

Mathematics of Finance

MATHEMATICS OF FINANCE

MATH-1057

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Fanshawe College Pressbooks
London Ontario



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ACKNOWLEDGEMENTS

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Collaborators

This project was a collaboration between the author and the team in the OER Design Studio at Fanshawe. The following staff and students were involved in the creation of this project:

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- Wilson Poulter, *Copyright*

ABOUT THIS BOOK

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Changes From Adapted Resource

The following formatting changes were made in this version of the textbook:

- Updates to LaTeX where possible.
- Formula Hub added.
- Formatting changes.
- Examples were reformatted into steps for ease of understanding

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CHAPTER 1: COMPOUND INTEREST

Chapter Outline

[1.0 Introduction](#)

[1.1 Compound Interest Terminology](#)

[1.2 Calculating the Future Value](#)

[1.3 Calculating the Present Value](#)

[1.4 Equivalent Payments](#)

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[1.8 Review Exercises](#)

[Chapter 1 Summary](#)

1.0 INTRODUCTION

Chapter Learning Objectives

By the end of the chapter you should be able to:

- Differentiate between the concept of compound interest and simple interest
- Define terms related to compound interest
- Calculate future value for compound interest
- Calculate present value for compound interest
- Calculate equivalent payments that replace another payment or a set of payments
- Calculate the interest rate of a loan or investment
- Calculate the number of compounding periods
- Calculate the term of a loan or investment
- Calculate effective interest rates
- Calculate equivalent interest rates

Do you dream of owning a home? What about a car or a home theatre system? Big purchases such as these require long-term financial planning. Compound interest means that you will pay substantially more money for your purchases or earn more money on investments than you would with simple interest.

Previously, you learned about simple interest. In simple interest, all interest is based solely on the original principal amount of the transaction, and the interest is converted to principal at the end of the transaction. But in compound interest, the interest also earns interest. **Compound interest** involves interest being periodically converted to principal throughout the transaction, with the result that the interest itself also accumulates interest. Compound interest is used for most transactions lasting at least one year.

But how significant is the interest when it compounds? Consider the following scenarios:

- A home theatre system with the big-screen TV, Blu-Ray player, stereo surround sound, and more can retail for \$5, 000. Did you know that if you put that amount on the retail store's 18% interest credit card and pay it off monthly for three years, you will pay over \$1, 500 of compound interest?
- You want to purchase a new car that retails for \$35, 000. Unless you have that amount of cash on hand, you will join the ranks of other Canadians who take out car loans to purchase their new car. If you take six years to pay-off this loan at an interest rate of 4.5% compounded monthly, you would have to make monthly payments of \$555.59 and pay over \$5, 000 in interest on the loan.
- To purchase a house that is listed for \$600, 000, you take out a 25 year mortgage at 6% compounded semi-annually. Over the course of the 25 years, you will pay over \$550, 000 in interest on the mortgage, which almost doubles the original cost of the house.

And it is not any different for businesses. Whether they are local companies or multinational conglomerates they must invest and borrow at compound interest rates in their attempt to achieve long-term financial strategies. Some examples of these business activities include the following:

- Borrowing \$480, 000 to open a new restaurant.
- Spending \$1, 000, 000 on a fleet of rigs and semi-trailers for product distribution.
- Constructing new production plants or warehouses costing \$10, 000, 000 or more.

The money for these types of transactions does not appear out of thin air. It must be borrowed or taken from savings, and either approach involves compound interest. Throughout the rest of this textbook, you will study compound interest as it relates to three distinct but interconnected concepts:

- Calculating interest on a single amount (called a lump-sum amount or single payment).
- Calculating interest on a series of regular, equal payments, called annuities.
- Specialized applications including amortization, mortgages, bonds, sinking funds, net present value, and internal rates of return.

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1.1 COMPOUND INTEREST TERMINOLOGY

Learning Objectives

- Differentiate between the concept of compound interest and simple interest
- Define terms related to compound interest

Formula & Symbol Hub

Symbols Used

- f or EFF = Effective interest rate
- i = Periodic interest rate
- j or I/Y = Nominal interest rate per year
- m or C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of compound periods for the term

Formulas Used

- **Formula 1.1 – Total Number of Compounds**

$$n = m \times \text{time in years}$$

- **Formula 1.2 – Periodic Interest Rate**

$$i = \frac{j}{m}$$

Introduction

Compound interest involves interest being periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest. This means that interest earned in the previous compounding period will earn interest in all subsequent compounding periods.

But how does compound interest compare to simple interest? The critical difference is the placement of interest into the account. Under simple interest, you convert the interest to principal at the end of the transaction's time frame. For example, in a six-month simple interest GIC the balance in its account at any point before the maturity date is the original principal and nothing more. Only upon maturity does the interest appear. In contrast, a five-year compound interest GIC receives an interest deposit into the account at periodic times every year. Once this interest is deposited into the account that interest will begin to earn interest for the remainder of the term of the GIC.

Suppose you invested \$1,000 into two different savings accounts: one account earns 10% simple interest and the other account earns 10% compounded annually. How much is in each account at the end of five years? Which account earns more interest?

From the previous chapter on simple interest, the simple interest savings account will earn \$500 in interest and have a maturity value of \$1,500 at the end of five years. Because this is simple interest, only the original \$1,000 principal earns interest over the five years. The table below shows the balance in the account at the end of each year.

Year	Principal at Start of Year	Accrued Interest	Balance at End of Year
1	\$1,000	\$100	\$1,000
2	\$1,000	\$200	\$1,000
3	\$1,000	\$300	\$1,000
4	\$1,000	\$400	\$1,000
5	\$1,000	\$500	\$1,000

For the compound interest savings account, the 10% interest is compounded annually, which means at the end of every year 10% of the current principal is added to the account. This balance at the end of each year becomes the principal for the next year and is the amount of money that will earn interest in the next year. The table below shows the balance in the account at the end of each year when the interest is calculated and added to the principal.

Year	Principal at Start of Year	Accrued Interest	Balance at End of Year
1	\$1,000	\$100	\$1,100
2	\$1,100	\$110	\$1,210
3	\$1,210	\$121	\$1,331
4	\$1,331	\$133.10	\$1,464.10
5	\$1,461.10	\$146.41	\$1,610.51

The simple interest account earns a total of \$500 in interest. But the compound interest account earns \$610.51 in interest (the sum of the accrued interest column), which is \$110.51 more interest than the simple interest account. This is the power of compounding interest. As an investor, you earn more interest on an investment when the interest is compounded than you would with simple interest. But this also means that as a borrower, you will pay more interest on a loan when the interest is compounded than with simple interest.

Compound Interest Terms

The **compounding period** is the interval of time between two consecutive interest calculations. That is, the compounding period is the period of time between the dates of successive conversions of interest to principal.

For example, an interest rate that compounds monthly means that the compounding period is monthly, and so the interest is calculated and added to the principal every month.

The **compounding frequency** (m) is the number of times the interest is compounded every year. That is, the compounding frequency is the number of compounding periods in one year. For example, an interest rate that compounds quarterly means that the interest is calculated and added to the principal four times every year, so $m = 4$.

Compounding Period	Number of Times per Year Interest is Compounded	Compounding Frequency
Annually	Every year/Once a year	1
Semi-annually	Every six months/Twice a year	2
Quarterly	Every three months/Four times a year	4
Monthly	Every month/Twelve times a year	12
Daily	Every day/365 times a year	365

The **term or time period** (t) is the length of time for the investment or loan. That is, the term is the period of time when the interest is calculated. Because the interest rate is given as an annual interest rate and the compounding frequency is the number of compoundings per year, the term must be in years in order to use the term in compound interest calculations. If the term is not in years, the term must be converted to years. If the term is given in months, divide by 12 to convert the term to years. If the term is given in days, divide by 365 to convert the term to years.

Most compound interest calculations require the **total number of compounding periods during the term** (n).

1.1 Total Number of Compounding Periods

$$n = m \times t$$

n is the Total Number of Compoundings: Most compound interest calculations require this value.

m is the Compounding Frequency: is the number of times the interest is compounded every year.

t is Time in Years: is the length of time for the investment or loan.

Example 1.1.1

For the following interest rates and terms, identify the compounding frequency and the term in years. Then calculate the total number of compounding periods during the term.

1. **3.5%** compounded monthly for five years.
2. **7%** compounded semi-annually for **18** months.
3. **4.9%** compounded quarterly for three years and nine months.

Solution

1. Compounding frequency = 12
Term = 5 years
 $n = 5 \times 12 = 60$
2. Compounding frequency = 2
Term = $\frac{18}{12} = 1.5$ years
 $n = 1.5 \times 2 = 3$
3. Compounding frequency = 4
Term = $3 + \frac{9}{12} = 3.75$ years
 $n = 3.75 \times 4 = 15$

Try It

1) Complete the following table.

Interest Rate and Time Period	Compounding Frequency (m)	Term (in years) (t)	Number of Compounding Periods for the Term (n)
7.2% compounded quarterly for 27 months			
3.7% compounded semi-annually for 6.5 years			
1.9% compounded daily for 657 days			

Solution

Interest Rate and Time Period	Compounding Frequency (m)	Term (in years) (t)	Number of Compounding Periods for the Term (n)
7.2% compounded quarterly for 27 months	4	$\frac{27}{12} = 2.25$	$2.25 \times 4 = 9$
3.7% compounded semi-annually for 6.5 years	2	6.5	$6.5 \times 2 = 13$
1.9% compounded daily for 657 days	365	$\frac{657}{365} = 1.8$	$1.8 \times 365 = 657$

The Nominal and Periodic Interest Rates

The **nominal interest rate** (j) is the quoted or stated interest rate annually. It is the rate, expressed as a percent, that precedes the word “compounded”. For example, if the interest rate is 5% compounded quarterly, the nominal interest rate is 5%.

The **periodic interest rate** (i) is interest rate earned or charged for a given compounding period.

1.2 Periodic Interest Rate

$$i = \frac{j}{m}$$

i is the **Periodic Interest Rate**: is interest rate earned or charged for a given compounding period.

j is the **Nominal Interest Rate**: is the quoted or stated interest rate annually.

m is the **Compounding Frequency**: is the number of times the interest is compounded every year.

Example 1.1.2

Calculate the periodic interest rate for the following nominal interest rates.

1. 9% compounded monthly.
2. 6% compounded quarterly.

Solution

1. For 9% compounded monthly.

$$j = 9\%$$
$$m = 12$$

$$i = \frac{j}{m}$$
$$= \frac{9\%}{12}$$
$$= 0.75\% \text{ per month}$$

This means that **9%** compounded monthly is equal to a periodic interest rate of **0.75%** per month. Interest is converted to principal **12** times throughout the year at the rate of **0.75%** each time.

2. For **6%** compounded quarterly.

$$j = 6\%$$
$$m = 4$$

$$i = \frac{j}{m}$$
$$= \frac{6\%}{4}$$
$$= 1.5\% \text{ per quarter}$$

This means that **6%** compounded quarterly is equal to a periodic interest rate of **1.5%** per quarter. Interest is converted to principal **4** times throughout the year at the rate of **1.5%** each time.

Example 1.1.3

Calculate the nominal interest rate for the following periodic interest rates.

1. **0.7%** per month.
2. **0.05%** per day.

Solution

1. For **0.7%** per month.

$$\begin{aligned} i &= 0.7\% \\ m &= 12 \end{aligned}$$

$$\begin{aligned} j &= i \times m \\ &= 0.7\% \times 12 \\ &= 8.4\% \text{ compounded monthly} \end{aligned}$$

This means that **0.7%** per month is equal to **8.4%** compounded monthly.

2. For **0.05%** per day.

$$\begin{aligned} i &= 0.05\% \\ m &= 365 \end{aligned}$$

$$\begin{aligned} j &= i \times m \\ &= 0.05 \times 365 \\ &= 18.25\% \text{ compounded daily} \end{aligned}$$

This means that **0.05%** per day is equal to **18.25%** compounded daily.

Example 1.1.4

Calculate the compounding frequency for the following nominal and periodic interest rates.

1. Nominal rate of **6%**, periodic rate of **3%**.
2. Nominal rate of **9%**, periodic rate of **2.25%**.

Solution

1. For **6%** nominal and **3%** periodic.

$$\begin{aligned} j &= 6\% \\ i &= 3\% \end{aligned}$$

$$\begin{aligned} m &= \frac{j}{i} \\ &= \frac{6\%}{3\%} \\ &= 2 \end{aligned}$$

For the nominal interest rate of **6%** to be equal to a periodic interest rate of **3%**, the compounding frequency is **2**, which means a compounding period of every six months, or semi-annually.

2. For **9%** nominal and **2.25%** periodic.

$$\begin{aligned} j &= 9\% \\ i &= 2.25\% \end{aligned}$$

$$\begin{aligned} m &= \frac{j}{i} \\ &= \frac{9\%}{2.25\%} \\ &= 4 \end{aligned}$$

For the nominal interest rate of **9%** to be equal to a periodic interest rate of **2.25%**, the compounding frequency is **4**, which means a compounded period of every three months, or quarterly.

Try It

2) Calculate the periodic interest rate if the nominal interest rate is **7.75%** compounded monthly.

Solution

$$\begin{aligned} i &= \frac{j}{m} \\ &= \frac{7.75\%}{12} \\ &= 0.65\% \text{ per month} \end{aligned}$$

Try It

3) Calculate the compounding frequency for a nominal interest rate of **9.6%** if the periodic interest rate is **0.8%**.

Solution

$$\begin{aligned} m &= \frac{j}{i} \\ &= \frac{9.6\%}{0.8} \\ &= 12 \end{aligned}$$

Try It

4) Calculate the nominal interest rate if the periodic interest rate is **2.0875%** per quarter.

Solution

$$\begin{aligned}j &= i \times m \\&= 2.0875\% \times 4 \\&= 8.35\% \text{ compounded quarterly}\end{aligned}$$

Section 1.1 Exercises

1. Calculate the periodic interest rate if the nominal interest rate is **7.5%** compounded monthly.

Solution

0.625%

2. Calculate the compounding frequency for a nominal interest rate of **9.6%** if the periodic interest rate is **0.8%**.

Solution

12

3. Calculate the nominal interest rate if the periodic interest rate is **2.0875%** per quarter.

Solution

8.35%

4. Lori hears her banker state, “We will nominally charge you **10.68%** on your loan, which works out to **0.89%** of your principal every time we charge you interest.” What is her compounding frequency?

Solution

12

5. You just received your monthly MasterCard statement and note at the bottom of the form that interest is charged at **11.9355%** compounded daily. What interest rate is charged to your credit card each day?

Solution

0.0327%

6. Calculate the total number of compoundings for an investment at **3.5%** compounded monthly for four years and seven months.

Solution

55

7. Calculate the total number of compoundings for an investment at **4.2%** compounded quarterly for **39** months.

Solution

13

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1.2 CALCULATING THE FUTURE VALUE

Learning Objectives

- Calculate future value for compound interest

Formula & Symbol Hub

Symbols Used

- f or EFF = Effective interest rate
- FV = Future value or maturity value
- i = Periodic interest rate
- j or I/Y = Nominal interest rate per year
- m or C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of compound periods for the term
- PV = Present value of principal

Formulas Used

- Formula 1.1 – **Total Number of Compounds**

$$n = m \times \text{time in years}$$

- **Formula 1.2 – Periodic Interest Rate**

$$i = \frac{j}{m}$$

- **Formula 1.3 – Future Value**

$$FV = PV \times (1 + i)^n$$

Introduction

Your company has an employee assistance plan through which employees can borrow funds at **12%** compounded semi-annually, much like a loan from a bank, then repay the money at their convenience. An employee who borrowed \$4,000 two years ago has been unable to repay the loan until today. As the human resources manager in charge of the assistance plan, you must tell him the exact amount he needs to pay to return his balance to zero. How do you do this?

In the above scenario, you need to tell the employee the future value of their loan. The **future value** (FV) is the accumulated value or maturity value of a loan or an investment at the end of the term of the loan or investment. The future value includes the principal of the loan or investment plus all of the interest the loan or investment has accumulated over the term. The principal of the loan or investment is called the **present value** (PV). The present value is the amount of money borrowed for a loan or the amount of money invested for an investment at the start of the term.

Calculating the Future Value

Continuing with the scenario described above, suppose \$4000 was borrowed two years ago at **12%** compounded semi-annually. At the end of the two years, the borrower will owe two years of compound interest **in addition** to the original principal of \$4,000. The amount of money borrowed for the loan is the present value, so $PV = \$4,000$. The compounding frequency is semi-annually, or twice per year, which makes the periodic interest rate $i = \frac{j}{m} = \frac{12\%}{2} = 6\%$. This means that every six months, the borrower accumulates 6% of current principal in interest charges. The total number of compoundings is $n = m \times t = 2 \times 2 = 4$, which means that interest will be calculated and added four times over the term of the loan.

At the end of the first six months (one compounding period) of the loan, the current balance or future value of the loan is

$$\begin{aligned}\text{Balance after first compounding period} &= \text{Principal} + \text{Interest} \\ FV &= PV + i \times PV \\ &= 4,000 + 0.06 \times 4,000 \\ &= \$4,240\end{aligned}$$

In the above calculation, note that the equation $FV = PV + i \times PV$ can be factored and rewritten as $FV = PV(1 + i)$, which can be used to calculate the future value for each compounding period.

The above process will continue for each of the compounding periods: 6% of the current balance will be added to the balance at the end of each compounding period. For the second compounding period, the present value is the future value from the previous compounding period and the future value is the balance on the loan at the end of the second compounding period.

$$\begin{aligned}FV_{\text{after two compounding periods}} &= PV \times (1 + i) \\ &= 4,240 \times (1 + 0.06) \\ &= \$4,494.40\end{aligned}$$

Recall that the $PV = \$4,240$ was the result of the future value calculation of the first compounding period. In fact, $4,240 = 4000 \times (1 + 0.06)$, which can be substituted into the previous calculation.

$$\begin{aligned}FV_{\text{after two compounding periods}} &= PV \times (1 + i) \times (1 + i) \\ &= 4,000 \times (1 + 0.06) \times (1 + 0.06) \\ &= \$4,494.40\end{aligned}$$

The equation $FV = PV \times (1 + i) \times (1 + i)$ can be simplified to the following

$$FV = PV \times (1 + i) \times (1 + i) = PV \times (1 + i)^2$$

Continuing the calculations for each of the four compounding periods results in the following.

Compounding Period	Present Value at the Beginning of the Compounding Period	Future Value at the End the Compounding Period
1	\$4,000	$4,000 \times (1 + 0.06) = \$4,240$
2	\$4,240	$4,240 \times (1 + 0.06) = 4,000 \times (1 + 0.06) \times (1 + 0.06)$ $= 4,000 \times (1 + 0.06)^2$ $= \$4,494.40$
3	\$4,494.40	$4,494.40 \times (1 + 0.06) = 4,000 \times (1 + 0.06) \times (1 + 0.06) \times (1 + 0.06)$ $= 4,000 \times (1 + 0.06)^3$ $= \$4,764.06$
4	\$4,764.06	$4,764.06 \times (1 + 0.06) = 4,000 \times (1 + 0.06) \times (1 + 0.06) \times (1 + 0.06) \times (1 + 0.06)$ $= 4,000 \times (1 + 0.06)^4$ $= \$5,059.91$

Do you notice a pattern in these calculations? With the first compounding period, the formula has only one $(1 + i)$, with two compounding periods, the formula has two $(1 + i)$'s in the form of $(1 + i)^2$, with three compounding periods, the formula has three $(1 + i)$'s in the form $(1 + i)^3$, and so on. In general, the exponent of $(1 + i)$ equals the number of the compounding period.

1.3 The Future Value Formula for Compound Interest

$$FV = PV \times (1 + i)^n$$

FV is the Future Value: the future value includes the principal plus all of the interest accumulated over the term.

PV is the Present Value or Principal: the present value is the starting amount upon which compound interest is calculated.

i is the Periodic Interest Rate: the periodic interest rate is the interest rate per compounding period:

$i = \frac{j}{m}$ where j is the nominal interest rate and m is the compounding frequency.

n is the Total Number of Compounding Periods Over the Term: $n = m \times t$ where m is the compounding frequency and t is the length of the term in years.

The amount of interest (I) accumulated by a loan or investment is

$$I = FV - PV$$

Example 1.2.1

If you invested \$5,000 for 10 years at 9% compounded quarterly, how much money would you have? What is the interest earned during the term?

Solution

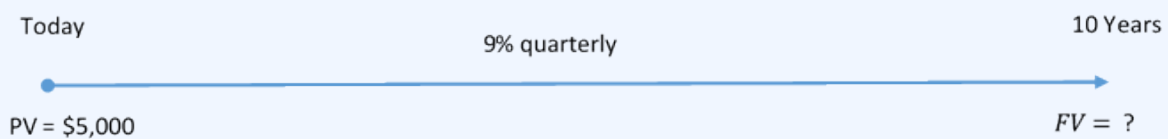


Figure 1.2.1

Image Description

The image shows a thin, straight blue line with an arrowhead at its right end, representing an investment timeline. The left side has the label “Today” above it and $PV = \$5,000$ below. At the line’s midpoint is a label reading 9% quarterly. The far right end of the line has labels “10 Years” above and $FV = ?$ below.

The timeline for the investment is below.

Step 1: Write what you get from the question.

$$\begin{aligned} PV &= \$5,000 \\ j &= 9\% \\ m &= 4 \\ t &= 10 \text{ years} \end{aligned}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned}
 i &= \frac{j}{m} \\
 &= \frac{9\%}{4} \\
 &= 2.25\% \\
 &= 0.0225
 \end{aligned}$$

Step 3: Calculate the total number of compoundings.

$$\begin{aligned}
 n &= m \times t \\
 &= 4 \times 10 \\
 &= 40
 \end{aligned}$$

Step 4: Calculate the future value.

$$\begin{aligned}
 FV &= PV \times (1 + i)^n \\
 &= 5,000 \times (1 + 0.0225)^{40} \\
 &= 5,000 \times (1.0225)^{40} \\
 &= \$12,175.94
 \end{aligned}$$

Step 5: Calculate the interest earned.

$$\begin{aligned}
 I &= FV - PV \\
 &= 12,175.94 - 5,000 \\
 &= \$7,175.94
 \end{aligned}$$

Step 6: Write as a statement.

After 10 years, the principal grows to **\$12,175.94**, which includes your **\$5,000** principal and **\$7,175.94** of interest.

Using a Financial Calculator

A financial calculator, like the TI BAII Plus, has built-in functions to solve compound interest problems. These functions use the “time value of money” buttons on the calculator. Throughout the remainder of this book, we will focus on using the financial calculator to solve problems involving compound interest, instead of using the formulas as illustrated above.

Using the TI BAII Plus Calculator to Find the Future Value for Compound Interest

The time value of money buttons are located in the ***TVM*** row (the third row from the top) of the calculator. The five buttons located on the third row of the calculator are five of the seven variables required for time value of money calculations. This row's buttons are different in color from the rest of the buttons on the keypad. The other two variables are in a secondary menu above the ***I/Y*** key and are accessed by pressing **2nd *I/Y***.

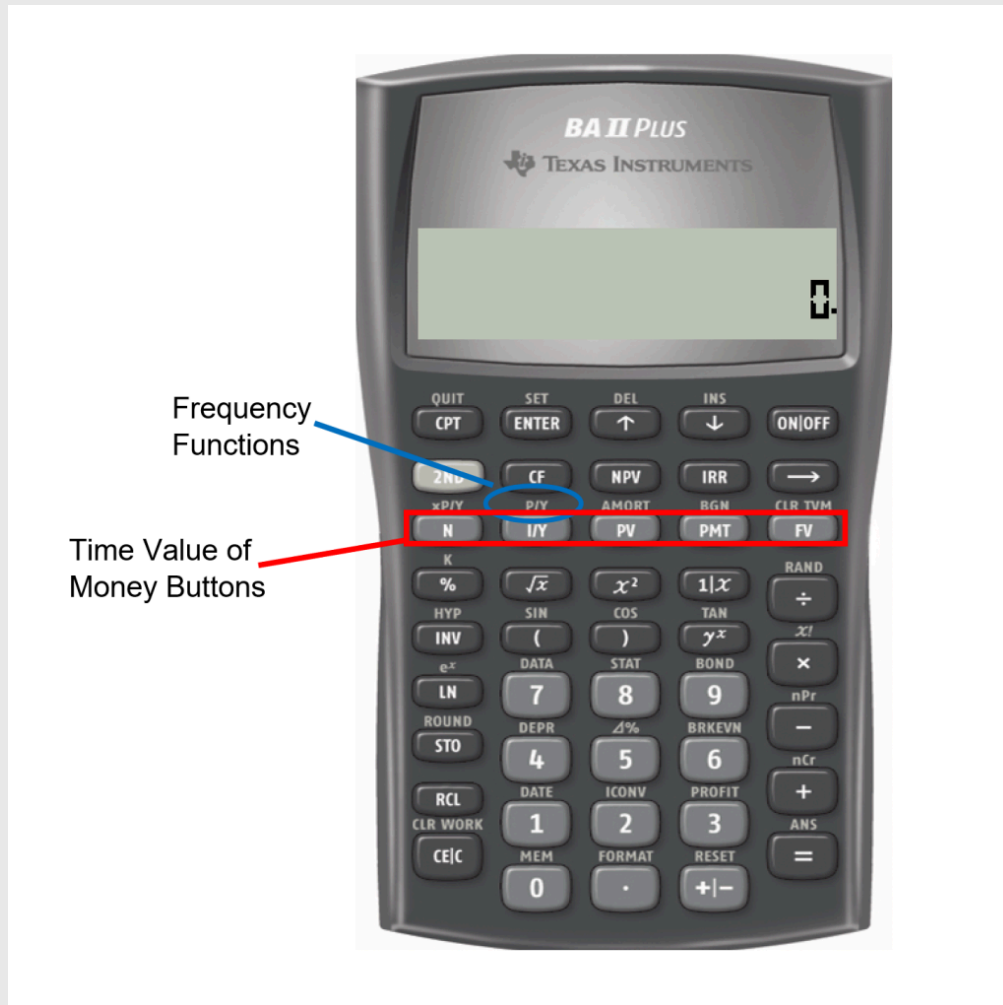


Figure 1.2.2

Image Description

Image of a Texas Instruments BA II Plus calculator. The calculator has a display screen at the top, followed by several rows of buttons below it. Two specific groups of buttons are highlighted and labeled: “Frequency Functions” and “Time Value of Money Buttons”.

The “Frequency Functions” buttons are indicated with blue arrows and include the following buttons:

- CPT
- 2ND

The “Time Value of Money Buttons” are enclosed within a red rectangle and include the following buttons:

- N
- I/Y
- PV
- PMT
- FV

Altogether, there are seven variables required to complete time value of money calculations. Note that P/Y and C/Y are not main button keys in the **TVM** row. The P/Y and C/Y variables are located in the secondary function accessed by pressing **2nd I/Y**.

Variable	Meaning
N	Total number of compounding periods. This is the same value as n in the future value formula. $N = \text{time in years} \times \text{compounding frequency}$
I/Y	Interest rate per year (i.e. the nominal interest rate). The interest rate is entered in percent form (without the % sign). For example, 5% is entered as 5.
PV	Present value or principal.
PMT	Periodic annuity payment. For compound interest only calculations, $PMT = 0$. (Note: in later chapters you will learn about annuities where PMT will not be 0.)
FV	Future value or maturity value.
P/Y	Payments frequency for annuity payment. For compound interest only calculations, P/Y is set to the same value as C/Y . (Note: in later chapters you will learn about annuities where P/Y will be set to the frequency of the payments.)
C/Y	Compounding frequency. This is the value of m .

To enter values into the calculator:

- For the main button keys in the **TVM** row (i.e. N , I/Y , PV , PMT , FV), enter the number first and then press the corresponding button.
 - For example, to enter $N = 34$, enter 34 on the calculator and then press N .
- For P/Y and C/Y , press **2nd I/Y**. At the P/Y screen, enter the value for P/Y and then press **ENTER**. Press the down arrow to access the C/Y screen. At the C/Y screen,

enter the value for C/Y and then press **ENTER**. Press **2nd QUIT** (the **CPT** button) to exit the menu.

- For example, to enter $P/Y = 4$ and $C/Y = 4$, press **2nd I/Y**. At the P/Y screen, enter **4** and press **ENTER**. Press the down arrow. At the C/Y screen, enter **4** and press **ENTER**. Press **2nd QUIT** to exit.

After all of the known quantities are loaded into the calculator, press **CPT** and then FV to solve for the future value.

Notes

1. The calculator automatically sets C/Y equal to whatever is entered for P/Y . That is, if you enter **4** for P/Y , the calculator will set $C/Y = 4$. This is what we need to complete compound interest only calculations. When we learn about annuities in later chapters, we will have instances where P/Y and C/Y will need to be different.
2. Your calculator has permanent memory. Once you enter data into any of the time value buttons it is permanently stored until
 - You override it by entering another piece of data and pressing the button; or
 - You clear the memory of the time value buttons by pressing **2nd CLR TVM** before proceeding with another question; or
 - The reset button on the back of the calculator is pressed.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=31#oembed-1>

Video: “[BA II Plus Calculator – Compound Interest \(Present & Future Values\)](#)” by [Joshua Emmanuel](#) [6:57] is licensed under the [Standard YouTube License](#). Transcript and closed captions available on YouTube.

Cash Flow Convention Signs

An investment and a loan are very different. An investment earns interest and the principal increases over time. A loan is charged interest but is usually paid off through payments, resulting in the principal decreasing over time. Notice that nowhere on the calculator is there a button to enter this critical piece of information. How does the calculator distinguish between the two? You must apply a principle called the **cash flow sign convention** to the PV , PMT , and FV buttons.

1. If you have money leaving your possession and going somewhere else (such as being put into an investment or making a payment against a loan), you must enter the number as a **NEGATIVE** number.
2. If you have money coming into your possession and you are receiving it (such as borrowing money from the bank or having an investment mature and pay out to you), you must enter the number as a **POSITIVE** number.

When doing financial calculations it is important to “be somebody” in the transaction. In any compound interest scenario, two parties are always involved—somebody is investing and somebody is borrowing. From this standpoint, think about whether the money leaves you or comes at you. This will help you place the correct sign in front of the PV , PMT , and FV when using your calculator. Who you are will not change the numbers of the transaction, just the cash flow sign convention.

1. If you borrow money from your friend and then pay it back, the initial loan is received by you and hence entered as a positive PV for you. As you pay back the loan, money leaves you and therefore the FV is

negative for you.

2. Taking the other side of the coin and being your friend, the loaning of money is a negative PV for him. When you repay the loan, your friend receives it and therefore results in a positive FV for him.

	Sign of FV	Sign of PV
Investments	When you receive your matured investment at the end of the term this is considered as a cash-inflow for you and the future value should be entered as a positive amount.	When money is invested (paid-out), this amount is considered as a cash-outflow and this amount has to be entered as a negative number for PV .
Loans	When the loan is repaid at the end of the term this is considered as a cash-outflow for you and the future value should be entered as a negative amount.	When money is received (loaned), this amount is considered as a cash-inflow and this amount has to be entered as a positive number for PV .

Things to Watch Out For

Notice that the PV and FV always have opposite signs.

When you compute solutions on the BAII Plus calculator, one of the most common error messages displayed is “Error 5.” This error indicates that the cash flow sign convention has been used in a manner that is financially impossible. Some examples of these financial impossibilities include loans with no repayment or investments that never pay out. In these cases, the PV and FV have been incorrectly set to the same cash flow sign.

Example 1.2.2

If you invested \$5,000 for 10 years at 9% compounded quarterly, how much money would you have?

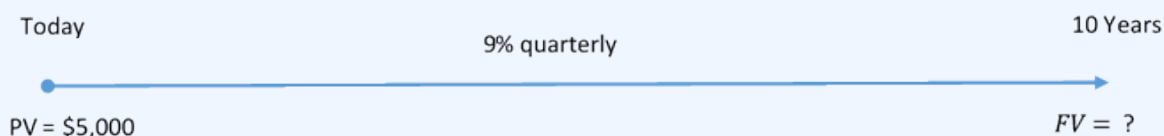
Solution

Figure 1.2.3

Image Description

The image shows a thin, straight blue line with an arrowhead at its right end, representing an investment timeline. The left side has the label “Today” above it and $PV = \$5,000$ below. At the line’s midpoint is a label reading **9% quarterly**. The far right end of the line has labels “10 Years” above and $FV = ?$ below.

The timeline for the investment is below.

N	$4 \times 10 = 40$
PV	$-5,000$
FV	$?$
PMT	0
I/Y	9
P/Y	4
C/Y	4

$$FV = \$12,175.94$$

Try It

1) Find the future value if **\$53,000** is invested at **6%** compounded monthly for **4** years and **3** months. How much interest did the investment make?

Solution

N	$12 \times \frac{51}{12} = 51$
PV	$-53,000$
FV	?
PMT	0
I/Y	6
P/Y	12
C/Y	12

$$FV = \$68,351.02$$

$$\begin{aligned}
 I &= FV - PV \\
 &= 68,351.02 - 53,000 \\
 &= \$15,351.02
 \end{aligned}$$

Future Value Calculations with Variable Changes

What happens if a variable such as the nominal interest rate, compounding frequency, or even the principal changes somewhere in the middle of the transaction? When any variable changes, you must break the timeline into separate time fragments at the point of the change. To arrive at the solution, you need to work from left to right one time segment at a time using the future value formula.

1. Read and understand the problem. Identify the present value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the length of the time segment in years.
2. Starting with the present value in the first time segment (starting on the left), solve for the future value.
3. Let the future value calculated in the previous step become the present value for the next step. If the principal changes, adjust the new present value accordingly.
4. Calculate the future value of the next time segment.
5. Repeat the previous steps until you obtain the final future value from the final time segment.

HOW TO

Transforming the future value from one time segment into the present value of the next time segment does not require re-entering the computed value. Instead, apply the following technique.

1. Load the calculator with all known compound interest variables for the first time segment.
2. Compute the future value at the end of the segment.
3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the \pm key, and then store the unrounded number back into the present value button by pressing **PV**. Change the **N**, **I/Y**, and **C/Y** as required for the next segment.
4. Return to step **2** for each time segment until you have completed all time segments.

Example 1.2.3

Five years ago Coast Appliances was supposed to upgrade one of its facilities at a quoted cost of \$48,000. The upgrade was not completed, so Coast Appliances delayed the purchase until now. The construction company that provided the quote indicates that prices rose 6% compounded quarterly for the first 1.5 years, 7% compounded semi-annually for the following 2.5 years, and 7.5% compounded monthly for the final year. If Coast Appliances wants to perform the upgrade today, what amount of money does it need?

Solution

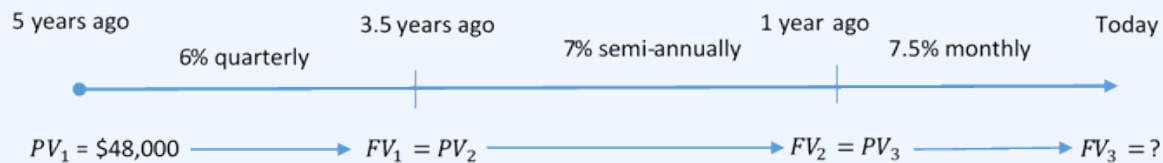


Figure 1.2.4

Image Description

Timeline: $PV_1 = \$48,000$ at 5 years ago moving to 3.5 years ago at 6% quarterly to become FV_1 . At 3.5 years ago, FV_1 becomes PV_2 which moves to 1 year ago at 7% semi-annually to become FV_2 . At 1 year ago, FV_2 becomes PV_3 which moves to Today at 7.5% monthly to become $FV_3 = ?$.

The timeline below shows the original quote from five years ago until today.

Step 1: Calculate the future value at the end of the first segment.

