify this differential equation by replacing $\sin(\theta)$ with its linearization at $a = 0$.

Letting $f(\theta) = \sin(\theta)$, we have $f'(\theta) = \cos(\theta)$ so

$$\sin(\theta) \approx L_0(\theta) = f(0) + f'(0)(\theta - 0) = \sin(0) + \cos(0)(\theta - 0) = \theta$$

So, if θ is small we can replace $f(\theta) = \sin(\theta)$ with the linear function $L_0(\theta) = \theta$.

This yields a simpler differential equation which is valid for describing small amplitude oscillations of the pendulum:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

This differential equation is actually solvable! Using higher level mathematics, the solution is found to be

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{g/L}$.

EXERCISE 3

World population in billions can be modelled by the exponential function

$$P(t) = 7.8e^{0.01t}$$

where t is the number of year elapsed since the year 2020. Find the linearization of $P(t)$ at $a = 0$ (i.e., the year 2020) and use it to estimate the population in the year 2025.

EXERCISE 4

The gravitational potential energy of a mass m a height h above the surface of the Earth is given by

$$E_g(h) = \frac{-mgR_E^2}{R_E + h}$$

where R_E is the radius of Earth and $g = \dfrac{GM}{R_E^2}$ is the gravitational acceleration at the surface of the Earth. Find the linearization of $E_g(h)$ about $a = 0$ to show that $E_g(h) \approx -mgR_E + mgh$.

Note: Since $-mgR_E$ is a constant, the *change* in gravitational potential energy by raising a mass to a height h is therefore approximately equal to simply mgh.

4.2.2 Binomial Approximation

Functions of the form $f(x) = (1 + g(x))^k$, for some function g and some real number k, appear commonly in science. We often need to approximate such functions when $|g(x)|$ is very close to 0. We can derive an approximation using the linearization. To see how this works, let's first consider the case when $g(x) = x$.

Given $f(x) = (1+x)^k$, we have $f'(x) = k(1+x)^{k-1}$ from which it follows that $f(0) = 1$ and $f'(0) = k$. Using these values, we can construct the linearization of f at $a = 0$.

$$L_0(x) = 1 + k(x - 0) = 1 + kx$$

THEOREM 1

Binomial Approximation for $(1 + x)^k$

Given a function of the form $(1 + x)^k$, when $|x|$ is very close to 0, the function can be approximated by

$$(1 + x)^k \approx 1 + kx$$

EXAMPLE 5

Apply the binomial approximation to $(1 + x)^{1/2}$.

Solution: Assuming $|x|$ is very close to 0, we get

$$(1 + x)^{1/2} \approx 1 + \frac{1}{2}x$$

EXAMPLE 6

Apply the binomial approximation to $(1 + x)^{10}$.

Solution: Assuming $|x|$ is very close to 0, we get

$$(1 + x)^{10} \approx 1 + 10x$$

EXAMPLE 7

Use the binomial approximation to estimate $\sqrt{1.03}$.

Solution: Let $f(x) = (1 + x)^{1/2}$ and observe that $\sqrt{1.03} = \sqrt{1 + 0.03} = f(0.03)$.

Applying the binomial approximation to $f(x)$ we get

$$f(x) = (1 + x)^{1/2} \approx 1 + \frac{1}{2}x$$

Therefore,

$$\sqrt{1.03} = f(0.03) \approx 1 + \frac{0.03}{2} = 1.015$$

Comparing this to the value produced by a calculator $\sqrt{1.03} \approx 1.0149$, we see that the binomial approximation yields a very good estimate of the actual value.

EXERCISE 5

Apply the binomial approximation to each function.

(a) $(1+x)^7$

(b) $(1+x)^{2/3}$

EXERCISE 6

Use the binomial approximation to estimate $(0.98)^{1/4}$.

The binomial approximation can be generalized to handle functions of the form $f(x) = (1+g(x))^k$ for some function g and real number k when $|g(x)|$ is very close to 0.

THEOREM 2

Binomial Approximation for $(1+g(x))^k$

Given a function of the form $(1+g(x))^k$, when $|g(x)|$ is very close to 0, the function can be approximated by

$$(1+g(x))^k \approx 1 + k\,g(x)$$

EXAMPLE 8

Apply the binomial approximation to $(1+x^3)^{1/3}$.

Solution: Assuming $|x^3|$ is very close to 0, we can apply the binomial approximation with $g(x) = x^3$ and $k = \dfrac{1}{3}$ to get

$$(1+x^3)^{1/3} \approx 1 + \frac{1}{3}x^3$$

EXAMPLE 9

Apply the binomial approximation to $(2+x^2)^4$.

Solution: Let $f(x) = (2+x^2)^4$. We first need to rewrite f in the form $c(1+g(x))^k$.

$$f(x) = (2+x^2)^4 = \left(2\left(1+\frac{x^2}{2}\right)\right)^4 = 2^4\left(1+\frac{x^2}{2}\right)^4 = 16\left(1+\frac{x^2}{2}\right)^4$$

Assuming $\left|\dfrac{x^2}{2}\right|$ is very close to 0, we can apply the binomial approximation to $\left(1+\dfrac{x^2}{2}\right)^4$ with $g(x) = \dfrac{x^2}{2}$ and $k = 4$. This gives

$$f(x) = 16\left(1+\frac{x^2}{2}\right)^4 \approx 16\left(1+4\left(\frac{x^2}{2}\right)\right) = 16 + 32x^2$$

REMARK

In Example 4.2.9, if we fully expanded $(2+x^2)^4$ we would get

$$(2+x^2)^4 = 16 + 32x^2 + 24x^4 + 8x^6 + x^8$$

The binomial approximation replaces this exact result with just the constant and quadratic term. It is reasonable to discard the higher powers of x^2 because when $|x^2|$ is very close to 0 these higher powers of x^2 will be very, very small.

EXAMPLE 10

Apply the binomial approximation to $(2-x)^{10}$.

Solution: We first need to rewrite $(2-x)^{10}$ in the form $c(1+g(x))^k$. We get

$$(2-x)^{10} = \left(2\left[1-\frac{x}{2}\right]\right)^{10} = 2^{10}\left(1-\frac{x}{2}\right)^{10}$$

Assuming $\left|-\frac{x}{2}\right|$ is very close to 0, we can apply the binomial approximation to $\left(1-\frac{x}{2}\right)^{10}$ with $g(x) = -\frac{x}{2}$ and $k = 10$. This gives

$$(1-2x)^{10} \approx 2^{10} \cdot \left(1 + 10 \cdot \left(-\frac{x}{2}\right)\right) = 2^{10}(1-5x)$$

EXAMPLE 11

Use the binomial approximation on $f(x) = (1-3x^2)^{2/3}$ to estimate $f(0.01)$.

Solution: Since $x = 0.01$ is close to 0, we get

$$f(x) \approx 1 + \frac{2}{3}(-3x^2) = 1 - 2x^2$$

Thus,

$$f(0.01) \approx 1 - 2(0.01)^2 = 0.9998$$

NOTE: The actual value is 0.999799990, so the approximation was extremely accurate!

EXERCISE 7

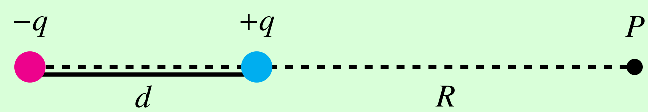
Apply the binomial approximation to each function.

(a) $\left(1-\frac{x}{2}\right)^{1/2}$

(b) $\left(\frac{1}{2}+x^2\right)^5$

EXERCISE 8 Use the binomial approximation on $f(x) = (1+2x)^{1/2}$ to estimate $\sqrt{1.02}$.

EXAMPLE 12 A positive charge q sits a distance d to the right of a negative charge $-q$. Another R units to the right of the positive charge is the point P.



The magnitude of the electric field E at point P due to the two charges is given as a function of R by

$$E(R) = \frac{kq}{R^2}\left(1 - \left(1 + \frac{d}{R}\right)^{-2}\right)$$

where k is a constant. Apply the binomial approximation on this expression.

Solution: Assuming $\frac{d}{R}$ is very close to 0, we can use the binomial approximation to rewrite

$$\left(1 + \frac{d}{R}\right)^{-2} \approx 1 - 2\left(\frac{d}{R}\right)$$

Substituting this back into the expression for $E(R)$ we find

$$E(R) \approx \frac{kq}{R^2}\left(1 - \left(1 - 2\left(\frac{d}{R}\right)\right)\right) = \frac{2kqd}{R^3}$$

EXERCISE 9 A spaceship of length L_0 flying by the Earth at close to the speed of light would appear contracted in length. The apparent length, L, depends on the speed v according to the formula

$$L(v) = L_0\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

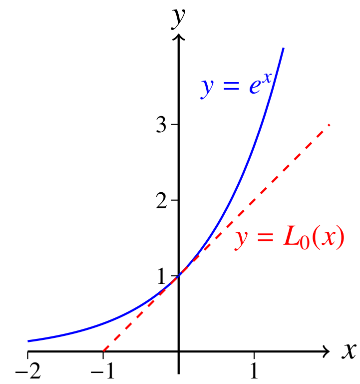
where c is the speed of light.

Apply the binomial approximation to this formula.

4.2.3 Quadratic Approximation

Consider the linearization of $f(x) = e^x$ at $a = 0$ which is given by $L_0(x) = 1 + x$. For values near $x = 0$, the linearization does a good job of approximating the actual function. For example, $f(0.1) = e^{0.1} \approx 1.105$ while $L_0(0.1) = 1+0.1 = 1.1$. However, the farther you move away from where the linearization is constructed, the worse the approximations becomes. For example, $f(0.5) = e^{0.5} \approx 1.649$ while
$L_0(0.5) = 1 + 0.5 = 1.5$.

We can see why this happens by graphing the $y = e^x$ and $y = L_0(x) = 1 + x$.



Because $y = e^x$ is a non-linear curve, it will bend away from any line we draw tangent to it.

So, if we want to construct an approximation which is useful over a wider interval, then we must use a non-linear function. The simplest non-linear function is a quadratic function!

Let $P_{2,a}(x) = c_0 + c_1(x-a) + c_2(x-a)^2$ denote such a quadratic function constructed to approximate a function f near $x = a$. We determine the coefficients of this quadratic function by requiring the quadratic function to have the same y-value, first derivative, and second derivative at $x = a$ as the original function. That is, we require

$$P_{2,a}(a) = f(a), \qquad P'_{2,a}(a) = f'(a), \qquad P''_{2,a}(a) = f''(a)$$

Note that the first two conditions also apply to the linearization. It is only the condition on the second derivative which is new and this condition basically makes sure that the quadratic function bends just the right amount at $x = a$.

Solving for c_0, c_1, and c_2 using the conditions above, we get

The **second degree Taylor polynomial of** f

$$P_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Observe that $P_{2,a}(x)$ is a refinement of the linearization (which, recall, can be called the first degree Taylor polynomial). In particular, the first two terms of $P_{2,a}(x)$ are just the terms from the linearization. In Chapter 7, we will extend this concept to even higher degree polynomials.

EXAMPLE 13

Determine the second degree Taylor polynomial of $f(x) = \ln(x)$ at $a = e$.

Solution: We have $f'(x) = x^{-1}$ and $f''(x) = -x^{-2}$. Thus,

$$\begin{aligned} P_{2,e}(x) &= f(e) + f'(e)(x-e) + \frac{f''(e)}{2}(x-e)^2 \\ &= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 \end{aligned}$$

EXERCISE 10

Find the second degree Taylor polynomial of $f(x) = \sin(x)$ at $a = \frac{\pi}{2}$.

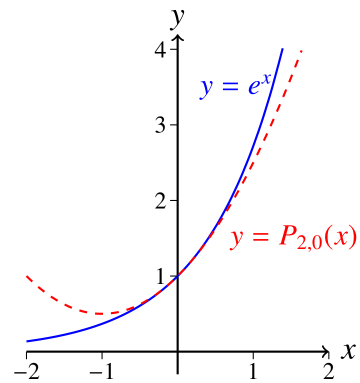
EXAMPLE 14 Determine the second degree Taylor polynomial of $f(x) = e^x$ centred at $a = 0$.

Solution: For $f(x) = e^x$ we have $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 1$. Therefore, the second degree Taylor polynomial of f at $a = 0$ is

$$\begin{aligned} P_{2,0}(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 \\ &= 1 + (1)(x-0) + \frac{1}{2}(x-0)^2 \\ &= 1 + x + \frac{x^2}{2} \end{aligned}$$

Graphing $y = P_{2,0}(x)$ alongside $y = e^x$ and comparing this to the figure above reveals that the second degree Taylor polynomial yields a good approximation over a wider interval compared to the linearization.

Moreover, $P_{2,0}(x)$ provides a better approximation than $L_0(x)$ for values of x near 0.



EXAMPLE 15 Compare the accuracy of the linearization $L_0(x) = 1 + x$ and the second degree Taylor polynomial $P_{2,0}(x) = 1 + x + \frac{x^2}{2}$ of $f(x) = e^x$ in approximating $f(0.5)$.

Solution: We have

$$f(0.5) = e^{0.5} \approx 1.649$$

The linearization gives

$$L_0(0.5) = 1 + 0.5 = 1.5$$

and the second degree Taylor polynomial gives

$$P_{2,0}(0.5) = 1 + 0.5 + \frac{0.5^2}{2} = 1.625$$

So, the quadraticization gets much closer to the actual value.

We get:

The **quadratic approximation**

$$f(x) \approx P_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

for values of x near a.

EXAMPLE 16

Use the quadratic approximation $P_{2,0}(x)$ for $f(x) = \dfrac{1}{1-x}$ to approximate $f(0.3)$.

Solution: For $f(x) = (1-x)^{-1}$, we have $f'(x) = (1-x)^{-2}$ and $f''(x) = 2(1-x)^{-3}$. Thus,

$$\begin{aligned} P_{2,0}(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 \\ &= 1 + x + \frac{2}{2}x^2 \\ &= 1 + x + x^2 \end{aligned}$$

Therefore,

$$f(0.3) \approx P_{2,0}(0.3) = 1 + 0.3 + (0.3)^2 = 1.39$$

EXERCISE 11

Use the quadratic approximation $P_{2,1}(x)$ of $f(x) = \sqrt{x}$ to approximate $\sqrt{1.1}$. Compare this approximation to the linear approximation.

EXERCISE 12

Use the quadratic approximation of $f(x) = \cos(x)$ at $a = 0$ to approximate $\cos(0.1)$.

4.2.4 Error Bound for the Linear Approximation

Whenever we approximate values, we would like to have some idea of how much error there is in our approximation. We now look at an error bound for the linear approximation.

In Example 4.2.2, we saw that the linear approximation gave a very good approximation. But, in Example 4.2.3, the approximation was not nearly as good. Why is one approximation more accurate than another?

The size of the error when the linear approximation is used depends on two different things:

First, it depends on the distance between where the linearization is centred and where the approximation is used, $|x-a|$. This is not too surprising since the farther we move away from where the linearization is centred, the more we expect the curve $y = f(x)$ to veer away from the tangent line $y = L_a(x)$.

This leads us to the second thing. It depends on how curvy the function is. How do we measure how curvy a function is? We use the second derivative! Therefore, the error also depends on how big the second derivative is between x and a.

The following theorem gives us a relatively easy way to approximate an upper bound on the error when using the linear approximation.

THEOREM 3

For a function f and its linearization L_a, we denote the error in the approximation of $f(x)$ by $L_a(x)$ by

$$R_a(x) = f(x) - L_a(x)$$

Then an upper bound for the error is given by

$$|R_a(x)| \leq \frac{M}{2}|x-a|^2$$

where M is the maximum value of $|f''(x)|$ between a and x.

REMARKS

1. A common mistake when using this theorem is to forget the absolute values signs around $f''(x)$ when calculating M. It is very important that you ensure that your M is positive.
2. We will see an extended version of this theorem, called Taylor's Remainder Theorem, in Chapter 7.

EXAMPLE 17

The linearization of $f(x) = \sqrt{x}$ at $a = 16$ is given by $L_{16}(x) = 4 + \frac{1}{8}(x-16)$. Using this linearization, we have $\sqrt{16.08} \approx L_{16}(16.08) \approx 4.01$. Determine an upper bound on the error of this approximation.

Solution: We first compute $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$.

Observe that since $|f''(x)| = \frac{1}{4x^{3/2}}$ is a decreasing function over the interval $[16, 16.08]$, it takes its maximum value at $x = 16$. The maximum value is

$$M = |f''(16)| = \frac{1}{4(16)^{3/2}} = \frac{1}{256}$$

Therefore, we have the following bound on the error in the approximation

$$|R_{16}(16.08)| \leq \frac{M}{2}|16.08-16|^2 = \frac{\left(\frac{1}{256}\right)}{2}(0.08)^2 = \frac{1}{80000} = 0.0000125$$

This means that the difference between the actual value of $\sqrt{16.08}$ and our approximate value of 4.01 is at most 0.0000125. For this problem, we can verify this result by using a calculator to compute $\left|\sqrt{16.08} - 4.01\right| \approx 0.00001247$.

EXAMPLE 18 Approximate $\frac{1}{0.98}$ using the linearization of $f(x) = \frac{1}{x}$ at $a = 1$, and find an upper bound for the error in the approximation.

Solution: We have $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$. Thus,

$$L_1(x) = f(1) + f'(1)(x-1) = 1 - (x-1)$$

Hence,

$$\frac{1}{0.98} \approx L_1(0.98) = 1 - (0.98 - 1) = 1 - (-0.02) = 1.02$$

Observe that since $|f''(x)| = \frac{2}{x^3}$ is a decreasing function over the interval $[0.98, 1]$, it takes is maximum at $x = 0.98$. The maximum value is

$$M = \frac{2}{(0.98)^3}$$

Therefore, we have the following bound on the error in the approximation

$$|R_1(0.98)| \leq \frac{M}{2}|0.98 - 1|^2 = \frac{\frac{2}{(0.98)^3}}{2}(0.02)^2 = \frac{(0.02)^2}{(0.98)^3}$$

EXAMPLE 19 Approximate $\sqrt[3]{9}$ using the linearization of $f(x) = \sqrt[3]{x}$ at $a = 8$, and find an upper bound for the error in the approximation.

Solution: We have $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = -\frac{2}{9}x^{-5/3}$. Thus,

$$L_8(x) = f(8) + f'(8)(x-8) = 82 + \frac{1}{12}(x-8)$$

Thus, we have that

$$\sqrt[3]{9} \approx L_8(9) = 2 + \frac{1}{12}(9-8) = \frac{25}{12}$$

Observe that since $|f''(x)| = \frac{2}{9}x^{-5/3}$ is a decreasing function, over the interval $[8, 9]$ it takes is maximum at $x = 8$. The maximum value is

$$M = \frac{2}{9(8)^{5/3}} = \frac{2}{9 \cdot 2^5} = \frac{1}{9 \cdot 16} = \frac{1}{144}$$

Therefore, we have the following bound on the error in the approximation

$$|R_8(9)| \leq \frac{M}{2}|9-8|^2 = \frac{\frac{1}{144}}{2}(1)^2 = \frac{1}{288}$$

EXERCISE 13

The linearization of $f(x) = x^{1/3}$ at $a = 1$ is $L_1(x) = 1+\frac{1}{3}(x-1)$. Use L_1 to approximate $(1.3)^{1/3}$ and then determine an upper bound on the error of this approximation.

EXERCISE 14

The gravitational potential energy of a mass m at a height h about the surface of the Earth is given by

$$E_g(h) = \frac{-mgR_E^2}{R_E + h}$$

where $R_E \approx 6.4 \times 10^6$ m is the radius of the Earth and $g \approx 10$ m/s^2 is the gravitational acceleration at the surface of the Earth. Recall that for calculations near the surface of the Earth, we can replace this with the linearization $L_0(h) = -mgR_E + mgh$. Use the linearization to determine the *change* in gravitational energy of a 1 kg mass raised to a height of 10 m above the surface of the Earth and then determine an upper bound on the error in this approximation.

Section 4.2 Problems

1. Find the linearization and second degree Taylor polynomial of each function at the given a.
   (a) $f(x) = x^2 + 3x - 2$, $a = 0$
   (b) $f(x) = x^2 + 3x - 2$, $a = 1$
   (c) $f(x) = \sin(x)$, $a = 0$
   (d) $f(x) = \cos(x)$, $a = 0$
   (e) $f(x) = \arctan(x)$, $a = 0$
   (f) $f(x) = \ln(x + 1)$, $a = 0$
   (g) $f(x) = (1 + x)^k$, $a = 0$ for any $k \in \mathbb{R}$
   (h) $f(x) = \ln(x)$, $a = 1$
   (i) $f(x) = \arcsin(x)$, $a = \frac{1}{2}$
   (j) $f(x) = \sqrt[3]{x + 1}$, $a = 7$
   (k) $f(x) = \sin(x)$, $a = \frac{\pi}{6}$
   (l) $f(x) = \tan(x)$, $a = \frac{\pi}{4}$

2. Apply the binomial approximation to each function.
   (a) $(1 + x)^{2/3}$
   (b) $(1 + x)^4$
   (c) $(1 + 3x)^{1/3}$
   (d) $(1 - 5x)^{1/5}$
   (e) $\sqrt{1 - x^2}$
   (f) $(2 + 3x)^5$
   (g) $(3 - x^2)^{1/4}$
   (h) $\frac{1}{\sqrt{x + 5}}$

3. Use the linear approximation and the quadratic approximation on the function at the given a to approximate the given value.
   (a) $f(x) = \sqrt{x}$, $a = 4$; $\sqrt{3.5}$
   (b) $f(x) = \sqrt[3]{x}$, $a = 1$; $\sqrt[3]{1.1}$
   (c) $f(x) = \sin(x)$, $a = 0$; $\sin(0.2)$
   (d) $f(x) = \cos(x)$, $a = 0$; $\cos(0.2)$
   (e) $f(x) = \sqrt[3]{1 + x^2}$, $a = 0$; $f(0.1)$

4. Find an upper bound for the error when the linearization of the given function at the given value of a is used to approximate the given value.
   (a) $f(x) = \sqrt{x}$, $a = 9$; $\sqrt{10}$
   (b) $f(x) = \frac{1}{x^2}$, $a = 1$; $\frac{1}{(1.1)^2}$
   (c) $f(x) = x^{2/3}$, $a = 1$; $(1.2)^{2/3}$
   (d) $f(x) = \cos(x)$, $a = 0$; $\cos(-0.2)$
   (e) $f(x) = \ln(2x + 1)$, $a = 1$; $\ln(2)$
   (f) $f(x) = \sqrt[4]{x}$; $a = 1$; $\sqrt[4]{1.2}$

5. Use the binomial approximation of the given function to approximate the given value.
   (a) $f(x) = (1 + x)^{2/3}$; $(1.1)^{2/3}$
   (b) $f(x) = (1 + x)^{1/4}$; $(1.01)^{1/4}$
   (c) $f(x) = (1 + x^2)^{1/4}$; $(1.01)^{1/4}$
   (d) $f(x) = \sqrt{2 + x}$; $\sqrt{1.9}$
   (e) $f(x) = \sqrt{2 - x}$; $\sqrt{1.9}$

End of Chapter Problems

1. Find the absolute maximum and minimum of f on the given closed interval.
   (a) $f(x) = x^3 - 3x^2 + 1$ on $[-1, 3]$
   (b) $f(x) = \cos(x) + 1$ on $[0, 2\pi]$
   (c) $f(x) = \dfrac{x^2 - 4x + 5}{x + 1}$ on $[1, 4]$
   (d) $f(x) = xe^{-x}$ on $[1, 3]$
   (e) $f(x) = x^3 - 6x^2 + 9x + 2$ on $[-1, 2]$
   (f) $f(x) = x^5 + 5x^3$ on $[-1, 1]$

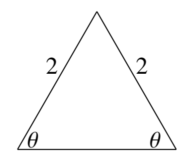
2. Find the linearization and second degree Taylor polynomial of each function at the given a.
   (a) $f(x) = x^3 + x^2 + x + 1$, $a = 0$
   (b) $f(x) = x^3 + x^2 + x + 1$, $a = 1$
   (c) $f(x) = \tan(x)$, $a = 0$
   (d) $f(x) = \arccos(x)$, $a = 0$
   (e) $f(x) = e^x$, $a = -1$
   (f) $f(x) = \ln(x^2 + 1)$, $a = 0$
   (g) $f(x) = \sec(x)$, $a = \dfrac{\pi}{4}$
   (h) $f(x) = \sqrt{x + 1}$, $a = 3$

3. Apply the binomial approximation to each function.
   (a) $(1 + x)^{2/5}$
   (b) $(1 + x)^7$
   (c) $(1 + 2x)^{2/3}$
   (d) $(1 - 3x)^{3/5}$
   (e) $\sqrt{1 - 3x^2}$
   (f) $(2 + x)^4$
   (g) $(3 - 2x)^{1/4}$

4. Use the linear approximation and the quadratic approximation on the function at the given a to approximate the given value.
   (a) $f(x) = \cos(x)$, $a = 0$; $\cos(-0.1)$
   (b) $f(x) = x^{2/3}$, $a = 1$; $(0.9)^{2/3}$
   (c) $f(x) = \dfrac{1}{\sqrt{1 - x}}$, $a = 0$; $\dfrac{1}{\sqrt{0.4}}$

5. Find an upper bound for the error when the linearization of the given function at the given value of a is used to approximate the given value.
   (a) $f(x) = \sqrt{x}$, $a = 4$; $\sqrt{4.1}$
   (b) $f(x) = x^{-1/3}$, $a = 1$; $(1.1)^{-1/3}$
   (c) $f(x) = \ln(\sec(x))$, $a = 0$; $\ln(\sec(0.1))$
   (d) $f(x) = (1 + x)^{2/3}$, $a = 0$; $(1.2)^{2/3}$

6. A tent is constructed using two 2 metre poles as shown in the diagram. What is the angle θ that will maximize the cross-sectional area in order to maximize the draft that goes through the tent?



7. A gardener wants to fence a rectangular plot of land with an area of 15 m^2 and then divide it in half with a fence parallel to one of the sides of the rectangle to create two side-by-side gardens. Let x represent the length of the fence that divides the plot in half. Find the value of x that minimizes the amount of fencing required.

Chapter 5: Integrals

In Section 1.2.1, we created definite integrals to find the net change of a quantity over an interval $[a, b]$ and the area under a curve. We saw how to evaluate a definite integral using the Fundamental Theorem of Calculus - Part 2. We also saw how to approximate a definite integral.

In this chapter, we will examine definite integrals a little more rigorously. Additionally, we will see how to set up definite integrals to calculate different kinds of quantities and our first technique of integration: integration by substitution.

Section 5.1: Integration by Substitution

LEARNING OUTCOMES

1. Know how to evaluate indefinite integrals using a substitution.
2. Know how to evaluate definite integrals using a substitution.

Through Chapters 1 and 2, you learned how to find the antiderivative of many basic functions. To try to find the antiderivative of more complicated functions, there are several techniques that we can try. For now, we look at the most commonly used technique, integration by substitution. In Chapter 8, we will look at more techniques.

You will likely find it very helpful to review your antiderivatives and the definition of differentials (see Section 1.1.4) before reading this section.

Integration by Substitution

The idea behind an integration by substitution is to perform a change of variables to convert the integrand into a function which we know the antiderivative of. This method is generally used when we have a composition of functions in the integrand, but can be used in other situations.

Let's begin by considering a couple of examples.

EXAMPLE 1

Find $\displaystyle\int \sin(2x)\, dx$.

Solution: Theoretically, this can be solved quite quickly by just thinking carefully about our derivative rules. However, our goal here is to figure out a general method that is going to work with more difficult integrals.

We identify that the integrand is a composition of functions: the inner function being $g(x) = 2x$ and the outer function being $f(x) = \sin(x)$.

Our goal is to perform a change of variables to turn the integrand into something that we immediately know the antiderivative of.

We define a new variable u to be the inner function. That is, we let

$$u = g(x) = 2x \tag{5.1}$$

To complete the change of variables, we also need to convert dx into terms of the new variable u. To do this we use differentials! That is, we use the formula

$$du = g'(x)\, dx$$

So, in this case, we get

$$du = 2\, dx$$

Solving for dx gives

$$\frac{1}{2}\, du = dx \tag{5.2}$$

Substituting equations (5.1) and (5.2) into the original integral gives

$$\int \sin(2x)\, dx = \int \sin(u) \cdot \frac{1}{2}\, du$$

We now have an integral that we know the antiderivative of. We get that

$$\int \frac{1}{2} \sin(u)\, du = -\frac{1}{2} \cos(u) + C$$

Since the original question was in terms of x, we must substitute back in for x using the substitution $u = 2x$. We get

$$\int \sin(2x)\, dx = -\frac{1}{2} \cos(2x) + C$$

EXAMPLE 2 Find $\int xe^{x^2}\, dx$.

Solution: We again identify that the integrand contains a composition of functions $f(g(x))$ with $g(x) = x^2$ and $f(x) = e^x$. We let u equal the inside function $g(x)$. That is, we let

$$u = x^2 \tag{5.3}$$

The differential is

$$du = 2x\, dx \tag{5.4}$$

Our substitution will convert e^{x^2} to e^u, but there will still be an $x\, dx$ to convert into terms of u. So, we rewrite equation (5.4) as

$$\frac{1}{2}\, du = x\, dx \tag{5.5}$$

Substituting equations (5.3) and (5.5) into the original integral gives

$$\begin{aligned}\int xe^{x^2}\,dx &= \int e^{x^2}\cdot x\,dx\\ &= \int e^u\cdot\frac{1}{2}\,du\\ &= \frac{1}{2}e^u + C\\ &= \frac{1}{2}e^{x^2} + C\end{aligned}$$

Let's look at several more examples to see how to use this in different situations.

EXAMPLE 3 Find $\int 2x^2\cos(x^3)\,dx$.

Solution: The integrand contains the composition of functions $f(g(x))$ where $f(x) = \cos(x)$ and $g(x) = x^3$. So, we let

$$u = x^3$$

and get that

$$du = 3x^2\,dx \tag{5.6}$$

After replacing x^3 with u, the integrand will still have an $2x^2\,dx$ that needs to be replaced. So, we rewrite equation (5.6) as

$$\frac{1}{3}\,du = x^2\,dx$$

and multiply both sides by 2 to get

$$\frac{2}{3}\,du = 2x^2\,dx$$

Therefore,

$$\begin{aligned}\int 2x^2\cos(x^3)\,dx &= \int \cos(x^3)\cdot 2x^2\,dx\\ &= \int \cos(u)\cdot\frac{2}{3}\,du\\ &= \frac{2}{3}\sin(u) + C\\ &= \frac{2}{3}\sin(x^3) + C\end{aligned}$$

EXAMPLE 4

Find $\displaystyle\int \frac{6x+3}{x^2+x}\,dx$.

Solution: Let $u = x^2 + x$. Then, $du = (2x+1)\,dx$.

Thus,

$$\begin{aligned}\int \frac{6x+3}{x^2+x}\,dx &= \int \frac{3(2x+1)}{x^2+x}\,dx \\ &= \int \frac{3}{u}\,du \\ &= 3\ln(|u|) + C \\ &= 3\ln(|x^2+x|) + C\end{aligned}$$

EXAMPLE 5

Evaluate $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$.

Solution: Let $u = 1 + x^2$. Then, $du = 2x\,dx$, so $\frac{1}{2}\,du = x\,dx$.

Thus,

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2}\,du \\ &= \int \frac{1}{2}u^{-1/2}\,du \\ &= u^{1/2} + C \\ &= \sqrt{1+x^2} + C\end{aligned}$$

EXERCISE 1

Evaluate the following indefinite integrals.

(a) $\displaystyle\int (3-5x)^7\,dx$

(b) $\displaystyle\int \sin^3(x)\cos(x)\,dx$

(c) $\displaystyle\int \frac{x}{3x^2-4}\,dx$

REMARK

Remember that you can always check your answer by taking the derivative and ensuring that you get back to the original integrand.

In all of the cases above, the integrand always had the form $cf(g(x))g'(x)$ which made substituting for x fairly easy. However, this is not always going to be the case.

EXAMPLE 6 Find $\int x^3(x^2+1)^5\,dx$.

Solution: We have a composition of functions $f(g(x))$ where $f(x) = x^5$ and $g(x) = x^2 + 1$.

So, we let $u = x^2 + 1$ and get $du = 2x\,dx$. Hence, $\frac{1}{2}\,du = x\,dx$.

We can now replace the $(x^2+1)^5$ in the integrand with u^5. However, we don't yet have a replacement for the x^3.

The trick in this case is to write x^3 as $x \cdot x^2$. We can use the equation $\frac{1}{2}\,du = x\,dx$ to replace the x, and we can solve the initial substitution, $u = x^2 + 1$, for x^2 to get $x^2 = u - 1$.

We now use all three of these substitutions to convert everything into terms of u.

$$\begin{aligned}
\int x^3(x^2+1)^5\,dx &= \int x^2 \cdot (x^2+1)^5 \cdot x\,dx \\
&= \int (u-1)\cdot u^5 \cdot \frac{1}{2}\,du \\
&= \int \frac{1}{2}(u^6 - u^5)\,du \\
&= \frac{1}{2}\left(\frac{1}{7}u^7 - \frac{1}{6}u^6\right) + C \\
&= \frac{1}{14}(x^2+1)^7 - \frac{1}{12}(x^2+1)^6 + C
\end{aligned}$$

EXERCISE 2 Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{x^3}{x^2+1}\,dx$

(b) $\displaystyle\int x^5(x^2-1)^4\,dx$

(c) $\displaystyle\int \sin^3(x)\cos^2(x)\,dx$ [Hint: Let $u = \cos(x)$ and use a trigonometric identity.]

EXAMPLE 7 $\displaystyle\int \tan^3(x)\,dx$

Solution: The natural thing to try is $u = \tan(x)$. This gives $du = \sec^2(x)\,dx$. But, we don't have a $\sec^2(x)$ term to replace! In this case, we can use the trigonometric identity

$$\tan^2(x) = \sec^2(x) - 1$$

to introduce a $\sec^2(x)$. We get

$$\begin{aligned}\int \tan^3(x)\,dx &= \int \tan(x)\tan^2(x)\,dx\\ &= \int \tan(x)(\sec^2(x)-1)\,dx\\ &= \int \big(\tan(x)\sec^2(x) - \tan(x)\big)\,dx\end{aligned}$$

We see that the substitution doesn't work for the entire integral. So, we can split this into two integrals.

$$\int \tan^3(x)\,dx = \int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx$$

We know the answer to the second integral, so we just need to do the first.

We get

$$\begin{aligned}\int \tan(x)\sec^2(x)\,dx &= \int u\,du\\ &= \frac{1}{2}u^2 + C\\ &= \frac{1}{2}\tan^2(x) + C\end{aligned}$$

Thus,

$$\int \tan^3(x)\,dx = \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C$$

Change of Variables in Definite Integrals

As with all of the techniques of integration that we will cover in this text, when evaluating a definite integral, we can always first find the indefinite integral and then use the Fundamental Theorem of Calculus.

Alternately, when doing a change of variables, we can also change the bounds on the definite integral by converting the bounds into terms of u as well. In particular, if we are integrating x over the interval $[a, b]$, then performing the change of variables $u = g(x)$ will mean that for the new definite integral we will be integrating u over the interval $[g(a), g(b)]$.

EXAMPLE 8

Evaluate $\displaystyle\int_1^2 \frac{(\ln(x))^2}{x}\, dx$.

Solution: Let $u = \ln(x)$. Then, we get $du = \dfrac{1}{x}\, dx$. In the original integral we are integrating x over $[1, 2]$. Performing $u = \ln(x)$ means that we are now integrating u over $[\ln(1), \ln(2)] = [0, \ln(2)]$. Thus, we get

$$\begin{aligned}
\int_1^2 \frac{(\ln(x))^2}{x}\, dx &= \int_0^{\ln(2)} u^2\, du \\
&= \frac{1}{3}u^3\Big|_0^{\ln(2)} \\
&= \frac{1}{3}(\ln(2))^3 - \frac{1}{3}(0)^3 \\
&= \frac{1}{3}(\ln(2))^3
\end{aligned}$$

EXAMPLE 9

Evaluate $\displaystyle\int_{\pi/4}^{\pi/2} \sin(x)\sqrt{\cos(x)}\, dx$.

Solution: Let $u = \cos(x)$. Then, $du = -\sin(x)\, dx$ and so $(-1)\, du = \sin(x)\, dx$.

When $x = \dfrac{\pi}{4}$, $u = \cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$, and when $x = \dfrac{\pi}{2}$, $u = \cos\left(\dfrac{\pi}{2}\right) = 0$. Thus, the interval of integration with respect to u is $\left[\dfrac{1}{\sqrt{2}}, 0\right]$.

Thus,

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \sin(x)\sqrt{\cos(x)}\, dx &= \int_{1/\sqrt{2}}^{0} \sqrt{u}\cdot(-1)\, du \\
&= -\frac{2}{3}u^{3/2}\Big|_{1/\sqrt{2}}^{0} \\
&= \left(-\frac{2}{3}(0)^{3/2}\right) - \left(-\frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^{3/2}\right) \\
&= \frac{2^{1/4}}{3}
\end{aligned}$$

REMARK

Observe in the last example that the interval of integration after the substitution was ’backwards’. That is, the left end-point was actually larger than the right end-point. This is okay. The answer we got still made sense (we got a positive net change as we would expect since the rate of change $\sin(x)\sqrt{\cos(x)}$ is positive on the interval).

EXERCISE 3 Evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 x(3x^2-2)^4\,dx$

(b) $\displaystyle\int_1^2 \frac{1}{x^2-6x+9}\,dx$

(c) $\displaystyle\int_0^2 \frac{x^2}{(x+1)^3}\,dx$

REMARK

It is important to note that a change of variables is not going to simplify all integrals and, in fact, in many cases it can even be unhelpful when we have a composition of functions. Recognizing when to use an integration by substitution as opposed to other integration techniques is learned from practice.

Section 5.1 Problems

1. Evaluate the following integrals. Check your answer using differentiation.

   (a) $\displaystyle\int 2x(1+x^2)^5\,dx$

   (b) $\displaystyle\int (11x-3)^{-3}\,dx$

   (c) $\displaystyle\int (5-3x)^4\,dx$

   (d) $\displaystyle\int \cos^3(\theta)\sin(\theta)\,d\theta$

   (e) $\displaystyle\int te^{t^2}\,dt$

   (f) $\displaystyle\int \frac{\ln(x)}{x}\,dx$

   (g) $\displaystyle\int \frac{1}{(2x-3)^4}\,dx$

   (h) $\displaystyle\int \frac{x}{1+x^2}\,dx$

   (i) $\displaystyle\int \frac{1}{1+4x^2}\,dx$

2. Evaluate the following integrals.

   (a) $\displaystyle\int (x+1)^4\,dx$

   (b) $\displaystyle\int \frac{x^2+1}{\sqrt{x}}\,dx$

   (c) $\displaystyle\int \frac{3x}{1-x^2}\,dx$

   (d) $\displaystyle\int \frac{x^2}{1+x^2}\,dx$

   (e) $\displaystyle\int \sin^5(\theta)\cos(\theta)\,d\theta$

   (f) $\displaystyle\int \frac{x^2}{x^3-2}\,dx$

   (g) $\displaystyle\int \frac{\sqrt{x}+\sqrt[3]{x}}{x}\,dx$

   (h) $\displaystyle\int (x-1)(x^2-2x)^3\,dx$

   (i) $\displaystyle\int (x^2-2x)(x^3-3x^2)^2\,dx$

   (j) $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$

   (k) $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

3. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{\arcsin(t)}{\sqrt{1-t^2}}\,dt$

   (b) $\displaystyle\int \frac{1}{\sqrt{1-9x^2}}\,dx$

   (c) $\displaystyle\int \frac{e^{\ln(x)}}{x}\,dx$

   (d) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$

   (e) $\displaystyle\int \frac{\ln(\sin(x))}{\tan(x)}\,dx$

   (f) $\displaystyle\int xe^{-x^2}\,dx$

   (g) $\displaystyle\int x\sec(x^2)\tan(x^2)\,dx$

   (h) $\displaystyle\int \frac{\arctan(2x)}{1+4x^2}\,dx$

   (i) $\displaystyle\int \frac{e^t}{1+e^{2t}}\,dt$

4. Evaluate the following integrals.

   (a) $\displaystyle\int_0^1 (2x+1)^5\,dx$

   (b) $\displaystyle\int_1^2 e^{1-x}\,dx$

   (c) $\displaystyle\int_4^5 \frac{1}{(x-3)^5}\,dx$

   (d) $\displaystyle\int_0^3 \frac{1}{9+x^2}\,dx$

   (e) $\displaystyle\int_0^1 x^2e^{-x^3}\,dx$

   (f) $\displaystyle\int_2^e \frac{1}{x(\ln(x))^2}\,dx$

   (g) $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$

   (h) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

   (i) $\displaystyle\int_0^{\pi/4} \frac{\sin(\theta)}{\cos^4(\theta)}\,d\theta$

Section 5.2: Riemann Sums

LEARNING OUTCOMES

1. Know how to use and evaluate sigma notation.
2. Know how to set up a right Riemann sum.
3. Know how to evaluate a definite integral using the definition.
4. Know how to use the properties of integrals.

Our goal is to make a more precise definition of the definite integral. As we will see, this is required in science so that we can formulate definite integrals to compute desired quantities.

Since a definite integral $\displaystyle\int_a^b f(x)\,dx$ represents the sum of infinitely many infinitely small quantities, we first look at notation that makes writing sums more compact.

5.2.1 Sigma Notation

Say we want to write the sum $3+5+7+9+11+13+15+17+19+21+23+25+27+29$ more compactly.

We might be tempted to write $3+5+7+\cdots+29$. But, someone might think you are adding odd primes instead of all odd numbers. Consequently, having convenient and precise notation is important.

NOTATION

We write $\displaystyle\sum_{i=j}^{n} f(i)$ to indicate that we are adding up the values values of $f(i)$ for all integer values of i starting at j and going up to and including n. That is,

$$\sum_{i=j}^{n} f(i) = f(j) + f(j+1) + f(j+2) + \cdots + f(n)$$

Note that we don't have to use the letter i as the index. For example,

$$\sum_{n=2}^{5} n^2 = 2^2 + 3^2 + 4^2 + 5^2$$

or

$$\sum_{k=1}^{n} \frac{1}{k+2} = \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+2}$$

EXAMPLE 1 Evaluate the following sums.

(a) $\displaystyle\sum_{i=0}^{5} i$

(b) $\displaystyle\sum_{i=3}^{3} i^2$

(c) $\displaystyle\sum_{i=1}^{4} 5$

(d) $\displaystyle\sum_{i=1}^{n} g(x_i)\Delta x$

Solution: (a) The function is $f(i) = i$. We start at $i = 0$ and go up by 1 until we reach $i = 5$. So, we get

$$\begin{aligned}\sum_{i=0}^{5} i &= f(0) + f(1) + f(2) + f(3) + f(4) + f(5)\\ &= 0 + 1 + 2 + 3 + 4 + 5\\ &= 15\end{aligned}$$

(b) The function is $f(i) = i^2$. We start at $i = 3$ and go up to $i = 3$. So, we get

$$\sum_{i=3}^{3} i^2 = f(3) = 3^2 = 9$$

(c) The function is $f(i) = 5$. We start at $i = 1$ and go up to $i = 4$. So, we get

$$\begin{aligned}\sum_{i=1}^{4} 5 &= f(1) + f(2) + f(3) + f(4)\\ &= 5 + 5 + 5 + 5\\ &= 20\end{aligned}$$

(d) The function is $f(i) = g(x_i)\Delta x$. We start at $i = 0$ and go up to $i = n - 1$. So, we get

$$\begin{aligned}\sum_{i=1}^{n} g(x_i)\Delta x &= f(1) + f(2) + \cdots + f(n-1) + f(n)\\ &= g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_{n-1})\Delta x + g(x_n)\Delta x\end{aligned}$$

EXAMPLE 2 Write the sum in sigma notation.

(a) $3 + 5 + 7 + \cdots + 29$

(b) $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

(c) $1 + x + x^2 + x^3 + x^4 + x^5$

(d) $(x_1)^2\Delta x + (x_2)^2\Delta x + \cdots + (x_{n-1})^2\Delta x + (x_n)^2\Delta x$

Solution: (a) $\sum_{i=1}^{14}(2i + 1)$

(b) $\sum_{i=5}^{8} \frac{1}{i}$

(c) $\sum_{i=0}^{5} x^i$

(d) $\sum_{i=1}^{n}(x_i)^2\Delta x$

REMARK

The answers in the example above are not the only possibilities. There are, in fact, infinitely many different correct answers for each one. For example, (b) could also be written as $\sum_{i=1}^{4} \frac{1}{i+4}$.

EXERCISE 1 Write the sum in sigma notation.

(a) $\sqrt{3} + \sqrt{4} + \sqrt{5}$

(b) $1 + 2 + 4 + 8 + 16$

(c) $1 - x + x^2 - x^3 + x^4$

(d) $g(x_1)\Delta x_1 + g(x_2)\Delta x_2 + g(x_3)\Delta x_3$

Since sigma notation is just adding a finite number of values, we get the following two properties.

THEOREM 1

Summation Properties

(1) $\displaystyle\sum_{i=m}^{n} cf(i) = c\sum_{i=m}^{n} f(i)$ for any real number c

(2) $\displaystyle\sum_{i=m}^{n} (f(i) + g(i)) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$

We also have the following formulas.

Summation Formulas:

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

EXAMPLE 3

Evaluate $\displaystyle\sum_{i=1}^{100} i$.

Solution: We use the second formula above with $n = 100$ to get

$$\sum_{i=1}^{100} i = \frac{100(100+1)}{2} = 50(101) = 5050$$

EXAMPLE 4

Evaluate the sum $\displaystyle\sum_{i=1}^{12} 3$.

Solution: Using the first formula above with $n = 12$ we get

$$\sum_{i=1}^{12} 3 = 3\sum_{i=1}^{12} 1 = 3(12) = 36$$

EXAMPLE 5 Evaluate the sum $\displaystyle\sum_{i=1}^{5}(-1)^i$.

Solution: None of the formulas apply for this one, but we can just write out the sum. We get

$$\sum_{i=1}^{5}(-1)^i = (-1) + 1 + (-1) + 1 + (-1) = -1$$

EXAMPLE 6 Evaluate the sum $\displaystyle\sum_{i=5}^{10} i^2$.

Solution: We *cannot* just use the third formula! The summation formulas above *only* work if we are starting at $i = 1$. Thus, we need to use property (2) in Theorem 5.2.1 to write this as the sum of the first 10 terms minus the sum of the first 4 terms.

$$\begin{aligned}\sum_{i=5}^{10} i^2 &= 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 \\ &= 1^2 + 2^2 + 3^2 + \cdots + 10^2 - \left(1^2 + 2^2 + 3^2 + 4^2\right) \\ &= \sum_{i=1}^{10} i^2 - \sum_{i=1}^{4} i^2\end{aligned}$$

Now we can use the third formula with $n = 10$ and $n = 4$ respectively.

$$\begin{aligned}\sum_{i=5}^{10} i^2 &= \frac{10(11)(21)}{6} - \frac{4(5)(9)}{6} \\ &= 5(11)7 - 2(5)(3) \\ &= 355\end{aligned}$$

EXERCISE 2 Determine the value of the sum.

(a) $\displaystyle\sum_{n=1}^{6}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

(b) $\displaystyle\sum_{n=1}^{4} 2n$

(c) $\displaystyle\sum_{i=1}^{5} i^2$

(d) $\displaystyle\sum_{i=1}^{4}(i^2 - i)$

There are cases in mathematics where we want to sum an infinite number of numbers. Of course, we cannot actually add up an infinite number of numbers. So, we use a limit!

EXAMPLE 7

Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$.

Solution: We have

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} &= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \qquad \text{since } n \text{ is constant as far as the sum is concerned}\\ &= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\\ &= \lim_{n\to\infty} \frac{1(1+1/n)(2+1/n)}{6}\\ &= \frac{1}{3}\end{aligned}$$

EXERCISE 3

Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2}$.

5.2.2 Riemann Sums

With sigma notation, we can now look at the definition of the definite integral more rigorously.

Given the rate of change $f(x) = F'(x)$ of a quantity $F(x)$, we want to create a definite integral that calculates the net change in F over an interval $[a, b]$. To do this, we repeat what we did back in Section 1.2.1.

We first divide the interval $[a, b]$ into n equal subintervals with width

$$\Delta x = \frac{\text{length of interval}}{\text{number of subintervals}} = \frac{b-a}{n}$$

Next, we label the x-coordinates of these subintervals by x_i where $x_0 = a$ is the beginning of the interval and $x_n = b$ is the end of the interval. That is,

$$x_i = a + i\Delta x, \quad 0 \le i \le n$$

Now, we approximate the net change over each sub-interval using the formula

$$\Delta y \approx F'(x)\Delta x$$

Since $F'(x) = f(x)$, this gives that the net change over the i-th sub-interval $[x_{i-1}, x_i]$ is

$$\Delta y_i \approx f(x_i)\Delta x$$

Summing all these approximations we get that the net change in F is approximately

$$\Delta F \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

REMARK

We have made a subtle change here from what we did early in the text. We have switched from using the right end point of the interval to get the approximation to the left end point of the interval. We have done this so that the sum is from $i = 1$ to $i = n$ rather than $i = 0$ to $i = n - 1$. We have done this to make it easier to use the summation formulas.

DEFINITION

Riemann Sum

For a function f that is continuous on a closed interval $[a, b]$, we define

$$\Delta x = \frac{b-a}{n}$$

and

$$x_i = a + i\Delta x, \qquad 0 \le i \le n$$

The sum

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

is called the **right Riemann sum** of f on $[a, b]$.

EXAMPLE 8

Set up the right Riemann sum R_n with n subdivisions which approximates the area under the graph of $f(x) = x$ over the interval $[-1, 1]$.

Solution: We have

$$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = -1 + \frac{2i}{n}, \quad 0 \le i \le n$$

Thus, the right Riemann sum is

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \left(-1 + \frac{2i}{n}\right)\frac{2}{n} \end{aligned}$$

EXAMPLE 9

Set up the right Riemann sum R_n with n subdivisions which approximates the net change of a quantity F with has rate of change $F'(t) = e^{t^2}$ over the interval $[0, 3]$.

Solution: We have

$$\Delta t = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$
$$t_i = a + i\Delta t = \frac{3i}{n}$$

Thus, the right Riemann sum is

$$R_n = \sum_{i=1}^{n} f(t_i)\Delta t$$
$$= \sum_{i=1}^{n} e^{(3i/n)^2} \frac{3}{n}$$

EXERCISE 4

Set up the right Riemann sum R_n with n subdivisions which approximates the area under the graph of $f(x) = x^2$ over the interval $[1, 3]$.

EXERCISE 5

Set up the right Riemann sum R_n with n subdivisions which approximates the net change of a quantity F with has rate of change $F'(t) = \sin(t^2)$ over the interval $[0, \pi]$.

By construction, a right Riemann sum is an approximation of the net change. In general, taking more subdivisions (a larger value of n), will give a more accurate approximation. If we take infinitely many subdivisions, then we will get the exact net change. This gives us a more formal definition of the definite integral.

DEFINITION

Definite Integral

For a function f that is continuous on a closed interval $[a, b]$, we define

$$\Delta x = \frac{b-a}{n}$$

and

$$x_i = a + i\Delta x, \qquad 0 \le i \le n$$

Then, the definite integral is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

To evaluate a definite integral using the definition, we simplify the expression $f(x_i)\Delta x$ using the definition of x_i and Δx, and then evaluate the limit of the sum just like we did in Example 5.2.7.

EXAMPLE 10 Evaluate $\int_0^2 x^2\,dx$ using the definition of the definite integral.

Solution: We have

$$\begin{aligned}
\Delta x &= \frac{2-0}{n} = \frac{2}{n} \\
x_i &= a + i\Delta x = \frac{2i}{n}, \quad 0 \le i \le n
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^2 x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} (x_i)^2 \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^2}{n^3} \\
&= \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} \frac{8}{6} \cdot \frac{n(n+1)(2n+1)}{n^3} \\
&= \lim_{n\to\infty} \frac{4}{3} \cdot \frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} \\
&= \lim_{n\to\infty} \frac{4}{3} \cdot 1 \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right) \\
&= 8 \cdot \frac{(1+0)(2+0)}{6} \\
&= \frac{8}{3}
\end{aligned}$$

EXAMPLE 11 Evaluate $\int_1^4 (x+3)\,dx$ using the definition of the definite integral.

Solution: We have

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$
$$x_i = a + i\Delta x = 1 + \frac{3i}{n}, \quad 0 \le i \le n$$

Thus,

$$\begin{aligned}
\int_1^4 (x+3)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} (x_i + 3)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1 + \frac{3i}{n}\right) + 3\right) \cdot \frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 + \frac{3i}{n}\right) \cdot \frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{12}{n} + \frac{9i}{n^2}\right) \\
&= \lim_{n\to\infty} \left[\frac{12}{n} \sum_{i=1}^{n} i + \frac{9}{n^2} \sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \left[\frac{12}{n} \cdot n + \frac{9}{n^2} \cdot \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[12 + \frac{9}{2}\left(1 + \frac{1}{n}\right)\right] \\
&= 12 + \frac{9}{2}(1+0) \\
&= \frac{33}{2}
\end{aligned}$$

EXERCISE 6 Evaluate $\int_1^3 5x\,dx$ using the definition of the definite integral.

We rarely evaluate a definite integral using the definition. However, right Riemann sums with a finite number of subdivisions can be used to approximate definite integrals. Thus, these are important to understand.

On the other hand, as we will see in Section 5.4, it is not uncommon to need to convert a given right Riemann sum into a definite integral.

EXAMPLE 12 Write $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^2 + x_i)\Delta x$ as a definite integral on the interval $[0,2]$.

Solution: The definition says that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

So, $f(x) = x^2 + x$. We are also given in the question that $a = 0$ and $b = 2$. Thus, we have that

$$\lim_{n\to\infty}\sum_{n=1}^{n}(x_i^2 + x_i)\Delta x = \int_0^2 (x^2 + x)\,dx$$

EXAMPLE 13 Write

$$\lim_{n\to\infty}\sum_{i=1}^{n}(2\cos(x_i) + e^{x_i})\Delta x$$

as a definite integral on the interval $[-\pi,\pi]$.

Solution: We have $\lim_{n\to\infty}\sum_{i=1}^{n}(2\cos(x_i) + e^{x_i})\Delta x = \int_{-\pi}^{\pi}(2\cos(x) + e^x)\,dx$.

EXERCISE 7 Write

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i + \ln(x_i))\Delta x$$

as a definite integral on the interval $[3,5]$.

5.2.3 Properties of the Definite Integral

We now look at some useful properties of the definite integral that follow from our definition.

Since the definite integral is a limit of a sum, we get by the limit laws and Theorem 5.2.1 the following theorem.

THEOREM 2 If $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist, then

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad \text{for any real number } c$$

$$\int_a^b \left(f(x) + g(x)\right)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

We also get the following useful properties.

THEOREM 3 If $\int_a^b f(x)\,dx$ exists, then

(1) $\int_a^a f(x)\,dx = 0$

(2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(3) If $f(x) < 0$ for all $a < x < b$, then $\int_a^b f(x)\,dx < 0$.

(4) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Proof: (1) We have $\Delta x = \dfrac{a-a}{n} = 0$. Thus,

$$\begin{aligned}
\int_a^a f(x)\,dx &= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\cdot 0\\
&= 0
\end{aligned}$$

(2) For $\int_b^a f(x)\,dx$ we have $\Delta x = \dfrac{a-b}{n} = -\dfrac{b-a}{n}$. Thus,

$$\begin{aligned}
-\int_b^a f(x)\,dx &= -\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x\\
&= -\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\left(-\frac{b-a}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\frac{b-a}{n}\\
&= \int_a^b f(x)\,dx
\end{aligned}$$

(3) If $f(x) < 0$ and $\Delta x = \dfrac{b-a}{n} > 0$, then $f(x_i)\Delta x < 0$. Hence,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x < 0$$

(4) For $\int_a^c f(x)\,dx$ we have $\Delta x = \dfrac{c-a}{n}$.

Using the property of fractions $\dfrac{x+y}{z} = \dfrac{x}{z} + \dfrac{y}{z}$, we can write Δx as

$$\Delta x = \frac{c-a}{n} = \frac{c-b+b-a}{n} = \frac{c-b}{n} + \frac{b-a}{n}$$

Thus,

$$\begin{aligned}
\int_a^c f(x)\,dx &= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\left(\frac{c-b}{n} + \frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\frac{c-b}{n} + \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\frac{b-a}{n} \\
&= \int_b^c f(x)\,dx + \int_a^b f(x)\,dx
\end{aligned}$$

□

EXAMPLE 14 If $\int_1^3 f(x)\,dx = 4$ and $\int_3^5 f(x)\,dx = -2$, what is $\int_1^5 f(x)\,dx$?

Solution: By property (4), we get

$$\begin{aligned}
\int_1^5 f(x)\,dx &= \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx \\
&= 4 + (-2) \\
&= 2
\end{aligned}$$

EXAMPLE 15 If $\int_1^3 f(x)\,dx = 4$ and $\int_5^1 f(x)\,dx = 2$, what is $\int_3^5 f(x)\,dx$?

Solution: Property (4) tells us that

$$\int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = \int_1^5 f(x)\,dx$$

By property (2), we get that

$$\int_1^5 f(x)\,dx = -\int_5^1 f(x)\,dx = -2$$

Thus,

$$\begin{aligned}\int_3^5 f(x)\,dx &= \int_1^5 f(x)\,dx - \int_1^3 f(x)\,dx \\ &= (-2) - 4 \\ &= -6\end{aligned}$$

EXAMPLE 16 If $\int_a^b f(x)\,dx = 5$ and $\int_a^b g(x)\,dx = 7$, what is $\int_a^b (2f(x) - g(x))\ dx$?

Solution: We have

$$\begin{aligned}\int_a^b (2f(x) - g(x))\ dx &= \int_a^b 2f(x)\,dx + \int_a^b -g(x)\,dx \\ &= 2\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= 2(5) - 7 \\ &= 3\end{aligned}$$

EXERCISE 8 Assume $\int_1^4 f(t)\,dt = -2$, $\int_0^4 f(t)\,dt = 5$, and $\int_0^4 g(t)\,dt = 7$.

Evaluate the following integrals.

(a) $\int_0^1 f(t)\,dt$

(b) $\int_0^4 (2f(t) - 3g(t))\,dt$

(c) $\int_4^1 f(t)\,dt$

Section 5.2 Problems

1. Evaluate the sum.
   (a) $\sum_{i=1}^{8} 1$
   (b) $\sum_{i=1}^{23} 1$
   (c) $\sum_{i=1}^{15} 3i$
   (d) $\sum_{i=1}^{12} i$
   (e) $\sum_{i=1}^{10} i^2$
   (f) $\sum_{i=3}^{10} i^2$
   (g) $\sum_{i=1}^{8} (i^2 - 2i)$
   (h) $\sum_{i=1}^{6} n$
   (i) $\sum_{i=1}^{10} \frac{i}{n}$
2. Write the sum in sigma notation.
   (a) $3 + 5 + 7 + 9 + 11$
   (b) $1 + 1 + 2 + 6 + 24$
   (c) $1 + 2x + 3x^2 + 4x^3 + 5x^4$
   (d) $2 + 6 + 24 + 120$
   (e) $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$
   (f) $f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x$
   (g) $(x_1^2 + 3x_1)\Delta x + (x_2^2 + 3x_2)\Delta x + (x_3^2 + 3x_3)\Delta x$
3. Evaluate the following
   (a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}$
   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i}{3n^2}$
   (c) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n^2} - \frac{i^2}{n^3}\right)$
4. Write $\lim_{n\to\infty} \sum_{i=1}^{n} (2x_i^3 - 3x_i^2)\Delta x$ as a definite integral on the interval $[-1, 3]$.
5. Write $\lim_{n\to\infty} \sum_{i=1}^{n} \sin(2x_i)\Delta x$ as a definite integral on the interval $[0, \pi]$.
6. Set up a right Riemann sum with n subdivisions which approximates the net change of F which has the given rate of change $F'(t) = f(t)$ over the specified interval.
   (a) $f(t) = t^2 + 1,\ 2 \le t \le 7$
   (b) $f(t) = \sin(t^2),\ 0 \le t \le \pi$
   (c) $f(t) = \frac{t}{1 + t^2},\ -3 \le t \le -1$
7. Assume $\int_0^2 f(x)\,dx = 5$ and $\int_0^4 f(x)\,dx = -2$. Evaluate the following.
   (a) $\int_2^0 f(x)\,dx$
   (b) $\int_2^4 f(x)\,dx$
   (c) $\int_3^3 f(x)\,dx$
   (d) $\int_0^2 3f(x)\,dx$
8. Assume $\int_{-2}^3 g(t)\,dt = 1$ and $\int_3^5 g(t)\,dt = 2$. Evaluate the following.
   (a) $\int_{-2}^5 g(t)\,dt$
   (b) $\int_1^1 g(t)\,dt$
   (c) $\int_5^{-2} g(t)\,dt$
   (d) $\frac{d}{dx}\int_3^5 g(t)\,dt$
9. Find $\int_0^3 g(x)\,dx$ given that $\int_0^3 f(x)\,dx = 7$ and $\int_0^3 (2f(x) - g(x))\,dx = 4$.

10. Assume $\int_0^4 f(x)\,dx = 5$, $\int_0^2 f(x)\,dx = -3$, $\int_0^4 g(x)\,dx = -1$, and $\int_0^2 g(x)\,dx = 2$. Evaluate the following.

    (a) $\int_0^4 (f(x) + g(x))\,dx$

    (b) $\int_2^0 (f(x) + g(x))\,dx$

    (c) $\int_0^2 (f(x) - g(x))\,dx$

    (d) $\int_2^4 (f(x) + g(x))\,dx$

    (e) $\int_0^2 (3f(x) - 2g(x))\,dx$

    (f) $\int_2^2 (f(x) + g(x))^3\,dx$

11. Explain why if $f(x) \geq g(x)$ for all $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

12. Evaluate the definite integral using the definition of the definite integral.

    (a) $\int_0^2 3x\,dx$

    (b) $\int_{-1}^4 (2x + 1)\,dx$

    (c) $\int_0^4 2x^2\,dx$

    (d) $\int_{-1}^2 x^2\,dx$

Section 5.3: Area Between Curves

LEARNING OUTCOMES

1. Know how to use a right Riemann sum to set up an integral to calculate the area between two curves.

We now look at how to use a right Riemann sum to set up an integral for the area between curves.

EXAMPLE 1 Use a right Riemann sum R_n to create an integral that calculates the area between the graphs of $h(x) = -x^2 + 1$ and $g(x) = -x - 1$.

Solution: We first determine where the curves intersect. We have

$$\begin{aligned} -x^2 + 1 &= -x - 1 \\ 0 &= x^2 - x - 2 \\ 0 &= (x+1)(x-2) \end{aligned}$$

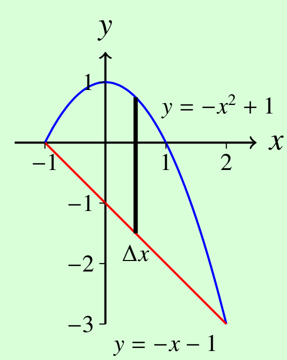
Thus, they intersect when $x = -1$ and $x = 2$. So, the interval is $[-1, 2]$.

As usual, we subdivide the interval into n equal pieces with width Δx and end points

$$x_i = -1 + i\Delta x, \qquad 0 \leq i \leq n$$

Since we are calculating area, the function f will be the height of the i-th rectangle.

To help us visualize this, we graph the functions on the interval and draw a thick black line to represent the i-th rectangle in the Riemann sum. From the figure, we see that the height is



$$f(x_i) = h(x_i) - g(x_i) = (-x_i^2 + 1) - (-x_i - 1) = -x_i^2 + x_i + 2$$

Thus, the Riemann sum is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} (-x_i^2 + x_i + 2)\Delta x$$

Therefore, the area between the curves is

$$\text{Area} = \int_{-1}^{2} (-x^2 + x + 2)\, dx$$

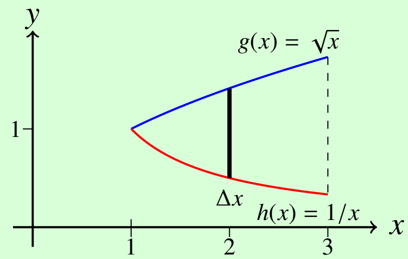
EXAMPLE 2

Use a right Riemann sum R_n to create an integral that calculates the area between the graphs of $g(x) = \sqrt{x}$ and $h(x) = \dfrac{1}{x}$ for $1 \le x \le 3$.

Solution: We subdivide the interval interval into n equal pieces with width Δx and end points

$$x_i = 1 + i\Delta x, \qquad 0 \le i \le n$$

Next, we sketch the curves and draw a thick black line to represent the i-th rectangle.



From the figure, we see that the height of the i-th rectangle is

$$f(x_i) = g(x_i) - h(x_i) = \sqrt{x_i} - \frac{1}{x_i}$$

Thus, the Riemann sum is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\sqrt{x_i} - \frac{1}{x_i} \right) \Delta x$$

Therefore, the area between the curves is

$$\text{Area} = \int_1^3 \left(\sqrt{x} - \frac{1}{x} \right) dx$$

EXERCISE 1

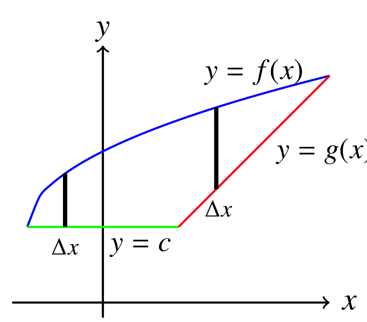
Use a right Riemann sum R_n to create an integral that calculates the area between the graphs of $g(x) = x^2$ and $h(x) = x$.

EXERCISE 2

Use a right Riemann sum R_n to create an integral that calculates the area between the graphs of $g(x) = \cos(x)$ and $h(x) = \sin(x)$ for $0 \le x \le \dfrac{\pi}{4}$.

When drawing the sample rectangle in the method above, it is necessary that the sample rectangle represents all possible rectangles. That is, every possible rectangle must have the same height function.

So, for example, in the figure to the right, we cannot use the method above as the two sample rectangles would have different height functions. Namely, the leftmost rectangle has height



$$h_1(x) = f(x) - c$$

and the rightmost rectangle has height

$$h_2(x) = f(x) - g(x)$$

Although there are cases where it is possible to simply split the picture into two regions and calculate the area of each region separately, there are many times where this is exceedingly difficult or even impossible. Instead, we realize that there is no requirement that the variable of integration has to be x. That is, in these cases, we can try to integrate with respect to y.

EXAMPLE 3 Use a right Riemann sum R_n to create an integral that calculates the area of the region bounded by $y = 2x$, $y = 3 - x$, and $y = 1$.

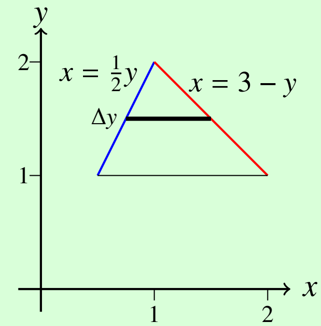
Solution: We first find where the curves intersect. The first two lines intersect when

$$\begin{aligned} 2x &= 3 - x \\ 3x &= 3 \\ x &= 1 \end{aligned}$$

The line $y = 2x$ intersects $y = 1$ when $x = \dfrac{1}{2}$.

The line $y = 3 - x$ intersects $y = 1$ when $x = 2$.

We get the figure to the right.



We see that not every vertical rectangle would have the same height function. Therefore, we instead solve the problem with respect to y instead of x.

To do this, we first solve the first two equations for x. We get $x = \dfrac{1}{2}y$ and $x = 3 - y$.

We can see from the figure that these two lines intersect when $y = 2$. So, the interval is now $1 \le y \le 2$.

We now think about doing the Riemann sum along the y-axis instead of the x-axis. That is, we will now subdivide the interval $1 \le y \le 2$ into n equal pieces of width Δy and end points

$$y_i = 1 + i\Delta y, \qquad 0 \le i \le n$$

Just as we did when solving the problem with respect to x, we draw the i-th rectangle as a thin strip. However, we now are drawing them horizontally instead of vertically.

From the figure, we get that the height of the rectangle is

$$h(y_i) = \text{top function} - \text{bottom function} = (3 - y_i) - \frac{1}{2}y_i$$

(Keep in mind that the values along the x-axis get larger as we move to the right, so the 'top' function of the rectangle is at the right end of the rectangle.) Hence, the Riemann sum is

$$R_n = \sum_{i=1}^{n} h(y_i)\Delta y = \sum_{i=1}^{n} \left((3 - y_i) - \frac{1}{2}y_i \right) \Delta y$$

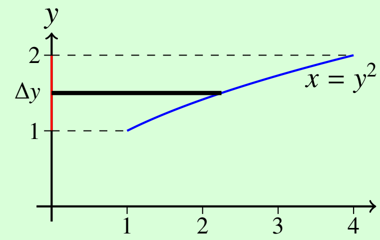
Thus, we get that the area is

$$\text{Area} = \int_1^2 \left((3 - y) - \frac{1}{2}y \right) \, dy$$

EXAMPLE 4

Use a right Riemann sum R_n to create an integral that calculates the area between $y = \sqrt{x}$ and $x = 0$ for $1 \leq y \leq 2$.

Solution: We begin by drawing the region.



Since not every vertical rectangle would have the same height function, we instead solve the problem with respect to y instead of x.

Solving the equations for x gives $x = y^2$ and $x = 0$ (that one was easy).

We draw the i-th rectangle horizontally with width Δy and end points

$$y_i = 1 + i\Delta y, \qquad 0 \leq i \leq n$$

From the figure, we get that the height of the i-th rectangle is

$$h(y_i) = y_i^2 - 0$$

Thus, we get that the area is

$$\text{Area} = \int_1^2 y^2 \, dy$$

EXERCISE 3

Use a right Riemann sum R_n to create an integral that calculates the area between $y = x - 4$ and $y = x^{2/3}$ for $0 \leq y \leq 2$.

EXERCISE 4

Consider the area between the graphs of $y = x^2$ and $y = x^{1/3}$.

(a) Use a right Riemann sum R_n to create an integral with respect to x that calculates the area. Use the integral to find the area.

(b) Use a right Riemann sum R_n to create an integral with respect to y that calculates the area.

Section 5.3 Problems

1. Use a right Riemann sum R_n to create an integral that calculates the area between $g(x) = 5 - x^2$ and $h(x) = x^2 - 3$. Use the integral to find the area.

2. Use a right Riemann sum R_n to create an integral that calculates the area between $g(x) = 2x + 2$ and $h(x) = 3x + 3$ for $1 \leq x \leq 4$. Use the integral to find the area.

3. Use a right Riemann sum R_n to create an integral that calculates the area between $g(x) = \cos(x)$ and $h(x) = \sin(x)$ for $\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$. Use the integral to find the area.

4. Use a right Riemann sum R_n to create an integral that calculates the area of the region bounded by $y = x$, $y = \frac{1}{x^2}$, and $x = 2$. Use the integral to find the area.

5. Use a right Riemann sum R_n to create an integral that calculates the area between $x = 2y$ and $x = y^2 - 3$. Use the integral to find the area.

6. Use a right Riemann sum R_n to create an integral that calculates the area of the region bounded by $y = \sqrt{x}$, $y = x - 2$, and $y = 0$. Use the integral to find the area.

7. Use a right Riemann sum R_n to create an integral that calculates the area of the region in the first quadrant that is bounded $g(x) = x$, $h(x) = 2 - x^2$, and $y = 0$. Use the integral to find the area.

8. Use a right Riemann sum R_n to create an integral that calculates the area of the region bounded by $y = x^{2/3}$, $x = 0$, and $y = 4$. Use the integral to find the area.

9. Use a right Riemann sum R_n to create an integral that calculates the area of the region bounded by $y = x$, $y = 2x$, and $x = 3$. Use the integral to find the area.

10. Consider the area of the region bounded by $y = 2x$, $y = 2$ and $x = 3$.
    (a) Use a right Riemann sum R_n to create an integral with respect to x that calculates the area. Use the integral to find the area.
    (b) Use a right Riemann sum R_n to create an integral with respect to y that calculates the area. Use the integral to find the area.

11. Consider the area between the graphs of $y = x^2$ and $y = 3$.
    (a) Use a right Riemann sum R_n to create an integral with respect to x that calculates the area. Use the integral to find the area.
    (b) Use a right Riemann sum R_n to create an integral with respect to y that calculates the area. Use the integral to find the area.

Section 5.4: Applications of Integration

LEARNING OUTCOMES

1. Know how to use a Riemann sum to set up an integral to calculate physical quantities.

As mentioned in the introduction, the power of mathematics is that a technique can be used to calculate a variety of different things by simply changing our interpretation of the functions involved. Keeping that in mind, we now look at how to use a Riemann sum to set up integrals for calculating a variety of quantities.

5.4.1 Mass

The density of a substance is the measurement of the distribution of mass per unit of volume. For example, pure water at 4°C has a density of 1 g/cm^3.

For a substance that has a uniform density ρ and volume v, we can calculate the substance's mass by

$$m = \rho v$$

If the density is not constant, then this formula does not apply. Instead, we can use an integral to calculate the mass of an object.

To set up such an integral, we first set up a Riemann sum.

Assume we want to calculate the mass of a substance that is b units long and the density of the substance is given by $\rho(x)$ for $0 \le x \le b$, where x is the distance from one-end of the substance. Further assume that the cross-sectional area of the substance at position is given by $A(x)$.

To set up a Riemann sum, we begin by dividing the interval $[0, b]$ into n equal pieces with width

$$\Delta x = \frac{b-0}{n} = \frac{b}{n}$$

The end points of the sub-intervals are given by

$$x_i = a + i\Delta x = \frac{bi}{n}, \quad 0 \le i \le n$$

We can approximate the mass of the substance from x_{i-1} to x_i by

$$m_i \approx \text{density at } x_i \times \text{volume at } x_i$$

Since the density at x_i is $\rho(x_i)$ and the volume is approximately

$$\begin{aligned} v_i &\approx \text{area of face at } x_i \times \text{thickness} \\ &= A(x_i)\Delta x \end{aligned}$$

we get

$$m_i \approx \rho(x_i)A(x_i)\Delta x$$

Hence, the mass of the substance from $x = 0$ to $x = b$ is approximately

$$\sum_{i=1}^{n} \rho(x_i)A(x_i)\Delta x$$

To get the exact mass m, we take the limit as $n \to \infty$. So,

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i)A(x_i)\Delta x$$

By definition of the definite integral, we get

$$m = \int_0^b \rho(x)A(x)\, dx$$

5.4.2 Arc Length

We now turn our attention to essentially a one dimensional problem; finding the length of a curve $y = f(x)$ from $x = a$ to $x = b$.

We begin, as usual, by subdividing the interval $[a, b]$ into n equal sub-intervals with width Δx and end points

$$x_i = a + i\Delta x, \qquad 0 \le i \le n$$

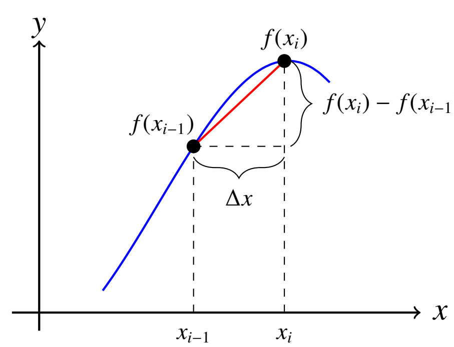
We approximate the length of the curve between the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ by drawing a straight line between the points.

From the figure, we observe that the length of the i-th line, by the Pythagorean Theorem, is

$$L_i = \sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2}$$

So, the length of the curve is approximately

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2}$$



We seem to have a problem. To convert this into an integral, we need a Δx term multiplied on the outside. We use some algebra to accomplish this.

Factoring $(\Delta x)^2$ out of both terms inside the square root gives

$$L_i = \sqrt{\left(1 + \frac{(f(x_i) - f(x_{i-1}))^2}{(\Delta x)^2}\right) \cdot (\Delta x)^2}$$

Using properties of square roots we get

$$\begin{aligned} L_i &= \sqrt{1 + \frac{(f(x_i) - f(x_{i-1}))^2}{(\Delta x)^2}}\sqrt{(\Delta x)^2} \\ &= \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{\Delta x}\right)^2}\,\Delta x, \qquad \text{since } \Delta x > 0 \end{aligned}$$

The term

$$\frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

should look a little familiar. We can also rewrite this to make it look even more familiar. Observe that

$$x_i = x_{i-1} + \Delta x$$

So,

$$\frac{f(x_i) - f(x_{i-1})}{\Delta x} = \frac{f(x_{i-1} + \Delta x) - f(x_{i-1})}{\Delta x}$$

If we just add a limit as $\Delta x \to 0$ in front, this would be the definition of f'. But, that is exactly what have! In particular, as $n \to \infty$ we get $\Delta x = \dfrac{b-a}{n} \to 0$.

Thus, we have

$$\begin{aligned} \text{Arc Length} &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{\Delta x}\right)^2}\,\Delta x \\ &= \int_a^b \sqrt{1 + (f'(x))^2}\,dx \end{aligned}$$

5.4.3 Fluid Flow

The flow rate of a fluid is defined to be the quantity of fluid flowing through a cross-section of piping per unit time. Our goal is to create an integral which calculates the flow rate of blood through a circular artery of radius R.

If the velocity of blood was constant throughout the artery, then the flow rate would simply be

$$F = \text{velocity} \times \text{cross-sectional area} = v \cdot \pi R^2$$

However, due to friction cause by the artery walls and the blood itself, the velocity of the blood is slower closer to the walls. In particular, the velocity for laminar blood at a distance r from the center of the artery is given by

$$v(r) = c(R^2 - r^2), \qquad 0 \le r \le R$$

where $c > 0$ is a constant.

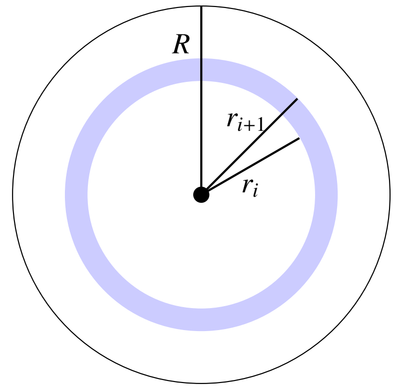
To calculate the flow rate F, we start as usual. We subdivide the interval $[0, R]$ into n equal pieces with width

$$\Delta r = \frac{R-0}{n} = \frac{R}{n}$$

We label the end points of the subintervals by

$$r_i = i\Delta r$$

Graphically, we are splitting the circular artery into n subrings.



We get that the flow rate through the i-th subring is approximately.

$$F_i \approx \text{velocity} \times \text{cross-sectional area} = v(r_i) \cdot \pi(r_{i+1}^2 - r_i^2)$$

Thus, the flow rate through the artery is approximately

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} v(r_i) \cdot \pi(r_{i+1}^2 - r_i^2)$$

Thus, the exact flow rate rate is

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} v(r_i) \cdot \pi(r_{i+1}^2 - r_i^2)$$

As we saw with arc length, we need this to be multiplied by a Δr if we are going to convert this into an integral.

This time the trick is to recognize that $\Delta r = r_{i+1} - r_i$ and to use the difference of squares formula. We get

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} v(r_i) \cdot \pi(r_{i+1} + r_i)(r_{i+1} - r_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} v(r_i) \cdot \pi(r_{i+1} + r_i)\Delta r \end{aligned}$$

We also above that as Δr gets smaller and smaller, r_{i+1} will get closer and closer to r_i. In particular,

$$\lim_{n\to\infty} r_{i+1} = r_i + \Delta r = r_i + 0 = r_i$$

Hence, we have

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} v(r_i) \cdot \pi(r_i + r_i)\Delta r \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} v(r_i) \cdot \pi 2 r_i \Delta r \\ &= \int_0^R v(r) \cdot 2\pi r \, dr \end{aligned}$$

Section 5.5: Approximating Definite Integrals

5.5.1 Right Riemann Sum Approximations

In many situations in the sciences, we do not need the exact value of a definite integral as our accuracy is restricted by the number of significant digits. This is doubly good since many of the integrals we encounter in science are very difficult to impossible to integrate. For these reasons, understanding how to approximate a definite integral is very useful.

REMARK

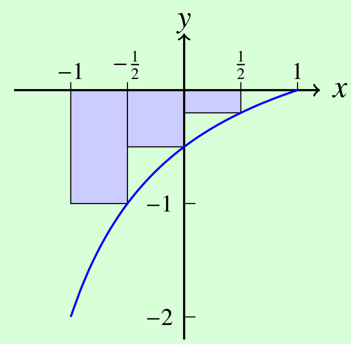
The majority of examples and exercises will involve definite integrals that can easily be evaluated exactly. The reason for this is twofold. First, it keeps the functions and calculations much simpler so that you can focus on understanding the procedure without being bogged down in the calculations. Second, and perhaps more importantly, it allows you to calculate the exact answer so that you can determine whether your approximation is reasonable (i.e. relatively good).

We begin with an example similar to what we have been doing earlier in this book. The main difference, as stated at the beginning of section 5.2.2, is that we are now using left-end points of the intervals instead of right-end points.

EXAMPLE 1 Approximate the area under $y = \dfrac{x-1}{x+2}$ for $-1 \le x \le 1$ using $n = 4$ subdivisions.

Solution: We have

$$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{1}{2}$$
$$x_i = a + i\Delta x = -1 + \frac{i}{2}, \qquad \text{for } 0 \le i \le 4$$



We get

$$\begin{aligned}
\text{Area} &\approx R_4 \\
&= \sum_{i=1}^{4} \frac{x_i - 1}{x_i + 2} \cdot \Delta x \\
&= \sum_{i=1}^{4} \frac{\left(-1+\frac{i}{2}\right) - 1}{\left(-1+\frac{i}{2}\right) + 2} \cdot \frac{1}{2} \\
&= \left[\frac{\frac{-3}{2}}{\frac{3}{2}} \cdot \frac{1}{2}\right] + \left[\frac{-1}{2} \cdot \frac{1}{2}\right] + \left[\frac{\frac{-1}{2}}{\frac{5}{2}} \cdot \frac{1}{2}\right] + \left[\frac{0}{3} \cdot \frac{1}{2}\right] \\
&= -\frac{1}{2} - \frac{1}{4} - \frac{1}{10} + 0 \\
&= -\frac{17}{20}
\end{aligned}$$

EXERCISE 1 Approximate the area under $f(x) = \dfrac{1}{x^2+2}$ for $0 \le x \le 3$ using $n = 3$ subdivisions.

EXERCISE 2 Set up a Riemann sum R_n with n subdivisions which approximates $\displaystyle\int_0^4 e^{x^2}\, dx$.

EXERCISE 3 In Example 2.4.8, we approximated ln(2) using $n = 4$ subdivisions whose heights were calculated from the left-end points of the intervals.

(a) Approximate ln(2) using $n = 4$ subdivisions with right-end points.

(b) Compare the approximation in (a) in with the actual value of ln(2) and the approximation in Example 2.4.8. Explain why the approximation in Example 2.4.8 was an overestimate and why the approximation in (a) was an underestimate.

Thanks to the summation formulas, we can now evaluate Riemann sums with larger values of n.

EXAMPLE 2 Given that $F'(x) = x$, approximate the net change in F over $1 \le x \le 3$ using $n = 20$ subdivisions.

Solution: We have

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{20} = \frac{1}{10}$$
$$x_i = a + i\Delta x = 1 + \frac{i}{10}, \qquad \text{for } 0 \le i \le 20$$

Thus,

$$\begin{aligned}
\text{Area} &\approx R_{20} \\
&= \sum_{i=1}^{20} x_i \Delta x \\
&= \sum_{i=1}^{20} \left(1 + \frac{i}{10}\right) \cdot \frac{1}{10} \\
&= \frac{1}{10}\sum_{i=1}^{20} 1 + \frac{1}{100}\sum_{i=1}^{20} i \\
&= \frac{1}{10} \cdot 20 + \frac{1}{100} \cdot \frac{20(21)}{2} \\
&= 2 + \frac{21}{10} \\
&= \frac{41}{10}
\end{aligned}$$

EXAMPLE 3

Approximate the area under $y = x^2$ for $0 \le x \le 2$ using $n = 10$ subdivisions.

Solution: We have

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{10} = \frac{1}{5}$$
$$x_i = a + i\Delta x = \frac{i}{5}, \qquad \text{for } 0 \le i \le 10$$

Thus,

$$\begin{aligned}
\text{Area} &\approx R_{10} \\
&= \sum_{i=1}^{10} x_i^2 \Delta x \\
&= \sum_{i=1}^{10} \left(\frac{i}{5}\right)^2 \cdot \frac{1}{5} \\
&= \frac{1}{5^3} \sum_{i=1}^{10} i^2 \\
&= \frac{1}{5^3} \cdot \frac{10(11)(21)}{6} \\
&= \frac{77}{25}
\end{aligned}$$

EXERCISE 4

Approximate $\int_2^4 x\,dx$ using $n = 10$ subdivisions.

EXERCISE 5

Given that $F'(x) = x^2$, approximate the net change in F over $-1 \le x \le 1$ using $n = 20$ subdivisions.

EXERCISE 6

Approximate the area under $f(x) = 2x^2 - x$ for $0 \le x \le 2$ using

(a) $n = 10$ subdivisions.
(b) $n = 20$ subdivisions.

5.5.2 Error Bound for the Right Riemann Sum

As discussed in Section 4.2.4, whenever we approximate a value, we would like to have a bound on the error. The following theorem gives us formula for an upper bound for the error when a right Riemann sum is used to approximation a definite integral.

THEOREM 1

For a definite integral $\int_a^b f(x)\,dx$, we denote the error in the approximation of the definite integral by a right Riemann sum R_n with n subdivisions by

$$E_n = \int_a^b f(x)\,dx - R_n$$

An upper bound for the error in the approximation is given by

$$|E_n| \leq \frac{M}{2n}(b-a)^2$$

where M is the maximum value of $|f'(x)|$ on $[a, b]$.

EXAMPLE 4

Find an upper bound for the error when R_{20} is used to approximate $\int_1^3 x\,dx$.

Solution: We have $f(x) = x$. So, $f'(x) = 1$. Therefore, the maximum value of $|f'(x)| = 1$ on $[1, 3]$ is $M = 1$. Thus,

$$|E_{20}| \leq \frac{M}{2n}(3-1)^2 = \frac{1}{2 \cdot 20}(2)^2 = \frac{1}{10}$$

EXERCISE 7

If we double the number of subdivisions in the example above to $n = 40$, what effect does that have on the upper bound for the error?

EXAMPLE 5

Find an upper bound for the error when R_{10} is used to approximate $\int_2^5 \frac{1}{x}\,dx$.

Solution: We have $f(x) = \frac{1}{x}$. So, $f'(x) = -\frac{1}{x^2}$.

Since $|f'(x)| = \frac{1}{x^2}$ is decreasing on $[2, 5]$, the maximum value is at $x = 2$. Therefore, the maximum value is

$$M = |f'(2)| = \frac{1}{4}$$

Thus,

$$|E_{10}| \leq \frac{M}{2n}(5-2)^2 = \frac{\frac{1}{4}}{2 \cdot 10}(3)^2 = \frac{9}{80}$$

EXAMPLE 6

Find an upper bound for the error when R_{30} is used to approximate $\int_{-3}^{1} (x^2 + x)\, dx$.

Solution: We have $f(x) = x^2 + x$. So, $f'(x) = 2x + 1$.

The function $|f'(x)| = |2x + 1|$ is neither increasing nor decreasing on $[-3, 1]$. So, we use the Closed Interval Method. We get

$$\frac{d}{dx}|2x + 1| = \frac{2(2x + 1)}{|2x + 1|} = \frac{4x + 2}{|2x + 1|}$$

So, the critical number on $[-3, 1]$ is $x = -\frac{1}{2}$. We get

$$|f'(-3)| = 5, \qquad \left|f'\left(-\frac{1}{2}\right)\right| = 0, \qquad |f'(1)| = 3$$

So, the maximum value is $M = 5$ at $x = -3$. Thus,

$$|E_{30}| \leq \frac{M}{2n}(1 - (-3))^2 = \frac{5}{2 \cdot 30}(4)^2 = \frac{4}{3}$$

EXAMPLE 7

Find an upper bound for the error when R_{20} is used to approximate $\int_{-\pi/4}^{0} \cos(x)\, dx$.

Solution: We have $f(x) = \cos(x)$. So, $f'(x) = -\sin(x)$.

Since $|f'(x)| = |\sin(x)|$ is decreasing on $\left[-\frac{\pi}{4}, 0\right]$, the maximum value is at $x = -\frac{\pi}{4}$. Therefore, the maximum value is

$$M = \left|f'\left(-\frac{\pi}{4}\right)\right| = \frac{1}{\sqrt{2}}$$

Thus,

$$|E_{20}| \leq \frac{M}{2n}\left(-\frac{\pi}{4} - 0\right)^2 = \frac{\frac{1}{\sqrt{2}}}{2 \cdot 20}\left(-\frac{\pi}{4}\right)^2 = \frac{\pi^2}{640\sqrt{2}}$$

EXERCISE 8

Find an upper bound for the error when R_{20} is used to approximate $\int_{1}^{5} x^2\, dx$.

EXERCISE 9

Find an upper bound for the error when R_{10} is used to approximate $\int_{1}^{2} \frac{1}{x^2}\, dx$.

Section 5.5 Problems

1. Approximate the definite integral using a right Riemann sum with the specified numbers of subdivisions.
   (a) $\int_0^{2\pi} \sin(x)\,dx$; $n = 8$
   (b) $\int_0^1 (3x + 1)\,dx$; $n = 10$
   (c) $\int_1^2 (1 - 2x)\,dx$; $n = 10$
   (d) $\int_0^4 3x^2\,dx$; $n = 20$
   (e) $\int_0^{2\pi} x\cos(x)\,dx$; $n = 4$
   (f) $\int_{-4}^{-1} (x - 2)\,dx$; $n = 10$
   (g) $\int_0^2 (x + 1)^2\,dx$; $n = 20$
   (h) $\int_{-1}^3 (x^2 - 2x)\,dx$; $n = 10$

2. Find an upper bound for the error when the definite integral is approximated using a right Riemann sum with the specified numbers of subdivisions.
   (a) $\int_0^{2\pi} \sin(x)\,dx$; $n = 10$
   (b) $\int_0^5 (3x^2 + 1)\,dx$; $n = 15$
   (c) $\int_1^4 \frac{1}{x}\,dx$; $n = 20$
   (d) $\int_0^2 \frac{1}{2}\ln(1 + x^2)\,dx$; $n = 25$

3. Approximate ln(3) using a right Riemann sum with $n = 4$. Find an upper bound for the error in the approximation.

4. Approximate ln(5) using a right Riemann sum with $n = 4$. Find an upper bound for the error in the approximation.

End of Chapter Problems

1. Evaluate the following integrals.
   (a) $\int 2^x \, dx$
   (b) $\int \tan(x) \, dx$
   (c) $\int \frac{1}{x} \, dx$
   (d) $\int \sec(x)\tan(x) \, dx$
   (e) $\int \frac{1}{\sqrt{1-x^2}} \, dx$
   (f) $\int \sec(x) \, dx$
   (g) $\int \frac{1}{1+x^2} \, dx$
   (h) $\int \csc(x) \, dx$
   (i) $\int x\cos(x^2) \, dx$
   (j) $\int 3x^2 e^{x^3} \, dx$
   (k) $\int \frac{1}{\sqrt{1-x}} \, dx$
   (l) $\int x\sqrt{x+1} \, dx$
   (m) $\int \frac{(\ln(x))^2}{x} \, dx$
   (n) $\int 2x\sqrt{1+x^2} \, dx$
   (o) $\int \frac{3-x}{2+x} \, dx$
   (p) $\int \frac{3x}{2-5x^2} \, dx$
   (q) $\int (2x+1)^5 \, dx$
   (r) $\int \frac{x+1}{x^2+1} \, dx$
   (s) $\int \frac{\arccos(x)}{\sqrt{1-x^2}} \, dx$
   (t) $\int \frac{3}{(2+x)^3} \, dx$
   (u) $\int x^3\sqrt{x^2+1} \, dx$

2. Evaluate the following integrals.
   (a) $\int_{\pi/4}^{2\pi/3} \sin(x) \, dx$
   (b) $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \, dx$
   (c) $\int_{-1}^{0} \frac{x}{1+x^2} \, dx$
   (d) $\int_{-1}^{0} \frac{1}{1+x^2} \, dx$
   (e) $\int_{-2}^{3} (3-x)^4 \, dx$
   (f) $\int_{e}^{e^4} \frac{1}{x\sqrt{\ln(x)}} \, dx$
   (g) $\int_0^1 \frac{1}{x-3} \, dx$
   (h) $\int_0^1 \frac{1}{(x-3)^2} \, dx$
   (i) $\int_0^1 3x\sin(x^2) \, dx$
   (j) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} \, dx$
   (k) $\int_1^4 \frac{x+1}{\sqrt{x}} \, dx$
   (l) $\int_0^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)} \, dx$
   (m) $\int_0^1 \frac{x+1}{x^2+2x+2} \, dx$

3. Evaluate the sum.
   (a) $\sum_{i=1}^{50} 1$
   (b) $\sum_{i=1}^{20} i$
   (c) $\sum_{i=1}^{15} 3i$
   (d) $\sum_{i=1}^{10} i^2$
   (e) $\sum_{i=1}^{5} n^2$

4. Write the sum in sigma notation.
   (a) $1 + x + 2x^2 + 6x^3 + 24x^4$
   (b) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
   (c) $f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$

5. Evaluate the following
   (a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}$
   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i}{3n^2}$
   (c) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n^2} - \frac{i^2}{n^3}\right)$

6. Set up a right Riemann sum with n subdivisions which approximates the net change of F which has the given rate of change $F'(t) = f(t)$ over the specified interval.
   (a) $f(t) = t^2 + 1$, $2 \le t \le 7$
   (b) $f(t) = \sin(t^2)$, $0 \le t \le \pi$
   (c) $f(t) = \frac{t}{1 + t^2}$, $-3 \le t \le -1$

7. Approximate the definite integral using a right Riemann sum with the specified numbers of subdivisions.
   (a) $\int_0^4 3x\, dx$; $n = 5$
   (b) $\int_1^3 (x - 1)\, dx$; $n = 10$
   (c) $\int_0^2 (1 - 4x)\, dx$; $n = 10$
   (d) $\int_0^2 x^2\, dx$; $n = 10$

8. Evaluate the definite integral using the definition of the definite integral.
   (a) $\int_0^3 (x - 1)\, dx$
   (b) $\int_{-2}^3 (2x + 1)\, dx$
   (c) $\int_2^1 x^2\, dx$

9. Use a right Riemann sum R_n to create an integral that calculates the area between $g(x) = x$ and $h(x) = x^2$. Use the integral to find the area.

10. Use a right Riemann sum R_n to create an integral that calculates the area between $g(x) = x^2$ and $h(x) = 4$. Use the integral to find the area.

11. Find an upper bound for the error when the definite integral is approximated using a right Riemann sum with the specified numbers of subdivisions.
    (a) $\int_0^\pi \cos(x)\, dx$; $n = 20$
    (b) $\int_{1/2}^1 \arctan(x)\, dx$; $n = 10$
    (c) $\int_1^4 \frac{1}{\sqrt{x}}\, dx$; $n = 10$
    (d) $\int_{-2}^{-1} \frac{1}{x^2}\, dx$; $n = 20$

Chapter 6: Sequences and Series

Section 6.1: Sequences

LEARNING OUTCOMES

1. Understand the definition and notation for sequences.
2. Know how to prove a sequence is increasing or decreasing.
3. Know how to determine whether a sequence converges or diverges.
4. Understand and know how to work with alternating sequences.

In grade school you saw arithmetic and geometric sequences and perhaps other sequences like the Fibonacci sequence. Here we will take a more detailed look at infinite sequences and their limits.

6.1.1 Introduction to Infinite Sequences

DEFINITION
Sequence

A **sequence** is an ordered list of numbers $\{a_0, a_1, a_2, \ldots\}$.

REMARK

In this text, unless specified otherwise, the index of the first term in the sequence is 0.

NOTATION

We typically denote a sequence $\{a_0, a_1, a_2, \ldots\}$ by writing $\{a_n\}$ where a_n is a formula for the n^{th} term of the sequence.

When we want to indicate that a sequence starts at a value other than $n = 0$, we use the notation

$$\{a_n\}_{n=k}^{\infty}$$

where k is the number where the sequence starts.

EXAMPLE 1

Write the first four terms of each sequence.

(a) $\left\{\dfrac{e^n}{n!}\right\}$

Solution: $a_0 = \dfrac{e^0}{0!} = \dfrac{1}{1} = 1$, $a_1 = \dfrac{e^1}{1!} = \dfrac{e}{1} = e$, $a_2 = \dfrac{e^2}{2!} = \dfrac{e^2}{2}$, $a_3 = \dfrac{e^3}{3!} = \dfrac{e^3}{6}$.

(b) $\left\{\frac{1}{n}\right\}_{n=2}^{\infty}$

Solution: $a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, a_4 = \frac{1}{4}, a_5 = \frac{1}{5}$.

EXAMPLE 2 Find a formula for the n^{th} term of the sequence.

(a) $\left\{\frac{7}{2}, \frac{7}{5}, \frac{7}{8}, \frac{7}{11}, \dots\right\}$

Solution: We see that the numerator is always 7 and the denominator equals 2 at $n = 0$ and increases by 3 each time. Thus, we have $a_n = \frac{7}{2 + 3n}$.

(b) $\left\{\frac{1}{2}, -\frac{1}{2}, \frac{3}{8}, -\frac{1}{4}, \frac{5}{32}, \dots\right\}$

Solution: We first notice that the terms are alternating in sign. In particular, the even terms (i.e. $n = 0, 2, 4, \dots$) of the sequence are positive and the odd terms (i.e. $n = 1, 3, 5, \dots$) are negative. To create this behaviour, we use $(-1)^n$.

At first glance, there doesn't seem to be a pattern. The trick is to notice that for the even terms the denominator is 2^{n+1}. So, if we un-simplify the odd terms to match the pattern with the even terms, we get

$$\frac{1}{2}, -\frac{2}{4}, \frac{3}{8}, -\frac{4}{16}, \frac{5}{32}, \dots$$

All of the denominators now have the form 2^{n+1}, and, ignoring the alternating sign, the numerator starts at 1 and increases by 1 each time. Thus, we get

$$a_n = \frac{(-1)^n(n+1)}{2^{n+1}}$$

(c) $\{-1, 1, -2, 6, -24, 120, \dots\}$

Solution: As in (b), we notice that the terms of the sequence alternating in sign. In this case, it is positive on the negative terms, so we include a factor of $(-1)^{n+1}$.

Ignoring the signs, we observe that the sequence is $n!$. We recall that $0! = 1$ and $1! = 1$, so this the sequence starts at $0!$.

So, we have $a_n = (-1)^{n+1} n!$.

In Chapter 7, we will need to find formulas for sequences that are 0 for all odd or even terms. We look at a couple of examples of how to do this.

EXAMPLE 3 Find a formula for the n^{th} term of the sequence.

(a) $\{1, 0, 2, 0, 3, 0, 4, 0, 5, 0, \ldots\}$

Solution: Observe that the sequence is 0 for all odd terms and is the sequence $\{1, 2, 3, 4, \ldots\}$ for even terms. To model this, we will use a piecewise defined function. Recall that even numbers can be represented by $2k$ and odd numbers can be represented by $2k + 1$.

The pattern for odd numbers is easy. For even numbers, we need to think more carefully about how they relate to the value of k where $n = 2k$. We can write out a table to make this easier.

k	n	a_n
0	$2(0) = 0$	1
1	$2(1) = 2$	2
2	$2(2) = 4$	3
3	$2(3) = 6$	4

From this, we see that we have

$$a_n = \begin{cases} 0 & \text{if } n = 2k+1 \text{ for some integer } k \\ k+1 & \text{if } n = 2k \text{ for some integer } k \end{cases}$$

(b) $\{0, 1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

Solution: Observe that the sequence is 0 for all even terms and is alternating between 1 and -1 for the odd terms.

In this case, the pattern for the even numbers is easy. For odd numbers, since the numbers are alternating, the pattern will use either $(-1)^k$ or $(-1)^{k+1}$. Testing $k = 0$ and 1 (corresponding to $n = 2(0) + 1 = 1$ and $n = 2(1) + 1 = 3$), we see that $(-1)^k$ works.

Thus, we get

$$a_n = \begin{cases} 0 & \text{if } n = 2k \text{ for some integer } k \\ (-1)^k & \text{if } n = 2k+1 \text{ for some integer } k \end{cases}$$

REMARK

Whenever you create a formula for a sequence, you should always ensure that your formula is correct by testing it for all the listed values of the sequence.

EXERCISE 1

Find a formula for the n^{th} term of each sequence.

(a) $\left\{\frac{1}{3}, \frac{4}{7}, \frac{9}{11}, \frac{16}{15}, \ldots\right\}$

(b) $\left\{-\frac{1}{1}, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, \ldots\right\}$

(c) $\left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \ldots\right\}$

(d) $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

In physics, sequences are used to describe behaviour such as quantum energy states. They can also be used to store data in the form of a string or list in computer science. In life sciences, it is often only possible to measure population size at discrete times (e.g. annually), so a sequence can be useful to develop a model.

EXERCISE 2

Suppose you measure the following data for the size of a population of insects:

t	0	1	2	3
a_t	10	30	90	270

Find a formula for a_t, the population of insects at time t in years.

6.1.2 Increasing/Decreasing Sequences

In this text, there will be situations where we need to show that a sequence is decreasing. So, we now look at the definition of increasing and of decreasing for sequences.

DEFINITION

Increasing Sequence

Decreasing Sequence

Let $\{a_n\}$ be a sequence.

If $a_n < a_{n+1}$ for all n, then we say that $\{a_n\}$ is **increasing**.

If $a_n > a_{n+1}$ for all n, then we say that $\{a_n\}$ is **decreasing**.

REMARKS

1. In the definition, when we say ‘for all n’, we mean for all n for which the sequence is defined ... remembering that, unless specified otherwise, we are starting sequences at $n = 0$.

2. Observe that the definition of increasing/decreasing for sequences matches that of increasing/decreasing for functions.

EXAMPLE 4

Show that $\left\{\dfrac{1}{\sqrt{n+4}}\right\}$ is decreasing.

Solution: We have $a_n = \dfrac{1}{\sqrt{n+4}}$. We need to compare a_n to a_{n+1}. Since we know that dividing a fixed numerator by a larger denominator makes a smaller fraction, we get

$$a_n = \frac{1}{\sqrt{n+4}} > \frac{1}{\sqrt{(n+1)+4}} = a_{n+1}$$

So, $\{a_n\}$ is decreasing.

EXAMPLE 5

Show that $\left\{\dfrac{n^2}{2}\right\}$ is increasing.

Solution: We have $a_n = \dfrac{n^2}{2}$. We need to compare a_n to a_{n+1}. We have

$$a_n = \frac{n^2}{2} < \frac{(n+1)^2}{2} = a_{n+1}$$

since we have just made the numerator larger. So, $\{a_n\}$ is increasing.

EXERCISE 3

Show that $\left\{\dfrac{1}{2^n}\right\}$ is decreasing.

EXERCISE 4

Show that $\left\{\dfrac{n}{n+1}\right\}$ is increasing.

For more complicated sequences, applying the definition is difficult. For example, intuitively we can guess that

$$\frac{n}{e^n} > \frac{n+1}{e^{n+1}}$$

However, we don't have good justification for this as both the numerator and the denominator are getting larger.

In such cases, we will use the fact that a function is decreasing when $f'(x) < 0$ and increasing when $f'(x) > 0$ to show that a sequence is either decreasing or increasing.

EXAMPLE 6

Show that $\left\{ \dfrac{n}{e^n} \right\}_{n=1}^{\infty}$ is decreasing.

Solution: Let $f(n) = a_n = \dfrac{n}{e^n}$. Then,

$$f'(n) = \frac{e^n - n \cdot e^n}{(e^n)^2} = \frac{1-n}{e^n}$$

Thus, $f'(n) < 0$ for $n > 1$. Hence, $\{a_n\}$ is decreasing for $n \geq 1$.

REMARK

Technically, a sequence is not a continuous function and therefore is not differentiable. So, to take the derivative, we should convert the sequence $\{a_n\}$ into a continuous function $f(n)$. We do this by defining $f(n) = a_n$ for all n and assuming f is continuous.

EXAMPLE 7

Show that $\left\{ \dfrac{\sqrt{n}}{n^2+1} \right\}_{n=1}^{\infty}$ is decreasing.

Solution: Let $f(n) = a_n = \dfrac{\sqrt{n}}{n^2+1}$. Then,

$$f'(n) = \frac{\frac{1}{2}n^{-1/2}(n^2+1) - n^{1/2}(2n)}{(n^2+1)^2}$$

To clear the fraction in the numerator, we multiply top and bottom by $2n^{1/2}$.

$$\begin{aligned} f'(n) &= \frac{(n^2+1) - 4n^2}{2n^{1/2}(n^2+1)^2} \\ &= \frac{1-3n^2}{2n^{1/2}(n^2+1)^2} \end{aligned}$$

Thus, $f'(n) < 0$ for $n \geq 1$. Thus, $\{a_n\}$ is decreasing for $n \geq 1$.

EXERCISE 5

Show that $\left\{ \dfrac{n}{3^n} \right\}_{n=1}^{\infty}$ is decreasing.

EXERCISE 6

Show that $\left\{ \dfrac{n}{1+n^2} \right\}_{n=1}^{\infty}$ is decreasing.

6.1.3 Limits of Sequences

In most applications of sequences, we are interested in the long term behavior of the sequence. In other words, we are interested in determining if the terms of the sequence are approaching some number L as n gets larger and larger.

DEFINITION

Limit of a Sequence

Convergent

Divergent

Let $\{a_n\}$ be a sequence. We say that L is the limit of $\{a_n\}$ if the values a_n can be made arbitrarily close to L by taking n sufficiently large. If such an L exists, then we say that $\{a_n\}$ is **convergent** and that it **converges** to L. We write

$$\lim_{n\to\infty} a_n = L$$

If no such L exists, then we say that $\{a_n\}$ is **divergent**.

REMARK

Just like limits of functions, if the limit of a sequence does not exist because the values of a_n get larger and larger as $n \to \infty$, then we write

$$\lim_{n\to\infty} a_n = \infty$$

rather than just saying the limit does not exist. We do this because it gives us more information about the sequence than simply saying the limit does not exist.

The next theorem implies that we can use all of the limit laws and formulas, including L'Hospital's Rule, that we had for limits of functions when trying to evaluate limits of sequences.

THEOREM 1

Let $f(x)$ be a function such that $f(n) = a_n$ for all positive integers n. If $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$.

EXAMPLE 8

Determine whether the sequence converges or diverges. If it converges, find its limit.

(a) $\left\{\dfrac{3n^2+1}{2n^2+5}\right\}$

Solution: We have

$$\lim_{n\to\infty} \frac{3n^2+1}{2n^2+5} = \lim_{n\to\infty} \frac{3+\frac{1}{n^2}}{2+\frac{5}{n^2}} = \frac{3}{2}$$

Hence, the sequence converges to $L = \dfrac{3}{2}$.

(b) $\{(-1)^n\}$

Solution: If we write out the first few terms of the sequence, we get

$$\{1, -1, 1, -1, 1, -1, \ldots\}$$

We can see that as n gets larger, the terms will not be approaching a single number, so this sequence diverges.

(c) $\left\{\dfrac{e^n}{n^2+2}\right\}$

Solution: Consider

$$\lim_{n\to\infty} \frac{e^n}{n^2+2}$$

This limit has the indeterminate form $\dfrac{\infty}{\infty}$. So, we can apply L'Hospital's Rule.

$$\begin{aligned}\lim_{n\to\infty} \frac{e^n}{n^2+2} &\overset{H}{=} \lim_{n\to\infty} \frac{e^n}{2n} \qquad \left(\text{form } \frac{\infty}{\infty}\right)\\ &\overset{H}{=} \lim_{n\to\infty} \frac{e^n}{2}\\ &= \infty\end{aligned}$$

Consequently, the sequence diverges.

(d) $\left\{\dfrac{e^n - e^{-n}}{e^n + e^{-n}}\right\}$

Solution: We get

$$\lim_{n\to\infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim_{n\to\infty} \frac{1 - e^{-2n}}{1 + e^{-2n}} = \frac{1-0}{1+0} = 1$$

So, the sequence converges to $L = 1$.

EXERCISE 7 Determine if the sequence converges or diverges. If it converges, find the limit.

(a) $\left\{\dfrac{3n^3 + 5n^2}{2n^3 + n + 1}\right\}$

(b) $\left\{\dfrac{2n^2 - n}{\sqrt{5n^4 + n^2 + 1}}\right\}$

(c) $\left\{\dfrac{n-1}{\ln(n+2)}\right\}$

6.1.4 Alternating Sequences

It is not uncommon to get sequences whose terms alternate between being positive and negative.

DEFINITION

Alternating Sequence

A sequence $\{a_n\}$ is called an **alternating sequence** if there exists a sequence $\{b_n\}$ where $b_n > 0$ for all n and either

$$a_n = (-1)^n b_n \quad \text{or} \quad a_n = (-1)^{n+1} b_n$$

for all n.

EXAMPLE 9

Show that each sequence is alternating.

(a) $\{(-1)^n\}$

Solution: We can take $b_n = 1$ and then $a_n = (-1)^n b_n$.

(b) $\left\{\dfrac{(-1)^{n+1} n}{2^n}\right\}_{n=1}^{\infty}$

Solution: We can take $b_n = \dfrac{n}{2^n}$ and then $a_n = (-1)^{n+1} b_n$.

(c) $\left\{\dfrac{\cos(\pi n)}{n}\right\}_{n=1}^{\infty}$

Solution: Observe that when n is odd, $\cos(\pi n) = -1$ and when n is even $\cos(\pi n) = 1$. Thus, $\cos(\pi n) = (-1)^n$. Consequently, we can take $b_n = \dfrac{1}{n}$ and then

$$a_n = (-1)^n b_n$$

EXAMPLE 10

Is the sequence $\{\sin(3n)\}_{n=1}^{\infty}$ alternating?

Solution: Writing out the first few terms (rounded to two decimals places) gives

$$\begin{aligned} a_1 &= 0.14 \\ a_2 &= -0.28 \\ a_3 &= 0.41 \\ a_4 &= -0.54 \\ a_5 &= 0.65 \end{aligned}$$

It looks like it is alternating. However, this is why we cannot trust just looking at a few values. This sequence is not alternating (it is impossible to find an appropriate sequence $\{b_n\}$). In particular, the terms a_{22} and a_{23} are both negative.

EXERCISE 8

Show that each sequence is alternating by finding an appropriate sequence b_n.

(a) $\left\{\dfrac{(-1)^n}{n^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\sin\left(\dfrac{\pi}{2} + n\pi\right)e^{n^2}\right\}$

The following theorem can sometimes help us evaluate the limit of an alternating sequence.

THEOREM 2

If $\{a_n\}$ is a sequence such that $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

EXAMPLE 11

Determine whether the sequence $\left\{\dfrac{(-1)^n}{n}\right\}$ converges or diverges. If it converges, find its limit.

Solution: We have that

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore, we also have

$$\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$$

EXERCISE 9

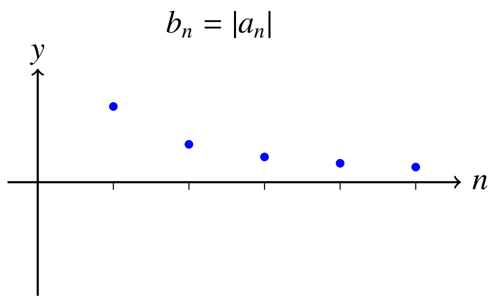
Show that $\lim\limits_{n\to\infty}\dfrac{(-1)^n n}{2^n} = 0$.

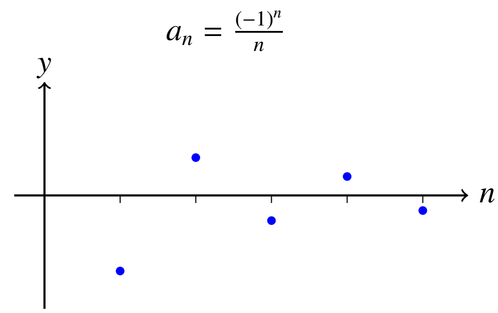
EXERCISE 10

Show that $\lim\limits_{n\to\infty}\dfrac{(-1)^{n+1}n^2}{n^3+1} = 0$.

Intuitively, we can see why Theorem 6.1.2 works. It says that if the magnitudes of the values of a sequence are getting smaller and smaller, then it doesn't matter if the values of the sequence are positive or negative, they are still getting closer and closer to 0.

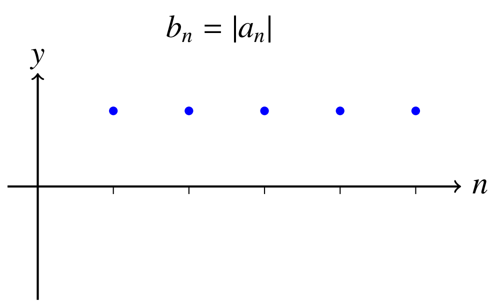
For example, let $a_n = \dfrac{(-1)^n}{n}$ and $b_n = |a_n|$. We can see from the picture below that the fact the values of b_n are getting closer and close to 0 means the values of a_n must also be getting closer and closer to 0.

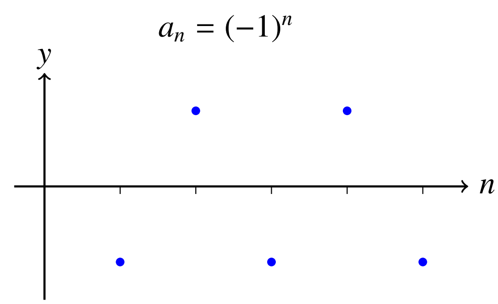




Compare this to the case where $\lim_{n\to\infty} |a_n| = L \neq 0$. In this case, the magnitude of the values of the sequence are approaching some value L. If all the a_n's are positive or all of the a_n's are negative, then the sequence $\{a_n\}$ will approach L or $-L$ respectively. But, if a_n is, for example, alternating, then the values of a_n will not approach a single number because they will bounce back and forth from being positive and negative. So, in this case, the sequence $\{a_n\}$ will diverge.

For example, let $a_n = (-1)^n$ and $b_n = |a_n|$. We can see from the picture below that even though $\lim_{n\to\infty} b_n = 1$, the sequence $\{a_n\}$ diverges.





Similarly, we can reason that if $\lim_{n\to\infty} |a_n|$ does not exist, that is, if the magnitude of the values of a_n are not getting closer and closer to some number as $n \to \infty$, then $\lim_{n\to\infty} a_n$ cannot exist.

This analysis gives us another theorem that will be useful later in this chapter.

THEOREM 3

If $\{a_n\}$ is a sequence such that $\lim_{n\to\infty} |a_n| \neq 0$, then $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist.

Additionally, if $\lim_{n\to\infty} |a_n|$ does not exist, then $\lim_{n\to\infty} a_n$ does not exist.

EXERCISE 11

Give an example of a sequence $\{a_n\}$ such that $\lim_{n\to\infty} |a_n| \neq 0$ and $\lim_{n\to\infty} a_n$ does not exist.

Section 6.1 Problems

1. Write out the first five terms of each sequence.
   (a) $\{a_n\} = \{n!\}$
   (b) $\{b_n\}_{n=1}^{\infty} = \left\{\dfrac{\sin(n)}{n}\right\}_{n=1}^{\infty}$
   (c) $\{c_n\} = \left\{\dfrac{(-1)^n}{n+1}\right\}$
   (d) $\{d_n\} = \left\{\dfrac{(x-1)^n}{2^n}\right\}$

2. Find a formula for the nth term of the sequence.
   (a) $\left\{\dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dfrac{1}{32}, \ldots\right\}$
   (b) $\left\{\dfrac{1}{\pi}, \dfrac{-1}{3\pi}, \dfrac{1}{5\pi}, \dfrac{-1}{7\pi}, \ldots\right\}$
   (c) $\left\{1, 1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots\right\}$
   (d) $\left\{\dfrac{-1}{6}, \dfrac{1}{24}, \dfrac{-1}{120}, \dfrac{1}{720}, \ldots\right\}$
   (e) $\left\{-\dfrac{3}{2}, \dfrac{3}{5}, -\dfrac{3}{8}, \dfrac{3}{11}, \ldots\right\}$
   (f) $\left\{\dfrac{3}{2}, \dfrac{7}{6}, \dfrac{11}{24}, \dfrac{15}{120}, \ldots\right\}$
   (g) $\left\{1, \dfrac{9}{7}, \dfrac{27}{11}, \dfrac{27}{5}, \ldots\right\}$

3. Determine whether the sequence converges or diverges. If it converges, find its limit.
   (a) $\left\{\dfrac{n^3+1}{2n^3+2}\right\}$
   (b) $\{-2e^{1/n}\}$
   (c) $\{n^2e^{-n}\}$
   (d) $\left\{\dfrac{n^2+n^4}{2n^2+5n^3}\right\}$
   (e) $\{\sin(n)\}$
   (f) $\left\{\dfrac{e^n+e^{2n}}{e^{2n}+e^{-n}}\right\}$
   (g) $\left\{\dfrac{2^n}{n^2}\right\}_{n=1}^{\infty}$
   (h) $\left\{(-1)^n\dfrac{n^2+1}{e^{2n}+n^2}\right\}$

4. Determine whether the sequence is increasing, decreasing, or neither.
   (a) $\{a_n\} = \left\{\dfrac{1}{3n+5}\right\}$
   (b) $\{a_n\} = \left\{\dfrac{3n+2}{n+1}\right\}$
   (c) $\{a_n\} = \{(-1)^n n^2\}$
   (d) $\{a_n\} = \left\{\dfrac{5n}{7n+3}\right\}$
   (e) $\{a_n\} = \left\{\dfrac{1}{n^2}\right\}_{n=1}^{\infty}$
   (f) $\{a_n\} = \left\{\dfrac{n^2}{\sqrt{n^3+2}}\right\}$
   (g) $\{a_n\} = \left\{ne^{-n^2}\right\}_{n=1}^{\infty}$
   (h) $\{a_n\} = \left\{\dfrac{1}{n\sqrt{\ln(n)}}\right\}_{n=2}^{\infty}$
   (i) $\{a_n\} = \left\{\dfrac{\ln(n)}{n^3}\right\}_{n=2}^{\infty}$
   (j) $\{a_n\} = \left\{\dfrac{n^2}{n^3+1}\right\}_{n=2}^{\infty}$

5. Show that each sequence $\{a_n\}$ is alternating by finding an appropriate sequence b_n. Then show that $\{a_n\}$ converges to 0.
   (a) $\left\{\dfrac{(-1)^{n+1}n}{n^2+1}\right\}_{n=1}^{\infty}$
   (b) $\left\{\dfrac{(-1)^n}{\ln((n+5)}\right\}$
   (c) $\left\{\dfrac{(-1)^n(n-1)}{e^n}\right\}$
   (d) $\left\{\dfrac{(-1)^{n+1}}{n+1}\right\}_{n=1}^{\infty}$
   (e) $\left\{\dfrac{\cos(n\pi)}{\sqrt{n+1}}\right\}$
   (f) $\left\{\dfrac{\sin\left(\frac{\pi}{2}+n\pi\right)}{n!}\right\}$

Section 6.2: Infinite Series

LEARNING OUTCOMES

1. Understand the notation for a series.
2. Understand the definition of the sum of a series.
3. Know how to use the Divergence Test.

We now look at adding all infinitely many terms of a sequence together. That is, given a sequence $\{a_n\}$ we wish to determine

$$a_0 + a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

6.2.1 Introduction to Series

DEFINITION
Infinite Series

An **infinite series** is a sum of an infinite number of terms of a sequence $\{a_n\}$. That is,

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

At first glance, one might think that the sum of an infinite number of positive numbers must always be infinity. A simple example such as

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots + n + \cdots = \infty$$

might reinforce this idea. However, as we will see, if the numbers that we are adding tend to zero suitably fast, then the sum of them can be finite.

EXAMPLE 1

Consider the sequence $\left\{\frac{9}{10^n}\right\}_{n=1}^{\infty} = \{.9, .09, .009, \ldots\}$. Evaluate the series $\sum_{n=1}^{\infty} \frac{9}{10^n}$.

Solution: Observe that we have

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots = 0.9999\ldots = 1$$

Because of the form of the sequence in the last example, it was easy to see how all infinitely many terms were going to add together. This is almost never the case. So, the strategy for trying to find the sum of a series is to look at the value the sum is approaching as we add more and more terms of the sequence. We make this more precise with a couple of definitions.

DEFINITION

Partial Sum

Sequence of Partial Sums

For a series $\sum_{n=\ell}^{\infty} a_n$, we define the ***k*-th partial sum** by

$$S_k = \sum_{n=\ell}^{k} a_n = a_\ell + a_{\ell+1} + \cdots + a_k$$

The sequence $\{S_k\}$ is called the **sequence of partial sums** for the series.

Observe that the k-th partial sum S_k is the sum from the first term of the sequence up to a_k. So, depending on the index of the first term of the sequence, it may not include k terms. See Example 6.2.2 and Example 6.2.4 below.

EXAMPLE 2

Find a formula for the sequence of partial sums of $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$.

Solution: We first write out the first several terms of the sequence of partial sums.

$$\begin{aligned}
S_1 &= a_1 = \frac{1}{2} \\
S_2 &= a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
S_3 &= a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
S_4 &= a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}
\end{aligned}$$

In general, we see that the denominator of S_k will be 2^k and the numerator is one less than the denominator. That is

$$S_k = \frac{2^k - 1}{2^k}$$

DEFINITION

Convergent

Divergent

Let $\sum_{n=0}^{\infty} a_n$ be a series with sequence of partial sums $\{S_k\}$. If

$$\lim_{k\to\infty} S_k = S$$

exists, then we says that the series is **convergent** and the sum of the series is S. We write

$$\sum_{n=0}^{\infty} a_n = S$$

If the limit does not exist, then we say the series is **divergent**.

EXAMPLE 3 Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent or divergent. If it is convergent, find the sum.

Solution: In Example 6.2.2, we found that the sequence of partial sums of the series is

$$S_k = \frac{2^k - 1}{2^k}$$

Therefore, by definition, we calculate the limit of S_k. We get

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{2^k - 1}{2^k} = \lim_{k\to\infty} \left(1 - \frac{1}{2^k}\right) = 1$$

Since the limit exists, we say that the series converges, and we write

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$$

EXAMPLE 4 Determine whether the series $\sum_{n=2}^{\infty} (-1)^n$ is convergent or divergent. If it is convergent, then find the sum.

Solution: We have

$$\begin{aligned} S_2 &= 1 \\ S_3 &= 1 - 1 = 0 \\ S_4 &= 1 - 1 + 1 = 1 \end{aligned}$$

In general we have $S_{2k} = 1$ and $S_{2k+1} = 0$. Therefore, $\lim_{k\to\infty} S_k$ does not exist and hence the series diverges.

EXERCISE 1 If a series $\sum_{n=1}^{\infty} a_n$ has sequence of partial sums $\{S_k\} = \left\{\frac{2^{k+1}}{2^k + 3}\right\}$, then determine if the series converges or diverges. If it converges, find the sum.

EXERCISE 2 If a series $\sum_{n=0}^{\infty} b_n$ has sequence of partial sums $\{S_k\} = \left\{\frac{(k+1)^{3/2}}{\sqrt{k} - 1}\right\}$, then determine if the series converges or diverges. If it converges, find the sum.

6.2.2 Divergence Test

For almost all series, finding a formula for S_k is unrealistic. However, just like we saw with integrals, in real world applications, if we know the series converges, then we are satisfied with a good approximation to the sum. In the next section, we will start looking at tests for convergence of series. For now, we look at an easy test for divergence that corresponds with the fact that the size of the terms of the sequence must eventually be getting smaller for the sum of the series to exist.

THEOREM 1 **Divergence Test**

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE 5 Prove that $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Solution: We have

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0$$

Thus, the series diverges by the Divergence Test.

EXAMPLE 6 Prove that $\sum_{n=1}^{\infty} \frac{2^n}{n}$ diverges.

Solution: We have

$$\lim_{n\to\infty} \frac{2^n}{n} \stackrel{H}{=} \lim_{n\to\infty} \frac{2^n \ln(2)}{1} = \infty$$

Since the limit does not exist, the series also diverges by the Divergence Test.

EXERCISE 3 Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n}$ diverges.

EXERCISE 4 Prove that $\sum_{n=0}^{\infty} \frac{e^n}{3e^n+5}$ diverges.

EXERCISE 5 Consider a series $\sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} a_n = 0$, then what does the Divergence Test tell us about whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent?

REMARKS

1. Whenever we are testing a series for convergence, the Divergence Test is the first test that should be considered.
2. The Divergence Test can sometimes be an effective way of proving that the limit of a sequence $\{a_n\}$ is 0. In particular, if we prove that the series $\sum_{n=1}^{\infty} a_n$ converges, then we have proven that $\lim_{n\to\infty} a_n = 0$. That is, we have proven the sequence $\{a_n\}$ converges to 0.

Section 6.2 Problems

1. Determine whether the series $\sum_{n=1}^{\infty} a_n$ with the given sequence of partial sums is convergent or divergent. If it converges, find the sum.
   (a) $\{S_k\} = \left\{\frac{1}{k}\right\}_{k=1}^{\infty}$
   (b) $\{S_k\} = \left\{\frac{3^k}{3^{k+2}+1}\right\}$
   (c) $\{S_k\} = \left\{\frac{k^2+3k}{2k^2+1}\right\}$
   (d) $\{S_k\} = \left\{\frac{e^k}{k}\right\}_{k=1}^{\infty}$
2. Determine whether the sequence converges or diverges. If it converges, find its limit.
   (a) $\left\{\frac{3n^2+2}{5n^2+3}\right\}$
   (b) $\{e^n\}$
   (c) $\left\{\frac{e^n-1}{e^{2n}+1}\right\}$
   (d) $\left\{\frac{(-1)^n(n+1)}{\ln(n+2)}\right\}$
3. Determine whether the following statements are true or false.
   (a) If $\lim_{n\to\infty} a_n = 2$, then $\sum_{n=1}^{\infty} a_n$ converges.
   (b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n$ exists.
   (c) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.
4. Either show that the series diverge or that the Divergence Test is inconclusive.
   (a) $\sum_{n=1}^{\infty} \frac{2n}{3n+4}$
   (b) $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$
   (c) $\sum_{n=1}^{\infty} \frac{1}{n}$
   (d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$
   (e) $\sum_{n=3}^{\infty} \frac{n}{\ln(n)}$
   (f) $\sum_{n=3}^{\infty} \frac{(-1)^n n^2}{e^n}$
   (g) $\sum_{n=1}^{\infty} n^{1/n}$
   (h) $\sum_{n=2}^{\infty} \left(\frac{1}{n}\right)^n$
5. Show that the sequence is always decreasing.
   (a) $\{a_n\} = \left\{\frac{1}{n!}\right\}_{n=1}^{\infty}$
   (b) $\{a_n\} = \left\{\frac{n}{e^n}\right\}_{n=1}^{\infty}$
   (c) $\{a_n\} = \{\arctan(-n)\}$
   (d) $\{a_n\} = \left\{\sqrt{n+2} - n\right\}$

Section 6.3: Geometric Series

LEARNING OUTCOMES

1. Know how to recognize a geometric series.
2. Know how to use the Geometric Series Test.
3. Know how to use geometric series to represent functions.

6.3.1 Introduction to Geometric Series

Recall that a geometric sequence is one where we get the next term of the sequence by multiplying the previous term by a number r called the common ratio. We use these to define geometric series.

DEFINITION

Geometric Series

A series of the form $\sum_{n=\ell}^{\infty} ar^n = ar^\ell + ar^{\ell+1} + ar^{\ell+2} + \cdots$ where a is a non-zero number is called a **geometric series**.

EXAMPLE 1

The series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$ is a geometric series since the sequence which is being summed is a geometric sequence with common ratio $r = \frac{1}{3}$.

We can write it as $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$.

EXERCISE 1

Determine whether the series is geometric. If it is, state the value of r.

(a) $\sum_{n=2}^{\infty} 5\frac{(-1)^n}{2^{2n}}$.

(b) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$.

The following theorem gives us not only an easy way to determine if a geometric series is convergent or divergent, but also gives us the sum of the series when it converges.

THEOREM 1

Geometric Series Test

For a geometric series $\sum_{n=\ell}^{\infty} ar^n$:

If $|r| \geq 1$, then the series diverges.

If $|r| < 1$, then the series converges and $\sum_{n=\ell}^{\infty} ar^n = \frac{ar^\ell}{1-r}$.

REMARKS

1. Notice that in the formula for the sum, the ar^ℓ in the numerator just represents the first term of the series. That is, it is what we get when we substitute the first value of n into the formula for the sequence being summed.

2. The formula for the sum of a convergent geometric series follows from the well known (and very useful) formula for the k-th partial sum of the geometric series:

$$S_k = \sum_{n=\ell}^{k} ar^n = \frac{ar^\ell(1-r^{k-\ell+1})}{1-r}$$

EXAMPLE 2 Determine whether $\displaystyle\sum_{n=2}^{\infty} 2^{-n}$ converges or diverges. If it converges, find the sum.

Solution: We first rewrite this into standard form for a geometric series $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^n$. Thus, we have $r = \frac{1}{2}$. Since $|r| = \frac{1}{2} < 1$, it converges by the Geometric Series Test. We get

$$\sum_{n=2}^{\infty} 2^{-n} = \frac{2^{-2}}{1-1/2} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

EXAMPLE 3 Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3(5)^n}$ converges or diverges. If it converges, find the sum.

Solution: We first rewrite this into standard form for a geometric series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{-2}{5}\right)^n$. So, we have $r = \frac{-2}{5}$. Since $|r| = \frac{2}{5} < 1$, the series converges by the Geometric Series Test. Then, the sum is

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-2)^n}{3(5)^n} &= \frac{\frac{(-2)^1}{3(5)^1}}{1-\frac{-2}{5}} \\ &= \frac{-2}{15}\cdot\frac{5}{7} \\ &= -\frac{2}{21}\end{aligned}$$

EXAMPLE 4 Determine whether $\sum_{n=1}^{\infty} \frac{2^{2n}}{5(-3)^{n+2}}$ converges or diverges. If it converges, find the sum.

Solution: We can rewrite this as

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{5(-3)^{n+2}} = \sum_{n=1}^{\infty} \frac{(2^2)^n}{5 \cdot (-3)^2 \cdot (-3)^n} = \sum_{n=1}^{\infty} \frac{1}{45}\left(-\frac{4}{3}\right)^n$$

Thus, it is a geometric series with $r = -\frac{4}{3}$. Since $|r| = \frac{4}{3} \geq 1$, we get that the series diverges by the Geometric Series Test.

EXAMPLE 5 Determine whether $\sum_{n=1}^{\infty} \frac{1+3^n}{2^{2n}}$ converges or diverges. If it converges, find the sum.