N	$4 \times 1.5 = 6$
PV	$-48,000$
FV	?
PMT	0
I/Y	6
P/Y	4
C/Y	4

$$FV_1 = \$52,485.27667....$$

Step 2: Calculate the future value at the end of the second segment. The future value from the first segment becomes the present value for the second segment:

$$FV_1 = \$52,485.27667.... = PV_2.$$

N	$2 \times 2.5 = 5$
PV	$-52,485.27667...$
FV	?
PMT	0
I/Y	7
P/Y	2
C/Y	2

$$FV_2 = \$62,336.04435....$$

Step 3: Calculate the future value at the end of the third segment. The future value from the second segment becomes the present value for the third segment:

$$FV_2 = \$62,336.04435.... = PV_3.$$

N	$12 \times 1 = 12$
PV	$-62,336.04435\dots$
FV	?
PMT	0
I/Y	7.5
P/Y	12
C/Y	12

$$FV_3 = \$67,175.35$$

Step 4: Write as a statement.

Coast Appliance requires **\$67,175.35** to perform the upgrade today.

Example 1.2.4

Two years ago Lorelei placed **\$2,000** into an investment earning **6%** compounded monthly. Today she makes a deposit to the investment in the amount of **\$1,500**. What is the maturity value of her investment three years from now?

Solution

The timeline for the investment is below.

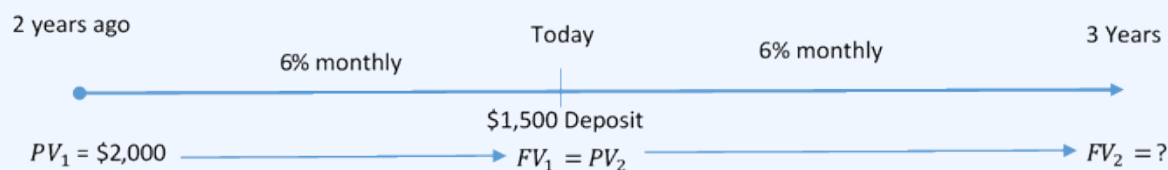


Figure 1.2.5

Image Description

Timeline: At 2 years ago, $FV_1 = \$2,000$ moves to Today at 6% monthly to become FV_1 . At Today, there is a \$1,500 deposit. At Today, FV_1 becomes PV_2 which moves to 3 years at 6% monthly to become $FV_2 = ?$.

Step 1: Calculate the future value at the end of the first segment.

N	$12 \times 2 = 24$
PV	$-2,000$
FV	$?$
PMT	0
I/Y	6
P/Y	12
C/Y	12

$$FV_1 = \$2,254.31955....$$

Step 2: Add the \$1,500 deposit to FV_1 to get the present value for the second segment.

$$PV_2 = 2,254.31955... + 1,500 = \$3,754.31955...$$

Step 3: Calculate the future value at the end of the second segment.

N	$12 \times 3 = 36$
PV	$-3,754.31955\dots$
FV	?
PMT	0
I/Y	6
P/Y	12
C/Y	12

$$FV_2 = \$4,492.72$$

Step 4: Write as a statement.

Three years from now Lorelei will have **\$4,492.72**. This represents **\$3,500** of principal and **\$992.72** of compound interest.

Try It

2) Find the future value if **\$24,500** is invested at **4.1%** compounded annually for **4** years; then **5.15%** compounded quarterly for **1** year, **9** months; then **5.35%** compounded monthly for **1** year, **3** months.

Solution

N	4	7	15
<i>PV</i>	−24,500	−28,771.930...	−31,467.335...
<i>FV</i>	28,771.930...	31,467.335...	33,638.67
<i>PMT</i>	0	0	0
<i>I/Y</i>	4.1	5.15	5.35
<i>P/Y</i>	1	4	12
<i>C/Y</i>	1	4	12

$$FV = \$33,638.67$$

Try It

3) Nirdosh borrowed \$9,3004.25 years ago at 6.35% compounded semi-annually. The interest rate changed to 6.5% compounded quarterly 1.75 years ago. What amount of money today is required to pay off this loan?

Solution

N	5	7
PV	9,300	10,873.1489...
FV	-10,873.1489...	12,171.92
PMT	0	0
I/Y	6.35	6.5
P/Y	2	4
C/Y	2	4

$$FV = \$12,171.92$$

Section 1.2 Exercises

1. What is the future value of a \$7,500 investment at 8% compounded quarterly for 3 years?

Solution

\$9,511.81

2. Ruth borrowed \$53,000 at 6% compounded quarterly 4 years and 3 months ago. How much does Ruth have to pay today to clear the loan? How much interest did Ruth pay?

Solution

$$FV = \$68,351.02, I = \$15,351.02$$

3. You invest **\$19,000** in a savings account at **5.75%** compounded semi-annually. How much is in your account **6.5** years from now?

Solution

$$\$27,465.13$$

4. You invest **\$3,750** in an investment that earns **4.75%** compounded annually for the first three years and then **5.5%** compounded semi-annually for the next two years. How much money do you have at the end of the five years? How much interest did your investment earn?

Solution

$$FV = \$4,804.20, I = \$1,054.20$$

5. James took out a **\$11,375** loan today. He will pay interest at **7.5%** compounded monthly for the first two years and nine months, and then **8.25%** compounded quarterly for the next three years and three months. How much money does James owe six years from now?

Solution

$$\$18,218.24$$

6. You are planning a **16**—day African safari to Rwanda to catch a rare glimpse of the **700** remaining mountain gorillas in the world. The estimated cost of this once-in-a-lifetime safari is **\$15,000** including the tour, permits, lodging, and airfare. Upon your graduation from college, your parents have promised you a **\$10,000** graduation gift. You intend to save this money for five years in a long-term investment earning **8.3%** compounded semi-annually. If the cost of the trip will be about the same, will you have enough money five years from now to pay for your trip?

Solution

$$\$15,017.33; \text{yes}$$

7. Your investment of **\$9,000** that you started six years ago earned **7.3%** compounded quarterly for the first $3\frac{1}{4}$ years, followed by **8.2%** compounded monthly after that. How much interest has your investment earned so far?

Solution**\$5,254.44**

8. What is the maturity value of your **\$7,800** investment after three years if the interest rate was **5%** compounded semiannually for the first two years, **6%** compounded quarterly for the last year, and **$2\frac{1}{2}$** years after the initial investment you made a deposit of **\$1,200**? How much interest is earned?

Solution **$FV = \$10,374.33$, $I = \$1,374.33$**

9. Cristy borrowed **\$4,800** from a family friend **$2\frac{1}{2}$** years ago at **7%** compounded annually for the first year and **8%** compounded semi-annually thereafter. She made a payment **$1\frac{1}{2}$** years into the loan for **\$2,500**. How much should Cristy pay today to clear her loan?

Solution**\$3,073.30**

10. You invested **\$5,000** 10 years ago and made two further deposits of **\$5,000** each after four years and eight years. Your investment earned **9.2%** compounded quarterly for the first two years, **8.75%** compounded monthly for the next six years, and **9.8%** compounded semi-annually for the remaining years. As of today, how much interest has your investment earned?

Solution**\$11,888.46**

11. Suppose you placed **\$10,000** into each of the following investments. Rank the maturity values after five years from highest to lowest.
- 8%** compounded annually for two years followed by **6%** compounded semi-annually.
 - 8%** compounded semi-annually for two years followed by **6%** compounded annually.

- c. 8% compounded monthly for two years followed by 6% compounded quarterly.
- d. 8% compounded semi-annually for two years followed by 6% compounded monthly.

Solution

c is largest with $FV = \$14,023.26$, d is second largest with $FV = \$13,999.47$, b is third largest with $FV = \$13,933.20$, a is smallest with $FV = \$13,927.43$

12. You made the following three investments:

- \$8,000 into a five-year fixed rate investment earning 5.65% compounded quarterly.
- \$6,500 into a five-year variable rate investment earning 4.83% compounded semi-annually for the first $2\frac{1}{2}$ years and 6.5% compounded monthly for the remainder.
- \$4,000 into a five-year variable rate investment earning 4.75% compounded monthly for the first two years and 5.9% compounded quarterly thereafter, with a \$4,000 deposit made 21 months into the investment.

What is the total maturity value of all three investments after the five years, and how much interest is earned?

Solution

\$29,270.56, \$6,770.56

1.3 CALCULATING THE PRESENT VALUE

Learning Objectives

- Calculate present value for compound interest

Formula & Symbol Hub

Symbols Used

- f or EFF = Effective interest rate
- FV = Future value or maturity value
- i = Periodic interest rate
- j or I/Y = Nominal interest rate per year
- m or C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of compound periods for the term
- PV = Present value of principal

Formulas Used

- Formula 1.1 – **Total Number of Compounds**

$$n = m \times \text{time in years}$$

- Formula 1.2 – Periodic Interest Rate

$$i = \frac{j}{m}$$

- Formula 1.3 – Future Value

$$FV = PV \times (1 + i)^n$$

- Formula 1.4 – Present Value

$$PV = FV \times (1 + i)^{-n}$$

Introduction

The principal of the loan or investment is called the **present value** (PV). The present value is the amount of money borrowed for a loan or the amount of money invested for an investment at the start of the term. The present value is the amount at some earlier point in time than when the future value is known, and so excludes the future interest.

1.4 The Present Value Formula

By solving for PV in the future value formula from the previous section, the **present value** for compound interest is

$$FV = PV \times (1 + i)^n$$

PV is the Present Value: the present value is the starting amount upon which compound interest is calculated.

FV is the Future Value: the future value includes the principal plus all of the interest accumulated over the term.

i is the **Periodic Interest Rate**: the periodic interest rate is the interest rate per compounding period:

$$i = \frac{j}{m} \text{ where } j \text{ is the nominal interest rate and } m \text{ is the compounding frequency.}$$

n is the **Total Number of Compounding Periods Over the Term**: $n = m \times t$ where m is the compounding frequency and t is the length of the term in years.

Example 1.3.1

Castillo's Warehouse will need to purchase a new forklift for its warehouse operations three years from now, when its new warehouse facility becomes operational. If the price of the new forklift is \$38,000 and Castillo's can invest its money at 7.25% compounded monthly, how much money should it put aside today to achieve its goal?

Solution

The timeline for the investment is shown below.

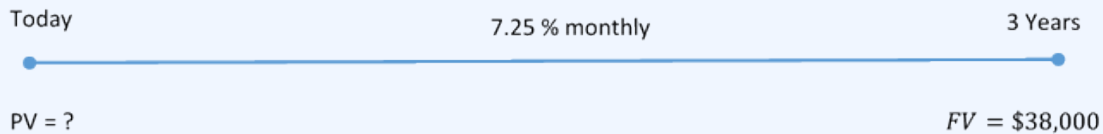


Figure 1.3.1

Image Description

The image depicts an investment timeline, labeled with values as described throughout the following example.

Step 1: Write what you get from the question.

$$FV = \$38,000$$

$$j = 7.25\%$$

$$m = 12$$

$$t = 3 \text{ years}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned}
 i &= \frac{j}{m} \\
 &= \frac{7.25\%}{12} \\
 &= 0.60416...\% \\
 &= 0.0060416...
 \end{aligned}$$

Step 3: Calculate the total number of compoundings.

$$\begin{aligned}
 n &= m \times t \\
 &= 12 \times 3 \\
 &= 36
 \end{aligned}$$

Step 4: Calculate the present value.

$$\begin{aligned}
 PV &= FV \times (1 + i)^{-n} \\
 &= 38,000 \times (1 + 0.0060416...)^{-36} \\
 &= \$30,592.06
 \end{aligned}$$

Step 5: Write as a statement.

If Castillo's Warehouse places **\$30,592.06** into the investment, it will earn enough interest to grow to **\$38,000** three years from now to purchase the forklift.

Using a Financial Calculator

As in the previous section, a financial calculator can be used to solve for the present value in compound interest problems. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is *PV* instead of *FV*. You must still load the other six variables into the calculator and apply the cash flow sign convention carefully.

Using the TI BAII Plus Calculator to Find the Present Value for Compound Interest

The time value of money buttons are located in the ***TVM*** row (the third row from the top) of the calculator. The five buttons located on the third row of the calculator are five of the seven variables required for time value of money calculations. This row's buttons are different in colour from the rest of the buttons on the keypad. The other two variables are in a secondary menu above the ***I/Y*** key and are accessed by pressing **2nd *I/Y***.

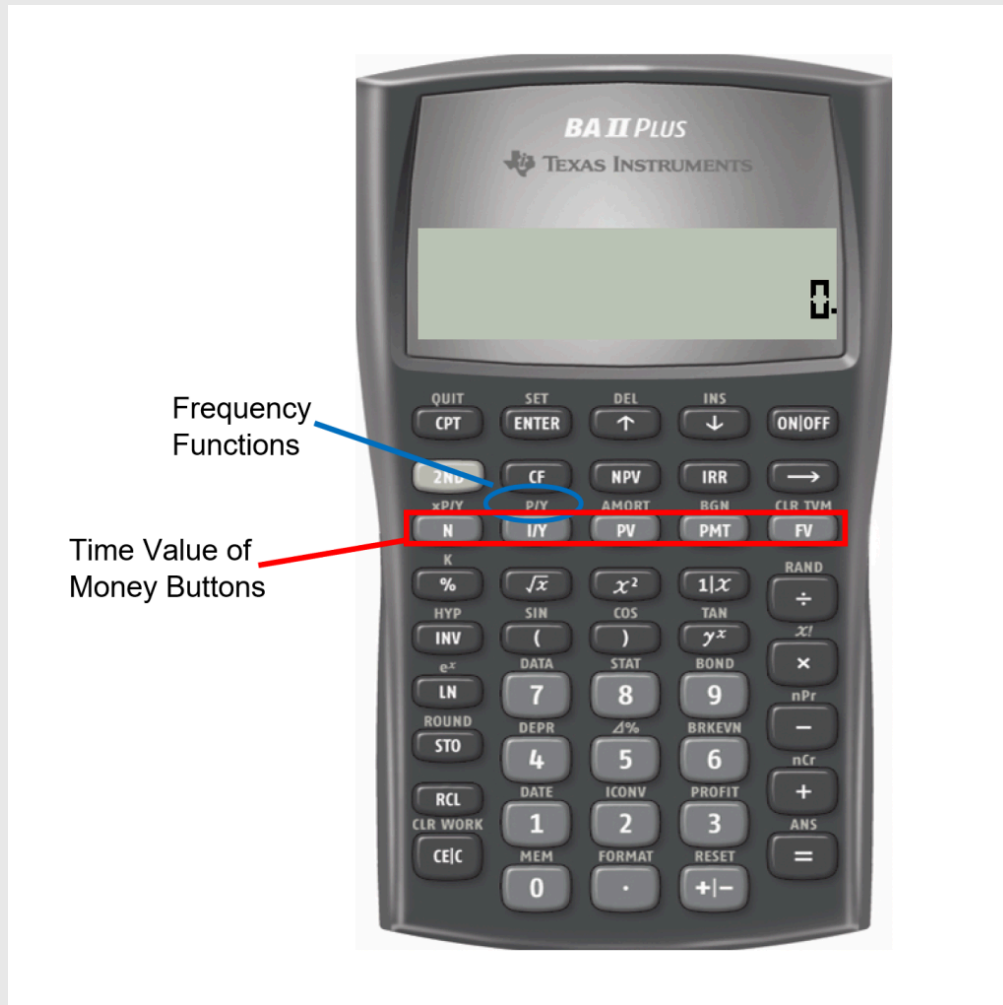


Figure 1.3.2

Image Description

The image shows a Texas Instruments BA II Plus financial calculator with a focus on two sets of buttons. The casing is dark grey, and it has a rectangular display screen at the top.

The labels indicate two groups of buttons:

- The “Frequency Functions” are indicated by a blue outline as the secondary functions of the buttons labeled “N,” “I/Y,” “PV,” “PMT,” and “FV.” The Frequency Functions available are “xP/Y,” “P/Y,” “AMORT,” “BGN,” and “CLR TVM”.
- The “Time Value of Money Buttons” are surrounded by a red outline, including the buttons labeled “N,” “I/Y,” “PV,” “PMT,” and “FV.”

Altogether, there are seven variables required to complete time value of money calculations. Note

that P/Y and C/Y are not main button keys in the **TVM** row. The P/Y and C/Y variables are located in the secondary function accessed by pressing **2nd I/Y**.

Variable	Meaning
N	Total number of compounding periods. This is the same value as n in the future value/present value formulas: $N = \text{time in years} \times \text{compounding frequency}$
I/Y	Interest rate per year (i.e. the nominal interest rate). The interest rate is entered in percent form (without the % sign). For example, 5% is entered as 5.
PV	Present value or principal.
PMT	Periodic annuity payment. For compound interest only calculations, $PMT=0$. (Note: in later chapters you will learn about annuities where PMT will not be 0.)
FV	Future value or maturity value.
P/Y	Payment frequency for annuity payment. For compound interest only calculations, P/Y is set to the same value as C/Y . (Note: in later chapters you will learn about annuities where P/Y will be set to the frequency of the payments.)
C/Y	Compounding frequency. This is the value of m .

To enter values into the calculator:

- For the main button keys in the **TVM** row (i.e. N , I/Y , PV , PMT , FV), enter the number first and then press the corresponding button.
 - For example, to enter $N = 34$, enter 34 on the calculator and then press N .
- For P/Y and C/Y , press **2nd I/Y**. At the P/Y screen, enter the value for P/Y and then press **ENTER**. Press the down arrow to access the C/Y screen. At the C/Y screen, enter the value for C/Y and then press **ENTER**. Press **2nd QUIT** (the **CPT** button) to exit the menu.
 - For example, to enter $P/Y = 4$ and $C/Y = 4$, press **2nd I/Y**. At the P/Y screen, enter 4 and press **ENTER**. Press the down arrow. At the C/Y screen, enter 4 and press **ENTER**. Press **2nd QUIT** to exit.

After all of the known quantities are loaded into the calculator, press **CPT** and then PV to solve for the present value.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=35#oembed-1>

Video: [Compound Interest \(Present and Future Values\)](#) by Joshua Emmanuel [6:56] (Transcript Available).

Example 1.3.2

Castillo's Warehouse will need to purchase a new forklift for its warehouse operations three years from now, when its new warehouse facility becomes operational. If the price of the new forklift is **\$38,000** and Castillo's can invest its money at **7.25%** compounded monthly, how much money should it put aside today to achieve its goal?

Solution

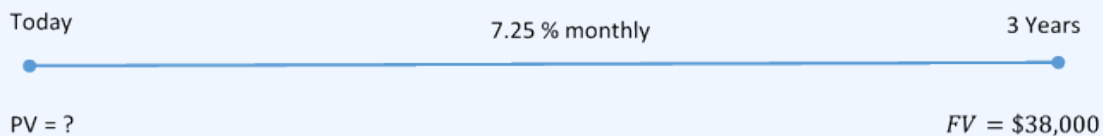


Figure 1.3.3

Image Description

The image depicts a horizontal blue line with small circles at its endpoints representing an investment timeline. The timeline is labeled with values that correspond to the values discussed in the following example.

The timeline for the investment is shown below.

N	$12 \times 3 = 36$
PV	?
FV	38,000
PMT	0
I/Y	7.25
P/Y	12
C/Y	12

$$PV = \$30,592.06$$

Try It

1) A debt of **\$37,000** is owed **21** months from today. If prevailing interest rates are **6.55%** compounded quarterly, what amount should the creditor be willing to accept today?

Solution

N	$4 \times 1.75 = 7$
PV	?
FV	$-37,000$
PMT	0
I/Y	6.55
P/Y	4
C/Y	4

$$PV = \$33,023.56$$

Present Value Calculations with Variable Changes

Addressing variable changes in present value calculations follows the same techniques as future value calculations discussed in the previous section. You must break the timeline into separate time segments, each of which involves its own calculations. Solving for the unknown PV at the left of the timeline means you must start at the right of the timeline. You must work from right to left, one time segment at a time using the formula for PV each time. Note that the present value for one time segment becomes the future value for the next time segment to the left.

1. Read and understand the problem. Identify the future value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the length of the time segment in years.
2. Starting with the future value in the last time segment (starting on the right), solve for the present value.
3. Let the present value calculated in the previous step become the future value for the next segment to the left. If the principal changes, adjust the future value accordingly.
4. Calculate the present value of the next time segment.
5. Repeat the previous steps until you obtain the final present value from the leftmost time segment.

HOW TO

To use your calculator efficiently in working through multiple time segments, follow a procedure similar to that for future value.

1. Load the calculator with all known compound interest variables for the last time segment (on the right).
2. Compute the present value at the beginning of the segment.
3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the \pm key, and then store the unrounded number back into the future value button by pressing **FV**. Change the **N**, **I/Y**, and **C/Y** as required for the next segment.
4. Return to step **2** for each time segment until you have completed all time segments.

Example 1.3.3

Sebastien needs to have **\$9,200** saved up three years from now. The investment he is considering pays **7%** compounded semi-annually, **8%** compounded quarterly, and **9%** compounded monthly in successive years. To achieve his goal, how much money does he need to place into the investment today?

Solution

The timeline shows today through to the future value three years from now

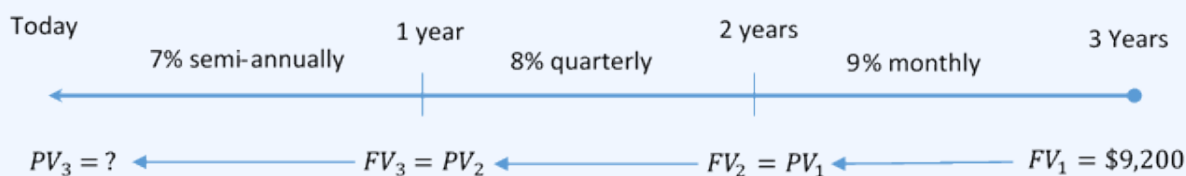


Figure 1.3.4

Image Description

The image depicts a horizontal blue line with small circles at its endpoints representing an investment timeline. The timeline is labeled with values that correspond to the values discussed in the rest of the example.

Step 1: Calculate the present value at the start of the last segment on the right.

N	$12 \times 1 = 12$
PV	?
FV	9,200
PMT	0
I/Y	9
P/Y	12
C/Y	12

$$PV_1 = \$8,410.991\dots$$

Step 2: Calculate the present value at the start of the second segment on the right. The present value from the first step becomes the future value for the second step:

$$PV_1 = \$8,410.991\dots = FV_2.$$

N	$4 \times 1 = 4$
PV	?
FV	8,410.991...
PMT	0
I/Y	8
P/Y	4
C/Y	4

$$PV_2 = \$7,770.4555...$$

Step 3: Calculate the present value at the end of the third segment on the right. The present value from the second step becomes the future value for the third step:

$$PV_2 = \$7,770.455... = FV_3.$$

N	$2 \times 1 = 2$
PV	?
FV	7,770.455..
PMT	0
I/Y	7
P/Y	2
C/Y	2

$$PV_3 = \$7,253.80$$

Step 4: Write as a statement.

Sebastien needs to place **\$7,253.80** into the investment today to have **\$9,200** three years from now.

Try It

2) For the first **4.5** years, a loan was charged interest at **4.5%** compounded semi-annually. For the next **4** years, the rate was **3.25%** compounded annually. If the maturity value was **\$45,839.05** at the end of the **8.5** years, what was the principal of the loan?

Solution

N	4	9
PV	40,334.378...	33,014.56
FV	−45,839.05	−40,334.378...
PMT	0	0
I/Y	3.25	4.5
P/Y	1	2
C/Y	1	2

$$PV = \$33,014.56$$

Section 1.3 Exercises

1. A loan is repaid with **\$14,000**. If the loan was taken out **14** years ago at **9%** compounded semi-annually, how much money was borrowed? How much interest was paid on the loan?

Solution

$$PV = \$4,081.99, I = \$9,9180.01$$

2. In **9** years and **3** months, you want to have **\$97,000** in your savings account. How much money must you invest today if the savings account earns **6%** compounded monthly?

Solution

$$\$55,762.07$$

3. Eight and a half years ago, Tom took out a loan. The interest rate on the loan was **4.5%** compounded semi-annually for the first four and half years and **3.25%** compounded annually for the last four years. Tom repaid the loan today with a payment of **\$45,839.05**. How much money did Tom borrow? How much interest did Tom pay?

Solution

$$PV = \$33,014.56, I = \$12,824.49$$

4. George wants to invest some money today. In **6.5** years, George wants to have **\$7,223.83** in his investment. The investment earns **8.05%** compounded semi-annually for the first **2** years and **6** months, then **7.95%** compounded quarterly for **1** year and **3** months, and then **7.8%** compounded monthly for **2** years and **9** months. How much money does George need to invest?

Solution

$$\$4,340$$

5. Dovetail Industries needs to save **\$1,000,000** for new production machinery that it expects will be needed six years from today. If money can earn **8.35%** compounded monthly, how much money should Dovetail invest today?

Solution

$$\$606,976.63$$

6. A debt of **\$37,000** is owed **21** months from today. If prevailing interest rates are **6.55%** compounded quarterly, what amount should the creditor be willing to accept

today?

Solution

\$33,023.56

7. Rene wants to invest a lump sum of money today to make a **\$35,000** down payment on a new home in five years. If he can place his money in an investment that will earn **4.53%** compounded quarterly in the first two years followed by **4.76%** compounded monthly for the remaining years, how much money does he need to invest today?

Solution

\$27,736.24

8. In August 2004, Google Inc. made its initial stock offering. The value of the shares grew to **\$531.99** by July 2011. What was the original value of a share in August 2004 if the stock has grown at a rate of **26.8104%** compounded monthly?

Solution

\$85

9. If a three-year and seven-month investment earned **\$8,879.17** of interest at **3.95%** compounded monthly, what amount was originally placed into the investment?

Solution

\$58,499.97

10. A lottery ticket advertises a **\$1** million prize. However, the fine print indicates that the winning amount will be paid out on the following schedule: **\$250,000** today, **\$250,000** one year from now, and **\$100,000** per year thereafter. If money can earn **9%** compounded annually, what is the value of the prize today?

Solution

\$836,206.54

- **Option A:** **\$520,000** today plus **\$500,000** one year from now.
- **Option B:** **\$200,000** today, **\$250,000** six months from now, and **\$600,000** 15 months from now.
- **Option C:** **\$70,000** today, **\$200,000** in six months, then four quarterly payments of **\$200,000** starting six months later. Your company is selling some

real estate and has received three potential offers:

11. Prevailing interest rates are expected to be **6.75%** compounded semi-annually in the next year, followed by **6.85%** compounded quarterly afterwards. Rank the three offers from best to worst based on their values today.

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1.4 EQUIVALENT PAYMENTS

Learning Objectives

- Calculate equivalent payments that replace another payment or a set of payments

Formula & Symbol Hub

Symbols Used

- f or EFF = Effective interest rate
- FV = Future value or maturity value
- i = Periodic interest rate
- j or I/Y = Nominal interest rate per year
- m or C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of compound periods for the term
- PV = Present value of principal

Formulas Used

- Formula 1.1 – **Total Number of Compounds**

$$n = m \times \text{time in years}$$

- **Formula 1.2 – Periodic Interest Rate**

$$i = \frac{j}{m}$$

- **Formula 1.3 – Future Value**

$$FV = PV \times (1 + i)^n$$

- **Formula 1.4 – Present Value**

$$PV = FV \times (1 + i)^{-n}$$

Introduction

Unforeseen events and circumstances can force you to rearrange your financial commitments. When this happens, the new deal has to be fair to all parties concerned.

For example, suppose you and your co-workers rely on your company's annual Christmas bonuses. However, the CEO just announced that because of tough economic conditions no one will get a bonus this year. You already earmarked that money to pay a \$5,000 debt due next week. You want to be financially responsible, but you cannot possibly make the payment. Before picking up the phone to call your creditor, you need to determine what course of action you should pursue. You need to make alternative arrangements that leave your creditor in the same financial position as the original agreement did. You saw in the previous section that if you were going to make an early payment, the payment should be reduced by an agreed-upon discount rate. In this case, though, you are going to make a late payment, so you must grant the creditor interest. Thus, if you propose paying the debt six months late and your creditor agrees to 9% compounded monthly as a fair rate, then you owe \$5,229.26.

This concept applies to all aspects of your personal and professional life. Except for gifts, personal debts to friends or family members should bear some interest. Everyone should be financially fair to each other. A business must be willing to work with its clients in the event they need to alter an agreement. A company that is inflexible tends to find itself writing off bad debt or pursuing unpleasant and sometimes expensive legal action. This section explores the concept of equivalent payment streams, which involves equating two or more alternative financial streams to ensure that neither party is penalized by any choice.

The Fundamental Concept of Equivalency

The **fundamental concept of equivalency** states that two or more payment streams are equal to each other if they have the **same economic value** on the **same focal date**. As illustrated in the figure below, the two alternative financial streams are equivalent if the total of Payment Stream 1 is equal to the total of Payment Stream 2 on the same focal date.

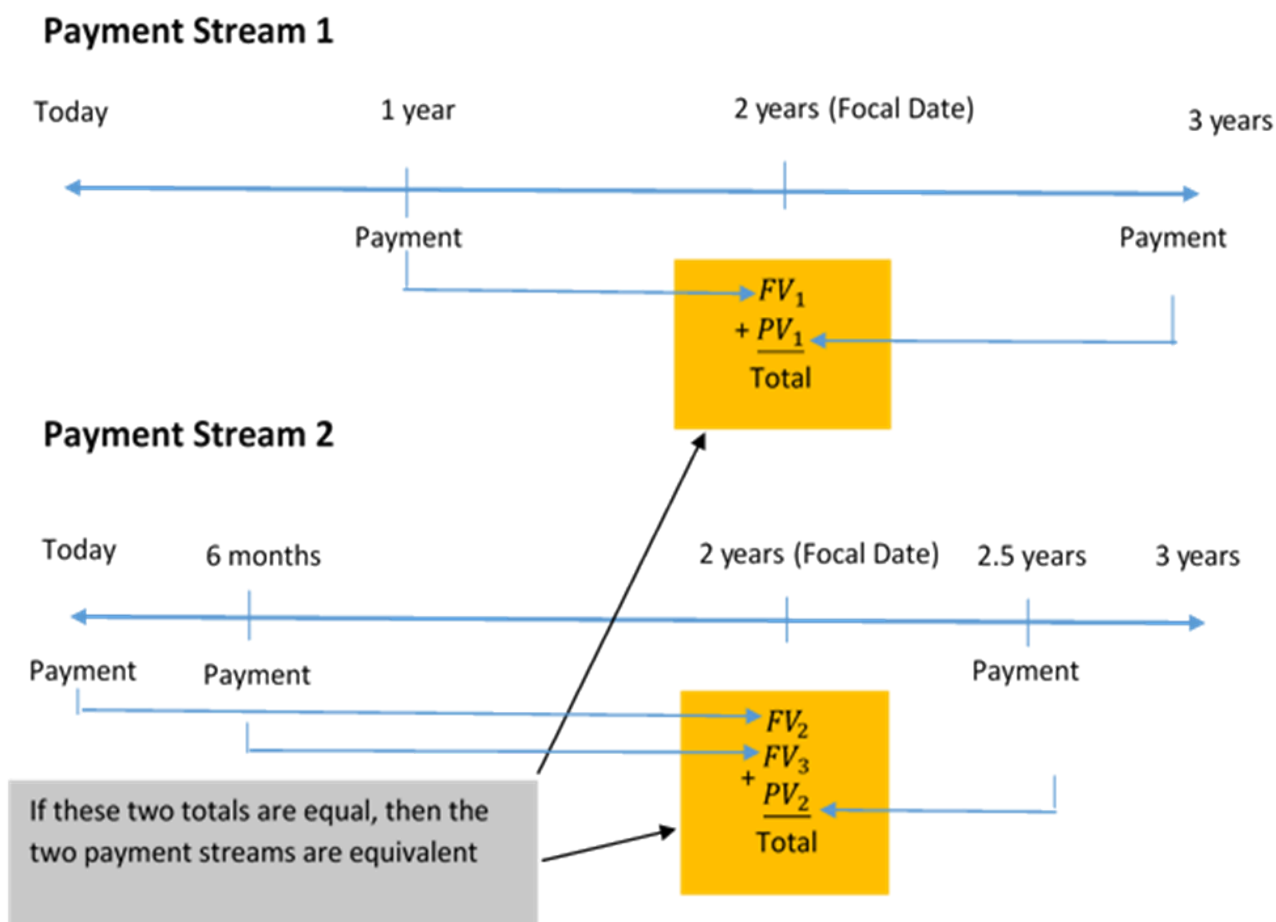


Figure 1.4.1

Image Description

This image consists of two timeline diagrams, each representing a different payment stream. Both timelines feature blue arrows denoting the direction of time from left to right.

The top section of the image illustrates a single payment stream:

- Vertical markers indicate four points in time: Today, 1 Year, 2 Years (Focal Date), and 3 Years. The markers at 1 Year and 3 Years each have a downward curved arrow, labeled “Payment”, that connects to a yellow box.
- The yellow box contains the expression $FV_1 + PV_1 = \text{Total}$

The bottom section of the image illustrates another payment stream:

- Vertical markers indicate five points in time: Today, 6 Months, 2 Years (Focal Date), 2.5 Years, 3 Years. The markers at Today, 6 Months, and 2.5 Years have downward curved arrows labeled “Payment” that connect to a yellow box.
- The yellow box contains the expression $FV_2 + FV_3 + PV_2 = \text{Total}$

At the bottom left corner of the image, there is a gray box containing the following text: “If these two totals are equal, then the two payment streams are equivalent.”

Replacing a Series of Payment with a Single Equivalent Payment

In this first situation, you are going to replace a series of payments with one single payment that is equivalent to the original series of payments. To find the single payment, follow these steps.

1. Draw a timeline, **in chronological order**, to show the original series of payments along with the times the payments were supposed to be made. Clearly indicate the dates of the payments and the payment amounts.
2. For a single equivalent payment calculation, choose the date of the single payment as the **focal date**.
3. Move all of the payments from the original series of payments to the focal date using a future value or present value calculation.
 - If the payment is **moving forward** in time to get to the focal date, the payment is a present value and you need to find its future value at the focal date.
 - If the payment is **moving backward** in time to get to the focal date, the payment is a future value and you need to find its present value at the focal date.
4. The single equivalent payment is the sum of the payments **at the focal date**.

Example 1.4.1

Assume you owe \$1,000 today and \$1,000 one year from now. You find yourself unable to make that payment today, so you indicate to your creditor that you want to make both payments six months from now instead. Prevailing interest rates are at 6% compounded semi-annually. What single payment six months from now (the new payment stream) is equivalent to the two payments (the original payment stream)?

Solution

Step 1: Draw the timeline.

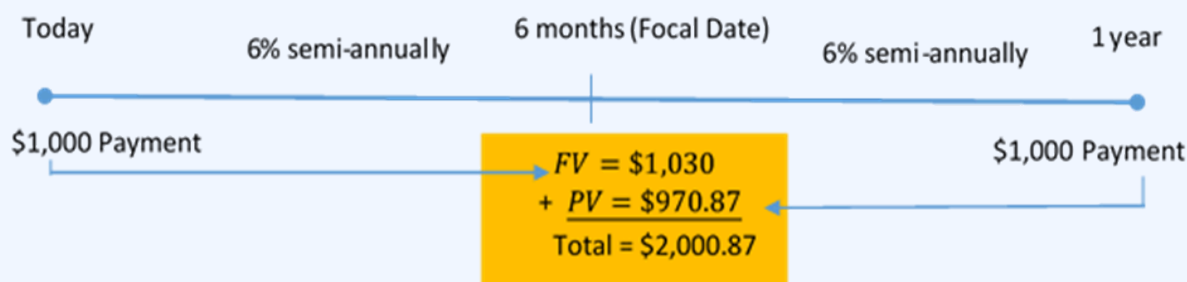


Figure 1.4.2

Image Description

This image shows a horizontal timeline with two endpoints and a yellow box placed in the middle of the timeline. The timeline is divided into two 6-month durations, with each half labeled **6% semi-annually** and the 6 month marker labeled as the focal date.