Solution: We can rewrite this as

$$\sum_{n=1}^{\infty} \frac{1+3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4^n} + \frac{3^n}{4^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

Both geometric series converge by the Geometric Series Test as we have $|r_1| = \frac{1}{4} < 1$ and $|r_2| = \frac{3}{4} < 1$. We find that the sum is

$$\sum_{n=1}^{\infty} \frac{1+3^n}{2^{2n}} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{1}{4}} + \frac{\frac{3}{4}}{1-\frac{3}{4}} = \frac{1}{3} + 3 = \frac{10}{3}$$

EXERCISE 2 Determine whether the series converges or diverges. If it converges, find the sum.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{(-5)^n}$

(c) $\sum_{n=2}^{\infty} \frac{2^{2n}}{3^{2n}}$

REMARK

Observe that the starting term of a geometric series does not affect whether the series converges or not. This is true of series in general. That is, for example, if the series $\sum_{n=5}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ will also converge. This is because if $\sum_{n=5}^{\infty} a_n = L$ is finite, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \sum_{n=5}^{\infty} a_n$$

must also be finite.

So, in general, neither convergence or divergence of a series will depend on the starting point of the series.

6.3.2 Geometric Series as Functions

One of our main purposes for looking at series in this chapter is so that we can use them to turn a complicated function into an infinite polynomial. Observe that an infinite polynomial is really just a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

We can use our knowledge of geometric series to start exploring this idea.

Observe that the geometric series with $a = 1$, $r = x$, and $\ell = 0$ is an infinite polynomial in the variable x:

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

We recall that normally the domain of a polynomial is all real numbers. However, that is not the case with an infinite polynomial! For example, $x = 1$ is not in the domain of f since

$$f(1) = \sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + \cdots = \infty$$

This is the problem with adding up an infinite number of numbers!

To find the domain of f, we need to find all values of x such that the series converges. Using the Geometric Series Test, we get that the series converges when $|r| = |x| < 1$. Moreover, the Geometric Series Test also tells us that for any such value of x we get

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } -1 < x < 1$$

The result of this is that we have written $\dfrac{1}{1-x}$ as an infinite polynomial... but it is only valid for $-1 < x < 1$. That is, to draw the graph of $f(x) = \sum_{n=0}^{\infty} x^n$, we would just draw the graph of $\dfrac{1}{1-x}$ except only for $-1 < x < 1$.

If you are thinking to yourself something like "so what?" or "who cares?", then give yourself a pat on the back. Writing $\dfrac{1}{1-x}$ as an infinite polynomial has limited uses especially considering it is only valid on such a small interval. However, it is not the specific example that is important. We will see later in the text that the ability to write complicated functions as infinite polynomials will allow us to turn many extremely difficult problems into easy problems. For now, just focus on understanding that we can write some functions as infinite polynomials and that one way of doing this is using geometric series.

EXAMPLE 6

Let $f(x) = \sum_{n=0}^{\infty} 3(2x)^n$. Determine the domain of f and what function the series converges to for values of x in its domain.

Solution: By the Geometric Series Test, the series converges when $|r| = |2x| < 1$. Dividing both sides by two gives

$$|x| < \frac{1}{2}$$

Therefore, the domain is $-\frac{1}{2} < x < \frac{1}{2}$. For values of x in this interval, we get

$$\sum_{n=0}^{\infty} 3(2x)^n = \frac{3(2x)^0}{1-2x} = \frac{3}{1-2x}$$

EXAMPLE 7

Let $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^n}{3^n}$. Determine the domain of f and what function the series converges to for values of x in its domain.

Solution: We first rewrite this as $\sum_{n=0}^{\infty} \left(\dfrac{-x}{3}\right)^n$.

By the Geometric Series Test, the series converges when

$$|r| = \left|\frac{-x}{3}\right| < 1$$
$$|x| < 3$$

So, the domain is $(-3, 3)$. For any $-3 < x < 3$, we get

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{-x}{3}\right)^n = \frac{1}{1-\left(-\frac{x}{3}\right)} = \frac{1}{1+\frac{x}{3}} = \frac{3}{3+x}$$

EXERCISE 3

Let $f(x) = \sum_{n=0}^{\infty} 5\left(\frac{x}{2}\right)^n$. Determine the domain of f and what function the series converges to for values of x in its domain. Draw the graph of f.

EXERCISE 4

Let $f(x) = \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{2^n}$. Determine the domain of f and what function the series converges to for values of x in its domain. Draw the graph of f.

It is worth looking at an example where we actually take a number x in the domain and show that the function and series representation are equal.

EXAMPLE 8

Show if $x = \frac{1}{3}$, then $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

Solution: On the left-hand side, we have

$$\frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

On the right-hand side, we have the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

where $a = 1$, $r = \frac{1}{3}$, and $\ell = 0$. Thus, according to the Geometric Series Test, we get

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1 \cdot \left(\frac{1}{3}\right)^0}{1-\frac{1}{3}} = \frac{3}{2}$$

We again stress that the formula will only work if $|x| < 1$. If we take any x such that $|x| \geq 1$, then the geometric series will diverge, so the left-hand side will not equal the right-hand side.

EXERCISE 5

Show if $x = 2$, then $\sum_{n=1}^{\infty} x^n$ is not equal to $\frac{1}{1-x}$.

Except for special series, such as geometric series, it is generally extremely difficult to find the exact sum of a series. However, in science applications, we typically only need a suitably accurate approximation of the sum. Of course, we need to know if a series actually converges before we put time and effort into trying to approximate it. Over this and the next few sections, we look at some more tests for convergence.

Section 6.3 Problems

1. Determine whether the series is geometric. If it is, state the value of r.
   (a) $\sum_{n=0}^{\infty} \frac{3}{5^n}$
   (b) $\sum_{n=1}^{\infty} n2^n$
   (c) $\sum_{n=0}^{\infty} \frac{(n+1)}{n!}3^n$
   (d) $\sum_{n=0}^{\infty} (-2)^n$

2. Determine whether the following converge or diverge.
   (a) $\left\{\frac{n^2}{2n^2+1}\right\}$
   (b) $\sum_{n=0}^{\infty} \frac{n^2}{2n^2+1}$
   (c) $\{2^{-n}\}$
   (d) $\sum_{n=0}^{\infty} 2^{-n}$

3. Determine whether the series converges or diverges. If it converges, find the sum.
   (a) $\sum_{n=1}^{\infty} (-1.1)^n$
   (b) $\sum_{n=2}^{\infty} e^{-n}$
   (c) $\sum_{n=2}^{\infty} \frac{2^n}{n}$
   (d) $\sum_{n=0}^{\infty} \frac{3 \cdot (-2)^n}{5^n}$
   (e) $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n}$
   (f) $\sum_{n=1}^{\infty} \frac{3}{(-5)^n}$
   (g) $\sum_{n=1}^{\infty} \left(-\frac{3}{2}\right)^n$
   (h) $\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n}$

4. Determine the domain of f and what function the series converges to for values of x in its domain.
   (a) $\sum_{n=0}^{\infty} 2x^n$
   (b) $\sum_{n=1}^{\infty} 2x^n$
   (c) $\sum_{n=0}^{\infty} \frac{1}{3}(2x)^n$
   (d) $\sum_{n=2}^{\infty} (-x)^n$
   (e) $\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$
   (f) $\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{3^n}$
   (g) $\sum_{n=2}^{\infty} \left(-\frac{3}{5}\right)^n x^n$
   (h) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{4^n}$
   (i) $\sum_{n=0}^{\infty} (2x^2)^n$
   (j) $\sum_{n=0}^{\infty} 2^n x^{3n}$

5. Determine the function the series converges to.
   (a) $3 + 6x + 12x^2 + 24x^3 + 48x^4 + \cdots$
   (b) $\frac{2}{3} + \frac{2}{9}x + \frac{2}{27}x^2 + \frac{2}{81}x^3 + \cdots$
   (c) $\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{8}x^6 - \frac{1}{16}x^8 + \cdots$

Section 6.4: The Integral Test

LEARNING OUTCOMES

1. Know how to use the Integral Test.
2. Know how to recognize a P-series.
3. Know how to use the P-Series Test.

6.4.1 The Integral Test

In Section 5.2.2, we used Riemann sums to approximate the area under a graph. In particular, we saw that

$$\sum_{n=1}^{t} f(x_n)\Delta x \approx \int_a^b f(x)\,dx \tag{6.1}$$

We can create a convergence test for a series by using this process in reverse! That is, rather than using a sum to approximate an integral, we use an integral to approximate a sum. Of course, since a series is the sum of an infinite number of terms, we will need to use a limit.

We get the following test.

THEOREM 1 **Integral Test**

Assume $f(x)$ is a continuous, positive, decreasing function for $x \geq i$.

(i) If $\displaystyle\lim_{t\to\infty}\int_i^t f(x)\,dx$ exists, then the series $\displaystyle\sum_{n=i}^{\infty} f(n)$ is convergent.

(ii) If $\displaystyle\lim_{t\to\infty}\int_i^t f(x)\,dx$ does not exist, then the series $\displaystyle\sum_{n=i}^{\infty} f(n)$ is divergent.

REMARK

To justify why the Integral Test works, we can use equation (6.1) to get that

$$\sum_{n=1}^{t} f(n) \approx \int_1^t f(x)\,dx$$

and then taking the limit as $n \to \infty$ of both sides. The precise details are beyond the scope of this course and not helpful when applying the Integral Test.

Whenever we use the Integral Test, we MUST show that the conditions of the Integral Test are satisfied. That is, we must always first show that the function $f(x)$ which defines the terms of the series is

- continuous for $x \geq i$
- positive for $x \geq i$
- decreasing for $x \geq i$

where i is the starting term of the series. To show that the function is decreasing we will show that its derivative is negative.

It is very important to note that when we calculate $\lim_{t\to\infty} \int_1^t f(x)\,dx$, we MUST fully evaluate the definite integral $\int_1^t f(x)\,dx$ before we take the limit as $t \to \infty$.

EXAMPLE 1 Show that $\sum_{n=1}^{\infty} \frac{6}{n^2}$ converges.

Solution: Let $f(x) = \frac{6}{x^2}$. Clearly $f(x)$ is continuous and positive for $x \geq 1$.

We have $f'(x) = -12x^{-3} < 0$ for $x \geq 1$. Thus, f is decreasing.

Next, we get that

$$\begin{aligned}
\lim_{t\to\infty} \int_1^t \frac{6}{x^2}\,dx &= \lim_{t\to\infty} -\frac{6}{x}\Big|_1^t \\
&= \lim_{t\to\infty} \left[-\frac{6}{t} + \frac{6}{1}\right] \\
&= 0 + 6 \\
&= 6
\end{aligned}$$

Therefore, since the limit exists, the series is convergent by the Integral Test.

REMARK

The limit of the integral does not tell us the sum of the series. So, our work in the example above only tells us that the series $\sum_{n=1}^{\infty} \frac{6}{n^2}$ converges. It does not tell us what number it converges to.

In Chapter 7, we will learn some methods for finding the exact sum of certain series. As an interesting fact, it can be shown that

$$\sum_{n=1}^{\infty} \frac{6}{n^2} = \pi^2$$

EXAMPLE 2 Determine if $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

Solution: Let $f(x) = xe^{-x^2}$. Clearly $f(x)$ is continuous and positive for $x \geq 1$. We have

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2) < 0$$

for all $x \geq 1$. Thus, $f(x)$ is also decreasing.

Next, we get that

$$\begin{aligned}
\lim_{t\to\infty} \int_1^t xe^{-x^2}\, dx &= \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^t \\
&= \lim_{t\to\infty} \left[-\frac{1}{2}e^{-t^2} + \frac{1}{2}e^{-1}\right] \\
&= 0 + \frac{1}{2}e^{-1} \\
&= \frac{1}{2}e^{-1}
\end{aligned}$$

Therefore, since the limit exist, the series is convergent by the Integral Test.

EXAMPLE 3 Determine if $\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ converges or diverges.

Solution: Let $f(x) = \frac{1}{x\sqrt{\ln(x)}}$. Clearly $f(x)$ is continuous and positive for $x \geq 3$.

We have

$$f'(x) = -\left(x\sqrt{\ln(x)}\right)^{-2} \cdot \left(\sqrt{\ln(x)} + x \cdot \frac{1}{2}(\ln(x))^{-1/2} \cdot \frac{1}{x}\right) < 0$$

since $x \geq 3$. Thus, f is decreasing for $x \geq 3$.

Consider

$$\lim_{t\to\infty} \int_3^t \frac{1}{x\sqrt{\ln(x)}}\, dx = \lim_{t\to\infty} \int_3^t \frac{1}{x\sqrt{\ln(x)}}\, dx$$

To evaluate the integral, we use integration by substitution. We let $u = \ln(x)$ and get $du = \ dx$. When $x = 3$, we get $u = \ln(3)$ and when $x = t$ we get $u = \ln(t)$. Substituting these into the integral gives

$$\begin{aligned}
\lim_{t\to\infty} \int_{\ln(3)}^{\ln(t)} \frac{1}{\sqrt{u}}\, du &= \lim_{t\to\infty} 2\sqrt{u}\Big|_{\ln(3)}^{\ln(t)} \\
&= \lim_{t\to\infty} \left[2\sqrt{\ln(t)} - 2\sqrt{\ln(3)}\right] \\
&= \infty
\end{aligned}$$

Therefore, since the limit does not exist, the series is divergent by the Integral Test.

EXERCISE 1

Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$

(d) $\displaystyle\sum_{n=4}^{\infty} \frac{n^2}{n^3+1}$

6.4.2 P-series

We now introduce another special type of series.

DEFINITION
P-**Series**

A series of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ for a positive number p is called a P-series.

We can use the Integral Test to create a simple test for determining whether a P-series converges or diverges.

THEOREM 2

***P*-Series Test**

A P-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

EXAMPLE 4

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^{1.01}}$ both converge by the P-Series Test since they have values of $p = 4$ and $p = 1.01$ respectively which are both greater than one.

EXAMPLE 5

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ both diverge by the P-Series Test since they have values of $p = \frac{1}{3}$ and $p = 1$ respectively which are less than or equal to one.

DEFINITION
Harmonic Series

The divergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **harmonic series**.

The P-Series Test shows that the harmonic series is the boundary between a P-series diverging or converging. It is worth memorizing the fact that it diverges as we will regularly use that fact in the next chapter.

EXERCISE 2 Determine if the following P-series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$

Section 6.4 Problems

1. Determine whether the following converge or diverge.
   (a) $\{\arctan(n)\}$
   (b) $\left\{\frac{\ln(n)}{n+1}\right\}$
   (c) $\sum_{n=1}^{\infty} \arctan(n)$
   (d) $\sum_{n=1}^{\infty} \frac{1}{n^\pi}$
   (e) $\sum_{n=0}^{\infty} \frac{2^n}{3^n}$
   (f) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
   (g) $\sum_{n=1}^{\infty} \frac{1}{n^{-1}}$
   (h) $\sum_{n=1}^{\infty} \frac{e^n}{n^2+1}$
   (i) $\sum_{n=1}^{\infty} \frac{e^n}{2^n}$
   (j) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$
   (k) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

2. Use the Integral Test to determine whether the following converge or diverge.
   (a) $\sum_{n=1}^{\infty} \frac{1}{n}$
   (b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$
   (c) $\sum_{n=1}^{\infty} e^{1-n}$
   (d) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$
   (e) $\sum_{n=9}^{\infty} \frac{(\ln(n))^2}{n}$
   (f) $\sum_{n=1}^{\infty} \frac{n}{(n^2+2)^3}$
   (g) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

3. Explain why you cannot use the Integral Test on the following series.
   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$
   (b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

Section 6.5: Limit Comparison Test

LEARNING OUTCOMES

1. Know how to use the Limit Comparison Test.
2. Know how to recognize when to use the Limit Comparison Test.

We have previously remarked that the starting point of a series does not dictate whether the series will converge or diverge. Indeed if we add up finitely many numbers, we are always going to get a finite number. So, whether a series $\sum a_n$ converges or diverges is always going to depend on the behaviour of the sequence $\{a_n\}$ as $n \to \infty$.

The Divergence Test tells us that the terms of the sequence must tend to 0 as $n \to \infty$ for it to be possible for the series to converge. However, in the last section, we saw examples will show that just the terms of the sequence tending to 0 is not enough information to conclude that the corresponding series converges. For example, even though $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, the P-Series Test says that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges. Thus, we can conclude that the terms of the sequence $\{a_n\}$ must approach 0 suitably quickly for the series $\sum a_n$ to converge.

We now use this fact to create another convergence/divergence test. The idea behind this next test is to compare the terms of the sequence $\{a_n\}$ with the terms of another sequence $\{b_n\}$ where we know how to determine whether the series $\sum b_n$ converges or not.

The idea is that if for large value of n the values of a_n are proportional to the values of b_n, then if the values of b_n are approaching 0 fast enough for $\sum b_n$ to converge, then the values of a_n must also be approaching 0 fast enough for $\sum a_n$ to converge.

Let's see how this works in an example.

EXAMPLE 1 Show the series $\sum\limits_{n=2}^{\infty} \dfrac{4 \cdot 2^n - 3}{3^n + 4}$ converges.

Solution: This series is not geometric, so we cannot use the Geometric Series Test. However, as $n \to \infty$ the terms

$$a_n = \frac{4 \cdot 2^n - 3}{3^n + 4}$$

are roughly proportional to

$$b_n = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

Since $|r| = \frac{2}{3} < 1$, the Geometric Series Test tells us that $\sum_{n=2}^{\infty} b_n$ converges. That is, the values of b_n are approaching 0 quickly enough for convergence. Since the values of a_n are roughly proportional to b_n, they must also be approaching 0 quickly enough for convergence. Thus, we expect that $\sum_{n=2}^{\infty} \frac{4 \cdot 2^n - 3}{3^n + 4}$ is also convergent.

To show that the terms a_n are roughly proportional to the terms of b_n, we use a limit as $n \to \infty$. We get the following test.

THEOREM 1

Limit Comparison Test

Assume $a_n > 0$, $b_n > 0$. Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

1. If $0 < L < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $0 < L < \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
4. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

EXERCISE 1

Explain why case 3 in the Limit Comparison Test make sense.

EXERCISE 2

When using the Limit Comparison Test, if we get $L = 0$ and that the series $\sum_{n=1}^{\infty} b_n$ is divergent, then what can we conclude?

We use the Limit Comparison Test when the terms a_n of the series roughly resemble a series $\sum_{n=1}^{\infty} b_n$ that we know how to determine the convergence/divergence of. This will most often be a geometric series or a P-series.

The key with using the Limit Comparison Test is to pick a simple sequence $\{b_n\}$ such that you can relatively easy compute $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ and determine whether $\sum_{n=1}^{\infty} b_n$ is convergent or divergent.

EXAMPLE 2 Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{3}{5n^2+n+1}$ converges.

Solution: As $n \to \infty$, the terms of $a_n = \dfrac{3}{5n^2+n+1}$ behave like $b_n = \dfrac{1}{n^2}$.

We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{\frac{3}{5n^2+n+1}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{3}{5n^2+n+1} \cdot \frac{n^2}{1} \\ &= \lim_{n\to\infty} \frac{3}{5+\frac{1}{n}+\frac{1}{n^2}} = \frac{3}{5} \end{aligned}$$

Observe that $\displaystyle\sum_{n=1}^{\infty} b_n$ is a P-Series with $p = 2 > 1$. Therefore, it converges by the P-Series Test. Since the limit satisfies $0 < L < \infty$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, we have that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges by the Limit Comparison Test.

EXAMPLE 3 Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}-1}{5^n+2}$ converges or diverges.

Solution: As $n \to \infty$, the terms $a_n = \dfrac{2^{3n}-1}{5^n+2}$ behave like $b_n = \dfrac{8^n}{5^n}$.

We have

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{8^n-1}{5^n+2} \cdot \frac{5^n}{8^n} \\ &= \lim_{n\to\infty} \frac{1-\frac{1}{8^n}}{1+\frac{2}{5^n}} = 1 \end{aligned}$$

Observe that $\displaystyle\sum_{n=1}^{\infty} b_n$ is a geometric series with $|r| = \dfrac{8}{5} \geq 1$. Thus, it diverges by the Geometric Series Test. Since the limit satisfies $0 < L < \infty$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, we get that $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}-1}{5^n+2}$ also diverges by the Limit Comparison Test.

EXAMPLE 4 Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^3+n+2}$ converges or diverges.

Solution: As $n \to \infty$ the terms $a_n = \dfrac{\sqrt{n^2+1}}{n^3+n+2}$ behave like

$$b_n = \frac{\sqrt{n^2}}{n^3} = \frac{n}{n^3} = \frac{1}{n^2}$$

We have

$$\begin{aligned} L = \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n^3+n+2} \cdot \frac{n^2}{1} \\ &= \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^2}}}{1+\frac{1}{n^2}+\frac{2}{n^3}} \\ &= 1 \end{aligned}$$

Observe that $\displaystyle\sum_{n=1}^{\infty} b_n$ is a P-Series with $p = 2 > 1$. Therefore, it converges by the P-Series Test. Since the limit satisfies $0 < L < \infty$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, we have that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges by the Limit Comparison Test.

EXERCISE 3 Determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2-1}{n^2+5}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2+1}$

EXAMPLE 5 Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$ converges or diverges.

Solution: Let $a_n = \dfrac{\ln(n)}{n^2}$. The difficult part with this question is figuring out what to compare the series to.

Since $\ln(n)$ grows much slower than n^2, it makes sense to try $b_n = \dfrac{1}{n^2}$. This gives

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln(n)}{n^2} \cdot \frac{n^2}{1} \\ &= \lim_{n\to\infty} \ln(n) \\ &= \infty\end{aligned}$$

But, this doesn't work as $\displaystyle\sum_{n=2}^{\infty} b_n$ is a convergent P-series ($p = 2 > 1$), but the Limit Comparison Test says that if the limit equals infinity, then we MUST have the series $\displaystyle\sum_{n=2}^{\infty} b_n$ diverge. Therefore, the Limit Comparison Test fails in this case and we must try again.

Given the results of our first test, it is natural to think that perhaps the series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$ is divergent. So, let's compare it with a divergent series. Let's try taking $b_n = \dfrac{1}{n}$.

We get

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \lim_{n\to\infty} \frac{\ln(n)}{n^2} \cdot \frac{n}{1} \\ &= \lim_{n\to\infty} \frac{\ln(n)}{n} \qquad \left(\text{form } \frac{\infty}{\infty}\right) \\ &\overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} \\ &= 0\end{aligned}$$

Again the Limit Comparison Test fails as the limit $L = 0$ case requires the series $\displaystyle\sum_{n=1}^{\infty} b_n$ to be convergent. But, we have that $\displaystyle\sum_{n=1}^{\infty} b_n$ is the harmonic series which we know is divergent.

Our problem is that we are just not making the right comparison. In particular, we can't ignore the $\ln(n)$ term in the numerator. Let's approximate $\ln(n)$ with $\sqrt{n}$. This is a bad approximation, but much better than approximating $\ln(n)$ with 1.

So, we take $b_n = \dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{3/2}}$. We get

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{\ln(n)}{n^2}}{\frac{1}{n^{3/2}}} \\
&= \lim_{n\to\infty} \frac{\ln(n)}{n^2} \cdot \frac{n^{3/2}}{1} \\
&= \lim_{n\to\infty} \frac{\ln(n)}{n^{1/2}} \qquad \left(\text{form } \frac{\infty}{\infty}\right) \\
&\overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2n^{1/2}}} \\
&= \lim_{n\to\infty} \frac{2}{n^{1/2}} \\
&= 0
\end{aligned}$$

Observe that $\displaystyle\sum_{n=1}^{\infty} b_n$ is a P-series with $p = \dfrac{3}{2} > 1$, so it converges by the P-Series Test. Therefore, since the limit equals 0 and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, we get that $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$ also converges by the Limit Comparison Test.

REMARK

The moral of this last example is that you are not always going to guess the correct b_n on the first try. When you are doing practice, focus on learning how to pick the correct b_n. However, also keep in mind that if you cannot seem to find an appropriate b_n, then perhaps the Limit Comparison Test is not the test you should be using.

Section 6.5 Problems

1. Which test should be used to determine whether the series is convergent or divergent?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt[3]{n}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1-3^n}{2^{2n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+5n}{5n^2-3}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^2+1}$

2. What sequence $\{b_n\}$ should be selected to use the Limit Comparison Test on the given series?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+3}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2-2n}{5n^4-3}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+7}}{2n^2+2}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+5^n}{3^n+4^n}$

3. Determine whether the series converges or diverges.

   (a) $\sum_{n=1}^{\infty} \frac{1}{2^n + 3}$

   (b) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

   (c) $\sum_{n=1}^{\infty} \frac{4n^2 - 2n}{5n^4 - 3}$

   (d) $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{4^n}$

   (e) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{5n + 2}}$

   (f) $\sum_{n=2}^{\infty} \ln(n)$

   (g) $\sum_{n=1}^{\infty} \frac{3n + 1}{\sqrt{n^5 + 1}}$

   (h) $\sum_{n=1}^{\infty} \frac{5}{(2n + 1)^2}$

   (i) $\sum_{n=1}^{\infty} \frac{3^n}{4^n + 1}$

4. Determine whether the series converges or diverges.

   (a) $\sum_{n=0}^{\infty} \frac{n^2 + 3n}{n^5 + n + 1}$

   (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3 + 1}}{n}$

   (c) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$

   (d) $\sum_{n=3}^{\infty} \frac{(\ln(n))^2}{n^3}$

   (e) $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$

   (f) $\sum_{n=3}^{\infty} \frac{2}{n(\ln(n))^6}$

   (g) $\sum_{n=0}^{\infty} \sqrt{n + 2}$

   (h) $\sum_{n=1}^{\infty} \frac{4^n}{3^n + 4^n}$

   (i) $\sum_{n=1}^{\infty} \frac{\sqrt{n + 3}}{\sqrt{n^3 + 2n^2}}$

Section 6.6: Alternating Series

LEARNING OUTCOMES

1. Know how to recognize and work with an alternating series.
2. Know how to use the Alternating Series Test.

Other than geometric series, all of our tests for convergence so far require that all of the terms of the series be positive. We begin looking at how to test a series which is the sum of an alternating sequence.

DEFINITION
Alternating Series

A series $\sum_{n=1}^{\infty} a_n$ is called an **alternating series** if $\{a_n\}$ is an alternating sequence.

EXAMPLE 1

The series $\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{n}$ and $\sum_{n=3}^{\infty}(-1)^{n+1}\frac{n}{\ln(n)}$ are both alternating series.

The series $\sum_{n=1}^{\infty}\sin(3n)$ is *not* an alternating series since $\{a_n\} = \{\sin(3n)\}$ is not an alternating sequence.

Observe that in an alternating series half of the terms are negative and the other half are positive. That is, every time we add one term, we subtract another term. As a result, it turns out that as long as the absolute value of the terms are decreasing to 0, then an alternating series will converge. We state this as the Alternating Series Test.

THEOREM 1

Alternating Series Test

For an alternating series $\sum_{n=1}^{\infty}(-1)^n b_n \left(\text{or } \sum_{n=1}^{\infty}(-1)^{n+1} b_n\right)$ where $b_n > 0$, if

1. $\lim_{n\to\infty} b_n = 0$
2. $\{b_n\}$ is decreasing

then the alternating series converges.

The Alternating Series Test can only be used to show convergence. If $\lim_{n\to\infty} b_n = 0$, but the sequence is not decreasing, then we *must* use another test. Note that if we have $\lim_{n\to\infty} b_n \neq 0$, then we would use the Divergence Test.

EXAMPLE 2 Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Solution: We have $b_n = \dfrac{1}{n}$. We see that $\displaystyle\lim_{n\to\infty} b_n = 0$ and b_n is decreasing since

$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$$

Therefore, the series converges by the Alternating Series Test.

DEFINITION

Alternating Harmonic Series

The convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is called the **alternating harmonic series**.

Memorizing the fact that the alternating harmonic series converges will be helpful in the next chapter.

EXAMPLE 3 Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}n^2}{n^3+1}$ is convergent or divergent.

Solution: We have $b_n = \dfrac{n^2}{n^3+1}$ and $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0$.

To show that b_n is decreasing we set $f(n) = \dfrac{n^2}{n^3+1}$ and take the derivative to get

$$f'(n) = \frac{2n(n^3+1) - n^2(3n^2)}{(n^3+1)^2} = \frac{n(2-n^3)}{(n^3+1)^2} < 0$$

for $n \geq 2$. Thus, $\{b_n\}$ is decreasing for $n \geq 2$. Hence, by the Alternating Series Test, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}n^2}{n^3+1}$ is convergent.

EXAMPLE 4 Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{n}$ is convergent or divergent.

Solution: Rather than trying to apply the Alternating Series Test, we realize we can just apply the Divergence Test. To do this, recall Theorem 6.1.3 which says that since

$$\lim_{n\to\infty} \left|\frac{(-1)^n 2^n}{n}\right| = \lim_{n\to\infty} \frac{2^n}{n} \overset{H}{=} \lim_{n\to\infty} \frac{2^n \ln(2)}{1} = \infty$$

does not exist, then $\displaystyle\lim_{n\to\infty} \frac{(-1)^n 2^n}{n}$ does not exist.

Thus, the series diverges by the Divergence Test.

EXERCISE 1 Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(b) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{\ln(n)}{n}$

(c) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{2^n}{5^n}$

Section 6.6 Problems

1. List all tests (Divergence Test, Geometric Series Test, Integral Test, *P*-Series Test, Limit Comparison Test, or Alternating Series Test) that you could use to determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(n)}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{3(-1)^n}{n\ln(n)}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+3}}{5^n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{\sqrt{n^3+n}}$

   (h) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{\sqrt{n^2+3}}{\sqrt{n^3+n}}\right)$

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{e^{n^2}}$

2. Determine whether the series converges or diverges.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2^n}{n^2}\right)$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)^{\frac{1}{4}}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{7^n}$

   (e) $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n n}{n^2+9}$

   (f) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{\sqrt{n^2+2}}{n+5}\right)$

   (g) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \left(\frac{2n-1}{n^4}\right)$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{3^n}$

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$

Section 6.7: Ratio Test

LEARNING OUTCOMES

1. Know how to use the Ratio Test.
2. Know how to recognize when to use the Ratio Test and when not to use it.

We now look at our last test for convergence, the Ratio Test. This test will be extremely useful in the next chapter.

THEOREM 1 **Ratio Test**

Consider $\sum_{n=1}^{\infty} a_n$.

1. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges.
2. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series diverges.

EXERCISE 1 What does the Ratio Test tell us if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$?

Because the Ratio Test involves taking a ratio of consecutive terms, we use it when:

- a_n contains factorials
- a_n contains a mixture of exponential functions (i.e. a^n), other products (i.e. things like $2 \cdot 4 \cdot 6 \cdots (2n)$) and power functions (i.e. n^k).

Note that if the series is geometric, it is better to use the Geometric Series Test than the Ratio Test.

Notice that the Ratio Test doesn't always give us an answer. In particular, if the limit does not satisfy either condition (typically, this is when the limits equals 1), then it means that we MUST use a different test.

Learning to recognize in advance when the limit will be 1 can save you a lot of time on tests.

REMARK

Make sure you always include the absolute value signs in $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. The absolute value signs will be absolutely necessary when we use the Ratio Test in the next chapter.

EXAMPLE 1 Determine whether $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or diverges.

Solution: We have $a_n = \frac{2^n}{n!}$. We find that

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| \\
&= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0 < 1
\end{aligned}$$

Hence, the series converges by the Ratio Test.

REMARK

When applying the Ratio Test, one can typically simplify the fraction in the first step as demonstrated in the rest of the examples.

EXAMPLE 2 Determine whether $\sum_{n=1}^{\infty} \frac{3n^2}{(-2)^n}$ converges or diverges.

Solution: We have $a_n = \frac{3n^2}{(-2)^n}$. We find that

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{3(n+1)^2}{(-2)^{n+1}} \cdot \frac{(-2)^n}{3n^2} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{1}{-2} \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^2 \cdot \frac{1}{-2} \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{1+\frac{1}{n}}{1} \right)^2 \cdot \frac{1}{-2} \right| \\
&= \frac{1}{2} < 1
\end{aligned}$$

So, the series converges by the Ratio Test.

EXAMPLE 3 Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2n}{e^n}$ converges or diverges.

Solution: We have $a_n = \dfrac{2 \cdot 4 \cdot 6 \cdots 2n}{e^n}$. We find that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2 \cdot 4 \cdot 6 \cdots 2n \cdot 2(n+1)}{e^{n+1}} \cdot \frac{e^n}{2 \cdot 4 \cdot 6 \cdots 2n}\right| \\ &= \lim_{n\to\infty} \frac{2(n+1)}{e} = \infty\end{aligned}$$

So, the series diverges by the Ratio Test.

EXAMPLE 4 Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$ converges or diverges.

Solution: We have $a_n = \dfrac{(-1)^{n+1}(n!)^2}{(2n)!}$. We find that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^{n+1}(n!)^2}\right| \\ &= \lim_{n\to\infty} \frac{1 \cdot 2 \cdots 2n \cdot (n+1)! \cdot (n+1)!}{1 \cdot 2 \cdots 2n(2n+1)(2n+2) \cdot n! \cdot n!} \\ &= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+1)(2n+2)} = \frac{1}{4} < 1\end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$ converges by the Ratio Test.

EXERCISE 2 Determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^3+1}$

EXERCISE 3 Show that the Ratio Test fails for the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$.

What series test should we use?

Recall from Section 6.3.2 that we used the Geometric Series Test to find the domain of a function defined in terms of series. For example, in Example 6.3.6 we found that $f(x) = \displaystyle\sum_{n=0}^{\infty} 3(2x)^n$ converges when $|x| < \frac{1}{2}$.

We can use the Ratio Test for the same purpose. Unfortunately, since the Ratio Test is inconclusive when $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, it can only show that either the function $f(x) = \displaystyle\sum_{n=1}^{\infty} a_n(x-a)^n$ converges for all x or that there is a number R such that the series converges when $|x-a| < R$.

We demonstrate this with some examples.

EXAMPLE 5 Use the Ratio Test to show that $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{n^2}$ converges when $|x| < \frac{1}{2}$.

Solution: We have $a_n = \dfrac{(-2)^n x^n}{n^2}$. We find that

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-2)^{n+1}x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-2)^n x^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(-2)n^2}{(n+1)^2} \cdot x\right| \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2\left(1+\frac{1}{n}\right)^2} \cdot |x| \\
&= \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^2} \cdot |x| \\
&= \frac{2}{(1+0)^2} \cdot |x| \\
&= 2|x|
\end{aligned}$$

The Ratio Test, says that the series converges when $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. So, $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{n^2}$ converges when

$$\begin{aligned}
2|x| &< 1 \\
|x| &< \frac{1}{2}
\end{aligned}$$

as required.

There are a couple of important things to notice about this example.

First, since x can be positive or negative, we had to keep the absolute value signs around x.

Second, if we take $x = \pm\frac{1}{2}$, then the limit would have worked out to be

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x| = 2\left|\frac{1}{2}\right| = 1$$

So, as was commented above the example, the Ratio Test would fail in this case. If we wanted to know whether the series converged when $x = \pm\frac{1}{2}$, then we would have to use another test. We will see this in the next chapter.

EXAMPLE 6

Use the Ratio Test to show that $f(x) = \sum_{n=0}^{\infty}\frac{(x+1)^n}{n!}$ converges for all x.

Solution: We have $a_n = \frac{(x+1)^n}{n!}$. We find that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(n+1)!}\cdot\frac{(n!)}{(x+1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\ &= 0 < 1\end{aligned}$$

so, by the Ratio Test, the series will converge for any value of x.

EXAMPLE 7

Use the Ratio Test to show that $f(x) = \sum_{n=2}^{\infty}\frac{(x-1)^n}{3^n}$ converges when $|x-1| < 3$.

Solution: We have $a_n = \frac{(x-1)^n}{3^n}$. We find that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{(x-1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x-1}{3}\right| \\ &= \frac{|x-1|}{3}\end{aligned}$$

The Ratio Test, says that the series converges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$. So, $\sum_{n=2}^{\infty}\frac{(x-1)^n}{3^n}$ converges when

$$\begin{aligned}\frac{|x-1|}{3} &< 1 \\ |x-1| &< 3\end{aligned}$$

as required.

Section 6.7 Problems

1. List all tests (Divergence Test, Geometric Series Test, Integral Test, P-Series Test, Limit Comparison Test, Alternating Series Test, or Ratio Test) that you could use to determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{3^{n+1}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

   (d) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n n}{\ln(n)}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

   (f) $\displaystyle\sum_{n=3}^{\infty} \frac{3(-1)^n}{n\ln(n)}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5^n}$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+3}}{5^n}$

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{\sqrt{n^3+n}}$

   (j) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{\sqrt{n^2+3}}{\sqrt{n^3+n}}\right)$

   (k) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{e^{n^2}}$

   (l) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^3}{n!}$

   (m) $\displaystyle\sum_{n=2}^{\infty} \frac{n^3+3n^2}{n^5-2n}$

   (n) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{n^4+n^2+1}}{n^2+2}\right)$

   (o) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n(n+1)}$

2. Use the Ratio Test to determine whether the series converges or diverges, or state why you cannot use the Ratio Test.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} 3^n$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{3-n}{3n+2}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^7}$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(-3)^n}$

   (i) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2-1}{n}$

3. Use the Ratio Test to find the number R such that the series converges when $|x-a| < R$.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} 3^n x^n$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{5^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{3^n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(x+2)^n}{3^n}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(x-2)^n}{4^n}$

End of Chapter Problems

1. Find the limit of the sequence.
   (a) $\{a_n\} = \{n+1\}$
   (b) $\{a_n\} = \left\{\dfrac{1}{n}\right\}$
   (c) $\{a_n\} = \left\{\dfrac{n^2+n}{3n^2+1}\right\}_{n=1}^{\infty}$
   (d) $\{a_n\} = \left\{\dfrac{\sqrt{n^2+1}}{3n}\right\}_{n=1}^{\infty}$
   (e) $\{a_n\} = \{ne^{-n}\}_{n=1}^{\infty}$
   (f) $\{a_n\} = \{\arctan(n)\}_{n=1}^{\infty}$

2. Determine whether the series converges or diverges.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{7^n}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$
   (f) $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n^4}$
   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{2^{3n-1}}$
   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$
   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n}$
   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$
   (k) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$
   (l) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n}$

3. Determine whether the series converges or diverges.
   (a) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{\sqrt{n}}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+6}{n^4-2n+3}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{e^n-1}$
   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n+1}{4^n+8}$
   (g) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{4^n}$
   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$
   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2+1}$
   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$
   (k) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+3}$
   (l) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+5}{3^{2n}+8}$
   (m) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+5}{(2n)!}\right)$
   (n) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{\ln(n)}$
   (o) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \arctan\left(\frac{1}{n}\right)$
   (p) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2}+\frac{1}{2^n}\right)$
   (q) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{5^n}$
   (r) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Chapter 7: Power Series

We saw in Section 6.3.2 that we could represent a function as an infinite polynomial. In particular, we saw that taking $a = 1$ and $r = x$ in the geometric series gives

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

for $-1 < x < 1$. This idea of extending series from infinite sums of numbers to infinite sums of functions is extremely powerful. As we will see, it will let us approximate complicated functions and solve problems that would otherwise be impossible.

Section 7.1: Power Series

LEARNING OUTCOMES

1. Understand the definitions of power series, interval of convergence, and radius of convergence.
2. Know how to use the Geometric Series Test and Ratio Test to find the radius of convergence of a power series.
3. Know how to find the interval of convergence of a power series.
4. Know how to use geometric series to create new power series.
5. Know how to integrate and differentiate power series.

7.1.1 Introduction to Power Series

DEFINITION

Power Series

A **power series centered at a** is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where x is the independent variable, a is a constant called the **center** of the power series, and the c_n are constants, called the **coefficients** of the power series.

REMARK

When working with power series, we always take $(x-a)^0 = 1$ even when $x = a$. The reason for this special case is to keep the notation from getting more complicated.

A power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is a function with independent variable x. If we pick any value of x, then the power series turns into an infinite series of numbers like in the last chapter. Consequently, to determine the domain of this function f (all value of x for which the series converges), we can apply the series tests from the last chapter.

EXAMPLE 1 Find all values of x such that $\sum_{n=0}^{\infty} n!(x-1)^n$ converges.

Solution: Since the power series contains products, we use the Ratio Test. We get

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!(x-1)^{n+1}}{n!(x-1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)\cdot n!\cdot(x-1)}{n!}\right| \\ &= \lim_{n\to\infty}(n+1)|x-1|\end{aligned}$$

We may be very tempted just to say that this limit is infinity. However, this is not true when $x = 1$. Indeed, if $x = 1$, then we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}(n+1)|1-1| = \lim_{n\to\infty} 0 = 0$$

So, by the Ratio Test, the power series converges when $x = 1$.

If $x \neq 1$, then we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}(n+1)|x-1| = \infty$$

Thus, by the Ratio Test, the power series diverges for all $x \neq 1$. Therefore, the power series converges only when $x = 1$.

EXAMPLE 2 Find all values of x such that $\sum_{n=0}^{\infty} \frac{1}{n!}(x+1)^n$ converges.

Solution: Using the Ratio Test gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{(x+1)^n}\right| = \lim_{n\to\infty}\frac{|x+1|}{n+1} = 0$$

regardless of the value of x. Therefore, since the limit is less than one for all values of x, by the Ratio Test, the power series converges for all x.

EXAMPLE 3 Find all values of x such that $\sum_{n=0}^{\infty} \left(\frac{x+1}{2}\right)^n$ converges.

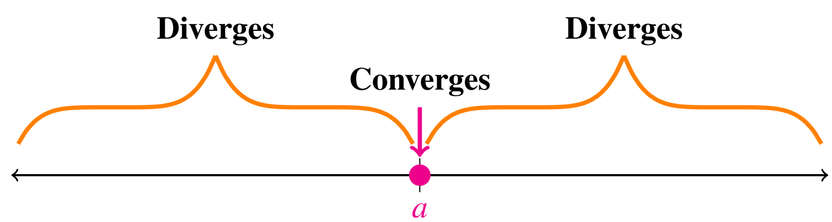
Solution: Since this is a geometric series, we use the Geometric Series Test. We get that the power series converges whenever

$$\begin{aligned} \left|\frac{x+1}{2}\right| &< 1 \\ |x+1| &< 2 \\ -2 < x+1 &< 2 \\ -3 < x &< 1 \end{aligned}$$

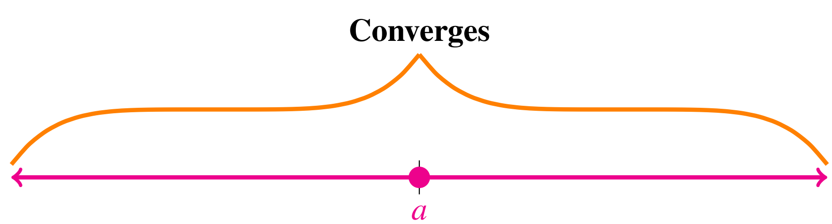
Thus, it converges for all x in the interval $(-3, 1)$.

In these examples we have seen three different possibilities for the domain of a power series.

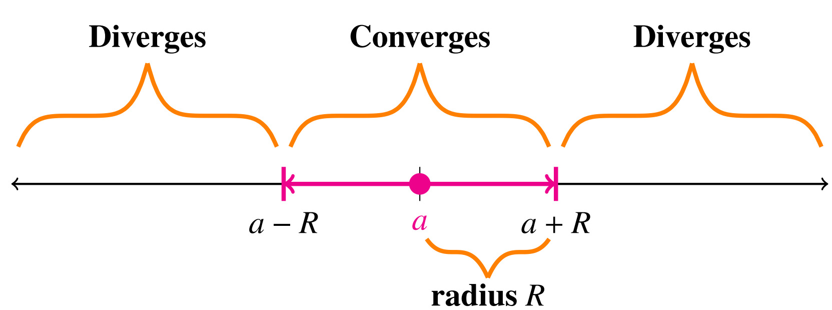
1. The power series only converges at its center a.



2. The power series converges for all x.



3. There exists an interval centered at a of radius R for which the power series converges for all x in the interval and diverges for all x outside the interval.



Before stating the theorem which shows that these are the only three possibilities, we make two definitions.

DEFINITION

Interval of Convergence

Radius of Convergence

The set of all values x for which the power series converges is called the **interval of convergence**.

The distance from the center a of the power series to either boundary of its interval of convergence is called the **radius of convergence**.

Essentially, the interval of convergence is the domain of the power series and the radius of convergence is half the length of the interval of convergence.

THEOREM 1

For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are three possibilities.

1. The power series converges only when $x = a$. Hence, the radius of convergence is $R = 0$ and the interval of convergence is $x = a$.
2. The power series converges for all x. Hence, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.
3. There exists a positive finite number R such that the power series converges if $|x-a| < R$ and diverges if $|x-a| > R$. Hence, the radius of convergence is R. The interval of convergence includes $(-R+a, R+a)$ and may contain none, one, or both of the end points of this interval.

Note that the third case gives us a short cut when the radius of convergence is a positive finite number. Rather than having to find the length of the interval and divide by 2, we immediately know the radius of convergence as soon as we get an equation saying that the power series converges when

$$|x-a| < R$$

In this text, we will always use either the Geometric Series Test or the Ratio Test to determine the radius of convergence. When using the Ratio Test, you must always remember that it is inconclusive when the limit equals 1. That is, in the third case, you must use another test to determine whether the power series converges at either endpoint of the interval. This is demonstrated in the next three examples.

EXAMPLE 4 Determine the interval and radius of convergence of $\sum_{n=1}^{\infty} \frac{1}{n5^{2n}}(x+2)^n$.

Solution: We use the Ratio Test. We get

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)5^{2(n+1)}}\cdot\frac{n5^{2n}}{(x+2)^n}\right| \\
&= \lim_{n\to\infty}\frac{n}{25(n+1)}|x+2| \\
&= \lim_{n\to\infty}\frac{1}{25\left(1+\frac{1}{n}\right)}|x+2| \\
&= \frac{1}{25}|x+2|
\end{aligned}$$

By the Ratio Test, the power series converges when

$$\begin{aligned}
\frac{|x+2|}{25} &< 1 \\
|x+2| &< 25
\end{aligned}$$

This has the form $|x-a|<R$. Thus, the radius of convergence is $R=25$.

To find the interval of convergence, we use properties of absolute values to solve this for x. We get

$$\begin{aligned}
-25 < x+2 < 25 \\
-27 < x < 23
\end{aligned}$$

Since the Ratio Test is inconclusive when the limit equals 1, the power series may or may not converge when

$$\frac{|x+2|}{25} = 1$$

Hence, to determine the interval of convergence we must also check the end points, $x=-27$ and $x=23$, of the interval.

When $x=-27$ we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n5^{2n}}(-27+2)^n &= \sum_{n=1}^{\infty}\frac{(-25)^n}{n5^{2n}} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^n(25)^n}{n\cdot 25^n} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}
\end{aligned}$$

This is the alternating harmonic series which we know converges. Therefore, $x=-27$ is in the interval of convergence.

When $x = 23$ we get

$$\sum_{n=1}^{\infty} \frac{1}{n5^{2n}}(23+2)^n = \sum_{n=1}^{\infty} \frac{(25)^n}{n \cdot 25^n}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the harmonic series which diverges. Thus, $x = 23$ is not in the interval of convergence.

Therefore, the interval of convergence for $\sum_{n=1}^{\infty} \frac{1}{n5^{2n}}(x+2)^n$ is $[-27, 23)$.

EXAMPLE 5 Determine the interval and radius of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n^3}(x-1)^n$.

Solution: We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}(x-1)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n(x-1)^n}\right|$$
$$= \lim_{n\to\infty} \frac{2n^3}{(n+1)^3}|x-1|$$
$$= \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^3}|x-1|$$
$$= 2|x-1|$$

By the Ratio Test, the power series converges when

$$2|x-1| < 1$$
$$|x-1| < \frac{1}{2}$$

Therefore, the radius of convergence is $R = \frac{1}{2}$. We have

$$|x-1| < \frac{1}{2}$$
$$-\frac{1}{2} < x-1 < \frac{1}{2}$$
$$\frac{1}{2} < x < \frac{3}{2}$$

When $x = \frac{1}{2}$ we get

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}\left(\frac{1}{2}-1\right)^n = \sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

This is an alternating series with $b_n = \dfrac{1}{n^3}$. We have that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^3} = 0$$

We also have

$$b_n = \frac{1}{n^3} > \frac{1}{(n+1)^3} = b_{n+1}$$

So, b_n is also decreasing, and hence, by the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges. Thus, $x = \dfrac{1}{2}$ is in the interval of convergence.

When $x = \dfrac{3}{2}$ we get

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}\left(\frac{3}{2} - 1\right)^n = \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

By the P-Series Test, this converges since $p = 3 > 1$. Thus, $x = \dfrac{3}{2}$ is also in the interval of convergence.

Therefore, the interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}(x-1)^n$ is $\left[\dfrac{1}{2}, \dfrac{3}{2}\right]$.

EXAMPLE 6 Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}x^{2n}$.

Solution: We have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{(n+1)^2}\cdot\frac{n^2}{x^{2n}}\right| \\ &= \lim_{n\to\infty}\frac{n^2}{(n+1)^2}\cdot|x^2| = x^2\end{aligned}$$

By the Ratio Test, the power series converges when

$$\begin{aligned} x^2 &< 1 \\ \sqrt{x^2} &= \sqrt{1} \\ |x| &< 1 \end{aligned}$$

Thus, the radius of convergence is $R = 1$. We have

$$\begin{aligned} |x| &< 1 \\ -1 < x &< 1 \end{aligned}$$

When $x = -1$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^{2n} = \sum_{n=1}^{\infty} \frac{((-1)^2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This converges by the P-Series Test since $P = 2 > 1$. So, the power series converges when $x = -1$.

Observe that the series for $x = 1$ is the same. So, the power series also converges when $x = 1$.

Therefore, the interval of convergence is $[-1, 1]$.

EXERCISE 1 Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{3^n}{n}(x-1)^n$.

EXERCISE 2 Find the radius and interval of convergence of $\sum_{n=0}^{\infty} n^2(x+2)^n$.

The Geometric Series Test does not have an inconclusive case, so we do not need to check the end points when using it.

EXAMPLE 7 Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-1)^n$.

Solution: We first put it into the standard form of a geometric series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-1)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{-(x-1)}{2}\right)^n$$

By the Geometric Series Test, we get that the power series converges if and only if

$$\begin{aligned} |r| = \left|\frac{-(x-1)}{2}\right| &< 1 \\ |x-1| &< 2 \end{aligned}$$

Thus, the radius of convergence is $R = 2$. Next, we get

$$\begin{aligned} -2 < x - 1 < 2 \\ -1 < x < 3 \end{aligned}$$

Therefore, the interval of convergence $(-1, 3)$.

EXAMPLE 8 Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{1}{3^n}(2x+4)^n$.

Solution: We first put it into the standard form of a geometric series:

$$\sum_{n=0}^{\infty} \frac{1}{3^n}(2x+4)^n = \sum_{n=0}^{\infty} \left(\frac{2x+4}{3}\right)^n$$

By the Geometric Series Test, we get that the power series converges if and only if

$$|r| = \left|\frac{2x+4}{3}\right| < 1$$
$$|x+2| < \frac{3}{2}$$

Therefore, the radius of convergence is $R = \frac{3}{2}$. We have

$$-\frac{3}{2} < x+2 < \frac{3}{2}$$
$$-\frac{7}{2} < x < -\frac{1}{2}$$

Hence, the interval of convergence is $\left(-\frac{7}{2}, -\frac{1}{2}\right)$.

EXERCISE 3 Find the radius and interval of convergence of $\sum_{n=0}^{\infty} 2^n(x-2)^n$.

EXERCISE 4 Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{1}{2^{3n+1}}(x-1)^{2n}$.

EXERCISE 5 Find the radius and interval of convergence of $\sum_{n=0}^{\infty} n!x^n$.

EXERCISE 6 Find the radius and interval of convergence of $\sum_{n=0}^{\infty} nx^n$.

7.1.2 Using a Geometric Series to Create a Power Series

In the near future, we will learn how to find a power series representation for many functions. For now, we can find a power series representation for some functions using the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

EXAMPLE 9

Find a power series for $\dfrac{1}{1+x^2}$ centered at $a = 0$ and its radius of convergence.

Solution: If we substitute $-x^2$ for x in the geometric series above, we get

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n, \quad |-x^2| < 1$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}, \quad |x| < 1$$

Thus, a power series representation for $\dfrac{1}{1+x^2}$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$. The radius of convergence is $R = 1$.

EXERCISE 7

Find a power series for $\dfrac{1}{1-2x}$ centered at $a = 0$ and its radius of convergence.

EXAMPLE 10

Find a power series for $\dfrac{x^2}{1-x}$ centered at $a = 0$ and its radius of convergence.

Solution: Multiplying both sides of $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ by x^2 gives

$$\frac{x^2}{1-x} = x^2\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+2}$$

Observe that multiplying by x^2 does not change the common ratio of the Geometric Series. So, by the Geometric Series Test, the power series converges when

$$|r| = |x| < 1$$

The radius of convergence is $R = 1$.

In general, multiplying a power series by a power function x^k, where k is a constant, does not change the radius of convergence.

EXAMPLE 11

Find a power series for $\dfrac{1}{2+x}$ centered at $a = 0$ and its radius of convergence.

Solution: First, we rewrite $\dfrac{1}{2+x}$ to make it resemble $\dfrac{1}{1-x}$. We get

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2} \cdot \frac{1}{1-(-\frac{x}{2})}$$

Thus, substituting in $-\dfrac{x}{2}$ into the geometric series and multiplying by $\dfrac{1}{2}$ gives

$$\frac{1}{2} \cdot \frac{1}{1-(-\frac{x}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n, \qquad \left|-\frac{x}{2}\right| < 1$$

$$\frac{1}{2+x} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n, \quad |x| < 2$$

Thus, a power series for $\dfrac{1}{2+x}$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$ and the radius of convergence is $R = 2$.

EXERCISE 8

Find a power series for $\dfrac{x^2}{3+2x}$ centered at $a = 0$ and its radius of convergence.

EXAMPLE 12

Find a power series for $\dfrac{1}{2+x}$ centered at $a = -1$ and its radius of convergence.

Solution: The difference between this example and the previous ones is that we are now centered at $a = -1$. By definition of a power series the power needs to be in the form $(x - a)$.

So, in this case, when we rewrite $\dfrac{1}{2+x}$, we need to ensure that we are going to have a factor $(x - (-1)) = (x + 1)$ in our substitution. We get

$$\frac{1}{2+x} = \frac{1}{1+x+1} = \frac{1}{1-(-(x+1))}$$

Therefore, we have

$$\frac{1}{2+x} = \sum_{n=0}^{\infty}(-(x+1))^n, \qquad |-(x+1)| < 1$$

$$= \sum_{n=0}^{\infty}(-1)^n(x+1)^n, \quad |x+1| < 1$$

Thus, a power series representation for $\dfrac{1}{2+x}$ centered at $a = -1$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n(x+1)^n$.
The radius of convergence is $R = 1$ and the interval of convergence is $(-2, 0)$.

EXAMPLE 13

Find a power series for $\dfrac{5}{3+2x}$ centered at $a = 1$ and its radius of convergence.

Solution: We have

$$\frac{5}{3+2x} = \frac{5}{3+2x-2+2} = \frac{5}{5+2(x-1)} = \frac{1}{1+\frac{2}{5}(x-1)} = \frac{1}{1-\left(-\frac{2}{5}(x-1)\right)}$$

So, we get

$$\begin{aligned}\frac{5}{3+2x} &= \sum_{n=0}^{\infty}\left(-\frac{2}{5}(x-1)\right)^n, \qquad \left|-\frac{2}{5}(x-1)\right| < 1\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{5^n}(x-1)^n, \quad |x-1| < \frac{5}{2}\end{aligned}$$

Thus, a power series representation for $\dfrac{5}{3+2x}$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{5^n}(x-1)^n$ and the radius of convergence is $R = \dfrac{5}{2}$.

EXERCISE 9

Find a power series for $\dfrac{1}{2+x}$ centered at $a = 1$ and its radius of convergence.

7.1.3 Working With Power Series

One of the many advantages of turning complicated functions into infinite polynomials (power series) is that it is easy to differentiate and integrate polynomials.

We begin by looking at an example.

EXAMPLE 14

Differentiate $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ to find a power series centered at $a = 0$ for $\dfrac{1}{(1-x)^2}$.

Solution: We know that

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n\\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots\end{aligned}$$

Differentiating both sides gives

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

So, $\displaystyle\sum_{n=1}^{\infty} nx^{n-1}$ is a power series representation for $\dfrac{1}{(1-x)^2}$.

In the last example, it is *very important* to not only notice that the new power series now starts at $n = 1$ instead of $n = 0$, but to understand why we made this change. It is because the first term of the original series was 1 (corresponding to $n = 0$). So, when we took the derivative, the 1 turned into a 0. Therefore, the new first term of the power series corresponds to $n = 1$.

Generalizing what we did in the example gives us the following theorem.

THEOREM 2

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with radius of convergence $R > 0$ or $R = \infty$, then

$$f'(x) = \sum_{n=1}^{\infty} c_n n(x-a)^{n-1}$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

both have radius of convergence R.

Notice that we are just differentiating or integrating the power series, which is just a polynomial, term by term. However, when differentiating, we must be careful about the first term series. If the first term is a constant, then the bottom index of the sum generally increases by 1 (as in the statement of the theorem and in Example 7.1.14). However, if the first term of the series is not a constant, then we do not increase the bottom index of the sum. This will be demonstrated in the next example.

EXAMPLE 15

Given that $f(x) = \dfrac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2}$ has radius of convergence $R = 1$, find a power series centered at $a = 0$ for $f'(x)$ and its radius of convergence.

Solution: We are given that

$$f(x) = \sum_{n=0}^{\infty} x^{n+2} = x^2 + x^3 + x^4 + \cdots$$

Thus, differentiating gives

$$f'(x) = \sum_{n=0}^{\infty} (n+2)x^{n+1} = 2x + 3x^2 + 4x^3 + \cdots$$

The theorem above tells us that since the radius of convergence of $f(x)$ is $R = 1$, the radius of convergence of $\sum_{n=0}^{\infty} (n+2)x^{n+1}$ is also $R = 1$.

EXAMPLE 16 Integrate $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ to find a power series centered at $a = 0$ for $\ln(1-x)$.

Solution: Integrating both sides of $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ gives

$$\int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n\,dx$$
$$-\ln(|1-x|) = C + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

Since the left hand side of this equation is a single function, the right hand side must also be a single function. That is, there must be a unique value of C that makes this equation true. To solve for C we can take any value of x in the interval of convergence. We take x to be the center of the series as it is easiest and guaranteed to be in the interval of convergence. In this case, taking $x = 0$ gives

$$-\ln(|1|) = C + \sum_{n=0}^{\infty} \frac{1}{n+1}(0)^{n+1}$$
$$0 = C + 0$$
$$0 = C$$

Since the original series has radius of convergence $R = 1$, the new series also has radius of convergence $R = 1$. Thus, we have

$$-\ln(|1-x|) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$
$$\ln(|1-x|) = \sum_{n=0}^{\infty} -\frac{1}{n+1} x^{n+1}$$

with radius of convergence $R = 1$. WAIT! The original question did not have absolute value signs inside of ln. We observe that $R = 1$ means $|x| < 1$. Thus, we have $-1 < x < 1$ and hence $1 - x > 0$. So we can drop the absolute value signs to get

$$\ln(1-x) = \sum_{n=0}^{\infty} -\frac{1}{n+1} x^{n+1}, \qquad |x| < 1$$

In the previous three examples, we stated what function/power series to take the derivative of to get the desired function. This will not generally be the case. And, in fact, the tricky part can be figuring out which function to begin with. The general idea for now is to use either a derivative or an integral to convert the function into something we know how find the power series of.

EXAMPLE 17

Determine the power series centered at $a = 0$ for $f(x) = \arctan(x)$ and its radius of convergence.

Solution: We want to relate $f(x)$ to a function which we know how to find a power series of. If we take the derivative of f we get

$$f'(x) = \frac{1}{1+x^2}$$

In Example 7.1.9, we found that

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

with radius of convergence $R = 1$. If we now integrate both sides of this equation we get

$$\begin{aligned} \int \frac{1}{1+x^2}\,dx &= \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx \\ \arctan(x) &= C + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \end{aligned}$$

and this new power series also has radius of convergence $R = 1$ by the theorem.

To solve for C take $x = 0$. This gives

$$\begin{aligned} \arctan(0) &= C + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(0)^{2n+1} \\ 0 &= C + 0 \end{aligned}$$

Thus, $C = 0$ and hence

$$\arctan(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

with radius of convergence $R = 1$.

EXAMPLE 18

Determine the power series centered at $a = 0$ for $f(x) = \dfrac{2x}{(1-x^2)^2}$ and its radius of convergence.

Solution: We want to relate $f(x)$ to a function which we know how to find a power series of. Taking the derivative will just make the function more complicated, so we try to integrate. Consider

$$\int \frac{2x}{(1-x^2)^2}\,dx$$

Let $u = 1 - x^2$. Then, $du = -2x\,dx$. Hence,

$$\begin{aligned}\int \frac{2x}{(1-x^2)^2}\,dx &= \int \frac{-1}{u^2}\,du \\ &= \frac{1}{u} + C \\ &= \frac{1}{1-x^2} + C\end{aligned}$$

Using the method of Section 7.1.2, we find that

$$\begin{aligned}\frac{1}{1-x^2} &= \sum_{n=0}^{\infty} (x^2)^n, \qquad |x^2| < 1 \\ &= \sum_{n=0}^{\infty} x^{2n}, \qquad |x| < 1\end{aligned}$$

Taking the derivative of both sides gives

$$\frac{2x}{(1-x^2)^2} = \sum_{n=1}^{\infty} 2nx^{2n-1}$$

and this new power series also has radius of convergence $R = 1$ by the theorem.

EXERCISE 10 Find a power series centered at $a = 0$ for $\dfrac{2}{(1-2x)^2}$ and its radius of convergence.

EXERCISE 11 Find a power series centered at $a = 0$ for $\ln(1 + x)$ and its radius of convergence.

EXERCISE 12 Find a power series centered at $a = 2$ for $\ln(x)$ and its radius of convergence.

Hint: You will need to think about what we did in Section 7.1.2.

We can use the same method for indefinite integrals.

EXAMPLE 19 Find a power series representation for $\displaystyle\int \frac{1}{1-x^3}\,dx$.

Solution: We have

$$\begin{aligned}\frac{1}{1-x^3} &= \sum_{n=0}^{\infty} (x^3)^n, \qquad |x^3| < 1 \\ &= \sum_{n=0}^{\infty} x^{3n}, \qquad |x| < 1\end{aligned}$$

Thus, integrating both sides gives

$$\int \frac{1}{1-x^3}\,dx = \int \sum_{n=0}^{\infty} x^{3n}\,dx$$
$$\int \frac{1}{1-x^3}\,dx = C + \sum_{n=0}^{\infty} \frac{1}{3n+1}x^{3n+1}$$

Since integrating does not change the radius of convergence, the radius of convergence is $R = 1$.

WARNING! The answer in the last example **must** have a $+C$ in it because it is the answer of an indefinite integral. It is very important that we always think about what we are calculating: a power series for an indefinite integral like in this question or the power series for a function like in Example 7.1.17.

EXERCISE 13

Find a power series for $\displaystyle\int \frac{1}{1+x^3}\,dx$.

We now look at how we can use power series to get infinite series representations of irrational numbers.

EXAMPLE 20

Find an infinite series representation for $\ln(2)$ using the power series for $\ln(1-x)$.

Solution: We know that

$$-\int \frac{1}{1-x}\,dx = \ln(|1-x|) + C$$

Hence, if we integrate both sides of

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

we get

$$-\ln(|1-x|) = C + \sum_{n=0}^{\infty} \frac{1}{n+1}x^{n+1}, \quad |x| < 1$$

To find C, we take $x = 0$ to get $0 = -\ln(1) = C + 0$. Hence,

$$-\ln(|1-x|) = \sum_{n=0}^{\infty} \frac{1}{n+1}x^{n+1}, \quad |x| < 1$$

To find $\ln(2)$ we use the identity that $\ln(2) = -\ln(2^{-1}) = -\ln\left(\frac{1}{2}\right)$. Therefore, since $x = \frac{1}{2}$ is in the interval of convergence, we get

$$\ln(2) = -\ln\left(\left|1-\frac{1}{2}\right|\right) = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)}$$

Section 7.1 Problems

1. Find the radius and interval of convergence of each power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3n} x^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} x^n$

   (c) $\displaystyle\sum_{n=0}^{\infty} 5^n (x-1)^n$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} (x+3)^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} n^2 x^n$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{2n+1}}{3^{2n+2}} (x-1)^n$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n} x^n$

   (h) $\displaystyle\sum_{n=3}^{\infty} \frac{x^n}{(n-2)!}$

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n$

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} (x+3)^n$

   (k) $\displaystyle\sum_{n=3}^{\infty} \frac{2^n}{n^3} (x-1)^n$

   (l) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x+2)^n$

   (m) $\displaystyle\sum_{n=0}^{\infty} n!(x-2)^{n+1}$

   (n) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

   (o) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^4}$

   (p) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n x^n}{5^n+1}$

2. Use a geometric series to find a power series centered at $a = 0$ for $f(x)$ and its radius of convergence.

   (a) $f(x) = \dfrac{x}{1-3x}$

   (b) $f(x) = \dfrac{1}{1-x^3}$

   (c) $f(x) = \dfrac{1}{1+2x^2}$

   (d) $f(x) = \dfrac{1}{2+3x}$

   (e) $f(x) = \dfrac{-3}{(2+3x)^2}$

   (f) $f(x) = \dfrac{2x}{(1+x^2)^2}$

3. For each function, find a power series centered at a and its radius of convergence.

   (a) $f(x) = x \arctan(x)$, $a = 0$

   (b) $f(x) = \ln(x)$, $a = 1$

   (c) $f(x) = \ln(x)$, $a = 3$

   (d) $f(x) = \dfrac{1}{(2-x)^2}$; $a = 0$

   (e) $f(x) = \dfrac{x}{(2-x)^3}$; $a = 0$

4. Find a power series centered at $a = 0$ for each indefinite integral and its radius of convergence.

   (a) $\displaystyle\int \frac{1}{1+x}\, dx$

   (b) $\displaystyle\int \frac{1}{1+x^5}\, dx$

   (c) $\displaystyle\int \frac{x}{1-x^4}\, dx$

   (d) $\displaystyle\int \frac{x-\arctan(x)}{x}\, dx$

5. Find an infinite series representation for π by using the power series for $\arctan(x)$ from Example 2.1.16.

Section 7.2: Taylor Series

LEARNING OUTCOMES

1. Know the how to find a Taylor series using the definition.
2. Know how to use the Binomial series.
3. Know how to find a Taylor series from known Taylor series.

So far, we have only been able to create power series by modifying the geometric series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. We now look at a formula for finding a power series representation for any function that has such a representation.

7.2.1 Introduction to Taylor Series

Assume that a function $f(x)$ has derivatives of all orders on an open interval centered at a point a. Further assume that $f(x)$ has a power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Our goal is to find a formula for the sequence $\{c_n\}$.

We first observe that we can solve for c_0 by taking $x = a$. We get

$$\begin{aligned} f(a) &= c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots \\ f(a) &= c_0 \end{aligned}$$

The trick to solving for c_1 is to take the derivative of both sides to get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

Now taking $x = a$ gives $c_1 = f'(a)$.

Repeating this procedure, we find that

$$\begin{aligned} c_2 &= \frac{1}{2} f''(a) \\ c_3 &= \frac{1}{3!} f^{(3)}(a) \end{aligned}$$

Following this pattern, we get that the general formula for c_n is

$$c_n = \frac{1}{n!} f^{(n)}(a)$$

REMARK

Observe that the formulas for the coefficients of c_0, c_1, and c_2 are exactly what we had for the linearization and the second degree Taylor polynomial in Section 4.2.

DEFINITION
Taylor Series

If a function f has derivatives of all orders on an open interval centered at a point a, then the **Taylor series for f centered at a** is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

The definition of the Taylor series requires us to find a formula for the sequence $\left\{f^{(n)}(a)\right\}$. To do this, we first generate the terms of the sequence by finding derivatives of f, starting at the 0-th derivative (which is just f itself), and substituting in the value of a. We generate terms until we can see the formula for $f^{(n)}(a)$.

We now demonstrate this with several examples.

EXAMPLE 1

Find the Taylor series centered at $a = 0$ for $f(x) = e^x$ and its radius of convergence.

Solution: We begin by computing derivatives of f and substituting in $a = 0$. We get

$$\begin{aligned} f(x) &= e^x, & f(0) &= 1 \\ f'(x) &= e^x, & f'(0) &= 1 \\ f''(x) &= e^x, & f''(0) &= 1 \end{aligned}$$

Here the pattern for the sequence $\left\{f^{(n)}(0)\right\}$ is easy. We have $f^{(n)}(0) = 1$.

Substituting this into the definition for a Taylor series, we get that the Taylor series centered at $a = 0$ for e^x is

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \end{aligned}$$

To find the radius of convergence, we use the Ratio Test. We get

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| &= \lim_{n\to\infty} \frac{|x|}{n+1} \\ &= 0 < 1 \end{aligned}$$

for any value of x. Hence, the radius of convergence is $R = \infty$.

EXAMPLE 2

Find the Taylor series of $f(x) = \frac{1}{x}$ centered at $a = 1$ and its radius of convergence.

Solution: We begin by computing derivatives of f and substituting in $a = 1$. We get

$$\begin{aligned} f(x) &= x^{-1}, & f(1) &= 1 \\ f'(x) &= -x^{-2}, & f'(1) &= -1 \\ f''(x) &= 2x^{-3}, & f''(1) &= 2 \\ f^{(3)}(x) &= -2(3)x^{-4}, & f^{(3)}(1) &= -2(3) \\ f^{(4)}(x) &= 2(3)(4)x^{-5}, & f^{(4)}(1) &= 2(3)(4) \end{aligned}$$

We see that a formula for the sequence $\left\{f^{(n)}(1)\right\}$ is $f^{(n)}(1) = (-1)^n n!$.

Thus, the Taylor series centered at $a = 1$ is

$$\begin{aligned} \frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(x-1)^n \\ &= \sum_{n=0}^{\infty} (-1)^n (x-1)^n \end{aligned}$$

To find the radius of convergence, we can use the Geometric Series Test. We get that the Taylor series converges when

$$\begin{aligned} |(-1)(x-1)| &< 1 \\ |x-1| &< 1 \end{aligned}$$

Thus, the radius of convergence is $R = 1$.

EXERCISE 1

Find the Taylor series for $f(x) = e^x$ centered at $a = 3$ and its radius of convergence.

EXERCISE 2

Find the Taylor series for $f(x) = \frac{1}{x^2}$ centered at $a = -1$ and its radius of convergence.

In the examples and exercises above, we were able to find a formula for $f^{(n)}(a)$ that worked for all values of n. Unfortunately, this is frequently not the case.

In some cases, a consistent formula for $f^{(n)}(a)$ will start for some value $n > 0$. In such cases, we just pull off the first few terms of the Taylor series as necessary. For example, if the pattern starts at $n = 3$, then we would write

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \sum_{n=3}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

EXAMPLE 3 Find the Taylor series centered at $a = 2$ for $f(x) = \ln(1 + x)$ and its radius of convergence.

Solution: We begin by computing derivatives of f and substituting in $a = 1$. We get

$$\begin{aligned}
f(x) &= \ln(1+x), & f(2) &= \ln(3)\\
f'(x) &= \frac{1}{1+x}, & f'(2) &= \frac{1}{3}\\
f''(x) &= -\frac{1}{(1+x)^2}, & f''(2) &= -\frac{1}{3^2}\\
f^{(3)}(x) &= \frac{2}{(1+x)^3}, & f^{(3)}(2) &= \frac{2}{3^3}\\
f^{(4)}(x) &= -\frac{2\cdot 3}{(1+x)^4}, & f^{(4)}(2) &= -\frac{2(3)}{3^4}\\
f^{(5)}(x) &= \frac{2\cdot 3\cdot 4}{(1+x)^5}, & f^{(5)}(2) &= -\frac{2(3)(4)}{3^5}
\end{aligned}$$

For $n \geq 1$, we get that

$$f^{(n)}(2) = (-1)^{n+1}\frac{(n-1)!}{3^n}$$

Since the pattern starts at $n = 1$, we get that

$$\begin{aligned}
\ln(1+x) &= f(1) + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n\\
&= \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{3^n n!}(x-2)^n\\
&= \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n}(x-2)^n
\end{aligned}$$

To find the radius of convergence we use the Ratio Test. We get

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{3^{n+1}(n+1)} \cdot \frac{3^n n}{(-1)^{n+1}(x-2)^n}\right| &= \lim_{n\to\infty} \frac{1}{3} \cdot \frac{n}{(n+1)}|x-2|\\
&= \lim_{n\to\infty} \frac{1}{3} \cdot \frac{1}{1+\frac{1}{n}}|x-2|\\
&= \frac{1}{3}|x-2|
\end{aligned}$$

By the Ratio Test, the series converges when

$$\begin{aligned}
\frac{1}{3}|x-2| &< 1\\
|x-2| &< 3
\end{aligned}$$

So, the radius of convergence is $R = 3$.

EXAMPLE 4 Find the Taylor series centered at $a = 0$ for $f(x) = \sin(x)$ and its radius of convergence.

Solution: We begin by computing derivatives of f and substituting in $a = 0$. We get

$$\begin{aligned} f(x) &= \sin(x), & f(0) &= \sin(0) = 0 \\ f'(x) &= \cos(x), & f'(0) &= \cos(0) = 1 \\ f''(x) &= -\sin(x), & f''(0) &= -\sin(0) = 0 \\ f^{(3)}(x) &= -\cos(x), & f^{(3)}(0) &= -\cos(0) = -1 \\ f^{(4)}(x) &= \sin(x), & f^{(4)}(0) &= \sin(0) = 0 \\ f^{(5)}(x) &= \cos(x), & f^{(5)}(0) &= \cos(0) = 1 \end{aligned}$$

Since there is a different pattern when n is even or odd, writing a formula for $f^{(n)}(0)$ is tricky. The good news is that since all the even terms (terms of the form $f^{(2n)}(0)$) are 0, we can ignore them. That means, we only really need to find a formula for

$$\left\{f^{(2n+1)}\right\} = \{1, -1, 1, -1, \ldots\}$$

We get

$$f^{(2n+1)}(0) = (-1)^n$$

Hence,

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!}(x-0)^{2n+1} \quad \text{since all the even terms are 0} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \end{aligned}$$

To find the radius of convergence we use the Ratio Test. We get

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| &= \lim_{n\to\infty} \frac{|x^2|}{(2n+2)(2n+3)} \\ &= 0 < 1 \end{aligned}$$

for any value of x. Hence, the radius of convergence is $R = \infty$.

EXERCISE 3 Verify the Taylor series centered at $a = 0$ in Example 7.2.4 is correct by expanding the sum $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ and ensuring that it is equal to

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

EXERCISE 4 Find the Taylor series of $f(x) = x\ln(x)$ centered at $a = 2$.

EXERCISE 5 Find the Taylor series of $f(x) = \sin(x)$ centered at $a = \frac{\pi}{2}$.

7.2.2 The Binomial Series

We now look at one of the most important Taylor series called the **binomial series**. The binomial series, which is an extension of the binomial approximation that we saw in Section 4.2.2, is the Taylor series centered at $a = 0$ for $f(x) = (1+x)^k$.

To construct the binomial series, we use the definition of a Taylor series on $f(x) = (1+x)^k$ with $a = 0$. We get

$$\begin{aligned} f(x) &= (1+x)^k, & f(0) &= 1 \\ f'(x) &= k(1+x)^{k-1}, & f'(0) &= k \\ f''(x) &= k(k-1)(1+x)^{k-2}, & f''(0) &= k(k-1) \\ f^{(3)}(x) &= k(k-1)(k-2)(1+x)^{k-3}, & f^{(3)}(0) &= k(k-1)(k-2) \end{aligned}$$

Hence, for $n \geq 1$ we get

$$f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

Thus, we get

$$\begin{aligned} (1+x)^k &= f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n \end{aligned}$$

DEFINITION
Binomial Series

For any real number k, the Taylor series centered at $a = 0$ of $(1+x)^k$, called the **binomial series**, is

$$\begin{aligned} (1+x)^k &= 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \end{aligned}$$

If k is a non-negative integer, then the radius of convergence is $R = \infty$. Otherwise, the radius of convergence $R = 1$.

If k is a non-negative integer, then we get $(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$ by the Binomial Theorem, where $\binom{k}{n}$ is the binomial coefficient we saw in Section 2.6.2.

EXAMPLE 5 Use the binomial series to find the Taylor series centered at $a = 0$ for $f(x) = \dfrac{1}{(1+x)^2}$ and its radius of convergence.

Solution: Observe that $\dfrac{1}{(1+x)^2} = (1+x)^{-2}$, so this is a binomial series with $k = -2$. Thus, we get

$$\begin{aligned}\frac{1}{(1+x)^2} &= 1 + \sum_{n=1}^{\infty} \frac{(-2)(-3)\cdots(-2-n+1)}{n!}x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-2)(-3)\cdots(-n-1)}{n!}x^n\end{aligned}$$

Factoring the negative out of all n terms in the numerator, we get

$$\begin{aligned}\frac{1}{(1+x)^2} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2 \cdot 3 \cdots n \cdot (n+1)}{n!}x^n \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n (n+1) x^n\end{aligned}$$

The radius of convergence is $R = 1$.

EXAMPLE 6 Use the binomial series to find the Taylor series centered at $a = 0$ for $f(x) = (1+x)^{-1/2}$ and its radius of convergence.

Solution: This is a binomial series with $k = -\dfrac{1}{2}$. Thus, we get

$$\begin{aligned}(1+x)^{-1/2} &= 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-n+\frac{1}{2}\right)}{n!}x^n\end{aligned}$$

Factoring out $\left(-\dfrac{1}{2}\right)$ from all n terms in the numerator, we get

$$\begin{aligned}(1+x)^{-1/2} &= 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}x^n\end{aligned}$$

The radius of convergence is $R = 1$.

EXERCISE 6

Use the binomial series to find the Taylor series centered at $a = 0$ for $f(x) = \dfrac{1}{(1+x)^3}$ and its radius of convergence.

EXERCISE 7

Use the binomial series to find the Taylor series centered at $a = 0$ for $f(x) = \dfrac{1}{(1+x)^{1/3}}$ and its radius of convergence.

7.2.3 Finding Taylor Series Using Known Series

As in Section 7.1.3, we can manipulate known Taylor series to create new ones.

EXAMPLE 7

Find a Taylor series centered at $a = 0$ for $f(x) = \dfrac{1}{1+x^2}$.

Solution: Wait! We've already done this in Example 7.1.9! Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is a Taylor series centered at $a = 0$, the power series we found,

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}, \qquad |x| < 1$$

is also a Taylor series centered at $a = 0$.

EXAMPLE 8

Show the Taylor series centered at $a = 0$ for $\cos(x)$ is $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!}x^{2n}$ by differentiating the Taylor series centered at $a = 0$ for $\sin(x)$ (see Example 7.2.4).

Solution: We have

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

So, taking the derivative of both sides gives

$$\begin{aligned}\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (2n+1)x^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}\end{aligned}$$

Moreover, since differentiating does not change the radius of convergence, the radius of convergence will be $R = \infty$.

EXAMPLE 9 Find a Taylor series centered at $a = 0$ for $f(x) = e^{x^2}$.

Solution: Since we know that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, we think of f as the composition of functions $f(x) = g(h(x))$ where $g(x) = e^x$ and $h(x) = x^2$. So, to calculate the Taylor series centered at $a = 0$ for f, we can just substitute $h(x) = x^2$ in for x in the Taylor series for e^x.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

EXAMPLE 10 Find a Taylor series centered at $a = 0$ for $f(x) = x\sin(2x)$.

Solution: We know that $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. Thus, to get $\sin(2x)$, we just need to substitute $2x$ in for x. This gives

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$$

Now, to get $x\sin(2x)$, we multiply both sides by x. We get

$$x\sin(2x) = x \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+2}$$

EXAMPLE 11 Find the Taylor series centered at $a = 0$ for $f(x) = (1+2x)^{-1/3}$ and its radius of convergence.

Solution: We start with the binomial series with $k = -\frac{1}{3}$. We get

$$\begin{aligned}(1+x)^{-1/3} &= 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\cdots\left(-\frac{1}{3} - n + 1\right)}{n!} x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\cdots\left(-n + \frac{2}{3}\right)}{n!} x^n\end{aligned}$$

Factoring out $\left(-\frac{1}{3}\right)$ from all n terms in the numerator, we get

$$\begin{aligned}(1+x)^{-1/3} &= 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{3}\right)^n \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)}{n!} x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)}{3^n n!} x^n\end{aligned}$$

Since the radius of convergence is $R = 1$, it converges for $|x| < 1$.

Now, substituting in $2x$ for x, we get

$$\begin{aligned}(1+2x)^{-1/3} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)}{3^n n!}(2x)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)}{3^n n!} x^n\end{aligned}$$

It convergence when

$$\begin{aligned}|2x| &< 1 \\ |x| &< \frac{1}{2}\end{aligned}$$

So, the radius of convergence is $R = \dfrac{1}{2}$.

EXERCISE 8 Find a Taylor series centered at $a = 0$ of $f(x) = e^{-2x}$.

EXERCISE 9 Find a Taylor series centered at $a = 0$ of $f(x) = x\cos(x^2)$.

EXAMPLE 12 Find the Taylor series centered at $a = 3$ for $f(x) = \dfrac{1}{x+2}$ and its radius of convergence.

Solution: By definition, a Taylor series centered at $a = 3$ will have powers of $(x-3)$. Thus, when we are changing this into the form of the geometric series, we need to make sure that our replacement for x in the geometric series will contain $x - 3$.

Observe that

$$\frac{1}{x+2} = \frac{1}{5+(x-3)} = \frac{1/5}{1+\frac{x-3}{5}} = \frac{1/5}{1-\left(-\frac{x-3}{5}\right)}$$

Hence,

$$\frac{1}{x+2} = \sum_{n=0}^{\infty} \frac{1}{5}\left(-\frac{x-3}{5}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}}(x-3)^n$$

It converges when

$$\begin{aligned}\left|-\frac{x-3}{5}\right| &< 1 \\ |x-3| &< 5\end{aligned}$$

Thus, the radius of convergence is $R = 5$.

EXERCISE 10

Find the Taylor series for $f(x) = \dfrac{1}{x+1}$ centered at $a = 1$ and its radius of convergence.

It is important to recognize that we now have two different methods for finding a Taylor series:

1. Use the definition of the Taylor series.
2. Manipulate a known series.

In some cases, only one of these methods will be feasible while in other cases, both methods will work, but one method might be considerable easier to use than the other. The only way to learn how to recognize which method to use is through practice. In particular, we recommend that you try both methods for a variety of functions to develop an intuition about which method is better in which circumstances.

EXERCISE 11

Let $f(x) = \ln(x)$.

(a) Find the Taylor series of $f(x)$ centered at $a = 1$ by integrating the answer from Example 7.2.2.

(b) Find the Taylor series of $f(x)$ centered at $a = 1$ using the definition of the Taylor series.

Useful Taylor Series to Know

Here is a list of the important Taylor series you need to know.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad R = \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad R = \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad R = \infty$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad R = 1$$

$$(1+x)^k = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = 1 + kx + \frac{k(k-1)}{2!} + \cdots, \quad R = 1$$

Section 7.2 Problems

1. For each function, find the Taylor series centered at $a = 0$ and state its radius of convergence.
   (a) $f(x) = \sin(x^2)$
   (b) $f(x) = x^2 \arctan(x)$
   (c) $f(x) = xe^{x^2}$
   (d) $f'(x)$ where $f(x) = xe^{x^2}$
   (e) $f(x) = \dfrac{1}{(1+x)^4}$
   (f) $f(x) = \dfrac{1}{(1+x)^5}$
   (g) $f(x) = x^3 - 2x + 3$
   (h) $f(x) = \dfrac{1}{x+3}$
   (i) $f(x) = \dfrac{1}{(1+x)^{1/4}}$
   (j) $f(x) = \dfrac{1}{(x-1)^2}$
   (k) $f(x) = \dfrac{1}{(2x-3)^2}$
   (l) $f(x) = \dfrac{x}{1-x^2}$
   (m) $f(x) = \dfrac{3x^2}{1+2x^2}$
   (n) $f(x) = -2x\cos(x^2)$
   (o) $f(x) = \sin(x^3)$

2. For each function, find the Taylor series centered at $a = 0$ and state its radius and interval of convergence.
   (a) $f(x) = x^4 - x^2 + 1$
   (b) $f(x) = \dfrac{1}{1-2x}$
   (c) $f(x) = \dfrac{1}{(1-3x)^3}$
   (d) $f(x) = \arctan(2x)$
   (e) $f(x) = \cos(3x)$
   (f) $f(x) = \dfrac{1}{2+3x}$
   (g) $f(x) = \dfrac{x}{(2+x)^{1/2}}$
   (h) $f(x) = \arctan(x^2)$
   (i) $f(x) = \dfrac{1}{1-x^3}$
   (j) $f(x) = -\ln(1-x)$

3. For each indefinite integral, find the Taylor series centered at a and state its radius of convergence.
   (a) $\displaystyle\int \arctan(x)\,dx;\ a = 0$
   (b) $\displaystyle\int \ln(1-x)\,dx;\ a = 0$
   (c) $\displaystyle\int \frac{x-1}{x+1}\,dx;\ a = 1$
   (d) $\displaystyle\int \ln(1+x)\,dx;\ a = 1$

4. For each function, find the Taylor series centered at a and state its radius of convergence.
   (a) $f(x) = \ln(x);\ a = 1$
   (b) $f(x) = \dfrac{1}{x};\ a = 4$
   (c) $f(x) = \dfrac{1}{x^3};\ a = 1$
   (d) $f(x) = e^x;\ a = -1$
   (e) $f(x) = 2^x;\ a = 0$
   (f) $f(x) = 2^x;\ a = 1$
   (g) $f(x) = x^3 - 2x + 3;\ a = 1$
   (h) $f(x) = x\ln(x);\ a = 1$
   (i) $f(x) = \dfrac{1}{x-3};\ a = 4$
   (j) $f(x) = \dfrac{1}{x+2};\ a = 2$
   (k) $f(x) = \dfrac{1}{x-1};\ a = 3$
   (l) $f(x) = \sin(x);\ a = \pi$

5. For each function, find the Taylor series centered at a and state its radius and interval of convergence.
   (a) $f(x) = x^4 + x^2 - x;\ a = 2$
   (b) $f(x) = \ln(x);\ a = 4$
   (c) $f(x) = \dfrac{1}{x^2};\ a = 1$
   (d) $f(x) = e^x;\ a = 2$

Section 7.3: Taylor Polynomials

LEARNING OUTCOMES

1. Know how to find Taylor polynomials.
2. Know how to approximate with Taylor polynomials.
3. Know how to find an upper bound for the error in an approximation using Taylor's Remainder Theorem.

In Section 4.2, we looked at the linearization and second degree Taylor polynomial of a function and saw how to use these to do approximations. We now use Taylor series to get higher degree Taylor polynomials.

7.3.1 Introduction to Taylor Polynomials

DEFINITION
Taylor Polynomial

If $f(x)$ has Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, then we define

$$\begin{aligned} P_{k,a}(x) &= \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

and call it the kth degree **Taylor polynomial of f centered at a**.

EXAMPLE 1

Write the third degree Taylor polynomial of e^x centered at $a = 0$.

Solution: We know $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$. To find the third-degree Taylor polynomial, we starting writing out the terms of the series until we get to the x^3 term. We get

$$P_{3,0}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

EXAMPLE 2

Write the fifth and sixth degree Taylor polynomial of $\sin(x)$ centered at $a = 0$.

Solution: We know that $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$. Thus,

$$P_{5,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Observe that since the coefficient of x^6 in the Taylor series of $\sin(x)$ centered at $a = 0$ is 0, we have that the 6th-degree Taylor polynomial of $\sin(x)$ centered at $a = 0$ is

$$\begin{aligned} P_{6,0}(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + 0x^6 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \end{aligned}$$

EXAMPLE 3

Write the second degree Taylor polynomial of $f(x) = (1 + x)^{1/3}$ centered at $a = 0$.

Solution: The binomial series tells us that

$$(1 + x)^k = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

Taking $k = \frac{1}{3}$ and writing out terms until we have a second degree polynomial gives

$$\begin{aligned} P_{2,0}(x) &= 1 + \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^2 \\ &= 1 + \frac{1}{3}x - \frac{1}{9}x^2 \end{aligned}$$

EXERCISE 1

Find the third degree Taylor polynomial $\frac{1}{1-x}$.

EXERCISE 2

Write the fourth and fifth degree Taylor polynomials of $f(x) = \cos(x)$ centered at $a = 0$.

EXERCISE 3

Write the second degree Taylor polynomial of $f(x) = (1 + x)^{1/2}$ centered at $a = 0$.

EXAMPLE 4

Write the sixth degree Taylor polynomial of $f(x) = \frac{1}{1-x^2}$ centered at $a = 0$.

Solution: We know that

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty} x^{2n}$$

Thus,

$$P_{6,0}(x) = 1 + x^2 + x^4 + x^6$$

EXAMPLE 5 Write the third degree Taylor polynomial of $f(x) = (1+3x)^{2/3}$ centered at $a = 0$.

Solution: Using the binomial series we get

$$(1+3x)^k = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}(3x)^n$$

Taking $k = \frac{2}{3}$ and writing out terms until we have a second degree polynomial gives

$$\begin{aligned} P_{1,0}(x) &= 1 + \frac{2}{3}(3x) + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)}{2!}(3x)^2 + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}(3x)^3 \\ &= 1 + 2x - x^2 + \frac{4}{3}x^3 \end{aligned}$$

EXERCISE 4 Write the third degree Taylor polynomial of $f(x) = e^{-x^2}$ centered at $a = 0$.

EXERCISE 5 Write the second degree Taylor polynomial of $f(x) = (1-2x)^{-1/3}$ centered at $a = 0$.

Of course, sometimes we won't have a known Taylor series to use to get a k-th degree Taylor polynomial. So, we will have to use the formula just as we did in the last section... but without having to figure out the pattern.

EXAMPLE 6 Write the third degree Taylor polynomial of $f(x) = \ln(x)$ centered at $a = 2$.

Solution: We have

$$\begin{aligned} f(x) &= \ln(x), & f(2) &= \ln(2) \\ f'(x) &= \frac{1}{x}, & f'(2) &= \frac{1}{2} \\ f''(x) &= -\frac{1}{x^2}, & f''(2) &= -\frac{1}{4} \\ f^{(3)}(x) &= \frac{2}{x^3}, & f^{(3)}(2) &= \frac{1}{4} \end{aligned}$$

Thus,

$$\begin{aligned} P_{3,2}(x) &= \frac{f(2)}{0!} + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3 \\ &= \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 \end{aligned}$$

EXAMPLE 7 Write the first three Taylor polynomials of $f(x) = \sqrt[3]{x}$ centered at $a = 8$.

Solution: We have

$$\begin{aligned} f(x) &= x^{1/3}, & f(8) &= 2 \\ f'(x) &= \frac{1}{3}x^{-2/3}, & f'(8) &= \frac{1}{12} \\ f''(x) &= -\frac{2}{9}x^{-5/3}, & f''(8) &= -\frac{1}{144} \end{aligned}$$

Hence, the first three Taylor polynomials centered at $a = 8$ are

$$\begin{aligned} P_{0,8}(x) &= 2 \\ P_{1,8}(x) &= \frac{2}{0!} + \frac{\frac{1}{12}}{1!}(x-8) = 2 + \frac{1}{12}(x-8) \\ P_{2,8}(x) &= \frac{2}{0!} + \frac{\frac{1}{12}}{1!}(x-8) - \frac{\frac{1}{144}}{2!}(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 \end{aligned}$$

EXERCISE 6 Find the third degree Taylor polynomial for $f(x) = x^3$ centered at $a = -1$.

Taylor Polynomial Approximations

Just as we saw in Chapter 4, we get that if $P_{k,a}(x)$ is the kth degree Taylor polynomial of $f(x)$ centered at a, then

$$f(x) \approx P_{k,a}(x)$$

for values of x close to a. Generally, if we use a higher degree Taylor polynomial, we will get a more accurate approximation. In the next section, we will learn how to find an upper bound for the error in the approximation.

EXAMPLE 8 Use the second degree Taylor polynomial $P_{2,4}(x)$ of $f(x) = \sqrt{x}$ centered at $a = 4$ to approximate $\sqrt{3}$.

Solution: We have

$$\begin{aligned} f(x) &= x^{1/2}, & f(4) &= 2 \\ f'(x) &= \frac{1}{2}x^{-1/2}, & f'(4) &= \frac{1}{4} \\ f''(x) &= -\frac{1}{4}x^{-3/2}, & f''(4) &= -\frac{1}{32} \end{aligned}$$

Hence,

$$\begin{aligned} P_{2,4}(x) &= \frac{2}{0!} + \frac{\frac{1}{4}}{1!}(x-4) + \frac{-\frac{1}{32}}{2!}(x-4)^2 \\ &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 \end{aligned}$$

We have

$$\begin{aligned} f(x) &\approx P_{2,4}(x) \\ f(3) &\approx P_{2,4}(3) \\ \sqrt{3} &\approx 2 + \frac{1}{4}(3-4) - \frac{1}{64}(3-4)^2 \\ &= \frac{113}{64} \end{aligned}$$

Note that $\frac{113}{64} \approx 1.734$ while $\sqrt{3} \approx 1.732$. So, we have quite an accurate approximation (better than the linear approximation) with minimal work.

EXAMPLE 9 Use the fourth degree Taylor polynomial $P_{4,0}(x)$ of $f(x) = e^x$ centered at $a = 0$ to approximate e.

Solution: We have

$$e^x \approx P_{4,0}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

So, taking $x = 1$ gives

$$e \approx 1 + 1 + \frac{1}{2}(1)^2 + \frac{1}{6}1^3 + \frac{1}{24}1^4 = \frac{65}{24}$$

Note that $\frac{65}{24} \approx 2.708$ while $e \approx 2.718$.

A natural question is to ask is why was the approximation in Example 7.3.8
better than the approximation if Example 7.3.9 when we used a higher degree Taylor polynomial in the latter? In the next section, we will look at factors that affect how accurate the approximation is.

EXERCISE 7 Use the second degree Taylor polynomial $P_{2,0}$ of $f(x) = \ln(1 + x)$ to approximate $\ln(2)$.

EXERCISE 8 Use the second degree Taylor polynomial $P_{2,0}$ for $f(x) = \arctan(x)$ to approximate π. Hint: Use $x = 1$ and solve for π.

7.3.2 Taylor's Remainder Theorem

When we are approximating, it is important to know how accurate the approximation is. In Section 4.2.4, we learned how to find an upper bound for the error in the linear approximation. We now extend what we did there so that we can find an upper bound for the error when approximating with an n-th degree Taylor polynomial.

THEOREM 1

Taylor's Remainder Theorem

For a function f and its kth degree Taylor polynomial $P_{k,a}$ centered at a, the error

$$R_{k,a}(x) = f(x) - P_{k,a}(x)$$

when $f(x)$ is approximated by $P_{k,a}(x)$ satisfies

$$|R_{k,a}(x)| \leq \frac{M}{(k+1)!}|x-a|^{k+1}$$

where M is the maximum value of $|f^{(k+1)}(x)|$ on the closed interval between a and x.

Taylor's Remainder Theorem shows that the accuracy when approximating $f(x)$ with a Taylor polynomial is affected by the degree k, how close x is to a, and the maximum value of $|f^{(k+1)}(x)|$ between a and x.

REMARK

To determine the maximum value of $|f^{(k+1)}(x)|$ on the closed interval between a and x we technically need to use the Closed Interval Method. However, if $|f^{(k+1)}(x)|$ is increasing or decreasing on the interval, then the the maximum value M must occur at one of the end points. In such cases, the Closed Interval Method is not necessary.

EXAMPLE 10

Find an upper bound for the error when $\sin(1)$ is approximated using the fourth degree Taylor polynomial of $f(x) = \sin(x)$ centered at $a = 0$.

Solution: To use Taylor's Remainder Theorem, we need the $k+1$-st derivative of f. We are given that $k = 4$, so we need to find the fifth derivative of $\sin(x)$. We get that

$$f^{(5)}(x) = \cos(x)$$

Next, since we need $x = 1$ and we are given that $a = 0$, we need to find the maximum M of $\left|f^{(5)}(x)\right| = |\cos(x)|$ on the interval $[0, 1]$.

Since $|\cos(x)|$ is decreasing on the interval $[0, 1]$, the maximum will be at the left end point of the interval. Therefore, we have $M = \cos(0) = 1$.

Then, Taylor's Remainder Theorem says that the absolute value of the error in the approximation $|R_{4,0}(1)|$ satisfies

$$|R_{4,0}(1)| \leq \frac{M}{(k+1)!}|x-a|^{k+1} = \frac{1}{5!}|1-0|^5 = \frac{1}{120}$$

EXAMPLE 11

Approximate $\frac{1}{e}$ using the fourth degree Taylor polynomial of $f(x) = e^x$ centered at $a = 0$, and find an upper bound for the error in the approximation.

Solution: We know that the fourth degree Taylor polynomial of $f(x) = e^x$ centered at $a = 0$ is

$$P_{4,0}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

So,

$$e^{-1} \approx P_{4,0}(-1) = 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} = \frac{3}{8}$$

We have $|f^{(5)}(x)| = e^x$. Since e^x is increasing on the interval $[-1, 0]$, the maximum of $|f^{(5)}(x)|$ on $[-1, 0]$ is $M = e^0 = 1$.

Thus, an upper bound for the error is

$$|R_{4,0}(-1)| \leq \left|\frac{1}{5!}(-1-0)^5\right| = \frac{1}{120}$$

EXAMPLE 12

Find an upper bound for the error when the second degree Taylor polynomial of $f(x) = \sqrt[3]{x}$ centered at $a = 1$ is used to approximate $\sqrt[3]{1.1}$.

Solution: We need the third derivative of f. We have

$$f'(x) = \frac{1}{3}x^{-2/3}$$
$$f''(x) = -\frac{2}{9}x^{-5/3}$$
$$f^{(3)}(x) = \frac{10}{27}x^{-8/3}$$

Since $\left|f^{(3)}(x)]\right|$ is decreasing on $[1, 1.1]$, the maximum will occur at the left end point $x = 1$. Thus,

$$M = \frac{10}{27(1)^{8/3}} = \frac{10}{27}$$

Hence, by Taylor's Remainder Theorem,

$$|R_{2,1}(1.1)| \leq \frac{\frac{10}{27}}{3!}|1.1-1|^3 = \frac{5}{81000}$$

EXERCISE 9

Approximate $\frac{1}{(1.1)^2}$ by using the second degree Taylor polynomial for $f(x) = \frac{1}{x^2}$ centered at $a = 1$ and find an upper bound for the error in the approximation.

Section 7.3 Problems

1. Find the kth degree Taylor polynomial centered at $a = 0$ of f for $k = 1, 2, 3$.
   (a) $f(x) = e^{-x}$
   (b) $f(x) = \cos(x)$
   (c) $f(x) = \sin(x^2)$
   (d) $f(x) = \arctan(x)$
   (e) $f(x) = \sqrt{1+x}$
   (f) $f(x) = \dfrac{1}{1+x}$
   (g) $f(x) = \ln(x+1)$
   (h) $f(x) = x^3$
   (i) $f(x) = 1 - 2x + 3x^2 + 2x^4$
   (j) $f(x) = \dfrac{1}{\sqrt{1+x}}$
   (k) $f(x) = \sin(\pi x)$
   (l) $f(x) = \tan(x)$
   (m) $f(x) = \dfrac{1}{1+2x}$
   (n) $f(x) = (1+x)^{1/3}$
   (o) $f(x) = (1+2x)^{-1/5}$
   (p) $f(x) = \arcsin(x)$
   (q) $f(x) = \displaystyle\int_0^x e^{t^2}\,dt$

2. Find the second degree Taylor polynomial centered at a of f.
   (a) $f(x) = x^3$; $a = 1$
   (b) $f(x) = 1 + x^2$; $a = 2$
   (c) $f(x) = \cos(x)$; $a = \dfrac{\pi}{4}$
   (d) $f(x) = \sin(x)$; $a = \dfrac{\pi}{2}$
   (e) $f(x) = e^{-x}$; $a = \ln(2)$
   (f) $f(x) = \dfrac{1}{x}$; $a = -1$
   (g) $f(x) = \sqrt{x}$; $a = 9$
   (h) $f(x) = \dfrac{1}{\sqrt{x}}$; $a = 4$
   (i) $f(x) = \arctan(x) + x^2$; $a = 1$

3. For each function, find the Taylor series centered at a and state is radius and interval of convergence.
   (a) $f(x) = \dfrac{1}{2-3x}$; $a = 0$
   (b) $f(x) = \dfrac{1}{(1+x)^3}$; $a = 0$
   (c) $f(x) = 2^{-x}$; $a = 1$
   (d) $f(x) = \displaystyle\int \frac{1}{x^2+2}\,dx$; $a = 0$

4. Approximate the given value using the second degree Taylor polynomial centered at a of f. Use Taylor's Remainder Theorem to find an upper bound on the error in the approximation.
   (a) $\cos(0.2)$ using $f(x) = \cos(x)$ at $a = 0$
   (b) $\ln(1.5)$ using $f(x) = \ln(x)$ at $a = 1$
   (c) $\sqrt{1.1}$ using $f(x) = \sqrt{1+x}$ at $a = 0$
   (d) $\sqrt[3]{2}$ using $f(x) = \sqrt[3]{x}$ at $a = 1$
   (e) $e^{-0.2}$ using $f(x) = e^{-2x}$ at $a = 0$
   (f) $\dfrac{1}{2.1}$ using $f(x) = \dfrac{1}{x}$ at $a = 2$
   (g) $\cos\left(\dfrac{2\pi}{5}\right)$ using $f(x) = \cos(x)$ at $a = \dfrac{\pi}{2}$
   (h) $0.1 + e^{-0.1}$ using $f(x) = x + e^{-x}$ at $a = 0$

5. Approximate the given value using the fourth degree Taylor polynomial centered at a of f. Use Taylor's Remainder Theorem to find an upper bound on the error in the approximation.
   (a) $\cos(2)$ using $f(x) = \cos(2x)$; $a = 0$
   (b) $(-2)^{5/3}$ using $f(x) = x^{5/3}$; $a = -1$

6. Use Taylor's Remainder Theorem with $k = 2$ to show that

$$\left|\sqrt{1+x} - \left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right)\right| \le \frac{x^3}{16}, \qquad \text{for all } x \ge 0$$

End of Chapter Problems

1. Find the radius and interval of convergence of each power series.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n^3} x^n$
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!} x^n$
   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n} (x+1)^n$
   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{3n}}{5^n} (x-4)^n$

2. Find the Taylor series of f centered at a and its radius of convergence.
   (a) $f(x) = \dfrac{x}{1+2x^2}$; $a = 0$
   (b) $f(x) = \dfrac{2}{3-x}$; $a = 0$
   (c) $f(x) = \dfrac{2}{3-x}$; $a = 1$
   (d) $f(x) = \dfrac{1}{(2+x)^4}$; $a = 0$
   (e) $f(x) = \cos(x^2)$; $a = 0$
   (f) $f(x) = \dfrac{1}{(1-2x)^{2/3}}$; $a = 0$
   (g) $f(x) = x^2 e^{x^2}$; $a = 0$
   (h) $f(x) = \ln(|x|)$; $a = -1$
   (i) $f(x) = \displaystyle\int_0^x \ln(1+2t)\,dt$; $a = 0$

3. Find the Taylor series of f centered at $a = 0$ for each indefinite integral and its radius of convergence.
   (a) $\displaystyle\int x\cos(x)\,dx$
   (b) $\displaystyle\int \frac{\sin(x)}{x}\,dx$
   (c) $\displaystyle\int e^{-x^2}\,dx$
   (d) $\displaystyle\int \frac{x^2}{1+x^4}\,dx$
   (e) $\displaystyle\int xe^x\,dx$

4. Find an upper bound for the error when the given value is approximated using the second degree Taylor polynomial of f centered at a.
   (a) $f(x) = \dfrac{1}{x^2}$; $a = 1$; $\dfrac{1}{(0.99)^2}$
   (b) $f(x) = \ln(x)$; $a = 1$; $\ln(1.2)$
   (c) $f(x) = \cos(x)$; $a = 0$; $\cos(-0.1)$
   (d) $f(x) = \sqrt{3+x}$; $a = 1$; $\sqrt{3.1}$
   (e) $f(x) = e^{2x}$; $a = 0$; $e^{-0.2}$

5. For each function
   (i) Find the Taylor series of f centered at a along with its radius and interval of convergence.
   (ii) Write the 2nd degree Taylor polynomial of f centered at a.
   (iii) Find an upper bound for the error when the given value is approximated using the second degree Taylor polynomial centered at a.

   (a) $f(x) = \sin(2x)$; $a = 0$; $\cos(0.1)$
   (b) $f(x) = \ln(x)$; $a = 1$; $\ln(0.8)$.
   (c) $f(x) = xe^x$; $a = 0$; $1.2e^{1.2}$.
   (d) $f(x) = xe^x$; $a = 1$; $1.2e^{1.2}$.
   (e) $f(x) = \dfrac{1}{2-x}$; $a = 0$; $\dfrac{1}{1.99}$

6. Use Taylor series to evaluate the limit.
   (a) $\displaystyle\lim_{x\to 0} \frac{x - \arctan(x)}{x^3}$
   (b) $\displaystyle\lim_{x\to 0} \frac{e^{x^2} - 1}{x^2}$
   (c) $\displaystyle\lim_{x\to 0} \frac{\cos(x) - 1}{x^2}$

7. Use a 2nd degree Taylor polynomial centered at $a = 0$ to approximate the definite integral.
   (a) $\displaystyle\int_0^{0.5} \frac{1}{1-x^2}\,dx$
   (b) $\displaystyle\int_1^2 \frac{\sin(x)}{x}\,dx$
   (c) $\displaystyle\int_0^1 (1+x^2)^{1/3}\,dx$

Chapter 8: Techniques of Integration

In Section 5.1, we looked at our first technique of integration: integration by substitution. In this chapter, we will examine some additional techniques of integration. As you proceed through this chapter, it is important that you not only focus on the techniques themselves, but also pay close attention to how to recognize when to use each technique. The most difficult part of many integration problems is identifying which technique(s) of integration to use.

You will see that integration by substitution occurs regularly with these new techniques. As a result, it is strongly recommended that you review Section 5.1 before proceeding with this chapter.

We begin with a list of integrals that you should know. Mastering these basic integrals will help you when you are solving more complicated integrals.

Integration Formulas

$$\int 0\,dx = C \qquad \int \cos(x)\,dx = \sin(x) + C$$

$$\int \frac{1}{x}\,dx = \ln(|x|) + C \qquad \int \sin(x)\,dx = -\cos(x) + C$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1 \qquad \int \tan(x)\,dx = \ln(|\sec(x)|) + C$$

$$\int e^x\,dx = e^x + C \qquad \int \sec(x)\,dx = \ln(|\sec(x) + \tan(x)|) + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \qquad \int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \qquad \int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C \qquad \int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C \qquad \int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

Section 8.1: Trigonometric Integrals

LEARNING OUTCOMES

1. Know how to solve integrals containing sin and cos.
2. Know how to solve integrals containing tan and sec.

The general strategy for solving integrals which only contain trigonometric functions is to use trigonometric identities to get it into a form where we can use integration by substitution.

We will frequently use the following trigonometric identities:

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ 1 + \tan^2(x) &= \sec^2(x) \\ \sin^2(x) &= \frac{1}{2}(1 - \cos(2x)) \\ \cos^2(x) &= \frac{1}{2}(1 + \cos(2x)) \end{aligned}$$

8.1.1 Sine-Cosine Integrals

We start by looking at integrals of the form $\int \sin^n(x) \cos^m(x)\, dx$. The general strategy will be to use the identity $\sin^2(x) + \cos^2(x) = 1$ so that we can use a substitution of either $u = \sin(x)$ or $u = \cos(x)$.

EXAMPLE 1

Evaluate $\int \cos^3(x)\, dx$.

Solution: Our first thought would probably be to try a substitution of $u = \cos(x)$. We get $du = -\sin(x)\, dx$. But, there is no $\sin(x)$ term to replace. So, this doesn't work.

So, we use the identity $\cos^2(x) = 1 - \sin^2(x)$ to get

$$\int \cos^3(x)\, dx = \int \cos^2(x) \cos(x)\, dx = \int (1 - \sin^2(x)) \cos(x)\, dx$$

Now, if we let $u = \sin(x)$, we get $du = \cos(x)\, dx$. We do have a $\cos(x)\, dx$ to replace! Doing so gives

$$\begin{aligned} \int \cos^3(x)\, dx &= \int (1 - \sin^2(x)) \cos(x)\, dx \\ &= \int (1 - u^2)\, du \\ &= u - \frac{1}{3}u^3 + c \\ &= \sin(x) - \frac{1}{3}\sin^3(x) + C \end{aligned}$$

From this example, we see that if we can get the integrand into the form $\sin^n(x)\cos(x)$, then we can use a substitution of $u = \sin(x)$. Similarly, if we can get the integrand into the form of $\cos^m(x)\sin(x)$, we will be able to use $u = \cos(x)$.

EXAMPLE 2 Evaluate $\int \sin^3(x)\cos^4(x)\,dx$.

Solution: Since the power of $\sin(x)$ is odd, we can factor out a $\sin(x)$ and use $\sin^2(x) = 1 - \cos^2(x)$ so that the rest of the integrand is in terms of powers of $\cos(x)$. This will let us use a substitution of $u = \cos(x)$.

$$\int \sin^3(x)\cos^4(x)\,dx = \int \sin(x)\sin^2(x)\cos^4(x)\,dx = \int \sin(x)\,(1-\cos^2(x))\cos^4(x)\,dx$$

Let $u = \cos(x)$, then $du = -\sin(x)\,dx$ and hence $-\,du = \sin(x)\,dx$. This gives

$$\begin{aligned}
\int \sin^3(x)\cos^4(x)\,dx &= \int (1-u^2)\cdot u^4\cdot(-1)\,du \\
&= \int (u^6 - u^4)\,du \\
&= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C \\
&= \frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + C
\end{aligned}$$

EXAMPLE 3 Evaluate $\int \sin^4(x)\cos^2(x)\,dx$.

Solution: Since the powers of both $\sin(x)$ and $\cos(x)$ are even, we will not be able to get the desired factor of $\sin(x)$ or $\cos(x)$ to make a substitution of $u = \cos(x)$ or $u = \sin(x)$ work. Thus, we instead use the identities

$$\begin{aligned}
\sin^2(x) &= \frac{1}{2}(1-\cos(2x)) \\
\cos^2(x) &= \frac{1}{2}(1+\cos(2x))
\end{aligned}$$

We get

$$\begin{aligned}
\int \sin^4(x)\cos^2(x)\,dx &= \int \left(\frac{1}{2}(1-\cos(2x))\right)^2\left(\frac{1}{2}(1+\cos(2x))\right)dx \\
&= \frac{1}{8}\int (1-2\cos(2x)+\cos^2(2x))(1+\cos(2x))\,dx \\
&= \frac{1}{8}\int (1-\cos(2x)-\cos^2(2x)+\cos^3(2x))\,dx
\end{aligned}$$

Let $u = 2x$, then $du = 2\,dx$. This gives

$$\int \sin^4(x)\cos^2(x)\,dx = \frac{1}{8}\int \frac{1}{2}\left[1 - \cos u - \cos^2(u) + \cos^3(u)\right]\,du$$
$$= \frac{1}{8}\int \frac{1}{2}\left[1 - \cos u - \left[\frac{1}{2}(1 + \cos(2u)\right] + \cos^3(u)\right]\,du$$

Integrating (using the result of Example 8.1.1 for the integral of $\cos^3(u)$) gives

$$\int \sin^4(x)\cos^2(x)\,dx = \frac{1}{16}\left[u - \sin(u) - \left[\frac{1}{2}\left(u + \frac{1}{2}\sin(2u)\right)\right] + \sin(u) - \frac{1}{3}\sin^3(u)\right] + C$$
$$= \frac{1}{16}\left[2x - \sin(2x) - \frac{2x}{2} - \frac{1}{4}\sin(2\cdot 2x) + \sin(2x) - \frac{1}{3}\sin^3(2x)\right] + C$$
$$= \frac{1}{16}\left[x - \frac{1}{4}\sin(4x) - \frac{1}{3}\sin^3(2x)\right] + C$$

We summarize the technique.

ALGORITHM

For $\int \sin^n(x)\cos^m(x)\,dx$:

- If n is positive and odd, then factor out $\sin(x)$, use $\sin^2(x) = 1 - \cos^2(x)$ as necessary, and let $u = \cos(x)$.
- If m is positive and odd, then factor out $\cos(x)$, use $\cos^2(x) = 1 - \sin^2(x)$ as necessary, and let $u = \sin(x)$.
- If both m and n are non-negative and even, then use

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \text{ and } \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

to rewrite the integral as powers of $\cos(2x)$.

EXERCISE 1 Evaluate $\int \sin^3(x)\cos^2(x)\,dx$.

EXERCISE 2 Evaluate $\int \sin^4(x)\cos^5(x)\,dx$.

EXERCISE 3 Evaluate $\int \sin^2(x)\cos^2(x)\,dx$.

8.1.2 Tangent-Secant Integrals

For integrals of the form $\int \sec^n(x)\tan^m(x)\,dx$, we essentially use the same strategy using the identity $\sec^2(x) = 1 + \tan^2(x)$. However, in this case, we need to remember that if we take $u = \sec(x)$, then $du = \sec(x)\tan(x)\,dx$, so we will need a $\sec(x)\tan(x)$ to replace, and if we take $u = \tan(x)$, then $du = \sec^2(x)\,dx$, so we will need a $\sec^2(x)$ to replace.

EXAMPLE 4 Evaluate $\int \sec^4(x)\tan(x)\,dx$.

Solution: Letting $u = \tan(x)$ gives $du = \sec^2(x)\,dx$. We can then replace the remaining $\sec^2(x)$ using the identity $\sec^2(x) = 1 + \tan^2(x)$. That is, we get

$$\begin{aligned}\int \sec^4(x)\tan(x)\,dx &= \int \sec^2(x)\tan(x)\sec^2(x)\,dx\\ &= \int (1+\tan^2(x))\tan(x)\sec^2(x)\,dx\\ &= \int (1+u^2)u\,du\\ &= \int (u+u^3)\,du\\ &= \frac{1}{2}u^2 + \frac{1}{4}u^4 + C\\ &= \frac{1}{2}\tan^2(x) + \frac{1}{4}\tan^4(x) + C\end{aligned}$$

EXAMPLE 5 Evaluate $\int \sec^3(x)\tan^3(x)\,dx$.

Solution: We can rewrite this using $\tan^2(x) = \sec^2(x) - 1$ to get

$$\begin{aligned}\int \sec^3(x)\tan^3(x)\,dx &= \int \sec^2(x)\tan^2(x)\cdot\sec(x)\tan(x)\,dx\\ &= \int \sec^2(x)\cdot(\sec^2(x)-1)\cdot\sec(x)\tan(x)\,dx\end{aligned}$$

Now, let $u = \sec(x)$ and get $du = \sec(x)\tan(x)\,dx$. Thus, we have

$$\begin{aligned}\int \sec^3(x)\tan^3(x)\,dx &= \int u^2(u^2-1)\,du\\ &= \int (u^4-u^2)\,du\\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C\\ &= \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C\end{aligned}$$

We again summarize the method.

ALGORITHM

For $\displaystyle\int \sec^n(x)\tan^m(x)\,dx$:

- If n is positive and even, then factor out a $\sec^2(x)$, use $\sec^2(x) = 1 + \tan^2(x)$ as necessary on remaining powers of $\sec^2(x)$, and let $u = \tan(x)$.
- If m is positive and odd, then factor out a $\sec(x)\tan(x)$, use $\tan^2(x) = \sec^2(x) - 1$ as necessary, and let $u = \sec(x)$.

EXERCISE 4 Evaluate $\displaystyle\int \sec(x)\tan^3(x)\,dx$.

EXERCISE 5 Evaluate $\displaystyle\int \sec^4(x)\tan^3(x)\,dx$.

Section 8.1 Problems

1. Evaluate the following integrals.
   (a) $\displaystyle\int \sin(x)\cos^2(x)\,dx$
   (b) $\displaystyle\int \sin^3(x)\cos(x)\,dx$
   (c) $\displaystyle\int \sec(x)\tan^5(x)\,dx$
   (d) $\displaystyle\int \sec^2(x)\tan^2(x)\,dx$
   (e) $\displaystyle\int \sin^2(x)\cos^5(x)\,dx$
   (f) $\displaystyle\int \sec^4(x)\tan^2(x)\,dx$
   (g) $\displaystyle\int \sin^3(x)\cos^3(x)\,dx$
   (h) $\displaystyle\int \sin^4(x)\cos^3(x)\,dx$
   (i) $\displaystyle\int \sec^5(x)\tan^3(x)\,dx$
   (j) $\displaystyle\int \sin^3(x)\,dx$

2. Evaluate the following integrals.
   (a) $\displaystyle\int \sin^2(x)\,dx$
   (b) $\displaystyle\int \cos^2(x)\,dx$
   (c) $\displaystyle\int \cos^{1/2}(x)\sin(x)\,dx$
   (d) $\displaystyle\int \cos^5(x)\,dx$
   (e) $\displaystyle\int \sec^4(x)\,dx$
   (f) $\displaystyle\int \frac{\sec^2(x)}{\tan(x)}\,dx$
   (g) $\displaystyle\int \csc^2(x)\cos^3(x)\,dx$
   (h) $\displaystyle\int \cos(x)\tan^3(x)\,dx$
   (i) $\displaystyle\int \frac{\sin^2(x)}{\cos^2(x)}\,dx$
   (j) $\displaystyle\int \sin^4(x)\,dx$

Section 8.2: Trigonometric Substitutions

LEARNING OUTCOMES

1. Know how to solve integrals containing $a^2 - b^2x^2$ using $\sin(\theta)$.
2. Know how to solve integrals containing $b^2x^2 - a^2$ using $\sec(\theta)$.
3. Know how to solve integrals containing $a^2 + b^2x^2$ using $\tan(\theta)$.

We now turn our attention to solving another common type of integral: integrals which involve $a^2 - b^2x^2$, $b^2x^2 - a^2$, or $a^2 + b^2x^2$.

We first demonstrate the idea behind the method with an example.

EXAMPLE 1 Show that $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$.

Solution: Observe that $1 - x^2$ matches the form of the trigonometric identity

$$1 - \sin^2(\theta) = \cos^2(\theta)$$

This and the fact that we know the answer is $\arcsin(x)$ gives us the idea to use the change of variables $x = \sin(\theta)$ or more precisely $\theta = \arcsin(x)$. This gives

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = |\cos(\theta)| \tag{8.1}$$

Since we are using the change of variables $\theta = \arcsin(x)$ and the range of $\arcsin(x)$ is $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, the domain is restricted to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. On this interval, we have that $\cos(\theta) \ge 0$, so from equation (8.1) we get

$$\sqrt{1-x^2} = |\cos(\theta)| = \cos(\theta) \tag{8.2}$$

We now treat this like a normal integration by substitution. We find that

$$dx = \cos(\theta)\,d\theta$$

Using this and equation (8.2) gives

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\cos(\theta)}\cos(\theta)\,d\theta \\
&= \int 1\,d\theta \\
&= \theta + c \\
&= \arcsin(x) + C
\end{aligned}$$

As demonstrated in the example, the idea is to use a substitution with an inverse trigonometric function. However, to make it easier, we always rewrite the substitution so that we don't have to work with the inverse trigonometric function.

For rigor, the three cases are outlined below. However, the summary table below these methods are the key things to remember.

ALGORITHM

For an integral containing $a^2 - b^2x^2$ we use the change of variables $\theta = \arcsin\left(\frac{b}{a}x\right)$ which we write in the form

$$x = \frac{a}{b}\sin(\theta)$$

This gives

$$a^2 - b^2x^2 = a^2 - a^2\sin^2(\theta) = a^2(1 - \sin^2(\theta)) = a^2\cos^2(\theta)$$

Additionally, since the change of variables restricts the domain to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, we have that $\cos(\theta) \geq 0$.

ALGORITHM

For an integral containing $b^2x^2 - a^2$ we use the change of variables $\theta = \operatorname{arcsec}\left(\frac{b}{a}x\right)$ which we write as

$$x = \frac{a}{b}\sec(\theta)$$

This gives us

$$b^2x^2 - a^2 = a^2\sec^2(\theta) - a^2 = a^2(\sec^2(\theta) - 1) = a^2\tan^2(\theta)$$

Additionally, since the change of variables restricts the domain to $0 \leq \theta < \frac{\pi}{2}$, $\pi \leq \theta < \frac{3\pi}{2}$, we have that $\tan(\theta) \geq 0$.

ALGORITHM

For an integral containing $a^2 + b^2x^2$ we use a change of variables $\theta = \arctan\left(\frac{b}{a}x\right)$ which we write as

$$x = \frac{a}{b}\tan(\theta)$$

Then we get

$$a^2 + b^2x^2 = a^2 + a^2\tan^2(\theta) = a^2(1 + \tan^2(\theta)) = a^2\sec^2(\theta)$$

Additionally, since the change of variables restricts the domain to $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we have that $\sec(\theta) \geq 0$.

We summarize these in a table

Type	Substitution	Replacement	Restriction Gives
$a^2 - b^2x^2$	$x = \frac{a}{b}\sin(\theta)$	$a^2\cos^2(\theta)$	$\cos(\theta) \geq 0$
$b^2x^2 - a^2$	$x = \frac{a}{b}\sec(\theta)$	$a^2\tan^2(\theta)$	$\tan(\theta) \geq 0$
$a^2 + b^2x^2$	$x = \frac{a}{b}\tan(\theta)$	$a^2\sec^2(\theta)$	$\sec(\theta) \geq 0$

We now look at some examples to see how this works.

EXAMPLE 2 Determine $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$.

Solution: This has the form $a^2+b^2x^2$ where $a = 2$ and $b = 1$. So, we let $x = \frac{2}{1}\tan(\theta)$. We get $dx = 2\sec^2(\theta)\,d\theta$ and the replacement

$$\sqrt{4+x^2} = \sqrt{4\sec^2(\theta)} = |2\sec(\theta)| = 2\sec(\theta)$$

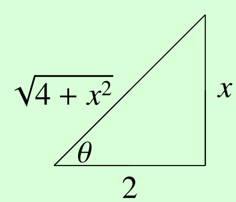
since $\sec(\theta) > 0$. Therefore, we have

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{\sqrt{4\sec^2(\theta)}} \cdot (2\sec^2(\theta))\,d\theta \\ &= \int \sec(\theta)\,d\theta \\ &= \ln(|\sec(\theta) + \tan(\theta)|) + C\end{aligned}$$

We now need to substitute back in for x. The change of variables $x = 2\tan(\theta)$ gives $\tan(\theta) = \frac{x}{2}$, but we also need to replace $\sec(\theta)$. So, we draw a triangle!

From the triangle, we get that

$$\sec(\theta) = \frac{\sqrt{4+x^2}}{2}$$



Hence,

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln\left(\left|\frac{\sqrt{4+x^2}+x}{2}\right|\right) + C = \ln(|\sqrt{4+x^2}+x|) + C$$

REMARK

To simplify the last line in the example, we used the property $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$, and that $-\ln(2) + C$ is still just an arbitrary constant.

EXAMPLE 3 Find $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

Solution: This has the form $b^2x^2 - a^2$ where $a = 1$ and $b = 1$. So, we let $x = \sec(\theta)$. We get $dx = \sec(\theta)\tan(\theta)\,d\theta$ and the replacement

$$\sqrt{x^2-1} = \sqrt{\tan^2(\theta)} = |\tan(\theta)| = \tan(\theta)$$

since $\tan(\theta) > 0$.

When changing the bounds of integration, we remember that we are technically using an inverse trig substitution. When $x = 1$, we have $1 = \sec(\theta)$, so $\theta = \operatorname{arcsec}(1) = 0$. When $x = 2$, we have $2 = \sec(\theta)$, so $\theta = \operatorname{arcsec}(2) = \dfrac{\pi}{3}$.

Therefore, we have

$$\begin{aligned}
\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx &= \int_0^{\pi/3} \frac{\tan(\theta)}{\sec(\theta)} \cdot \sec(\theta)\tan(\theta)\,d\theta \\
&= \int_0^{\pi/3} \tan^2(\theta)\,d\theta \\
&= \int_0^{\pi/3} (\sec^2(\theta) - 1)\,d\theta \\
&= (\tan(\theta) - \theta)\Big|_0^{\pi/3} \\
&= \tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3} - [\tan(0) - 0] \\
&= \sqrt{3} - \frac{\pi}{3}
\end{aligned}$$

EXAMPLE 4 Evaluate $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$.

Solution: This has the form $a^2 - b^2x^2$ where $a = 3$ and $b = 2$. So, we let $x = \dfrac{3}{2}\sin(\theta)$. We get $dx = \dfrac{3}{2}\cos(\theta)\,d\theta$ and the replacement

$$\sqrt{9-4x^2} = \sqrt{9\cos^2(\theta)} = |3\cos(\theta)| = 3\cos(\theta)$$

since $\cos(\theta) > 0$. Therefore, we have

$$\begin{aligned}
\int \frac{1}{9-4x^2}\,dx &= \int \frac{1}{3\cos(\theta)} \cdot \frac{3}{2}\cos(\theta)\,d\theta \\
&= \int \frac{1}{2}\,d\theta \\
&= \frac{1}{2}\theta + C
\end{aligned}$$

We now need to substitute back in for x. The change of variables $x = \frac{3}{2}\sin(\theta)$ gives

$$\frac{2}{3}x = \sin(\theta)$$
$$\arcsin\left(\frac{2}{3}x\right) = \theta$$

Thus,

$$\int \frac{1}{9 - 4x^2}\,dx = \frac{1}{2}\arcsin\left(\frac{2}{3}x\right) + C$$

EXAMPLE 5 Find $\int x^3\sqrt{4x^2 - 1}\,dx$.

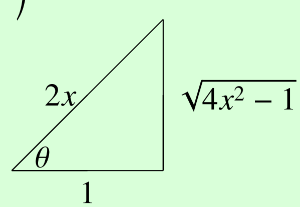
Solution: This has the form $b^2x^2 - a^2$ where $a = 1$ and $b = 2$. So, we let $x = \frac{1}{2}\sec(\theta)$. This gives $dx = \frac{1}{2}\sec(\theta)\tan(\theta)\,d\theta$ and $\sqrt{4x^2 - 1} = \sqrt{\tan^2(\theta)} = |\tan(\theta)| = \tan(\theta)$ since $\tan(\theta) \geq 0$. We now have

$$\begin{aligned}\int x^3\sqrt{4x^2 - 1}\,dx &= \int \left(\frac{1}{2}\sec(\theta)\right)^3 \cdot \tan(\theta) \cdot \frac{1}{2}\sec(\theta)\tan(\theta)\,d\theta \\ &= \int \frac{1}{16}\sec^4(\theta)\tan^2(\theta)\,d\theta\end{aligned}$$

We now have a trigonometric integral to solve. Since the power of $\sec(\theta)$ is even, we let $u = \tan(\theta)$. Then $du = \sec^2(\theta)\,d\theta$. This gives

$$\begin{aligned}\int x^3\sqrt{4x^2 - 1}\,dx &= \frac{1}{16}\int (1 + \tan^2(\theta)) \cdot \tan^2(\theta) \cdot \sec^2(\theta)\,d\theta \\ &= \frac{1}{16}\int (1 + u^2) \cdot u^2\,du \\ &= \frac{1}{16}\int (u^2 + u^4)\,du \\ &= \frac{1}{16}\left(\frac{1}{3}u^3 + \frac{1}{5}u^5\right) + C \\ &= \frac{1}{16}\left(\frac{1}{3}\tan^3(\theta) + \frac{1}{5}\tan^5(\theta)\right) + C\end{aligned}$$

We have that $\sec(\theta) = \frac{2x}{1}$. From the triangle we get that $\tan(\theta) = \frac{\sqrt{4x^2 - 1}}{1}$. Hence,



$$\int x^3\sqrt{4x^2 - 1}\,dx = \frac{1}{48}(4x^2 - 1)^{3/2} + \frac{1}{80}(4x^2 - 1)^{5/2} + C$$

EXERCISE 1 Evaluate $\displaystyle\int \frac{1}{\sqrt{1+9x^2}}\,dx$.

EXERCISE 2 Evaluate $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\,dx$.

EXERCISE 3 Evaluate $\displaystyle\int \sqrt{4-x^2}\,dx$.

Section 8.2 Problems

1. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$

   (b) $\displaystyle\int \frac{1}{(1+x^2)^{3/2}}\,dx$

   (c) $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx$

   (d) $\displaystyle\int \frac{1}{(1-x^2)^{3/2}}\,dx$

   (e) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

   (f) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+9}}\,dx$

   (g) $\displaystyle\int \frac{\sqrt{x^2-1}}{x^2}\,dx$

   (h) $\displaystyle\int \sin^3(x)\cos^2(x)\,dx$

   (i) $\displaystyle\int \frac{x}{\sqrt{4+x^2}}\,dx$

   (j) $\displaystyle\int \frac{1}{\sqrt{1-2x^2}}\,dx$

   (k) $\displaystyle\int \frac{\sqrt{4x^2-9}}{x}\,dx$

   (l) $\displaystyle\int \frac{x^2}{(25+x^2)^2}\,dx$

   (m) $\displaystyle\int \sqrt{9-4x^2}\,dx$

2. Evaluate the following integrals.

   (a) $\displaystyle\int \cos^3(x)\,dx$

   (b) $\displaystyle\int_0^4 \frac{1}{\sqrt{16+x^2}}\,dx$

   (c) $\displaystyle\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$

   (d) $\displaystyle\int_0^{\pi/4} \frac{\sin(x)}{\cos^2(x)}\,dx$

   (e) $\displaystyle\int_{10/\sqrt{3}}^{10} \frac{1}{\sqrt{x^2-25}}\,dx$

   (f) $\displaystyle\int \sec^4(x)\sqrt{\tan(x)}\,dx$

   (g) $\displaystyle\int_2^{2\sqrt{3}} \frac{x^2}{(x^2+4)^2}\,dx$

   (h) $\displaystyle\int \tan^3(x)\,dx$

   (i) $\displaystyle\int \frac{1}{x(x^2-1)^{3/2}}\,dx$

   (j) $\displaystyle\int \frac{x^3}{(16-x^2)^{3/2}}\,dx$

   (k) $\displaystyle\int_0^1 \frac{x^4}{1+x^2}\,dx$

   (l) $\displaystyle\int_{1/\sqrt{3}}^{1} \frac{1}{x^2\sqrt{1+x^2}}\,dx$

   (m) $\displaystyle\int_{2\sqrt{2}}^{4} \frac{1}{x^2(x^2-4)}\,dx$

Section 8.3: Integration by Partial Fractions

LEARNING OUTCOMES

1. Know how to find a partial fraction decomposition.
2. Know how to integrate by partial fractions.

We now turn our attention to integrals of the form $\int \frac{N(x)}{D(x)}\,dx$ where $N(x)$ and $D(x)$ are polynomials and the degree of $N(x)$ is less than the degree of $D(x)$.