The left side of the yellow box has a blue arrow labeled **\$1,000 Payment** pointing to the right side. The text indicated by the arrow is: $FV = \$1,030$

The right side of the yellow box has a blue arrow labeled **\$1,000 Payment** pointing to the left side. The text indicated by the arrow is: $+ PV = \$970.87$

Below the FV and PV values being added together is their sum: $Total = \$2,000.87$

Step 2: Choose six months from today as the focal date.

Step 3: Calculate the future value of the \$1,000 payment today at the focal date.

Because this payment must move forward in time to get to the focal date, the \$1,000 is a present value and its value at the focal date is a future value. This payment must move 6 months (from today to 6 months from today) to get to the focal date, so the time for the future value calculation is 6 months or 0.5 year.

N	$2 \times 0.5 = 1$
PV	-1,000
FV	?
PMT	0
I/Y	6
P/Y	2
C/Y	2

$$FV = \$1,030$$

Step 4: Calculate the present value of the \$1,000 payment one year from now at the focal date.

Because this payment must move backward in time to get to the focal date, the \$1,000 is a future value and its value at the focal date is a present value. This payment must move 6 months (from 1 year from today to 6 months from today) to get to the focal date, so the time for the present value calculation is 6 months or 0.5 year.

N	$2 \times 0.5 = 1$
PV	?
FV	-1,000
PMT	0
I/Y	6
P/Y	2
C/Y	2

$$PV = \$970.87$$

Step 5: The single equivalent payment is the sum of the payments at the focal date.

$$\begin{aligned}\text{Single Payment} &= 1,030 + 970.87 \\ &= \$2,000.87\end{aligned}$$

Step 6: Write as a statement.

A single payment of **\$2,000.87** six months from now is equivalent to the original payments of **\$1,000** today and **\$1,000** one year from now.



Paths to Success

A properly drawn timeline, in chronological order, can help you see what you need to do, where the payments need to go (in relation to the focal date), and whether you need to do a future value or present value calculation to move the payments to the focal date.

Any date can be chosen as the focal date. But in a single equivalent payment calculation you need to know the value of the single payment at the focal date, so it makes sense to choose the time of the single payment as the focal date.

To find the time a payment has to move from its current date to the focal date, use the following technique.

- If the date of the payment and the focal date are on the **same side** as today, subtract the times. For example, if the original payment is at **2** years from today and the focal date is at **5** years from today, the time the original payment has to move to get to the focal date is $5 - 2 = 3$ years.
- If the date of the payment and the focal date are on **opposite sides** of today, add the times. For example, if the original payment is at **3** years ago and the focal date is **4** years from today, the time the original payment has to move to get to the focal date is $3 + 4 = 7$ years.
- Treat today as time **0**.

Example 1.4.2

Johnson's Garden Centre has recently been unprofitable and concludes that it cannot make two debt payments of \$4,500 due today and another \$6,300 due in three months. After discussions between Johnson's Garden Centre and its creditor, the two parties agree that both payments could be made nine months from today, with interest at 8.5% compounded monthly. What total payment does Johnson's Garden Centre need nine months from now to clear its debt?

Solution

Step 1: Draw the timeline.

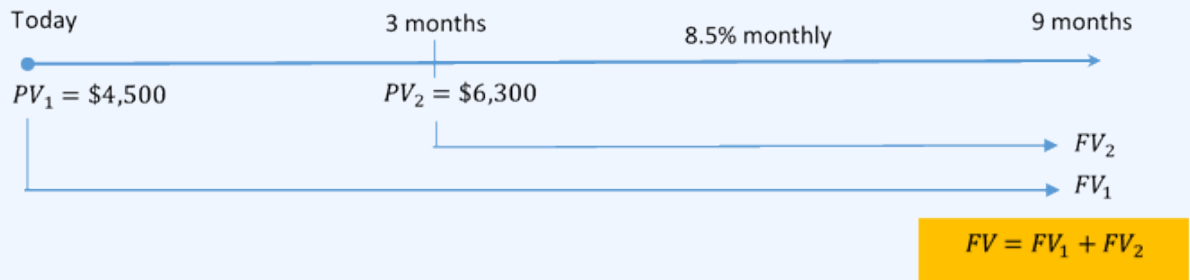


Figure 1.4.3

Image Description

The image depicts a financial timeline with three horizontal arrows labeled with different time periods. The arrows are described as follows:

- The first arrow is labeled with time intervals from left to right (Today, 3 Months, 9 Months) as well as the interest rate **8.5% monthly**
- There is an arrow indicating FV_1 that starts from the Today marker and is labeled with the equation $PV_1 = \$4,500$.
- Another arrow indicating FV_2 starts from the 3 month point and is labeled with the equation $PV_2 = \$6,300$.

On the right side of the image, there is a rectangle with a yellow background containing the equation $FV = FV_1 + FV_2$.

Step 2: Choose nine months from today as the focal date.

Step 3: Calculate the future value of the \$4,500 payment today at the focal date.

Because this payment must move forward in time to get to the focal date, the \$4,500 is a present value and its value at the focal date is a future value. This payment must move 9 months (from today to 9 months from today) to get to the focal date, so the time for the future value calculation is 9 months or 0.75 year.

N	$12 \times 0.75 = 9$
PV	-4,500
FV	?
PMT	0
I/Y	8.5
P/Y	12
C/Y	12

$$FV_1 = \$4,795.14$$

Step 4: Calculate the future value of the \$6,300 payment today at the focal date.

Because this payment must move forward in time to get to the focal date, the \$6,300 is a present value and its value at the focal date is a future value. This payment must move 6 months (from 3 months from today to 9 months from today) to get to the focal date, so the time for the future value calculation is 6 months or 0.5 year.

N	$12 \times 0.5 = 6$
PV	$-6,300$
FV	?
PMT	0
I/Y	8.5
P/Y	12
C/Y	12

$$FV_2 = \$6,572.54$$

Step 5: The single equivalent payment is the sum of the payments at the focal date.

$$\begin{aligned}\text{Single Payment} &= 4,795.14 + 6,572.54 \\ &= \$11,367.68\end{aligned}$$

Step 6: Write as a statement.

A single payment of **\$11,367.68** nine months from now is equivalent to the original payments of **\$4,500** today and **\$6,300** three months from now.

Try It

1) What single payment in **2** years would replace payments of **\$5,000** due **1.5** years ago, **\$2,500** due in **1** year, and **\$1,000** due in **4** years if money earns **3.5%** compounded quarterly?

Solution

Using **2** years as the focal date:

N	14	4	8
PV	−5,000	−2,500	932.68
FV	5,648.59	2,588.66	−1,000
PMT	0	0	0
I/Y	3.5	3.5	3.5
P/Y	4	4	4
C/Y	4	4	4

$$\begin{aligned}\text{Single Payment} &= 5,648.59 + 2,588.66 + 932.68 \\ &= \$9,169.93\end{aligned}$$

HOW TO

Replace a Series of Payments with an Equivalent Series of Payments

In this situation, you are going to replace a series of payments with another series of payments that is equivalent to the original series of payments. To find the new series payment, follow these steps.

1. Draw a timeline, **in chronological order**, to show the original series of payments and the new series of payments, along with the times the payments were supposed to be made. Clearly indicate the dates of the payments and the payment amounts. To keep track of which payment belongs to which payment stream, draw the timeline in different colours or draw multiple timelines, one for each payment stream.
2. Select a **focal date**.

3. Move all of the payments (from both the original payment stream and the new payment stream) to the focal date using a future value or present value calculation.
 - If the payment is **moving forward** in time to get to the focal date, the payment is a present value and you need to find its future value at the focal date.
 - If the payment is **moving backward** in time to get to the focal date, the payment is a future value and you need to find its present value at the focal date.
4. The sum of the payments in the original payment stream at the focal date equals the sum of the payments in the new payment stream at the focal date. This creates an equation that allows you to solve for the unknown in the new payment stream.

Example 1.4.3

You have three debts to the same creditor: **\$3,000** due today, **\$2,500** due in **2.25** years, and **\$4,250** due in **3** years **11** months. Unable to fulfill this obligation, you arrange with your creditor to make two alternative payments: **\$3,500** in nine months and a second payment due in two years. You agree upon an interest rate of **9.84%** compounded monthly. What is the amount of the second payment?

Solution

Step 1: Draw the timeline.

With two payment streams and multiple amounts all on different dates, visualize two timelines, one for the original payment stream and the other for the new payment stream. Alternatively, draw one timeline with the payment streams in different colours. The unknown payment in the new payment stream is represented by x .

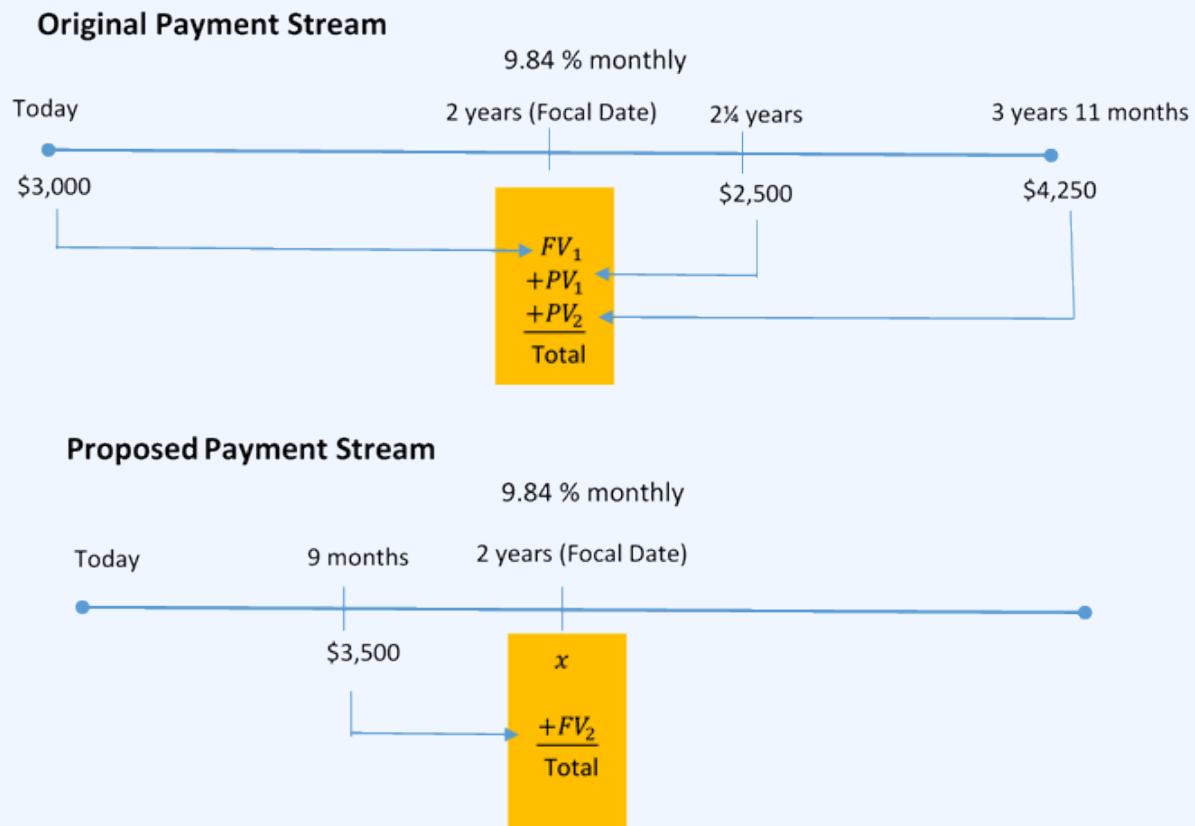


Figure 1.4.4

Image Description

This image consists of two block diagrams. Each diagram represents a payment stream.

The top diagram, labeled “Original Payment Stream” consists of:

- A horizontal timeline labeled divided at the following points: Today, 2 Years (Focal Date), 2.25 Years, 3 Years 11 Months. This line is also labeled with the interest rate **9.84% monthly**.
- A yellow box containing the equation $FV_1 + PV_1 + PV_2 = \text{Total}$
- There is an arrow extending from the Today point on the timeline, labeled **\$3, 000** and pointing to the FV_1 term in the yellow box.
- There is an arrow extending from the 2.25 Years point on the timeline, labeled **\$2, 500** and pointing to the PV_1 term in the yellow box.
- There is an arrow extending from the 3 Years 11 Months point on the timeline, labeled **\$4, 250** and pointing to the PV_2 term in the yellow box.

The bottom diagram, labeled “Proposed Payment Stream”, consists of:

- A horizontal timeline labeled divided at the following points: Today, 9 Months, 2 Years (Focal Date). This line is also labeled with the interest rate **9.84% monthly**.
- A yellow box containing the equation $x + FV_2 = \text{Total}$
- There is an arrow extending from the 9 Months point on the timeline, labeled **\$3,500** and pointing to the yellow box.

Step 2: Choose two years from today as the focal date.

Step 3: Calculate the future value of the \$3,000 payment today (in the original payment stream) at the focal date.

Because this payment must move forward in time to get to the focal date, the **\$3,000** is a present value and its value at the focal date is a future value. This payment must move **2** years (from today to **2** years from today) to get to the focal date, so the time for the future value calculation is **2** years.

N	$12 \times 2 = 24$
PV	$-3,000$
FV	?
PMT	0
I/Y	9.84
P/Y	12
C/Y	12

$$FV_1 = \$3,649.57$$

Step 4: Calculate the present value of the \$2,500 payment 2.25 years from today (in the original payment stream) at the focal date.

Because this payment must move backward in time to get to the focal date, the **\$2,500** is a future value and its value at the focal date is a present value. This payment must move **0.25** years (from **2.25** years from today to **2** years from today) to get to the focal date, so the time for the present value calculation is **0.25** years.

N	$12 \times 0.25 = 3$
PV	?
FV	$-2,500$
PMT	0
I/Y	9.84
P/Y	12
C/Y	12

$$PV_1 = \$2,439.49$$

Step 5: Calculate the present value of the \$4,250 payment 3 years and 11 months from today (in the original payment stream) at the focal date.

Because this payment must move backward in time to get to the focal date, the \$4,250 is a future value and its value at the focal date is a present value. This payment must move 1 year, 11 months (from 3 years, 11 months from today to 2 years from today) to get to the focal date, so the time for the present value calculation is 1 year, 11 months.

N	$12 \times \frac{23}{12} = 23$
PV	?
FV	$-4,250$
PMT	0
I/Y	9.84
P/Y	12
C/Y	12

$$PV_2 = \$3,522.21$$

Step 6: Calculate the future value of the \$3,500 payment 9 months from today (in the new payment stream) at the focal date.

Because this payment must move forward in time to get to the focal date, the **\$3,500** is a present value and its value at the focal date is a future value. This payment must move **1** year, **3** months (from **9** months from today to **2** years from today) to get to the focal date, so the time for the future value calculation is **1** year, **3** months.

N	$12 \times \frac{15}{12} = 15$
PV	$-3,500$
FV	?
PMT	0
I/Y	9.84
P/Y	12
C/Y	12

$$FV_2 = \$3,956.11$$

Step 7: At the focal date, the sum of the payments in the original payment stream equal the sum of the payments in the new payment stream.

$$3,649.57 + 2,439.49 + 3,522.21 = 3,956.11 + x$$

$$9,611.27 = 3,956.11 + x$$

$$9,611.27 - 3,956.11 = x$$

$$\$5,644.16 = x$$

Step 8: Write as a statement.

The amount of the second payment is **\$5,644.16**.

Things to Watch Out For

It does not matter what focal date you choose because two values that are equal when moved

to one date in common will still be equal when both are moved together to another date. But you should simplify your calculations by selecting a focal date corresponding to the date of an unknown variable. It is always easier to move actual amounts of money than it is to move an unknown variable x . For example, in the previous question, suppose you selected today as the focal date (instead of **2** years). Then the x at **2** years would need to be moved to today (the focal date). Regardless of the chosen focal date, the final answer will be the same.

Try It

2) Loan payments of **\$5,000** due two years ago and **\$3,000** due today have not been made. Instead, the payments are rescheduled with two payments of **\$1,500** in **1** year and **2.5** years, and a final payment in **4** years. What is the size of the final payment if money earns **3.7%** compounded semi-annually?

Solution

Using **4** years as the focal date:

N	12	8	6	3
PV	−5,000	−3,000	−1,500	−1,500
FV	6,230.21	3,473.84	1,674.39	1,584.80
PMT	0	0	0	0
I/Y	3.7	3.7	3.7	3.7
P/Y	2	2	2	2
C/Y	2	2	2	2

$$\begin{aligned}
 6,230.21 + 3,473.84 &= 1,674.39 + 1,584.80 + x \\
 9,704.05 &= 3,259.19 + x \\
 9,704.05 - 3,259.19 &= x \\
 \$6,444.86 &= x
 \end{aligned}$$

Example 1.4.4

A \$10,000 loan is to be repaid with three equal payments made in 1 year, 3.5 years and 6 years. If money earns interest at 4.7% compounded quarterly, what is the size of the payments?

Solution

Step 1: Draw the timeline. The \$10,000 from the original payment stream is in black.

The unknown payments in the new payments stream, represented by x , are in red.

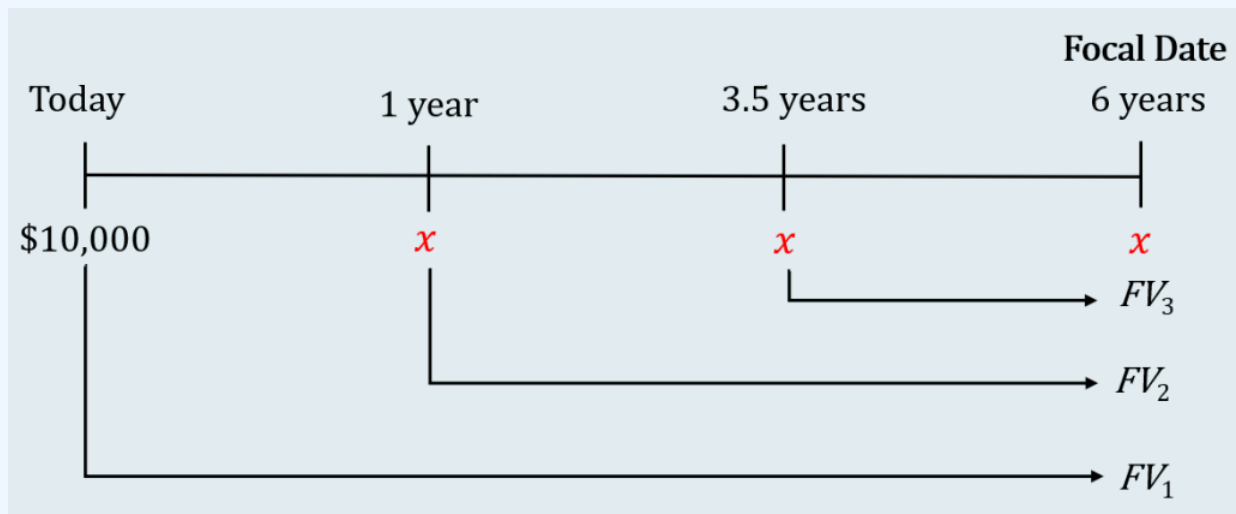


Figure 1.4.5

Image Description

The image shows a financial timeline diagram designed to illustrate the future value of an

investment over a 6-year period. The timeline begins with “Today” and progresses through “1 year,” “3.5 years,” and culminates at “6 years,” labeled as the “Focal Date.”

At the “Today” point, an amount of \$10,000 is indicated. The timeline has three points marked with a red “x” at “1 year,” “3.5 years,” and “6 years.”

From these red “x” points, three arrows denote the projection of future values:

- From the “Today” mark, an arrow points to the right and indicates FV_1 .
- From the “1 year” mark, an arrow points to the right and indicates FV_2 .
- From the “3.5 years” mark, another arrow points to the right and indicates FV_3 .

Step 2: Choose six years from today as the focal date.

Step 3: Calculate the future value of the \$10,000 payment today (in the original payment stream) at the focal date.

Because this payment must move forward in time to get to the focal date, the \$10,000 is a present value and its value at the focal date is a future value. This payment must move 6 years (from today to 6 years from today) to get to the focal date, so the time for the future value calculation is 6 years.

N	$4 \times 6 = 24$
PV	$-10,000$
FV	?
PMT	0
I/Y	4.7
P/Y	4
C/Y	4

$$FV_1 = \$13,236.01$$

Step 4: Calculate the future value of the x at 1 year (in the new payment stream) at the focal date.

Because this payment must move forward in time to get to the focal date, the x is a present value and its value at the focal date is a future value. This payment must move 5 years (from

1 year from today to 6 years from today) to get to the focal date, so the time for the future value calculation is 5 years.

Because x cannot be entered in the calculator, you need to find an expression for the future value of x at the focal date. You can think of the unknown payment x as $1 \times x$. The future value of x is the (future value of 1) $\times x$. You can find the future value of 1 using the calculator by entering 1 for the present value and calculating out the future value.

N	$4 \times 5 = 20$
PV	-1
FV	1.2631...
PMT	0
I/Y	4.7
P/Y	4
C/Y	4

$$FV_2 = 1.2631... \times x$$

Step 5: Calculate the future value of the x at 3.5 year (in the new payment stream) at the focal date.

Because this payment must move forward in time to get to the focal date, the x is a present value and its value at the focal date is a future value. This payment must move 2.5 years (from 3.5 years from today to 6 years from today) to get to the focal date, so the time for the future value calculation is 2.5 years.

Because x cannot be entered in the calculator, you need to find an expression for the future value of x at the focal date. You can think of the unknown payment x as $1 \times x$. The future value of x is the (future value of 1) $\times x$. You can find the future value of 1 using the calculator by entering 1 for the present value and calculating out the future value.

N	$4 \times 2.5 = 10$
PV	-1
FV	1.1239...
PMT	0
I/Y	4.7
P/Y	4
C/Y	4

$$FV_3 = 1.1239... \times x$$

Step 6: At the focal date, the sum of the payments in the original payment stream equal the sum of the payments in the new payment stream.

$$13,236.01 = x + (1.2631...) \times x + (1.1239...) \times x$$

$$13,236.01 = (1 + 1.2631... + 1.1239...) \times x$$

$$13,236.01 = x \times 3.38708...$$

$$\frac{13,236.01}{3.38708...} = x$$

$$\$3,907.78 = x$$

Step 7: Write as a statement.

The payments are \$3,907.78.

Key Takeaways



1. You can choose any of the dates on the timeline in the previous example as the focal date. A different focal date will require different calculations to move the payments to the focal date, but the final answer will be the same.
2. Keep all of the decimal places throughout the calculation, especially for the future or present value expressions for any unknown payments x , to avoid any round-off error in the final answer.
3. To move an unknown payment x to the focal date, find the value of **1** at the focal date and then multiply this value by x .
 - If x moves forward in time to get to the focal date, you need to find the future value of x . To do this, find the future value of **1** and then the value of x at the focal date is **(future value of 1) $\times x$** .
 - If x moves backward in time to get to the focal date, you need to find the present value of x . To do this, find the present value of **1** and then the value of x at the focal date is **(present value of 1) $\times x$** .

Try It

3) Payments of \$570 due 1.5 years ago and \$890 due 6 months ago have not been made. Instead, these payments are to be replaced with two equal payments due 9 months and 2 years from today. If money earns 5% compounded monthly, what is the size of the payments?

Solution

Using 2 years from today as the focal date:

N	42	30	15
PV	−5,70	−890	−1
FV	678.76	1,008.24	1.06435...
PMT	0	0	0
I/Y	5	5	5
P/Y	12	12	12
C/Y	12	12	12

$$678.76 + 1,008.24 = x + (1.06435...) \times x$$

$$1,678 = (1 + 1.06435...) \times x$$

$$1,678 = (2.06435...) \times x$$

$$\frac{1,678}{2.06435...} = x$$

$$\$817.21 = x$$

Section 1.4 Exercises

1. If money earns **6%** compounded monthly, what single payment due in **27** months is equivalent to payments of **\$5,000** today and **\$5,000** in three years?

Solution

\$10,501.28

2. If money earns **8.5%** compounded quarterly, what single payment due in **2** years is equivalent to payments of **\$4,385** due **1** year ago and **\$6,000** due in **4** years?

Solution

\$10,714.59

3. If money earns **6.65%** compounded monthly, what single payment due in **16** months is equivalent to payments of **\$16,000** due today, **\$8,000** due in **10** months, and **\$19,000** due in **33** months?

Solution

\$43,045.16

4. A **\$35,000** loan taken out today is to be repaid with a **\$15,000** payment due in **6** months and a second unknown payment due in **2** years. If money earns **5.95%** compounded semi-annually, find the size of the unknown payment due in **2** years.

Solution

\$22,975.60

5. A **\$51,000** loan borrowed a today is to be repaid with a **\$15,000** payment due in **3** months, a **\$10,000** payment due in **1** year, and an unknown payment due in **1.5** years. If money earns **6.32%** compounded quarterly, what is the size of the unknown payment?

Solution

\$29,488.31

6. A **\$44,000** loan borrowed today is repaid with two equal payments due in **6** months and **15** months. If money earns **5.55%** compounded monthly, what is the size of the payments.

Solution

\$23,087.18

7. If money earns **8.88%** compounded monthly, what two equal payments, one due today and one due in **18** months would be equivalent to two equal payments of **\$2,500**, one due **1** year ago and one due in **6** months?

Solution

\$2,731.26

8. Payments of **\$38,000** due **1.5** years ago, **\$17,000** due **6** months ago and **\$45,000** due in **1** year are replaced with three equal payments due today, in **18** months and in **2** years. If money earns **9.5%** compounded semi-annually, find the size of the payments.

Solution

\$37,952.43

9. Payments of **\$3,750** due **2.5** years ago, **\$2,400** due in **3.25** years and **\$1,950** in **5** years are replaced with three equal payments due in **1** year, **2** years and **4.5** years. If money earns **10.75%** compounded quarterly, find the size of the payments.

Solution

\$3,322.29

10. A winning lottery ticket offers the following two options:

- Option A: A single payment of **\$1,000,000** today.
- Option B: **\$250,000** today followed by annual payments of **\$300,000** for the next three years.

If money can earn **9%** compounded annually, which option should the winner select? How much better is that option in current dollars?

Solution

Option B by **\$9,388.40**

11. Darwin is a young entrepreneur trying to keep his business afloat. He has missed two payments to a creditor. The first was for **\$3,485** seven months ago and the second was for **\$5,320** last month. Darwin has had discussions with his creditor, who is willing to accept **\$4,000** one month from now and a second payment in full six months from now.

If the agreed upon interest rate is **7.35%** compounded monthly, what is the amount of the second payment?

Solution

\$5,201.24

12. James is a debt collector. One of his clients has asked him to collect an outstanding debt from one of its customers. The customer has failed to pay three amounts: **\$1,600** 18 months ago, **\$2,300** nine months ago, and **\$5,100** three months ago. In discussions with the customer, James finds she desires to clear up this situation and proposes a payment of **\$1,000** today, **\$4,000** nine months from now, and a final payment two years from now. The client normally charges **16.5%** compounded quarterly on all outstanding debts. What is the amount of the third payment?

Solution

\$7,465.59

13. Working in the accounting department, Ariel has noticed that a customer is having trouble paying its bills. The customer has missed a payment of **\$2,980** two years ago, **\$5,150** 14 months ago, and **\$4,140** four months ago. The customer proposes making two equal payments in six months and in 18 months. If the agreed-upon interest rate is **6.89%** compounded monthly, what are the amounts of each payment?

Solution

\$7,083.74

14. A **\$30,000** loan at **4.9%** compounded semi-annually is to be repaid with four equal semi-annual payments. The first payment is one year after the loan. Calculate the amount of each payment.

Solution

\$8,160.07

15. The Ontario Labour Relations Board is reviewing a human resource complaint. At the time of filing, a construction worker indicated that her employer failed to pay her monthly wages of **\$3,400**, **\$3,200**, **\$3,600**, and **\$3,475** starting four months prior. It has taken the Ontario Labour Relations Board nine months since the time of filing to gather the needed information and make a judgment in favour of the complainant. If the standard interest rate used in its judgment is **9%** compounded monthly, what amount is

awarded to the construction worker?

Solution

\$14,900.01

16. Larry is a financial adviser helping a client save up **\$100,000** in five years' time. The client has the financial means to pursue either of the following two alternatives:
- Option A: Make three equal deposits due today, in **2** years, and **4** years.
 - Option B: Make four equal deposits due today, in **1** year, in **3** years, and **4½** years.

If Larry can invest the funds at **9%** compounded semi-annually, which option is in the best interests of the client? In current dollars, how much better for the client is that recommended option?

Solution

Option A by **\$735.14**

17. Yi-Fang is the store owner of a franchise. In flipping through her records, she notices the following debts to the same supplier: **\$2,389.56** due eight months ago, **\$3,478.34** due six months ago, **\$1,694.32** due four months ago, **\$6,497.98** due two months ago, **\$4,611.03** due today, **\$5,784.39** due in two months, and **\$5,177.44** due in four months. She would like to clear all of these debts with a single payment next month. If the supplier charges **18.1%** compounded monthly on overdue balances and provides a credit of **9%** compounded monthly on early payments, calculate the amount of the payment.

Solution

\$30,703.92

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1.5 CALCULATING THE INTEREST RATE

Learning Objectives

- Calculate the interest rate of a loan or investment

Formula & Symbol Hub

Symbols Used

- f or EFF = Effective interest rate
- FV = Future value or maturity value
- i = Periodic interest rate
- j or I/Y = Nominal interest rate per year
- m or C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of compound periods for the term
- PV = Present value of principal

Formulas Used

- Formula 1.1 – **Total Number of Compounds**

$$n = m \times \text{time in years}$$

- **Formula 1.2 – Periodic Interest Rate**

$$i = \frac{j}{m}$$

- **Formula 1.3 – Future Value**

$$FV = PV \times (1 + i)^n$$

- **Formula 1.4 – Present Value**

$$PV = FV \times (1 + i)^{-n}$$

Introduction

You need to calculate the nominal interest rate under many circumstances including (but not limited to) the following.

- Determining the interest rate on a single payment loan.
- Understanding what interest rate is needed to achieve a future savings goal.
- Calculating the interest rate that generated a specific amount of interest.
- Finding a fixed interest rate that is equivalent to a variable interest rate.

Using a Financial Calculator

Although it is possible to find the interest rate by using the future value or present value formulas, it is much more practical to use a financial calculator. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is I/Y (the nominal interest rate). You must still load the other six variables into the calculator and apply the cash flow sign conventions carefully.

Using the TI BAII Plus Calculator to Find the Interest Rate for Compound Interest

Enter values for the known variables (PV , FV , N , PMT , P/Y and C/Y) following the steps below and paying close attention to the cash flow sign convention for PV and FV .

- For the main button keys in the TVM row (i.e. N , I/Y , PV , PMT , FV), enter the number first and then press the corresponding button.
 - For example, to enter $N = 34$, enter **34** on the calculator and then press N .
- For P/Y and C/Y , press **2nd** I/Y . At the P/Y screen, enter the value for P/Y and then press **ENTER**. Press the down arrow to access the C/Y screen. At the C/Y screen, enter the value for C/Y and then press **ENTER**. Press **2nd** **QUIT** (the **CPT** button) to exit the menu.
 - For example, to enter $P/Y = 4$ and $C/Y = 4$, press **2nd** I/Y . At the P/Y screen, enter **4** and press **ENTER**. Press the down arrow. At the C/Y screen, enter **4** and press **ENTER**. Press **2nd** **QUIT** to exit.

After all of the known quantities are loaded into the calculator, press **CPT** and then I/Y to solve for the interest rate.

Things to Watch Out For

When entering both PV and FV into the calculator, ensure proper application of cash flow sign convention to PV and FV . One number must be negative and the other must be positive. An **ERROR** message will appear on the calculator display if PV and FV are entered with the same signs (i.e. both are negative or both are positive).

Example 1.5.1

When Sandra borrowed \$7, 100 from Sanchez, she agreed to reimburse him \$8, 615.19 three years from now including interest compounded quarterly. What nominal quarterly compounded rate of interest is being charged?

Solution

The timeline for the loan is shown below.

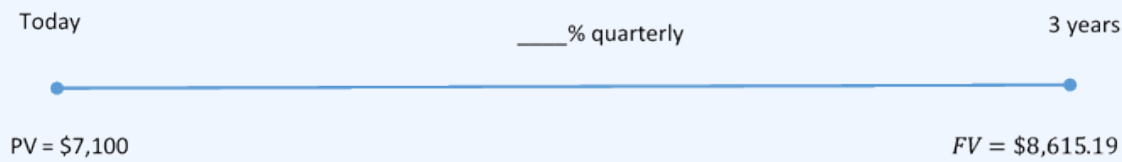


Figure 1.5.1

Image Description

The image shows a horizontal, light blue line representing a loan timeline with labels corresponding to values discussed in the rest of the example.

N	$4 \times 3 = 12$
PV	7, 100
FV	−8, 615.19
PMT	0
I/Y	?
P/Y	4
C/Y	4

$I/Y = 6.5\%$

Sanchez is charging an interest rate of **6.5%** compounded quarterly on the loan to Sandra.

Example 1.5.2

Five years ago, Taryn placed **\$15,000** into an RRSP that earned **\$6,799.42** of interest compounded monthly. What was the nominal interest rate for the investment?

Solution

The timeline for the investment is shown below.

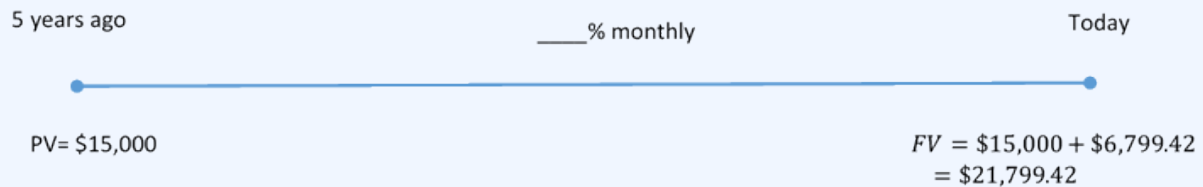


Figure 1.5.2

Image Description

The image shows a horizontal, light blue line representing an investment timeline with labels corresponding to values discussed in the rest of the example.

Step 1: Calculate the future value.

$$\begin{aligned}
 FV &= PV + I \\
 &= 15,000 + 6,799.42 \\
 &= \$21,799.42
 \end{aligned}$$

Step 2: Calculate I/Y .

N	$12 \times 5 = 60$
PV	15,000
FV	-21,799.42
PMT	0
I/Y	?
P/Y	12
C/Y	12

$$I/Y = 7.5\%$$

Step 3: Write as a statement.

Taryn's investment in his RRSP earned **7.5%** compounded monthly over the five years.

Try It

1) Your company paid an invoice five months late. If the original invoice was for **\$6,450** and the amount paid was **\$6,948.48**, what monthly compounded interest rate is your supplier charging on late payments?

Solution

N	$12 \times \frac{5}{12} = 5$
PV	6,450
FV	-6,948.48
PMT	0
I/Y	?
P/Y	12
C/Y	12

$$I/Y = 18\%$$

Try It

2) At what monthly compounded interest rate does it take five years for an investment to double?

Solution

N	$12 \times 5 = 60$
PV	-1
FV	2
PMT	0
I/Y	$?$
P/Y	12
C/Y	12

$$I/Y = 13.94\%$$

Section 1.5 Exercises

1. What is the interest rate compounded monthly if a \$101,000 loan is repaid 10 years later with a payment of \$191,981.42?

Solution

6.44%

2. You invested \$59,860.48 five and half years ago. Today the investment is worth \$78,500. What interest rate compounded semi-annually did your investment earn?

Solution

4.99%

3. If a \$5,000 investment grew to \$20,777.73 in five years, what interest rate

compounded daily did the investment earn?