As with all of our previous techniques of integration, the idea is to convert the integrand into something that we can find the antiderivative of. In this case, we will do this using a method that is essentially the opposite of adding fractions, called the **partial fraction decomposition**.

8.3.1 Partial Fraction Decomposition (PFD)

First, let's look at an example of adding fractions.

EXAMPLE 1 Add $\frac{-1/5}{x+3} + \frac{1/5}{x-2}$.

Solution: We begin by making a common denominator. We do this by multiplying the first fraction by the denominator of the second fraction and by multiplying the second fraction by the denominator of the first fraction.

$$\frac{-1/5}{x+3} + \frac{1/5}{x-2} = \frac{-\frac{1}{5}(x-2)}{(x+3)(x-2)} + \frac{\frac{1}{5}(x+3)}{(x+3)(x-2)}$$

If we now add the fractions on the right hand side together, we get

$$\begin{aligned}
\frac{-1/5}{x+3} + \frac{1/5}{x-2} &= \frac{-\frac{1}{5}(x-2)}{(x+3)(x-2)} + \frac{\frac{1}{5}(x+3)}{(x+3)(x-2)} \\
&= \frac{-\frac{1}{5}(x-2) + \frac{1}{5}(x+3)}{(x+3)(x-2)} \\
&= \frac{-\frac{1}{5}x + \frac{2}{5} + \frac{1}{5}x + \frac{3}{5}}{(x+3)(x-2)} \\
&= \frac{1}{(x+3)(x-2)}
\end{aligned}$$

We can use this example to see why we are interested in doing this process in reverse. In particular, say we were asked to find

$$\int \frac{1}{(x+3)(x-2)}\,dx$$

If we could do the process in reverse to get that $\dfrac{1}{(x+3)(x-2)} = \dfrac{-1/5}{x+3} + \dfrac{1/5}{x-2}$, then we could easily evaluate the integral. In particular,

$$\begin{aligned} \int \frac{1}{(x+3)(x-2)}\,dx &= \int \left(\frac{-1/5}{x+3} + \frac{1/5}{x-2}\right)dx \\ &= -\frac{1}{5}\ln(|x+3|) + \frac{1}{5}\ln(|x-2|) + C \end{aligned}$$

Now that we know why we want to do this, we just need to figure out how to do it. We again use a couple of examples to figure this out.

EXAMPLE 2 Write $\dfrac{2x}{(x+1)(x+2)}$ as a sum of fractions.

Solution: From our work in the first example, we see that if we add fractions of the form

$$\frac{A}{x+1} + \frac{B}{x+2}$$

then by making a common denominator we are going to get

$$\frac{A(x+2)}{(x+1)(x+2)} + \frac{B(x+1)}{(x+1)(x+2)}$$

We want this to equal $\dfrac{2x}{(x+1)(x+2)}$. That is, we want

$$\frac{2x}{(x+1)(x+2)} = \frac{A(x+2)}{(x+1)(x+2)} + \frac{B(x+1)}{(x+1)(x+2)}$$

Multiplying both sides by $(x+1)(x+2)$ to clear the denominators gives

$$2x = A(x+2) + B(x+1)$$

We now just need to solve for A and B. We can do this by picking clever values of x. In particular, if we take $x = -1$, we get

$$\begin{aligned} 2(-1) &= A(-1+2) + B(-1+1) \\ -2 &= A + 0 \\ -2 &= A \end{aligned}$$

If we now take $x = -2$, we get

$$\begin{aligned} 2(-2) &= (-2)(-2+2) + B(-2+1) \\ -4 &= 0 - B \\ 4 &= B \end{aligned}$$

Thus,

$$\frac{2x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{-2}{x+1} + \frac{4}{x+2}$$

As usual, we can check our answer by adding the fractions on the right and making sure we get back to the original fraction.

In the example above, it was easy to guess that we could write $\dfrac{2x}{(x+1)(x+2)}$ in the form of $\dfrac{A}{x+1}+\dfrac{B}{x+2}$. Of course, in more difficult cases, it would not be so easy.

Thus, we now list the rules for creating the form of the partial fraction decomposition.

ALGORITHM

Writing the Form of a Partial Fraction Decomposition (PFD):

For $\dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials, the degree of $N(x)$ is less than the degree of $D(x)$, and $D(x)$ can be factored into powers of linear terms and irreducible quadratic terms. Then:

1. For each factor of the form $(ax+b)^n$ in $D(x)$ include the following terms in the decomposition:
$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$$
2. For each factor of the form $(ax^2+bx+c)^n$ in $D(x)$ include the following terms in the decomposition:
$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

EXAMPLE 3 Write the form of the PFD of $f(x)=\dfrac{5x}{(x+3)(x-2)(x+1)}$.

Solution: We have three factors $(x+3)$, $(x-2)$, and $(x+1)$. By the first rule we get the form
$$\frac{5x}{(x+3)(x-2)(x+1)}=\frac{A_1}{x+3}+\frac{A_2}{x-2}+\frac{A_3}{x+1}$$

EXAMPLE 4 Write the form of the PFD of $f(x)=\dfrac{x^2+1}{(x+1)^2(x-1)}$.

Solution: We have two factors $(x+1)^2$ and $(x-1)$. Since $(x+1)$ is to the power of two, by the first rule, it gets two terms, one for each power, while $(x-1)$ just gets one term since it is to the power of one. Thus, we get
$$\frac{x^2+1}{(x+1)^2(x-1)}=\frac{A_1}{x+1}+\frac{A_2}{(x+1)^2}+\frac{A_3}{x-1}$$

EXAMPLE 5

Write the form of the PFD of $f(x) = \dfrac{2}{x(x-1)^3(x^2+1)}$.

Solution: We have three factors x, $(x-1)^3$, and (x^2+1). Since x is to the power of one, it gets one term. $(x-1)$ is to the power of three, so it gets three terms, one for each power. The (x^2+1) term is quadratic and to the power of one, so we use the second rule to get one term. We get

$$\frac{2}{x(x-1)^3(x^2+1)} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2} + \frac{A_4}{(x-1)^3} + \frac{B_1x+C_1}{x^2+1}$$

EXAMPLE 6

Write the form of the PFD of $f(x) = \dfrac{3x^2-2x+5}{x^3(x+1)(x^2+2)^2}$.

Solution: Using the first rule for x^3 and $(x+1)$, and the second rule for $(x^2+2)^2$, we get

$$\frac{3x^2-2x+5}{x^3(x+1)(x^2+2)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{A_4}{x+1} + \frac{B_1x+C_1}{x^2+2} + \frac{B_2x+C_2}{(x^2+2)^2}$$

EXERCISE 1

Write the form of the PFD of $f(x) = \dfrac{1}{(x-2)(x+3)}$.

EXERCISE 2

Write the form of the PFD of $f(x) = \dfrac{x}{(x+5)^2(x-4)}$.

EXERCISE 3

Write the form of the PFD of $f(x) = \dfrac{3x^2+x+1}{x^2(x^2+5)^3}$.

To complete the partial fraction decomposition, we need to solve for all of the unknowns. To do this, we first clear the fractions by multiplying both sides of the form of the PFD by the denominator of $f(x)$. We then solve for as many unknowns as possible by substituting convenient values for x into the new equation. If we are not able to solve for all unknowns in this manner, we then gather like powers of x and equate coefficients to get a system of linear equations in the remaining unknowns.

This is best demonstrated with some examples.

EXAMPLE 7 Find a PFD for $f(x) = \dfrac{2x}{(x-1)(x+2)}$.

Solution: We first find the form of the PFD. We get

$$\frac{2x}{(x-1)(x+2)} = \frac{A_1}{x-1} + \frac{A_2}{x+2}$$

Next, we clear the fractions by multiplying both sides by the denominator $(x-1)(x+2)$. This gives

$$2x = A_1(x+2) + A_2(x-1)$$

We now choose values of x that will help us solve for A_1 and A_2. We can see the best values of x will be $x = -2$ and $x = 1$.

Taking $x = -2$ gives

$$\begin{aligned} 2(-2) &= A_1(-2+2) + A_2(-2-1) \\ -4 &= 0 - 3A_2 \\ \frac{4}{3} &= A_2 \end{aligned}$$

Taking $x = 1$ gives

$$\begin{aligned} 2(1) &= A_1(1+2) + A_2(1-1) \\ 2 &= 3A_1 + 0 \\ \frac{2}{3} &= A_1 \end{aligned}$$

Thus, the partial fraction decomposition is

$$\frac{2x}{(x-1)(x+2)} = \frac{\frac{2}{3}}{x-1} + \frac{\frac{4}{3}}{x+2}$$

REMARK

This is perhaps the only case where we generally we leave fractions inside the numerator of a fraction. We do this as it will make the next step, integrating, easier.

EXAMPLE 8

Find a PFD of $f(x) = \dfrac{6}{x(x-3)^2}$.

Solution: The form of the PFD is

$$\frac{6}{x(x-3)^2} = \frac{A_1}{x} + \frac{A_2}{x-3} + \frac{A_3}{(x-3)^2}$$

Multiplying both sides $x(x-3)^2$ gives

$$6 = A_1(x-3)^2 + A_2x(x-3) + A_3x$$

Taking $x = 3$ gives

$$\begin{aligned} 6 &= A_1(0)^2 + A_2(3)(0) + A_3(3) \\ 2 &= A_3 \end{aligned}$$

Taking $x = 0$ gives

$$\begin{aligned} 6 &= A_1(0-3)^2 + A_2(0) + 2(0) \\ 6 &= 9A_1 \\ \frac{2}{3} &= A \end{aligned}$$

Since we only have one value left to solve for, we take any other value of x to find A_2. We take $x = 4$ to get

$$\begin{aligned} 6 &= \frac{2}{3}(4-3)^2 + A_2(4)(4-3) + 2(4) \\ 6 &= \frac{2}{3} + 4A_2 + 8 \\ -\frac{8}{3} &= 4A_2 \\ -\frac{2}{3} &= A_2 \end{aligned}$$

Thus, the partial fraction decomposition is

$$\frac{6}{x(x-3)^2} = \frac{\frac{2}{3}}{x} + \frac{-\frac{2}{3}}{x-3} + \frac{2}{(x-3)^2}$$

EXERCISE 4

Find the PDF of $f(x) = \dfrac{3}{x(x+2)}$.

EXERCISE 5 Find the PDF of $f(x) = \dfrac{x}{(x-1)(x-2)}$.

EXAMPLE 9 Find a PFD of $g(x) = \dfrac{4x^2}{(2x-1)(x^2+1)}$.

Solution: The form of the PFD is

$$\frac{4x^2}{(2x-1)(x^2+1)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(2x-1)(x^2+1)$ gives

$$4x^2 = A(x^2+1) + (Bx+C)(2x-1)$$

Taking $x = \dfrac{1}{2}$ we get

$$\begin{aligned} 1 &= A\left(\frac{5}{4}\right) + 0 \\ \frac{4}{5} &= A \end{aligned}$$

Taking $x = 0$ gives

$$\begin{aligned} 0 &= \frac{4}{5}(1) + (0+C)(-1) \\ -\frac{4}{5} &= -C \\ \frac{4}{5} &= C \end{aligned}$$

Taking $x = 1$ gives

$$\begin{aligned} 4 &= \frac{4}{5}(2) + \left(B + \frac{4}{5}\right)(1) \\ \frac{12}{5} &= B + \frac{4}{5} \\ \frac{8}{5} &= B \end{aligned}$$

Hence, the partial fraction decomposition is

$$\frac{4x^2}{(2x-1)(x^2+1)} = \frac{\frac{4}{5}}{2x-1} + \frac{\frac{8}{5}x + \frac{4}{5}}{x^2+1}$$

EXERCISE 6 Find the PDF of $f(x) = \dfrac{5x^2 - 6x + 7}{(x-1)(x^2+1)}$.

EXAMPLE 10 Find a PFD of $f(x) = \dfrac{x^3 + x + 1}{(x^2+1)(x^2+5)}$.

Solution: The form of the PFD is

$$\frac{x^3 + x + 1}{(x^2+1)(x^2+5)} = \frac{B_1x + C_1}{x^2+1} + \frac{B_2x + C_2}{x^2+5}$$

Multiplying both sides by $(x^2+1)(x^2+5)$ gives

$$x^3 + x + 1 = (B_1x + C_1)(x^2+5) + (B_2x + C_2)(x^2+1)$$

In this case, we gather like powers of x and equate coefficients.

$$x^3 + x + 1 = (B_1 + B_2)x^3 + (C_1 + C_2)x^2 + (5B_1 + B_2)x + 5C_1 + C_2$$

Comparing the coefficients of x^3 on both sides gives $1 = B_1 + B_2$.

Comparing the coefficients of x^2 on both sides gives $0 = C_1 + C_2$.

Comparing the coefficients of x on both sides gives $1 = 5B_1 + B_2$.

Comparing the constant terms on both sides gives $1 = 5C_1 + C_2$.

Subtracting the first equation from the third gives

$$\begin{aligned} 0 &= 4B_1 + 0 \\ 0 &= B_1 \end{aligned}$$

Substituting this into the first equation gives $B_2 = 1$.

Subtracting the second equation from the fourth gives

$$\begin{aligned} 1 &= 4C_1 + 0 \\ \frac{1}{4} &= C_1 \end{aligned}$$

Substituting this into the second equation gives $C_2 = -\dfrac{1}{4}$.

Thus, we get the partial fraction decomposition

$$\frac{x^3 + x + 1}{(x^2+1)(x^2+5)} = \frac{0x + \frac{1}{4}}{x^2+1} + \frac{x - \frac{1}{4}}{x^2+5} = \frac{\frac{1}{4}}{x^2+1} + \frac{x - \frac{1}{4}}{x^2+5}$$

EXERCISE 7

Find the PDF of $f(x) = \dfrac{x+1}{x(x^2+1)^2}$.

8.3.2 Integration by Partial Fractions

We will now look at how partial fraction decompositions help us evaluate integrals. The most common forms of integrals we will see are:

$$\int \frac{1}{x-a}\,dx = \ln(|x-a|) + C$$
$$\int \frac{1}{(x-a)^2}\,dx = -\frac{1}{x-a} + C$$
$$\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$$
$$\int \frac{x}{x^2+a}\,dx = \frac{1}{2}\ln(|x^2+a|) + C$$

This is NOT an exhaustive list. For integrals that you are not sure of, you can often use a change of variables to make it easier.

EXAMPLE 11

Evaluate $\displaystyle\int \frac{6}{x(x-3)^2}\,dx$.

Solution: We saw in Example 8.5.8 that

$$\frac{6}{x(x-3)^2} = \frac{\frac{2}{3}}{x} + \frac{-\frac{2}{3}}{x-3} + \frac{2}{(x-3)^2}$$

Thus,

$$\begin{aligned}\int \frac{6}{x(x-3)^2}\,dx &= \int\left(\frac{\frac{2}{3}}{x} + \frac{-\frac{2}{3}}{x-3} + \frac{2}{(x-3)^2}\right)dx\\ &= \frac{2}{3}\ln(|x|) - \frac{2}{3}\ln(|x-3|) - \frac{2}{x-3} + C\end{aligned}$$

EXAMPLE 12

Evaluate $\displaystyle\int \frac{4x^2}{(2x-1)(x^2+1)}\,dx$.

Solution: We saw in Example 8.5.9 that

$$\frac{4x^2}{(2x-1)(x^2+1)} = \frac{\frac{4}{5}}{2x-1} + \frac{\frac{8}{5}x+\frac{4}{5}}{x^2+1} = \frac{\frac{4}{5}}{2x-1} + \frac{\frac{8}{5}x}{x^2+1} + \frac{\frac{4}{5}}{x^2+1}$$

Thus,

$$\int \frac{4x^2}{(2x-1)(x^2+1)}\,dx = \int \left(\frac{\frac{4}{5}}{2x-1} + \frac{\frac{8}{5}x}{x^2+1} + \frac{\frac{4}{5}}{x^2+1} \right) dx$$
$$= \frac{2}{5}\ln(|2x-1|) + \frac{4}{5}\ln(x^2+1) + \frac{4}{5}\arctan(x) + C$$

EXERCISE 8

Determine $\int \frac{3x}{(x-1)(x^2+1)}\,dx$ given that

$$\frac{3x}{(x-1)(x^2+1)} = \frac{\frac{3}{2}}{x-1} - \frac{\frac{3x}{2}}{x^2+1} + \frac{\frac{3}{2}}{x^2+1}$$

EXERCISE 9

Determine $\int \frac{x-1}{x^2(x^2+1)}\,dx$ given that

$$\frac{x-1}{x^2(x^2+1)} = \frac{1}{x} - \frac{1}{x^2} + \frac{1-x}{x^2+1}$$

We conclude with a couple examples showing the entire process.

EXAMPLE 13

Determine $\int \frac{x}{(x+2)(x+3)}\,dx$.

Solution: The form of the PFD is

$$\frac{x}{(x+2)(x+3)} = \frac{A_1}{x+2} + \frac{A_2}{x+3}$$

Multiplying both sides by $(x+2)(x+3)$ gives

$$x = A_1(x+3) + A_2(x+2)$$

Taking $x = -3$ gives $-3 = 0 + A_2(-1)$, so $A_2 = 3$.

Taking $x = -2$ gives $-2 = A_1$.

Therefore,

$$\frac{x}{(x+2)(x+3)} = \frac{-2}{x+2} + \frac{3}{x+3}$$

Hence,

$$\int \frac{x}{(x+2)(x+3)}\,dx = \int \left(\frac{-2}{x+2} + \frac{3}{x+3} \right) dx$$
$$= -2\ln(|x+2|) + 3\ln(|x+3|) + C$$

EXAMPLE 14 Determine $\int \frac{5}{(x^2+4)(x-1)}\,dx$.

Solution: Using our rules for the PFD we get

$$\frac{5}{(x^2+4)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

Multiplying both sides by $(x^2+4)(x-1)$ gives

$$\begin{aligned} 5 &= (x^2+4)A + (Bx+C)(x-1) \\ &= (A+B)x^2 + (C-B)x + (4A-C) \end{aligned}$$

Equating coefficients of like powers of x gives

$$\begin{aligned} 0 &= A+B \\ 0 &= C-B \\ 5 &= 4A-C \end{aligned}$$

Adding all three equations together gives $5 = 5A$. Thus, $A = 1$. Thus, we get $B = -1$ and $C = -1$. Therefore, we have

$$\begin{aligned} \int \frac{5}{(x^2+4)(x-1)}\,dx &= \int \left(\frac{1}{x-1} + \frac{-x-1}{x^2+4}\right) dx \\ &= \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx \end{aligned}$$

For the second integral we use the substitution $u = x^2+4$. We get $du = 2x\,dx$, so $\frac{1}{2}\,du = x\,dx$. Thus,

$$\int \frac{x}{x^2+4}\,dx = \int \frac{1}{2}\cdot\frac{1}{u}\,du = \frac{1}{2}\ln(|u|) + C = \frac{1}{2}\ln(x^2+4) + C$$

For the third integral, we use the trigonometric substitution. Let $x = 2\tan(\theta)$. We get $dx = 2\sec^2(\theta)\,d\theta$ and the replacement

$$x^2+4 = 4\sec^2(\theta)$$

This gives

$$\begin{aligned} \int \frac{1}{x^2+4}\,dx &= \int \frac{2\sec^2(\theta)}{4\sec^2(\theta)}\,d\theta \\ &= \int \frac{1}{2}\,d\theta \\ &= \frac{1}{2}\theta + C \\ &= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C \end{aligned}$$

Therefore,

$$\int \frac{5}{(x^2+4)(x-1)}\,dx = \ln(|x-1|) - \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

Section 8.3 Problems

1. Write the form of the PFD of each function.
   (a) $f(x) = \dfrac{3x}{(x-2)(x+1)}$
   (b) $f(x) = \dfrac{x}{(x+1)^2(x+2)}$
   (c) $f(x) = \dfrac{x+3}{x^3(x-3)^2}$
   (d) $f(x) = \dfrac{x-1}{x(x^2+1)}$
   (e) $f(x) = \dfrac{1}{x^2(x^2+2)^2}$
   (f) $f(x) = \dfrac{2x+5}{(x+1)^2(x^2+3)}$
   (g) $f(x) = \dfrac{1}{(x^2-1)^2}$
2. Find a PFD of each function.
   (a) $f(x) = \dfrac{1}{(x+1)(x+2)}$
   (b) $f(x) = \dfrac{x}{(x+2)(x-3)}$
   (c) $f(x) = \dfrac{-2}{(x+3)(x-5)}$
   (d) $f(x) = \dfrac{x-1}{x(2x+1)}$
   (e) $f(x) = \dfrac{1}{(x+1)^2(x-1)}$
   (f) $f(x) = \dfrac{3}{x^2(x+2)}$
   (g) $f(x) = \dfrac{x}{x^2+1}$
   (h) $f(x) = \dfrac{x^2}{(x+1)(x^2+1)}$
   (i) $f(x) = \dfrac{5x}{(x^2+1)(x+2)}$
   (j) $f(x) = \dfrac{x^2+2x+1}{(x^2+2)(x-2)}$
   (k) $f(x) = \dfrac{x^2-1}{x(x^2+2)}$
   (l) $f(x) = \dfrac{1}{x(x^2+4)}$
   (m) $f(x) = \dfrac{x^2}{(x^2+1)(x^2+2)}$
   (n) $f(x) = \dfrac{x^2+2}{(x-2)^2(x+1)^2}$
3. Evaluate the following integrals.
   (a) $\displaystyle\int \frac{1}{x-1}\,dx$
   (b) $\displaystyle\int \frac{1}{(x+1)^2}\,dx$
   (c) $\displaystyle\int \frac{1}{1+x^2}\,dx$
   (d) $\displaystyle\int \frac{x+1}{1+x^2}\,dx$
   (e) $\displaystyle\int \frac{1}{x(x+1)}\,dx$
   (f) $\displaystyle\int \frac{1}{(x-3)(x-1)}\,dx$
   (g) $\displaystyle\int \frac{x}{(3x+1)(x+1)}\,dx$
   (h) $\displaystyle\int \frac{3x}{(x-3)^2(x+3)}\,dx$
   (i) $\displaystyle\int \frac{x^2}{1+x^2}\,dx$
   (j) $\displaystyle\int \frac{1}{x(x-1)^2}\,dx$
   (k) $\displaystyle\int \frac{1}{4+x^2}\,dx$
   (l) $\displaystyle\int \tan^3(x)\sec^3(x)\,dx$
   (m) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$
   (n) $\displaystyle\int \frac{6x-3}{x(x-1)(x+2)}\,dx$
   (o) $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$
   (p) $\displaystyle\int \frac{x-1}{(x+1)(x^2+1)}\,dx$
   (q) $\displaystyle\int \frac{x}{(x-2)(x^2+1)}\,dx$
   (r) $\displaystyle\int \frac{x^2-4}{(2x+1)(x^2+1)}\,dx$
   (s) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$
   (t) $\displaystyle\int \frac{3x^2}{(x-2)(x^2+2)}\,dx$
   (u) $\displaystyle\int \frac{4x}{(x+2)(x^2+4)}\,dx$

Section 8.4: Integration by Parts

LEARNING OUTCOMES

1. Know how to use integration by parts to evaluate definite and indefinite integrals.

Integration by parts is typically used when we have a product of functions in the integrand, but none of the other methods we have covered will work. For example, we would use integration by parts for the following integrals:

$$\int x\cos(x)\,dx$$
$$\int e^x \sin(x)\,dx$$
$$\int \ln(x)\,dx$$

You are probably thinking to yourself that the last one isn't actually a product of functions. Well, we can think of it as a product of functions by writing it as

$$\int \ln(x)\cdot 1\,dx$$

In general, if you have a single function and don't know the antiderivative of it, then integration by parts is a good method to try.

The integration by parts formula is derived from the product rule for derivatives. In particular, if $g(x)$ and $h(x)$ are differentiable functions, then we have that

$$\frac{d}{dx}[g(x)h(x)] = g'(x)h(x) + g(x)h'(x)$$

Thus, we have that

$$\int [g'(x)h(x) + g(x)h'(x)]\,dx = g(x)h(x) + C$$

Using properties of integrals, we can rearrange this to get

$$\int g(x)h'(x)\,dx = g(x)h(x) - \int g'(x)h(x)\,dx \tag{8.3}$$

Notice that we have dropped the $+C$ since it can be included in the constant of integration of $\int g(x)h'(x)\,dx$.

Performing two change of variables $u = g(x)$ and $v = h(x)$ gives $du = g'(x)\,dx$ and $dv = h'(x)\,dx$. Substituting these into equation (8.3) gives:

DEFINITION

Integration by Parts

The **integration by parts formula** is

$$\int u\, dv = uv - \int v\, du$$

To use the integration by parts formula to solve an integral $\int f(x)\, dx$ we need to pick u and dv so that

$$f(x)\, dx = u\, dv$$

It is important to note that dv must contain a factor of dx.

To apply the formula, we need determine du and v. To calculate du, we use the exact same method we used for integration by substitution: if $u = g(x)$, then

$$du = g'(x)\, dx$$

Since we pick dv, we need to use the reverse process to find v. That is, if $dv = j(x)$, then v is any antiderivative of $j(x)$.

In general, the difficult part of using integration by parts is figuring out how to pick $u = g(x)$ and $dv = j(x)\, dx$. In particular, we need to ensure we can find an antiderivative of $j(x)$ so that we can determine v.

We demonstrate this with some examples.

EXAMPLE 1

Find $\int x \ln(x)\, dx$.

Solution: We have two main choices: we can pick $u = x$ and $dv = \ln(x)\, dx$, or we can pick $u = \ln(x)$ and $dv = x\, dx$. However, if we don't know the antiderivative of $\ln(x)$, then we will not able to find v. Thus, we should use

$$u = \ln(x) \quad \text{and} \quad dv = x\, dx$$

We get that $du = \frac{1}{x}\, dx$.

Since an antiderivative of x is $\frac{1}{2}x^2$, we get that $v = \frac{1}{2}x^2$.

Thus, the integration by parts formula gives

$$\begin{aligned}
\int x \ln(x)\, dx &= uv - \int v\, du \\
&= \ln(x) \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx \\
&= \frac{x^2 \ln(x)}{2} - \int \frac{1}{2}x\, dx \\
&= \frac{x^2 \ln(x)}{2} - \frac{1}{4}x^2 + C
\end{aligned}$$

EXAMPLE 2 Find $\int x \sin(x)\, dx$.

Solution: This time our choices are $u = x$ and $dv = \sin(x)\, dx$, or $u = \sin(x)$ and $dv = x\, dx$. In either case, we can easily find v, so let's try both.

Taking $u = \sin(x)$ and $dv = x\, dx$ gives $du = \cos(x)\, dx$ and, as in the last example, $v = \frac{1}{2}x^2$. In this case, the integration by parts formula gives

$$\begin{aligned}\int x \sin(x)\, dx &= uv - \int v\, du \\ &= \sin(x) \cdot \left(\frac{1}{2}x^2\right) - \int \frac{1}{2}x^2 \cdot \cos(x)\, dx\end{aligned}$$

The problem here is that the new integral is actually more complicated than the original integral. Consequently, this is a poor choice of u and dv.

Instead, we take $u = x$ and $dv = \sin(x)\, dx$. We get $du = dx$, and, since an antiderivative of $\sin(x)$ is $-\cos(x)$, we get $v = -\cos(x)$. The integration by parts formula gives

$$\begin{aligned}\int x \sin(x)\, dx &= uv - \int v\, du \\ &= x(-\cos(x)) - \int (-\cos(x))\, dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

To avoid having to use trial and error, we would love to have a method telling us how to pick u and dv. We do! Well, almost. The guidelines below works for most... but not all... cases where we use integration by parts.

TIP 1 Pick dv to be the largest part of the integrand that you can easily find the antiderivative of.

TIP 2 Choose u to be whichever function comes first in the following list:

(a) **L**ogarithmic functions ($\ln(x)$)
(b) **I**nverse trigonometric functions ($\arcsin(x)$, $\arccos(x)$, etc)
(c) **A**lgebraic functions (x, $\sqrt{x^2+1}$, etc)
(d) **T**rigonometric functions ($\sin(x), \cos(x)$ etc)
(e) **E**xponential functions (e^x, 2^x, etc)

REMARK

TIP 2 is often referred to by its acronym LIATE.

Observe in Example 8.4.2, that LIATE would have told us to take $u = x$ since 'Algebraic functions' comes before 'Trigonometric functions'.

EXAMPLE 3

Evaluate $\int xe^x \, dx$.

Solution: We know the antiderivative of both x and e^x, so we consider the LIATE rule. Because 'Algebraic functions' come before 'Exponential functions', we take $u = x$ and $dv = e^x \, dx$. Thus, we get $du = dx$ and $v = e^x$. Therefore,

$$\begin{aligned}\int xe^x \, dx &= uv - \int v \, du \\ &= xe^x - \int e^x \, dx \\ &= xe^x - e^x + C\end{aligned}$$

EXERCISE 1

Try to evaluate $\int xe^x \, dx$ using $u = e^x$ and $dv = x \, dx$. What problem do you encounter?

EXAMPLE 4

Evaluate $\int \sqrt{x} \ln(2x) \, dx$.

Solution: Since we know the antiderivative of $\sqrt{x}$, we take $dv = \sqrt{x} \, dx$, and hence $u = \ln(2x)$. We get

$$du = \frac{2}{2x} \, dx = \frac{1}{x} \, dx \quad \text{and} \quad v = \frac{2}{3} x^{3/2}$$

Thus,

$$\begin{aligned}\int \sqrt{x} \ln(2x) \, dx &= uv - \int v \, du \\ &= \ln(2x) \cdot \frac{2}{3} x^{3/2} - \int \frac{2}{3} x^{3/2} \cdot \frac{1}{x} \, dx \\ &= \frac{2}{3} x^{3/2} \ln(2x) - \int \frac{2}{3} x^{1/2} \, dx \\ &= \frac{2}{3} x^{3/2} \ln(2x) - \frac{4}{9} x^{3/2} + C\end{aligned}$$

REMARK

Observe that the LIATE would have also told us to take $u = \ln(2x)$ and $dv = \sqrt{x} \, dx$ in the last example.

EXAMPLE 5

Evaluate $\int \arcsin(x)\,dx$.

Solution: Since we don't know the antiderivative of $\arcsin(x)$, the largest part of the integrand that we know the antiderivative of is $dv = dx$. Thus, we must take $u = \arcsin(x)$. We get

$$du = \frac{1}{\sqrt{1-x^2}}\,dx, \quad \text{and} \quad v = x$$

Thus,

$$\begin{aligned}\int \arcsin(x)\,dx &= uv - \int v\,du \\ &= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx\end{aligned}$$

For this new integral, we see that we have a composition of functions, so we use a substitution. Let $w = 1 - x^2$ and get $dw = -2x\,dx$. Thus,

$$\begin{aligned}\int \arcsin(x)\,dx &= x\arcsin(x) - \int -\frac{1}{2}\frac{1}{\sqrt{w}}\,dw \\ &= x\arcsin(x) + \int \frac{1}{2}w^{-1/2}\,dw \\ &= x\arcsin(x) + w^{1/2} + C \\ &= x\arcsin(x) + \sqrt{1-x^2} + C\end{aligned}$$

EXERCISE 2

Evaluate $\int x\sec^2(x)\,dx$.

EXERCISE 3

Evaluate $\int \frac{x}{\sqrt{x+1}}\,dx$ using:

(a) integration by parts.
(b) integration by substitution.

EXERCISE 4

Evaluate $\int x^2\ln(x)\,dx$.

EXERCISE 5

Evaluate $\int \ln(x)\,dx$.

EXAMPLE 6 Evaluate $\int x^2 e^x \, dx$.

Solution: Since we know the antiderivative of both x^2 and e^x, we use the LIATE rule. Because 'Algebraic functions' comes before 'Exponential functions', we take $u = x^2$ and $dv = e^x \, dx$. This gives

$$du = 2x \, dx \quad \text{and} \quad v = e^x$$

Thus,

$$\begin{aligned}\int x^2 e^x \, dx &= uv - \int v \, du \\ &= x^2 \cdot e^x - \int e^x \cdot 2x \, dx\end{aligned}$$

We see that the new integral is simpler than the original, but does require another integration by parts. Using LIATE, we take $u_1 = 2x$ and $dv_1 = e^x \, dx$. This gives

$$du_1 = 2 \, dx \quad \text{and} \quad v_1 = e^x$$

Hence,

$$\begin{aligned}\int x^2 e^x \, dx &= x^2 e^x - \int e^x \cdot 2x \, dx \\ &= x^2 e^x - \left(u_1 v_1 - \int v_1 \, du_1\right) \\ &= x^2 e^x - \left(2x \cdot e^x - \int e^x \cdot 2 \, dx\right) \\ &= x^2 e^x - 2x e^x + \int e^x \cdot 2 \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C\end{aligned}$$

EXAMPLE 7 Evaluate $\int x^3 \cos(x^2) \, dx$.

Solution: We observe that we have a composition of functions. So, the first thing we think of is a change of variables. In particular, we take $w = x^2$. Then, $dw = 2x \, dx$ and hence $\frac{1}{2} \, dw = x \, dx$. This gives

$$\begin{aligned}\int x^3 \cos(x^2) \, dx &= \int x^2 \cdot \cos(x^2) \cdot x \, dx \\ &= \int w \cdot \cos(w) \cdot \frac{1}{2} \, dw \\ &= \int \frac{1}{2} w \cos(w) \, dw\end{aligned}$$

Now, we have an integration by parts. Since 'Algebraic functions' comes before 'Trigonometric functions', we take $u = \frac{1}{2}w$ and $dv = \cos(w)\,dw$. Then we get

$$du = \frac{1}{2}\,dw \quad \text{and} \quad v = \sin(w)$$

Hence,

$$\begin{aligned}\int x^3\cos(x^2)\,dx &= \int \frac{1}{2}w\cos(w)\,dw \\ &= \frac{1}{2}w\cdot\sin(w) - \int \sin(w)\cdot\frac{1}{2}\,dw \\ &= \frac{1}{2}w\sin(w) + \frac{1}{2}\cos(w) + C \\ &= \frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C\end{aligned}$$

EXERCISE 6 Evaluate $\int \arctan(x)\,dx$ by using integration by parts followed by integration by a change of variables.

EXERCISE 7 Evaluate $\int e^{\sqrt{x}}\,dx$ by first using the change of variables $w = \sqrt{x}$.

In the next example we will see another case where we have to use integration by parts twice. This is a special case where the second integration by parts will return us to the original integral which we will then be able to solve for.

EXAMPLE 8 Evaluate $\int e^x\sin(x)\,dx$.

Solution: According to the LIATE rule, we pick $u = \sin(x)$ and $dv = e^x\,dx$. We get $du = \cos(x)\,dx$ and $v = e^x$. Thus,

$$\int e^x\sin(x)\,dx = \sin(x)\cdot e^x - \int e^x\cdot\cos(x)\,dx$$

We use integration by parts again for this new integral. Again, LIATE tells us to take $u_1 = \cos(x)$ and $dv_1 = e^x\,dx$. Then $du_1 = -\sin(x)\,dx$ and $v_1 = e^x$. Hence, we get

$$\int e^x\sin(x)\,dx = \sin(x)\cdot e^x - \left[\cos(x)\cdot e^x - \int -\sin(x)\cdot e^x\,dx\right]$$
$$\int e^x\sin(x)\,dx = e^x\sin(x) - e^x\cos(x) - \int e^x\sin(x)\,dx$$

Notice that we have now reproduced the original integral. Thus, we can add it to both sides to to get

$$\begin{aligned}
2\int e^x \sin(x)\,dx &= e^x \sin(x) - e^x \cos(x) + C \\
\int e^x \sin(x)\,dx &= \frac{1}{2}\left[e^x \sin(x) - e^x \cos(x) + C\right] \\
&= \frac{1}{2}e^x \sin(x) - \frac{1}{2}e^x \cos(x) + C
\end{aligned}$$

REMARKS

1. Observe that when we moved the integral to the left hand side, we no longer had any integrals on the right hand side and thus had to add in the '+ C'.
2. Also recall that when we multiply the '+ C' by a non-zero constant, like we did in the last example, we can leave it as '+ C' because it still can be any number.

8.4.1 Integration by Parts in Definite Integrals

DEFINITION

Integration by Parts for Definite Integrals

For a definite integral, the Integration by Parts formula is

$$\int_a^b u\,dv = (uv)\Big]_a^b - \int_a^b v\,du$$

EXAMPLE 9

Evaluate $\displaystyle\int_{-\pi/3}^{\pi/4} x\sec^2(x)\,dx$.

Solution: Taking $u = x$ and $dv = \sec^2(x)\,dx$ gives $du = dx$ and $v = \tan(x)$. Hence,

$$\begin{aligned}
\int_{-\pi/3}^{\pi/4} x\sec^2(x)\,dx &= x\tan(x)\Big]_{-\pi/3}^{\pi/4} - \int_{-\pi/3}^{\pi/4} \tan(x)\,dx \\
&= x\tan(x)\Big]_{-\pi/3}^{\pi/4} - \ln(|\sec(x)|)\Big]_{-\pi/3}^{\pi/4} \\
&= \left[\frac{\pi}{4}\tan\left(\frac{\pi}{4}\right) - \frac{-\pi}{3}\tan\left(\frac{-\pi}{3}\right)\right] - \left[\ln\left(\left|\sec\left(\frac{\pi}{4}\right)\right|\right) - \ln\left(\left|\sec\left(\frac{-\pi}{3}\right)\right|\right)\right] \\
&= \frac{\pi}{4} - \frac{\pi}{\sqrt{3}} - \ln(\sqrt{2}) + \ln(2) \\
&= \frac{\pi}{4} - \frac{\pi}{\sqrt{3}} + \frac{1}{2}\ln(2)
\end{aligned}$$

EXAMPLE 10 Evaluate $\displaystyle\int_1^2 \frac{\ln(x)}{x^2}\,dx$.

Solution: Taking $u = \ln(x)$ and $dv = x^{-2}\,dx$ gives

$$du = \frac{1}{x}\,dx \quad \text{and} \quad v = -\frac{1}{x}$$

Thus,

$$\begin{aligned}
\int_1^2 \frac{\ln(x)}{x^2}\,dx &= \ln(x)\cdot\left(-\frac{1}{x}\right)\Bigg|_1^2 - \int_1^2 \left(-\frac{1}{x}\right)\cdot\frac{1}{x}\,dx \\
&= -\frac{1}{x}\ln(x)\Bigg|_1^2 - \int_1^2 \left(-\frac{1}{x^2}\right)dx \\
&= -\frac{1}{x}\ln(x)\Bigg|_1^2 - \left(\frac{1}{x}\right)\Bigg|_1^2 \\
&= \left[-\frac{1}{2}\ln(2) + 1\ln(1)\right] - \left[\frac{1}{2} - 1\right] \\
&= -\frac{1}{2}\ln(2) + \frac{1}{2}
\end{aligned}$$

EXAMPLE 11 Evaluate $\displaystyle\int_0^{\pi/4} x\arctan(x)\,dx$.

Solution: Taking $u = \arctan(x)$ and $dv = x\,dx$ gives

$$du = \frac{1}{1+x^2}\,dx \quad \text{and} \quad v = \frac{1}{2}x^2$$

Thus,

$$\begin{aligned}
\int_0^{\pi/4} x\arctan(x)\,dx &= \arctan(x)\cdot\frac{1}{2}x^2\Bigg|_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{2}x^2\cdot\frac{1}{1+x^2}\,dx \\
&= \frac{1}{2}x^2\arctan(x)\Bigg|_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{2}\cdot\frac{x^2}{1+x^2}\,dx \\
&= \frac{1}{2}x^2\arctan(x)\Bigg|_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{2}\cdot\frac{1+x^2-1}{1+x^2}\,dx \\
&= \frac{1}{2}x^2\arctan(x)\Bigg|_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{2}\left(1-\frac{1}{1+x^2}\right)dx \\
&= \frac{1}{2}x^2\arctan(x)\Bigg|_0^{\pi/4} - \frac{1}{2}(x-\arctan(x))\Bigg|_0^{\pi/4} \\
&= \left[\frac{\pi^2}{32}\arctan\left(\frac{\pi}{4}\right) - 0\right] - \left[\frac{\pi}{8} - \frac{1}{2}\arctan\left(\frac{\pi}{4}\right) - \left(0 - \frac{1}{2}\arctan(0)\right)\right] \\
&= \frac{\pi^2}{32}\arctan\left(\frac{\pi}{4}\right) - \frac{\pi}{8} + \frac{1}{2}\arctan\left(\frac{\pi}{4}\right)
\end{aligned}$$

EXERCISE 8 Evaluate $\displaystyle\int_0^1 \frac{x}{e^x}\,dx$.

EXERCISE 9 Evaluate $\displaystyle\int_0^{\pi/3} x\cos(2x)\,dx$.

Section 8.4 Problems

1. Evaluate the indefinite integral.

   (a) $\displaystyle\int x\cos(x)\,dx$

   (b) $\displaystyle\int x\sec(x)\tan(x)\,dx$

   (c) $\displaystyle\int x^2\arctan(x)\,dx$

   (d) $\displaystyle\int \sin(x)\sec(x)\,dx$

   (e) $\displaystyle\int e^x\cos(x)\,dx$

   (f) $\displaystyle\int x\sin(x^2)\,dx$

   (g) $\displaystyle\int (x+1)e^x\,dx$

   (h) $\displaystyle\int \frac{\ln(x)}{x^3}\,dx$

   (i) $\displaystyle\int (x+x^2)\sin(2x)\,dx$

   (j) $\displaystyle\int \sin(\ln(x))\,dx$

   (k) $\displaystyle\int x^3e^{2x}\,dx$

   (l) $\displaystyle\int x(\ln(x))^2\,dx$

   (m) $\displaystyle\int \arctan\left(\frac{1}{x}\right)dx$

   (n) $\displaystyle\int \cos(x)\ln(\sin(x))\,dx$

   (o) $\displaystyle\int \sqrt{x}\sin(\sqrt{x})\,dx$

   (p) $\displaystyle\int x\arcsin(x)\,dx$

2. Evaluate the definite integral.

   (a) $\displaystyle\int_0^{\pi/2} (2x+1)\sin(x)\,dx$

   (b) $\displaystyle\int_0^{\pi/3} 2x\cos(3x)\,dx$

   (c) $\displaystyle\int_0^1 xe^{-2x}\,dx$

   (d) $\displaystyle\int_0^1 \frac{1}{(3x+1)(x+2)}\,dx$

   (e) $\displaystyle\int_1^e \ln(x^2)\,dx$

   (f) $\displaystyle\int_0^{\pi/2} x^2\cos(x)\,dx$

   (g) $\displaystyle\int_1^2 (\ln(x))^2\,dx$

   (h) $\displaystyle\int_0^1 x\arccos(x)\,dx$

   (i) $\displaystyle\int_0^1 \ln(1+x^2)\,dx$

3. Assume that n is a positive integer. Use integration by parts to show that the formula, called a **reduction formula**, is true.

   (a) $\displaystyle\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$

   (b) $\displaystyle\int (\ln(x))^n\,dx = x(\ln(x))^n - n\int (\ln(x))^{n-1}\,dx$

   (c) $\displaystyle\int x^n\cos(x)\,dx = x^n\sin(x) - n\int x^{n-1}\sin(x)\,dx$

   (d) $$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx, \quad n \ge 2$$

Section 8.5: Improper Integrals

LEARNING OUTCOMES

1. Know how to evaluate infinite improper integrals.
2. Know how to evaluate definite integrals with infinite discontinuities.

Consider $\int_0^\infty e^{-x}\,dx$ and $\int_{-2}^2 \frac{1}{x^4}\,dx$. How do we evaluate these integrals? We can not use the Fundamental Theorem of Calculus on the first integral as the upper bound is not a number. We also can not use the Fundamental Theorem of Calculus on the second integral as x^{-4} is not continuous on $[-2, 2]$. We call these **improper integrals**. We now look at how to solve each of these types of problems.

8.5.1 Infinite Improper Integrals

For $\int_0^\infty e^{-x}\,dx$ the only problem is the upper bound. If the integral was $\int_0^t e^{-x}\,dx$, then we could evaluate it. Thinking about the integrals in terms of area, we can see that as t gets larger and larger the second integral gets closer and closer to the area under the graph of the original integral. So, it makes sense that

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx$$

Thus,

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big|_0^t = \lim_{t\to\infty}[-e^{-t}+1] = 1$$

We, in fact, use this to define what we mean by an infinite improper integral.

DEFINITION

Infinite Improper Integrals

1. If $f(x)$ is continuous for $x \geq a$, then
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx, \quad \text{if the limit exists}$$
2. If $f(x)$ is continuous for $x \leq a$, then
$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx, \quad \text{if the limit exists}$$
3. If $f(x)$ is continuous for all x, then for any number c
$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to-\infty}\int_t^c f(x)\,dx + \lim_{s\to\infty}\int_c^s f(x)\,dx$$
provided both limits exist.

If the limit(s) exist, then we say the improper integral **converges**. If any one of the limits do not exist, then we say that the improper integral **diverges**.

Just like in Section 6.4 when using the Integral Test, we must fully evaluate the definite integral before we take the limit.

EXAMPLE 1 Determine whether $\displaystyle\int_0^\infty \frac{1}{x+3}\,dx$ converges or diverges.

Solution: Since $f(x) = \dfrac{1}{x+3}$ is continuous on $[0,\infty)$, we have

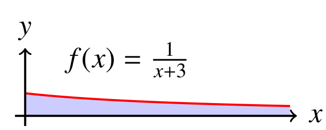
$$\begin{aligned}\int_0^\infty \frac{1}{x+3}\,dx &= \lim_{t\to\infty}\int_0^t \frac{1}{x+3}\,dx\\ &= \lim_{t\to\infty} \ln(|x+3|)\Big|_0^t\\ &= \lim_{t\to\infty}[\ln(|t+3|)-\ln(3)] = \infty\end{aligned}$$

Since the limit does not exist, the improper integral diverges.

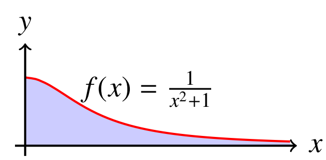
EXAMPLE 2 Determine whether $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx$ converges or diverges.

Solution: Since $f(x) = \dfrac{1}{x^2+1}$ is continuous on $(-\infty,\infty)$, we have

$$\begin{aligned}\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx &= \lim_{t\to-\infty}\int_t^0 \frac{1}{x^2+1}\,dx + \lim_{s\to\infty}\int_0^s \frac{1}{x^2+1}\,dx\\ &= \lim_{t\to-\infty}\arctan(x)\Big|_t^0 + \lim_{s\to\infty}\arctan(x)\Big|_0^s\\ &= \lim_{t\to-\infty}[\arctan(0)-\arctan(t)] + \lim_{s\to\infty}[\arctan(s)-\arctan(0)]\\ &= 0-\left(-\frac{\pi}{2}\right)+\frac{\pi}{2}-0 = \pi\end{aligned}$$



Example 8.5.1 is saying that the area under the graph of $f(x) = \dfrac{1}{x+3}$ from $x = 0$ to ∞ is infinite. This makes intuitive sense as we have an infinite width.



But, then Example 8.5.2 shows that we can get a finite area under the graph even though we actually go to infinity in both directions (the width of the area is doubly infinite!). How can this be a finite area?

We can relate this back to series and the Integral Test. In particular, we saw that a series will converge if the terms a_n are tending to 0 quickly enough. So, similarly, if the values of the function $f(x)$ are tending to 0 quickly enough, then the improper integral will converge to a finite number.

EXAMPLE 3 Determine whether $\int_{-\infty}^{\infty} x\,dx$ converges or diverges.

Solution: Since $f(x) = x$ is continuous on $(-\infty, \infty)$, we have

$$\int_{-\infty}^{\infty} x\,dx = \lim_{t\to-\infty} \int_t^0 x\,dx + \lim_{s\to\infty} \int_0^s x\,dx$$

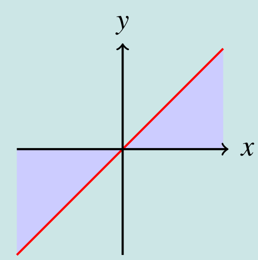
Observe that

$$\lim_{t\to-\infty} \int_t^0 x\,dx = \lim_{t\to-\infty} \frac{1}{2}x^2 \Big|_t^0 = \lim_{t\to-\infty} \left[\frac{1}{2}(0)^2 - \frac{1}{2}t^2\right] = -\infty$$

Because one of the limits does not exist, we do not have to try to evaluate the other limit. We automatically have, by definition, that the improper integral diverges.

REMARK

At first glance, the fact that $\int_{-\infty}^{\infty} x\,dx$ diverges is counter intuitive. From the picture, it seems 'obvious' that the answer should be 0. And, in fact, there are other areas of mathematics where the answer is 0. It all depends on how you deal with the concept of infinity. In this course, we will treat this strictly as in the definition of an infinite improper integral.



EXERCISE 1 Determine whether $\int_4^{\infty} \frac{1}{(x-2)^2}\,dx$ converges or diverges.

EXERCISE 2 Determine whether $\int_1^{\infty} \frac{x}{\sqrt{1+x^2}}\,dx$ converges or diverges.

EXERCISE 3 Determine whether $\int_{-\infty}^{\infty} \frac{x}{e^x}\,dx$ converges or diverges.

EXERCISE 4 Show that $\int_0^{\infty} \sin(x)\,dx$ diverges.

8.5.2 Discontinuous Integrals

Observe that the problem with $\int_{-2}^{2} \frac{1}{x^4}\,dx$ is that $f(x) = x^{-4}$ has an infinite discontinuity at $x = 0$. We could, however, find the integral over the interval $[-2, 0)$ and over the interval $(0, 2]$. To find the integral over these intervals we will use the same trick we did above. That is, since we want to get closer and closer to 0 we can define the integral over the interval $[-2, 0)$ as

$$\lim_{t \to 0^-} \int_{-2}^{t} x^{-4}\,dx$$

and over the interval $(0, 2]$ as

$$\lim_{s \to 0^+} \int_{s}^{2} x^{-4}\,dx$$

Then, since the function is not defined at $x = 0$, it makes sense to define

$$\int_{-2}^{2} \frac{1}{x^4}\,dx = \lim_{t \to 0^-} \int_{-2}^{t} x^{-4}\,dx + \lim_{s \to 0^+} \int_{s}^{2} x^{-4}\,dx$$

DEFINITION

Discontinuous Improper Integral

1. If $f(x)$ is continuous on the interval $(a, b]$, but not at $x = a$, then

$$\int_{a}^{b} f(x)\,dx = \lim_{t \to a^+} \int_{t}^{b} f(x)\,dx$$

   if the limit exists.

2. If $f(x)$ is continuous on the interval $[a, b)$, but not at $x = b$, then

$$\int_{a}^{b} f(x)\,dx = \lim_{t \to b^-} \int_{a}^{t} f(x)\,dx$$

   if the limit exists.

3. If $f(x)$ is continuous on the interval $[a, b]$ except at some point c in the interval, then

$$\int_{a}^{b} f(x)\,dx = \lim_{t \to c^-} \int_{a}^{t} f(x)\,dx + \lim_{s \to c^+} \int_{s}^{b} f(x)\,dx$$

   if both limits exist.

If the limit(s) exist, then we say the improper integral **converges**. If any one of the limits do not exist, then we say that the improper integral **diverges**.

From now on, we must be very careful to check that the integrand is continuous (or not) on the interval of integration when we are trying to evaluate definite integrals. The trickiest part of discontinuous improper integrals is recognizing when we have one.

EXAMPLE 4 Determine whether $\int_{-2}^{2} \frac{1}{x^4}\,dx$ converges or diverges.

Solution: Since $f(x) = \frac{1}{x^4}$ is discontinuous at $x = 0$, we have

$$\int_{-2}^{2} \frac{1}{x^4}\,dx = \lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x^4}\,dx + \lim_{s\to 0^+} \int_{s}^{2} \frac{1}{x^4}\,dx$$

However, observe that

$$\begin{aligned}\lim_{s\to 0^+} \int_{s}^{2} x^{-4}\,dx &= \lim_{s\to 0^+} -\frac{1}{3}x^{-3}\Big|_{s}^{2} \\ &= -\frac{1}{3}\lim_{s\to 0^+}\left[\frac{1}{8} - \frac{1}{s^3}\right] = \infty\end{aligned}$$

Since the limit does not exist, the improper integral diverges.

EXAMPLE 5 Determine whether $\int_{0}^{3} \frac{1}{\sqrt{x}}\,dx$ converges or diverges.

Solution: Since $f(x) = \frac{1}{\sqrt{x}}$ is discontinuous at $x = 0$, we have

$$\begin{aligned}\int_{0}^{3} \frac{1}{\sqrt{x}}\,dx &= \lim_{t\to 0^+} \int_{t}^{3} \frac{1}{\sqrt{x}}\,dx \\ &= \lim_{t\to 0^+} 2\sqrt{x}\Big|_{t}^{3}\,dx \\ &= \lim_{t\to 0^+}\left[2\sqrt{3} - 2\sqrt{t}\right] \\ &= 2\sqrt{3}\end{aligned}$$

So, the improper integrals converges to $2\sqrt{3}$.

EXERCISE 5 Determine whether $\int_{0}^{1} \frac{1}{\sqrt{1-x}}\,dx$ converges or diverges. If it converges, evaluate it.

EXERCISE 6 Determine whether $\int_{-3}^{-1} \frac{1}{x+2}\,dx$ converges or diverges. If it converges, evaluate it.

We can have more than one discontinuity in the interval of integration. In these cases, we treat each discontinuity as above, and remember that if any of the limits do not exist, then the improper integral diverges.

EXAMPLE 6 Determine whether $\displaystyle\int_{-1}^{2} \frac{x}{(x^2-1)^{2/3}}\,dx$ converges or diverges.

Solution: We first observe that the integrand is discontinuous at $x = \pm 1$. Thus, by definition, we get

$$\int_{-1}^{2} \frac{x}{(x^2-1)^{2/3}}\,dx = \lim_{s\to -1^+}\int_{s}^{0} \frac{x}{(x^2-1)^{2/3}}\,dx + \lim_{t\to 1^-}\int_{0}^{t} \frac{x}{(x^2-1)^{2/3}}\,dx + \lim_{w\to 1^+}\int_{w}^{2} \frac{x}{(x^2-1)^{2/3}}\,dx$$

We now need to find an antiderivative of the integrand. Let $u = x^2 - 1$. We get $du = 2x\,dx$ and hence $\frac{1}{2}\,du = x\,dx$. This gives

$$\begin{aligned}\int \frac{x}{(x^2-1)^{2/3}}\,dx &= \int \frac{1}{2}u^{-2/3}\,du\\ &= \frac{3}{2}u^{1/3} + c\\ &= \frac{3}{2}(x^2-1)^{1/3} + c\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{-1}^{2} \frac{x}{(x^2-1)^{2/3}}\,dx &= \lim_{s\to -1^+}\frac{3}{2}(x^2-1)^{1/3}\Big]_{s}^{0} + \lim_{t\to 1^-}\frac{3}{2}(x^2-1)^{1/3}\Big]_{0}^{t} + \lim_{w\to 1^+}\frac{3}{2}(x^2-1)^{1/3}\Big]_{w}^{2}\\ &= \lim_{s\to -1^+}\left[\frac{3}{2}(s^2-1)^{1/3} - \frac{3}{2}(-1)^{1/3}\right] + \lim_{t\to 1^-}\left[\frac{3}{2}(-1)^{1/3} - \frac{3}{2}(t^2-1)^{1/3}\right]\\ &\quad + \lim_{w\to 1^+}\left[\frac{3}{2}(3)^{1/3} - \frac{3}{2}(w^2-1)^{1/3}\right]\\ &= \left[0 - \left(-\frac{3}{2}\right)\right] + \left[-\frac{3}{2} - 0\right] + \left[\frac{3^{4/3}}{2} - 0\right]\\ &= \frac{3^{4/3}}{2}\end{aligned}$$

So, the improper integral converges to $\frac{3^{4/3}}{2}$.

We may also have improper integrals containing both discontinuities and $\pm\infty$ as one of the bounds.

EXAMPLE 7 Determine whether $\displaystyle\int_0^\infty \frac{1}{x^2-9}\,dx$ converges or diverges.

Solution: Observe that the integrand is discontinuous at $x = \pm 3$, but $x = -3$ is not in the range of integration, so we ignore it. Thus, we get

$$\int_0^\infty \frac{1}{x^2-9}\,dx = \lim_{t\to 3^-}\int_0^t \frac{1}{x^2-9}\,dx + \lim_{s\to 3^+}\int_s^5 \frac{1}{x^2-9}\,dx + \lim_{w\to\infty}\int_5^w \frac{1}{x^2-9}\,dx$$

It is always important to remember that if any one of these limits does not exist, then the entire improper integral diverges. Which means that once you evaluate such a limit, you do not have to evaluate the other limits.

We will consider the first limit: $\displaystyle\lim_{t\to 3^-}\int_0^t \frac{1}{x^2-9}\,dx.$

The partial fraction decomposition is

$$\begin{aligned}\frac{1}{(x+3)(x-3)} &= \frac{A}{x+3} + \frac{B}{x-3}\\ 1 &= A(x-3) + B(x+3)\end{aligned}$$

Taking $x = 3$ gives $B = \dfrac{1}{6}$ and taking $x = -3$ gives $A = -\dfrac{1}{6}$. Thus, we have

$$\begin{aligned}\lim_{t\to 3^-}\int_0^t \frac{1}{x^2-9}\,dx &= \lim_{t\to 3^-}\int_0^t \left(\frac{-\frac{1}{6}}{x+3} + \frac{\frac{1}{6}}{x-3}\right)dx\\ &= \lim_{t\to 3^-}\left[-\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3|\right]_0^t\\ &= \lim_{t\to 3^-}\frac{1}{6}\left[\ln\left|\frac{x-3}{x+3}\right|\right]_0^t\\ &= \frac{1}{6}\lim_{t\to 3^-}\left(\ln\left|\frac{t-3}{t+3}\right| - \ln|(-1)|\right)\\ &= -\infty\end{aligned}$$

Thus, this limit does not exist, so the improper integral diverges.

Section 8.5 Problems

1. Determine whether the integral is improper or not.

   (a) $\displaystyle\int_{-\infty}^{1} (x^2 + 2)\, dx$

   (b) $\displaystyle\int_{-2}^{3} \frac{1}{x+1}\, dx$

   (c) $\displaystyle\int_{-5}^{4} \frac{1}{x-5}\, dx$

   (d) $\displaystyle\int_{0}^{2\pi} \tan(x)\, dx$

   (e) $\displaystyle\int_{1}^{\infty} \sin(x)\, dx$

   (f) $\displaystyle\int_{0}^{5} \frac{1}{x^2+1}\, dx$

   (g) $\displaystyle\int_{-3}^{1} \frac{1}{x^2-4}\, dx$

   (h) $\displaystyle\int_{0}^{4} \sqrt{x}\, dx$

   (i) $\displaystyle\int_{0}^{4} \frac{1}{\sqrt{x}}\, dx$

2. Evaluate the definite integral.

   (a) $\displaystyle\int_{1}^{\infty} \frac{1}{x+1}\, dx$

   (b) $\displaystyle\int_{1}^{\infty} \frac{x}{1+x^2}\, dx$

   (c) $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt{x}}\, dx$

   (d) $\displaystyle\int_{2}^{\infty} \frac{1}{x^3}\, dx$

   (e) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{x}}\, dx$

   (f) $\displaystyle\int_{0}^{1} \frac{1}{x^2}\, dx$

   (g) $\displaystyle\int_{0}^{\infty} \frac{1}{x^3}\, dx$

   (h) $\displaystyle\int_{0}^{\infty} \frac{1}{x^{1/3}}\, dx$

3. Evaluate the definite integral.

   (a) $\displaystyle\int_{2}^{4} \frac{1}{x-4}\, dx$

   (b) $\displaystyle\int_{-\infty}^{0} e^{2x}\, dx$

   (c) $\displaystyle\int_{1}^{5} \frac{3x-1}{x(x-2)}\, dx$

   (d) $\displaystyle\int_{-1}^{3} |x|\, dx$

   (e) $\displaystyle\int_{4}^{\infty} \frac{1}{x\ln(x)}\, dx$

   (f) $\displaystyle\int_{1}^{\infty} \frac{1}{x(x+1)}\, dx$

   (g) $\displaystyle\int_{0}^{\pi/2} \sec^2(x)\, dx$

   (h) $\displaystyle\int_{0}^{2} \frac{x}{\sqrt{4-x^2}}\, dx$

   (i) $\displaystyle\int_{1}^{2} \frac{1}{\sqrt{x^2-1}}\, dx$

   (j) $\displaystyle\int_{0}^{\infty} 2xe^{-3x}\, dx$

   (k) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx$

   (l) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{1-x}}\, dx$

   (m) $\displaystyle\int_{0}^{2} x^2\ln(x)\, dx$

   (n) $\displaystyle\int_{0}^{2} \frac{1}{(x-1)^{1/3}}\, dx$

   (o) $\displaystyle\int_{0}^{1} \frac{\ln(x)}{\sqrt{x}}\, dx$

   (p) $\displaystyle\int_{-2}^{\infty} \frac{1}{x^2+2x}\, dx$

End of Chapter Problems

1. Which technique of integration should you use to evaluate the integral?
   (a) $\int x \sin(x)\, dx$
   (b) $\int \sin^2(x) \cos^2(x)\, dx$
   (c) $\int \frac{x}{\sqrt{4-x^2}}\, dx$
   (d) $\int \frac{1}{\sqrt{4+x^2}}\, dx$
   (e) $\int \frac{1}{(x+2)(x-3)}\, dx$
   (f) $\int \frac{1}{x^2\sqrt{4x^2-1}}\, dx$
   (g) $\int e^x \sin(x)\, dx$
   (h) $\int \frac{3x}{(x^2+1)(x-2)}\, dx$
   (i) $\int \tan^3(x) \sec^3(x)\, dx$
2. Evaluate the definite integral.
   (a) $\int_0^1 x\sqrt{3-x}\, dx$
   (b) $\int_{-\infty}^{\infty} \frac{1}{x^2}\, dx$
   (c) $\int_0^{\pi/4} \sin^3(x)\cos^2(x)\, dx$
   (d) $\int_0^1 \sqrt{1-x^2}\, dx$
   (e) $\int_1^2 \frac{1}{(x-1)^{3/2}}\, dx$
   (f) $\int_0^{\pi} \tan^2(x)\sec^2(x)\, dx$
   (g) $\int_0^{1/2} \frac{1}{\sqrt{1-4x^2}}\, dx$
   (h) $\int_0^{\pi/2} \sin^3(x)\, dx$
   (i) $\int_1^{\infty} \frac{\ln(x)}{x^2}\, dx$
   (j) $\int_0^{\infty} \frac{1}{x^2-4}\, dx$
   (k) $\int_0^{\infty} \frac{x\arctan(x)}{(1+x^2)^2}\, dx$
3. Evaluate the integral.
   (a) $\int \sin^3(x)\cos^2(x)\, dx$
   (b) $\int \sec^2(x)\tan(x)\, dx$
   (c) $\int \sin^4(x)\, dx$
   (d) $\int \frac{\sin^3(x)}{\cos^2(x)}\, dx$
   (e) $\int \frac{x-2}{(x+1)(2x-3)}\, dx$
   (f) $\int \frac{1}{\sqrt{x^2+1}}\, dx$
   (g) $\int x\cos(x)\, dx$
   (h) $\int e^x\cos(x)\, dx$
   (i) $\int \sqrt{1-x^2}\, dx$
   (j) $\int \frac{1}{(x+1)^2(x-1)}\, dx$
   (k) $\int \sec(x)\tan^5(x)\, dx$
   (l) $\int x(\ln(x))^2\, dx$
   (m) $\int \frac{x^3}{1+x^2}\, dx$
   (n) $\int \frac{1}{x^2\sqrt{1+x^2}}\, dx$
   (o) $\int \frac{x^2}{\sqrt{1-4x^2}}\, dx$
   (p) $\int \frac{x}{4+9x^2}\, dx$
   (q) $\int \frac{x^2-x+4}{(x^2+1)(x+2)^2}\, dx$
   (r) $\int \frac{\sqrt{25x^2-4}}{x}\, dx$
   (s) $\int \frac{2x^2+1}{(x-1)(x+2)^2}\, dx$
   (t) $\int \frac{x^2+x+1}{(x^2+1)^2}\, dx$
   (u) $\int \frac{x^2+2}{(x+1)(2x^2+1)}\, dx$

Chapter 9: Volume

Section 9.1: Volume

LEARNING OUTCOMES

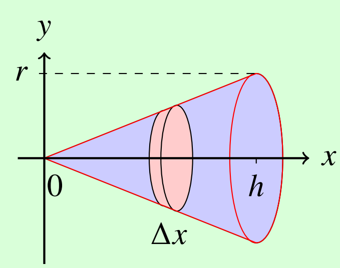
1. Know how to use an integral to find the volume of a three-dimensional region.

We have seen that we can use an integral to calculate the area of a two-dimensional region and that this is often used to calculate the net change of a quantity. We now look at how to use integrals to find the volume of three-dimensional regions.

Since we are going to use the same methods we saw in Section 5.3, it is, again, highly recommended that you review Section 5.3 before proceeding with this chapter. In particular, ensure you understand how we set up a Riemann sum to calculate the area between curves, and how to recognize when we wanted to integrate with respect to y instead of x.

EXAMPLE 1 Consider a cone with height h and radius r. Create a right Riemann sum with n-subdivisions to approximate the volume of the cone.

Solution: To create a right Riemann sum, we slice the object into n pieces each with width $\Delta x = \dfrac{b-a}{n} = \dfrac{h}{n}$. One of these slices is shown in the figure.



The closest nice geometric shape to a slice is a right circular cylinder. So, we will approximate, the volume of each slice with a right circular cylinder.