Solution

28.5%

4. In a civil lawsuit, a plaintiff was awarded damages of \$15,000 plus \$4,621.61 in interest for a period of $3\frac{1}{4}$ years. What quarterly compounded rate of interest was used in the settlement?

Solution

8.35%

5. Muriel just received \$4,620.01 including \$840.01 of interest as payment in full for a sum of money that was loaned 2 years and 11 months ago. What monthly compounded rate of interest was charged on the loan?

Solution

6.9%

6. At what monthly compounded interest rate does it take five years for an investment to double?

Solution

13.94%

7. In 2003, a home in Winnipeg was purchased for \$214,000. In 2011, the same home was appraised at \$450,000. What annually compounded rate of growth does this reflect?

Solution

9.74%

8. On October 1, 1975, the minimum wage in Manitoba was \$2.60 per hour. It rose to \$10 per hour by October 1, 2011. What is the annually compounded growth rate for minimum wage in Manitoba during this period?

Solution

3.81%

9. Jean-Luc's first month's gross salary in June 1994 was \$800. By June 2012 his monthly gross salary was \$1,969.23 higher. What monthly compounded rate did his salary

increase by over the period?

Solution

6.92%

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1.6 CALCULATING THE TERM

Learning Objectives

- Calculate the number of compounding periods
- Calculate the term of a loan or investment

Formula & Symbol Hub

Symbols Used

- f or EFF = Effective interest rate
- FV = Future value or maturity value
- i = Periodic interest rate
- j or I/Y = Nominal interest rate per year
- m or C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of compound periods for the term
- PV = Present value of principal

Formulas Used

- **Formula 1.1 – Total Number of Compounds**

$$n = m \times \text{time in years}$$

- **Formula 1.2 – Periodic Interest Rate**

$$i = \frac{j}{m}$$

- **Formula 1.3 – Future Value**

$$FV = PV \times (1 + i)^n$$

- **Formula 1.4 – Present Value**

$$PV = FV \times (1 + i)^{-n}$$

Introduction

How long will it take to reach a financial goal? At a casual get-together at your house, a close friend discusses saving for a 14-day vacation to the Blue Bay Grand Esmeralda Resort in the Mayan Riviera of Mexico upon graduation. The estimated cost from Travelocity.ca is **\$1,998.94** including fares and taxes. He has already saved **\$1,775** into a fund earning 8% compounded quarterly. Assuming the costs remain the same and he makes no further contributions, can you tell him how soon he will be basking in the sun on the beaches of Mexico?

Using a Financial Calculator

Although it is possible to find the number of compounding periods by using the future value or present value formulas, it is much more practical to use a financial calculator. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is N (the number of compoundings). You must still load the other six variables into the calculator and apply the cash flow sign conventions carefully.

Using the TI BAII Plus Calculator to Find the Number of Compoundings for Compound Interest

Enter values for the known variables (PV , FV , I/Y , PMT , P/Y and C/Y) following the steps below and paying close attention to the cash flow sign convention for PV and FV .

- For the main button keys in the TVM row (i.e. N , I/Y , PV , PMT , FV), enter the number first and then press the corresponding button.
 - For example, to enter $N = 34$, enter **34** on the calculator and then press **N**.
- For P/Y and C/Y , press **2nd I/Y**. At the P/Y screen, enter the value for P/Y and then press **ENTER**. Press the down arrow to access the C/Y screen. At the C/Y screen, enter the value for C/Y and then press **ENTER**. Press **2ndQUIT** (the **CPT** button) to exit the menu.
 - For example, to enter $P/Y = 4$ and $C/Y = 4$, press **2nd I/Y**. At the P/Y screen, enter **4** and press **ENTER**. Press the down arrow. At the C/Y screen, enter **4** and press **ENTER**. Press **2ndQUIT** to exit.

After all of the known quantities are loaded into the calculator, press **CPT** and then N to solve for the number of compounding periods.

Things to Watch Out For

When entering both PV and FV into the calculator, ensure proper application of cash flow sign convention to PV and FV . One number must be negative and the other must be positive. An ERROR message will appear on the calculator display if PV and FV are entered with the same signs (i.e. both are negative or both are positive).

Finding the Term

The value of N is the **number of compounding periods**. For example, $N = 9$ means that there are 9 compounding periods. N does not represent the time period or term. Because, $N = C/Y \times t$, where t is the time in years, the number of years can be found by

$$\text{Number of Years} = \frac{N}{C/Y}$$

The whole number portion of the number of years (the part on the left-side of the decimal) represents the number of years. As needed, take the decimal number portion (the part on the right-side of the decimal point) and multiply it by 12 to convert it to months.

HOW TO

Find the Term

1. To determine the time period, the value of N must be calculated first.
2. In compound interest, the value of N should **not** be rounded to an integer. Because it is possible to have a fraction of a compounding period, N is not rounded.
3. After calculating the number of months, the months are **rounded up** to the next month.

Example 1.6.1

Convert each of the following number of years to years and months.

1. 8.25 years

2. **11.375985** years

Solution

1. The number of years is **8**. The number of months is $0.25 \times 12 = 3$. So **8.25** years is **8** years and **3** months.
2. The number of years is **11**. The number of months is $0.375985 \times 12 = 4.51182$, which is rounded up to **5**. So **11.375985** years is **11** years and **5** months.

Example 1.6.2

Jenning Holdings invested **\$43,000** at **6.65%** compounded quarterly. A report from the finance department shows the investment is currently valued at **\$67,113.46**. How long has the money been invested?

Solution

Step 1: Calculate the value of N .

N	?
PV	−43,000
FV	67,113.46
PMT	0
I/Y	6.65
P/Y	4
C/Y	4

$$N = 26.9999....$$

Step 2: Convert N to years and months.

$$\begin{aligned}\text{Number of Years} &= \frac{N}{C/Y} \\ &= \frac{26.999...}{4} \\ &= 6.74999... \\ &\rightarrow 6 \text{ years}\end{aligned}$$

$$\begin{aligned}\text{Number of Months} &= 0.74999... \times 12 \\ &= 8.999... \\ &\rightarrow 9 \text{ months}\end{aligned}$$

Step 3: Write as a statement.

Jenning Holdings has had the money invested for six years and nine months.

Example 1.5.3

Tabitha estimates that she will need at least \$20,000 for her daughter's postsecondary education when she turns 18. If Tabitha is able to save up \$8,500, how far in advance of her daughter's 18th birthday would she need to invest the money at 7.75% compounded semi-annually? Answer in years and months.

Solution

The timeline for the investment is shown below.

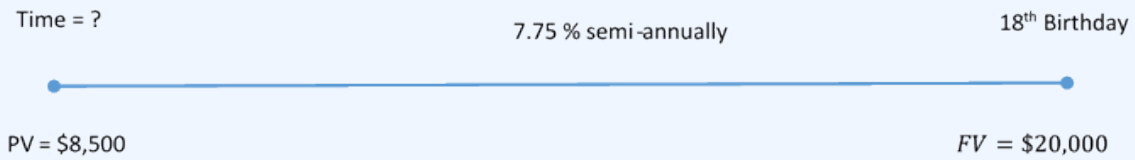


Figure 1.6.1

Image Description

The image shows a horizontal, light blue line representing a loan timeline with labels corresponding to values discussed in the rest of the example.

Step 1: Calculate the value of N .

N	?
PV	−8,500
FV	20,000
PMT	0
I/Y	7.75
P/Y	2
C/Y	2

$$N = 22.50682....$$

Step 2: Convert N to years and months.

$$\begin{aligned}
 \text{Number of Years} &= \frac{N}{C/Y} \\
 &= \frac{22.50682...}{2} \\
 &= 11.2534... \\
 &\rightarrow 11 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.2534... \times 12 \\
 &= 3.0409... \\
 &\rightarrow 4 \text{ months}
 \end{aligned}$$

Step 3: Write as a statement.

If Tabitha invests the \$8,500 11 years and 4 months before her daughter's 18th birthday, it will grow to \$20,000.

Try It

1) You just took over another financial adviser's account. A client invested \$15,500 at 6.92% compounded monthly and now has \$24,980. How long (in years and months) has this client had the money invested?

Solution

N	?
PV	−15,500
FV	24,980
PMT	0
I/Y	6.92
P/Y	12
C/Y	12

$$N = 82.9959\dots$$

$$\begin{aligned}
 \text{Number of Years} &= \frac{N}{C/Y} \\
 &= \frac{82.9959\dots}{12} \\
 &= 6.91633\dots \\
 &\rightarrow 6 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.9163\dots \times 12 \\
 &= 10.995\dots \\
 &\rightarrow 11 \text{ months}
 \end{aligned}$$

The money has been invested for **6** years and **11** months.

Section 1.6 Exercises

1. How long will it take an investment of \$68,000 to mature to at least \$89,032 if the interest rate is 4.91% compounded monthly?

Solution

5 years, 6 months

2. What is the term of an investment of \$41,790 that has a maturity value of \$120,000 at 8.36% compounded quarterly?

Solution

12 years, 9 months

3. How long did it take a \$111,244 investment to earn \$888,756 in interest if the interest rate is 8.8% compounded semi-annually?

Solution

25 years, 6 months

4. A debt of \$7,500 is owed. Suppose prevailing interest rates are 4.9% compounded quarterly. How far in advance was the debt paid if the creditor accepted a payment of \$6,721.25?

Solution

2 years, 4 months

5. Wayne was late in making a \$3,500 payment to Dora. If Dora accepted a payment of \$3,801 and charged 5.59% compounded semi-annually, how late was the payment?

Solution

1 year, 6 months

6. How long will it take \$5,800 to become \$10,000 at 6.25% compounded quarterly?

Solution

8 years, 10 months

7. A friend of yours just won the 6/7 category on the Lotto Max (matching six out of seven numbers), and her share of the prize was \$275,000. She wants to pay cash for a new home that sells for \$360,000. If she can invest the money at 7.45% compounded semi-annually, how long will she have to wait to purchase the home assuming its sale price remains the same?

Solution

3 years, 9 months

8. Lakewood Properties anticipates that the City of Edmonton in the future will release some land for a development that costs \$30 million. If Lakewood can invest \$17.5 million today at 9.5% compounded monthly, how long will it take before it will have enough money to purchase the land?

Solution

5 years, 9 months

9. As marketing manager, you want to pursue a new product development for which you require \$1 million for research. However, budgetary constraints mean you can only receive \$850,000. If you take your budget and invest it at 8.7% compounded monthly, how long will it be before you can pursue the necessary research for the project?

Solution

1 year, 11 months

10. How long will it take money to triple if it earns 8.2% compounded quarterly?

Solution

13 years, 7 months

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1.7 EFFECTIVE AND EQUIVALENT INTEREST RATES

Learning Objectives

- Calculate effective interest rates
- Calculate equivalent interest rates

Formula & Symbol Hub

Symbols Used

- f or EFF = Effective interest rate
- FV = Future value or maturity value
- i = Periodic interest rate
- j or I/Y = Nominal interest rate per year
- m or C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of compound periods for the term
- PV = Present value of principal

Formulas Used

- **Formula 1.1 – Total Number of Compounds**

$$n = m \times \text{time in years}$$

- **Formula 1.2 – Periodic Interest Rate**

$$i = \frac{j}{m}$$

- **Formula 1.3 – Future Value**

$$FV = PV \times (1 + i)^n$$

- **Formula 1.4 – Present Value**

$$PV = FV \times (1 + i)^{-n}$$

- **Formula 1.5 – Effective Interest Rate**

$$f = (1 + i)^m - 1$$

- **Formula 1.6 – Equivalent Interest Rate**

$$i_2 = (1 + i_1)^{\frac{m_1}{m_2}} - 1$$

Introduction

How can you compare interest rates posted with different compounding? For example, let's say you are considering the purchase of a new home, so for the past few weeks you have been shopping around for financing. You have spoken with many banks as well as onsite mortgage brokers in the show homes. With semi-annual compounding, the lowest rate you have come across is 6.6%. In visiting another show home, you encounter a mortgage broker offering a mortgage for 6.57% compounded quarterly. You remember from your business math class that the compounding is an important component of an interest rate and wonder which one you should choose—6.6% compounded semi-annually or 6.57% compounded quarterly.

When considering interest rates on loans, you clearly want the best rate. If all of your possible loans are compounded in the same manner, selecting the best interest rate is a matter of picking the lowest number. However, when interest rates are compounded differently the lowest number may in fact not be your best choice. For investments, on the other hand, you want to earn the most interest. However, the highest nominal rate may not be as good as it appears depending on the compounding frequency.

To compare interest rates fairly and select the best, they all have to be expressed with the same compounding frequency. This section explains the concept of an effective interest rate, and you will learn to convert interest rates from one compounding frequency to a different frequency.

Effective Interest Rates

The **effective interest rate** is the interest rate compounded annually that has the same future value of a given present value for a fixed term as an interest rate compounded at some other (non-annual) frequency. In other words, the amount of interest accrued at the effective interest rate once in an entire year exactly equals the amount of interest accrued at the periodic interest rate successively compounded the stated number of times in a year. For example, if you invest \$1,000 at 10.25% effective (compounded annually) for one year you will have the same future value (\$1,102.50) as investing the \$1,000 for one year at 10% compounded semi-annually. Because 10.25% effective and 10% compounded semi-annually result in the same future value, these interest rates are called **equivalent**.

The corresponding effective interest rate (f) for a given nominal interest can be calculated using the following formula.

1.5 Effective Interest Rate

$$f = (1 + i)^m - 1$$

f is the **Effective Interest**: the corresponding effective interest rate.

i is the **Periodic Interest Rate**: for the given nominal interest rate.

m is the **Compounding Frequency**: for the given nominal interest rate.

Example 1.7.1

Convert 10% compounded semi-annually to its corresponding effective interest rate.

Solution

Step 1: Calculate the periodic interest rate.

$$\begin{aligned} i &= \frac{j}{m} \\ &= \frac{10\%}{2} \\ &= 5\% \\ &= 0.05 \end{aligned}$$

Step 2: Calculate the effective interest rate.

$$\begin{aligned} f &= (1 + i)^m - 1 \\ &= (1 + 0.05)^2 - 1 \\ &= 0.1025 \\ &= 10.25\% \end{aligned}$$

Using the TI BAII Plus Calculator to Find and Effective Interest Rate

The TI BAII Plus calculator has a built-in effective interest rate converter called **ICONV**. The interest rate converter involves three variables.

Variable	Meaning
NOM	Nominal interest rate, I/Y .
C/Y	Compounding frequency, C/Y .
EFF	Effective interest rate.

To find the effective interest rate:

1. Press **2nd ICONV** (the number **2** key).
2. At the **NOM** screen, enter the nominal interest rate I/Y and press **ENTER**. The interest rate is entered as a percent (without the percent sign). For example **10%** is entered as **10**.
3. Press the up arrow.
4. At the **C/Y** screen, enter the compounding frequency for the nominal interest rate entered in the previous step and press **ENTER**.
5. Press the up arrow.
6. At the **EFF** screen, press **CPT** to calculate the effective interest rate.

The interest converter allows you to solve for any of these three variables, not just the effective interest rate.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=53#oembed-1>

Video: [Nominal and Effective Rate Conversions](#) by Joshua Emmanuel [5:50] (Transcript Available).

Example 1.7.2

Convert **10%** compounded semi-annually to its corresponding effective interest rate.

Solution

NOM	10
I/Y	2
EFF	?

1. Press **2nd ICONV**.
2. At the **NOM** screen, enter **10** and press **ENTER**.
3. Press the up arrow.
4. At the **C/Y** screen, enter **2** and press **ENTER**.
5. Press the up arrow.
6. At the **EFF** screen, press **CPT**.

$$EFF = 10.25\%$$

Example 1.7.3

You are offered loans at two different interest rates: **6.6%** compounded semi-annually and **6.57%** compounded quarterly. Which loan should you accept?

Solution**Step 1: Convert 6.6% compounded semi-annually to its effective interest rate.**

NOM	6.6
I/Y	2
EFF	?

$$EFF = 6.71$$

Step 2: Convert 6.75% compounded quarterly to its effective interest rate.

NOM	6.57
I/Y	4
EFF	?

$$EFF = 6.73$$

Step 3: Write as a statement.

You should select the loan at **6.6%** compounded semi-annually because it has the lower effective interest rate.

**Paths to Success**

To compare interest rates that compound at different frequencies, you must convert the rates so that they compound at the same frequency. As in the above example, both interest rates were converted to their effective interest rates and then their effective rates are compared.

- When choosing an interest rate for a loan, you want to pay the least amount of interest. So, select the rate with the lowest effective interest rate.
- When choosing an interest rate for an investment, you want to receive the most interest. So, select the rate with the highest effective interest rate.

Try It

1) If your investment earns **5.5%** compounded monthly, what is the effective rate of interest?

Solution

NOM	5.5
I/Y	12
EFF	?

$$EFF = 5.64$$

Try It

2) As you search for a car loan, all banks have quoted you monthly compounded rates (which are typical for car loans), with the lowest being **8.4%** compounded monthly. At your last stop, the credit union agent says that by taking out a car loan with them, you would be charged **8.65%** effective. Should you go with the bank loan or the credit union loan?

Solution

Convert the bank's rate to its effective rate:

NOM	8.4
I/Y	12
EFF	?

$$EFF = 8.73$$

The credit union's 8.65% effective is the better choice because it is a lower effective rate than the bank's.

Equivalent Interest Rates

Equivalent interest rates are interest rates with different compounding frequencies that result in the same future value for the same given present value and term. For example, if you invest \$1,000 at 10.25% effective (compounded annually) for one year you will have the same future value (\$1,102.50) as investing the \$1,000 for one year at 10% compounded semi-annually. Because 10.25% effective and 10% compounded semi-annually result in the same future value, these interest rates are **equivalent**.

To convert a given nominal interest (j_1) to an equivalent nominal interest (j_2) for a given nominal interest use the following formula.

1.6 Equivalent Interest

$$i_2 = \left(1 + i_1\right)^{\frac{m_1}{m_2}} - 1$$

i_1 is the Periodic Interest Rate: for the given Nominal Interest Rate j_1 .

m_1 is the Compounding Frequency: for the given Nominal Interest Rate j_1 .

m_2 is the Compounding Frequency: for the New Nominal Interest Rate j_2 .

i_2 is the Periodic Interest Rate: for the New Nominal Interest Rate j_2 .

Key Takeaways



Because Formula 1.5 only gives the periodic interest rate for the new nominal interest rate, the nominal interest rate must be calculated from the periodic interest rate:

$$j_2 = m_2 \times i_2$$

Using the TI BAII Plus Calculator to Find an Equivalent Interest Rate

To find an equivalent interest rate:

1. Press **2nd ICONV** (the number **2** key).
2. At the **NOM** screen, enter the nominal interest rate I/Y for the old interest rate and press **ENTER**. The interest rate is entered as a percent (without the percent sign). For example **10%** is entered as **10**.
3. Press the up arrow.
4. At the **C/Y** screen, enter the compounding frequency for the old interest rate entered in the previous step and press **ENTER**.
5. Press the up arrow.

6. At the **EFF** screen, press **CPT** to calculate the effective interest rate.
7. Press the down arrow.
8. At the **C/Y** screen, enter the compounding frequency for the new interest rate and press **ENTER**.
9. Press the down arrow.
10. At the **NOM** screen, press **CPT** to calculate the new interest rate.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=53#oembed-2>

Video: [Nominal and Effective Rate Conversions](#) by Joshua Emmanuel [5:50] (Transcript Available).

Note

Two equivalent interest rates have the same effective rate. The calculator uses this fact to find an equivalent rate. The first step to finding an equivalent rate on the calculator requires the calculation of the effective rate for the old interest rate. Using this effective rate, the calculator finds the new interest rate.

Example 1.7.4

What rate compounded quarterly is equivalent to 8% compounded monthly?

Solution

Step 1: Convert 8% compounded monthly to its effective rate.

NOM	8.8
I/Y	12
EFF	?

$$EFF = 8.2999...\%$$

Step 2: Convert the effective rate from the previous step to the new interest rate compounding quarterly.

Make sure that you keep all of the decimal places from the effective interest rate found in the previous step to avoid any round off error in calculating the new interest rate.

NOM	?
I/Y	4
EFF	8.2999....

$$NOM = 8.05\%$$

Things to Watch Out For

When converting interest rates, the most common source of error lies in confusing the two

values of the compounding frequency, or C/Y . When working through the steps, clearly distinguish between the old compounding that you want to convert from and the new compounding that you want to convert to. A little extra time spent on double-checking these values helps avoid mistakes.

Example 1.7.5

You are looking at three different investments bearing interest rates of **7.75%** compounded semi-annually, **7.7%** compounded quarterly, and **7.76%** compounded semi-annually. Which investment offers the highest interest rate?

Solution

Notice that two of the three interest rates are compounded semi-annually while only one is compounded quarterly. Although you could convert all three to effective rates (requiring three calculations), it is easier to convert the quarterly compounded rate to a semi-annually compounded rate. Then all rates are compounded semi-annually and are therefore comparable.

Step 1: Convert 7.7% compounded quarterly to its effective rate.

NOM	7.7
I/Y	4
EFF	?

$$EFF = 7.9252...\%$$

Step 2: Convert the effective rate in the previous step to the new interest rate compounding semi-annually.

NOM	?
I/Y	2
EFF	7.9252...

$$NOM = 7.77\%$$

Step 3: Write as a statement.

7.7% compounded quarterly is equivalent to 7.77% compounded semi-annually. In comparison to the semi-annually compounded rates of 7.75% and 7.76%, the 7.7% quarterly rate is the highest interest rate for the investment.

Try It

3) What nominal interest rate compounded semi-annually would result in the same financial position as 9.5% compounded monthly?

Solution

NOM	9.5	9.69
C/Y	12	2
EFF	9.9247...	9.9247...

Try It

4) You are offered three different investment opportunities at three different interest rates: **6.2%** compounded semi-annually, **6.19%** compounded quarterly, and **6.07%** compounded daily. Which investment should you pick?

Solution

Convert **6.09%** compounded quarterly to a rate compounded semi-annually.

NOM	6.19	6.24
C/Y	4	2
EFF	6.335...	6.335...

6.19% compounded quarterly is equivalent to **6.24%** compounded semi-annually.

Convert **6.07%** compounded daily to a rate compounded semi-annually.

NOM	6.07	6.16
C/Y	365	2
EFF	6.257...	6.257...

6.07% compounded daily is equivalent to **6.16%** compounded semi-annually.

Select the investment at **6.19%** compounded quarterly.

Note: There are other ways to solve this problem. For example, all of the interest rates could be converted to rates that compound quarterly, or all the interest rates could be converted to their effective rates.

Section 1.7 Exercises

1. For each of the following, find their effective rates.

- a. **4.75%** compounded quarterly.
- b. **7.2%** compounded monthly.
- c. **3.95%** compounded semi-annually.

Solution

a. **4.84%**, b. **7.44%**, c. **3.99%**

2. Convert **10%** effective to the equivalent rate compounded semi-annually.

Solution

9.76%

3. Convert **12%** effective to the equivalent rate compounded monthly.

Solution

11.39%

4. Convert **8%** effective to the equivalent rate compounded quarterly.

Solution

7.77%

5. Convert **6%** compounded monthly to the equivalent rate compounded semi-annually.

Solution

6.08%

6. Convert **4.5%** compounded quarterly to the equivalent rate compounded monthly.

Solution

4.48%

7. Convert **8%** compounded semi-annually to the equivalent rate compounded quarterly.

Solution

7.92%

8. The HBC credit card has a nominal interest rate of **26.44669%** compounded monthly. What effective rate is being charged?

Solution

29.9%

9. What is the effective rate on a credit card that charges interest of **0.049315%** per day?

Solution

19.72%

10. RBC offers two different investment options to its clients. The first option is compounded monthly while the latter option is compounded quarterly. If RBC wants both options to have an effective rate of **3.9%**, what nominal rates should it set for each option?

Solution

3.832% compounded monthly, **3.844%** compounded quarterly

11. Louisa is shopping around for a loan. TD Canada Trust has offered her **8.3%** compounded monthly, Conexus Credit Union has offered **8.34%** compounded quarterly, and ING Direct has offered **8.45%** compounded semi-annually. Rank the three offers and show calculations to support your answer.

Solution

Rank	Company	Rate
1	ING	8.629%
2	TD	8.623%
3	Conexus	8.605%

12. Your three-year monthly compounded investment just matured and you received **\$9,712.72**, of which **\$2,212.72** was interest. What effective rate of interest did you earn?

Solution

9%

13. The TD Emerald Visa card wants to increase its effective rate by **1%**. If its current interest rate is **19.067014%** compounded daily, what new daily compounded rate should it advertise?

Solution

19.89%

14. Five investors reported the following results. Rank the effective rates of return realized by each investor from highest to lowest and show your work.

- **Investor 1:** \$4,000 principal earned \$1,459.10 of interest compounded monthly over four years.
- **Investor 2:** \$9,929.85 maturity value including \$1,429.85 of interest compounded quarterly for two years.
- **Investor 3:** \$14,750 principal maturing at \$19,370.83 after $3\frac{1}{2}$ years of semi-annually compounded interest.
- **Investor 4:** \$3,194.32 of interest earned on a \$6,750 principal after five years of daily compounding.
- **Investor 5:** \$5,321.65 maturity value including \$1,421.65 of interest compounded annually over four years.

Solution

Investor	Rate
3	8.098%
1	8.085%
2	8.084%
5	8.08%
4	8.057%

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1.8 REVIEW EXERCISES

Chapter 1 Review Exercises

1. If you invest \$10,000 at 7.74% compounded quarterly for 10 years, what is the maturity value? How much interest did the investment earn?

Solution

$$FV = \$21,524.50, I = \$11,524.50$$

2. Ford Motor Company is considering an early retirement buyout package for some employees. The package involves paying out today's fair value of the employee's final year of salary. Shelby is due to retire in one year. Her salary is at the company maximum of \$72,000. If prevailing interest rates are 6.75% compounded monthly, what buyout amount should Ford offer to Shelby today?

Solution

$$PV = \$67,313.13$$

3. Polo Park Bowling Lanes owes the same supplier \$3,000 today and \$2,500 one year from now. The owner proposed to pay both bills with a single payment four months from now. If interest rates are 8.1% compounded monthly, what amount should the supplier be willing to accept?

Solution

$$\$5,450.83$$

4. What is the effective rate of interest on your credit card if you are being charged 24.5% compounded daily?

Solution

$$27.7516\%$$

5. Your $2\frac{1}{2}$ -year investment of \$5,750 just matured for \$6,364.09. What weekly compounded rate of interest did you earn?

Solution

4.0604% compounded weekly

6. Vienna just paid \$9,567.31 for an investment earning 5.26% compounded semi-annually that will mature for \$25,000. What is the term of the investment (in years and months)?

Solution

18 years, 6 months

7. Merryweather's union just negotiated a new four-year contract with her employer. The terms of the contract provide for an immediate wage increase of 3.3%, followed by annual increases of 3.5%, 4.25%, and 2.75%. If she currently earns \$61,500, what will her salary be in the final year of the contract?

Solution

$FV = \$70,432.59$

8. Bronco's four-year investment just matured at \$26,178.21. If the investment earned semi-annually compounded interest rates of 4.5% and 4.75% in the first two years, followed by monthly compounded interest rates of 5% and 5.1% in the last two years, how much money did Bronco initially invest? How much interest did the investment earn?

Solution

$PV = \$21,600, I = \$4,578.21$

9. Jay's Pharmacy owes the same creditor two debts of \$6,000 due one year ago and \$7,500 due in two years. Jay has proposed making two alternative payments of \$10,000 due in three months and a final payment in $2\frac{1}{2}$ years. If the creditor is agreeable to this proposal and wants an interest rate of 9% compounded quarterly, what is the amount of the final payment? Use $2\frac{1}{2}$ years as the focal date.

Solution

\$3,817.05

10. Over a 10-year period, the Growth Fund of America had annual returns of

0.02%, 29.8%, 14.84%, 26.86%, 31.78%, 45.7%, 7.49%, -12.2%, -22.02%, and 32.9%. What fixed annually compounded rate of return did the fund realize over the 10 years?

Click to see Answer

13.5131% compounded annually

11. A sum of \$84,100 was invested for $2\frac{1}{2}$ years and matured at \$101,268 using quarterly compounding. What quarterly compounded interest rate was realized? What is this effectively?

Solution

7.5% compounded quarterly, 7.714% effective

12. A \$10,000 loan at 8.15% compounded quarterly is to be repaid by two equal payments due in 9 months and in 33 months. Determine the size of the payments.

Solution

\$5,739.43

13. Exactly how long will it take for your money to quadruple at 6.54% compounded monthly? Express answer in years, months, days.

Solution

21 years, 4 months

14. Six months ago Old Dutch Foods purchased some new machinery for a new product line that they just developed. The supplier agreed to three payments on the machinery of \$40,000 due today, \$85,000 due in six months, and \$75,000 due in 15 months. The new product line has not been as successful as initially planned, so Old Dutch Foods has proposed an alternative agreement involving three equal payments, each due at 3 months, 9 months, and 21 months. If the supplier is agreeable to this and wants an interest rate of 8.55% compounded monthly, determine the payments required in the proposed agreement.

Solution

\$67,978.25

15. Louisa owns a furniture store and decided to help a friend out by allowing him to purchase \$5,000 of furniture using her credit at 6.25% compounded semi-annually.

The furniture loan is to be repaid in four years. However, after $2\frac{1}{2}$ years Louisa can no longer have the loan outstanding and needs the money. Avco Financial has agreed to purchase the maturity amount of this loan from Louisa using a discount rate of **17.1%** compounded monthly. If Louisa proceeds with selling the loan contract to Avco Financial, what sum of money can she expect to receive?

Solution

$$PV = \$4,957.60$$

16. Jack is considering alternative three-year investment proposals from different banks for his **\$20,000** principal. RBC says that if he invests his money with them, he will have **\$23,841.78** using quarterly compounding. CIBC indicates that he would have **\$23,607.15** using monthly compounding. What nominal and effective rates are being offered by each bank?

Solution

RBC: **5.925%** compounded quarterly, **6.0318%** effective; CIBC: **5.54%** compounded monthly, **5.6829%** effective

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CHAPTER 1 SUMMARY

Key Concepts Summary

- **1.1 Compound Interest Terminology**

- How compounding works.
- How to calculate the periodic interest rate.

- **1.2 Calculating the Future Value**

- The basics of taking a single payment and moving it to a future date.
- Moving single payments to the future when variables change.

- **1.3 Calculating the Present Value**

- The basics of taking a single payment and moving it to an earlier date.
- Moving single payments to the past when variables change.

- **1.4 Equivalent Payments**

- The concept of equivalent payments.
- The fundamental concept of time value of money.
- The fundamental concept of equivalency.
- Applying single payment concepts to loans and payments.

- **1.5 Calculating the Interest Rate**

- Solving for the nominal interest rate.
- How to convert a variable interest rate into its equivalent fixed interest rate.

- **1.6 Calculating the Term**

- Figuring out the term when n is an integer.

- Figuring out the term when n is a non-integer.
- **1.7 Equivalent and Effective Interest Rates**
 - The concept of effective rates.
 - Taking any nominal interest rate and finding its equivalent nominal interest rate.

Glossary of Terms

Compound Interest. A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more. Interest is periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

Compounding Frequency. The number of compounding periods in a complete year.

Compounding Period. The amount of time that elapses between the dates of successive conversions of interest to principal.

Discount Rate. An interest rate used to remove interest from a future value.

Effective Interest Rate. The true annually compounded interest rate that is equivalent to an interest rate compounded at some other (non-annual) frequency.

Equivalent Payment Streams. Equating two or more alternative financial streams such that neither party receives financial gain or harm by choosing either stream.

Equivalent Interest Rates. Interest rates with different compounding that produce the same effective rate and therefore are equal to each other.

Focal Date. A point in time to which all monies involved in all payment streams will be moved using time value of money calculations.

Fundamental Concept of Equivalency. Two or more payment streams are equal to each other if they have the same economic value on the same focal date.

Fundamental Concept of Time Value of Money. All monies must be brought to the same focal date before any mathematical operations, decisions, or equivalencies can be determined.

Nominal Interest Rate. A nominal number for the annual interest rate, which is commonly followed by words that state the compounding frequency.

Periodic Interest Rate. The percentage of interest earned or charged at the end of each compounding period.

Present Value Principle for Loans. The present value of all payments on a loan is equal to the principal that was borrowed.

Formula & Symbol Hub

Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- m or C/Y = Number of compounds per year or compounding frequency
- i = Periodic interest rate
- j or I/Y = Nominal interest rate per year
- n or N = Total number of compound periods for the term
- f or EFF = Effective interest rate

Formulas Used

- Formula 1.1 – **Total Number of Compounds**

$$n = m \times \text{time in years}$$

- Formula 1.2 – **Periodic Interest Rate**

$$i = \frac{j}{m}$$

- **Formula 1.3 – Future Value**

$$FV = PV \times (1 + i)^n$$

- **Formula 1.4 – Present Value**

$$PV = FV \times (1 + i)^{-n}$$

- **Formula 1.5 – Effective Interest Rate**

$$f = (1 + i)^m - 1$$

- **Formula 1.6 – Equivalent Interest Rate**

$$i_2 = (1 + i_1)^{\frac{m_1}{m_2}} - 1$$

Calculator

- **Time Value of Money Functions**

- The time value of money buttons are the five buttons located on the third row of your calculator.

Calculator Symbol	Characteristic	Data Entry Requirements
N	The number of compounding periods.	An integer or decimal number; no negatives.
I/Y	The nominal interest rate per year.	Percent format without the % sign (i.e., 7% is just 7).
PV	Present value or principal.	An integer or decimal number.
PMT	Used for annuity payment amounts (covered in Chapter 11) and is not applicable to lump-sum amounts; it needs to be set to zero for lump-sum calculations.	An integer or decimal number.
FV	Future value or maturity value.	An integer or decimal number.

To enter any information into any one of these buttons or variables, called loading the calculator, key in the information first and then press the appropriate button.

- The frequency function is logically placed above the I/Y button and is labeled P/Y . This function addresses compound interest frequencies, such as the compounding frequency. Access the function by pressing **2nd** P/Y to find the following entry fields, through which you can scroll using your arrow buttons.

Calculator Symbol	Characteristic	Data Entry Requirements
P/Y	Annuity payments per year (payment frequency is introduced in Chapter 11); when working with lump-sum payments and not annuities, the calculator requires this variable to be set to match the C/Y .	A positive, nonzero number only.
C/Y	Compounds per year (compounding frequency).	A positive, nonzero number only.

To enter any information into one of these fields, scroll to the field on your screen, key in the data, and press **ENTER**. When you enter a value into the P/Y field, the calculator will automatically copy the value into the C/Y field for you. If in fact the C/Y is different, you can change the number manually. To exit the P/Y window, press **2nd QUIT**.

- To key in a question, you must load the calculator with six of the seven variables. To solve for the missing variable, press **CPT** followed by the variable.
- The cash flow sign convention is used for the PV , PMT , and FV buttons. If money leaves you, you must enter it as a negative. If money comes at you, you must enter it as a positive.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=57#oembed-1>

[Compound Interest \(Present and Future Values\)](#) by Joshua Emmanuel [6:56] (transcript available).

• Interest Conversions

- To find the effective interest rate.
 1. Press **2nd ICONV** (the number 2 key).

2. At the **NOM** screen, enter the nominal interest rate I/Y and press **ENTER**. The interest rate is entered as a percent (without the percent sign). For example 10% is entered as 10.
 3. Press the up arrow.
 4. At the C/Y screen, enter the compounding frequency for the nominal interest rate entered in the previous step and press **ENTER**.
 5. Press the up arrow.
 6. At the **EFF** screen, press **CPT** to calculate the effective interest rate.
- To find the equivalent rate.
 1. Find the effective interest rate for the original interest rate following the steps above.
 2. Press the down arrow.
 3. At the C/Y Screen, enter the new compounding frequency and press **ENTER**.
 4. Press the down arrow.
 5. At the **NOM** screen, press **CPT** to calculate the new interest rate.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=57#oembed-2>

[Nominal and Effective Rate Conversions](#) by Joshua Emmanuel [5:50] (transcript available).

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CHAPTER 2: ANNUITIES

Chapter Outline

[2.0 Introduction](#)

[2.1 Fundamentals of Annuities](#)

[2.2 Future Value of Annuities](#)

[2.3 Present Value of Annuities](#)

[2.4 Calculating the Payment](#)

[2.5 Calculating the Term of an Annuity](#)

[2.6 Calculating the Interest Rate for an Annuity](#)

[2.7 Deferred Annuities](#)

[2.8 Perpetuities](#)

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[Chapter 2 Summary](#)

2.0 INTRODUCTION

Chapter Learning Objectives

By the end of the chapter you should be able to:

- Define terminology associated with annuities
- Classify annuities based on the frequency of the payments and the compounding frequency
- Calculate the future value of an ordinary annuity
- Calculate the future value of an annuity due
- Calculate the present value of an ordinary annuity
- Calculate the present value of an annuity due
- Calculate the payment for an annuity
- Calculate the number of payments in an annuity
- Calculate the term of an annuity
- Calculate the interest rate for an annuity
- Solve problems involving a deferred annuity
- Solve problems involving a perpetuity

Payments show up everywhere. For example, you make payments on bank loans or you make payments into your RRSP. An annuity is anything where you make regularly scheduled payments. Here are some real-world examples of annuities.

- Last week you completed the purchase of your \$150,000 starter home and finished all of the paperwork on your mortgage with the bank. You snagged a great mortgage rate of 5% compounded semi-annually, locking it in for a five-year term with monthly payments of \$872.41 (for the next 25 years) starting one month after you move in.
- You excitedly cruise to the Honda dealership to pick up your new \$22,475 Honda Civic Sport, on which you have a four-year lease. Pulling out your iPhone (for which you pay \$60 monthly), you ensure

that you have enough funds in your chequing account to make the \$2,000 down payment today along with the first of your \$242.16 monthly lease payments at 2.9% compounded monthly.

- Driving away in your Civic, you stop in at Sleep Country Canada and purchase a queen size mattress for your new bedroom. You recall a TV ad promoting an \$895 Sealy mattress plus 12% taxes with no money down and 12 easy, interest-free monthly payments of \$83.53. It will be nice to have a comfortable bed to sleep in!

This chapter explores the concept of making regular payments toward savings goals or debt extinguishment. First, it introduces key payment concepts and reviews different types of payment plans. You will then calculate the future and present value of a stream of payments. You extend the application of the concepts further to solve for other variables such as the payment amount, the term, and the nominal interest rate.

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2.1 FUNDAMENTALS OF ANNUITIES

Learning Objectives

- Define terminology associated with annuities
- Classify annuities based on the frequency of the payments and the compounding frequency

Formula & Symbol Hub

Symbols Used

- n or N = Total number of annuity payments
- P/Y = Number of payments per year or payment frequency
- t = Time in years

Formulas Used

- Formula 2.1 – **Total Number of Payments (Annuity)**

$$n = P/Y \times t$$

Introduction

An **annuity** is a continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation. An **annuity payment** is the dollar amount of the equal periodic payment in an annuity environment. The payments are continuous, equal, periodic, and occur over a fixed time frame. If any one of these four characteristics is not satisfied, then the financial transaction fails to meet the definition of a singular annuity and requires other techniques and formulas to solve.

Annuity Terms

The **payment interval** is the interval of time between two consecutive payments. For example, if payments are made every month, then the payment interval is monthly and the time period between two successive payments is one month.

The **payment frequency** (P/Y) is the number of times payments are made every year. That is, the payment frequency is the number of payment periods in one year.

Payment Interval	Number of Times per Year Payments are Made	Payment Frequency
Annually	Once a year	1
Semi-annually	Twice a year/Every six months	2
Quarterly	Four times a year/Every three months	4
Monthly	Twelve times a year/Every month	12
Semi-monthly	24 times a year	24
Bi-weekly	26 times a year	26
Weekly	52 times a year	52
Daily	Every day/365 times a year	365

The **term or time period** (t) of an annuity is the length of time from the beginning of the first payment interval to the end of the last payment interval.

Annuity calculations require the **total number of payments during the term** (n). As with compound interest, to calculate the total number of payments, the time must be in years. If the term is not in years, the

term must be converted to years. If the term is given in months, divide by **12** to convert the term to years. If the term is given in days, divide by **365** to convert the term to years.

2.1 Total Number of Payments (Annuity)

$$n = P/Y \times t$$

n is **Total Number of Payments**: total number of payments during the term.

P/Y is **Payment Frequency**: the number of times payments are made every year.

t is **Time in Years**: the length of time from the beginning of the first payment interval to the end of the last payment interval.

Example 2.1.1

For each of the following, calculate the number of payments.

1. \$200 deposited every month for five years.
2. \$750 deposited every quarter for three years and nine months.
3. \$300 loan payments semi-annually for 54 months.

Solution

1. $n = P/Y \times t = 12 \times 5 = 60$
2. $n = P/Y \times t = 4 \times \frac{45}{12} = 15$

$$3. \ n = P/Y \times t = 2 \times \frac{54}{12} = 9$$

Things to Watch Out For

Annuity calculations do **not** require the total number of compoundings and the calculation of n for an annuity does not involve the compounding frequency.

It now becomes critical to ensure the proper application of the cash flow sign convention on the calculator—failure to do so will result in an incorrect answer. For example, if you borrow money and then make annuity payments on it, you enter the present value (PV) as a positive (you received the money) while you enter the annuity payments as negatives (you paid the money to the bank). This results in future balances getting smaller and you owing less money. If you inadvertently enter the annuity payment as a positive number, this would mean you are borrowing more money from the bank so your future balance would increase and you would owe more money. These two answers are very different!

Types of Annuities

Annuities are classified in two ways—by the timing of the payment (at the end or beginning of the payment interval) and by whether or not the payment frequency and the compounding frequency are equal.

In an **ordinary annuity**, the payments are made at the end of the payment interval. The first payment occurs one interval after the beginning of the annuity and the last payment occurs on the same date as the end of the annuity. For example, if you took out a car loan today that requires you to make monthly payments to repay the loan, your first payment would occur one month from today at the end of the first payment interval and all of your subsequent payments would occur at the end of each month. Common applications of ordinary annuities include bank loans, mortgages, bonds, and Canada Pension Plan payments.

In an **annuity due**, the payments are made at the beginning of the payment interval. The first payment occurs on the same date as the beginning of the annuity and the last payment occurs one payment interval before the end of the annuity. For example, if you rent an apartment, your rent payment forms an annuity due because

the your rent is paid at the beginning of every month. Common applications of annuities due include any kind of lease, any kind of rental, membership dues, and insurance payments.

In a **simple annuity**, the frequency of the payments and the compounding frequency for the interest rate are equal. For example, an annuity with quarterly payments and an interest rate that compounds quarterly is a simple annuity.

In a **general annuity**, the frequency of the payments and the compounding frequency for the interest rate are not equal. For example, an annuity with quarterly payments and an interest rate that compounds monthly is a general annuity.

Altogether, there are four types of annuities—ordinary simple annuity, ordinary general annuity, simple annuity due, general annuity due. The table below summarizes the four types of annuities and their characteristics.

Annuity Type	Timing of Payments in a Payment Interval	Payment Frequency and Compounding Frequency	Start of Annuity and First Payment Same Date?	End of Annuity and Last Payment Same Date?
Ordinary Simple Annuity	End	Equal	No, first payment one interval later	Yes
Ordinary General Annuity	End	Unequal	No, first payment one interval later	Yes
Simple Annuity Due	Beginning	Equal	Yes	No, last payment one interval earlier
General Annuity Due	Beginning	Unequal	Yes	No, last payment one interval earlier

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2.2 FUTURE VALUE OF ANNUITIES

Learning Objectives

- Calculate the future value of an ordinary annuity
- Calculate the future value of an annuity due

Formula & Symbol Hub

Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of annuity payments

Formulas Used

- Formula 2.1 – **Total Number of Payments (Annuity)**

$$n = P/Y \times t$$

- Formula 2.2 – **Future Value of Ordinary Annuity**

$$FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- Formula 2.3 – **Future Value of Annuity Due**

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

Introduction

The **future value of any annuity** equals the sum of the future values for all of the annuity payments when they are moved to the **end** of the last payment interval. For example, assume you will make \$1,000 contributions at the end of every year for the next three years to an investment earning 10% compounded annually. To find out how much money you have in your investment after the three years, you would apply the fundamental concept of the time value of money to move each payment amount to the future date (the focal date) and sum the values at the focal date to arrive at the future value.

Although you could use this technique to solve all future value of an annuity situations, the computations become increasingly cumbersome as the number of payments increases. In the above example, what if the person made quarterly contributions of \$250 for three years? That is 12 payments over three years, resulting in 11 separate future value calculations. Or if they made monthly payments, the 36 payments over three years would result in 35 separate future value calculations! Clearly, solving this would be tedious and time consuming—not to mention prone to error.

2.2 Future Value of Ordinary Annuities

$$FV = PMT \times \left[\frac{(1 + i_2)^{n-1}}{i_2} \right]$$

FV is the Future Value of the Annuity: the future value is the sum of the future value of all of the annuity payments at the end of the last payment interval.

PMT is the Annuity Payment: the amount of money that is invested or paid after each payment interval.

i_2 is the Periodic Interest Rate: for the interest rate whose compounding frequency matches the payment frequency. If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.

n is the Total Number of Payments made during the Annuity: $n = P/Y \times t$ where P/Y is the payment frequency and t is the time in years.

For investment annuities, the amount of interest (I) earned by the annuity is

$$I = FV - n \times PMT$$



Key Takeaways

In a **simple annuity**, the payment frequency and the compounding frequency are equal ($P/Y = C/Y$). In this situation the periodic interest rate $i_2 = \frac{j}{m}$ where j is the nominal interest rate given in the question and m is the compounding frequency (and the payment frequency).

In a **general annuity**, the payment frequency and the compounding frequency are not equal ($P/Y \neq C/Y$). In this situation, the given interest rate must first be converted to the equivalent interest rate where the new compounding frequency equals the payment frequency. Using the equivalent interest rate, calculate the periodic interest rate i_2 .

Example 2.2.1

A financial adviser is reviewing one of her client's accounts. The client has been investing \$1,000 at the end of every quarter for the past 11 years in a fund that has averaged 7.3% compounded quarterly. How much money does the client have in his account today? How much interest did the client earn?

Solution

The timeline shows the client's account.

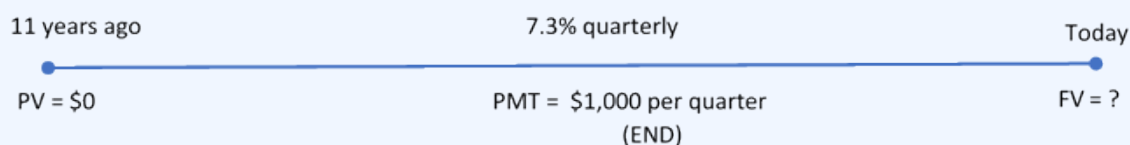


Figure 2.2.1

Step 1: The given information is

The payments are at the end of the payment intervals, and both the compounding frequency and the payment frequency are the same (both quarterly). This is an ordinary simple annuity. Because this is a simple annuity, an interest rate conversion is not required.

$$\begin{aligned}
 PMT &= \$1,000 \\
 P/Y &= 4 \\
 I/Y &= 7.3\% \\
 C/Y &= 4 \\
 t &= 11 \text{ years}
 \end{aligned}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned}
 i_2 &= \frac{I/Y}{C/Y} \\
 &= \frac{7.3\%}{4} \\
 &= 1.825\%
 \end{aligned}$$

Step 3: Calculate the number of payments.

$$\begin{aligned}
 n &= P/Y \times t \\
 &= 4 \times 11 \\
 &= 44
 \end{aligned}$$

Step 4: Calculate the future value.

$$\begin{aligned}
 FV &= PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right] \\
 &= 1,000 \times \left[\frac{(1 + 0.01825)^{44} - 1}{0.01825} \right] \\
 &= \$66,637.03
 \end{aligned}$$

Step 5: Calculate the interest earned.

$$\begin{aligned}
 I &= FV - n \times PMT \\
 &= 66,637.03 - 44 \times 1,000 \\
 &= \$22,637.03
 \end{aligned}$$

Step 6: Write as a statement.

After 11 years, the client has \$66,637.03 in the account and has earned \$22,637.03 in interest.

Using a Financial Calculator

The above example was a fairly straightforward annuity question. But even this simple example, which did not require an interest conversion, is cumbersome, and time-consuming, to solve using the formula. A financial calculator can quickly solve annuity problems, with the added bonus of not requiring an interest conversion in situations where the payment frequency and compounding frequency are not equal.

Using the TI BAII Plus Calculator to Find the Future Value for Ordinary Annuities

The time value of money buttons are located in the **TVM** row (the third row from the top) of the calculator. The five buttons located on the third row of the calculator are five of the seven variables required for time value of money calculations. This row's buttons are different in colour from the rest of the buttons on the keypad. The other two variables are in a secondary menu above the **I/Y** key and are accessed by pressing **2nd I/Y**.

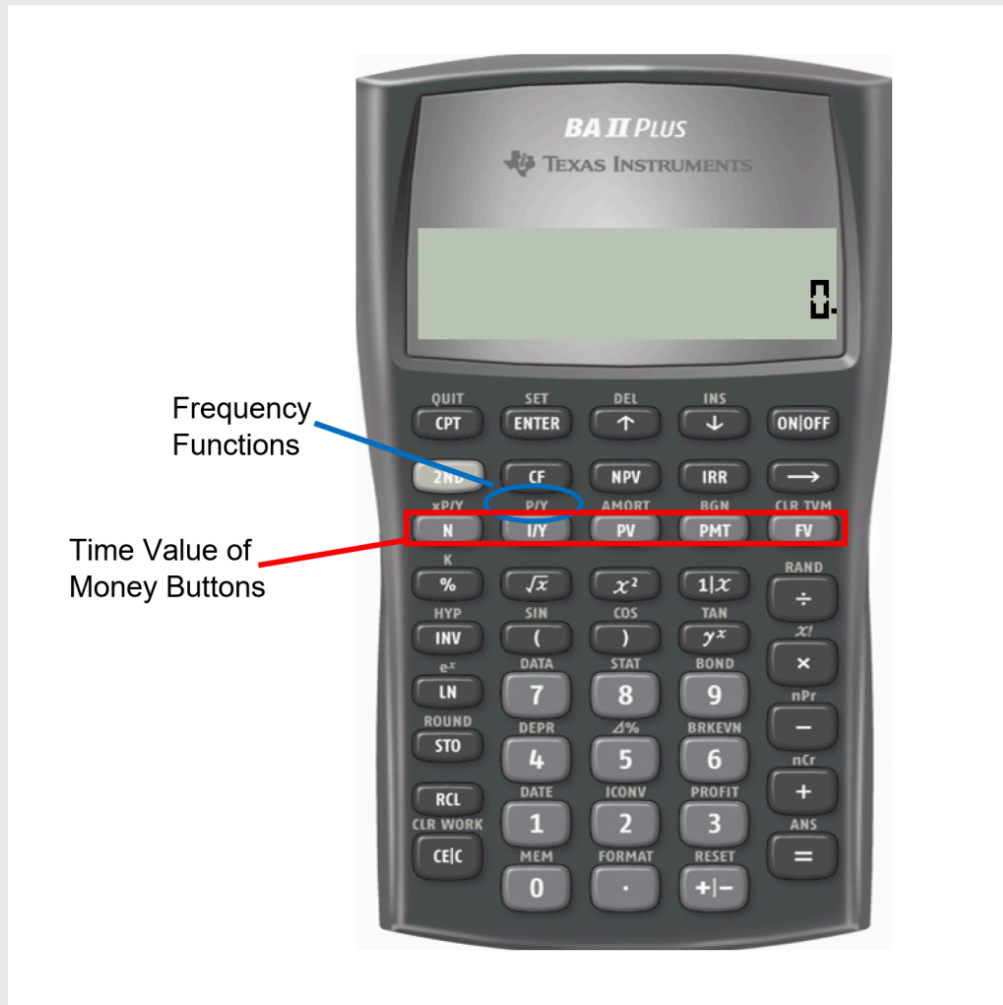


Figure 5.2.2

Image Description

The image is of a black Texas Instruments BA II Plus financial calculator. The calculator has a large LCD screen at the top which is displaying the number "0.". Below the screen, there is a keypad with numerous buttons divided into several rows. The buttons provide various financial calculations and standard calculator functions.

There are labels indicating two key sections of the calculator:

- The Frequency Functions are indicated in blue as the secondary functions of the third row of buttons from the top:
 - xP/Y
 - P/Y

- AMORT
 - BGN
 - CLR TVM
- Time Value of Money Buttons are indicated in red as the primary functions of the third row of buttons from the top:
 - N
 - I/Y
 - PV
 - PMT
 - FV

Altogether, there are seven variables required to complete time value of money calculations. Note that P/Y and C/Y are not main button keys in the **TVM** row. The P/Y and C/Y variables are located in the secondary function accessed by pressing **2nd I/Y**.

Variable	Meaning
N	Total number of payments. This is the same value as n in the future value formula. $N = P/Y \times \text{time in years}$
I/Y	Interest rate per year (i.e. the nominal interest rate). The interest rate is entered in percent form (without the % sign). For example, 5% is entered as 5
PV	Present value or principal
PMT	Periodic annuity payment
FV	Future value or maturity value
P/Y	Payment frequency
C/Y	Compounding frequency

To enter values into the calculator:

- For the main button *null* keys in the **TVM** row (i.e. N , I/Y , PV , PMT , FV), enter the number first and then press the corresponding button.
- For P/Y and C/Y , press **2nd I/Y**. At the P/Y screen, enter the value for P/Y and then press **ENTER**. Press the down arrow to access the C/Y screen. At the C/Y screen, enter

the value for C/Y and then press **ENTER**. Press **2nd QUIT** (the CPT button) to exit the menu.

After all of the known quantities are loaded into the calculator, press CPT and then FV to solve for the future value.

Notes

- Be sure to enter PV , FV , and PMT with the correct cash flow sign convention. When you invest, PMT has the same sign as the PV (both negative). When you borrow, PMT has the opposite sign of PV (PMT is negative and PV is positive).
- Because you can enter P/Y and C/Y , there is no need to do an interest conversion for a general annuity. You simply have to tell the calculator the P/Y and C/Y given in the question, and the calculator will handle any interest conversion internally.
- If there is no money in the account or investment to start, then $PV = 0$.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=70#oembed-1>

Video: [Ordinary Annuity Calculations \(PV, PMT, FV\)](#) by Joshua Emmanuel [4:31] (Transcript Available).

Example 2.2.2

A financial adviser is reviewing one of her client's accounts. The client has been investing **\$1,000** at the end of every quarter for the past **11** years in a fund that has averaged **7.3%** compounded quarterly. How much money does the client have in his account today?

Solution

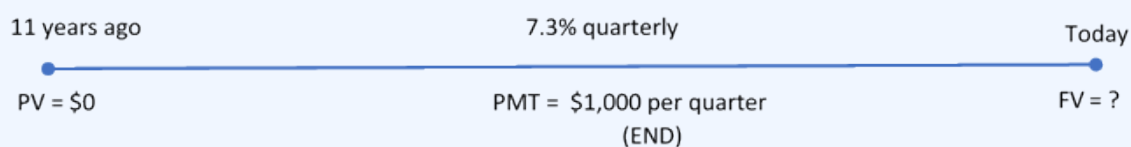


Figure 2.2.3

The timeline shows the client's account.

PMT Setting	END
N	$4 \times 11 = 44$
PV	0
FV	?
PMT	-1,000
I/Y	7.3
P/Y	4
C/Y	4

$$FV = \$66,637.03$$

After 11 years, the client has **\$66,637.03** in the account.

Try It

1) You are a financial adviser. Your client is thinking of investing **\$600** at the end of every six months for the next six years with the invested funds earning **6.4%** compounded semi-annually. Your client wants to know how much money she will have after six years. What do you tell your client?

Solution

PMT Setting	END
N	$2 \times 12 = 24$
PV	0
FV	?
PMT	-600
I/Y	6.4
P/Y	2
C/Y	2

$$FV = \$8,612.62$$

Example 2.2.4

A savings annuity already contains **\$10,000**. If an additional **\$250** is invested at the end of every month at **9%** compounded semi-annually for a term of **20** years, what will be the maturity value of the investment?

Solution

The timeline is shown below.

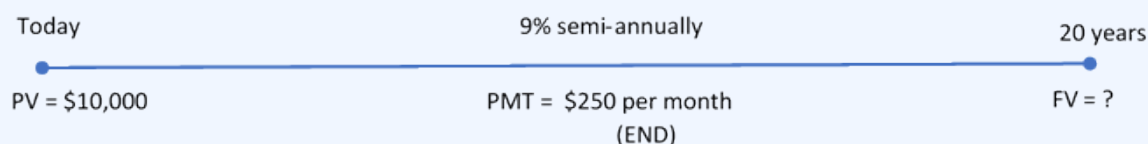


Figure 2.2.4

$$\begin{aligned} \text{PMT Setting} & \text{END} \quad N12 \times 20 = 240 \quad PV - 10,000 \quad FV? \quad PMT - 250 \quad I/Y9 \quad P/Y12 \quad C/Y2 \\ & FV = \$221,693.59 \end{aligned}$$

The savings annuity will have a balance of **\$221,693.59** after the **20** years.

Try It

2) McDonald's major distribution partner, The Martin-Brower Company, needs at least **\$1,000,000** to build a new warehouse in Medicine Hat two years from today. To date, it has invested **\$500,000**. If it continues to invest **\$50,000** at the end of every quarter into a fund earning **6%** quarterly, will it have enough money to build the warehouse two years from now?

Solution

PMT Setting	END
N	$4 \times 2 = 8$
PV	$-500,000$
FV	?
PMT	$-50,000$
I/Y	6
P/Y	4
C/Y	4

$$FV = \$984,888.25$$

The fund is **\$15,111.75** short of the \$1 million target.

If any of the variables, including I/Y , C/Y , PMT , or P/Y change between the starting point and ending point of the annuity, or if any additional single payment deposit or withdrawal is made, a new time segment is created and must be treated separately. In such cases, there will be multiple time segments that require you to work from left to right through the timeline in order to find the future value at the end of the annuity. The future value at the end of one time segment becomes the present value in the next time segment.

Example 2.2.5

Genevieve has decided to start saving up for a vacation in two years, when she graduates from university. She already has **\$1,000** saved today. For the first year, she plans on making end-of-month contributions of **\$300** and then switching to end-of-quarter contributions of **\$1,000** in the second year. If the account can earn **5%** compounded semi-annually in the first

year and 6% compounded quarterly in the second year, how much money will she have saved when she graduates?

Solution

The timeline for her vacation saving appears below.

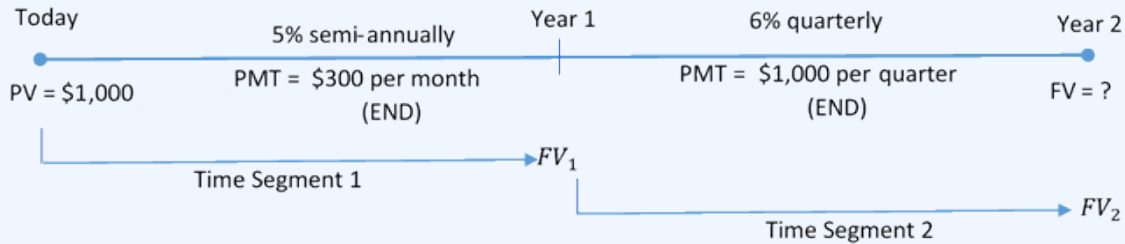


Figure 2.2.5

Step 1: Find the future value at the end of the first segment.

PMT Setting	END
N	$12 \times 1 = 12$
PV	$-1,000$
FV	$?$
PMT	-300
I/Y	5
P/Y	12
C/Y	2

$$FV_1 = \$4,733.411\dots$$

Step 2: Calculate the future value at the end of the second segment.

The future value from the first segment becomes the present value for the second segment:

$$FV_1 = \$4,733.411\dots = PV_2.$$

PMT Setting	END
N	$4 \times 1 = 4$
PV	$-4,733.411\dots$
FV	?
PMT	$-1,000$
I/Y	6
P/Y	4
C/Y	4

$$FV_2 = \$9,114.77$$

Step 3: Write as a statement.

When Genevieve graduates she will have saved **\$9,114.77** toward her vacation.

Try It

3) Carlyle plans to make month-end contributions of **\$400** to his RRSP from age 20 to age 40. From age 40 to age 65, he plans to make no further contributions to his RRSP. The RRSP can earn **9%** compounded annually from age 20 to age 60, and then **5%** compounded annually from age 60 to age 65. Under this plan, what is the maturity value of his RRSP when he turns 65?

Solution

PMT Setting	END	END	END
N	240	20	5
PV	0	-255,540.68...	1,432,254.94...
FV	255,540.68...	1,432,254.94...	1,827,832.95
PMT	-400	0	0
I/Y	9	9	5
P/Y	12	1	1
C/Y	1	1	1

$$FV = \$1,827,832.95$$

2.3 Future Value of Annuities Due

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

FV is the future value of the annuity: the future value is the sum of the future value of all of the annuity payments at the end of the last payment interval.

PMT is the annuity payment: the amount of money that is invested or paid after each payment interval.

i_2 is the periodic interest rate: for the interest rate whose compounding frequency matches the payment frequency. If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest

conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.

n is the total number of payments made during the annuity: $n = P/Y \times t$ where P/Y is the payment frequency and t is the time in years.

For investment annuities, the amount of interest (I) earned by the annuity is

$$I = FV - n \times PMT$$

Key Takeaways



The steps required to solve the future value of an annuity due are identical to those you use for an ordinary annuity except you use the formula for the future value of an annuity due.

If the payment setting is NOT specified in the question, it is assumed that the payments come at the end of the interval.

Using the TI BAII Plus Calculator to Find the Future Value for Annuities Due

To adapt your calculator to an annuity due, you must toggle the payment setting from **END** to **BGN**. The calculator default is **END**, which is the ordinary annuity. The payment setting is found on the second shelf above the PMT key (because it is related to the PMT !).

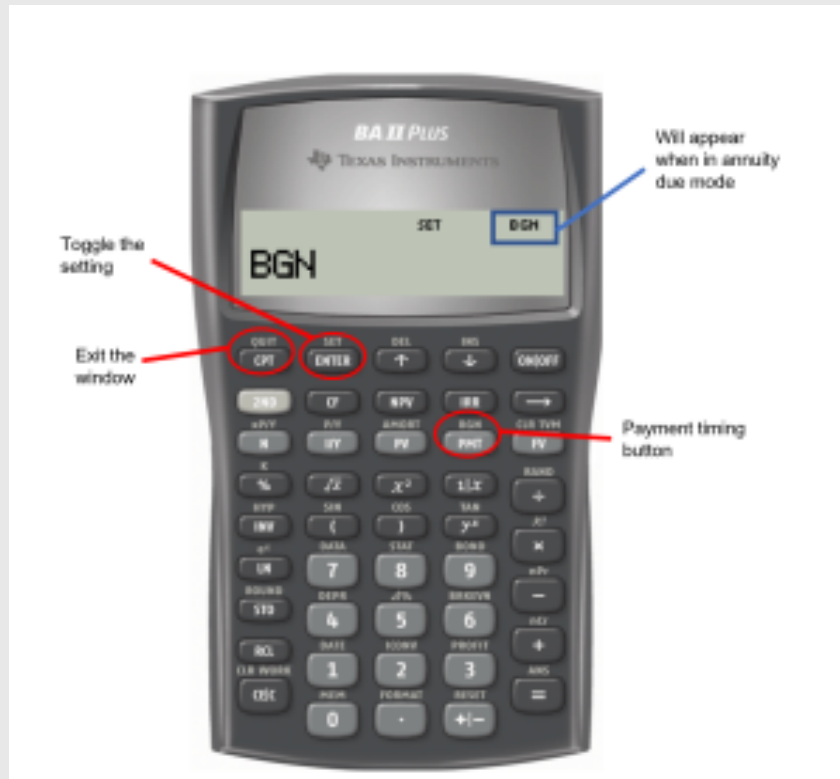


Figure 5.2.6

Image Description

The image depicts a Texas Instruments BA II Plus financial calculator with certain buttons and display elements highlighted and annotated for instructional purposes.

- **Top of calculator:** The display screen shows “BGN”, “SET”, and “END”. An annotation points to “BGN” indicating that it will appear when in annuity due mode.
- **Left side of keypad:** The button labeled “ENTER” is highlighted, and an annotation points out that it is used to toggle the setting.
- **Middle of keypad:** The button labeled “CPT” is highlighted, with an annotation indicating that it is used to exit the window.
- **Right side of keypad:** The button labeled “PMT” is highlighted, with an annotation indicating it is the payment timing button.

To change the payment setting, complete the following sequence.

1. Press **2nd BGN** (the *PMT* button).

2. Press **2nd SET** (the **ENTER** button).
3. Press **2nd QUIT** (the **CPT** button).

When the calculator is in annuity due mode, a tiny **BGN** appears in the upper right-hand corner of your calculator. To return the calculator to ordinary mode, repeat the above keystrokes. When the calculator is in ordinary annuity mode there is nothing in the upper right-hand corner.

To calculate the future value of annuity due, make sure the calculator is in **BGN** mode. Then enter the values of N , I/Y , PV , PMT , P/Y and C/Y into the **TVM** money keys on the calculator and compute FV.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=70#oembed-2>

[Annuity Due Calculations Using the BAII Plus](#) by Joshua Emmanuel [2:51] (transcript available).

Example 2.2.6

The Set for Life instant scratch n' win ticket offers players a chance to win \$1,000 per week for the next 25 years starting immediately upon validation. If a winner was to invest all of his money into an account earning 5% compounded annually, how much money would he have at the end of his 25-year term? Assume each year has exactly 52 weeks.

Solution

The timeline for the lottery savings is below.

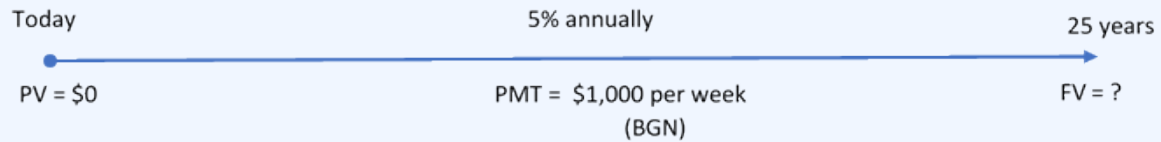


Figure 2.2.7

The payments start immediately, and the compounding period and payment intervals are different. Therefore, this is a general annuity due.

PMT Setting	BGN
N	$52 \times 25 = 1,300$
PV	0
FV	?
PMT	-1,000
I/Y	5
P/Y	52
C/Y	1

$$FV = \$2,544,543.22$$

If the winner was to invest all of his lottery prize money, he would have **\$2,544,543.22** after **25** years.

Try It

4) The Saskatchewan Roughriders started a rainy-day savings fund three-and-a-half years ago to help pay for stadium improvements. At the beginning of every quarter the team deposited \$20,000 into the fund, which has been earning 4.85% compounded semi-annually. How much money is in the fund today? How much interest did the fund earn?

Solution

PMT Setting	BGN
N	$4 \times 3.5 = 14$
PV	0
FV	?
PMT	-20,000
I/Y	4.85
P/Y	4
C/Y	2

$$FV = \$306,680.93$$

$$\begin{aligned}
 I &= FV - n \times PMT \\
 &= 306,680.93 - 14 \times 20,000 \\
 &= \$26,680.93
 \end{aligned}$$

Example 2.2.7

When Roberto's son was born, Roberto started making payments of **\$1,000** at the beginning of every six months to a trust fund earning **5.75%** compounded monthly. After five years, he changed his contributions and started depositing **\$500** at the beginning of every quarter. How much money will be in his son's trust fund when his son turns **18**?

Solution

The timeline for the trust fund is shown below.

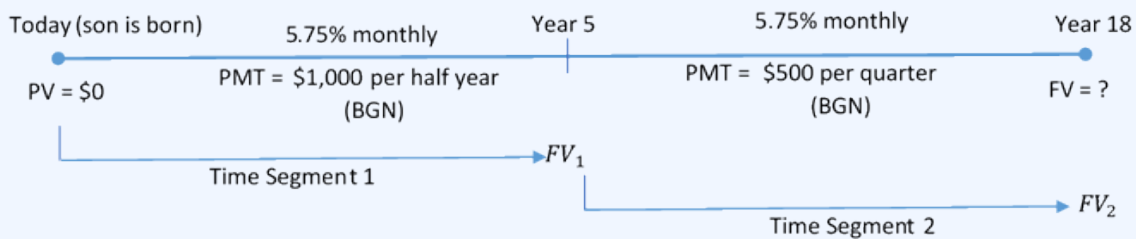


Figure 2.2.8

Step 1: Find the future value at the end of the first segment.

PMT Setting	BGN
N	$2 \times 5 = 10$
PV	0
FV	?
PMT	-1,000
I/Y	5.75
P/Y	2
C/Y	12

$$FV_1 = \$11,748.474\dots$$

Step 2: Calculate the future value at the end of the second segment.

The future value from the first segment becomes the present value for the second segment:

$$FV_1 = \$11,748.474... = PV_2.$$

PMT Setting	BGN
N	$4 \times 13 = 52$
PV	$-11,748.474...$
FV	?
PMT	-500
I/Y	5.75
P/Y	4
C/Y	12

$$FV_2 = \$63,672.39$$

Step 3: Write as a statement.

When Roberto's son turns 18, the trust fund will have a balance of **\$63,672.39**.

Section 2.2 Exercises

1. You invest **\$2,000** at the beginning of every quarter into a savings account earning **7%** compounded quarterly. How much money is in your account in **10** years? How much interest did the account earn?

Solution

\$116,471.46, \$36,471.46

2. For **20** years, Joan deposits **\$375** every quarter into her RRSP. If the RRSP earns **9%** compounded monthly, how much money does Joan have in her RRSP at the end of the **20** years?

Solution

\$250,457.58

3. Your savings account currently has **\$15,000**. For the next **30** years, you plan to deposit **\$3,000** at the end of every year. If the account earns **5.6%** compounded quarterly, how much money will you have in the account at the end of the **30** years?

Solution

\$305,305.23

4. Jesse's RRSP has a balance of **\$38,000**. Over the next **8** years, Jesse will invest **\$1,500** at the beginning of every month. The RRSP earns **8%** compounded semi-annually. How much money will be in Jesse's RRSP in **8** years?

Solution

\$272,152.25

5. You open a savings account to save money for a down payment on a house. You deposit **\$1,000** every quarter for **10** years. For the first four years, the account earns **6%** compounded annually. For the last six years, the account earns **7%** compounded semi-annually. How much money can you save for your down payment? How much interest did you earn?

Solution

\$56,486.35, \$16,486.35

6. Phil invests **\$100** at the beginning of every month into an RRSP for **10** years. For the first **6.5** years, the RRSP earns **12%** compounded quarterly. For the last **3.5** years, the RRSP earns **11%** compounded semi-annually. How much money is in Phil's RRSP at the end of the **10** years? How much interest did Phil's RRSP earn?

Solution

\$22,278.17, \$10,278.17

7. Nikola is currently **47** years old and planning to retire at age **60**. She has already saved **\$220,000** in her RRSP. If she continues to contribute **\$200** at the beginning of every month, how much money will be in her RRSP at retirement if it can earn **8.1%** compounded monthly? No deposit is made the day she turns **60**.