The width of the right circular cylinder is Δx. To get the radius, we pick the left end point of the slice,

$$x_i = a + i\Delta x = 0 + i\Delta x = \frac{ih}{n}$$

and find the distance from the x-axis to the top of the slice.

The top of the slice lies upon the line stretching from $(0,0)$ to (h,r). Thus, the equation of this line is

$$y = mx + b = \frac{r}{h}x + 0 = \frac{r}{h}x$$

Therefore, the right circular cylinder the slice at location x_i has radius $r_i = \frac{r}{h}x_i$ and so the area of the face is

$$A(x_i) = \pi\left(\frac{r}{h}x_i\right)^2$$

and so the volume the i-th cylinder is

$$V_i = \text{ area of the face} \times \text{thickness} = A(x_i)\Delta x = \pi\left(\frac{r}{h}x_i\right)^2 \Delta x$$

We can then approximate the volume of the cone by summing the volume of the cylinders. That is, a Riemann sum R_n which approximates the volume of the cone is

$$R_n = \sum_{i=1}^{n} \pi \left(\frac{r}{h} x_i\right)^2 \Delta x$$

Recall that we defined the definite integral by

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Consequently, we can find the exact volume of the cone using an integral!

Generalizing what we did in the example above gives the following theorem.

THEOREM 1 The volume of a region which has cross-sectional area $A(x)$ for $a \leq x \leq b$ is given by

$$\text{Volume} = \int_a^b A(x)\, dx$$

EXAMPLE 2 Find the volume of a region which has cross-sectional area $A(x) = \cos(x) + 1$ for $0 \leq x \leq \pi$.

Solution: We have

$$\begin{aligned}
\text{Volume} &= \int_a^b A(x)\, dx \\
&= \int_0^{\pi} (\cos(x) + 1)\, dx \\
&= (\sin(x) + x)\Big|_0^{\pi} \\
&= \sin(\pi) + \pi - (\sin(0) + 0) \\
&= \pi
\end{aligned}$$

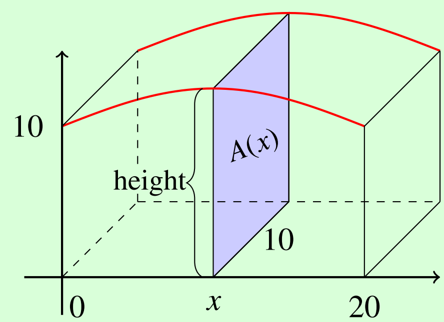
EXERCISE 1 Find the volume of a region which has cross-sectional area $A(x) = e^{2x}$ for $1 \leq x \leq 2$.

EXERCISE 2 Find the volume of a region which has cross-sectional area $A(x) = \pi(2x - x^2)$ for $0 \leq x \leq 2$.

EXAMPLE 3 Find the volume of a loaf of bread that has width 10 cm, length 20 cm, and whose top is given by the curve

$$f(x) = \frac{1}{2}\sin\left(\frac{\pi x}{20}\right) + 10 \text{ cm}$$

for $0 \le x \le 20$.



Solution: We need to find the cross-sectional area of a slice.

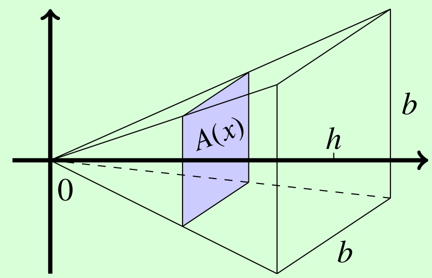
The base of the slice is 10 cm long, and, at any position x, the height will be $f(x)$. Hence,

$$\begin{aligned} A(x) &= \text{base} \times \text{height} \\ &= 10 \cdot \left(\frac{1}{2}\sin\left(\frac{\pi x}{20}\right) + 10\right) \\ &= 5\sin\left(\frac{\pi x}{20}\right) + 100 \end{aligned}$$

Therefore, we have

$$\begin{aligned} \text{Volume} &= \int_a^b A(x)\,dx \\ &= \int_0^{20} \left(5\sin\left(\frac{\pi x}{20}\right) + 100\right)\,dx \\ &= \left(-\frac{100}{\pi}\cos\left(\frac{\pi x}{20}\right) + 100x\right)\Bigg|_0^{20} \\ &= -\frac{100}{\pi}\cos(\pi) + 2000 - \left(-\frac{100}{\pi}\cos(0) - 0\right) \\ &= \frac{100}{\pi} + 2000 + \frac{100}{\pi} \\ &= \frac{200}{\pi} + 2000 \text{ cm}^3 \end{aligned}$$

EXAMPLE 4 Find the volume of a pyramid whose base is a square of side length b cm and whose height is h cm.



Solution: We need to find the cross-sectional area of a slice. Since the cross-sections will be squares, we just need to find the length of one side.

We observe that slope of the line directly above the x-axis is

$$m = \frac{\text{rise}}{\text{run}} = \frac{b/2}{h}$$

Therefore, the equation of that line is $f(x) = \dfrac{b}{2h}x$.

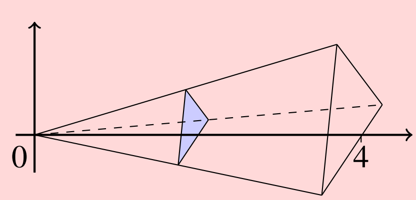
Since this is half the length of one side of the square, we get that the cross-sectional area at any position x is

$$\begin{aligned} A(x) &= (\text{length})^2 \\ &= \left(\frac{b}{h}x\right)^2 \\ &= \frac{b^2}{h^2}x^2 \end{aligned}$$

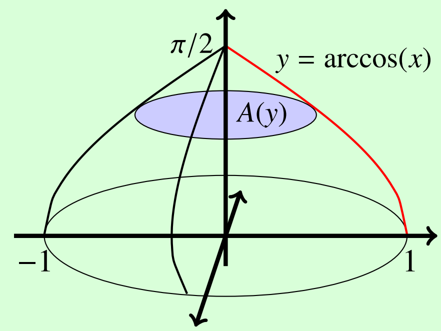
Therefore, we have

$$\begin{aligned} \text{Volume} &= \int_a^b A(x)\,dx \\ &= \int_0^h \frac{b^2}{h^2}x^2\,dx \\ &= \frac{b^2}{h^2}\cdot\frac{1}{3}x^3\Big|_0^h \\ &= \frac{b^2}{h^2}\cdot\frac{1}{3}h^3 - 0 \\ &= \frac{b^2h}{3} \end{aligned}$$

EXERCISE 3 Find the volume of the shape in the figure whose cross-sections are triangles, is 4 cm long, and the base is a triangle of width 2 cm and height 1 cm.



EXAMPLE 5 Find the volume of the mound given in the figure.



Solution: Making slices with thickness Δx as usual is going to be problematic.

In particular, that will mean we will likely have to integrate $\arccos(x)$. However, what is worse, is that the cross-sections are not going to have a nice geometric shape. Hence, we will have difficulty finding a formula for the cross-sectional $A(x)$.

However, if we do everything in terms of y, then not only will the cross-sections be circles, but the radius of the circles will be $x = g(y) = \cos(y)$ and so we don't have to work with $\arccos(x)$.

We get

$$\begin{aligned} A(y) &= \pi \times (\text{radius})^2 \\ &= \pi(\cos(y))^2 \end{aligned}$$

The y-values run from $y = 0$ to $y = \arccos(0) = \dfrac{\pi}{2}$.

Therefore, we have

$$\begin{aligned} \text{Volume} &= \int_a^b A(y)\, dy \\ &= \int_0^{\pi/2} \pi \cos^2(y)\, dy \\ &= \int_0^{\pi/2} \frac{\pi}{2}\left(1 + \cos(2y)\right)\, dy \\ &= \int_0^{\pi/2} \left(\frac{\pi}{2} + \frac{\pi}{2}\cos(2y)\right) dy \\ &= \left(\frac{\pi}{2}y + \frac{\pi}{4}\sin(2y)\right)\Bigg|_0^{\pi/2} \\ &= \frac{\pi^2}{4} + \frac{\pi}{4}\sin(\pi) - \left(0 + \frac{\pi}{4}\sin(0)\right) \\ &= \frac{\pi^2}{4} \end{aligned}$$

REMARK

When we look at the Cylindrical Shell Method below, we will see how to do this problem using x instead of y.

Section 9.1 Problems

1. Find the volume of a region which has the given cross-sectional area.
   (a) $A(x) = \sqrt{x}$, for $1 \le x \le 4$.
   (b) $A(x) = 3x^2 + 1$, for $-1 \le x \le 2$.
   (c) $A(x) = xe^x$, for $0 \le x \le 2$.
   (d) $A(x) = \dfrac{1}{(x+1)(x+2)}$, for $1 \le x \le 3$.
   (e) $A(x) = \sin^2(x)\cos^3(x)$, for $0 \le x \le \dfrac{\pi}{2}$.
   (f) $A(x) = \dfrac{1}{(9x^2+1)^{3/2}}$ for $0 \le x \le \dfrac{1}{3}$.
   (g) $A(x) = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$ for $4 \le x \le 9$
   (h) $A(x) = x^2 \ln(x)$ for $1 \le x \le e$
   (i) $A(x) = \dfrac{1}{1+2x}$, for $1 \le x \le \dfrac{3}{2}$
   (j) $A(x) = x^2\sqrt{x^3+1}$, for $0 \le x \le 2$
   (k) $A(x) = \tan(x)\sec^4(x)$, for $\dfrac{\pi}{3} \le x \le \dfrac{2\pi}{3}$
   (l) $A(x) = \dfrac{1+2x}{(1+x^2)(1+x)}$, for $0 \le x \le 1$

2. Determine if the given cross-sectional area will result in a convergent integral when used to determine a volume. If the integral converges, find the volume.
   (a) $A(x) = \dfrac{1}{(1+x)^2}$, for $x \ge 0$.
   (b) $A(x) = \dfrac{1}{1+x^2}$, for $x \ge 0$.
   (c) $A(x) = \dfrac{1}{\sqrt{1+x^2}}$, for $x \ge 0$.
   (d) $A(x) = x\ln(x)$, for $0 < x \le 1$.
   (e) $A(x) = \dfrac{2}{2-\sqrt{x}}$, for $0 \le x < 4$.

3. The following are volumes represented by using the integral of a cross-sectional area. Find a 3D shape that can be described by this volume.
   (a) $V = \displaystyle\int_a^b 4^2\,dx$. (Hint: think of base times height.)
   (b) $V = \displaystyle\int_a^b \pi 2^2\,dx$ (Hint: think of circles.)
   (c) $V = \displaystyle\int_0^h \pi x^2\,dx$
   (d) $V = \displaystyle\int_0^h x^2\,dx$

4. A wooden toy placed on a desk occupies an area defined by the region enclosed by the curves $x = 0$, $y = -\dfrac{1}{2}x^2 + 10$, and $y = \dfrac{x}{2}$. The toy has a uniform height (rising off the desk) of 5 units. Find the volume of the toy by integrating cross-sectional areas defined perpendicular to the x-axis.

5. Find the volume of a tetrahedron (pyramid-like shape) that has an equilateral triangle with side lengths 6 m as its base and a height of 10 m.

6. Let R be the region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$. A solid has R as its base and cross sections through the solid perpendicular to the base are parallel to the y-axis are squares. Find the volume of the solid.

7. A solid has a base that is bounded by the curves $y = x^2$ and $y = 2 - x^2$ in the xy-plane. Cross sections through the solid perpendicular to the base and parallel to the y-axis are circular discs. Find the volume of the solid.

Section 9.2: Volumes of Solids of Revolution

LEARNING OUTCOMES

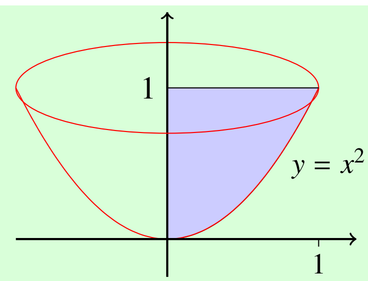
1. Know how to use the disc method to find the volume of a solid of revolution.
2. Know how to use the washer method to find the volume of a solid of revolution.
3. Know how to use the cylindrical shell method to find the volume of a solid of revolution.
4. Know how to determine which method to use when revolving a region around a certain axis.

We now look at some methods for finding the volume of a solid formed by rotating a two-dimensional region around an axis, called a **solid of revolution**. We will look at three different methods for doing this.

We first look at a few examples of generating a solid of revolution.

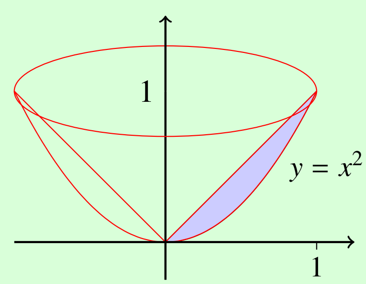
EXAMPLE 1 Sketch the solid of revolution formed by rotating the region bounded by $y = x^2$, $y = 1$, and the y-axis around the y-axis.

Solution: Taking the shaded region and rotating it around the y-axis we get the solid object in the figure.



EXAMPLE 2 Sketch the solid of revolution formed by rotating the region bounded by $y = x^2$ and $y = x$ around the y-axis.

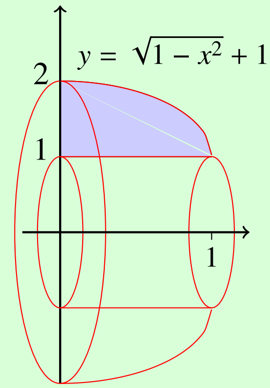
Solution: Taking the shaded region and rotating it around the y-axis we get a bowl with a curved outside and hollow conical inside.



When we find the volume of these solids of revolution, we will essentially be finding the amount of material required to produce the object. That is, since the object in Example 9.2.1 is solid while the object in Example 9.2.2 could hold water, the object in Example 9.2.1 will have a greater volume.

EXAMPLE 3 Sketch the solid of revolution formed by rotating the region bounded by $y = 1$, $y = \sqrt{1-x^2}+1$ and the y-axis around the x-axis.

Solution: Taking the shaded region and rotating it around the x-axis we get a donut like object with a cylindrical hole in the middle.



We now look at three different methods for finding the volume of a solid of revolution. Technically, these are not actually different methods and are really just doing what we did in the last section. We refer to them as different methods as the shape of the cross-sectional area is different in each case.

9.2.1 Disc Method

We have already used this method in Example 9.1.1 and Example 9.1.5. In particular, this is the case where the cross-sectional area is a circle and hence the slices used to approximate the volume are discs (right circular cylinders).

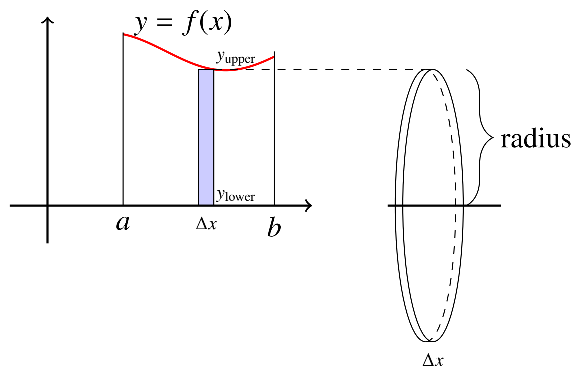
For example, say we want to determine the volume of a solid of revolution formed by rotating the region bounded by $y = f(x)$, $x = a$, $x = b$, and $y = 0$ about the x-axis. We begin by considering a typical rectangle, of width Δx, at position x. When we rotate the region about the x-axis, this rectangle will generate a disc of thickness Δx and radius

$$r(x) = y_{\text{upper}} - y_{\text{lower}}$$

as in the figure.

The cross-sectional area of such a disc will be

$$A(x) = \pi(r(x))^2$$



Therefore, using the same process we did in Section 9.1 to find volume (i.e. the limit of a Riemann sum), we get the volume of the solid of revolution will be

$$\text{Volume} = \int_a^b A(x)\,dx = \int_a^b \pi(r(x))^2\,dx$$

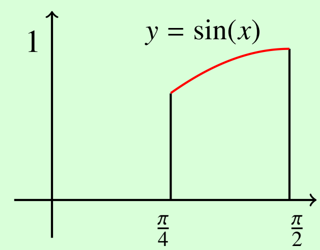
REMARKS

1. It is very important to ensure that every possible rectangle will have the same bottom function, y_{lower}, and the same top function, y_{upper}. If this is not possible, try drawing the rectangles horizontally instead of vertically.

2. Labelling the thickness of the rectangle (Δx if you draw the rectangle vertically, or Δy if you draw the rectangle horizontally) is highly recommended as it indicates which variable you are integrating with respect to.

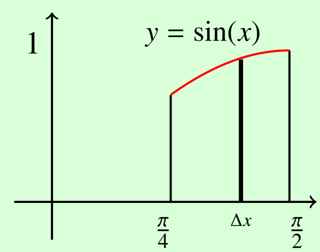
3. When solving problems, it is common to draw the rectangle as a thick line rather than actually taking the time to draw a rectangle.

EXAMPLE 4 Find the volume of the solid of revolution formed by rotating the region bounded by $y = \sin(x)$, $x = \dfrac{\pi}{4}$, $x = \dfrac{\pi}{2}$ and the x-axis around the x-axis.



Solution: We begin by drawing a vertical rectangle (thick line) at position x. The disc that this rectangle will generate when rotated around the x-axis has radius

$$r(x) = y_{\text{upper}} - y_{\text{lower}} = \sin(x) - 0 = \sin(x)$$



Thus, the cross-sectional area is

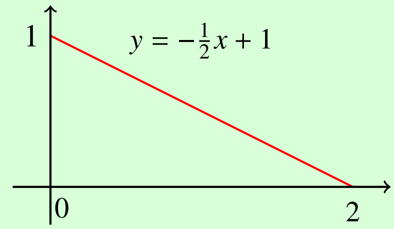
$$A(x) = \pi(r(x))^2 = \pi \sin^2(x)$$

Hence,

$$\begin{aligned}
\text{volume} &= \int_a^b A(x)\,dx \\
&= \int_{\pi/4}^{\pi/2} \pi \sin^2(x)\,dx \\
&= \int_{\pi/4}^{\pi/2} \pi \cdot \frac{1}{2}\left(1 - \cos(2x)\right)\,dx \\
&= \frac{\pi}{2}\left[\left(x - \frac{1}{2}\sin(2x)\right)\Bigg|_{\pi/4}^{\pi/2}\right] \\
&= \frac{\pi}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin(\pi) - \left(\frac{\pi}{4} - \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right)\right] \\
&= \frac{\pi^2}{8} + \frac{\pi}{4}
\end{aligned}$$

EXAMPLE 5

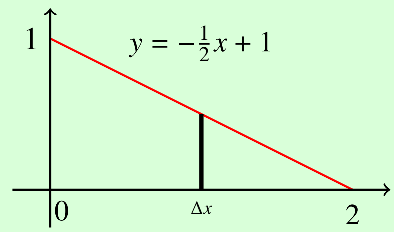
Find the volume of the solid of revolution formed by rotating the region bounded by $y = -\dfrac{1}{2}x + 1$, the x-axis and the y-axis around the x-axis.



Solution: We begin by drawing a vertical rectangle (thick line) at position x.

The disc that this rectangle will generate when rotated around the x-axis has radius

$$r(x) = y_{\text{upper}} - y_{\text{lower}} = -\frac{1}{2}x + 1 - 0 = -\frac{1}{2}x + 1$$



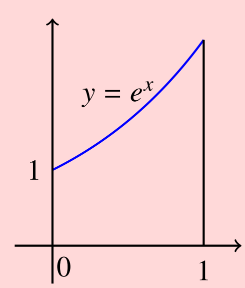
Thus, the cross-sectional area is $A(x) = \pi(r(x))^2 = \pi\left(-\dfrac{1}{2}x + 1\right)^2$.

The x values range from 0 to 2, so the volume is

$$\begin{aligned}
\text{volume} &= \int_a^b A(x)\,dx \\
&= \int_0^2 \pi\left(-\frac{1}{2}x + 1\right)^2\,dx \\
&= \int_0^2 \left(\frac{\pi}{4}x^2 - \pi x + \pi\right)dx \\
&= \left(\frac{\pi}{12}x^3 - \frac{\pi}{2}x^2 + \pi x\right)\Big|_0^2 \\
&= \left(\frac{8\pi}{12} - \frac{4\pi}{2} + 2\pi\right) - 0 \\
&= \frac{2\pi}{3}
\end{aligned}$$

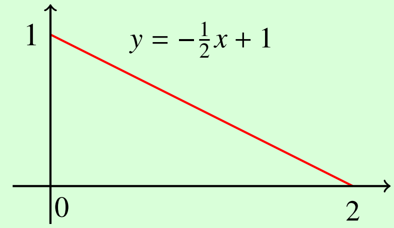
EXERCISE 1

Find the volume of the solid of revolution formed by rotating the region bounded by $y = e^x$, $x = 1$, the y-axis and the x-axis around the x-axis.



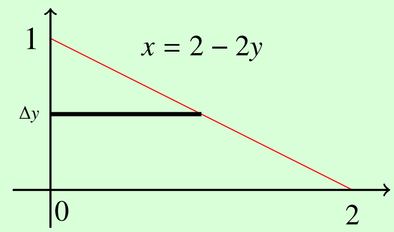
EXAMPLE 6

Find the volume of the solid of revolution formed by rotating the region bounded by $y = -\frac{1}{2}x + 1$, the x-axis and the y-axis around the y-axis.



Solution: Since we are revolving around the y-axis, we now draw a horizontal rectangle at position y and width Δy. Since we are going to be integrating with respect to y, we solve the equation $y = -\frac{1}{2}x + 1$ for x to get

$$x = 2 - 2y$$



Hence, the disc that will be generated when the rectangle is rotated around the y-axis has radius

$$r(y) = x_{\text{upper}} - x_{\text{lower}} = 2 - 2y - 0 = 2 - 2y$$

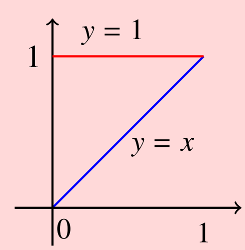
Thus, the cross-sectional area is $A(y) = \pi(r(y))^2 = \pi(2 - 2y)^2$.

The y values range from 0 to 1, so we have

$$\begin{aligned}
\text{volume} &= \int_a^b A(y)\, dy \\
&= \int_0^1 \pi(2 - 2y)^2\, dy \\
&= \pi \int_0^1 \left(4 - 8y + 4y^2\right)\, dy \\
&= \pi \left(4y - 4y^2 + \frac{4}{3}y^3\right)\Big|_0^1 \\
&= \pi \left(4 - 4 + \frac{4}{3}\right) - 0 \\
&= \frac{4\pi}{3}
\end{aligned}$$

EXERCISE 2

Use the Disc Method to find the volume of the solid of revolution formed by rotating the region bounded by $y = x$, $y = 1$, and the y-axis around the y-axis.



The disc method is only applicable when every rectangle is bounded by the axis of rotation. We now look at a method for dealing with the situation when that is not the case.

9.2.2 Washer Method

Say we want to determine the volume of a solid of revolution formed by rotating the region bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ about the line $y = c$. We begin by drawing a thin strip just like we did in the disc method. However, when we rotate the region about the x-axis, this rectangle will now generate a washer (a disc with a hole in the middle). The cross-sectional area of the washer is the area of the large circle minus the area of the hole. Therefore, the radius of the large circle (the outer radius) is

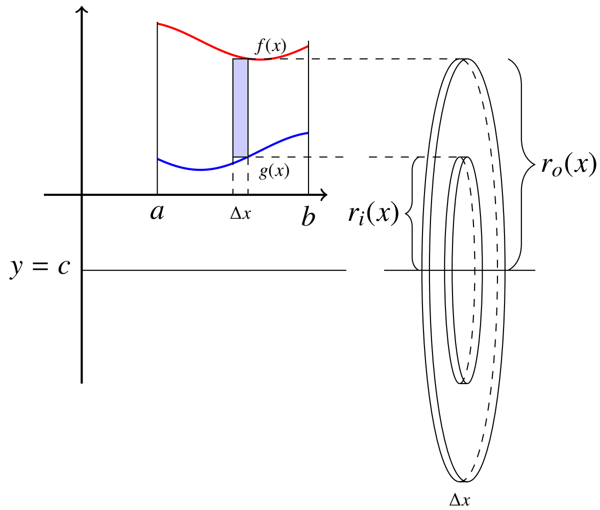
$$r_o(x) = f(x) - c$$

and the radius of the hole (the inner radius) is

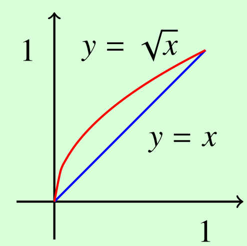
$$r_i = g(x) - c$$

Hence, the cross-sectional area is

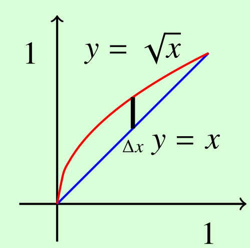
$$A(x) = \pi\left((r_o(x))^2 - (r_i(x))^2\right)$$



EXAMPLE 7 Find the volume of the solid of revolution formed by rotating the region bounded by $y = \sqrt{x}$ and $y = x$ for $0 \le x \le 1$, about the x-axis.



Solution: Drawing a thin strip at position x vertically we see that we have top function $f(x) = \sqrt{x}$ and bottom function $g(x) = x$.
Since we are rotating around $y = 0$, we get washers with



$$r_o = f(x) - c = \sqrt{x} - 0 = \sqrt{x}$$
$$r_i = g(x) - c = x - 0 = x$$

Thus, the cross-sectional area at position x is

$$A(x) = \pi(\sqrt{x})^2 - \pi(x)^2$$

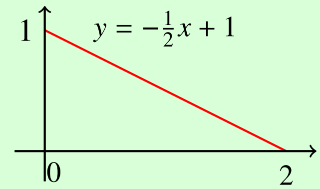
Hence, the volume is

$$\begin{aligned} V &= \int_0^1 \left(\pi(\sqrt{x})^2 - \pi(x)^2\right)\,dx \\ &= \pi\int_0^1 (x - x^2)\,dx \\ &= \pi\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 \\ &= \pi\left(\frac{1}{2} - \frac{1}{3} - 0\right) \\ &= \frac{\pi}{6} \end{aligned}$$

EXAMPLE 8

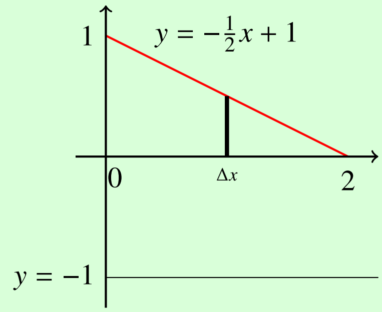
Find the volume of the solid of revolution formed by rotating the region bounded by

$y = -\dfrac{1}{2}x + 1$, the x-axis and the y-axis around $y = -1$.



Solution: We begin by drawing the axis of rotation and a vertical thin strip. We get top function $f(x) = -\dfrac{1}{2}x + 1$ and bottom function $g(x) = 0$.

The washer generated by this rectangle when it is rotated around the line $y = -1$ has



$$r_0 = f(x) - c = -\frac{1}{2}x + 1 - (-1) = 2 - \frac{1}{2}x$$
$$r_i = g(x) - c = 0 - (-1) = 1$$

Thus, the cross-sectional area is

$$A(x) = \pi\left(2 - \frac{1}{2}x\right)^2 - \pi(1)^2 = \pi\left(3 - 2x + \frac{1}{4}x^2\right)$$

The x values range from 0 to 2, so the volume is

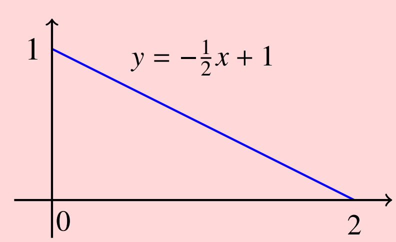
$$\begin{aligned}
\text{volume} &= \int_0^2 \pi\left(3 - 2x + \frac{1}{4}x^2\right)\,dx \\
&= \pi\left(3x - x^2 + \frac{1}{12}x^3\right)\Bigg|_0^2 \\
&= \pi\left(6 - 4 + \frac{8}{12}\right) - 0 \\
&= \frac{8\pi}{3}
\end{aligned}$$

EXERCISE 3

Find the volume of the solid of revolution formed by rotating the region bounded by

$y = -\dfrac{1}{2}x + 1$, the x-axis and the y-axis around

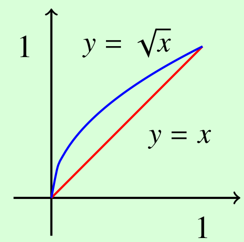
$y = 1$.



As with the Disc Method, the Washer Method can also be used when we are rotating around a vertical line $x = c$.

EXAMPLE 9

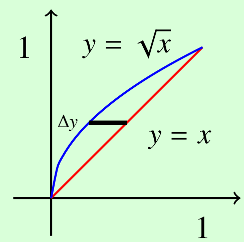
Find the volume of the solid of revolution formed by rotating the region bounded by $y = \sqrt{x}$ and $y = x$ for $0 \le x \le 1$, about the y-axis.



Solution: We are drawing a horizontal thin strip at position y. Since we are going to integrate with respect to y, we solve the equations for x to get

$$y = x \Rightarrow x = y$$
$$y = \sqrt{x} \Rightarrow x = y^2$$

Thus, the thin strip has top function $f(y) = y$ and bottom function $g(y) = y^2$.
Since we are rotating around $x = 0$, we get washers with



$$r_o = y$$
$$r_i = y^2$$

Thus, the cross-sectional area at position x is

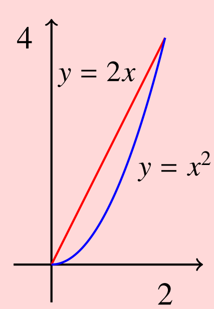
$$A(y) = \pi(y)^2 - \pi(y^2)^2$$

Hence, the volume is

$$\begin{aligned} V &= \int_0^1 \left(\pi y^2 - \pi y^4\right)\, dy \\ &= \pi\left(\frac{1}{3}y^3 - \frac{1}{5}y^5\right)\Big|_0^1 \\ &= \pi\left(\frac{1}{3} - \frac{1}{5} - 0\right) \\ &= \frac{2\pi}{15} \end{aligned}$$

EXERCISE 4

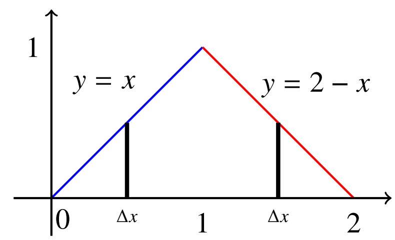
Find the volume of the solid of revolution formed by rotating the region bounded by $y = x^2$ and $y = 2x$ for $0 \le x \le 2$, about the line $x = -2$.



9.2.3 Cylindrical Shell Method

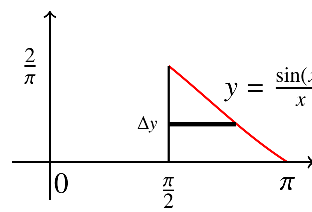
In both the Disc Method and the Washer Method, we integrate with respect to x (drawing the thin strip vertically) if we are rotating around the line $y = c$ and we integrate with respect to y (drawing the thin strip horizontally) if we are rotating around the line $x = c$.

However, this is not always going to work. For example, what if we wanted to find the volume of the solid of revolution obtained by rotating the region bounded by $y = x$, $y = 2 - x$ and the x-axis around the x-axis? This would be a problem as not every vertical rectangle would have the same top function.



We would also have a problem trying to find the volume of the solid of revolution obtained by rotating the region bounded by $y = \dfrac{\sin(x)}{x}$, $x = \dfrac{\pi}{2}$, and the x-axis around the y-axis.

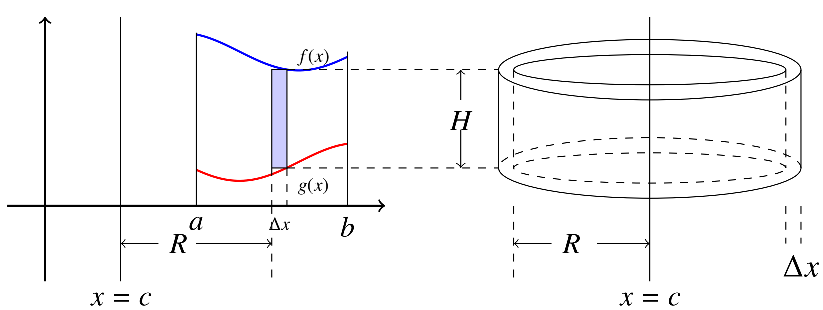
In particular, if we draw the thin strip horizontally, then we would need to solve $y = \dfrac{\sin(x)}{x}$ for x, which is impossible.



Therefore, we need to be able to integrate with respect to x (drawing the thin strip vertically) when we are rotating around a line $x = c$. Similarly, we need to be able to integrate with respect to y (drawing the thin strip horizontally) when we are rotating around a line $y = d$.

Let's look at how to solve this in the case where the solid is obtained by rotating the region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ around the line $x = c$.

To ensure that we are going to integrate with respect to x, we draw the thin strip at position x vertically with width Δx. Rotating the rectangle around the line $x = c$ produces a cylinder with height $H = f(x) - g(x)$ and radius $R = x - c$ (the distance from the position x to the axis of rotation).



The cross-sectional area is the surface area of the side of this cylinder. That is,

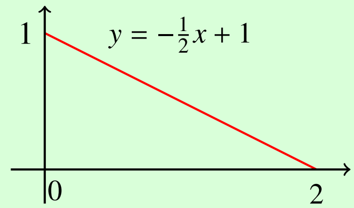
$$A(x) = 2\pi R \cdot H = 2\pi(x - c) \cdot (f(x) - g(x))$$

Hence, the volume of the solid of revolution is

$$\int_a^b 2\pi(x-c)\cdot(f(x)-g(x))\,dx$$

This is called the **Cylindrical Shell Method** since the cross-sectional area is the area of the 'shell' of the cylinders.

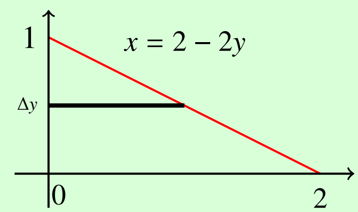
EXAMPLE 10 Use the Cylindrical Shell Method to find the volume of the solid of revolution formed by rotating the region bounded by $y = -\dfrac{1}{2}x + 1$, the x-axis and the y-axis around the x-axis.



Solution: To use the Cylindrical Shell Method, we need the thin strip to be parallel to the axis of rotation. So, we begin by drawing a horizontal thin strip at position y with width Δy.

Since we are integrating with respect to y, we solve the equation $y = -\dfrac{1}{2}x + 1$ for x to get

$$x = 2 - 2y$$



The height H of the cylindrical shell is the length of the thin strip. So,

$$H(y) = (2-2y) - 0 = 2 - 2y$$

The radius R of the cylindrical shell is the distance from the thin strip to the axis of rotation. So,

$$R(y) = y - 0 = y$$

Thus, the cross-sectional area is

$$A(y) = 2\pi R\cdot H = 2\pi y\cdot(2-2y)$$

The y values range from 0 to 1, so the volume is

$$\begin{aligned}
\text{volume} &= \int_0^1 2\pi y\cdot(2-2y)\,dy \\
&= \pi\int_0^1 \left(4y - 4y^2\right)\,dy \\
&= \pi\left(2y^2 - \frac{4}{3}y^3\right)\Big|_0^1 \\
&= \pi\left(2 - \frac{4}{3}\right) - 0 \\
&= \frac{2\pi}{3}
\end{aligned}$$

REMARK

In Example 9.2.10 we solved the same problem as in Example 9.2.5, but with a different method. Which method do you think was easier?

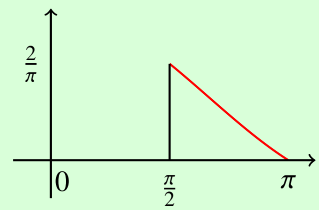
For all future 'find the volume of the solid of revolution' problems, we will need to decide which of the three methods to use. In some cases we will be able to use more than one method, while in other cases only one of the methods will work.

The primary consideration is usually the region which we are rotating. In particular, we need to ensure that every possible rectangle (thin strip) will have the same bottom function and the same top function. If drawing the thin strips one way does not accomplish this, they you know to draw the thin strip the other way. Remember that how the thin strip is drawn (horizontally or vertically) indicates the variable that you are integrating with respect to (y or x).

Other considerations include how easy it is to solve for the top and bottom function and how easy it will be to integrate the function.

We now look at a few examples.

EXAMPLE 11 Find the volume of the solid of revolution formed by rotating the region bounded by $y = \dfrac{\sin(x)}{x}$, $x = \dfrac{\pi}{2}$ and the x-axis around the y-axis.

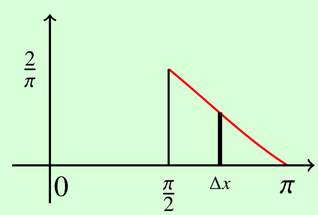


Solution: As we discussed above, we cannot draw the rectangles horizontally, so we must draw them vertically with width Δx.

Since the rectangles are now parallel to the axis of rotation, it means that we are using the Cylindrical Shell Method.

The height H of the cylindrical shell is

$$H(x) = f(x) - g(x) = \frac{\sin(x)}{x} - 0 = \frac{\sin(x)}{x}$$



and the radius R is

$$R(x) = x - 0 = x$$

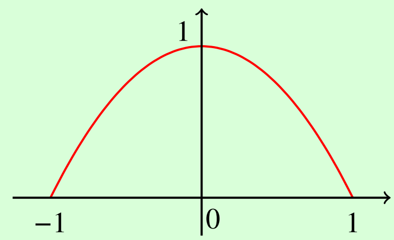
Thus, the cross-sectional area is

$$A(x) = 2\pi R \cdot H = 2\pi x \cdot \frac{\sin(x)}{x} = 2\pi \sin(x)$$

The x values range from $\frac{\pi}{2}$ to π, so the volume is

$$\begin{aligned}\text{volume} &= \int_{\pi/2}^{\pi} 2\pi \sin(x)\,dx \\ &= (2\pi - \cos(x))\Big|_{\pi/2}^{\pi} \\ &= 2\pi\left(-\cos(\pi) - \left[-\cos\left(\frac{\pi}{2}\right)\right]\right) \\ &= 2\pi\end{aligned}$$

EXAMPLE 12 Find the volume of the solid of revolution formed by rotating the region bounded by $y = 1 - x^2$ and the x-axis around the the line $x = 2$.



Solution: If we draw the thin strip horizontally with width Δy, then we would have to solve $y = 1 - x^2$ for y. Although that would work, it will be easier if we instead draw the thin strip vertically at position x with width Δx.

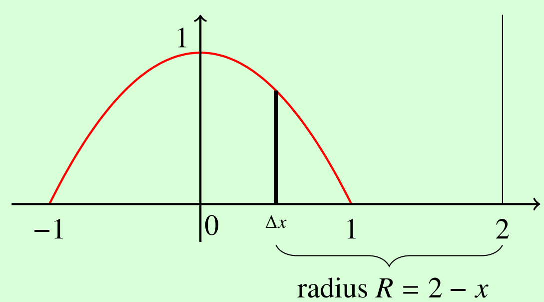
Since the thin strip is parallel to the axis of rotation, we are using the Cylindrical Shell Method.

The height H of the cylindrical shell is

$$H(x) = f(x) - g(x) = 1 - x^2$$

and the radius R is

$$R(x) = 2 - x$$



Thus, the cross-sectional area is

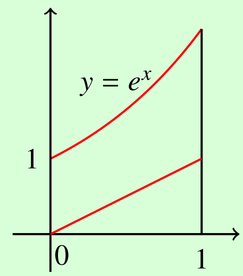
$$A(x) = 2\pi R \cdot H = 2\pi(2 - x)\cdot(1 - x^2) = 2\pi(x^3 - 2x^2 - x + 2)$$

The x values range from -1 to 1, so the volume is

$$\begin{aligned}\text{volume} &= \int_{-1}^{1} 2\pi(x^3 - 2x^2 - x + 2)\,dx \\ &= 2\pi\left(\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right)\Big|_{-1}^{1} \\ &= \frac{16\pi}{3}\end{aligned}$$

EXERCISE 5 Find the volume of the solid of revolution formed by rotating the region bounded by $y = 1 - x^2$ and the x-axis around the the line $x = 2$ using the Washer Method.

EXAMPLE 13 Find the volume of the solid of revolution formed by rotating the region bounded by $y = e^x$, $y = x$, $x = 1$, and the y-axis around $x = -1$.



Solution: We draw the thin strip vertically at position x with width Δx.

Since the thin strip is parallel to the axis of rotation, we are using the Cylindrical Shell Method.

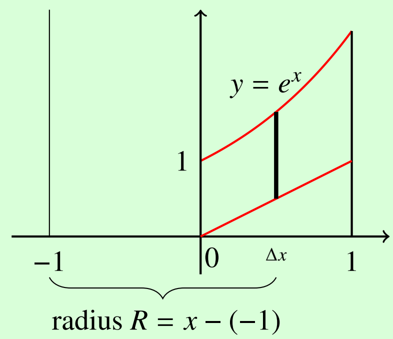
The height H of the cylindrical shell is

$$H(x) = f(x) - g(x) = e^x - x$$

and the radius R is

$$R(x) = x - (-1) = x + 1$$

Thus, the cross-sectional area is



$$A(x) = 2\pi R \cdot H = 2\pi(x + 1) \cdot (e^x - x)$$

The x values range from -1 to 1, so the volume is

$$\text{volume} = \int_0^1 2\pi(x + 1) \cdot (e^x - x)\, dx$$

To solve we use integration by parts. Let $u = x + 1$ and $dv = (e^x - x)\, dx$. Then, $du = dx$ and $v = e^x - \frac{1}{2}x^2$. Hence,

$$\begin{aligned}
\text{volume} &= 2\pi(x + 1)\left(e^x - \frac{1}{2}x^2\right)\Big|_0^1 - 2\pi \int_0^1 \left(e^x - \frac{1}{2}x^2\right) dx \\
&= 2\pi\,[2e - 1 - (1 - 0)] - 2\pi\left(e^x - \frac{1}{6}x^3\right)\Big|_0^1 \\
&= 2\pi\,[2e - 2] - 2\pi\left[e - \frac{1}{6} - (1 - 0)\right] \\
&= 2\pi e - \frac{5\pi}{3}
\end{aligned}$$

Section 9.2 Problems

1. Find the volume of the solid formed by rotating the region bounded by the given curves around the x-axis.
   (a) $y = 2x$, $y = 0$, $x = 3$
   (b) $y = x$, $y = x + 2$, $x = 0$, $x = 4$
   (c) $y = 2 - 2x$, $y = 0$, $x = 0$
   (d) $y = \sec(x)$, $y = 0$, $x = 0$, $x = \frac{\pi}{4}$
   (e) $y = x$, $y = 2x$, $y = 3$
   (f) $y = x$, $y = \sqrt{x}$, $y = 1$, $y = 2$
   (g) $y = 1 - \sin(x)$, $y = \sin(x)$, $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$
   (h) $y = \sin(x)$, $y = 0$, $x = 0$, $x = \pi$

2. Find the volume of the solid formed by rotating the region bounded by the given curves around the y-axis.
   (a) $y = x$, $y = 0$, $x = 1$
   (b) $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 2$
   (c) $y = 2 - 2x$, $y = 0$, $x = 0$
   (d) $y = x$, $y = 2x$, $y = 3$
   (e) $y = x$, $y = 2x$, $x = 4$
   (f) $y = \sqrt{x - 1}$, $y = 1$, $x = 0$
   (g) $y = x$, $y = x^2$, $1 \le y \le 2$

3. Set up, but do not evaluate, an integral to determine the volume of the solid formed by rotating the region bounded by the given curves around the given axis.
   (a) $y = x^3$, $y = 10 - x$, $y = 1$; around the x-axis
   (b) $y = x^3$, $y = 10 - x$, $y = 1$; around the line $y = 1$
   (c) $y = x^3$, $y = 10 - x$, $y = 1$; around the line $y = -1$
   (d) $y = \sin(x)$, $y = \sqrt{x}$, $x = 0$, $x = \frac{\pi}{2}$; around the line $y = 2$
   (e) $y = 2\sin(x)$, $y = 0$, $0 \le x \le \pi$; around the line $x = -1$
   (f) $y = 2\sin(x)$, $y = 0$, $0 \le x \le \pi$; around the line $y = -1$
   (g) $y = 1 - |x|$, $y = 0$; around the line $x = -1$
   (h) $y = (x - 2)^4$, $y = 8x - 16$, around the line $x = 8$
   (i) $y = \cos(x)$, $y = x + 1$, $x = \frac{\pi}{2}$; around the line $x = 2$
   (j) $y = \sqrt{1 - x^2}$, $y = 1 - x$; around the line $y = 1$
   (k) $y = \tan^3(x)$, $y = 2$, $x = 0$; around the line $y = 2$

End of Chapter Problems

1. Find the volume of a region which has the given cross-sectional area.
   (a) $A(x) = \sin(x)$ for $0 \le x \le \pi$.
   (b) $A(x) = x + 1$ for $3 \le x \le 5$.
   (c) $A(x) = e^x$ for $-2 \le x \le 0$.
   (d) $A(x) = \sin^2(x)\cos(x)$ for $0 \le x \le \dfrac{\pi}{2}$.
   (e) $A(x) = \tan^2(x)\sec^4(x)$ for $0 \le x \le \dfrac{\pi}{4}$.
   (f) $A(x) = \dfrac{1}{(x+1)(x+4)}$ for $0 \le x \le 2$.
   (g) $A(x) = \dfrac{1}{x^2\sqrt{x^2+1}}$ for $1 \le x \le 2$.
   (h) $A(x) = x\ln(x)$ for $1 \le x \le 4$.

2. Set up, but do not evaluate, an integral to compute the length of the curve.
   (a) $f(x) = 2x^3 + 1$ for $1 \le x \le 2$.
   (b) $f(x) = e^x$ for $0 \le x \le 1$.
   (c) $f(x) = \ln(\cos(x))$ for $0 \le x \le \pi/4$.
   (d) $f(x) = \dfrac{1}{2}x^2$ for $0 \le x \le 1$.

3. Find the volume of the solid formed by rotating the region bounded by the given curves about the given axis.
   (a) $y = x^{3/2}$, $x = 1$, $y = 0$; around the x-axis.
   (b) $y = x^{1/3}$, $y = 1$, $x = 0$; around the y-axis.
   (c) $y = (x-1)^2$, $y = 1$; around the x-axis.
   (d) $y = (x-1)^2$, $y = 1$; around the y-axis.
   (e) $y = (x-1)^2$, $y = 1$; around the line $x = -2$.
   (f) $y = (x-1)^2$, $y = 1$; around the line $y = 2$.
   (g) $y = \dfrac{x}{2}$, $y = \sqrt{x}$, $y = 1$; around the line $x = -1$
   (h) $y = \dfrac{x}{2}$, $y = \sqrt{x}$, $y = 1$; around the line $x = 2$
   (i) $y = \dfrac{x}{2}$, $y = \sqrt{x}$, $y = 1$; around the line $y = 2$
   (j) $y = \sqrt{x}$, $y = x^2$; around the line $y = -1$.
   (k) $y = \sqrt{x}$, $y = x^2$; around the line $x = 2$.
   (l) $xy = 1$, $x = 0$, $y = 1$, $y = e$; around the line $y = -1$.
   (m) $y = x^2$, $y = 2 - x$, $y = 0$; around the line $x = 2$.
   (n) $y = x\sqrt{1-x^2}$, $y = 0$; around the y-axis.

Chapter 10: Differential Equations

In Section 1.1.5, we defined differential equations along with the order of a differential equation. Then, in Section 1.2.3, we learned how to solve simple differential equations and initial value problems. We now take a more detailed look at differential equations. As usual, we recommend reviewing Sections 1.1.5 and 1.2.3 before proceeding with this Chapter.

Section 10.1: Direction Fields and Euler's Method

LEARNING OUTCOMES

1. Know how to draw a direction field for a differential equation.
2. Know how to match a direction field with a differential equation.
3. Know how to use a direction field to sketch the graph of a solution of a differential equation.
4. Know how to use Euler's Method to approximate a solution of a differential equation.

Unfortunately, most differential equations that occur in real world problems are not solvable. This should not be surprising given that we have no known solutions to even some relatively easy looking integrals $\left(\text{e.g. } \int e^{-x^2}\, dx\right)$. In this section, we will first look at how to make a rough sketch of solutions to a first order differential equation... even if we don't know what the solutions are. We will then use that approach to develop a method for roughly approximating a particular solution of the differential equation.

10.1.1 Direction Fields

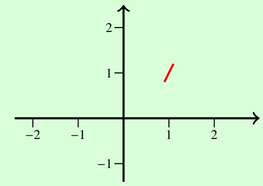
One of the most fundamental ways to get a better understanding of a function is to graph it. The exciting part here is that we can actually make rough sketches of the solutions to a first order differential equation without actually knowing what the solutions are!

We achieve this by thinking of the differential equation $\dfrac{dy}{dx} = f(x, y)$ as specifying the slope of the graph of a solution that passes through the point (x, y). To create a **direction field** for the differential equation, we draw a short line segment at each chosen point (x, y) with slope $f(x, y)$. Of course, the more points we select, the more accurate the direction field will be.

EXAMPLE 1 Draw a direction field for $y' = x + y^2$.

Solution: When $x = 1$ and $y = 1$, we have

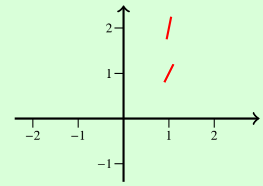
$$y' = 1 + (1)^2 = 2$$



So, we draw a short line segment with slope $m = 2$ at the point $(1, 1)$.

When $x = 1$ and $y = 2$, we have

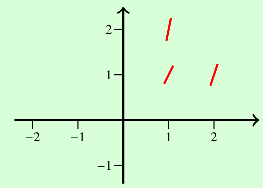
$$y' = 1 + (2)^2 = 5$$



So, we draw a short line segment with slope $m = 5$ at the point $(1, 2)$.

When $x = 2$ and $y = 1$, we have

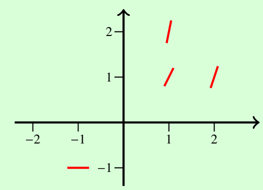
$$y' = 2 + (1)^2 = 3$$



So, we draw a short line segment with slope $m = 3$ at the point $(2, 1)$.

When $x = -1$ and $y = -1$, we have

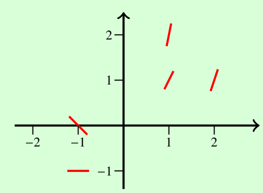
$$y' = -1 + (-1)^2 = 0$$



So, we draw a short line segment with slope $m = 0$ at the point $(-1, -1)$.

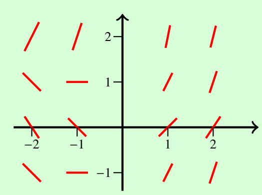
When $x = -1$ and $y = 0$, we have

$$y' = -1 + (0)^2 = -1$$



So, we draw a short line segment with slope $m = -1$ at the point $(-1, 0)$.

Continuing in this way, we get:



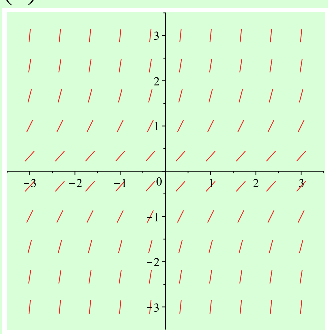
EXERCISE 1 Draw the line segments on the direction field for $y' = x - 2y$ at the points $(0, 1)$, $(1, 0)$, $(1, 1)$, $(-1, 1)$ and $(1, -1)$.

EXERCISE 2 Draw the line segments on the direction field for $y' = y^2$ at the points $(0, 1)$, $(1, 0)$, $(1, 1)$, $(-1, 1)$ and $(1, -1)$.

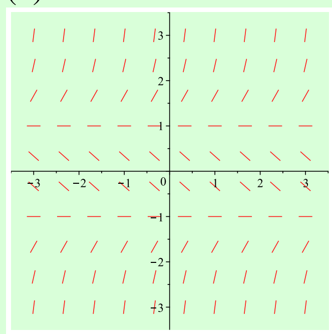
In addition to having you draw a few line segments of a direction field, we can also test your ability to match direction fields with differential equations.

EXAMPLE 2 Which of the following direction fields corresponds to $\dfrac{dy}{dx} = 1 - y^2$?

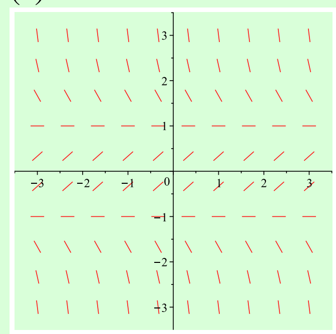
(a)



(b)



(c)

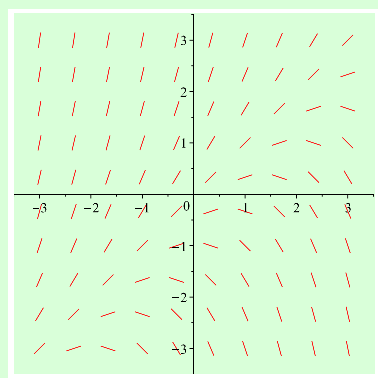


Solution: If $y = 1$, then $\dfrac{dy}{dx} = 1 - (1)^2 = 0$. In (a), the slope of the line segments for $y = 1$ are not 0, so it cannot be (a).

If $-1 < y < 1$, then $\dfrac{dy}{dx} > 0$. In (b), the graph has negative slope for these values of y. So, it cannot be (b).

Thus, the answer must be (c).

EXAMPLE 3 Which of the following differential equations corresponds to the given direction field?



(a) $y' = x + y + 1$

(b) $y' = -x + y + 1$

(c) $y' = -x^2 + y$

Solution: Consider the point $(x, y) = (2, 1)$.

For (a) we would have slope $y' = 2 + 1 + 1 = 4$.

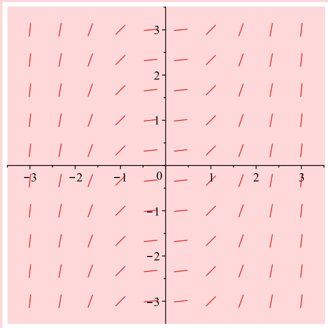
For (b) we would have slope $y' = -2 + 1 + 1 = 0$.

For (c) we would have slope $y' = -(2)^2 + 1 = -3$.

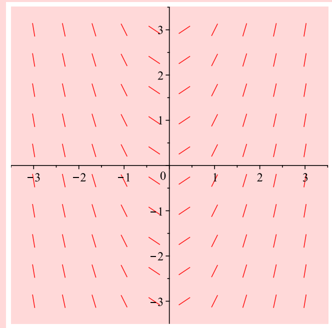
In the direction field, the slope at the point $(2, 1)$ looks closest to 0, so the answer is (b).

EXERCISE 3 Which of the following direction fields corresponds to $\dfrac{dy}{dx} = 2x$?

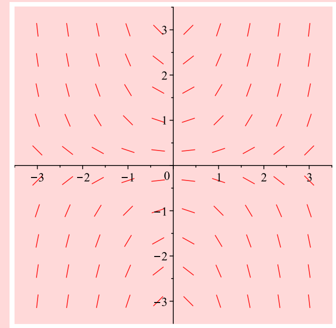
(a)



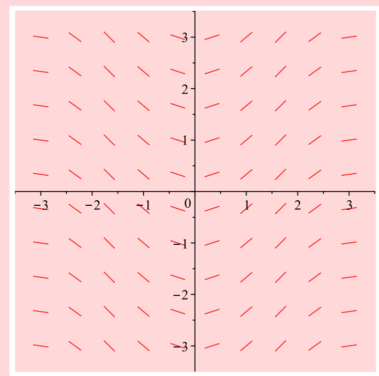
(b)



(c)



EXERCISE 4 Which of the following differential equations corresponds to the given direction field?

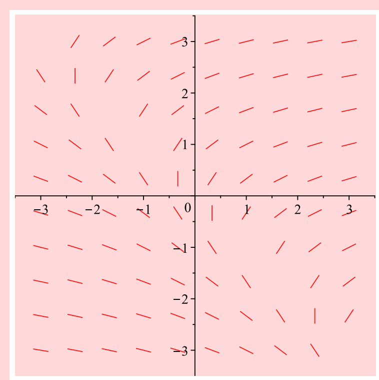


(a) $y' = \cos(x)$

(b) $y' = \sin(x)$

(c) $y' = \sin(y)$

EXERCISE 5 Which of the following differential equations corresponds to the given direction field?



(a) $y' = \dfrac{1}{x + y}$

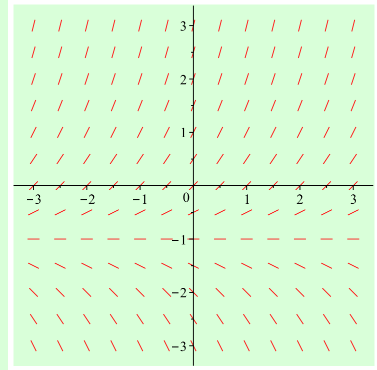
(b) $y' = \dfrac{x}{x^2 + y}$

(c) $y' = \dfrac{y}{x + y}$

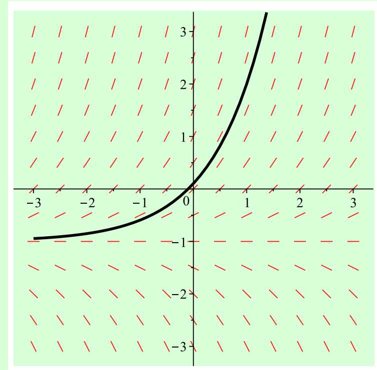
We now look at how to use a direction field to sketch a solution curve of an initial value problem that passes through the point (x_0, y_0) specified by the initial condition. Starting at the point (x_0, y_0) we start drawing the curve in the direction of the line segment in the direction field. As we approach another line segment in the direction field, we change the direction we are drawing to correspond to this new line segment. By following the direction of the line segments, we can get a rough approximation of the shape of the graph of that particular solution.

EXAMPLE 4

The direction field of a differential equation is given. Sketch the graph of the particular solution with initial condition $y(1) = 2$.

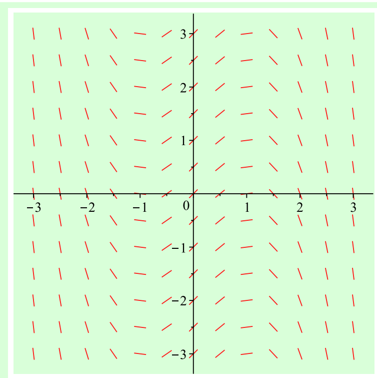


Solution: We start with the point $(1, 2)$. We then draw the graph, in both directions, by following the direction indicated by the line segment. When we approach a new line segment, we switch to the direction of that new line segment.

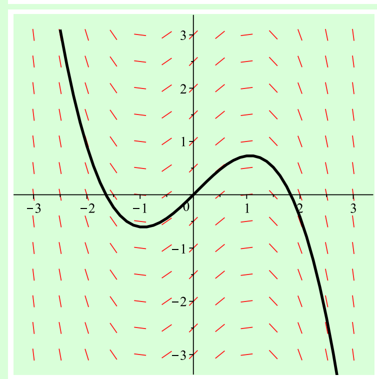


EXAMPLE 5

The direction field of a differential equation is given. Sketch the graph of the particular solution with initial condition $y(0) = 0$.



Solution: We get the following graph.



REMARK

This is a good example of why going to class is important. Many procedures like this one are demonstrated better in live action.

10.1.2 Euler's Method

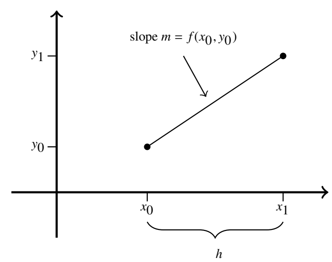
Observe that the short line segments we draw on a direction field are really tangent lines to a solution curve of the differential equation passing through the point. Now, rather than doing this graphically, we will do it algebraically to get a numerical approximation. In particular, instead of trying to follow the short line segments in a direction field, we will generate a sequence of points (x_i, y_i) that approximate points on the solution curve.

Consider an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

We begin by choosing how long we will follow each tangent line. We call this the **step-size** and label it as h.

Next, we find the tangent line at (x_0, y_0) and follow it to the point (x_1, y_1) where $x_1 = x_0 + h$. We now need to figure out what y_1 is. We know the slope of the tangent line at (x_0, y_0) is



$$y' = f(x_0, y_0)$$

Thus, we have

$$\begin{aligned} \frac{\text{rise}}{\text{run}} &= \text{slope} \\ \frac{y_1 - y_0}{x_1 - x_0} &= f(x_0, y_0) \\ \frac{y_1 - y_0}{h} &= f(x_0, y_0) \\ y_1 - y_0 &= f(x_0, y_0) \cdot h \\ y_1 &= y_0 + f(x_0, y_0) \cdot h \end{aligned}$$

Now, we have the coordinates of the new point (x_1, y_1). We, then repeat the procedure to get the next point (x_2, y_2), and continue as many times as desired.

The method above gives us the following algorithm called **Euler's Method**.

ALGORITHM

To approximate n points on a solution for an initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

using a step-size h:

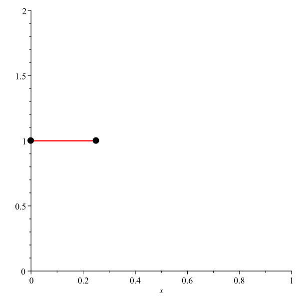
Step 1: Calculate $x_i = x_{i-1} + h$ for $1 \le i \le n$.

Step 2: Calculate $y_i = y_{i-1} + f(x_{i-1}, y_{i-1}) \cdot h$ for $1 \le i \le n$.

Let's first look at Euler's Method graphically using the initial value problem

$$\frac{dy}{dx} = xy^3, \quad y(0) = 1$$

and a step size of $h = \frac{1}{4}$.

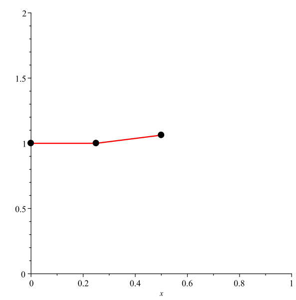


The initial condition tells us the initial point is $(x_0, y_0) = (0, 1)$.
The slope of the tangent line at this point is

$$y' = x_0(y_0)^3 = 0(1)^3 = 0$$

We then draw the tangent line for $0 \le x \le h$.

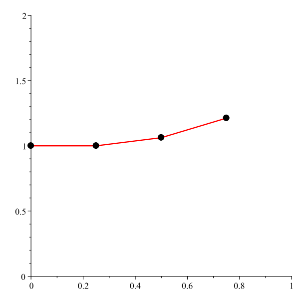
The end point of this line is $(x_1, y_1) = \left(\frac{1}{4}, 1\right)$.



The slope of the tangent line at this new point is

$$y' = x_1(y_1)^3 = \frac{1}{4}(1)^3 = \frac{1}{4}$$

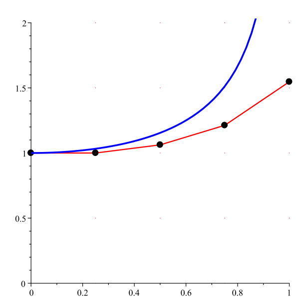
We then draw the tangent line for $h \le x \le 2h$.
The end point of this line is $(x_2, y_2) = \left(\frac{1}{2}, \frac{17}{16}\right)$.



The slope of the tangent line at this new point is

$$y' = x_2(y_2)^3 = \frac{1}{2}\left(\frac{17}{16}\right)^3 \approx 0.6$$

We then draw the tangent line for $2h \le x \le 3h$.



Repeating this for the next two points

$$(x_3, y_3) = \left(\frac{3}{4}, 1.2124\right)$$
$$(x_4, y_4) = (1, 1.5466)$$

we get the figure to the right. The graph of the actual solution to the initial value problem is also given (in blue).

Let's now look at Euler's Method algebraically using the algorithm above.

EXAMPLE 6 Approximate $n = 3$ points on a solution for the initial value problem

$$\frac{dy}{dx} = y - 2x, \qquad y(0) = 1$$

using a step size of $h = 0.2$.

Solution: We have $f(x, y) = y - 2x$ and the initial condition $y(0) = 1$ corresponds to the point $(x_0, y_0) = (0, 1)$. Thus,

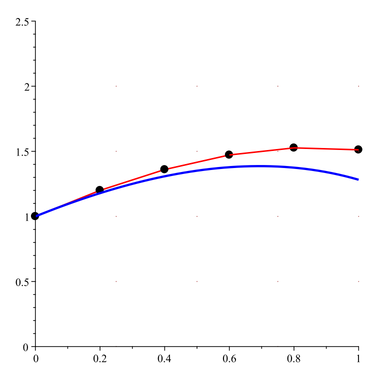
$$\begin{aligned} x_1 &= x_0 + h = 0 + 0.2 = 0.2 \\ x_2 &= x_1 + h = 0.2 + 0.2 = 0.4 \\ x_3 &= x_2 + h = 0.4 + 0.2 = 0.6 \end{aligned}$$

Next, we get

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0) \cdot h \\ &= 1 + (1 - 2(0)) \cdot 0.2 = 1 + 0.2 = 1.2 \\ y_2 &= y_1 + f(x_1, y_1) \cdot h \\ &= 1.2 + (1.2 - 2(0.2)) \cdot 0.2 = 1.2 + (0.8)(0.2) = 1.36 \\ y_3 &= y_2 + f(x_2, y_2) \cdot h \\ &= 1.36 + (1.36 - 2(0.4)) \cdot 0.2 = 1.36 + (0.56)(0.2) = 1.472 \end{aligned}$$

If we continued the calculations in the example above to $n = 5$, we would get the additional points

$$\begin{aligned} (x_4, y_4) &= (0.8, 1.5264) \\ (x_5, y_5) &= (1, 1.51168) \end{aligned}$$



The graph to the right shows a plot of the points (x_0, y_0) through (x_5, y_5) along with the connecting tangent lines. The blue curve is graph of the actual solution of the differential equation.