Solution

\$683,712.33

8. The human resource department helps employees save by taking preauthorized RRSP deductions from employee paycheques and putting them into an investment. For the first five years, Margaret has had **\$50** deducted at the beginning of every biweekly pay period. Then for the next five years, she increased the deduction to **\$75**. The company has been able to average **8.85%** compounded monthly for the first seven years, and then **7.35%** compounded monthly for the last three years. What amount has Margaret accumulated in her RRSP after **10** years? Assume there are **26** biweekly periods in a year.

Solution

\$24,035.26

9. Joshua is opening up a Builder GIC that allows him to make regular contributions to his GIC throughout the term. He will initially deposit **\$10,000**, then at the end of every month for the next five years he will make **\$100** contributions to his GIC. The annually compounded interest rates on the GIC in each year are **0.75%**, **1.5%**, **2.5%**, **4.5%**, and **7.25%**. What is the maturity value of his GIC?

Solution

\$18,452.55

10. How much more money would an individual who makes **\$300** contributions to her RRSP at the beginning of the month have compared to another individual who makes **\$300** contributions to his RRSP at the end of the month? Assume a term of **30** years and that both RRSPs earn **9%** annually.

Solution

\$3680.30

11. Assume a **10**-year ordinary annuity earning **10%** compounded annually.

- a. If **\$5,000** is deposited annually, what is the maturity value?
- b. What is the maturity value if the deposits are doubled to **\$10,000**? Compared to (a), what is the relationship between the size of the deposit and the maturity value, all other conditions being held equal?
- c. What is the maturity value if the **\$5,000** deposits are made semi-annually? Compared to (b), what is the relationship between the frequency of payments and the maturity value, all other conditions being held equal?

Solution

a. \$79,687.12, b. \$159,377.24, c. \$163,263.68

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2.3 PRESENT VALUE OF ANNUITIES

Learning Objectives

- Calculate the present value of an ordinary annuity
- Calculate the present value of an annuity due

Formula & Symbol Hub

Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of annuity payments

Formulas Used

- Formula 2.1 – **Total Number of Payments (Annuity)**

$$n = P/Y \times t$$

- Formula 2.2 – **Future Value of Ordinary Annuity**

$$FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- Formula 2.3 – **Future Value of Annuity Due**

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- Formula 2.4 – **Present Value of Ordinary Annuity**

$$PV = PMT \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

- Formula 2.5 – **Present Value of Annuity Due**

$$PV = PMT \times (1 + i_2) \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

Introduction

The **present value of any annuity** is equal to the sum of the present values of all the annuity payments when they are moved to the beginning of the first payment interval. For example, assume you will receive \$1,000 annual payments at the end of every payment interval for the next three years from an investment earning 10% compounded annually. How much money needs to be in the annuity at the start to make this happen? To find out how much money you need in the investment to sustain these payments, you would apply the fundamental concept of the time value of money to move each payment amount to the start of the first payment interval (the focal date) and sum the values to arrive at the present value.

Although you could use this technique to solve all present value of an annuity situations, the computations

become increasingly cumbersome as the number of payments increases. In the above example, what if the person instead received quarterly payments of \$250 for three years? That is 12 payments over three years, resulting in 12 separate present value calculations. Or if they received monthly payments, the 36 payments over three years would result in 36 separate present value calculations! Clearly, solving this would be tedious and time consuming—not to mention prone to error.

2.4 Present Value of Ordinary Annuities

$$PV = PMT \times \left[1 - \frac{(1 + i_2)^{-n}}{i_2} \right]$$

PV is the Present Value of the Annuity: The present value is the sum of the present value of all of the annuity payments at the start of the first payment interval.

PMT is the Annuity Payment: The amount of money that is invested or paid after each payment interval.

i_2 is the Periodic Interest Rate: for the interest rate whose compounding frequency matches the payment frequency}. If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.

n is the Total Number of Payments made during the Annuity: $n = P/Y \times t$ where P/Y is the payment frequency and t is the time in years.

The amount of interest (I) earned by the annuity is

$$I = n \times PMT - PV$$



Key Takeaways

In a **simple annuity**, the payment frequency and the compounding frequency are equal ($P/Y = C/Y$). In this situation the periodic interest rate $i_2 = \frac{j}{m}$ where j is the nominal interest rate given in the question and m is the compounding frequency (and the payment frequency).

In a **general annuity**, the payment frequency and the compounding frequency are not equal ($P/Y \neq C/Y$). In this situation, the given interest rate must first be converted to the equivalent interest rate where the new compounding frequency equals the payment frequency. Using the equivalent interest rate, calculate the periodic interest rate i_2 .

Example 2.3.1

Rodriguez is planning on having an annual gross income of **\$50,000** at the end of every year when he retires at age **65**. He is planning for the account to be emptied by age **78**, which is the average life expectancy for a Canadian man. If the account earns **5.1%** compounded annually, how much money needs to be in the account when he retires? How much interest will he receive from the fund?

Solution

The timeline for Rodriguez's account appears below.

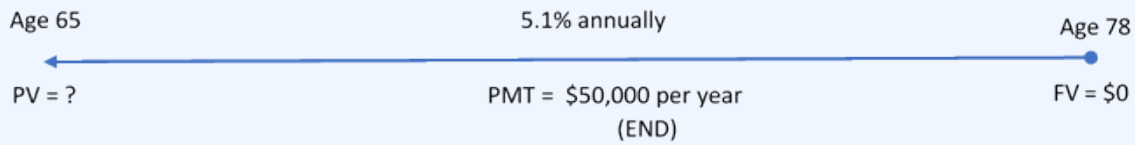


Figure 2.3.1

Step 1: The given information is

The payments are at the end of the payment intervals, and both the compounding frequency and the payment frequency are the same (both annually). This is an ordinary simple annuity. Because this is a simple annuity, an interest rate conversion is not required.

$$PMT = \$50,000$$

$$P/Y = 1$$

$$I/Y = 5.1\%$$

$$C/Y = 1$$

$$t = 13 \text{ years}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned} i_2 &= \frac{I/Y}{C/Y} \\ &= \frac{5.1\%}{1} \\ &= 5.1\% \end{aligned}$$

Step 3: Calculate the number of payments.

$$\begin{aligned} n &= P/Y \times t \\ &= 1 \times 13 \\ &= 13 \end{aligned}$$

Step 4: Calculate the present value.

$$\begin{aligned} PV &= PMT \times \left[1 - \frac{(1 + i_2)^{-n}}{i_2} \right] \\ &= 50,000 \times \left[\frac{1 - (1 + 0.051)^{-13}}{0.051} \right] \\ &= \$466,863.69 \end{aligned}$$

Step 5: Calculate the interest earned.

$$\begin{aligned}
 I &= n \times PMT - PV \\
 &= 13 \times 50,000 - 466,863.69 \\
 &= \$183,136.31
 \end{aligned}$$

Step 6: Write as a statement.

Rodriguez will need to have **\$466,863.69** in his account when he turns **65** if he wants to receive **13** years of **\$50,000** payments. He will receive **\$183,136.31** in interest from the fund.

2.5 Present Value of Annuities Due

$$PV = PMT \times (1 + i_2) \times \left[1 - \frac{(1 + i_2)^{-n}}{i_2} \right]$$

PV is the Present Value of the Annuity: The present value is the sum of the present value of all of the annuity payments at the start of the first payment interval.

PMT is the Annuity Payment: The amount of money that is invested or paid after each payment interval.

i₂ is the Periodic Interest Rate: for the interest rate whose compounding frequency matches the payment frequency. If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.

n is the Total Number of Payments made during the Annuity: $n = P/Y \times t$ where P/Y is the payment frequency and t is the time in years.

The amount of interest (*I*) earned by the annuity is

$$I = n \times PMT - PV$$



Key Takeaways

The steps required to solve the present value of an annuity due are identical to those you use for an ordinary annuity except you use the formula for the present value of an annuity due. If the payment setting is NOT specified in the question, it is assumed that the payments come at the end of the interval.

Example 2.3.2

Tom has a savings account that earns 6% compounded monthly. How much money does Tom need in his account today in order to receive beginning-of-month payments of \$250 for the next two years? How much interest does Tom receive?

Solution

Step 1: The given information is

The payments are at the beginning of the payment intervals, and both the compounding frequency and the payment frequency are the same (both monthly). This is a simple annuity due. Because this is a simple annuity, an interest rate conversion is not required.

$$\begin{aligned}
 PMT &= \$250 \\
 P/Y &= 12 \\
 I/Y &= 6\% \\
 C/Y &= 12 \\
 t &= 2 \text{ years}
 \end{aligned}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned}
 i_2 &= \frac{I/Y}{C/Y} \\
 &= \frac{6\%}{12} \\
 &= 0.5\%
 \end{aligned}$$

Step 3: Calculate the number of payments.

$$\begin{aligned}
 n &= P/Y \times t \\
 &= 12 \times 2 \\
 &= 24
 \end{aligned}$$

Step 4: Calculate the present value.

$$\begin{aligned}
 PV &= PMT \times (1 + i_2) \times \left[1 - \frac{(1 + i_2)^{-n}}{i_2} \right] \\
 &= 250 \times \left[\frac{1 - (1 + 0.005)^{-24}}{0.005} \right] \\
 &= \$5,668.92
 \end{aligned}$$

Step 5: Calculate the interest earned.

$$\begin{aligned}
 I &= n \times PMT - PV \\
 &= 24 \times 250 - 5,668.92 \\
 &= \$331.08
 \end{aligned}$$

Step 6: Write as a statement.

Tom needs **\$5,668.92** in his account to receive the **24** payments of **\$250**. He will receive **\$331.08** in interest.

Using a Financial Calculator

The above examples were fairly straightforward annuity questions. But even these simple examples, which did not require interest conversions, are cumbersome, and time-consuming, to solve using the formulas. A financial calculator can quickly solve annuity problems, with the added bonus of not requiring an interest conversion in situations where the payment frequency and compounding frequency are not equal.

Using the TI BAII Plus Calculator to Find the Present Value for Annuities

- Set the calculator to the correct payment setting (**END** or **BGN**).
- Enter values for the known variables (FV , PMT , N , I/Y , P/Y and C/Y), paying close attention to the cash flow sign convention for PMT and FV .
- After all of the known quantities are loaded into the calculator, press **CPT** and then PV to solve for the present value.

Notes

1. Ensure that the calculator is set to the required payment setting. The present value will be different for **END** and **BGN**.
2. Because you can enter P/Y and C/Y , there is no need to do an interest conversion for a general annuity. You simply have to tell the calculator the P/Y and C/Y given in the question, and the calculator will handle any interest conversion internally.
3. If there is no money in the account or investment at the end of the term then $FV = 0$.
4. The values you enter for PV , FV , and PMT must adhere to the cash flow sign

convention.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=75#oembed-1>

[Ordinary Annuity Calculations \(PV, PMT, FV\)](#) by Joshua Emmanuel [4:31] (transcript available).



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=75#oembed-2>

[Annuity Due Calculations Using BAII Plus-Part 2](#) by Joshua Emmanuel [2:05] (transcript available).

Example 2.3.3

Rodriguez is planning on having an annual gross income of **\$50,000** at the end of every year

when he retires at age **65**. He is planning for the account to be emptied by age **78**, which is the average life expectancy for a Canadian man. If the account earns **5.1%** compounded annually, how much money needs to be in the account when he retires?

Solution

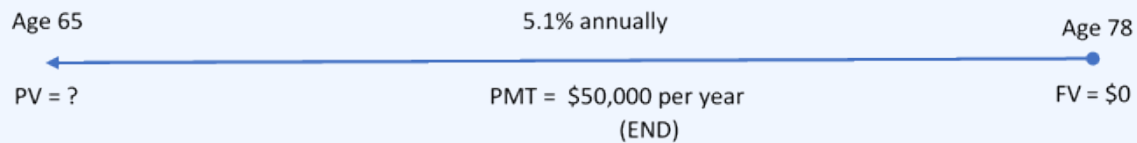


Figure 2.3.2

The timeline for the Rodriguez's account appears below.

PMT Setting	END
N	$1 \times 13 = 13$
PV	?
FV	0
PMT	50,000
I/Y	5.1
P/Y	1
C/Y	1

$$PV = \$466,863.69$$

Rodriguez will need to have **\$466,863.69** in his account when he turns **65** if he wants to receive **13** years of **\$50,000** payments.

Try It

1) When Sinbad retires, he expects his RRSP to pay him **\$2,000** at the end of every month for **25** years. If his retirement annuity earns **3.8%** compounded quarterly, how much money does he need to have in his RRSP when he retires? How much interest will he receive from the RRSP?

Solution

PMT Setting	END
N	300
PV	?
FV	0
PMT	2,000
I/Y	3.8
P/Y	12
C/Y	4

$$PV = \$387,444.19$$

$$\begin{aligned}
 I &= n \times PMT - PV \\
 &= 300 \times 2,000 - 387,444.19 \\
 &= \$212,555.81
 \end{aligned}$$

When you purchase an expensive or large-ticket item such as a car or house, you often need to take out a loan to make the purchase. After all, most people do not have large sums of cash available to pay for these items outright. With these expensive items, you often make an up-front partial payment on the purchase price called a **down payment**, and then take out a loan for the outstanding purchase price.

For example, suppose you want to purchase a \$100,000 luxury sports car. You pay \$10,000 as a down

payment and take out a loan for the balance. The \$10,000 down payment is cash you are paying up-front on the purchase price. You would then need to take out a loan for the remaining \$90,000 of the purchase price. The \$90,000 is a loan, so you would be charged interest for borrowing the money and then you would need to make regularly scheduled payments to pay back the loan. Because the \$10,000 down payment is paid up-front, there is no interest charged on this \$10,000. By making the \$10,000 down payment, you are reducing the amount of money you need to borrow, and consequently reducing the amount of interest you need to pay.

In general

$$\text{Purchase Price} = \text{Down Payment} + \text{Loan Amount}$$

Example 2.3.4

Charlie purchased a new car. To pay for the car, she made a \$5,000 down payment on the purchase price and took out a loan for the balance at an interest rate of 2.9% compounded monthly. Charlie made \$550 monthly payments for six years to settle the loan.

1. What is the purchase price of the car?
2. How much interest did Charlie pay to purchase the car?

Solution

Step 1: Calculate the purchase price of the car.

PMT Setting	END
N	$12 \times 6 = 72$
PV	?
FV	0
PMT	-550
I/Y	2.9
P/Y	12
C/Y	12

$$PV = \$36,306.08$$

$$\begin{aligned}\text{Purchase Price} &= \text{Down Payment} + \text{Loan Amount} \\ &= 5,000 + 36,306.08 \\ &= \$41,306.08\end{aligned}$$

Step 2: Calculate the interest.

Because the \$5,000 is paid upfront, there is no interest charged on the down payment. Interest is only charged on the loan amount.

$$\begin{aligned}I &= n \times PMT - PV \\ &= 72 \times 550 - 36,306.08 \\ &= \$3,293.92\end{aligned}$$

Step 3: Write as a statement.

The purchase price of the car was \$41,306.08. Charlie paid \$3,293.92 in interest to purchase the car.

Try It

2) You want to purchase a piece of vacant land on the river so that you can build your dream home when you retire. The bank agrees to lend you the money but you have to make a **\$25,000** down payment and make payments of **\$800** every month for ten years. The bank charges you interest at **3.5%** compounded semi-annually.

1. What is the purchase price of the land?
2. How much interest will you pay?

Solution

PMT Setting	END
<i>N</i>	120
<i>PV</i>	?
<i>FV</i>	0
<i>PMT</i>	−800
<i>I/Y</i>	3.5
<i>P/Y</i>	12
<i>C/Y</i>	2

$$PV = \$80,998.2$$

$$\begin{aligned}\text{Purchase Price} &= \text{Down Payment} + \text{Loan Amount} \\ &= 25,000 + 80,998.20 \\ &= \$105,998.20\end{aligned}$$

$$\begin{aligned}I &= n \times PMT - PV \\ &= 120 \times 800 - 80,998.20 \\ &= \$15,001.80\end{aligned}$$

Example 2.3.5

Rodriguez is planning on having an annual gross income of **\$50,000** at the beginning of every year when he retires at age **65**. He wants to leave a **\$100,000** inheritance for his children, assuming he dies at age **78**. If the account earns **5.1%** compounded semi-annually, how much money needs to be in the account when he retires?

Solution

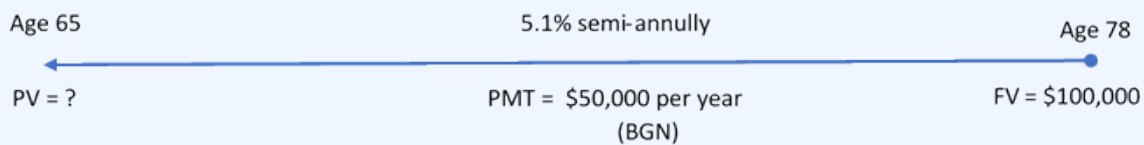


Figure 2.3.3

The timeline for the Rodriguez's account appears below.

PMT Setting	BGN
N	$1 \times 13 = 13$
PV	?
FV	100,000
PMT	50,000
I/Y	5.1
P/Y	1
C/Y	2

$$PV = \$541,027.07$$

Rodriguez will need to have **\$541,027.07** in his account when he turns **65** if he wants to receive **13** years of **\$50,000** payments and leave a **\$100,000** inheritance for his children.

Try It

3) Sandy's parents would like to have an annuity pay her **\$500** at the beginning of every month from September 1, 2012 to April 1, 2017 to help with her university tuition and living expenses. On May 1, 2017, they would like to give her a graduation gift of **\$5,000**. If the annuity can earn **6.15%** compounded quarterly, how much money must be in the account on September 1, 2012? (Use years and months in the calculations.)

Solution

PMT Setting	BGN
<i>N</i>	56
<i>PV</i>	?
<i>FV</i>	5,000
<i>PMT</i>	500
<i>I/Y</i>	6.15
<i>P/Y</i>	12
<i>C/Y</i>	4

$$PV = \$28,188.42$$

Example 2.3.6

Rodriguez is planning on having an annual gross income of **\$50,000** at the beginning of every year when he retires at age **65** until age **71**. Realizing he needs to make some type of adjustment to his annual gross income to account for the rising cost of living, starting at age **71** he wants his annual payments to increase to **\$60,000** until age **78**. He wants to leave a **\$100,000** inheritance for his children, assuming he dies at age **78**. If the account earns **5.1%** compounded semi-annually, how much needs to be in the account when he retires?

Solution

The timeline for the Rodriguez's account appears below.

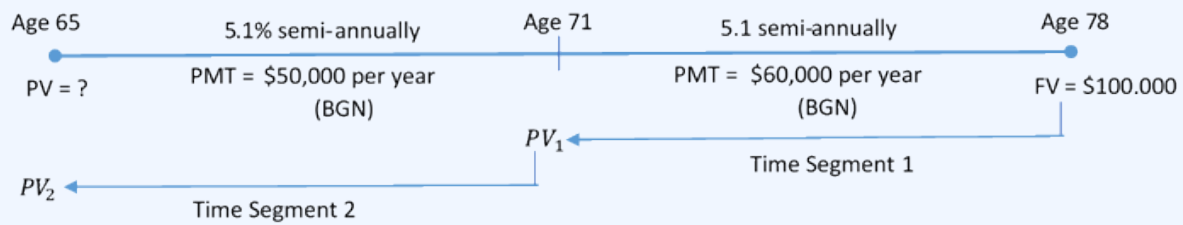


Figure 2.3.4

Step 1: Find the present value at the start of time segment 1.

PMT Setting	BGN
N	$1 \times 7 = 7$
PV	?
FV	100,000
PMT	60,000
I/Y	5.1
P/Y	1
C/Y	2

$$PV_1 = \$433,232.035\dots$$

Step 2: Calculate the present value at the start of time segment 2.

The present value from time segment 1 becomes the future value for time segment 2:

$$PV_1 = \$433,232.035\dots = FV_2.$$

PMT Setting	BGN
N	$1 \times 6 = 6$
PV	?
FV	433,232.035...
PMT	50,000
I/Y	5.1
P/Y	1
C/Y	2

$$PV_2 = \$585,742.42$$

Step 3: Write as a statement.

Rodriguez needs to have **\$585,742.42** invested in his retirement fund at age **65**.

Try It

4) You want to purchase an annuity that will provide you with payments over the next twelve years. The annuity will pay you **\$1,000** at the end of every quarter for the first seven years and **\$3,000** at the end of every six months for the next five years. At the end of the twelve years the annuity is exhausted. If the annuity earns interest at **4.7%** compounded monthly, what is the purchase price of the annuity? How much interest does the annuity pay?

Solution

PMT Setting	END	END
<i>N</i>	10	28
<i>PV</i>	-26,429.159...	42,756.53
<i>FV</i>	0	26,429.159...
<i>PMT</i>	3,000	1,000
<i>I/Y</i>	4.7	4.7
<i>P/Y</i>	2	4
<i>C/Y</i>	12	12

$$PV = \$42,756.53$$

$$\begin{aligned} I &= 10 \times 3,000 + 28 \times 1,000 - 42,756.53 \\ &= \$15,240.47 \end{aligned}$$

Section 2.3 Exercises

1. You bought a new luxury sports car for **\$5,000** down and payments of **\$2,000** at the beginning of every quarter for **10** years. If money earns **7%** compounded quarterly, what was the purchase price of the car? How much interest did you pay on the loan?

Solution

\$63,189.28, \$21,810.74

2. You have an investment that pays you **\$3,000** every year for **30** years. How much money must you invest today at **5.6%** compounded quarterly?

Solution

\$42,567.80

3. You took out a loan that requires you to pay **\$1,500** every month for **8** years. How much money did you borrow if the interest on the loan is **8%** compounded semi-annually? How much interest did you pay?

Solution

\$106,605.18, \$37,394.82

4. Your RIF pays you **\$1,000** at the end of every quarter for ten years. For the first six years the RIF earns **7%** compounded semi-annually and for the last four years the RIF earns **6%** compounded annually. How much money is in your RIF today? How much interest does the RIF pay?

Solution

\$28,879.80, \$11,129.20

5. You want to purchase an annuity that will pay you **\$1,000** every quarter for the first **10** years and then **\$750** every month for the next **5** years. If the annuity earns **11%** compounded quarterly, what is the purchase price of the annuity? How much interest does the annuity pay?

Solution

\$35,758.83, \$49,241.17

6. You invest some money into an investment at **10%** effective that pays you **\$4,000** a year for seven years. At the end of the seven years you take the balance in the investment and place it in a savings account at **8%** effective and you will withdraw **\$5,000** a year from the account for three years at which point the account is empty. How much money do you need to invest?

Solution

\$26,085.97

7. When Sinbad retires, he expects his RRSP to pay him **\$2,000** at the end of every month for **25** years. If his retirement annuity earns **3.8%** compounded quarterly, how much

money does he need to have in his RRSP when he retires?

Solution

\$387,444.19

8. The Workers' Compensation Board has determined that an injury in the workplace was your company's responsibility. As a result, your company has been ordered to pay the employee **\$3,000** at the end of every month for the next four years. Your human resource manager wants to set up an annuity to fund this obligation. If the proposed annuity can earn **5.7%** compounded monthly for the first two-and-a-half years and then **6%** compounded quarterly for the remaining one-and-a-half years, how much money should your company set aside today to meet its responsibilities?

Solution

\$128,398.17

9. Lynne acquired a Sea Ray Sundancer boat and put **\$4,000** down. For the past two years, her end-of-month payments have been **\$1,049.01** including **9.32%** compounded monthly. If she still owes **\$22,888.78** today, what was the purchase price of the boat?

Solution

\$45,898.34

10. Gerald has been granted power of attorney and is now responsible for setting up his aging parents in a seniors' home. The rent will be **\$2,490** at the beginning of every month for the first year, then increase by **5%** the following year and **4%** in the third year. Gerald wants to take money from his parents' estate and set up an annuity to pay their monthly rent. If he can get an annuity that earns **3.75%** semi-annually, how much money from his parents' estate needs to be invested today to meet the rental payments over the next three years?

Solution

\$88,880.80

11. Compare the amount of money that needs to be invested today to provide the required payments from the investment fund annuities earning **9%** compounded semi-annually.

- a. Payments of \$1,000 quarterly for 40 years.
- b. Payments of \$1,600 quarterly for 25 years.
- c. Payments of \$4,000 quarterly for 10 years.

Note that in all three of these annuities the same total payout occurs. Explain your results and comment on your findings.

Solution

a. \$43,610.57; b. \$63,942.03; c. \$105,221.32

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2.4 CALCULATING THE PAYMENT

Learning Objectives

- Calculate the payment for an annuity

Formula & Symbol Hub

Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of annuity payments

Formulas Used

- Formula 2.1 – **Total Number of Payments (Annuity)**

$$n = P/Y \times t$$

- **Formula 2.2 – Future Value of Ordinary Annuity**

$$FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.3 – Future Value of Annuity Due**

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.4 – Present Value of Ordinary Annuity**

$$PV = PMT \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

- **Formula 2.5 – Present Value of Annuity Due**

$$PV = PMT \times (1 + i_2) \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

Introduction

Whether you are acquiring merchandise, property or saving up toward some future goal such as retirement, you will deal with annuity payments. When you graduate college and land that promising entry-level position with your employer, a lot of demands are going to get placed on your limited income. If you do not already own a place of your own, perhaps you will get one. This means the purchase of a starter home for which you will make monthly mortgage payments. To fill that home, acquiring some furniture and electronics might take you to The Brick, Sleep Country Canada, Best Buy, or The Home Depot. Then you may be staggered by all the home maintenance items you need. If you make a lot of purchases all at once, you will probably take advantage of various payment plans. These place even more demands on your monthly income. Do not forget that you will need some wheels too. You can either lease or purchase a car. Great, another payment to make! Finally, you remember what your math instructor taught you about the importance of getting started early on your RRSP, so you should begin making those monthly contributions soon, too.

Similarly, businesses also make annuity payments for a wide variety of purposes, such as saving up for future corporate goals or acquiring products and property, businesses have regular bills, too. Marketers develop

payment plans for their consumers. Financial agents make investments involving periodic payments. Companies issue marketable bonds that require regular interest payments to investors. Human resource personnel look after employee benefits, including RRSP contributions and pension plan payments. Production departments need expensive machinery, so they must keep payment plans within operating budgets. No matter your choice of profession, as a business manager you will encounter annuity payment calculations.

You need to calculate an annuity payment in many situation:

- Figuring out loan or mortgage payments.
- Determining membership or product payment plans.
- Calculating lease payments.
- Determining the periodic payment necessary to achieve a savings goal.
- Determining the maximum payment that an investment annuity can sustain over period of time.

Using a Financial Calculator

Recall that the annuity payment, PMT , is one of the variables in the future value and present value formulas for annuities. Calculating the payment requires you to determine which formula to use (depending on the type of annuity and whether you have the future value or present value), and then rearranging the formula to solve for PMT .

As we have seen, a financial calculator can quickly find the payment amount. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is PMT (the payment). You must still load the other six variables into the calculator, set the calculator to the correct payment setting, and apply the cash flow sign conventions carefully.

Using the TI BAII Plus Calculator to Find an Annuity Payment

- Set the calculator to the correct payment setting (**END** or **BGN**).

- Enter values for the known variables (PV , FV , N , I/Y , P/Y and C/Y), paying close attention to the cash flow sign convention for PV and FV .
- After all of the known quantities are loaded into the calculator, press CPT and then PMT to solve for the payment.

HOW TO

1. Ensure that the calculator is set to the required payment setting. The payment will be different for **END** and **BGN**.
2. The values you enter for PV and FV must adhere to the cash flow sign convention.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=81#oembed-1>

Video: [Ordinary Annuity Calculations \(PV, PMT, FV\)](#) by Joshua Emmanuel [4:31] *Transcript Available.*



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=81#oembed-2>

Video: [BAIL Plus Down Payment Questions|PV and PMT](#) by Joshua Emmanuel [3:54] *Transcript Available.*

Example 2.4.1

Morgan wants to consolidate a lot of smaller debts into a single three-year loan for \$25,000. If the loan is charged interest at 7.8% compounded monthly, what is her payment amount at the end of every month?

Solution

The timeline for the loan appears below.

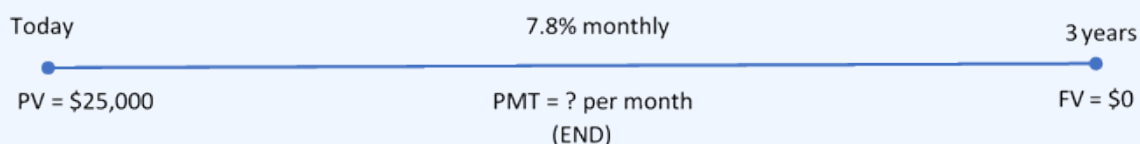


Figure 2.4.1

PMT Setting	END
N	$12 \times 3 = 36$
PV	25,000
FV	0
PMT	?
I/Y	7.8
P/Y	12
C/Y	12

$$PMT = \$781.10$$

To pay off her consolidated loan, Morgan's month-end payments for the next three years are \$781.10.

Example 2.4.2

You plan to retire in **20** years. When you do, you want to have saved **\$200,000** in an RRSP. To accomplish this goal, you will make quarterly payments into an RRSP that earns **3.1%** compounded semi-annually.

1. Calculate the size of the quarterly payments.
2. How much interest does the RRSP earn?

Solution

Step 1: Calculate the quarterly payments.

PMT Setting	END
N	$4 \times 20 = 80$
PV	0
FV	200,000
PMT	?
I/Y	3.1
P/Y	4
C/Y	2

$$PMT = \$1,816.29$$

Step 2: Calculate the interest.

$$\begin{aligned}
 I &= FV - n \times PMT \\
 &= 200,000 - 80 \times 1,816.29 \\
 &= \$54,696.80
 \end{aligned}$$

Step 3: Write as a statement.

You need to make quarterly payments of **\$1,816.29** to accomplish your goal. You will earn **\$54,696.80** in interest over the **20** years.

Example 2.4.3

Franco has placed **\$10,000** into an investment fund with the goal of receiving equal amounts at the beginning of every month for the next year while he backpacks across Europe. If the investment fund can earn **5.25%** compounded quarterly, how much money can Franco expect to receive each month?

Solution

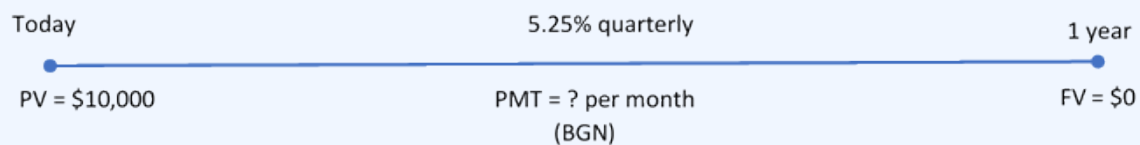


Figure 2.4.2

The timeline for the vacation money appears below.

PMT Setting	BGN
N	$12 \times 1 = 12$
PV	$-10,000$
FV	0
PMT	?
I/Y	5.25
P/Y	12
C/Y	4

$$PMT = \$853.40$$

While backpacking across Europe, Franco's annuity will pay him **\$853.40** at the beginning of every month.

Try It

1) To save approximately **\$30,000** for a down payment on a home four years from today, what amount needs to be invested at the end of every month at **4.5%** compounded semi-annually?

Solution

PMT Setting	END
N	48
PV	0
FV	30,000
PMT	?
I/Y	4.5
P/Y	12
C/Y	2

$$PMT = \$572.08$$

Example 2.4.4

Kingsley's financial adviser has determined that when he reaches age **65**, he will need **\$1.7 million** in his RRSP to fund his retirement. Kingsley is currently **22** years old and has saved up **\$10,000** already. His adviser thinks that his RRSP will average **9%** compounded annually throughout the years. To meet his RRSP goal, how much does Kingsley need to invest every month starting today?

Solution

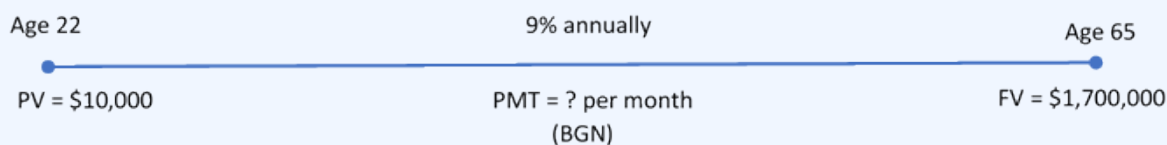


Figure 2.4.3

The timeline for Kingsley's RRSP contributions appears below.

PMT Setting	BGN
N	$12 \times 43 = 516$
PV	$-10,000$
FV	$1,700,000$
PMT	?
I/Y	9
P/Y	12
C/Y	1

$$PMT = \$233.24$$

To meet his retirement goals, Kingsley needs to invest **\$233.24** at the beginning of every month for the next **43** years.

Example 2.4.5

The production department just informed the finance department that in five years' time the robotic systems on the production line will need to be replaced. The estimated cost of the replacement is **\$10** million. To prepare for this purchase, the finance department immediately deposits **\$1,000,000** into a savings annuity earning **6.15%** compounded semi-annually, and it plans to make semi-annual contributions starting in six months. How large do those contributions need to be?

Solution

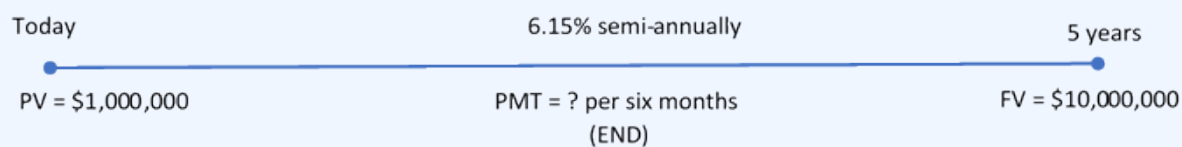


Figure 2.4.4

The timeline for the machinery fund appears below.

PMT Setting	END
N	$2 \times 5 = 10$
PV	-1,000,000
FV	10,000,000
PMT	?
I/Y	6.15
P/Y	2
C/Y	2

$$PMT = \$751,616.87$$

To have adequate funding for the production line machinery replacement five years from now, the finance department needs to deposit **\$751,616.87** into the fund every six months.

Try It

2) Sinclair does not believe in debt and will only pay cash for all purchases. He has already saved up **\$140,000** toward the purchase of a new home with an estimated cost of **\$300,000**. Suppose

his investments earn **7.5%** compounded monthly. How much does he need to contribute at the beginning of each quarter if he wants to purchase his home in five years?

Solution

PMT Setting	BGN
<i>N</i>	20
<i>PV</i>	−140,000
<i>FV</i>	300,000
<i>PMT</i>	?
<i>I/Y</i>	7.5
<i>P/Y</i>	4
<i>C/Y</i>	12

$$PMT = \$3,943.82$$

Example 2.4.6

Over the next **15** years, you will invest **\$900** every month into an RRSP that earns **4.3%** compounded semi-annually. At the end of the **15** years, you plan to retire and will transfer the money in the RRSP to a RIF earning **2.9%** compounded quarterly. Over the next **20** years, you will receive quarterly payments from the RIF. Calculate the size of the quarterly RIF payments.

Solution

Step 1: Calculate the future value at the end of the RRSP.

PMT Setting	END
N	$12 \times 15 = 180$
PV	0
FV	?
PMT	-900
I/Y	4.3
P/Y	12
C/Y	2

$$FV = \$226,289.88\dots$$

Step 2: Calculate the payment for the RIF. The future value from the RRSP becomes the present value for the RIF: $PV = \$226,289.88\dots$

PMT Setting	END
N	$4 \times 20 = 80$
PV	$-226,289.88\dots$
FV	0
PMT	?
I/Y	2.9
P/Y	4
C/Y	4

$$PMT = \$3,737.74$$

Step 3: Write as a statement.

You will receive quarterly payments of **\$3,737.74** from the RIF.

Example 2.4.7

When you retire, you want to receive beginning-of-month payments of **\$750** for **30** years from a RIF earning **4.1%** compounded quarterly. To plan for this, you want to make beginning-of-quarter payments for the next **25** years leading up to your retirement into your RRSP that earns **2.7%** compounded semi-annually. What payments do you need to make to your RRSP?

Solution

Step 1: Calculate the present value at the start of the RIF.

PMT Setting	BGN
N	$12 \times 30 = 360$
PV	?
FV	0
PMT	750
I/Y	4.1
P/Y	12
C/Y	4

$$PV = \$156,005.04\dots$$

Step 2: Calculate the payment for the RRSP. The present value from the RIF becomes the future value for the RRSP: $FV = \$156,005.04\dots$

PMT Setting	BGN
N	$4 \times 25 = 100$
PV	0
FV	156,005.04...
PMT	?
I/Y	2.7
P/Y	4
C/Y	2

$$PMT = \$1,091.40$$

Step 3: Write as a statement.

You will need to make quarterly payments of **\$1,091.40** into the RRSP.

Try It

3) The Kowalskis' only child is eight years old. They want to start saving into an RESP so that their son will be able to receive **\$5,000** at the end of every quarter for four years once he turns **18** and starts attending postsecondary school. When the annuity is paying out, it is forecast to earn **4%** compounded monthly. While they make contributions at the end of every month to the RESP, it will earn **8%** compounded semi-annually. What is the monthly contribution payment by the Kowalskis?

Solution

PMT Setting	END	END
N	16	120
PV	$-73,569.219\dots$	0
FV	0	$73,569.219\dots$
PMT	5,000	405.06
I/Y	4	8
P/Y	4	12
C/Y	12	2

$$PMT = \$405.06$$

Section 2.4 Exercises

1. In 5 years, Joan wants to have \$25,000 in her savings account. What beginning-of-month payments does she need to make if the account earns 8.25% compounded monthly? How much interest does the account earn over the 5 years?

Solution

\$335.72, \$4,856.80

2. You have \$500,000 in an investment fund that earns 5.9% compounded semi-annually. Over the next 15 years, you will receive quarterly payments from the fund. Calculate the size of the payments. How much interest does the fund pay?

Solution

\$12,580.44, \$254,826.40

3. You retire today with **\$1,000,000** saved up in your RRSP. You immediately transfer the funds to an RIF earning **4.75%** compounded semi-annually. You will receive beginning-of-month payments from the RIF for **25** years. What is the size of the monthly payments you will receive?

Solution

\$5,652.40

4. How much does Alex need to deposit every year into his savings account if he wants to have **\$1,500,000** in **35** years? The savings account earns interest at **9%** compounded annually.

Solution

\$6,953.76

5. You purchase a **\$58,000** car. You pay **\$4,500** as a down payment and take out a loan for the balance at **3.65%** quarterly. You repay the loan with monthly payments for **6.5** years. What is the size of your monthly loan payment? How much interest did you pay to buy the car?

Solution

\$771.25, \$6,657.50

6. Your business needs to save up for a major capital expense planned in **4** years that will cost **\$5,000,000**. Currently the business has **\$450,000** saved in an investment fund earning **4.35%** compounded monthly. How much does the business need to deposit into the fund at the end of every year to have enough money saved for the project?

Solution

\$1,044,548.68

7. At age **60**, Tiger has managed to save **\$850,000** and decides to retire. He wants to receive equal payments at the beginning of each month for the next **25** years. The annuity can earn **5.4%** compounded quarterly.
- If he plans on depleting the annuity, how much are his monthly payments?

- b. If he wants to have **\$50,000** left over at the end of the annuity, how much are his monthly payments?

Solution

a. **\$5,133.93**; b. **\$5,054.93**

8. To purchase his new **\$50,000** car, Scooby-Doo pays **\$10,000** down and obtains a six-year loan for the balance at **8.8%** compounded semi-annually.
- Determine the monthly payments required on the loan.
 - How much interest does he pay on the loan?

Solution

a. **\$713.95**; b. **\$11,404.40**

9. Gold's Gym wants to offer its clients a monthly payment option on its annual membership dues of **\$490**. If the gym charges **7.75%** compounded quarterly on its membership fee, what beginning-of-month payments should it advertise?

Solution

\$42.29

10. Carling Industries needs to acquire some real estate to expand its operations. In negotiations with the Province of Nova Scotia, it will be allowed to purchase the **\$15** million parcel of land today and start making payments at the end of every six months for the next **10** years. If interest will be charged at **7.6%** compounded semi-annually, what will be the required payments? (Round to the nearest dollar.)

Solution

\$1,084,268

11. You want to have saved **\$100,000** in your savings account in **12** years. For the first **5** years, when the account earns **2.3%** compounded monthly you make monthly payments of **\$235**. For the last **7** years the account earns **3.62%** compounded quarterly. What monthly payments do you need to make during the last **7** years to accomplish your goal? How much interest does your savings account earn?

Solution

\$846.83, \$14,766.28

12. In ten years, your daughter will go to university. During the four years she is in university, you want the RESP you set-up for your daughter to pay her **\$3,000** at the start of every six months. What beginning-of-month deposits do you need to make into the RESP over the next ten years if the RESP earns **3.89%** compounded semi-annually?

Solution

\$153.16

13. For the next **20** years, you deposit **\$1500** every quarter into your RRSP that earns **2.67%** compounded semi-annually. At the end of the **20** years, you retire and convert the money saved in your RRSP to a RIF that earns **3.52%** compounded quarterly. You want to receive monthly payments from the RIF for **25** years. What are the size of the monthly RIF payments?

Solution

\$790.63

14. Santana wants his retirement money to pay him **\$3,000** at the beginning of every month for **20** years. He expects the annuity to earn **6.15%** compounded monthly during this time. If his RRSP can earn **10.25%** compounded annually and he contributes for the next **30** years, how much money does he need to invest into his RRSP at the end of every month? He has already saved **\$15,000** to date.

Solution

\$62.65

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2.5 CALCULATING THE TERM OF AN ANNUITY

Learning Objectives

- Calculate the number of payments in an annuity
- Calculate the term of an annuity

Formula & Symbol Hub

Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of annuity payments

Formulas Used

- **Formula 2.1 – Total Number of Payments (Annuity)**

$$n = P/Y \times t$$

- **Formula 2.2 – Future Value of Ordinary Annuity**

$$FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.3 – Future Value of Annuity Due**

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.4 – Present Value of Ordinary Annuity**

$$PV = PMT \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

- **Formula 2.5 – Present Value of Annuity Due**

$$PV = PMT \times (1 + i_2) \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

Introduction

When saving up for future goals, many people and businesses simply determine how much they can afford to invest each time period and then try to be patient until they meet their savings goal. What they do not know is how long it will take them. For example, if you can put \$75 per month into your vacation fund, how long will it take to save up the \$1,000 needed for a spring break vacation in Puerto Vallarta? How long it takes to fulfill the goal of your annuity depends on your annuity payment, interest rate, and the amount of money involved.

You must calculate the number of annuity payments, and the corresponding term, in a variety of scenarios such as:

- Savings planning.
- Debt extinguishment.
- Sustaining withdrawals from an investment.

Using a Financial Calculator

Although it is possible to find the number of payments by using the appropriate future value or present value formula, it is much more practical to use a financial calculator. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is N (the number of payments in the annuity). You must still load the other six variables into the calculator and apply the cash flow sign conventions carefully.

Using the TI BAII Plus Calculator to Find the Number of Payments for an Annuity

- Set the calculator to the correct payment setting (**END** or **BGN**).
- Enter values for the known variables (PV , FV , PMT , I/Y , P/Y and C/Y), paying close attention to the cash flow sign convention for PV , PMT , and FV .
- After all of the known quantities are loaded into the calculator, press **CPT** and then N to solve for the number of payments.

Things to Watch Out For

Because N is the number of payments made during the annuity, N must be a whole number. However, it is unlikely that the calculation of N will result in a whole number. After N is calculated, the value of N must be rounded UP to the next whole number. For example, if

$N = 10.1397$ after working out the value of N on the calculator, then this value is rounded up to 11.

Finding the Term

The value of N is the **number of payment periods for the annuity**, after the value obtained from the calculator is rounded up to the next whole number. For example, $N = 8.3174$ means that there are 9 payments in the annuity. N does not represent the time period or term. Because, $N = P/Y \times t$, where t is the time in years, the number of years can be found by

$$\text{Number of Years} = \frac{\text{rounded up value of } N}{P/Y}$$

The whole number portion of the number of years (the part on the left-side of the decimal) represents the number of years. As needed, take the decimal number portion (the part on the right-side of the decimal point) and multiply it by 12 to convert it to months.

Key Takeaways



1. To determine the term of the annuity, the value of N must be calculated first.
2. In annuity calculations, the value of N is rounded UP to the next whole number *before* converting to years and months.
3. Because the value of N is rounded up to the next whole number, the last payment in the annuity will typically be smaller than the other payments. In a later section, you will learn

how to calculate the size of the last payment.

Example 2.5.1

Samia has \$500,000 accumulated in her retirement savings when she decides to retire at age 60. If she wants to receive beginning-of-month payments of \$3,000 and her retirement annuity can earn 5.2% compounded monthly, how old is Samia when the fund is depleted?

Solution

The timeline for the retirement annuity appears below.

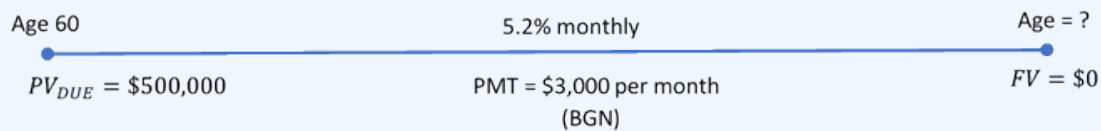


Figure 2.5.1

Step 1: Calculate the value of N .

PMT Setting	BGN
N	?
PV	−500,000
FV	0
PMT	3,000
I/Y	5.2
P/Y	12
C/Y	12

$$N = 293.6601...$$

Rounding up, Samia receives **294** monthly payments.

Step 2: Convert (rounded UP) N to years and months.

$$\begin{aligned}
 \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\
 &= \frac{294}{12} \\
 &= 24.5 \\
 &\rightarrow 24 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.5 \times 12 \\
 &= 6
 \end{aligned}$$

Step 3: Write as a statement.

Samia can receive payments for **24** years and **6** months. If Samia is currently **60** years old and the annuity endures for **24** years and **6** months, then she will be **84.5** years old when the annuity is depleted.

Example 2.5.2

Brendan is purchasing a brand new Mazda MX—5 GT. Including all options, accessories, and fees, the total amount he needs to finance is **\$47,604.41** at the dealer's special interest financing of **2.4%** compounded monthly. If he makes payments of **\$1,000** at the end of every month, how long will it take to pay off his car loan?

Solution

The timeline for the car payments appears below.

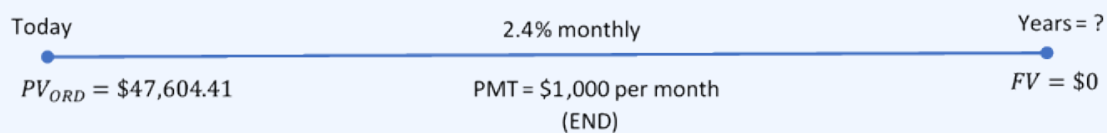


Figure 2.5.2

Step 1: Calculate the value of N .

PMT Setting	END
N	?
PV	47,604.41
FV	0
PMT	−1,000
I/Y	2.4
P/Y	12
C/Y	12

$$N = 50.0755\dots$$

Rounding up, Brendan makes **51** monthly payments.

Step 2: Convert (rounded UP) N to years and months.

$$\begin{aligned}\text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\ &= \frac{51}{12} \\ &= 4.25 \\ &\rightarrow 4 \text{ years}\end{aligned}$$

$$\begin{aligned}\text{Number of Months} &= 0.25 \times 12 \\ &= 3\end{aligned}$$

Step 3: Write as a statement.

To own his vehicle, Brendan will make payments for **4** years and **3** months.

Example 2.5.3

Trevor wants to save up \$1,000,000. He will contribute \$2,500 every six months to an investment earning 5.2% compounded quarterly. How long will it take for Trevor to reach his goal?

Solution

Step 1: Calculate the value of N .

PMT Setting	END
N	?
PV	0
FV	1,000,000
PMT	-2,500
I/Y	5.2
P/Y	2
C/Y	4

$$N = 94.436...$$

Rounding up, Trevor will need to make **95** payments.

Step 2: Convert (rounded UP) N to years and months.

$$\begin{aligned}
 \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\
 &= \frac{95}{2} \\
 &= 47.5 \\
 &\rightarrow 47 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.5 \times 12 \\
 &= 6
 \end{aligned}$$

Step 3: Write as a statement.

Trevor will need to make the semi-annual payments for **47** years and **6** months to reach his goal.

Try It

1) Amarjit wants to save up for a down payment on his first home. A typical starter home in his area sells for **\$250,000** and the bank requires a **10%** down payment. If he starts making **\$300** month-end contributions to an investment earning **4.75%** compounded monthly, how long will it take for Amarjit to have the necessary down payment?

Solution

PMT Setting	END
N	?
PV	0
FV	25,000
PMT	−300
I/Y	4.75
P/Y	12
C/Y	12

$$N = 72.1612... \rightarrow 73 \text{ payments}$$

$$\begin{aligned}
 \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\
 &= \frac{73}{12} \\
 &= 6.0833... \\
 &\rightarrow 6 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.833... \times 12 \\
 &= 1
 \end{aligned}$$

Final Statement: It will take **6** years and **1** month.

Try It

2) Hi-Tec Electronics is selling a **52"** LG HDTV during a special “no sales tax” event for **\$1,995** with end of month payments of **\$100** including interest at **15%** compounded semi-annually. How long will it take a consumer to pay off her new television?

Solution

PMT Setting	END
N	?
PV	1,995
FV	0
PMT	−100
I/Y	15
P/Y	12
C/Y	2

$$N = 22.9783... \rightarrow 23 \text{ payments}$$

$$\begin{aligned}
 \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\
 &= \frac{23}{12} \\
 &= 1.9166... \\
 &\rightarrow 1 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.9166... \times 12 \\
 &= 11
 \end{aligned}$$

Final statement: It will take **1** year and **11** months.

Section 2.5 Exercises

1. You make **\$3,000** quarterly payments into a savings account earning **6.35%** compounded quarterly. How long will it take you to accumulate **\$100,000**?

Solution

6 years, 9 months

2. You make **\$2,500** contributions at the beginning of every six months into your RRSP. If the RRSP earns interest at **7%** effective, how long will it take your RRSP to reach an accumulated value of **\$175,000**?

Solution

18 years

3. You took a **\$50,000** loan at **7.2%** compounded quarterly. The loan agreement requires you to make monthly payments of **\$500** until the loan is paid off. How long will it take you to repay the loan?

Solution

12 years, 9 months

4. You have **\$1,000,000** in your RIF. You want to receive **\$40,000** beginning-of-quarter payments for as long as possible from the RIF. If the RIF earns **9%** compounded monthly, how long will it take to exhaust the RIF?

Solution

9 years, 3 months

5. An investment of **\$100,000** today will make advance quarterly payments of **\$4,000**. If the annuity can earn **7.3%** compounded semi-annually, how long will it take for the annuity to be depleted?

Solution

8 years, 3 months

6. The neighbourhood grocery store owned by Raoul needs **\$22,500** to upgrade its fixtures and coolers. If Raoul contributes **\$3,000** at the start of every quarter into a

fund earning **5.4%** compounded quarterly, how long will it take him to save up the needed funds for his store's upgrades?

Solution

2 years

7. Andre has stopped smoking. If he takes the **\$80** he saves each month and invests it into a fund earning **6%** compounded monthly, how long will it take for him to save **\$10,000**?

Solution

8 years, **2** months

8. How much longer will a **\$500,000** investment fund earning **4.9%** compounded annually last if beginning-of-month payments are **\$3,500** instead of **\$4,000**?

Solution

3 years, **2** months

9. Consider a **\$150,000** loan with month-end payments of **\$1,000**. How much longer does it take to pay off the loan if the interest rate is **6%** compounded monthly instead of **5%** compounded monthly?

Solution

3 years, **6** months

10. You make **\$250** month-end contributions to your RRSP, which earns **9%** compounded annually.
- How much less time will it take to reach **\$100,000** if you increase your payments by **10%**?
 - Which alternative requires less principal and by how much? (Assume all payments are equal.)

Solution

a. **9** months; b. regular payments require **\$2250** less principal

11. Most financial institutions tout the benefits of “topping up” your mortgage payments—that is, increasing from the required amount to any higher amount. Assume a

25-year mortgage for \$200,000 at a fixed rate of 5% compounded semi-annually.

- a. How many fewer payments does it take to pay off your mortgage if you increased your monthly payments by 10%?
- b. How much money is saved by “topping up” the payments? Assume that all payments are equal amounts in your calculations.

Solution

- a. 48 fewer payments; b. \$26,521.44

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2.6 CALCULATING THE INTEREST RATE FOR AN ANNUITY

Learning Objectives

- Calculate the interest rate for an annuity

Formula & Symbol Hub

Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of annuity payments

Formulas Used

- **Formula 2.1 – Total Number of Payments (Annuity)**

$$n = P/Y \times t$$

- **Formula 2.2 – Future Value of Ordinary Annuity**

$$FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.3 – Future Value of Annuity Due**

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.4 – Present Value of Ordinary Annuity**

$$PV = PMT \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

- **Formula 2.5 – Present Value of Annuity Due**

$$PV = PMT \times (1 + i_2) \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

Introduction

You must calculate the interest rate on an annuity in a variety of situations:

- To determine the interest rate being charged on any debt.
- To determine the interest rate that an investment is earning.
- To calculate the required interest rate for savings to reach a goal within a certain time period.
- To calculate the required interest rate needed for a series of payments to be sustained over a certain time period.

Using a Financial Calculator

In the future value and present value formulas for annuities, there is no way to algebraically isolate the periodic interest rate. Consequently, calculating the interest rate for an annuity cannot be found by using the formulas. However, a financial calculator can quickly find the required interest rate. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is I/Y (the nominal interest rate). You must still load the other six variables into the calculator, set the calculator to the correct payment setting, and apply the cash flow sign conventions carefully.

Using the TI BAII Plus Calculator to Find the Interest Rate for Annuities

- Set the calculator to the correct payment setting (**END** or **BGN**).
- Enter values for the known variables (PV , FV , PMT , I/Y , P/Y and C/Y), paying close attention to the cash flow sign convention for PV , PMT , and FV .
- After all of the known quantities are loaded into the calculator, press **CPT** and then I/Y to solve for the interest rate.

Notes

1. Ensure that the calculator is set to the required payment setting. The interest rate will be different for **END** and **BGN**.
2. Because the calculator is using a trial-and-error technique to calculate the interest rate, it may take your calculator a few seconds to compute the answer. If your screen goes blank and your calculator hesitates, this is normal.
3. The values you enter for PV , FV , and PMT must adhere to the cash flow sign

convention.

Example 2.6.1

Smartchoice, a rent-to-own store, offers a Dell 10" Mini Inspiron Netbook for a cash n' carry price of **\$399**. Alternatively, under its rent-to-own plan you could make **\$59.88** monthly payments in advance and own the laptop after one year. What effective interest rate is being charged on the rent-to-own plan?

Solution

The timeline for the annuity appears below.

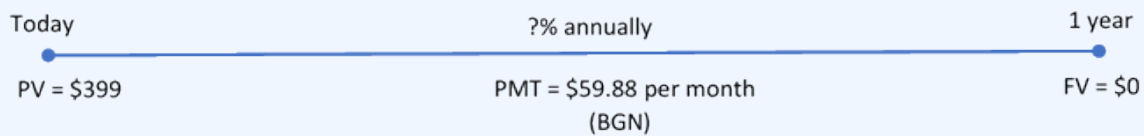


Figure 2.6.1

PMT Setting	BGN
N	$12 \times 1 = 12$
PV	399
FV	0
PMT	-59.88
I/Y	?
P/Y	12
C/Y	1

$$I/Y = 337.98\%$$

The effective interest rate being charged under the rent-to-own payment plan is **337.9759%**.

Example 2.6.2

Cubonic Industries deposits **\$30,000** at the end of every quarter to save up **\$550,000** for a capital project in four years. To achieve its goal, what nominal interest rate compounded quarterly does Cubonic Industries require on its investment?

Solution

PMT Setting	END
N	$4 \times 4 = 16$
PV	0
FV	550,000
PMT	-30,000
I/Y	?
P/Y	4
C/Y	4

$$I/Y = 7.15\%$$

For Cubonic Industries to achieve its savings goal, the savings annuity must earn **7.15%** compounded quarterly.

Try It

1) Francisco just changed occupations. Unfortunately, he is not able to transfer his company pension with him to his new company. The administrators of the pension plan offer him the choice of a lump-sum payout of **\$103,075** today or beginning-of-month payments of **\$535** for the next **25** years. What semi-annually compounded rate of return are the pension administrators using in their calculations?

Solution

PMT Setting	BGN
N	$12 \times 25 = 300$
PV	103,075
FV	0
PMT	-535
I/Y	?
P/Y	12
C/Y	2

$$I/Y = 3.90\%$$

Example 2.6.3

Amadeus has already saved \$5,000 in his RRSP today. Suppose he continues to make \$250 contributions at the beginning of each month for the next 14 years. For him to achieve his goal of having \$100,000, what effective rate of return must his investment earn?

Solution

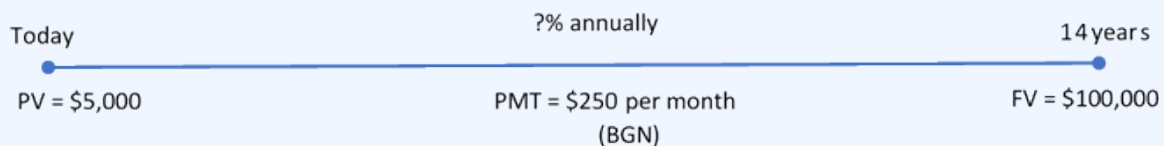


Figure 2.6.2

The timeline for RRSP contributions appears below.

PMT Setting	BGN
N	$12 \times 14 = 168$
PV	$-5,000$
FV	$100,000$
PMT	-250
I/Y	?
P/Y	12
C/Y	1

$$I/Y = 9.17\%$$

The nominal rate of interest that Amadeus must earn on his investment is **9.17%** effective.

Example 2.6.4

Gemma is looking to purchase a new Nissan Pathfinder for **\$54,904.64** including all fees and sales taxes. She can afford to pay no more than **\$1,500** at the end of every month, and she wants to have the balance owing reduced to **\$30,000** after two years, when she can pay off the vehicle with her trust fund. What is the maximum effective rate of interest she could be charged on the car loan to meet her goals?

Solution

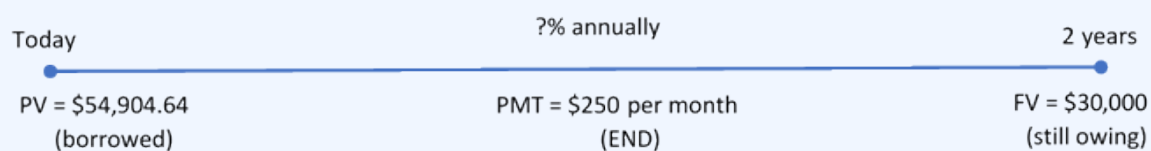


Figure 2.6.3

The timeline for the car loan appears below.

PMT Setting	END
N	$12 \times 2 = 24$
PV	54,904.64
FV	−30,000
PMT	−1,500
I/Y	?
P/Y	12
C/Y	1

$$I/Y = 13.53\%$$

Gemma will be able to purchase the car if she can obtain a car loan that has an effective interest rate lower than **13.53%**.

Try It

2) An investment today requires **\$1, 125.51** to purchase. In return, the investment pays out **\$30** after every six months for the next **20** years, along with an additional final lump-sum payout of **\$1, 000**. What semi-annually compounded interest rate is being earned on the investment?

Solution

PMT Setting	END
N	$2 \times 20 = 40$
PV	$-1, 125.51$
FV	$1, 000$
PMT	30
I/Y	$?$
P/Y	2
C/Y	2

$$I/Y = 5\%$$

Try It

3) When you buy a car, a cash rebate is usually available if you finance the vehicle through your bank instead of the dealership. If you finance the vehicle through the dealership, you are not eligible

for the cash rebate. Assume you can purchase a vehicle for **\$24,960** and finance it for four years with month-end payments at **0%** through the dealership. Alternatively, you could get a loan from a bank and pay cash for your vehicle, which would entitle you to receive a **\$3,500** cash rebate. What monthly compounded interest rate would the bank have to charge to arrive at the same monthly payment as the dealership alternative? What decision rule can you create from this calculation?

Solution

Payments for financing through the dealership at **0%** interest:

PMT Setting	END
<i>N</i>	$12 \times 4 = 48$
<i>PV</i>	$-24,960$
<i>FV</i>	0
<i>PMT</i>	?
<i>I/Y</i>	0
<i>P/Y</i>	12
<i>C/Y</i>	12

$$PMT = \$520$$

Interest rate for the bank with monthly payments of **\$520** and applying the **\$3,500** rebate to the present value:

PMT Setting	END
N	$12 \times 4 = 48$
PV	$-21,460$
FV	0
PMT	-520
I/Y	?
P/Y	12
C/Y	12

$$I/Y = 7.61\%$$

Section 2.6 Exercises

1. You purchase an annuity for \$500,000. The annuity will pay you \$8,000 every quarter for 21.5 years. What rate compounded quarterly does the annuity earn?

Solution

3.12%

2. You invest \$300 at the beginning of every month for 33.5 years. At the end of the 33.5 years, you have saved \$750,000. What rate compounded semi-annually did your investment earn?

Solution

9.04%

3. You have the opportunity to make an investment of **\$100,000** today and receive beginning-of-the-month payments of **\$1,200** for **10** years. Calculate the rate compounded monthly on the investment.

Solution

7.90%

4. Today, your RRSP has **\$25,000**. Over the next **8** years you plan to deposit **\$6,250** every six months into the RRSP with the goal of having **\$200,000** in the account. Calculate the rate compounded semi-annually the RRSP must earn for you to achieve your goal.

Solution

9.76%

5. You have **\$300,000** in your RIF. Over the next **13** years and **3** months, you receive beginning-of-quarter payments of **\$4,975** from the RIF. At the end of the **13** years and **3** months, the balance in your RIF is **\$150,000**. What rate compounded quarterly does the RIF earn?

Solution

3.77%

6. Following his financial adviser's recommendations, Sanchez starts monthly contributions of **\$375** today to his RRSP. The plan is to have **\$240,000** in his RRSP after **20** years of monthly compounding. What nominal interest rate does the financial adviser think Sanchez's RRSP will be able to realize?

Solution

8.61%

7. Helen's husband recently passed away. The life insurance company is offering her a lump-sum payout of **\$250,000** today, or month-end payments of **\$1,585** for **20** years.
- What monthly compounded rate is the life insurance company using in its calculations?
 - Helen thinks she can take the lump sum and invest it herself at **4.75%** effectively. How much will her monthly payment increase?

Solution

a. **4.53%**; b. **\$16.89**

8. Under a wrongful dismissal suit, a court awarded a former employee **\$100,835.25** for end-of-month wages of **\$4,500** for the past **21** months. What effective interest rate is the court using in the judgment?

Solution

8%

9. Jake's Electronics wants to match a competitor's advertised payment plan on an identical stereo system. If the system retails for **\$1,011.35** including all sales taxes and Jake wants to advertise six equal end-of-month payments of **\$174**, what effective rate of interest does he need to charge?

Solution

11.56%

10. The marketing manager for Gold's Gym offers members a two-year membership for **\$650** in advance or beginning-of-month payments of **\$29**. What monthly compounded interest is the marketing manager using in his pricing?

Solution

7.25%

11. A retail store wants to offer its clients different two-year ordinary payment plans on their product purchases. The marketing manager understands the importance of odd-number pricing in these plans, where **\$499** is better than stating **\$500**. On a typical **\$5,000** purchase, the marketing manager wants to offer payments of **\$229** monthly, **\$699** quarterly, or **\$1,399** semi-annually. The Competition Act of Canada requires full disclosure of the annual interest rate being charged on any plan. What interest rate must be published for each plan?

Solution

monthly plan: **9.95%**, quarterly plan: **10.62%**, semi-annual plan: **9.54%**

12. On a **\$3,500** purchase, a company is thinking of offering a year-long month-end payment plan that requires payments of **\$299**, **\$319**, **\$334**, or **\$349**.
- a. If the goal of the plan is to offer a competitive interest rate comparable to a bank

credit card that averages **18%** effectively, which payment plan should be chosen?

- b. If the goal of the plan is to offer a competitive interest rate comparable to a department store credit card that averages **28%** effectively, which payment plan should be chosen?

Solution

\$299 payments: **4.71%**, **\$319** payments: **18.24%**, **\$334** payments: **29.07%**, **\$349** payments: **40.51%**; a. choose **\$319** plan with **18.24%** effective; b. choose **\$334** plan with **29.07%** effective

13. Margarite's goal is to save up **\$100,000** after **10** years of monthly contributions into an investment annuity starting today. Depending on the level of risk she chooses, her adviser tells her that under low-risk conditions she would need to contribute **\$645.19**, under medium-risk conditions her contribution needs to be **\$523.32**, and if she puts her money into high-risk investments she would need **\$401.14** per month. Based on the adviser's calculations, what are the effective interest rates on the low-, medium-, and high-risk investments?

Solution

low: **5%**, medium: **9%**, high: **14%**

Attribution

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2.7 DEFERRED ANNUITIES

Learning Objectives

- Solve problems involving a deferred annuity

Formula & Symbol Hub

Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of annuity payments

Formulas Used

- Formula 2.1 – **Total Number of Payments (Annuity)**

$$n = P/Y \times t$$

- Formula 2.2 – **Future Value of Ordinary Annuity**

$$FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- Formula 2.3 – **Future Value of Annuity Due**

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- Formula 2.4 – **Present Value of Ordinary Annuity**

$$PV = PMT \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

- Formula 2.5 – **Present Value of Annuity Due**

$$PV = PMT \times (1 + i_2) \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

Introduction

Suppose at the age of 21, you took \$50,000 and invested it into an RRSP. You leave the money in the RRSP and do not make any further contributions to the RRSP. When you retire in 40 years, you will take the accumulated money in the RRSP to provide you with monthly payments for 20 years during your retirement. During the 40 years leading up to your retirement, the \$50,000 you invested at age 21 is left to earn interest, with no payments or withdrawals occurring during that time. The payments begin after this 40 year period of interest. This is an example of a **deferred annuity**, where the payments begin after an extended period of time.

What is a Deferred Annuity?

A **deferred annuity** is a financial transaction where annuity payments are delayed until a certain period of time has elapsed. Usually the annuity has two stages, as depicted in this figure.

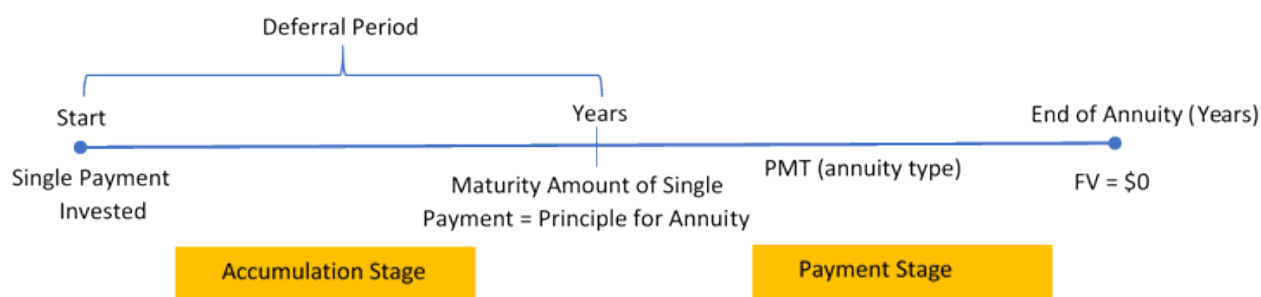


Figure 2.7.1

Image Description

The image is a horizontal timeline visualizing two stages: the Accumulation Stage and the Payment Stage. The timeline starts on the left with a single payment invested and extends to the right to the end of the annuity. It is divided into two main sections by a vertical marker labeled “Maturity Amount of Single Payment = Principle for Annuity”. The portion before the vertical marker represents the “Accumulation Stage” and is labeled accordingly with a yellow box containing the text “Accumulation Stage” below this part of the timeline. The portion after the vertical marker represents the “Payment Stage” and is similarly labeled with a yellow box containing the text “Payment Stage” below this part of the timeline. The distinction between these two stages is indicated by an arrowed bracket that spans across showing the “Deferral Period.”

- **Accumulation Stage:** A single payment is allowed to earn interest for a specified duration. There are no annuity payments during this period of time, which is commonly referred to as the **period of deferral**.
- **Payment Stage:** The annuity takes the form of any of the four annuity types and starts at the beginning of this stage as per the financial contract. Note that the maturity value of the accumulation stage is the same as the principal for the payments stage.

Working with Deferred Annuities

For deferred annuities, the most common unknown variables are either the present value, the length of the period of deferral, the annuity payment amount, or the number of annuity payments that are sustainable for a fixed income payment. Follow this sequence of steps for each of these variables.

	Solving for the Present Value	Solving for the Period of Deferral	Solving for the Annuity Payment	Solving for the Number of Annuity Payments
Step 1	Draw a timeline. Identify the known quantities. Identify the locations of the present value of the annuity and the first payment and place the corresponding times on the timeline.			
Step 2	Starting at the end of your timeline, calculate the present value of the annuity.		Starting at the beginning of the timeline, calculate the future value at the end of the period of deferral.	
Step 3	Calculate the present value at the start of the period of deferral.	Calculate the number of compounding periods and convert to years and months.	Calculate the annuity payment.	Calculate the number of annuity payments and convert to years and months.

Things to Watch Out For

- It is an error to treat the period of deferral and the term of the annuity as simultaneous time periods. For example, if a deferred annuity has a three-year period of deferral and a **10**—year annuity term, this is sometimes interpreted, mistakenly, as an annuity ending **10** years from today. These time segments are **separate** and **consecutive** on the timeline! The correct interpretation is that the annuity term ends **13** years from today, since the **10**—year term does not start until the three-year deferral terminates.
- A common mistake is to incorrectly determine when the period of deferral ends and the annuity starts. This error usually results from forgetting that the payments on ordinary annuities start **one payment interval after the annuity starts**, whereas annuity due payments start immediately. Thus, if the first quarterly payment on an ordinary annuity is to be paid **6** years from today, then the period of deferral is **5** years and **9** months. If the deferral is for an annuity due, then the period of deferral is **6** years.

Example 2.7.1

You borrow **\$10,000** for your business to purchase new computer software. The loan agreement requires you make quarterly payments for **5 years** at **3%** compounded quarterly. The first loan payment is due in **2 years**. Calculate the size of the quarterly loan payments.

Solution

Step 1: Draw the timeline for the deferred annuity. Because the payment setting is not specified, the payments occur at the end of the payment interval.