Observe that Euler's Method does not give us a function that approximates the solution of the differential equation. It only gives us a set of points that lie approximately on the graph of the solution. We call such a set of points a *numerical simulation* of the differential equation.

EXAMPLE 7 Approximate $n = 3$ points on a solution for the initial value problem

$$\frac{dy}{dx} = x^2 + y, \qquad y(-1) = 2$$

using a step size of $h = 1$.

Solution: We have $f(x, y) = x^2 + y$ and the initial condition $y(-1) = 2$ corresponds to the point $(x_0, y_0) = (-1, 2)$. Thus,

$$\begin{aligned} x_1 &= x_0 + h = -1 + 1 = 0 \\ x_2 &= x_1 + h = 0 + 1 = 1 \\ x_3 &= x_2 + h = 1 + 1 = 2 \end{aligned}$$

Next, we get

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0) \cdot h \\ &= 2 + ((-1)^2 + 2) \cdot 1 = 2 + 3 \cdot 1 = 5 \\ y_2 &= y_1 + f(x_1, y_1) \cdot h \\ &= 5 + ((0)^2 + 5) \cdot 1 = 5 + 5 \cdot 1 = 10 \\ y_3 &= y_2 + f(x_2, y_2) \cdot h \\ &= 10 + ((1)^2 + 10) \cdot 1 = 10 + 11 \cdot 1 = 21 \end{aligned}$$

EXERCISE 6 Approximate $n = 2$ points on a solution for the initial value problem

$$\frac{dy}{dx} = x + y, \qquad y(1) = 1$$

using a step size of $h = 0.1$.

EXERCISE 7 Approximate $n = 2$ points on a solution for the initial value problem

$$\frac{dy}{dx} = 3y, \qquad y(0) = 2$$

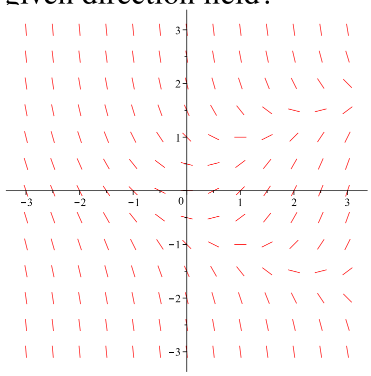
using a step size of $h = 0.2$.

There are many more efficient and more accurate methods than Euler's Method for approximating points on a solution curve for an initial value problem. However, understanding how Euler's Method works will help if you need to learn these more advanced methods in the future.

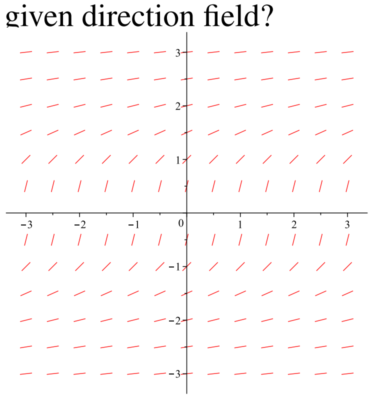
Section 10.1 Problems

1. Draw the line segments on the direction field for each differential equation at the points $(1, 1)$, $(2, 1)$, $(1, 2)$, and $(2, 2)$.
   (a) $y' = 3x - 2y$
   (b) $y' = \dfrac{x-1}{y}$
   (c) $y' = x^2 y$
   (d) $y' = y - 2 - (x-2)^2$

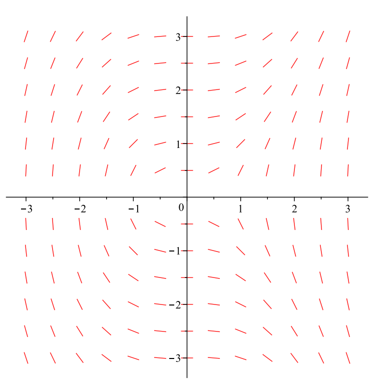
2. Which of the following differential equations corresponds to the given direction field?
   (a) $y' = x^2 - y$
   (b) $y' = x - y^2$
   (c) $y' = -x^2 - y$
   (d) $y' = -y^2$
   (e) $y' = x - y$



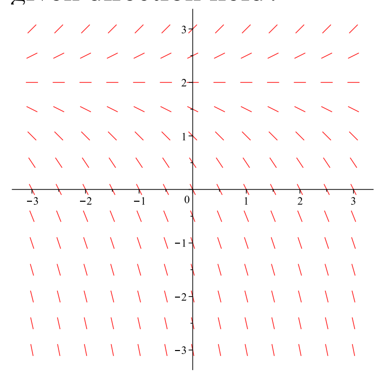
3. Which of the following differential equations corresponds to the given direction field?
   (a) $y' = xy^2$
   (b) $y' = \dfrac{x}{y}$
   (c) $y' = \dfrac{1}{y}$
   (d) $y' = \dfrac{1}{y^2}$



4. Which of the following differential equations corresponds to the given direction field?
   (a) $y' = \dfrac{y^2}{x}$
   (b) $y' = \dfrac{x}{y^3}$
   (c) $y' = \dfrac{x^2}{y}$
   (d) $y' = \dfrac{x}{y^2}$



5. Which of the following differential equations corresponds to the given direction field?
   (a) $y' = y - 2$
   (b) $y' = (y-2)^2$
   (c) $y' = 2 - y$
   (d) $y' = x - y - 2$
   (e) $y' = (x-2)^2$



6. For each initial value problem:
   (i) Solve the initial value problem.
   (ii) Use Euler's Method with step size h to approximate $n = 4$ values of the solution to the given initial value problem.
   (iii) On the same graph, plot the points found in part (ii) along with the tangent line between the points, and the solution curve found in part (i) (for (c) you will likely need a calculator to plot points on the solution curve).

   (a) $y' = 2x$, $y(-1) = 2$; $h = 0.5$
   (b) $y' = \dfrac{x}{3}$, $y(0) = 0$; $h = 0.5$
   (c) $y' = \dfrac{1}{x^2+1}$, $y(0) = 0$; $h = 1$

7. Use Euler's Method with step size h to approximate $n = 2$ values of the solution to the given initial value problem.
   (a) $y' = y$, $y(0) = 1$; $h = 1$
   (b) $y' = x + y$, $y(0) = 1$; $h = 0.1$
   (c) $y' = 2x$, $y(1) = 2$; $h = 1$
   (d) $y' = x - 2y$, $y(0) = 1$; $h = 0.5$
   (e) $y' = y - 2x$, $y(1) = 0$; $h = 0.5$
   (f) $y' = xy - y^2$, $y(0) = 1$; $h = 0.5$
   (g) $y' = x + y^2$, $y(1) = -1$; $h = 1$
   (h) $y' = \dfrac{x}{y^2}$, $y(1) = 1$; $h = 1$

Section 10.2: Separable Differential Equations

LEARNING OUTCOMES

1. Know how to recognize first order separable differential equations.
2. Know how to solve first order separable differential equations.

We now look at how to solve a special kind of first order differential equation.

DEFINITION

Separable Differential Equation

A first order differential equation that can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

is said to be **separable**.

These differential equations are called 'separable' since we can write the right hand side as a product of function of x multiplied by a function of y. In fact, as we will see in the theorem below, we will even think of the $\frac{dy}{dx}$ on the left hand side as $dy \cdot \frac{1}{dx}$ so that we can fully separate the y's from the x's.

EXAMPLE 1

Determine which of the following differential equations are separable.

(a) $\dfrac{dy}{dx} = \dfrac{x^2}{y+1}$

(b) $\dfrac{dy}{dx} = x - y$

(c) $(x^2+1)\dfrac{dy}{dx} = \dfrac{\cos(x)}{y}$

Solution: (a) $\dfrac{dy}{dx} = \dfrac{x^2}{y+1}$ is separable as we can write it in the form

$$\frac{dy}{dx} = x^2 \cdot \frac{1}{y+1}$$

(b) $\dfrac{dy}{dx} = x - y$ is not separable as it cannot be written in the form $\dfrac{dy}{dx} = f(x)g(y)$.

(c) $(x^2+1)\dfrac{dy}{dx} = \dfrac{\cos(x)}{y}$ is separable since we can write it in the form

$$\frac{dy}{dx} = \left(\frac{\cos(x)}{x^2+1}\right) \cdot \left(\frac{1}{y}\right)$$

As stated in the next theorem, the strategy for solving separable differential equations is to get all of the y's on the left hand side, all of the x's on the right hand side, and then integrate both sides. This gives us the solutions y as functions of x (possibly implicitly defined). To justify why this method works, we include the proof of the theorem.

THEOREM 1 The solutions for a separable differential equation of the form

$$\frac{dy}{dx} = f(x)g(y)$$

are given by

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

and all values of y such that $g(y) = 0$.

Proof: If $y = k$ is a constant such that $g(y) = 0$, then the differential equation is satisfied as $\dfrac{dy}{dx} = 0$ and $f(x)g(y) = f(x) \cdot 0 = 0$. Thus, these are all solutions.

If $g(y) \neq 0$, then we can rewrite the differential equation as

$$\frac{1}{g(y)} y'(x) = f(x)$$

Taking the integral with respect to x of both sides gives

$$\int \frac{1}{g(y)} y'(x)\, dx = \int f(x)\, dx$$

To solve the integral on the left, we perform a change of variables $u = y(x)$. We get $du = y'(x)\, dx$ and hence

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$

Since, $u = y$, we get

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

as required. $\square$

When we integrate both sides of $\displaystyle\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$ we only need to include one constant of integration. Since we generally want to try to solve for y, we always put the constant of integration on the x side. This will be demonstrated in the examples below.

EXAMPLE 2 Find all solutions of $\dfrac{dy}{dx} = (y^2 + 1)\cos(x)$.

Solution: Since $y^2 + 1 \neq 0$, we can divide both sides by it to get

$$\frac{1}{y^2+1}\,\frac{dy}{dx} = \cos(x)$$

Next, we multiply both side by dx.

$$\frac{1}{y^2+1}\,dy = \cos(x)\,dx$$

Thus, the solutions are given by

$$\begin{aligned}\int \frac{1}{y^2+1}\,dy &= \int \cos(x)\,dx\\ \arctan(y) &= \sin(x) + C\\ y &= \tan(\sin(x) + C)\end{aligned}$$

Thus, the solutions are $y = \tan(\sin(x) + C)$.

EXAMPLE 3 Find all solutions of $y' = e^x y^2$.

Solution: Observe that $y = 0$ is a solution.

Assuming $y \neq 0$, and rewrite y' using the notation $\dfrac{dy}{dx}$ we get

$$\begin{aligned}\frac{1}{y^2}\,\frac{dy}{dx} &= e^x\\ \frac{1}{y^2}\,dy &= e^x\,dx\end{aligned}$$

Thus, we have

$$\begin{aligned}\int \frac{1}{y^2}\,dy &= \int e^x\,dx\\ -\frac{1}{y} &= e^x + C\\ -y &= \frac{1}{e^x + C}\\ y &= \frac{-1}{e^x + C}\end{aligned}$$

Thus, the solutions are $y = \dfrac{-1}{e^x + C}$ and $y = 0$.

REMARK

In the example above, the constant solution $y = 0$ is sometimes called a **singular solution** since it cannot be obtained from the formula $y = \dfrac{-1}{e^x + C}$.

EXAMPLE 4 Find all solutions of $y' = \dfrac{y}{x}$.

Solution: Observe that $y = 0$ is a solution.

Assuming $y \neq 0$, we get

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx$$

Thus, we have

$$\begin{aligned}
\int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\
\ln(|y|) &= \ln(|x|) + C \\
|y| &= e^{\ln(|x|)+C} \\
|y| &= e^C \cdot e^{\ln(|x|)} && \text{(by properties of exponents)} \\
|y| &= e^C \cdot |x| && \text{(by properties of logarithms)} \\
y &= \pm e^C \cdot |x|
\end{aligned}$$

Having a term like $\pm e^C$ would make solving for C in an initial value problem complicated. To simplify this, we observe that since C is any number, e^C can be any positive number and hence $\pm e^C$ can be any non-zero number. Thus, we can replace $\pm e^C$ simplify with k where $k \neq 0$.

That is, this part of the solution gives

$$y = k|x|, \qquad k \neq 0$$

Observe that the solution $y = 0$ corresponds to the case where $k = 0$. Therefore, we can add the $k = 0$ case to the formula above to get that all solutions are

$$y = k|x|$$

for any real number k.

EXAMPLE 5 Find all solutions of $x + yy' = 0$.

Solution: We can rewrite this as

$$\begin{aligned} y\frac{dy}{dx} &= -x \\ y\,dy &= -x\,dx \end{aligned}$$

Next we find that

$$\begin{aligned} \int y\,dy &= \int -x\,dx \\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + C \\ y^2 &= -x^2 + C \end{aligned}$$

Thus, the solutions of the differential equation satisfy the implicitly defined function $y^2 = -x^2 + C$.

Observe in the last example that we technically should have got $2C$ in the last line, but, we left it just as C since two times an arbitrary number is still any arbitrary number. This is different than the case in Example 10.2.4 when we had $\pm e^C$ as $\pm e^C$ is no longer any arbitrary number... it couldn't be 0. In the cases where we add a restriction to the arbitrary number, we usually substitute in a new variable name and indicate the restriction.

EXERCISE 1 Find all solutions of $y' = -2y^3(2x + 1)$.

EXERCISE 2 Find all solutions of $y' = \dfrac{y}{x^2}$.

Here is an example where we need to use techniques of integration when solving the integrals.

EXAMPLE 6 Find all solutions of $\sec(x)\dfrac{dy}{dx} = x(y^2 - y)$.

Solution: Observe that $y = 0$ and $y = 1$ are solutions.

Assuming $y^2 - y \neq 0$, we get

$$\frac{1}{y^2 - y}\,dy = x\cos(x)\,dx$$

Thus, we have

$$\int \frac{1}{y(y-1)}\,dy = \int x\cos(x)\,dx \tag{10.1}$$

For the integral on the left, we use partial fractions. The form of the PFD is

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

Multiplying by the denominator gives

$$1 = A(y-1) + By$$

Taking $y = 1$, we get $1 = B$.

Taking $y = 0$, we get $1 = A(-1)$, so $A = -1$. Hence, we have

$$\begin{aligned}\int \frac{1}{y(y-1)}\,dy &= \int \left(\frac{-1}{y} + \frac{1}{y-1}\right) dy\\ &= -\ln(|y|) + \ln(|y-1|) + C_1\\ &= \ln\left(\left|\frac{y-1}{y}\right|\right) + C_1\end{aligned}$$

For the integral on the right, we use integration by parts. Let $u = x$ and $dv = \cos(x)\,dx$. Then, $du = dx$ and $v = \sin(x)$. Hence,

$$\begin{aligned}\int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx\\ &= x\sin(x) + \cos(x) + C_2\end{aligned}$$

Substituting these into equation (10.1) gives

$$\begin{aligned}\ln\left(\left|\frac{y-1}{y}\right|\right) &= x\sin(x) + \cos(x) + C_3\\ \left|\frac{y-1}{y}\right| &= e^{x\sin(x)+\cos(x)+C_3}\\ \frac{y-1}{y} &= \pm e^{C_3}e^{x\sin(x)+\cos(x)}\\ y-1 &= y\cdot ke^{x\sin(x)+\cos(x)} \qquad \text{where } k = \pm e^{C_3} \neq 0\\ y - y\cdot ke^{x\sin(x)+\cos(x)} &= 1\\ y(1-ke^{x\sin(x)+\cos(x)}) &= 1\\ y &= \frac{1}{1-ke^{x\sin(x)+\cos(x)}}, \quad k \neq 0\end{aligned}$$

If $k = 0$, then we get $y = 1$. That is, the solution $y = 1$ can be combined with $y = \dfrac{1}{1-ke^{x\sin(x)+\cos(x)}}$.

So, the solutions are $y = 0$ and $y = \dfrac{1}{1-ke^{x\sin(x)+\cos(x)}}$, for any real number k.

EXERCISE 3 Find all solutions of $\frac{dy}{dx} = \frac{xy^2}{\sqrt{1+x^2}}$.

EXERCISE 4 Find all solutions of $\frac{dy}{dx} = \frac{e^y \sin^2(x)}{y \sec(x)}$.

We now look at a couple of examples of initial values problems involving separable differential equations.

EXAMPLE 7 Solve the initial value problem $\frac{1}{x}\frac{dy}{dx} = 1 + y$, $y(1) = 2$.

Solution: The initial condition says $y = 2$, thus $1 + y \neq 0$ and hence we can rewrite the equation as

$$\frac{1}{1+y}\, dy = x\, dx$$

Thus, we have

$$\begin{aligned}
\int \frac{1}{1+y}\, dy &= \int x\, dx \\
\ln(|1+y|) &= \frac{1}{2}x^2 + C \\
|1+y| &= e^{\frac{1}{2}x^2 + C} \\
1 + y &= \pm e^C e^{\frac{1}{2}x^2} \\
1 + y &= C_1 e^{x^2/2} \qquad \text{where } C_1 = \pm e^C \neq 0 \\
y &= -1 + C_1 e^{x^2/2}
\end{aligned}$$

The initial condition says when $x = 1$ we have $y = 2$. Consequently,

$$\begin{aligned}
2 &= -1 + C_1 e^{(1)^2/2} \\
3 &= C_1 e^{1/2} \\
3e^{-1/2} &= C_1
\end{aligned}$$

Therefore, the solution is

$$y = -1 + 3e^{-1/2} e^{x^2/2}$$

Observe that, as mentioned in Example 10.2.4, replacing the $\pm e^C$ with C_1 made it much easier to solve for the constant of integration.

EXAMPLE 8

Solve the initial value problem $xy' = e^y$, $y(-1) = 0$.

Solution: Since $e^y \neq 0$, we can separate to get

$$e^{-y}\,dy = \frac{1}{x}\,dx$$

Thus, we have

$$\begin{aligned}
\int e^{-y}\,dy &= \int \frac{1}{x}\,dx \\
-e^{-y} &= \ln(|x|) + C \\
e^{-y} &= -\ln(|x|) + C \\
-y &= \ln(-\ln(|x|) + C) \\
y &= -\ln(-\ln(|x|) + C)
\end{aligned}$$

The initial condition says when $x = -1$ we have $y = 0$. Consequently,

$$\begin{aligned}
0 &= -\ln(-\ln(|-1|) + C) \\
0 &= -\ln(C) \\
e^0 &= C \\
1 &= C
\end{aligned}$$

Therefore, the solution is

$$y = -\ln(-\ln(|x|) + 1)$$

EXERCISE 5

Solve the initial value problem $y' = y^2\sin(x)$, $y(0) = 2$.

EXERCISE 6

Solve the initial value problem $\dfrac{dy}{dt} = (1 + y)(2 + t)$, $y(0) = 3$.

EXERCISE 7

Solve the initial value problem $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{\sqrt{9 + x^2}}$, $y(0) = 1$.

Section 10.2 Problems

1. Find all solutions of the differential equation.
   (a) $y' = 5y$
   (b) $xy' = 5$
   (c) $y^2y' = 1$
   (d) $y' = 3(y - 10)$
   (e) $y' = y^2 \sin(x)$
   (f) $y' = 2(y + 5)$
   (g) $y' = 6y^2x$
   (h) $y' = (x^2 + 1)y$
   (i) $\frac{1}{y}y' = 2x + 1$
   (j) $(1 + x^2)y' = xy$
   (k) $y' = -\frac{1}{2}(y - 20)$
   (l) $y' = \frac{x^2 + 6x}{y^2}$
   (m) $y' = e^{x+y}$
   (n) $y' = \frac{5 - y}{3}$
   (o) $y' = \frac{x^2}{y}$
   (p) $y' = 2 + 2y + 2x + 2xy$
   (q) $\frac{1}{y + 2}y' = \cos(x)$
   (r) $y' = -\frac{1 + y^2}{x^3y + xy}$
   (s) $\frac{dr}{d\theta} = \frac{r^2}{\theta}$

2. Solve the initial value problem.
   (a) $y' = xe^x$, $y(1) = 0$
   (b) $y' = 3x^2y^2$, $y(1) = 1$
   (c) $y' = 10(y - 50)$, $y(0) = 20$
   (d) $y' = \frac{\cos(x)}{y^2}$, $y(0) = 1$
   (e) $y' = 3(x + 1)^2y$, $y(0) = 1$
   (f) $y' = 5(y - 20)$, $y(0) = 100$
   (g) $y' = \frac{4x^2}{y}$, $y(0) = -2$
   (h) $y' = \frac{4y}{x + 3}$, $y(-2) = 1$
   (i) $y' = 2x(1 + y^2)$, $y(2) = 1$
   (j) $y' = -20(y - 100)$, $y(0) = 40$
   (k) $y' = \frac{1 - 10y}{100}$, $y(0) = 0.2$
   (l) $y' = (y^2 - 1)x^2$, $y(0) = 0$

3. Consider the initial value problem

$$y' = xy^2, \qquad y(0) = 1$$

   (a) Use Euler's method to approximate $n = 2$ values of the solution with step-size $h = \frac{1}{4}$.
   (b) Solve the initial value problem and compare $y\left(\frac{1}{4}\right)$ and $y\left(\frac{1}{2}\right)$ with the approximated values in part (a).

Section 10.3: Linear Differential Equations

LEARNING OUTCOMES

1. Know how to recognize first order linear differential equations.
2. Know how to solve first order linear differential equations using an integrating factor.

DEFINITION
Linear DE

A differential equation which can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called a **first order linear differential equation**.

EXAMPLE 1

Determine if the differential equation is linear, separable, or both.

(a) $y' = -2xy + x^2$

(b) $\frac{dy}{dx} + \left(-\frac{3x}{\cos(x)}\right)y = 0$

Solution: (a) $y' = -2xy + x^2$ is linear since it can be written in the form

$$y' + 2xy = x^2$$

It is not separable as it cannot be written in the form $y' = f(x)g(y)$.

(b) $\cos(x)\frac{dy}{dx} = 3xy$ is both linear since it can be written in the form

$$\frac{dy}{dx} + \left(-\frac{3x}{\cos(x)}\right)y = 0$$

It is also separable since it can be written in the form

$$\frac{dy}{dx} = \frac{3x}{\cos(x)} \cdot y$$

It is important to remember that in the standard form for a first order linear differential equation the coefficient of $\frac{dy}{dx}$ must be 1.

To solve first order linear differential equations, we use the method outlined in the following theorem.

THEOREM 1 Let $\frac{dy}{dx} + P(x)y = Q(x)$ be a first order linear differential equation. Let $F(x)$ be an antiderivative of $P(x)$ and define

$$I(x) = e^{F(x)}$$

The solution of the differential equation can be found by solving

$$I(x)y = \int I(x)Q(x)\,dx$$

Proof: We begin by multiplying the differential equation by $I(x)$ to get

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x) \tag{10.2}$$

Our goal now is to show that the left hand side is equal to $\frac{d}{dx}I(x)y$. We do this by calculating $\frac{d}{dx}I(x)y$.

By implicit differentiation we get

$$\frac{d}{dx}I(x)y = I(x)\frac{dy}{dx} + I'(x)y$$

and, since $F(x)$ is an antiderivative of $P(x)$, we get

$$I'(x) = F'(x)e^{F(x)} = P(x)e^{F(x)} = I(x)P(x)$$

Therefore,

$$\frac{d}{dx}I(x)y = I(x)\frac{dy}{dx} + I(x)P(x)y$$

as we wanted. Substituting this into equation (10.2) gives

$$\frac{d}{dx}I(x)y = I(x)Q(x)$$

Taking the integral of both sides we get

$$I(x)y = \int I(x)Q(x)\,dx$$

as required. $\square$

We now look at some examples to see how this is used.

EXAMPLE 2 Find all solutions of $\frac{dy}{dx} - \frac{2}{x}y = 3$.

Solution: We have $P(x) = -\frac{2}{x}$. An antiderivative of $P(x)$ is $F(x) = -2\ln(|x|)$. Thus, we define

$$\begin{aligned} I(x) &= e^{-2\ln(|x|)} \\ &= e^{\ln(x^{-2})} \\ &= x^{-2} \end{aligned}$$

We get

$$\begin{aligned} I(x)y &= \int I(x)Q(x)\,dx \\ x^{-2}y &= \int 3x^{-2}\,dx \\ x^{-2}y &= -3x^{-1} + C \\ y &= -3x + Cx^2 \end{aligned}$$

Thus, the solutions are $y = -3x + Cx^2$.

EXAMPLE 3 Find all solutions of $\frac{dy}{dx} = -2x(y-1)$.

Solution: We can rewrite this as

$$\begin{aligned} \frac{dy}{dx} &= -2xy + 2x \\ \frac{dy}{dx} + 2xy &= 2x \end{aligned}$$

Thus, we have $P(x) = 2x$. An antiderivative of $P(x)$ is $F(x) = x^2$. Therefore,

$$I(x) = e^{x^2}$$

We get

$$e^{x^2}y = \int 2xe^{x^2}\,dx$$

Let $u = x^2$. Then, $du = 2x\,dx$. Hence, we have

$$\begin{aligned} e^{x^2}y &= \int e^u\,du \\ e^{x^2}y &= e^u + C \\ e^{x^2}y &= e^{x^2} + C \\ y &= 1 + Ce^{-x^2} \end{aligned}$$

Thus, the solutions are $y = 1 + Ce^{-x^2}$.

EXERCISE 1

Solve $\frac{dy}{dx} = -2x(y-1)$ as a separable differential equation and compare the answer to that found in Example 10.3.3.

EXERCISE 2

Find all solutions of $\frac{dy}{dx} = 3y + 5$.

EXERCISE 3

Find all solutions of $\frac{dy}{dt} + y = t$.

EXERCISE 4

Find all solutions of $x\frac{dy}{dx} = -y + x$ with $x > 0$.

For completeness here is an example of an initial value problems involving a linear differential equation.

EXAMPLE 4

Solve the initial value problem $\frac{dy}{dx} = y + e^x$, $y(1) = 5$.

Solution: We rewrite this as
$$\frac{dy}{dx} - y = e^x$$
Thus, we have $P(x) = -1$, so an antiderivative of $P(x)$ is $F(x) = -x$. Therefore,
$$I(x) = e^{-x}$$
We get
$$\begin{aligned} e^{-x}y &= \int e^{-x}e^x\,dx \\ e^{-x}y &= \int 1\,dx \\ e^{-x}y &= x + C \\ y &= xe^x + Ce^x \end{aligned}$$
Substituting in the initial condition gives
$$5 = 1e^1 + Ce^1 \Rightarrow C = \frac{5-e}{e}$$
Hence, the solution is
$$y = xe^x + \frac{5-e}{e}e^x$$

EXERCISE 5 Solve the initial value problem $\frac{dy}{dx} + \frac{y}{2x} = 3$, $y(4) = 8$.

EXERCISE 6 Solve the initial value problem $x\frac{dy}{dx} - 2y = 3x^4$, $y(-1) = 2$.

EXERCISE 7 Consider the initial value problem $\frac{dy}{dx} = xy$, $y(-2) = 3$.

(a) Solve the problem as a separable differential equation.

(a) Solve the problem as a linear differential equation.

Section 10.3 Problems

1. For the given differential equation:
   (a) solve it as a separable differential equation.
   (b) solve it as a linear differential equation.

   (a) $y' = -2y$
   (b) $y' = (x+1)y$
   (c) $y' = y + 2$
   (d) $y' = \frac{4}{x}y$

2. Find all solutions of the differential equation.
   (a) $y' = y + x$
   (b) $y' = 10 - \frac{y}{2}$
   (c) $y' + \frac{2}{x}y = 5x^2$
   (d) $(x+1)y' - 2y = 3(x+1)$
   (e) $xy' + y = \sqrt{x}$
   (f) $y' = xy^2 + x$
   (g) $xy' + 4y = \frac{e^x}{x^2}$
   (h) $x\ln(x)y' + y = xe^x$, $x > 1$
   (i) $y' + \frac{1}{5}y = 25$
   (j) $y' = e^y$
   (k) $y' + \frac{1}{x}y = \ln(x)$

3. Solve the initial value problem.
   (a) $y' = 2 - 3y$, $y(1) = 2$
   (b) $y' = x^2 + \frac{2}{x}y$, $y(1) = 1$
   (c) $y' = xy^2 + y^2$, $y(1) = 2$
   (d) $y' + 2y = xe^{-2x}$, $y(0) = 1$
   (e) $xy' + 2y = x^2 - 1$, $y(-1) = 2$
   (f) $y' + \frac{1}{10}y = 25$, $y(0) = 10$
   (g) $y' = e^{x+y}$, $y(0) = 0$
   (h) $y' = 3y + e^{2x}$, $y(0) = 5$
   (i) $2y' + y = x$, $y(0) = 2$
   (j) $y' - y^2 = 1$, $y(0) = 1$
   (k) $xy' + (x+1)y = x$, $x > 0$, $y(1) = 0$
   (l) $xy' - y = x^2e^{-x}$, $x > 0$, $y(1) = -1$
   (m) $xy' + 2y = \sin(x)$, $y\left(\frac{\pi}{2}\right) = 0$
   (n) $y' = (x^2 - 1)y(x)$, $y(1) = 0$
   (o) $y' - \frac{1}{x}y = \ln(x)$, $y(1) = 2$
   (p) $y' = 1 - x + y - xy$, $y(0) = 1$

Section 10.4: Applications of Differential Equations

LEARNING OUTCOMES

1. Know how to set up a differential equation from a word problem.
2. Know how to solve rate-in/rate-out problems.

We now look briefly at some applications of differential equations.

10.4.1 Newton's Law of Cooling

EXAMPLE 1 Newton's Law of Cooling states that the rate of change of temperature of a substance is proportional to the difference between the temperature of the object and the temperature surrounding it. That is, if $T(t)$ is the temperature of the object at time t and A is the temperature surrounding it, then $T(t)$ satisfies $\frac{dT}{dt} = k(T - A)$.

A calculator has initial temperature of 20° C and is placed in boiling water that is 110° C. If the temperature of the calculator has reached 65° C after 3 minutes, then find the temperature of the calculator an any time t in minutes.

Solution: Solving the separable DE $\frac{dT}{dt} = k(T - A)$ we get

$$\begin{aligned} \int \frac{1}{T - 110}\, dT &= \int k\, dt \\ \ln(|T - 110|) + C_1 &= kt + C_2 \\ |T - 110| &= e^{kt+C} \qquad \text{where } C = C_2 - C_1 \\ T - 110 &= De^{kt} \qquad \text{where } D = \pm e^C \end{aligned}$$

Since $T(0) = 20$, we have $-90 = D$, thus

$$T = 110 - 90e^{kt}$$

Next, since $T(3) = 65$, we get

$$\begin{aligned} 65 &= 110 - 90e^{3k} \\ -45 &= -90e^{3k} \\ \frac{1}{2} &= e^{3k} \\ \ln\left(\frac{1}{2}\right) &= 3k \\ -\frac{1}{3}\ln(2) &= k \end{aligned}$$

Therefore, the temperature in degrees Celsius is

$$T(t) = 110 - 90e^{-\ln(2)t/3}$$

10.4.2 Rate in/Rate out Problems

EXAMPLE 2

A tank initially contains 100 L of pure water. Brine containing 10 g/L of salt is poured into the tank at a rate of 5 L/s. The tank is kept well mixed and drained at 5 L/s. Write an initial value problem for the amount of salt in the tank.

Solution: Let $S(t)$ be the amount of salt in the tank.

The rate of change of salt in the tank $\frac{dS}{dt}$ will be the amount of salt going into the tank minus the amount of salt leaving the tank. That is

$$\frac{dS}{dt} = \text{rate in} - \text{rate out}$$

The rate in is

$$\text{rate in} = \text{concentration} \times \text{amount} = 10\, g/L \cdot 5\, L/s = 50\, g/s$$

Similarly, the rate out is

$$\text{rate out} = \text{concentration} \times \text{amount}$$

What is the concentration in the tank at any time t?

$$\text{concentration} = \frac{\text{amount of salt}}{\text{amount of liquid}} = \frac{S(t)}{100}\, g/L$$

Thus,

$$\text{rate out} = \frac{S}{100}\, g/L \cdot 5\, L/s = \frac{S}{20}\, g/s$$

In the question, we are given that $S(0) = 0$. Thus, the initial value problem is

$$\frac{dS}{dt} = 50 - \frac{S}{20}, \quad S(0) = 0$$

EXERCISE 1

A tank initially contains 20 L of pure water. Brine containing 5 g/L of salt is poured into the tank at a rate of 3 L/s. The tank is kept well mixed and drained at the same rate. Write an initial value problem for the amount of salt in the tank.

EXERCISE 2

A tank initially contains 20 L water which has 1 g of salt dissolved in it. Brine containing 3 g/L of salt is poured into the tank at a rate of 4 L/s. The tank is kept well mixed and drained at the same rate. Write an initial value problem for the amount of salt in the tank.

EXAMPLE 3 A tank initially contains 10 L of water which has 3 g of salt dissolved in it. Brine containing 2 g/L of salt is poured into the tank at a rate of 4 L/s. The tank is kept well mixed and drained at 3 L/s. Find the amount of salt in the tank.

Solution: Let $S(t)$ be the amount of salt in the tank. Observe that the amount of the water in the tank is not constant since 4 L/s is entering while only 3 L/s is draining out. So, the volume of water in the tank at any time t is

$$v(t) = 10 + 4t - 3t = 10 + t$$

We know that

$$\frac{dS}{dt} = \text{rate in} - \text{rate out}$$

The rate in is

$$\text{rate in} = 2\,g/L \cdot 4\,L/s = 8\,g/s$$

The concentration in the tank at any time t is

$$\frac{S(t)}{v(t)} = \frac{S(t)}{10+t}\ \text{g/L}$$

Hence, the rate out is

$$\text{rate out} = \frac{S(t)}{10+t}\ \text{g/L} \cdot 3\text{L/s} = \frac{3S}{10+t}\ \text{g/s}$$

Hence, we have the linear differential equation

$$\begin{aligned}\frac{dS}{dt} &= 8 - \frac{3S}{10+t}\\ \frac{dS}{dt} + \frac{3}{10+t}S &= 8\end{aligned}$$

An antiderivative of $P(t) = \dfrac{3}{10+t}$ is $F(t) = 3\ln(10+t)$. Therefore,

$$I(t) = e^{3\ln(10+t)} = e^{\ln((10+t)^3)} = (10+t)^3$$

Thus,

$$\begin{aligned}(10+t)^3 S &= \int 8(10+t)^3\,dt\\ (10+t)^3 S &= 2(10+t)^4 + C\\ S &= 2(10+t) + \frac{C}{(10+t)^3}\end{aligned}$$

In the question, we are given that $S(0) = 3$. This gives

$$\begin{aligned}3 &= 20 + \frac{C}{(10)^3}\\ -17000 &= C\end{aligned}$$

Therefore, the amount of salt in the tank is given by

$$S(t) = 20 + 2t - \frac{17000}{(10+t)^3}\ \text{g}$$

EXERCISE 3

A tank initially contains 20L of pure water. Brine containing 5 g/L of salt is poured into the tank at a rate of 3 L/s. The tank is kept well mixed and drained at 4 L/s rate. Write a differential equation for the amount of salt in the tank. What is the domain of the problem?

EXAMPLE 4

A tank contains 1000 L of water. Brine containing 6 g/L of salt is added at a rate of 40 L/s and another brine containing 10 g/L of salt is added at a rate of 60 L/s. At the same time, the tank is being mixed constantly and 100 L of the mixture is being drained per second. Find the amount of salt in the tank.

Solution: Let $S(t)$ be the amount of salt in the tank at time t. We get

$$\begin{aligned}\frac{dS}{dt} &= \text{rate in} - \text{rate out}\\ &= (40\cdot 6 + 60\cdot 10) - \left(\frac{S}{1000}\cdot 100\right)\\ &= 840 - \frac{S}{10}\end{aligned}$$

This differential equation is separable. We get

$$\begin{aligned}\frac{dS}{dt} &= \frac{8400 - S}{10}\\ \frac{1}{8400 - S}\,dS &= \frac{1}{10}\,dt\\ \int \frac{1}{8400 - S}\,dS &= \int \frac{1}{10}\,dt\\ -\ln(|8400 - S|) &= \frac{1}{10}t + C\\ \ln(|8400 - S|) &= -\frac{1}{10}t + C\\ |8400 - S|) &= e^{-t/10+C}\\ 8400 - S &= ke^{-t/10}, \qquad \text{where } k = \pm e^{C}\\ -S &= -8400 + ke^{-t/10}\\ S &= 8400 + ke^{-t/10}\end{aligned}$$

We are given that $S(0) = 0$. Hence,

$$\begin{aligned}0 &= 8400 + ke^{0}\\ -8400 &= k\end{aligned}$$

Therefore, the amount of salt is given by

$$S(t) = 8400 - 8400e^{-t/10}\ \text{g}$$

EXERCISE 4 A tank contains 100 L of water. Brine containing 2 g/L of salt is added at a rate of 4 L/s and another brine containing 3 g/L of salt is added at a rate of 5 L/s. At the same time, the tank is being mixed constantly and 10 L of the mixture is being drained per second. Find how much salt is in the tank as a function of time.

EXAMPLE 5 The air in a room with volume 200 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Find the amount of carbon dioxide in the room as a function of time. What will the level of carbon dioxide be in the long run?

Solution: Let $y = y(t)$ be the amount of carbon dioxide in the room after t minutes. Then

$$y(0) = 0.0015 \times 200 = 0.3$$

The amount of air in the room is 200 m^3 at all times, so the concentration at time t is $\frac{y}{200}$, and the change in the amount of carbon dioxide is

$$\begin{aligned}
\frac{dy}{dt} &= \text{the rate coming in } - \text{ the rate going out}\\
&= 0.0005 \times 2 - \frac{y}{200} \times 2\\
&= \frac{1}{1000} - \frac{y}{100}\\
&= \frac{1-10y}{1000}
\end{aligned}$$

This differential equation is separable. We get

$$\begin{aligned}
\frac{1}{1-10y}\,dy &= \frac{1}{1000}\,dt\\
\int \frac{1}{1-10y}\,dy &= \int \frac{1}{1000}\,dt\\
-\frac{1}{10}\ln(|1-10y|) &= \frac{1}{1000}t + C\\
\ln(|1-10y|) &= -\frac{1}{100}t + C\\
|1-10y| &= e^{-t/100+C}\\
1-10y &= ke^{-t/100} \qquad \text{where } k = \pm e^{C_1} \neq 0
\end{aligned}$$

Using the initial condition $y(0) = 0.3$, we get that

$$\begin{aligned}
1-3 &= ke^{0}\\
-2 &= k
\end{aligned}$$

Therefore,

$$
\begin{aligned}
1 - 10y &= -2e^{-t/100} \\
10y &= 1 + 2e^{-t/100} \\
y &= \frac{1}{10} + \frac{1}{5}e^{-t/100}
\end{aligned}
$$

Thus, the amount of carbon dioxide in the room at any time t is

$$y(t) = \frac{1}{10} + \frac{1}{5}e^{-t/100}\ m^3$$

So, in the long run, the amount of carbon dioxide will be

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\frac{1}{10} + \frac{1}{5}e^{-t/100}\right) = \frac{1}{10}\ m^3$$

This corresponds to a concentration of $\frac{\frac{1}{10}}{200} = 0.05\%$ of carbon dioxide.

EXERCISE 5

A chemical will enter a cell at a rate proportional to the concentration D of the chemical outside the cell, and will leave a cell at a rate proportional to its concentration C inside the cell. The constants of proportionality depend on the properties of the substance and the membrane.

(a) Write a differential equation for the rate of change of the concentration of a certain chemical inside the cell . You can assume the constant of proportionality k is for both entering and leaving the cell.

(b) If $\frac{dC}{dt} < 0$ what does that tell us about C and D?

(c) If no other factors are changing the concentrations inside or outside the cell, then what do you expect to happen in the long run?

Section 10.4 Problems

1. A tank initially contains 10 L pure water. Salt water containing 5 g/L of salt enters the tank at a rate of 2 L/min. If a well mixed solution leaves the tank at a rate of 2 L/min, how much salt is in the tank at time t?

2. A tank initially contains 60 L of water with 4 g of salt dissolved in it. Salt water with a concentration of 2 g/L of salt enters the tank at a rate of 5 L/min. If a well mixed solution leaves the tank at a rate of 5 L/min, how much salt is in the tank at time t?

3. When Weiwei grabs a can of sparkling water from the fridge, it is $5°C$. They place it on the counter in a room which is $20°C$ and then begins a 50 minute yoga video. Halfway into the yoga video, the temperature of the can is $10°C$. Given that the temperature of the can satisfies the differential equation $\frac{dT}{dt} = k(T - 20)$ for some number k, what is the temperature of the can at the end of their yoga video?

4. A tank initially contains 10 L of water with 5 mg of salt dissolved in it. Salt water with a concentration of 1 mg/L of salt enters the tank at a rate of 3 L/min. If a well mixed solution leaves the tank at a rate of 3 L/min, how much salt is in the tank at time t?

5. A tank initially contains 100 L of water with 100 g of salt dissolved in it. Pure water enters the tank at a rate of 5 L/min. If a well mixed solution leaves the tank at a rate of 5 L/min, how much salt is in the tank at time t?

6. A tank initially contains 7 mg of salt dissolved in 100 L of water. Salt water containing 3 mg/L salt enters the tank at a rate of 8 L/min, and the mixed solution is drained from the tank at a rate of 8 L/min. How much salt is in the tank at time t?

7. A tank contains 4 g of salt dissolved in 100L of water. Salt water containing 2 g/L of salt enters the tank at a rate of 5 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank at any time t?

8. A dead body is found at 12 pm in a room that is maintained at 20°C. The body is 28°C when it is found and has cooled to 27°C at 1 pm. Given that the temperature of the body from time of death satisfies

$$\frac{dT}{dt} = k(T - 20)$$

estimate the time of death.

9. A tank initially contains 10 grams of salt dissolved in 100 L of water. Brine containing 5 g/L salt enters the tank at a rate of 10 L/min, and the mixed solution is drained from the tank at a rate of 2 L/min. How much salt is in the tank at any time t?

10. A tank has pure water flowing into it at a rate of 6 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at a rate of 5 L/min. Initially, the tank contains 1 kg of salt in 10 litres of water. Find the amount of salt in the tank at any time t.

11. A tank initially contains 10 L of pure water. Salt water containing 2 g/L salt enters the tank at a rate of 3 L/min, and the mixed solution is drained from the tank at a rate of 2 L/min. How much salt is in the tank at any time t?

12. A tank initially contains 20 L of water with 3 g of salt dissolved in it. Salt water containing 4 g/L salt enters the tank at a rate of 1 L/min, and the mixed solution is drained from the tank at a rate of 2 L/min. How much salt is in the tank at any time t?

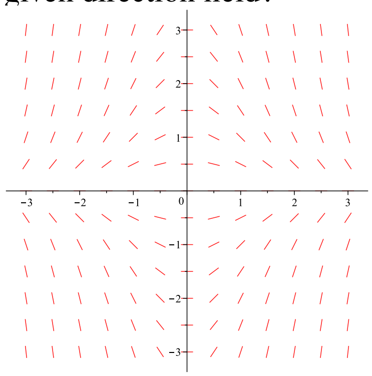
13. A tank initially contains 100 L of pure water. Salt is added to the tank at a rate 2 g/min. The mixed solution is drained from the tank at a rate of 1 L/min. How much salt is in the tank at any time t?

14. The air in a room with volume 10 m^3 contains 0.1% carbon dioxide initially. Fresher air with only 0.06% carbon dioxide flows into the room at a rate of 0.1 m^3/min and the mixed air flows out at the same rate. Find the amount of carbon dioxide in the room as a function of time.

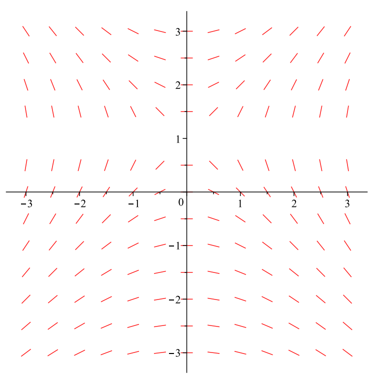
End of Chapter Problems

1. Draw the line segments on the direction field for each differential equation at the points $(1, 1)$, $(-1, 1)$, $(1, -1)$, and $(-1, -1)$.
   (a) $y' = x + 2y$
   (b) $y' = (x - y)^2$
   (c) $y' = \dfrac{x + 2}{y}$

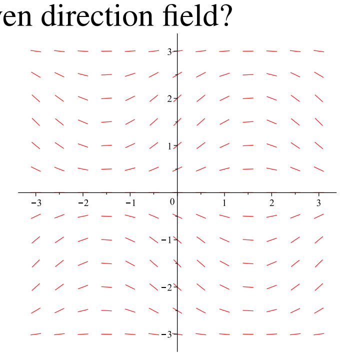
2. Which of the following differential equations corresponds to the given direction field?
   (a) $y' = x^2$
   (b) $y' = y^2$
   (c) $y' = -x^2$
   (d) $y' = xy$
   (e) $y' = -xy$



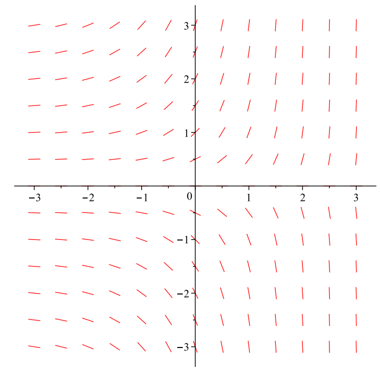
3. Which of the following differential equations corresponds to the given direction field?
   (a) $y' = \dfrac{1}{1 - y}$
   (b) $y' = \dfrac{x}{1 - y}$
   (c) $y' = \dfrac{x}{y - 1}$
   (d) $y' = \dfrac{x^2}{y - 1}$



4. Which of the following differential equations corresponds to the given direction field?
   (a) $y' = x\tan(y)$
   (b) $y' = y\tan(x)$
   (b) $y' = x\sin(y)$
   (d) $y' = \cos(y)\sin(x)$
   (e) $y' = \cos(x)\sin(y)$



5. Which of the following differential equations corresponds to the given direction field?
   (a) $y' = yx^2$
   (b) $y' = ye^x$
   (b) $y' = y\sqrt{|x|}$
   (d) $y' = x^2$
   (e) $y' = y^2$



6. For each initial value problem:
   (i) Solve the initial value problem.
   (ii) Use Euler's Method with step size h to approximate $n = 4$ values of the solution to the given initial value problem.
   (iii) On the same graph, plot the points found in part (ii) along with the tangent line between the points, and the solution curve found in part (i).
   (a) $y' = \dfrac{y(x) + 1}{x}$, $y(1) = 0$; $h = 0.5$
   (b) $y' = \dfrac{2}{x}y + 1$, $y(1) = 0$; $h = 0.5$

7. Use Euler's Method with step size h to approximate $n = 2$ values of the solution to the given initial value problem.
   (a) $y' = x^2$, $y(1) = 0$; $h = 0.5$
   (b) $y' = \dfrac{1}{xy}$, $y(1) = 2$; $h = 1$
   (c) $y' = y^2 + 1$, $y(1) = -1$; $h = 0.5$
   (d) $y' = 2xy$, $y(0) = 1$; $h = 0.5$
   (e) $y' = \dfrac{1}{y}$, $y(-2) = -1$; $h = 1$

8. Identify each of the following differential equations as separable, linear, both, or neither. If the differential equation is separable or linear, write it in standard form.
   (a) $y' = xy - 2y + x - 2$
   (b) $3xy' - y = x\cos(x)$
   (c) $xy' = \cos^2(y) + x(1 - \sin^2(y))$
   (d) $x^2y' + xy = 1$
   (e) $y' = e^{x^2+y^2}$
   (f) $y' = \ln(x + y)$

9. Find all solutions of the differential equation.
   (a) $\frac{dy}{dx} = \frac{x}{y}$
   (b) $\frac{dy}{dt} = e^y \sin^2(t)$
   (c) $\theta \frac{dr}{d\theta} + 4 = \theta \sin(\theta)$
   (d) $y' = -y + e^{2x}$
   (e) $xy' = (1 + 2\ln(x))\cos^2(y)$
   (f) $\frac{dy}{dt} + ty = t$
   (g) $y' = 5(y - 13)$
   (h) $\frac{dy}{dt} + y = \cos(e^t)$
   (i) $\frac{dy}{dx} = \frac{2}{x}y + x$
10. Solve the initial value problem.
   (a) $\frac{dy}{dt} = y^2(1 + t)$, $y(1) = 2$
   (b) $\frac{dy}{dx} = e^{x-y}$, $y(1) = 0$
   (c) $\frac{dy}{dx} - \frac{y}{x} = 1$, $x > 0$, $y(1) = 1$
   (d) $\cos(t)\frac{dy}{dt} + y\sin(t) = 0$, $y\left(\frac{\pi}{4}\right) = \sqrt{2}$
   (e) $(1 + x^2)y' = y$, $y(0) = 1$
   (f) $\frac{dy}{dx} + \frac{2}{x}y = 6x^3$, $y(1) = -1$
   (g) $\frac{dy}{dx} = \frac{(1 + y^2)\cos(x)}{y}$, $y(0) = 1$
   (h) $\frac{dy}{dx} - y = \sin(x)$, $y(0) = 0$
11. A 20 L tank initially contains pure water. Salt water containing 3 g/L of salt enters the tank at a rate of 4 L/min. If a well mixed solution leaves the tank at a rate of 4 L/min, how much salt is in the tank at time t?
12. A 50 L tank initially contains water with 10 g of salt dissolved in it. Salt water with a concentration of 5 g/L of salt enters the tank at a rate of 2 L/min. If a well mixed solution leaves the tank at a rate of 2 L/min, how much salt is in the tank at time t?
13. A decorated egg at temperature 20°C is placed in is placed in a warming oven which has a temperature of 80°C . Given that the egg warms to a temperature of 40°C in 10 minutes and the temperature of the egg satisfies the differential equation
$$\frac{dT}{dt} = k(T - 80)$$
find the temperature of the egg at any time t.
14. A tank initially contains 40 g of salt dissolved in 10L of water. Brine containing 3 g/L salt enters the tank at a rate of 4 L/min, and the mixed solution is drained from the tank at a rate of 5 L/min. How much salt is in the tank at any time $t \geq 0$?

Chapter 11: Parametric Equations

Section 11.1: Parametric Equations

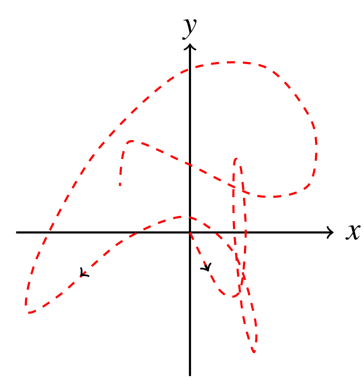
LEARNING OUTCOMES

1. Know how to sketch a parametric curve including its orientation.
2. Know how to convert parametric equations into a function $y = f(x)$ by eliminating the parameter.
3. Know how to parameterize a curve $y = f(x)$.

11.1.1 Parametric Equations

In many real world applications, we encounter functions that do not pass the vertical line test... that is, cannot be written in the form of $y = f(x)$.

For example, suppose a conservation officer tags a deer with a gps tracking device. Recording the position of the deer for 1 day, they get the graph to the right. It isn't the graph of a function $y = f(x)$ and it would be exceedingly difficult to try to find an implicitly defined function that has this graph. Yet, the conservation officer would like to use calculus to determine things like how many kilometers the deer walked in the day, and what it's maximum speed was. So, they do need a representation that we can perform calculus on.



To accomplish this, we represent the deer's path in a different way. Rather than trying to relate y and x together, we will look at the horizontal and vertical positions of the deer individually with respect to time t. That is, find a function

$$x = f(t)$$

that represents the horizontal component (the x-value) of the deer's position at any time t, and a function

$$y = g(t)$$

that represents the vertical component (the y-value) at any time t. This motivates the following definition.

DEFINITION

Parametric Equations

Parametric Curve

Parametrization

A pair of equations

$$x = f(t), \qquad y = g(t)$$

in the same variable t, called the **parameter**, are called **parametric equations**.

The graph that is represented by the set of points $(x, y) = (f(t), g(t))$ is called a **parametric curve**, and the parametric equations is called a **parametrization** of the curve.

EXAMPLE 1 Show that

$$x = 4\sin(t) + 3$$
$$y = 2\cos(t)$$

are a parametrization of the ellipse

$$\frac{y^2}{2^2} + \frac{(x-3)^2}{4^2} = 1 \tag{11.1}$$

Plot the points on the parametric curves corresponding to $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}$.

Solution: To verify

$$x = 4\sin(t) + 3$$
$$y = 2\cos(t)$$

is a parametrization of the ellipse, we substitute the equations into equation (11.1).

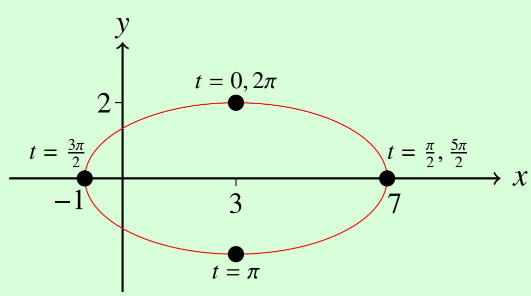
We get

$$\frac{(2\cos(t))^2}{4} + \frac{(4\sin(t)+3-3)^2}{16} = \frac{4\cos^2(t)}{4} + \frac{16\sin^2(t)}{16} = \cos^2(t) + \sin^2(t) = 1$$

Since the equation is satisfied (both sides are equal), it is a parametrization of the ellipse.

To plot points, we will make a table of values. For each value of the parameter t, which we can think of as time, we get an x and y value which we can plot.

t	x	y
0	3	2
$\frac{\pi}{2}$	7	0
π	3	-2
$\frac{3\pi}{2}$	-1	0
2π	3	2
$\frac{5\pi}{2}$	7	0



Observe in the example that the parametric curve will not only go around the ellipse more than once (it will in fact go around infinitely many times since we have no restriction on t), but it also has a direction associated with it. In particular, if we imagine that t is measuring time, then we can trace the direction of motion along the curve by following the points with increasing values of t.

EXAMPLE 2

Show that

$$x = -|t-3| + 2 \tag{11.2}$$
$$y = (-|t-3| + 2)^2 \tag{11.3}$$

for $0 \le t \le 6$ is a parametrization for $y = x^2$ for $-1 \le x \le 2$.

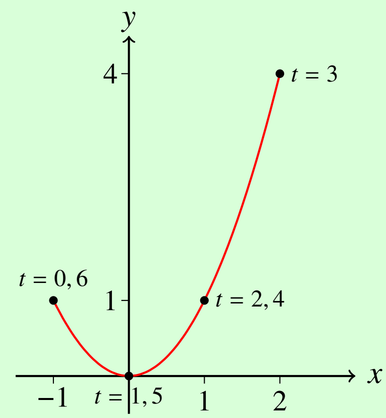
Plot the points on the parametric curve corresponding to $t = 0, 1, 2, 3, 4, 5, 6$.

Solution: To show that this is a parametrization for $y = x^2$, we just need to show that it satisfies $y = x^2$. Substituting equation (11.2) into equation (11.3) gives

$$y = (-|t-3| + 2)^2 = x^2$$

To plot the points, we make a table of values.

t	x	y
0	−1	1
1	0	0
2	1	1
3	2	4
4	1	1
5	0	0
6	−1	1



Observe in this example, the parametric curve starts at $(-1, 1)$, follows the parabola to $(2, 4)$ and then moves backwards along the parabola until it gets back to $(-1, 1)$.

Since the direction is important, we make the following definition.

DEFINITION

Orientation

The direction in which the points on a parametric curve are plotted as the parameter increases is called the **orientation** of the parametric curve. It is indicated by arrows on the curve pointing in the direction of the orientation.

REMARK

From this point forward, whenever we sketch a parametric curve, we will indicate its orientation with an arrow.

Due to the addition of the parameter t, we can represent even more than just the direction of the curve.

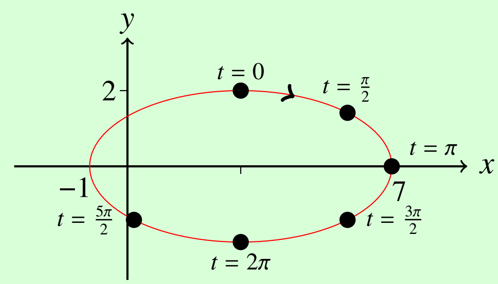
EXAMPLE 3

Plot the points corresponding to $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}$ on the parametric curve

$$x = 4\sin(t/2) + 3$$
$$y = 2\cos(t/2)$$

Solution: We begin by making a table of values. We then plot the points and indicate the orientation of the curve with an arrow.

t	x	y
0	3	2
$\frac{\pi}{2}$	$2\sqrt{2}+3$	$\sqrt{2}$
π	7	0
$\frac{3\pi}{2}$	$2\sqrt{2}+3$	$-\sqrt{2}$
2π	3	-2
$\frac{5\pi}{2}$	$-2\sqrt{2}+3$	$-\sqrt{2}$



This is the exact same curve as in Example 11.1.1 except that we move around the curve at half the speed. That is, parametric curves can encode not only the direction of movement, but how fast we are moving along the curve!

EXERCISE 1

Plot the points corresponding to $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}$ on the parametric curve

$$x = 4\sin(2t) + 3$$
$$y = 2\cos(2t)$$

How did replacing t with $2t$ in the parametric equations change the parametric curve?

What change could we make to switch the orientation of the parametric curve?

EXERCISE 2

Plot the points corresponding to $t = 0, 1, 2, 2.5$ on the parametric curve

$$x = -|2t - 3| + 2$$
$$y = (-|2t - 3| + 2)^2$$

Compare the result to Example 11.1.2.

Let's try to plot another parametric curve.

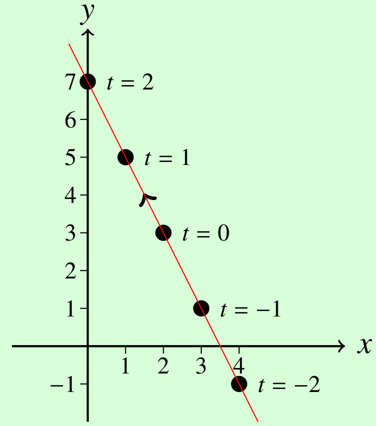
EXAMPLE 4 Plot the points corresponding to $t = -2, -1, 0, 1, 2$ on the parametric curve

$$x = 2 - t$$
$$y = 3 + 2t$$

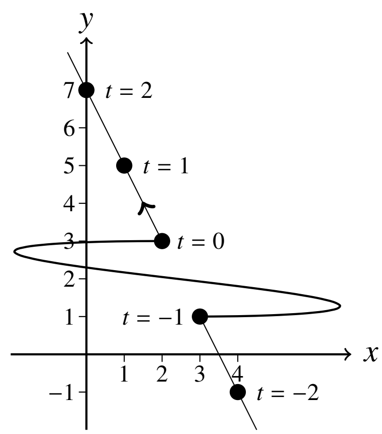
Solution: We begin by making a table of values.

Plotting the points, the parametric curve looks like a straight line with an upwards orientation.

t	x	y
-2	4	-1
-1	3	1
0	2	3
1	1	5
2	0	7



WAIT! How did we know the graph in Example 11.1.4 was a straight line? We only plotted 5 points! Maybe, the graph looks like:



We now look at a method for converting parametric curves into the form $y = f(x)$ so that we can identify the exact shape of the graph.

11.1.2 Eliminating the Parameter

To get a better understanding of a parametric curve defined by $x = f(t)$ and $y = g(t)$, we can try to combine the two equations together to get a single function $y = F(x)$.

The general procedure for this is to solve one of the equations for t and substitute that into the other equation.

EXAMPLE 5 Show that the parametric curve defined by

$$x = 2 - t$$
$$y = 3 + 2t$$

is a line by eliminating the parameter.

Solution: Solving the first equation for t gives

$$t = 2 - x$$

Substituting this into the second equation, we get

$$\begin{aligned} y &= 3 + 2(2 - x) \\ &= 7 - 2x \end{aligned}$$

Hence, it is the line $y = 7 - 2x$.

EXERCISE 3 Show that the parametric curve defined by

$$x = 3t + 1$$
$$y = t^2 - 2$$

is a parabola by eliminating the parameter.

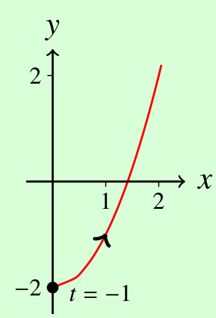
EXAMPLE 6 Sketch the parametric curve defined by

$$x = \sqrt{t + 1}$$
$$y = t - 1$$

by eliminating the parameter.

Solution: Solving the second equation for t gives $t = y + 1$. Substituting this into the first equations gives

$$\begin{aligned} x &= \sqrt{(y + 1) + 1} \\ x^2 &= y + 2, \quad x \geq 0 \\ x^2 - 2 &= y, \quad x \geq 0 \end{aligned}$$



Thus, the parametric curve is the half of the parabola $y = x^2 - 2$ for which $x \geq 0$.

We determine the orientation of the curve by plotting a couple points.

EXAMPLE 7

Sketch the parametric curve

$$x = (2-t)^2$$
$$y = -t + 2$$

for $0 \le t \le 2$ by eliminating the parameter.

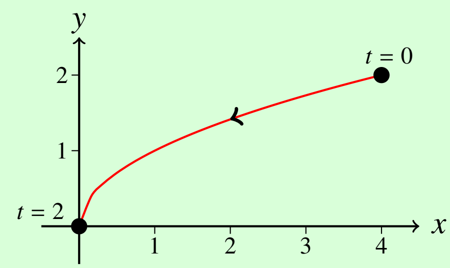
Solution: In this case, we do not actually need to solve one of the equations for t. Instead we observe that $y = -t + 2 = 2 - t$. Thus, the first equation is

$$x = y^2$$
$$\sqrt{x} = y$$

When $t = 0$, we get $x = (2-0)^2 = 4$ and $y = -0 + 2 = 2$.

When $t = 2$, we get $x = (2-2)^2 = 0$ and $y = -2 + 2 = 0$.

Thus, we get the graph to the right.



There are cases where we use a different approach for eliminating the parameter. This is most common when we have trigonometric functions. In these cases, we can look for a trigonometric identity that corresponds to our functions.

EXAMPLE 8

Sketch the parametric curve represented by $x = \cos(t)$, $y = \sin(t)$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ by eliminating the parameter.

Solution: We know that

$$\cos^2(t) + \sin^2(t) = 1$$

Hence, we have

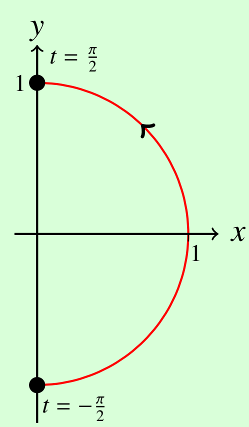
$$x^2 + y^2 = 1$$

So, this parametric curve is part of a circle of radius 1.

When $t = -\frac{\pi}{2}$, we get $x = \cos\left(-\frac{\pi}{2}\right) = 0$ and $y = \sin\left(-\frac{\pi}{2}\right) = -1$.

When $t = \frac{\pi}{2}$, we get $x = \cos\left(\frac{\pi}{2}\right) = 0$ and $y = \sin\left(\frac{\pi}{2}\right) = 1$.

Thus, we get the graph to the right.



EXERCISE 4

Sketch the parametric curve represented by $x = 2\cos(t)$, $y = 2\sin(t)$, $0 \le t \le \pi$.

EXERCISE 5 Sketch the parametric curve represented by $x = 2\cos(2t)$, $y = 2\sin(2t)$, $0 \le t \le \pi$.

EXERCISE 6 Sketch the parametric curve represented by $x = 2\cos(2t)$, $y = 2\sin(2t)$, $0 \le t \le 2\pi$.

11.1.3 Parameterizing a Curve

There are many times in mathematics where it is very useful to take a function defined in terms of x and y (perhaps implicitly) and find parametric equations which represent it. That is, do the reverse of eliminating the parameter.

In some situations, this is extremely easy as the next example demonstrates. In other cases, you essentially need to have memorized what to do as Example 11.1.10 shows.

EXAMPLE 9 Parametrize the line $y = 3x + 5$.

Solution: Take $x = t$. Then, we get $y = 3t + 5$. So, the parametric equations are

$$\begin{aligned} x &= t \\ y &= 3t + 5 \end{aligned}$$

EXAMPLE 10 Parametrize the circle $x^2 + y^2 = 9$.

Solution: Since parameterizing is the reverse of eliminating the parameter, we do the reverse of what we did in Example 11.1.8. That is, we take

$$\begin{aligned} x &= 3\cos(t) \\ y &= 3\sin(t) \end{aligned}$$

Since we typically think of the circle $x^2 + y^2 = 9$ as being only traversed once, we make a restriction on t so that the parametric curve only goes around the circle once. We make the restriction $0 \le t < 2\pi$.

EXAMPLE 11 Parametrize the straight line segment that lies along the x-axis from $(-1, 0)$ to $(3, 0)$.

Solution: We can see that the y value is always 0, so we take $y = 0$. The x values run from -1 to 3, so we take

$$x = t$$

and make the restriction $-1 \le t \le 3$.

That is, the parametric equations representing the line segment are

$$\begin{aligned} x &= t \\ y &= 0 \end{aligned}$$

for $-1 \le t \le 3$.

REMARK

Notice that there are infinitely many different correct answers for Example 11.1.11 as neither the orientation nor the 'speed' matters.

Of course, we should always pick as simple of a representation as possible. The point of parameterizing the curve is that we will want to use the equations for other purposes... so making them more complicated will just make future calculations with them harder.

EXERCISE 7 Parametrize the straight line segment from $(1, -1)$ to $(3, 5)$.

EXERCISE 8 Find parametric equations that represent the circle $x^2 + y^2 = 5$ traversed twice in the counterclockwise direction.

Section 11.1 Problems

1. Sketch each parametric equations by eliminating the parameter. Plot the points corresponding to $t = 0$, $t = 1$, and $t = 2$ on the curve and indicate the curve's orientation.
   (a) $x = t + 2$, $y = t + 3$
   (b) $x = 1 - t$, $y = t + 1$
   (c) $x = t^2 + 1$, $y = \frac{t}{2}$
   (d) $x = t + 1$, $y = t^2 + 2$
   (e) $x = \sqrt{t} + 4$, $y = 2\sqrt{t}$, $0 \le t \le 4$
   (f) $x = e^{2t}$, $y = e^t + 1$
2. Sketch each parametric equations by eliminating the parameter. Plot the points corresponding to $t = 0$, $t = \frac{\pi}{2}$, and $t = \pi$ on the curve and indicate the curve's orientation.
   (a) $x = 3\cos(t)$, $y = 3\sin(t)$, $0 \le t \le 2\pi$
   (b) $x = 3\cos(2t)$, $y = 3\sin(2t)$, $0 \le t \le 2\pi$
   (c) $x = 3\cos(t/2)$, $y = 3\sin(t/2)$, $0 \le t \le 2\pi$
   (d) $x = \cos(t)$, $y = \sin^2(t)$, $0 \le t \le \pi$
   (e) $x = \cos(t)$, $y = 1 + \sin(t)$, $0 \le t \le 2\pi$
3. Parametrize the following curves.
   (a) The line $y = 2x + 2$.
   (b) The line $y = \frac{1}{2}x + \frac{3}{2}$.
   (c) The line $x = 3y - 2$.
   (d) The circle $x^2 + y^2 = 1$.
   (e) The circle $x^2+y^2 = 2$ that is traversed twice in the clockwise direction.
   (f) The circle $x^2+y^2 = 4$ that is traversed once in the counterclockwise direction.
   (g) The parabola $y = x^2 + 1$.
   (h) The parabola $y = 2(x - 1)^2 - 1$.
4. Find parametric equations for the following curves.
   (a) The line segment from $(1, -1)$ to $(1, 3)$.
   (b) The line segment from $(0, 3)$ to $(3, 0)$.
   (c) The line segment from $(1, -1)$ to $(-2, 3)$.
   (d) The part of the circle $x^2 + y^2 = 1$ from $(-1, 0)$ to $(1, 0)$ in the counterclockwise direction.
   (e) The part of the circle $x^2+y^2 = 1$ from $(0, 1)$ to $(0, -1)$ in the clockwise direction.
   (f) The part of the parabola $y = (x - 1)^2 + 2$ that runs from $(-1, 6)$ to $(2, 3)$.
   (g) The part of the parabola $y = \frac{1}{2}(x + 1)^2$ that runs from $\left(2, \frac{9}{2}\right)$ to $\left(0, \frac{1}{2}\right)$.

Section 11.2: Calculus of Parametric Equations

LEARNING OUTCOMES

1. Know how to find the derivative of a parametric curve.
2. Know how to find area under a parametric curve.
3. Know how to find area the arc length a parametric curve.

When sketching the graphs of more complicated parametric equations, it is useful to perform many of the same steps we did when trying to sketch more complicated functions $y = f(x)$. For example, it is very helpful to know for what values of t the graph has positive or negative slope.

11.2.1 Slope of a Parametric Curve

THEOREM 1 If we have a parametric curve represented by parametric equations $x = x(t)$, $y = y(t)$ such that $x(t)$ and $y(t)$ are both differentiable and $x'(t) \neq 0$, then the slope of the tangent line to the parametric curve is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Proof: Assume that the parameter t can be eliminated and that we get a differentiable function $y = F(x) = F(x(t))$. Differentiating both sides with respect to t using the Chain Rule (in Leibniz notation) gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Assuming $\frac{dx}{dt} \neq 0$, we can solving this for $\frac{dy}{dx}$. We get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

□

This theorem allows us to find the slope of the tangent line of a parametric curve without having to eliminate the parameter.

EXAMPLE 1 Find the slope of the tangent line to the curve $x = 3t - 1$, $y = t^2 - t$ for any t.

Solution: We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 1}{3}$$

EXAMPLE 2 Find the equation of the tangent line to the curve $x = t - t^2$, $y = t + t^3$ at $t = 1$.

Solution: We have $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{1+3t^2}{1-2t}$.

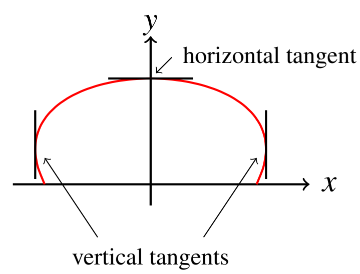
Hence, the equation of the tangent line at $t = 1$ is

$$\begin{aligned} y &= y(1) + \left.\frac{dy}{dx}\right|_{t=1} (x - x(1)) \\ &= \left(1 + (1)^3\right) + \frac{1+3(1)^2}{1-2(1)} \left(x - (1 - (1)^2)\right) \\ &= 2 - 4x \end{aligned}$$

EXERCISE 1 Find the equation of the tangent line to the curve $x = t^2 - 1$, $y = \sqrt{t+1}$ at $t = 3$.

When analyzing parametric curves, we often looks for values of the parameter where the tangent line to the curve is horizontal or is vertical.

A horizontal tangent line occurs when the slope is 0, while a vertical tangent line occurs when the slope is infinite.



From the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

we get that the tangent line will be horizontal when $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$, and the tangent line will be vertical when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$.

What about when both $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} = 0$? Since a derivative is technically a limit, we use limits in this case to determine whether $\dfrac{dy}{dx}$ is 0, ∞, or neither.

Similarly, if either $\dfrac{dy}{dt}$ or $\dfrac{dx}{dt}$ do not exist at the point, then we use limits.

Thus, when looking for horizontal and/or vertical tangent lines of parametric curves, we need to find all places where $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ equal 0 or do not exist... that is, we need to find their critical points. After we have all the critical points, we use the cases outlined above. This is best demonstrated with some examples.

EXAMPLE 3

Consider the parametric curve $x = 2\cos(t)$, $y = \sin(2t)$ for $0 \le t \le 2\pi$. Find all points where the tangent line is horizontal and all points where the tangent line is vertical.

Solution: We have

$$\frac{dx}{dt} = -2\sin(t)$$
$$\frac{dy}{dt} = 2\cos(2t)$$

The critical points of $\frac{dy}{dt}$ on $0 \le t \le 2\pi$ are $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. At these points, $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. Therefore, these values of t correspond to horizontal tangent lines.

The critical points of $\frac{dx}{dt}$ on $0 \le t \le 2\pi$ are $t = 0, \pi, 2\pi$. At these points, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. Hence, these values of t correspond to vertical tangent lines.

EXAMPLE 4

Consider the parametric curve $x = t + 2$, $y = \sqrt{t}$. Find all points where the tangent line is horizontal, and all points where the tangent line is vertical.

Solution: We have

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = \frac{1}{2\sqrt{t}}$$

The only critical point of $\frac{dy}{dt}$ is at $t = 0$. At this point, $\frac{dy}{dt}$ does not exist, so we use a limit to find $\frac{dy}{dx}$. We have

$$\lim_{t \to 0^+} \frac{dy}{dx} = \lim_{t \to 0^+} \frac{\frac{1}{2\sqrt{t}}}{1} = \infty$$

Thus, there is a vertical tangent line at $t = 0$.

Since $\frac{dx}{dt}$ does not have any critical points, there are no other vertical or horizontal tangent lines.

EXERCISE 2

Consider the parametric curve $x = t^3 - 2t + 1$, $y = t^2 + 5$. Find all points where the tangent line is horizontal, and all points where the tangent line is vertical.

EXERCISE 3 Consider the parametric curve $x = t^2$, $y = \cos(t)$ for $-1 \le t \le 1$. Find all points where the tangent line is horizontal, and all points where the tangent line is vertical.

11.2.2 Area Under a Parametric Curve

We have seen that there are a variety of times in math and science where we want to find the area under a curve. For example, if the curve is the rate of change of a quantity, then the area under the graph represents the net change in the quantity. So, we also want to know how to find the area under a curve that is defined by parametric equations.

We will set up the integral for calculating the area under a parametric curve using a Riemann sum.

Assume we have a parametric curve defined by the parametric equations

$$x = x(t)$$
$$y = y(t)$$

Further assume that the parametric curve does not intersect itself for $a \le t \le b$.

As usual, we begin by subdividing the interval $a \le t \le b$ into n equal pieces of width $\Delta t = \dfrac{b-a}{n}$, and we define

$$t_i = a + i\Delta t$$

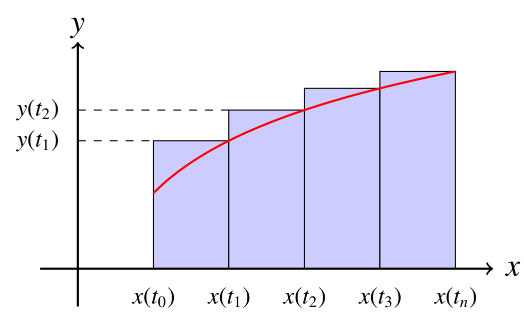
As we did for finding the area under a curve $y = f(x)$, we create rectangles. The width of the i-th rectangle will be $x(t_i) - x(t_{i-1})$ while the height is the distance from the x-axis to the curve. That is, it is the y-coordinate of the point on the curve corresponding to t_i which is $y(t_i)$.

So, the area of the i-th rectangle will be

$$A_i = y(t_i) \cdot (x(t_i) - x(t_{i-1}))$$

Hence, we can approximate the area under the graph over $a \le t \le b$ by

$$\text{Area} \approx \sum_{i=1}^{n} y(t_i) \cdot (x(t_i) - x(t_{i-1}))$$



Just like we did with arc length in Section 5.4.2, we need to modify this to get a Δt at the end so that we can turn this into an integral.

Since $\Delta t = t_i - t_{i-1}$, we get

$$\begin{aligned}
\text{Area} &\approx \sum_{i=1}^{n} y(t_i) \cdot (x(t_i) - x(t_{i-1})) \cdot \frac{\Delta t}{t_i - t_{i-1}} \\
&= \sum_{i=1}^{n} y(x_i) \cdot \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \cdot \Delta t
\end{aligned}$$

Again, just like we saw for arc length, taking $n \to \infty$ gives $\dfrac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \to x'(t)$.

Hence, we have that

$$\begin{aligned}\text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} y(t_i) \cdot \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \cdot \Delta t \\ &= \int_a^b y(t)x'(t)\,dt\end{aligned}$$

We have derived the following theorem.

THEOREM 2

If the parametric curve defined by $x = x(t)$ and $y = y(t)$ does not intersect itself for $a \leq t \leq b$, then the area under the parametric curve for $a \leq t \leq b$ is

$$\text{Area} = \int_a^b y(t)x'(t)\,dt$$

REMARK

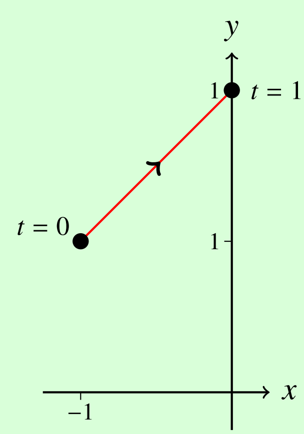
Since we have defined the area by measuring the distance above the x-axis, any area underneath the x-axis will be measured as negative area.

EXAMPLE 5

Find the area under $x = t^2 - 1$, $y = t^2 + 1$ for $0 \leq t \leq 1$.

Solution: Since $x'(t) = 2t$, we have

$$\begin{aligned}\text{Area} &= \int_0^1 (t^2 + 1) \cdot (2t)\,dt \\ &= \int_0^1 (2t^3 + 2t)\,dt \\ &= \left(\frac{1}{2}t^4 + t^2\right)\Bigg|_0^1 \\ &= \frac{1}{2} + 1 - 0 \\ &= \frac{3}{2}\end{aligned}$$



EXAMPLE 6 Find the area under $x = \cos(t)$, $y = \sin(2t)$ for $\pi \le t \le \frac{3\pi}{2}$.

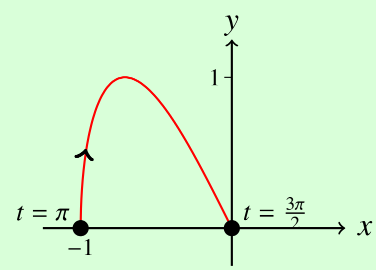
Solution: Since $x'(t) = -\sin(t)$, we have

$$\begin{aligned}\text{Area} &= \int_{\pi}^{3\pi/2} (\sin(2t)) \cdot (-\sin(t))\, dt \\ &= \int_{\pi}^{3\pi/2} 2\sin(t)\cos(t) \cdot (-\sin(t))\, dt \\ &= \int_{\pi}^{3\pi/2} -2\sin^2(t)\cos(t)\, dt\end{aligned}$$

Let $u = \sin(t)$, then $du = \cos(t)\, dt$.

When $t = \pi$, $u = 0$. When $t = \frac{3\pi}{2}$, $u = -1$. Hence,

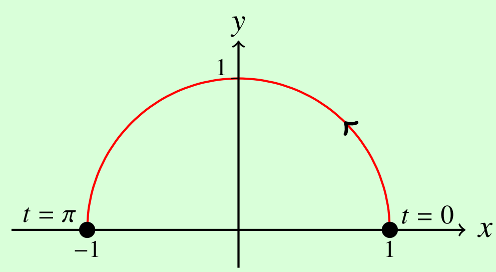
$$\begin{aligned}\text{Area} &= \int_{0}^{-1} -2u^2\, du \\ &= -\frac{2}{3}u^3 \Big|_0^{-1} \\ &= \frac{2}{3} - 0 \\ &= \frac{2}{3}\end{aligned}$$



EXAMPLE 7 Find the area under $x = \cos(t)$, $y = \sin(t)$ for $0 \le t \le \pi$.

Solution: Since $x'(t) = -\sin(t)$, we have

$$\begin{aligned}\text{Area} &= \int_{0}^{\pi} \sin(t) \cdot (-\sin(t))\, dt \\ &= \int_{0}^{\pi} -\sin^2(t)\, dt \\ &= \int_{0}^{\pi} -\frac{1}{2}(1 - \cos(2t))\, dt \\ &= \left(-\frac{1}{2}t + \frac{1}{4}\sin(2t)\right) \Big|_0^{\pi} \\ &= -\frac{\pi}{2} + 0 - (0 + 0) \\ &= -\frac{\pi}{2}\end{aligned}$$



Wait! Why did we get a negative area? The graph is entirely above the x-axis! It is because the orientation of the curve is 'backwards'. In particular, the values of $x(t_i) - x(t_{i-1})$ will be negative.

EXERCISE 4 Find the area under $x = \cos(t)$, $y = \sin(t)$ for $0 \leq t \leq 2\pi$.

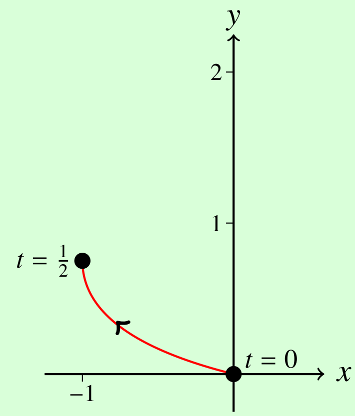
EXAMPLE 8 Find the area under the parametric curve $x = 4t^2 - 4t$, $y = t^2 + t$ for

(a) $0 \leq t \leq \frac{1}{2}$

(b) $\frac{1}{2} \leq t \leq 1$

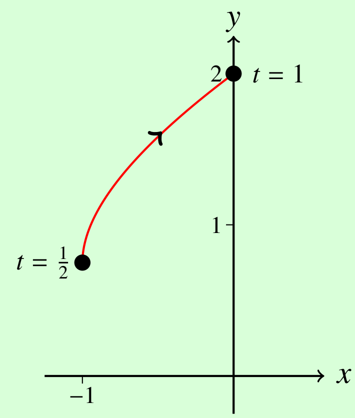
Solution: (a) We have $x'(t) = 8t - 4$, so

$$\begin{aligned} \text{Area}_1 &= \int_0^{1/2} (t^2 + t) \cdot (8t - 4)\, dt \\ &= \int_0^{1/2} (8t^3 + 4t^2 - 4t)\, dt \\ &= \left(2t^4 + \frac{4}{3}t^3 - 2t^2\right)\Big|_0^{1/2} \\ &= \frac{1}{8} + \frac{1}{6} - \frac{1}{2} - 0 = -\frac{5}{24} \end{aligned}$$



(b) We have

$$\begin{aligned} \text{Area}_2 &= \int_{1/2}^{1} (t^2 + t) \cdot (8t - 4)\, dt \\ &= \left(2t^4 + \frac{4}{3}t^3 - 2t^2\right)\Big|_{1/2}^{1} \\ &= 2 + \frac{4}{3} - 2 - \left[\frac{1}{8} + \frac{1}{6} - \frac{1}{2}\right] \\ &= \frac{4}{3} - \left[-\frac{5}{24}\right] = \frac{37}{24} \end{aligned}$$



EXERCISE 5 Find the area under the parametric curve $x = 4t^2 - 4t$, $y = t^2 + t$ for $0 \leq t \leq 1$.

EXERCISE 6 Find the area under $x = e^t$, $y = t$ for $0 \leq t \leq 1$.

11.2.3 Arc Length of a Parametric Curve

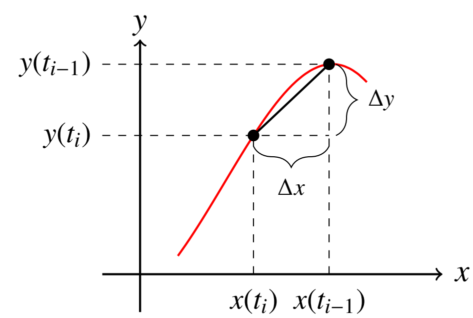
We now look at how to find the arc length of a parametric curve $x = x(t)$, $y = y(t)$ over $a \leq t \leq b$. We will find the integral for this using ... a Riemann sum! This will be very similar to what we saw in Section 5.4.2.

We subdivide the interval $[a, b]$ into n equal sub-intervals with width $\Delta t = \dfrac{b-a}{n}$ and define $t_i = a + i\Delta t$.

We approximate the length of the curve by summing the length of the straights lines from $(x(t_{i-1}), y(t_{i-1}))$ to $(x(t_i), y(t_i))$.

From the figure, we observe that the length of the i-th line, by the Pythagorean Theorem, is

$$L_i = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$



We multiply by $\dfrac{(\Delta t)^2}{(\Delta t)^2}$ inside the square root to get

$$L_i = \sqrt{((\Delta x)^2 + (\Delta y)^2) \cdot \frac{(\Delta t)^2}{(\Delta t)^2}}$$

Simplifying we get

$$\begin{aligned} L_i &= \sqrt{\frac{(\Delta x)^2}{(\Delta t)^2} + \frac{(\Delta y)^2}{(\Delta t)^2}} \cdot \sqrt{(\Delta t)^2} \\ &= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \cdot \Delta t \end{aligned}$$

Hence,

$$\begin{aligned} \text{Length} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \cdot \Delta t \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \end{aligned}$$

THEOREM 3 The arc length of a parametric curve $x = x(t)$ and $y = y(t)$ over $a \leq t \leq b$ is

$$\text{Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

EXAMPLE 9 The position of a particle at time t (in seconds) is given by the parametric equations

$$x = \frac{1}{2}t^2 + 1$$
$$y = \frac{1}{3}(2t+1)^{3/2}$$

Find the distance travelled by the particle from time $t = 0$ to time $t = 2$.

Solution: The distance travelled is

$$\begin{aligned}
\text{Length} &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^2 \sqrt{(t)^2 + (\sqrt{2t+1})^2}\,dt \\
&= \int_0^2 \sqrt{t^2 + 2t + 1}\,dt \\
&= \int_0^2 \sqrt{(t+1)^2}\,dt \\
&= \int_0^2 (t+1)\,dt \\
&= \left(\frac{1}{2}t^2 + t\right)\Big|_0^2 \\
&= 2 + 2 - 0 \\
&= 4
\end{aligned}$$

EXAMPLE 10 Find the length of the parametric curve $x = t^2$, $y = t^3$, for $0 \le t \le 1$.

Solution: The length is

$$\begin{aligned}
\text{Length} &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\,dt \\
&= \int_0^1 \sqrt{4t^2 + 9t^4}\,dt \\
&= \int_0^1 \sqrt{t^2(4 + 9t^2)}\,dt \\
&= \int_0^1 \sqrt{t^2}\,\sqrt{4 + 9t^2}\,dt \\
&= \int_0^1 t\sqrt{4 + 9t^2}\,dt
\end{aligned}$$

Let $u = 4 + 9t^2$. Then, $du = 18t\,dt$, so $\frac{1}{18}\,du = t\,dt$.

When $t = 0$, $u = 4$. When $t = 1$, $u = 13$. Hence,

$$\begin{aligned}
\text{Length} &= \int_4^{13} \frac{1}{18}\sqrt{u}\,du \\
&= \frac{1}{27}u^{3/2}\Big|_4^{13} \\
&= \frac{1}{27}(13)^{3/2} - \frac{1}{27}(4)^{3/2} \\
&= \frac{(13)^{3/2}}{27} - \frac{8}{27}
\end{aligned}$$

EXAMPLE 11 Write parametric equations for a circle of radius r and use them to show that the circumference of a circle of radius $r > 0$ is $2\pi r$.

Solution: Parametric equations for a circle of radius r are

$$\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta)
\end{aligned}$$

When using these parametric equations to find the circumference, we only want to count one loop of the circle, so we take $0 \leq \theta \leq 2\pi$.

We get

$$\begin{aligned}
\text{Length} &= \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta \\
&= \int_0^{2\pi} \sqrt{(-r\sin(\theta))^2 + (r\cos(\theta))^2}\,d\theta \\
&= \int_0^{2\pi} \sqrt{r^2(\sin^2(\theta) + \cos^2(\theta))}\,d\theta \\
&= \int_0^{2\pi} \sqrt{r^2}\,d\theta \\
&= \int_0^{2\pi} r\,d\theta \\
&= r\theta\Big|_0^{2\pi} \\
&= 2\pi r - 0 \\
&= 2\pi r
\end{aligned}$$

EXERCISE 7

Find the arc length of the curve $y = \frac{1}{3}x^{3/2}$ from $(0, 0)$ to $\left(4, \frac{8}{3}\right)$ by:

(a) Using the formula from Section 5.4.2.

(b) By parameterizing the curve and using the formula from this section.

EXERCISE 8

The particle in Example 11.1.9 traversed the distance from $\left(1, \frac{1}{3}\right)$ to $\left(3, \frac{5^{3/2}}{3}\right)$ in 2 seconds. We represented this with the parametric equations

$$x = \frac{1}{2}t^2 + 1, \qquad y = \frac{1}{3}(2t + 1)^{3/2}$$

(a) Create parametric equations so that the particle follows the same path but covers the distance in 1 second (i.e. travels twice as fast).

(b) Show that the curve defined by the parametric equations you created for part (a) also has an arc length of 4, confirming that the particle indeed covers the same distance.

Section 11.2 Problems

1. Find the equation of the tangent to the curve at the given value of t.
   (a) $x = 2 - \frac{3}{2}t$, $y = \frac{3}{5} + \frac{1}{3}t$; $t = 2$
   (b) $x = \sin(t)$, $y = \cos(t)$; $t = \frac{\pi}{3}$
   (c) $x = 1 + 2t$, $y = t^2$; $t = -1$
   (d) $x = 2t$ $y = (t + 1)^2$; $t = 1$
   (e) $x = \cos(t)$, $y = 2\sin(t)$; $t = \frac{\pi}{2}$.
   (f) $x = t$, $y = \cos(t)$; $t = 0$
   (g) $x = t^2$, $y = t^3$; $t = 1$
2. For each curve, find all values of t where the tangent line is horizontal and all values of t where the tangent line is vertical.
   (a) $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le 2\pi$
   (b) $x = t$, $y = 2t + 4$
   (c) $x = \sec(t)$, $y = \tan(t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$
   (d) $x = t + 3$, $y = t^2$
   (e) $x = t - 2\sin(t)$, $y = 4 - 3\cos(t)$, $0 \le t \le 2\pi$
   (f) $x = \sqrt{t}$, $y = 2t + 5$
   (g) $x = \cos(2t)$, $y = \sin(t)$, $0 \le t \le \pi$
3. Determine the area under each parametric curve.
   (a) $x = t + 5$, $y = t - 2$, $0 \le t \le 3$
   (b) $x = 2t - 1$, $y = 1 - t$, $-2 \le t \le 3$
   (c) $x = t^2$, $y = t^3$, $-1 \le t \le 1$
   (d) $x = t^2 - 2t$, $y = \sqrt{t}$, $0 \le t \le 2$
   (e) $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le 2\pi$
   (f) $x = \cos^2(t)$, $y = \sin^2(t)$, $0 \le t \le \frac{\pi}{2}$
4. Determine the length of the parametric curve.
   (a) $x = 3t + 5$, $y = 3 - t$, $0 \le t \le \sqrt{2}$
   (b) $x = 1 + 3t^2$, $y = 2 + t^3$, $0 \le t \le 1$
5. Set up, but do not evaluate, an integral to calculate the length of the parametric curve.
   (a) $x = t$, $y = t^2$, $0 \le t \le 2$
   (b) $x = \frac{1}{t + 1}$, $y = t^2 - 2$, $-3 \le t \le -2$
   (c) $x = e^t$, $y = e^{-t}$, $0 \le t \le 2$
   (d) $x = \sqrt{t} - 2$, $y = 2t^{3/4}$, $1 \le t \le 4$
   (e) $x = \arctan(t)$, $y = t^2$, $0 \le t \le 1$
   (f) $x = \ln(1 + t)$, $y = \frac{1}{t + 1}$, $1 \le t \le 2$

End of Chapter Problems

1. Sketch each parametric equations by eliminating the parameter. Plot the points corresponding to $t = 0$, $t = 1$, and $t = 2$ on the curve and indicate the curve's orientation.
   (a) $x = 3t + 5$, $y = t - 1$
   (b) $x = 2t + 3$, $y = 3t + 1$
   (c) $x = t + 2$, $y = t^3 - 2t$
   (d) $x = \sin(\pi t)$, $y = t$
   (e) $x = 3\cos\left(\frac{\pi}{2}t\right)$, $y = 3\sin\left(\frac{\pi}{2}t\right)$
   (f) $x = \cos\left(\frac{\pi}{4}t\right)$, $y = \sin\left(-\frac{\pi}{4}t\right)$

2. Parametrize the following curves.
   (a) The line $y = 3x + 5$.
   (b) The line $y = \frac{1}{3}x + 1$.
   (c) The circle $x^2 + y^2 = 5$.
   (d) The circle $x^2+y^2 = 1$ that is traversed twice in the clockwise direction.
   (e) The parabola $y = 3x^2 + x + 1$.

3. Find parametric equations for the following curves.
   (a) The line segment from $(0, 4)$ to $(2, 5)$.
   (b) The line segment from $(1, 3)$ to $(2, 0)$.
   (c) The line segment from $(2, 1)$ to $(-2, -1)$.
   (d) The part of the circle $x^2+y^2 = 1$ from $(1, 0)$ to $(0, -1)$ in the counterclockwise direction.
   (e) The part of the circle $x^2+y^2 = 1$ from $(1, 0)$ to $(0, -1)$ in the clockwise direction.
   (f) The part of the parabola $y = x^2 + 1$ that runs from $(-1, 2)$ to $(2, 5)$.

4. Find the equation of the tangent to the curve at the given value of t.
   (a) $x = \frac{3}{2}t + 1$, $y = \frac{1}{2}t + 2$; $t = 2$
   (b) $x = t^2 + 3t + 1$, $y = t^2 - 2t$; $t = -2$
   (c) $x = 2\cos(t)$, $y = -2\sin(t)$; $t = \frac{\pi}{6}$
   (d) $x = \tan(t)$ $y = \sec(t)$; $t = 0$
   (e) $x = t^2$, $y = \sqrt{t}$; $t = 0$

5. For each curve, find all values of t where the tangent line is horizontal and all values of t where the tangent line is vertical.
   (a) $x = 5 - t$, $y = t^2 + 1$
   (b) $x = t^3 - \frac{3}{2}t^2$, $y = t^3 + 1$
   (c) $x = t^2 - 2t$, $y = \sqrt{t}$

6. Determine the area under each parametric curve.
   (a) $x = 2t^2$, $y = t + 1$, $0 \leq t \leq 3$
   (b) $x = t^2 - 1$, $y = t^2 + t$, $-1 \leq t \leq 1$
   (c) $x = 1 + e^t$, $y = t - t^2$, $0 \leq t \leq 1$
   (d) $x = 3\cos(t)$, $y = 2\sin(t)$, $0 \leq t \leq 2\pi$

7. Set up, but do not evaluate, an integral to calculate the length of the parametric curve.
   (a) $x = 3t + 1$, $y = t^2 + 2t$, $-1 \leq t \leq 2$
   (b) $x = \ln(t)$, $y = t - 1$, $1 \leq t \leq 3$
   (c) $x = 1 + \cos(t)$, $y = 2 + \sin(t)$, $0 \leq \pi$
   (d) $x = t + \ln(t)$, $y = t - \ln(t)$, $1 \leq t \leq 4$
   (e) $x = \arcsin(t)$, $y = t$, $0 \leq t \leq 1$

Chapter 12: Polar Coordinates

A **coordinate system** is a system for representing the location of a point in a space by an ordered n-tuple $(c_1, c_2, \ldots, c_n)$. We call the elements of the n-tuple the **coordinates** of the point.

We are used to using the Cartesian coordinate system in which the location of the point is represented by the directed distance from a set of perpendicular axes which all intersect at a point O. However, you may also be used to other coordinate systems. For example, the geographic coordinate system represents a location on the earth by longitude, latitude and altitude.

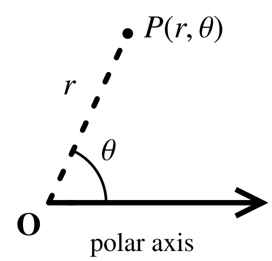
We will now look at another important coordinate system called polar coordinates.

Section 12.1: Polar Coordinates

LEARNING OUTCOMES

1. Know how to plot points in polar coordinates.
2. Know how to convert between polar and Cartesian coordinates.

As in all coordinate systems, we must have a frame of reference for our coordinate system. So, in a plane we choose a point O called the **pole** (or origin). From O we draw a ray called the **polar axis**. Generally, the polar axis is drawn horizontally to the right to match the positive x-axis in Cartesian coordinates.



Let P be any point in the plane. We will represent the position of P by the ordered pair

$$(r, \theta)$$

where:

$r \geq 0$, called the **radial coordinates**, is the length of the line OP, and

θ, called the **angular coordinate**, is the angle between the polar axis and OP.

We call r and θ the **polar coordinates** of P.

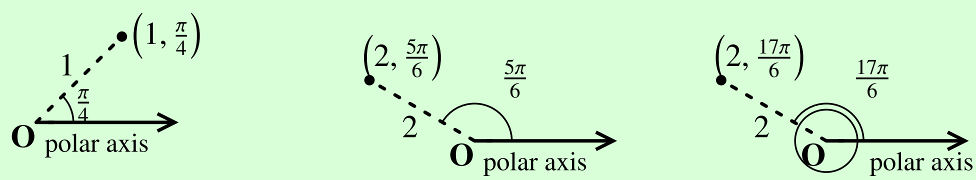
Polar coordinates are particularly helpful when dealing with functions such as circles, lines through the origin, or when the distance from the origin is a function of the angle.

REMARKS

1. We assume, as usual, that an angle θ is considered positive if measured in the counterclockwise direction from the polar axis and negative if measured in the clockwise direction.
2. We represent the point O by the polar coordinates $(0, \theta)$ for any value of θ.
3. We are restricting r to be non-negative to coincide with the interpretation of r as distance. Many textbooks do not put this restriction on r.

EXAMPLE 1 Plot the points $\left(1, \frac{\pi}{4}\right)$, $\left(2, \frac{5\pi}{6}\right)$, and $\left(2, \frac{17\pi}{6}\right)$ in polar coordinates.

Solution:

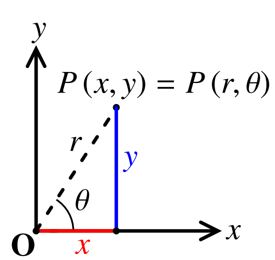


The example demonstrates one important difference between polar coordinates and Cartesian coordinates. In Cartesian coordinates each point has a unique representation (x, y). However, observe that a point P can have infinitely many representations in polar coordinates. In particular,

$$(r, \theta) = (r, \theta + 2\pi k), \qquad \text{for any integer} k$$

12.1.1 Relationship to Cartesian Coordinates

If we now place the pole O at the origin of the Cartesian plane and lie the polar axis along the positive x-axis, we can find a relationship between the coordinates of a point P in the two coordinate systems. From the diagram, we get:



$$\boxed{\begin{aligned} x &= r\cos(\theta), & r &= \sqrt{x^2 + y^2} \\ y &= r\sin(\theta), & \tan(\theta) &= \frac{y}{x} \end{aligned}} \qquad (12.1)$$

EXAMPLE 2 Convert $\left(2, -\frac{\pi}{3}\right)$ and $\left(1, \frac{3\pi}{4}\right)$ from polar coordinates to Cartesian coordinates.

Solution: For $\left(2, -\frac{\pi}{3}\right)$, we have $r = 2$ and $\theta = -\frac{\pi}{3}$, so we get

$$x = 2\cos\left(-\frac{\pi}{3}\right) = 1, \qquad y = 2\sin\left(-\frac{\pi}{3}\right) = -\sqrt{3}$$

Hence, $\left(2, -\frac{\pi}{3}\right)$ in Cartesian coordinates is $(1, -\sqrt{3})$.

For $\left(1, \frac{3\pi}{4}\right)$ we have $r = 1$ and $\theta = \frac{3\pi}{4}$, so we get

$$x = \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \qquad y = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

So, this point has Cartesian coordinates $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

EXERCISE 1 Convert $\left(2, \frac{11\pi}{3}\right)$ and $\left(0, \frac{\sqrt{3}\,\pi}{5}\right)$ from polar coordinates to Cartesian coordinates.

EXAMPLE 3 Convert $(1, 1)$ from Cartesian coordinates to polar coordinates.

Solution: We have $x = 1$ and $y = 1$, so $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan(\theta) = 1$.

Since x and y are both positive the point is in quadrant 1, and hence

$$\theta = \frac{\pi}{4} + 2\pi k, \text{ for any integer} k$$

Thus, we get the polar coordinate representations $\left(\sqrt{2}, \frac{\pi}{4} + 2\pi k\right)$, for any integer k.

Often we do not need to find all possible polar representations for a point. Thus, we further restrict ourselves to a range of θ (such as $0 \le \theta < 2\pi$ or $-\pi < \theta \le \pi$) which gives unique representation.

EXAMPLE 4 Convert $(2, 2)$ from Cartesian coordinates to polar coordinates with $0 \le \theta < 2\pi$.

Solution: $x = 2$ and $y = 2$ gives $r = \sqrt{2^2 + 2^2} = \sqrt{8}$ and $\tan(\theta) = 1$.

Since θ is in the first quadrant we get $\theta = \frac{\pi}{4}$.

Hence, the point has polar representation $\left(\sqrt{8}, \frac{\pi}{4}\right)$.

It is important to keep in mind that even with the restriction $0 \leq \theta \leq 2\pi$ the equation $\tan(\theta) = \frac{y}{x}$ does not uniquely determine θ. When finding θ using this equation we must be careful to choose the θ which lies in the correct quadrant.

EXAMPLE 5

Convert $(-1, \sqrt{3})$ from Cartesian coordinates to polar coordinates with $0 \leq \theta < 2\pi$.

Solution: $x = -1$ and $y = \sqrt{3}$ gives $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$ and $\tan(\theta) = -\sqrt{3}$.

Since θ is in the second quadrant we get $\theta = \frac{2\pi}{3}$.

Hence, the point has polar representation $\left(2, \frac{2\pi}{3}\right)$.

EXERCISE 2

Convert $\left(1, \frac{1}{\sqrt{3}}\right)$ from Cartesian coordinates to polar coordinates.

EXERCISE 3

Convert $(-\sqrt{3}, 1)$ from Cartesian coordinates to polar coordinates.

12.1.2 Functions in Polar Coordinates

In the same way that a graph in Cartesian coordinates can be represented by a function involving x and y, we can represent a graph in polar coordinates by using a function involving r and θ. The standard form of functions in polar coordinates is $r = f(\theta)$. However, as we will see, we can have functions of the form $\theta = g(r)$ or implicitly defined functions $h(r, \theta) = 0$.

To help us understand functions in polar coordinates, it is often helpful to convert them to Cartesian coordinates using equations (12.1).

EXAMPLE 6

Convert the polar equation $r = 2$ to Cartesian coordinates.

Solution: Since $r = \sqrt{x^2 + y^2}$, we get

$$\begin{aligned} 2 &= \sqrt{x^2 + y^2} \\ 4 &= x^2 + y^2 \end{aligned}$$

EXAMPLE 7 Convert the polar equation $r = \cos(\theta)$ to Cartesian coordinates.

Solution: We start by multiplying both sides by r so that we can use the formulas $r^2 = x^2 + y^2$ and $x = r\cos(\theta)$. We get

$$\begin{aligned} r &= \cos(\theta) \\ r^2 &= r\cos(\theta) \\ x^2 + y^2 &= x \end{aligned}$$

EXAMPLE 8 Convert the polar equation $\theta = \frac{\pi}{4}$ to Cartesian coordinates.

Solution: We first notice that by definition of polar coordinates, if $\theta = \frac{\pi}{4}$, then the graph is in the first quadrant. That is, we must have $x \geq 0$ and $y \geq 0$.

Next, from the equation $\tan(\theta) = \frac{y}{x}$ we get

$$\begin{aligned} \tan\left(\frac{\pi}{4}\right) &= \frac{y}{x} \\ 1 &= \frac{y}{x} \\ x &= y, \qquad x \geq 0 \end{aligned}$$

EXERCISE 4 Convert the equation of the curve $\theta = -\frac{\pi}{3}$ to polar coordinates.

EXERCISE 5 Convert the equation of the curve $r = \sin(\theta)$ to polar coordinates.

In other cases, it can be very helpful to convert functions in Cartesian coordinates into polar coordinates.

EXAMPLE 9 Convert the equation $(x^2 + y^2)^{3/2} = 2xy$ to polar coordinates.

Solution: Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$ we get

$$\begin{aligned} (x^2 + y^2)^{3/2} &= 2xy \\ r^3 &= 2r\cos(\theta) \cdot r\sin(\theta) \\ r^3 &= r^2\sin(2\theta) \\ r &= \sin(2\theta) \end{aligned}$$

Notice that the last simplification is only valid since the pole, $r = 0$, is still included in the graph (the case where $\theta = \pi$).

It is valuable to think about the domain of the equation in the last example. Since we have the restriction $r \geq 0$ we must also have $\sin(2\theta) \geq 0$. Hence, we find that a domain of the function in polar coordinates is

$$0 \leq \theta \leq \frac{\pi}{2}, \quad \pi \leq \theta \leq \frac{3\pi}{2}$$

Notice that the matches the domain in Cartesian coordinates. In particular, since $(x^2 + y^2)^{3/2} \geq 0$, the equation $(x^2 + y^2)^{3/2} = 2xy$ implies that $xy \geq 0$. Thus, we must have either $x, y \geq 0$ or $x, y \leq 0$.

EXAMPLE 10

Convert the equation $x^2 + y^2 = y$ to polar coordinates.

Solution: We have

$$\begin{aligned} x^2 + y^2 &= y \\ r^2 &= r\sin(\theta) \\ r &= \sin(\theta) \end{aligned}$$

since $r = 0$ is still included in the graph.

EXERCISE 6

Convert the equation of the curve $x^2 - y^2 = 1$ to polar coordinates.

EXERCISE 7

Convert the equation of the curve $y = -x$ to polar coordinates.

Section 12.1 Problems

1. Convert the following points from Cartesian coordinates to polar coordinates with $0 \leq \theta < 2\pi$.

   (a) $(3, 0)$

   (b) $(2, -2)$

   (c) $(0, -1)$

   (d) $(-1, -\sqrt{3})$

   (e) $(-3, -3)$

   (f) $(-2\sqrt{3}, 2)$

   (g) $\left(1, -\frac{1}{\sqrt{3}}\right)$

   (h) $(-2\sqrt{2}, 2\sqrt{2})$

   (i) $(\sqrt{3}, -1)$

   (j) $(-1, \sqrt{3})$

2. Convert the following points from polar coordinates to Cartesian coordinates. Illustrate with a sketch.

   (a) $\left(2, \frac{\pi}{4}\right)$

   (b) $\left(0, \frac{5\pi}{6}\right)$

   (c) $\left(2, \frac{4\pi}{3}\right)$

   (d) $\left(3, -\frac{\pi}{6}\right)$

   (e) $\left(1, \frac{3\pi}{4}\right)$

   (f) $\left(2, -\frac{2\pi}{3}\right)$

   (g) $\left(2, \frac{7\pi}{6}\right)$

3. Convert the following equations from Cartesian coordinates to polar coordinates.

   (a) $x^2 + y^2 = 9$

   (b) $y = \sqrt{3}x$

   (c) $x^2 + y^2 = 2x$

   (d) $y = -x$

   (e) $x^2 + y^2 = 5$

   (f) $x^2 + y^2 = \sqrt{x^2 + y^2} - x$

   (g) $(x^2 + y^2)^2 = y$

   (h) $x^2 + 2y^2 = 1$

   (i) $x = -y^2$

   (j) $xy = 1$

4. Convert the following equations from polar coordinates to Cartesian coordinates.

   (a) $r = \sec(\theta)$

   (b) $r = \sqrt{3}$

   (c) $\theta = \dfrac{\pi}{6}$

   (d) $r^2 = \sin(2\theta)$

   (e) $\theta = \dfrac{3\pi}{4}$

   (f) $r = 1 - \sin(\theta)$

   (g) $r = -2\sin(\theta) + 2\cos(\theta)$

   (h) $r = \tan(\theta)\sec(\theta)$

Section 12.2: Graphs in Polar Coordinates

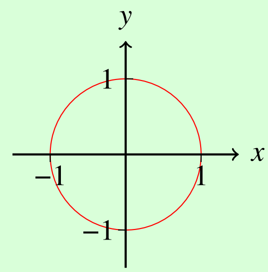
LEARNING OUTCOMES

1. Know how to sketch curves defined in polar coordinates.

The graph of an explicitly defined polar equation $r = f(\theta)$ or $\theta = f(r)$, or an implicitly defined polar equation $f(r, \theta) = 0$, is a curve that consists of all points that have *at least one* polar representation (r, θ) that satisfies the equation of the curve.

EXAMPLE 1

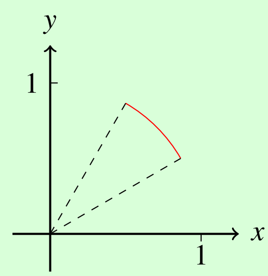
Sketch the polar equation $r = 1$.



Solution: This is the curve which consists of all points $(r, \theta) = (1, \theta)$ for all real numbers θ. Observe that this is all points that have distance 1 from the origin. Hence, we get a circle of radius 1.

EXAMPLE 2

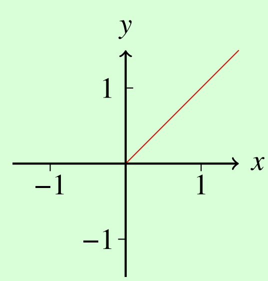
Sketch the polar equation $r = 1$ for $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$.



Solution: This is the curve containing all points that are 1 units from the origin that have angular coordinates $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$. So, this will be the part of the circle of radius 1 between $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$.

EXAMPLE 3

Sketch the polar equation $\theta = \frac{\pi}{4}$.



Solution: This is the curve which consists of all points $(r, \theta) = \left(r, \frac{\pi}{4}\right)$, $r \geq 0$. So, the curve consists of all points that have angle $\frac{\pi}{4}$ and any non-negative distance from the origin. Thus, we get the part of the line $y = x$ that is in the first quadrant.

EXERCISE 1

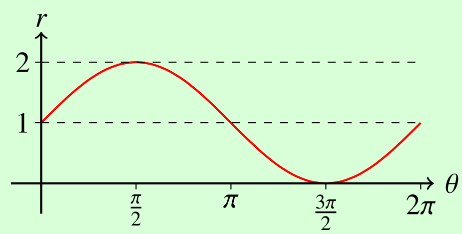
Sketch the polar equation $r = 2$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

EXERCISE 2 Sketch the polar equation $r = 2$ for $0 \le \theta \le \frac{\pi}{4}$.

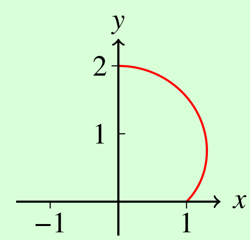
To sketch more complicated curves, we can plot points and "connect the dots". However, as we saw in the last chapter, just plotting a few points may not give us an accurate representation of the graph. A thorough treatment of polar curve sketching would include an analysis of intervals of increase/decrease together with intervals of concavity. Unfortunately, we do not have time to cover these in this course. So, we will instead will produce a sketch by plotting infinitely many points while keeping in mind that graphs in polar coordinates are likely circle-like. To get a 'table of values' that contains infinitely many points we simply sketching the function as if it were in Cartesian coordinates. We demonstrate this with a couple of example.

EXAMPLE 4 Sketch the polar equation $r = 1 + \sin(\theta)$.

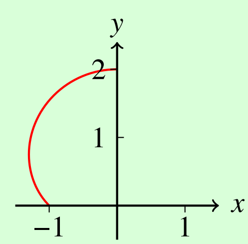
Solution: We first sketch the curve in Cartesian coordinates in the θr-plane (that is, the plane where the horizontal axis is θ and the vertical axis is r).



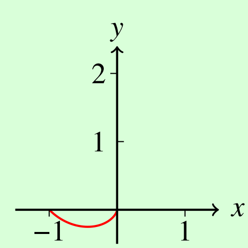
We read from the Cartesian graph that as θ increases from 0 to $\frac{\pi}{2}$ the radius increases from 1 to 2. That is, we start with the point $(r, \theta) = (1, 0)$ and as the angle changes to $\frac{\pi}{2}$, the radius is slowly increasing to 1 until we reach the point $(r, \theta) = \left(2, \frac{\pi}{2}\right)$.



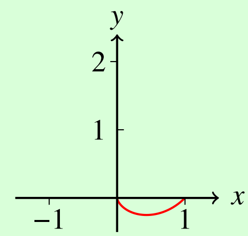
As θ increases from $\frac{\pi}{2}$ to π the radius decreases from 2 to 1. That is, we start with the point $(r, \theta) = \left(2, \frac{\pi}{2}\right)$ and as the angle changes to π, the radius is slowly decreasing to 1 until we reach the point $(r, \theta) = (1, \pi)$.



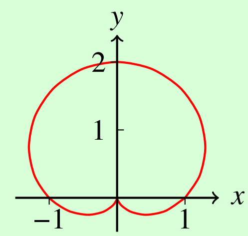
As θ increases from π to $\frac{3\pi}{2}$ the radius decreases from 1 to 0. That is, we start with the point $(r, \theta) = (1, \pi)$ and as the angle changes to $\frac{3\pi}{2}$, the radius is slowly decreasing to 0 until we reach the point $(r, \theta) = \left(0, \frac{3\pi}{2}\right)$.



As θ increases from $\frac{3\pi}{2}$ to 2π the radius increases from 0 to 1. That is, we start with the point $(r, \theta) = \left(0, \frac{3\pi}{2}\right)$ and as the angle changes to 2π, the radius is slowly increasing to 1 until we reach the point $(r, \theta) = (1, 2\pi)$.

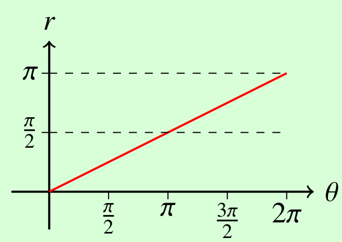


Putting these all together we get the figure to the right which is called a **cardioid**.

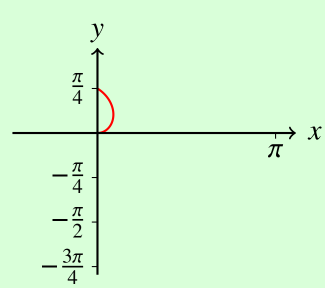


EXAMPLE 5 Sketch the polar equation $r = \frac{1}{2}\theta$, $0 \le \theta \le 2\pi$.

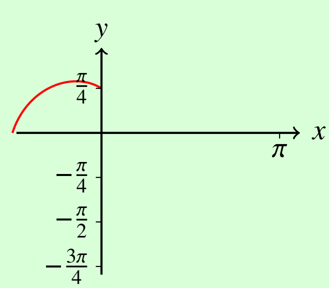
Solution: We first sketch $r = \frac{1}{2}\theta$, $0 \le \theta \le 2\pi$ in the $r\theta$-plane in Cartesian coordinates.



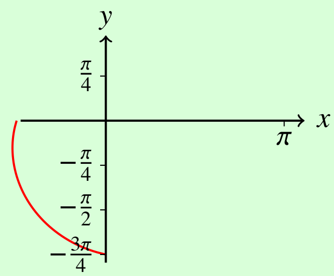
As θ increases from 0 to $\frac{\pi}{2}$ the radius increases from 0 to $\frac{\pi}{4}$. That is, we start with the point $(r, \theta) = (0, 0)$ and as the angle changes to $\frac{\pi}{2}$, the radius is slowly increasing to $\frac{\pi}{4}$ until we reach the point $(r, \theta) = \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.



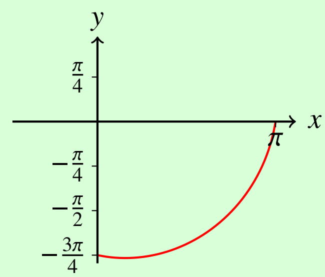
As θ increases from $\frac{\pi}{2}$ to π the radius increases from $\frac{\pi}{4}$ to $\frac{\pi}{2}$. That is, we start with the point $(r, \theta) = \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and as the angle changes to π, the radius is slowly increasing to $\frac{\pi}{2}$ until we reach the point $(r, \theta) = \left(\frac{\pi}{2}, \pi\right)$.



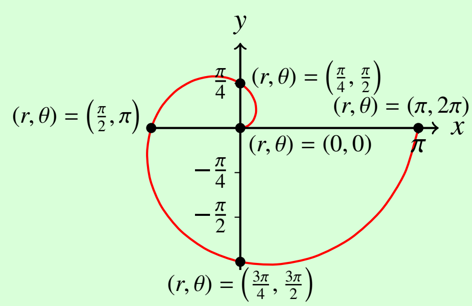
As θ increases from π to $\dfrac{3\pi}{2}$ the radius increases from $\dfrac{\pi}{2}$ to $\dfrac{3\pi}{4}$. That is, we start with the point $(r, \theta) = \left(\frac{\pi}{2}, \pi\right)$ and as the angle changes to $\dfrac{3\pi}{2}$, the radius is slowly increasing to $\dfrac{3\pi}{4}$ until we reach the point $(r, \theta) = \left(\dfrac{3\pi}{4}, \dfrac{3\pi}{2}\right)$.



As θ increases from $\dfrac{3\pi}{2}$ to 2π the radius increases from $\dfrac{3\pi}{4}$ to π. That is, we start with the point $(r, \theta) = \left(\dfrac{3\pi}{4}, \dfrac{3\pi}{2}\right)$ and as the angle changes to 2π, the radius is slowly increasing to π until we reach the point $(r, \theta) = (\pi, 2\pi)$.

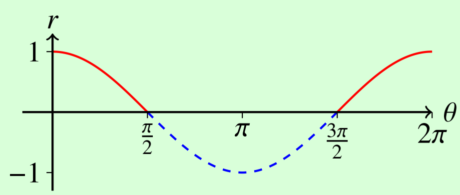


Putting these together gives the graph to the right.

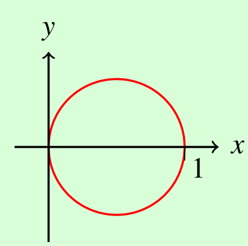


EXAMPLE 6 Sketch the polar equation $r = \cos(\theta)$.

Solution: We first sketch the curve in Cartesian coordinates in the $r\theta$-plane.



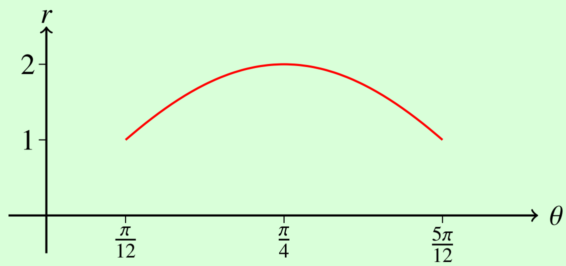
We see that as θ increases from 0 to $\dfrac{\pi}{2}$ the radius decreases from 1 to 0 (the top half of the circle). For values of θ from $\dfrac{\pi}{2}$ to $\dfrac{3\pi}{2}$ the radius is negative (the dashed part), thus we do not draw any points since we have made the restriction that $r \geq 0$. As θ moves from $\dfrac{3\pi}{2}$ to 2π the radius increases from 0 to 1 (the bottom half of the circle).



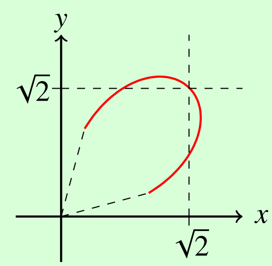
This graph matches what we found when we converted $r = \cos(\theta)$ into Cartesian coordinates in Example 12.1.7. That is, this is the graph of $x^2 + y^2 = x$.

EXAMPLE 7 Sketch the polar equation $r = 2\sin(2\theta)$ for $\frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12}$.

Solution: We first sketch the curve in Cartesian coordinates in the $r\theta$-plane for $\frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12}$.



We see that as θ increases from $\frac{\pi}{12}$ to $\frac{\pi}{4}$ the radius increase from 1 to 2. For values of θ from $\frac{\pi}{4}$ to $\frac{5\pi}{12}$ the radius decreases from 2 to 1.



EXERCISE 3 Sketch the polar equation $r = \sin(\theta)$.

EXERCISE 4 Sketch the polar equation $r = 2 + \sin(\theta)$.

EXERCISE 5 Sketch the polar equation $r = 2 + \sin(\theta)$ for $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$.

Section 12.2 Problems

1. Sketch the polar equation.
   (a) $r = 2$
   (b) $\theta = \frac{\pi}{3}$
   (c) $\theta = \frac{7\pi}{3}$
   (d) $r^2 - 3r + 2 = 0$
   (e) $r = \sin(2\theta)$
   (f) $r = 1 + \cos(\theta)$
   (g) $r = \cos(2\theta)$

2. Sketch the polar equation.
   (a) $r = 1 - 2\cos(\theta)$
   (b) $r = 2 - \cos(\theta)$
   (c) $r = 2 - 2\sin(\theta)$
   (d) $r = 1 + 2\cos(2\theta)$
   (e) $r^2 = 9\sin(2\theta)$
   (f) $r = \sin(3\theta)$
   (g) $r = \ln(\theta)$

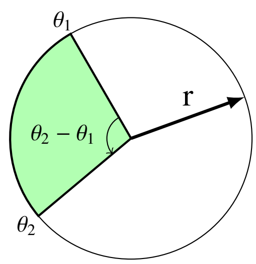
Section 12.3: Area in Polar Coordinates

LEARNING OUTCOMES

1. Know how to find area bounded by a polar curve.
2. Know how to find area between curves in polar coordinates.

We now wish to derive the formula for computing area between curves in Polar coordinates. Clearly this will be a little different than before as it does not make sense to use rectangles to find the area. In polar coordinates, it is natural to use sectors of a circle. To do this, we will need the formula for the area of a sector of a circle.

Assume θ_1 and θ_2 are two angles in a circle of radius r with $\theta_2 > \theta_1$. The area of the sector from θ_1 to θ_2 is the area of a circle multiplied by the ratio of the change in the angle and the angle of one complete revolution (2π).



$$\pi r^2 \cdot \frac{\theta_2 - \theta_1}{2\pi} = \frac{1}{2} r^2 (\theta_2 - \theta_1)$$

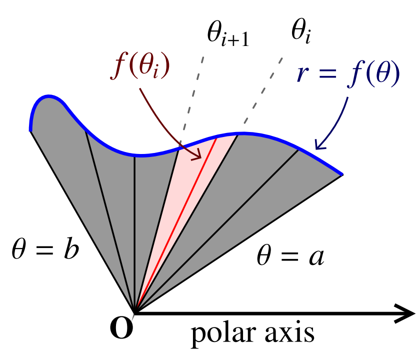
We now derive the area as before. We divide the region bounded by $\theta = a$, $\theta = b$ and $r = f(\theta)$ into sub-regions $\theta_0, \ldots, \theta_n$ of equal difference $\Delta\theta$. We get the area of i-th sector is

$$A_i = \frac{1}{2}[f(\theta_i)]^2 \Delta\theta$$

Hence, the area is approximately

$$\sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2 \Delta\theta$$

As usual, we let the number of subdivisions go to infinity and get



$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2 \Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta$$

THEOREM 1 If a and b are angles such that

$$a < b \leq a + 2\pi$$

and $f(\theta) \geq 0$ is continuous for $a \leq \theta \leq b$, then the area of the region bounded by $r = f(\theta)$, $\theta = a$ and $\theta = b$ is

$$\text{Area} = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta$$

EXAMPLE 1

Find the area inside the circle $r = a$ for any constant a.

Solution: We saw when sketching graphs in polar coordinates that to create one loop of the circle $r = a$ in polar coordinates we require that $0 \leq \theta \leq 2\pi$. Thus, we get

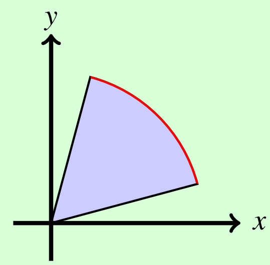
$$A = \int_0^{2\pi} \frac{1}{2}a^2\, d\theta = \frac{1}{2}a^2\theta\Big|_0^{2\pi} = \frac{1}{2}a^2[2\pi - 0] = \pi a^2$$

EXAMPLE 2

Find the area inside the part of the circle $r = 2$ for $\frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12}$.

Solution: We get

$$A = \int_{\pi/12}^{5\pi/12} \frac{1}{2}(2)^2\, d\theta = 2\theta\Big|_{\pi/12}^{5\pi/12} = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3}$$

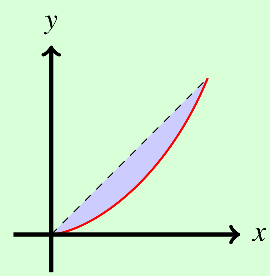


EXAMPLE 3

Find the area of the part of $r = 1 - \cos(\theta)$ for $0 \leq \theta \leq \frac{\pi}{4}$.

Solution: We get

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} \frac{1}{2}(1 - \cos(\theta))^2\, d\theta \\
&= \int_0^{\pi/4} \frac{1}{2}(1 - 2\cos(\theta) + \cos^2(\theta))\, d\theta \\
&= \int_0^{\pi/4} \left(\frac{1}{2} - \cos(\theta) + \frac{1}{4}(1 + \cos(2\theta))\right) d\theta \\
&= \int_0^{\pi/4} \left(\frac{3}{4} - \cos(\theta) + \frac{1}{4}\cos(2\theta)\right) d\theta \\
&= \left(\frac{3}{4}\theta - \sin(\theta) + \frac{1}{8}\sin(2\theta)\right)\Big|_0^{\pi/4} \\
&= \frac{3\pi}{16} - \frac{1}{\sqrt{2}} + \frac{1}{8} - (0 - 0 + 0) \\
&= \frac{3\pi}{16} - \frac{1}{\sqrt{2}} + \frac{1}{8}
\end{aligned}$$



EXERCISE 1

Find the area inside $r = 2\sin(2\theta)$ for $\frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12}$. Illustrate the area with a sketch.

EXERCISE 2 Find the area inside the lemniscate $r = 2\sqrt{\sin(2\theta)}$.

ALGORITHM

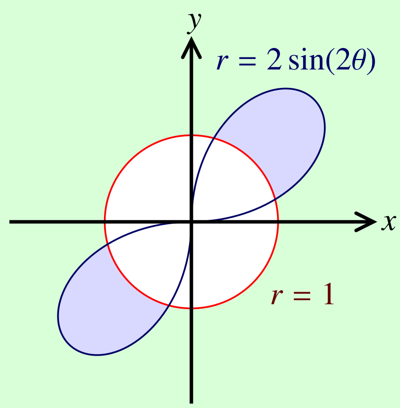
To find the area between two curves in polar coordinates, we use the same method we used for doing this in Cartesian coordinates.

1. Find the points of intersection.
2. Graph the curves and split the desired region into easily integrable regions.
3. Integrate.

EXAMPLE 4 Write an integral for the area inside $r = 2\sin(2\theta)$, but outside $r = 1$.

Solution:

Setting the curves equal to each other we get $1 = 2\sin(2\theta)$, hence $2\theta = \dfrac{\pi}{6}$ or $2\theta = \dfrac{5\pi}{6}$. Therefore, we want to integrate over the region $\dfrac{\pi}{12}$ to $\dfrac{5\pi}{12}$. To find the shaded area, we will find the area inside the lemniscate in the first quadrant and subtract off the area of the region that is inside both the circle and the lemniscate. Finally, we will multiply by 2 for the symmetric region in the third quadrant. We get



$$A = 2\left(\int_{\pi/12}^{5\pi/12} \frac{1}{2}(2\sin(2\theta))^2\,d\theta - \int_{\pi/12}^{5\pi/12} \frac{1}{2}(1)^2\,d\theta\right)$$

REMARK

Finding points of intersection can be tricky, especially at the pole/origin which does not have a unique representation: $(0, \theta)$ for any θ represents the origin, so simply setting expressions equal to each other may 'miss' that point. It is essential to sketch the region when finding points of intersection.

EXERCISE 3 Find the area between the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Section 12.3 Problems

1. Find the area inside $r = \sin(\theta)$.
2. Find the area inside $r = 2$ for $0 \le \theta \le \dfrac{\pi}{6}$.
3. Find the area inside $r^2 = \cos(\theta)$.
4. Find the area inside $r = 2 - 2\cos(\theta)$.
5. Find the area inside one loop of $r = \cos(2\theta)$.
6. Find the area by the part of $r = 1 + \sin(\theta)$ with $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$.
7. Find the area inside one loop of $r = 2\sin(3\theta)$.
8. Find the area inside one loop of $r = \sqrt{\cos(2\theta)}$.
9. Find the area inside one loop of $r = 2\cos(3\theta)$.
10. Find the area inside $r = 2\cos(\theta)$, but outside $r = 1$.
11. Find the area inside both $r = \cos(2\theta)$ and $r = \sin(2\theta)$ for $0 \le \theta \le \dfrac{\pi}{4}$.
12. Find the area inside $r = 1 - \sin(\theta)$, but outside $r = 1$.
13. Find the area inside both $r = 2 + \cos(\theta)$ and $r = 2 + \sin(\theta)$.
14. Find the area inside $r = 3\cos(\theta)$, but outside $r = 1 + \cos(\theta)$.

End of Chapter Problems

1. Convert the following points from Cartesian coordinates to polar coordinates with $0 \le \theta < 2\pi$.
   (a) $(0, -2)$
   (b) $(-3\sqrt{3}, 3)$
   (c) $(-2, -2\sqrt{3})$
   (d) $(1, -1)$
2. Convert the following points from polar coordinates to Cartesian coordinates. Illustrate with a sketch.
   (a) $\left(2, \dfrac{\pi}{3}\right)$
   (b) $\left(1, \dfrac{7\pi}{6}\right)$
   (c) $\left(2, \dfrac{3\pi}{4}\right)$
   (d) $\left(3, -\dfrac{2\pi}{3}\right)$
3. Convert the following equations from Cartesian coordinates to polar coordinates.
   (a) $(x^2 + y^2)^2 = 4$
   (b) $x = \sqrt{3}y$
   (c) $x^2 + y^2 = 2y$
   (d) $y = x^{-2}$
4. Convert the following equations from polar coordinates to Cartesian coordinates.
   (a) $r = 3$
   (b) $r = 2\cos(\theta)$
   (c) $\theta = \dfrac{7\pi}{6}$
   (d) $r = 1 - \sin(\theta)$
5. Sketch the polar equation.
   (a) $r = 2$
   (b) $\theta = \dfrac{2\pi}{3}$
   (c) $r = 1 + \sin(\theta)$
   (d) $r = \cos(2\theta)$
   (e) $r = 1 - 2\cos(\theta)$
   (f) $r = \sin(4\theta)$
6. Find the area inside $r = 3$ for $0 \le \theta \le \dfrac{\pi}{3}$.
7. Find the area inside $r = \cos(\theta)$.
8. Find the area inside $r^2 = \sin(2\theta)$.
9. Find the area inside both $r = 1$ and $r = 1 - \sin(\theta)$.
10. Find the area inside $r = 2\sin(2\theta)$ and outside $r = 1$ in the first quadrant.