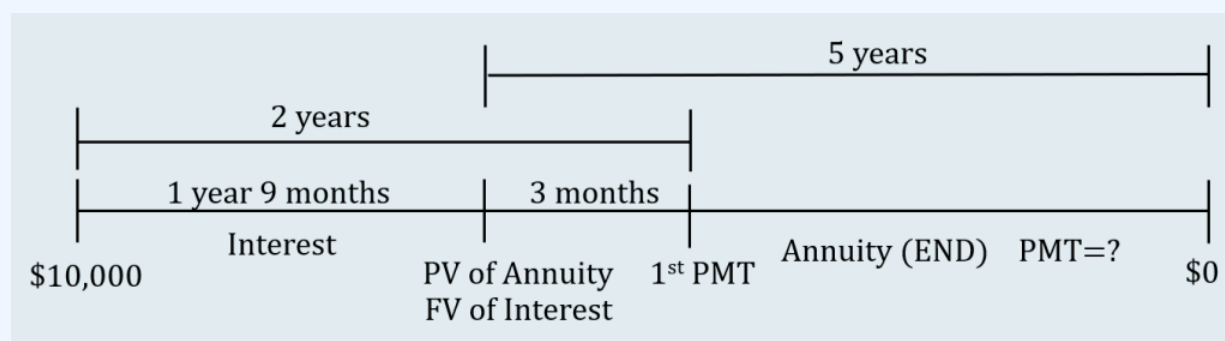


Figure 2.7.2

Image Description

The image depicts a financial timeline illustrating the breakdown of an investment over different time periods. The timeline is as follows:

- The total payment duration is 5 years, indicated by a long horizontal line at the top labeled “5 years”.
- Beneath this, there is a shorter horizontal line representing a period of “2 years”.
- Within these 2 years, a segment of “1 year 9 months” is highlighted, labeled “Interest”.
- Following this is a “3 months” segment, marked as “1st PMT (First Payment)”.
- The timeline ends at the 5-year mark, labeled “Annuity (END) PMT=?”, which signifies the end of the annuity period and the payment amount to be determined.
- At the start of the timeline, there is an initial investment amount of “\$10,000”.
- Throughout the timeline, several financial terms are mentioned, including “PV of Annuity”

(Present Value of Annuity), “FV of Interest” (Future Value of Interest).

- The timeline concludes with “\$0” at the end, indicating the final value.

Some notes about the timeline.

1. The **\$10,000** is the present value at the start of the interest period.
2. There is a **2** year time gap between the start of the interest period and first payment. Because the payments are at the end of the payment interval, the first payment does NOT occur at the same time as the present value of the annuity, which occurs at the beginning of the first payment interval.
3. Because the payments are at the end of the payment interval, the present value of the annuity occurs one payment interval before the first payment. The payments are quarterly (or **3** months), so the present value of the annuity occurs **3** months before the first payment.
4. The **\$10,000** at the start of the interest period must be moved (with a future value calculation) to the location of the present value of the annuity. Because the present value of the annuity occurs **3** months before the first payment and it is **2** years from the start of the interest period to the first payment, the **\$10,000** must move **1** year and **9** months (**2** years minus **3** months) to get the location of the present value of the annuity.
5. The payments for the annuity last for **5** years.

Step 2: Calculate the future value of the \$10,000. As noted above, the \$10,000 must move to the location of the present value of the annuity, which is 1 year and 9 months from the start of the timeline.

N	$4 \times 1.75 = 7$
PV	10,000
FV	?
PMT	0
I/Y	3
P/Y	4
C/Y	4

$$FV = \$10,536.961\dots$$

Step 3: Calculate the payment for the annuity. The future value calculated in the previous step becomes the present value for the annuity: $PV = \$10,536.961\dots$

PMT Setting	END
N	$4 \times 5 = 20$
PV	10,536.961...
FV	0
PMT	?
I/Y	3
P/Y	4
C/Y	4

$$PMT = \$569.32$$

Step 4: Write as a statement.

Your quarterly loan payments are **\$569.32**.

As you can see with the previous example, sometimes adjustments need to be made to the timing of the period of deferral, depending on the wording of the question and the type of annuity. Because the previous question

was an ordinary annuity (payments at the end of the interval) and the timing for the period of deferral was to the first payment, we had to adjust the timing of the interest period so that the money was moved to the location that corresponded to the present value of the annuity.

We are going to do the previous example again, but this time we will treat the annuity as an annuity due (payments at the beginning of the interval). We will do this despite the wording in the question which implies that the annuity has payments at the end of the interval.

Example 2.7.2

You borrow \$10,000 for your business to purchase new computer software. The loan agreement requires you make quarterly payments for 5 years at 3% compounded quarterly. The first loan payment is due in 2 years. Calculate the size of the quarterly loan payments.

Solution

Step 1: Draw the timeline for the deferred annuity. Treat the annuity as an annuity due with the payments at the beginning of the interval.

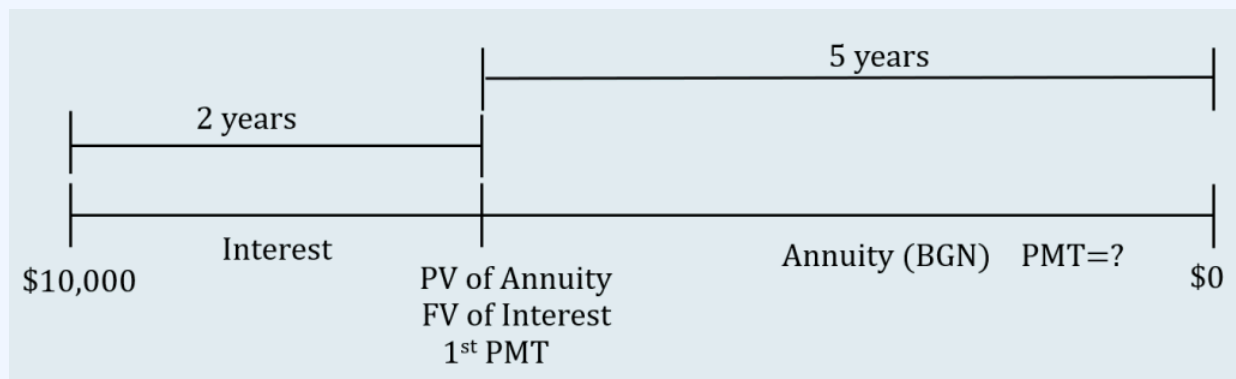


Figure 2.7.3

Image Description

This is a timeline diagram illustrating a financial calculation. The timeline is divided into two segments. The first segment spans 2 years and is labeled “Interest”. The second segment spans 5 years and is labeled “Annuity (BGN)”. At the start of the timeline, there is a value of \$10,000. At the end of the timeline, the value is \$0. The timeline highlights important points such as “PV of

Annuity,” “FV of Interest,” and “1st PMT” at the 2-year mark. The final point on the timeline is labeled “PMT = ?”.

Some notes about the timeline.

1. The **\$10,000** is the present value at the start of the interest period.
2. Because the payments are at the beginning of the payment interval, the present value of the annuity and the first payment occur at the same time, at the start of the annuity. Note that this eliminates the gap between the present value of the annuity and the first payment seen in the previous example.
3. The **\$10,000** at the start of the interest period must be moved (with a future value calculation) to the location of the present value of the annuity. Because the present value of the annuity occurs at the time of the first payment, the **\$10,000** must move **2** years to get the location of the present value of the annuity.
4. The payments for the annuity last for **5** years.

Step 2: Calculate the future value of the \$10,000. As noted above, the \$10,000 must move to the location of the present value of the annuity, which is 2 years from the start of the timeline.

N	$4 \times 2 = 8$
PV	10,000
FV	?
PMT	0
I/Y	3
P/Y	4
C/Y	4

$$FV = \$10,615.988\dots$$

Step 3: Calculate the payment for the annuity. The future value calculated in the previous step becomes the present value for the annuity: $PV = \$10,615.988\dots$

PMT Setting	BGN
N	$4 \times 5 = 20$
PV	10,615.988...
FV	0
PMT	?
I/Y	3
P/Y	4
C/Y	4

$$PMT = \$569.32$$

Step 4: Write as a statement.

Your quarterly loan payments are **\$569.32**.

Note: Treating the annuity as an annuity due (payments at the beginning of the interval) had no impact on the final answer. But, by treating the annuity as an annuity due, we eliminated the time adjustments required in the first example to get the future value of the interest aligned with the timing of the present value of the annuity. You can always treat a deferred annuity as an annuity due, regardless of the information in the question.



You can treat any deferred annuity as an ordinary deferred annuity or a deferred annuity due. Regardless of the approach, you will get the same *final* answer, provided you make the correct adjustments based on your approach. However, it is much easier to treat any deferred annuity as a deferred annuity due (payments at the beginning of the interval). The deferred annuity due approach lines up the future value of the interest, the present value of the annuity and the first payment at the same moment in time, which eliminates any time adjustments to the period of deferral.

Try It

1) You are going to invest **\$80,000** today at **5.5%** compounded quarterly. You want to receive month-end payments from this investment for **25** years. You will receive your first payment in **10** years. What are the size of your monthly payments?

Solution

PMT Setting		BGN
<i>N</i>	40	300
<i>PV</i>	−80,000	−138,141.661...
<i>FV</i>	138,141.661...	0
<i>PMT</i>	0	842.40
<i>I/Y</i>	5.5	5.5
<i>P/Y</i>	4	12
<i>C/Y</i>	4	4

$$PMT = \$842.40$$

Example 2.7.3

You want to invest some money today to help fund your retirement. When you retire, you want to receive **\$5,000** at the end of every month for **15** years. You want to receive your first payment in

20 years. If your retirement fund earns interest at 5% compounded semi-annually, how much money do you need to invest?

Solution

Step 1: Draw the timeline for the deferred annuity. Using the trick identified above, treat the annuity as an annuity due with the payments at the beginning of the interval (despite the wording in the question that identifies the payments at the end of the interval).

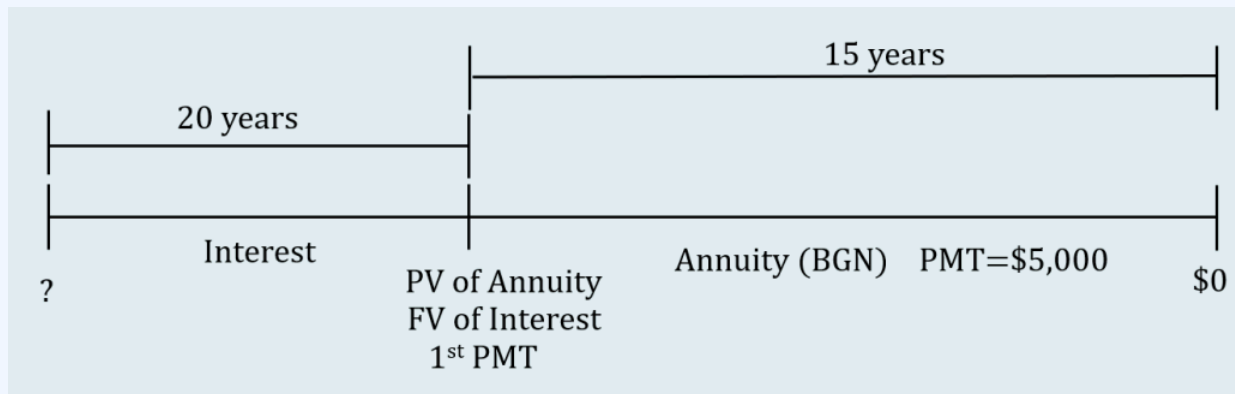


Figure 2.7.4

Image Description

The image contains a horizontal timeline divided into two main segments represented by horizontal lines.

- The first segment spans 20 years and is labeled “Interest.” Below this segment, there is text indicating three financial terms: “PV of Annuity,” “FV of Interest,” and “1st PMT.”
- The second segment spans 15 years and is labeled “Annuity (BGN).” At the end of this segment, it is indicated that “PMT=\$5,000” will be paid.

Both segments combined indicate a total timeline of 35 years, starting from an unspecified initial point (“?”) to the endpoint marked as “\$0.”

Some notes about the timeline.

1. The present value at the start of the interest period is unknown.
2. Because the payments are at the beginning of the payment interval, the present value of the annuity and the first payment occur at the same time, at the start of the annuity. There is a

20 year gap from the start of the interest period and the present value of annuity/future value of interest/first payment.

3. The **\$5,000** payments for the annuity last for **15** years.
4. The present value of the annuity becomes the future value of the interest period. Then, the future value of the interest period will be moved back **20** years to the start of the interest period.

Step 2: Calculate the present value of the annuity.

PMT Setting	BGN
N	$12 \times 15 = 180$
PV	?
FV	0
PMT	5,000
I/Y	5
P/Y	12
C/Y	2

$$PV = \$637,034.376\dots$$

Step 3: Calculate the present value for the interest period. The present value calculated in the previous step becomes the future value for the interest period:

$$FV = \$637,034.376\dots$$

N	$2 \times 20 = 40$
PV	?
FV	637,034.376...
PMT	0
I/Y	5
P/Y	2
C/Y	2

$$PV = \$237,251.11$$

Step 4: Write as a statement.

You need to invest **\$237,251.11** today to fund your retirement.

Try It

2) You need to borrow some money for your business. You can afford to pay **\$1,250** every month for **10** years. You are charged interest at **4.78%** compounded semi-annually and will make your first payment in **4** years. How much money can you borrow?

Solution

PMT Setting	BGN	
N	120	8
PV	119,784.5266..	99,160.88
FV	0	-118,784.5266...
PMT	-1,250	0
I/Y	4.78	4.78
P/Y	12	2
C/Y	2	2

$$PV = \$99,160.88$$

Example 2.7.4

Bashir has \$50,000 to invest. Eventually, Bashir wants the annuity to pay him \$2,500 at the beginning of every month for 10 years. When can Bashir receive his first payment if the investment earns 8.25% compounded quarterly?

Solution

Step 1: Draw the timeline for the deferred annuity.

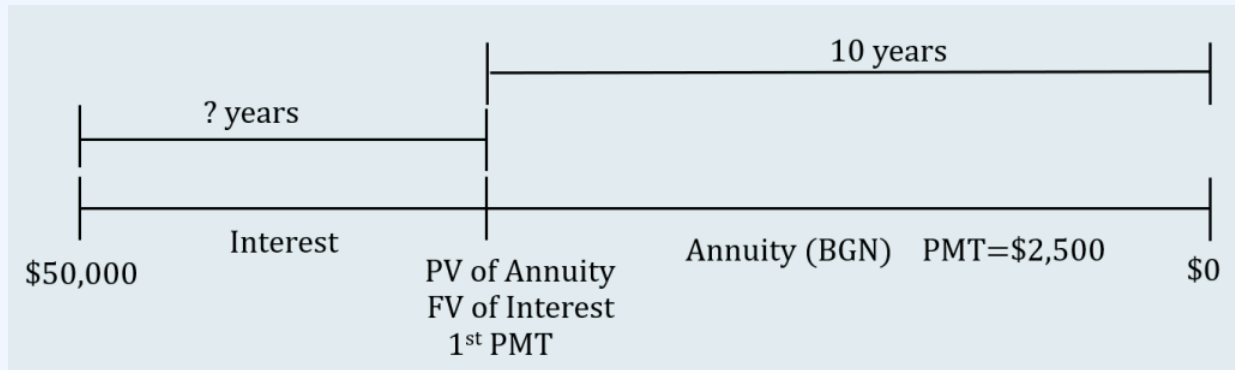


Figure 2.7.5

Image Description

The image illustrates a timeline associated with financial calculations for an annuity. The timeline is divided into two primary sections:

The first section starts at the left with an amount of \$50,000. This section is labeled “Interest” and extends for an unknown number of years, represented with a “? years” label.

The second section is labeled as “Annuity (BGN)” and spans 10 years. It begins after the first section and ends at the rightmost side of the image at \$0. This part of the timeline details an annuity with annual payments (PMT) of \$2,500.

Additionally, at the junction where these two sections meet, three financial terms are noted: PV of Annuity, FV of Interest, and 1st PMT.

Some notes about the timeline.

1. The **\$50,000** is at the start of the interest period.
2. Because the payments are at the beginning of the payment interval, the present value of the annuity and the first payment occur at the same time, at the start of the annuity.
3. The time from the start of the interest period to the present value of the annuity/future value of the interest period/first payment is unknown.
4. The **\$2,500** payments for the annuity last for **10** years.

Step 2: Calculate the present value of the annuity.

PMT Setting	BGN
N	$12 \times 10 = 120$
PV	?
FV	0
PMT	2,500
I/Y	8.25
P/Y	12
C/Y	4

$$PV = \$205,719.154\dots$$

Step 3: Calculate the number of compounding periods for the interest period and convert to years and months. The present value calculated in the previous step becomes the future value for the interest period: $FV = \$205,719.154\dots$

N	?
PV	-50,000
FV	205,719.154...
PMT	0
I/Y	8.25
P/Y	4
C/Y	4

$$N = 69.286\dots$$

$$\begin{aligned}
 \text{Number of Years} &= \frac{N}{C/Y} \\
 &= \frac{69.286...}{4} \\
 &= 17.321... \\
 &\rightarrow 17 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.321... \times 12 \\
 &= 3.858... \\
 &\rightarrow 4 \text{ months}
 \end{aligned}$$

Step 4: Write as a statement.

Bashir can receive his first payment **17** years and **4** months after he invests the **\$50,000**.

Try It

3) You want to receive **\$5,000** semi-annual payments for **5.5** years from a fund earning **6.82%** compounded monthly. If you invest **\$20,000** in the fund today, when can you receive your first payment. Give your answer in years and months.

Solution

PMT Setting	BGN	
N	11	149.5185...
PV	-46,669.472...	-20,000
FV	0	46,669.472...
PMT	5,000	0
I/Y	6.82	6.82
P/Y	2	12
C/Y	12	12

$$N = 149.5185....$$

$$\begin{aligned}
 \text{Number of Years} &= \frac{N}{C/Y} \\
 &= \frac{149.518...}{12} \\
 &= 12.4598... \\
 &\rightarrow 12 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.598... \times 12 \\
 &= 5.5185... \\
 &\rightarrow 6 \text{ months}
 \end{aligned}$$

Example 2.7.5

Emile received a \$25,000 one-time bonus from his employer today, and he immediately invested

it at **3.25%** compounded semi-annually. Fourteen years from now, he plans to withdraw **\$2,300** every quarter to use as his retirement income. How long can Emile receive these payments?

Solution

Step 1: Draw the timeline for the deferred annuity. Treat the annuity as an annuity due with the payments at the beginning of the interval (despite the wording in the question that identifies the payments at the end of the interval).

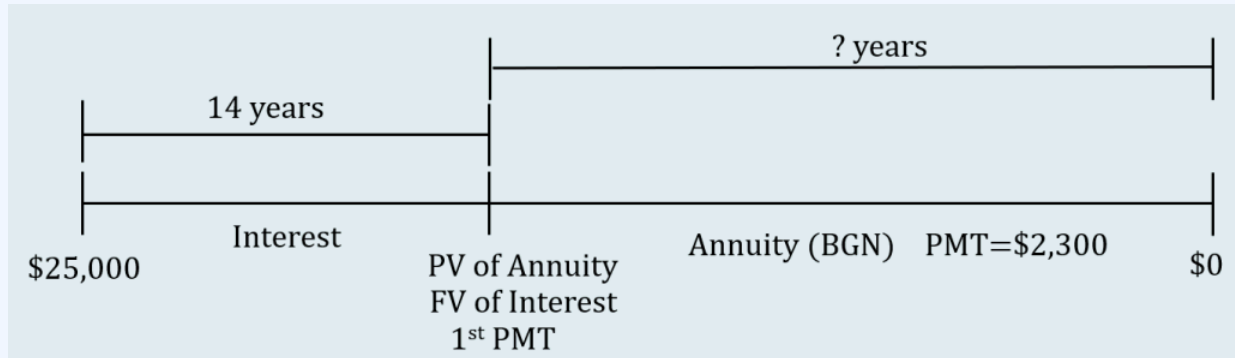


Figure 2.7.6

Image Description

The image is a financial timeline diagram depicting the process of calculating annuities. It consists of two horizontal timelines placed one above the other.

- The top timeline starts with a point marked \$25,000 on the left and ends with a point labeled “\$0” on the right.
- The timeline is divided into two segments:
 - The first segment is 14 years long and is labeled “Interest.”
 - The second segment is labeled with “?” years. There is an arrow pointing to the text “\$0”.
- In the middle of the image below the 14-year mark, there are three terms stacked vertically:
 - PV of Annuity
 - FV of Interest
 - 1st PMT

Alongside the bottom timeline is the phrase: Annuity (BGN) PMT = \$2,300

Some notes about the timeline.

1. The **\$25,000** is at the start of the interest period.
2. Because the payments are at the beginning of the payment interval, the present value of the annuity and the first payment occur at the same time, at the start of the annuity. There is a **14** year gap from the start of the interest period and the present value of annuity/future value of interest/first payment.
3. The **\$2,300** payments for the annuity last for an unknown number of years.

Step 2: Calculate the future value of the \$25,000.

N	$2 \times 14 = 28$
PV	$-25,000$
FV	?
PMT	0
I/Y	3.25
P/Y	2
C/Y	2

$$FV = \$39,260.484\dots$$

Step 3: Calculate the number of payment periods for the annuity and convert to years and months. The future value calculated in the previous step becomes the present value for the annuity: $PV = \$39,260.484\dots$

PMT Setting	BGN
N	?
PV	$-39,260.484\dots$
FV	0
PMT	2,300
I/Y	3.25
P/Y	4
C/Y	2

$N = 18.284\dots \rightarrow 19$ payments

$$\begin{aligned}
 \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\
 &= \frac{19}{4} \\
 &= 4.75 \\
 &\rightarrow 4 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.75 \times 12 \\
 &= 9
 \end{aligned}$$

Step 4: Write as a statement.

Emile can receive payments for **4** years and **9** months.

Try It

4) You borrowed \$15,000 today at 4.5% compounded monthly. The loan agreement requires

semi-annual payments of \$750 with the first payment due in 3 years. How long will you need to make payments to repay the loan?

Solution

PMT Setting		BGN
N	36	31.602...
PV	15,000	17,163.717...
FV	-17,163.717...	0
PMT	0	-750
I/Y	4.5	4.5
P/Y	12	2
C/Y	12	12

$$N = 31.602... \rightarrow 32 \text{ payments}$$

$$\begin{aligned} \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\ &= \frac{32}{2} \\ &= 16 \text{ years} \end{aligned}$$

Section 2.7 Exercises

1. How much money should a company borrow at **7.7%** compounded semi-annually in order to repay **\$4,200** at the end of every quarter for **12** years. The first payment is due **15.5** years from now.

Solution

\$41,484

2. You purchase an annuity for **\$40,000**. The annuity will pay you beginning-of-the-month payments for **10** years and you will receive your first payment in **5** years. If the annuity earns **6.1%** compounded quarterly, what is the size of the payments?

Solution

\$599.92

3. Your company took out a **\$100,000** business loan at **8.5%** effective. The loan agreement calls for payments of **\$20,000** at the beginning of every six months with the first payment made in **7** years. How long will the company need to make payments to repay the loan?

Solution

5 years, 6 months

4. You invest **\$17,500** in an account at **7.5%** compounded monthly. You want to withdraw annual payments from the account for **5** years. If you receive the first payment in **14.5** years, what is the size of the payments you will receive?

Solution

\$11,951.16

5. You want to receive quarterly payments of **\$750** from your investment fund for a period of **13** years. The fund earns **2.8%** compounded semi-annually. Calculate the amount of money you need to deposit in the fund if the first payment is to be received in **7** years and **9** months.

Solution

\$26,476.06

6. You invest **\$55,000** in a fund earning **6.6%** compounded quarterly. You want to withdraw **\$900** from the fund at the end of every month with the first withdrawal to be

made 4 years from now. How long will it take for the fund to be depleted?

Solution

8 years, 8 months

7. What is the present value of a deferred annuity with a deferral period of 17 years at 6.7% compounded semi-annually followed by a 10—year annuity due paying \$1,250 every month at 4.78% compounded semi-annually?

Solution

\$39,070.09

8. If \$38,000 is invested for 15 years at 9.4% compounded quarterly and then pays out \$10,000 at the beginning of each year while earning 2.4% compounded annually, how far from today would the last payment occur?

Solution

19 years

9. Jeff and Sarah want to invest some money in an RESP earning 4% compounded quarterly for their daughter's education. They want the annuity to pay their daughter \$1,000 monthly for three years and nine months for the duration of her educational studies. What lump-sum amount do they need to invest today if they want their daughter to receive the first payment in 10 years?

Solution

\$28,123.56

10. Parker invested \$80,000 in a fund earning 3.85% effective. He wants to receive month-end payments from the fund for 25 years. Calculate the size of Parker's payments if he receives his first payment in 10 years.

Solution

\$600.36

11. Amber plans to retire in 40 years. When she does retire, she would like her RRSP to pay her \$2,500 every month for 20 years. How much money does Amber need to invest today if the RRSP earns 5.9% compounded semi-annually?

Solution

\$34,735.26

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2.8 PERPETUITIES

Learning Objectives

- Solve problems involving a perpetuity

Formula & Symbol Hub

Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- n or N = Total number of annuity payments

Formulas Used

- Formula 2.1 – **Total Number of Payments (Annuity)**

$$n = P/Y \times t$$

- **Formula 2.2 – Future Value of Ordinary Annuity**

$$FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.3 – Future Value of Annuity Due**

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.4 – Present Value of Ordinary Annuity**

$$PV = PMT \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

- **Formula 2.5 – Present Value of Annuity Due**

$$PV = PMT \times (1 + i_2) \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

- **Formula 2.6 – Present Value of Ordinary Perpetuity**

$$PV = \frac{PMT}{i_2}$$

- **Formula 2.7 – Present Value of Perpetuity Due**

$$PV = (1 + i_2) \times \frac{PMT}{i_2}$$

Introduction

Some of the scholarships offered to students at your college were created decades ago, yet these scholarships continue to pay out money to students each and every year. Where does all of the money come from? Did somebody bestow a tremendously large endowment fund all those many years ago to sustain the scholarship over decades? Or does an individual or supporting organization simply pay it out of pocket each year? How is it possible for these scholarships to pay out through all the past years and all of the foreseeable years?

Scholarships, like the one described above, are examples of **perpetuities**, which are annuities that have an infinite term. In other words, the payments last forever. At your individual level, any sum of money you invest as a perpetuity can be used to generate income forever. At the professional level, many companies and not-for-profit organizations such as sports clubs establish scholarship programs and bursaries for their employees, members, or clients. In some contracts, payments such as royalties continue forever. Some dividend stocks have their price determined by indefinite future dividend amounts.

What are Perpetuities?

A **perpetuity** is a special type of annuity that has fixed, regular payments that continue indefinitely. If the principal of the investment is never withdrawn, then the interest earned each period can be withdrawn without affecting the future interest earnings of the investment. Therefore, the annuity continues to earn the same amount of interest each and every future interval and can pay out interest forever.

A perpetuity can be an ordinary perpetuity or a perpetuity due. As well, the payment frequency and compounding frequency create either a simple or general annuity structure. Perpetuities can even occur after a deferral period, and hence deferred perpetuities are possible.

The Perpetuity Formulas

2.6 Present Value for an Ordinary Perpetuity

$$PV = \frac{PMT}{i_2}$$

2.7 Present Value for a Perpetuity Due

$$PV = (1 + i_2) \times \frac{PMT}{i_2}$$

PV is the Present Value of the Perpetuity: the present value is the sum of the present value of all of the annuity payments at the start of the first payment interval.

PMT is the Annuity Payment: the amount of money that is invested or paid after each payment interval.

i_2 is the Periodic Interest Rate: for the interest rate whose compounding frequency matches the payment frequency. If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.

Key Takeaways



- There is no future value formula for a perpetuity. Because a perpetuity lasts forever, the future value is an undefined quantity.
- There is no N value in perpetuity formulas. Because a perpetuity lasts forever, the term of a perpetuity is an undefined quantity.
- For perpetuity questions, there are only three possible quantities to solve for: PV , PMT or I/Y .
- In a simple perpetuity, the payment frequency and the compounding frequency are equal ($P/Y = C/Y$). In this situation the periodic interest rate $i_2 = \frac{j}{m}$ where j is the nominal interest rate given in the question and m is the compounding frequency (and the payment frequency).
- In a general perpetuity, the payment frequency and the compounding frequency are not equal ($P/Y \neq C/Y$). In this situation, the given interest rate must first be converted to the equivalent interest rate where the new compounding frequency equals the payment

frequency. Using the equivalent interest rate, calculate the periodic interest rate i_2 .

The steps required to solve for the present value or payment of a perpetuity remain essentially unchanged from solving for the present value or payment of any annuity.

Example 2.8.1

Your college wants to provide a \$2,000 scholarship at the end of every quarter. How much money does the college need to invest today at 4% compounded quarterly to sustain this scholarship indefinitely?

Solution

Step 1: The given information is

The payments are at the beginning of the payment intervals, and both the compounding frequency and the payment frequency are the same (both quarterly). This is an ordinary simple perpetuity. Because this is a simple perpetuity, an interest rate conversion is not required.

$$PMT = \$2,000$$

$$P/Y = 4$$

$$I/Y = 4\%$$

$$C/Y = 4$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned} i_2 &= \frac{I/Y}{C/Y} \\ &= \frac{4\%}{4} \\ &= 1\% \end{aligned}$$

Step 3: Calculate the present value.

$$\begin{aligned}
 PV &= \frac{PMT}{i_2} \\
 &= \frac{2,000}{0.01} \\
 &= \$200,000
 \end{aligned}$$

Step 4: Write as a statement.

The college needs to invest **\$200,000** today to sustain the scholarship indefinitely.



Paths to Success

In Example 2.8.1 above, the \$2,000 payments from the perpetuity come from the interest generated by the \$200,000 investment. At 4% compounded quarterly, \$2,000 in interest is accumulated at the end of each quarter. The \$2,000 in interest becomes the payment. None of the \$200,000 principal is ever used to make up the payment. Consequently, the \$2,000 payments will last forever.

Using a Financial Calculator

The TI BAII Plus calculator is set up for **fixed term** annuities only. That is, the calculator can only deal with annuities that have a definite end date. Therefore, the calculator has no specific built-in function or manner in which to handle a perpetuity. However, because a perpetuity is a specialized version of a regular annuity, a few minor adaptations to the annuity inputs will allow you to trick the calculator into calculating answers for perpetuities.

Using the TI BAII Plus Calculator on Perpetuities

- Set the calculator to the correct payment setting (**END** or **BGN**).
- Set FV to 0.
- Set N to 10,000.
- Enter the values for the other known variables as normal.
- After all of the known quantities are loaded into the calculator, calculate the unknown quantity (PV , PMT or I/Y depending on the question.)

HOW TO

Solve the Future Value of a Perpetuity that is Undefined

Step 1: You must enter something into the calculator for FV , so we set the convention as $FV = 0$. This will not affect the perpetuity calculation.

Step 2: Setting N to 10,000 will trick the calculator into generating the answer to a perpetuity question. A large value of N places the future value far into the distance future, creating an annuity that lasts a very, very long time. There is nothing special about 10,000, except that it is a very large number. Any very large number for N will work, but it has to be a large enough number to get the calculator to solve the perpetuity question. Using 10,000 is a large enough number to solve the perpetuity question but not so large that it exceeds the computational abilities of the calculator (i.e. sending the calculator into an indefinite loop or producing an error). If you experience an error, try lowering the value of N a little bit. However, try to avoid using a number less than 10,000 if possible to ensure that the calculator finds the answer to the perpetuity question, and not just a fixed term annuity that lasts a long time.

Step 3: Ensure that the calculator is set to the required payment setting. As with any

annuity, the answer for a perpetuity will be different for payments at the end or beginning of the interval.

Step 4: Because you can enter P/Y and C/Y , there is no need to do an interest conversion for a general perpetuity. You simply have to tell the calculator the P/Y and C/Y given in the question, and the calculator will handle any interest conversion internally.

Step 5: The values you enter for PV and PMT must adhere to the cash flow sign convention.

Example 2.8.2

In memory of his late father, who was a highly successful marketing manager, Brian would like to set up a marketing scholarship program with his university's school of business. He is able to donate \$100,000 from his father's estate to set up the fund. If the perpetuity fund can earn 6.3% compounded semi-annually and the first scholarship is to be provided immediately, what semi-annual scholarship amount can the fund offer?

Solution

PMT Setting	BGN
<i>N</i>	10,000
<i>PV</i>	−100,000
<i>FV</i>	0
<i>PMT</i>	?
<i>I/Y</i>	6.3
<i>P/Y</i>	2
<i>C/Y</i>	2

$$PMT = \$3,053.81$$

The fund can provide semi-annual scholarships for **\$3,053.81**.

Example 2.8.3

One of the methods for valuing common shares is to determine the present value of its future dividends. If IBM Corporation is forecast to have year-end dividends of **\$1.50** per share in perpetuity, how much should an investor be willing to pay for a share if he requires a **12%** compounded quarterly rate of return?

Solution

PMT Setting	END
N	10,000
PV	?
FV	0
PMT	1.50
I/Y	12
P/Y	1
C/Y	4

$$PV = \$11.95$$

An investor should pay **\$11.95** per share.

Example 2.8.4

A college invests **\$500,000** in an endowment fund to provide month-end scholarships of **\$800**. What interest rate compounded semi-annually does the fund need to earn to provide the scholarships indefinitely?

Solution

PMT Setting	END
N	10,000
PV	−500,000
FV	0
PMT	800
I/Y	?
P/Y	12
C/Y	2

$$I/Y = 1.44$$

The fund must earn **1.44%** compounded semi-annually.

Try It

1) What amount do you need to invest today at **2.9%** compounded quarterly to sustain payments of **\$450** at the end of every month indefinitely?

Solution

PMT Setting	END
N	10,000
PV	?
FV	0
PMT	450
I/Y	2.9
P/Y	12
C/Y	4

$$PV = \$186,656.17$$

Try It

2) You invest **\$150,000** today into a investment earning **3.72%** compounded monthly. You want to receive beginning-of quarter payments in perpetuity. Calculate the size of the payments.

Solution

PMT Setting	BGN
<i>N</i>	10,000
<i>PV</i>	−150,000
<i>FV</i>	0
<i>PMT</i>	?
<i>I/Y</i>	3.72
<i>P/Y</i>	4
<i>C/Y</i>	12

$$PMT = \$1,386.40$$

Try It

3) An investor purchased shares in a business when the market price per share was **\$100**. Each share pays quarterly dividends of **\$2.50**. What effective rate of return does the investor realize on her investment?

Solution

PMT Setting	END
N	10,000
PV	-100
FV	0
PMT	2.50
I/Y	?
P/Y	4
C/Y	1

$$I/Y = 10.38\%$$

Example 2.8.5

You purchase an investment that will pay you \$1,500 every quarter in perpetuity. You will receive your first payment in seven years. If the perpetuity earns 2.3% compounded semi-annually, what is the purchase price of the investment?

Solution

Step 1: Draw the timeline for the deferred perpetuity. Treat the perpetuity as a perpetuity due with the payments at the beginning of the interval.

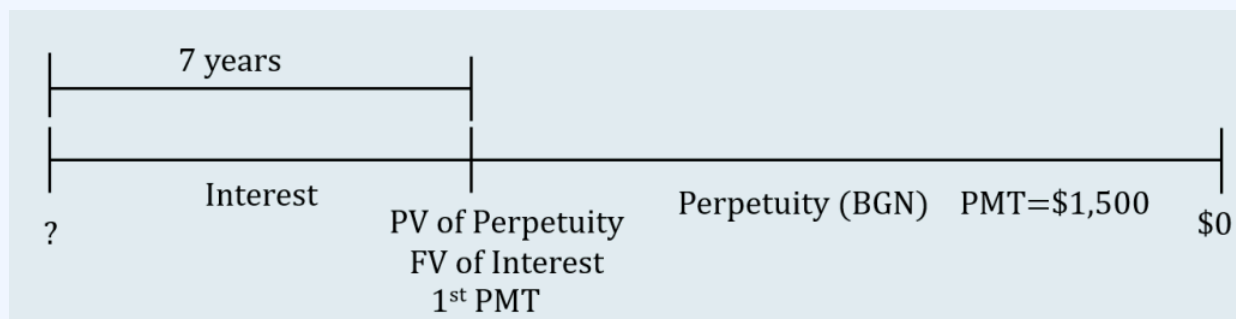


Figure 2.8.1

Image Description

The image represents a financial timeline diagram depicting cash flows and related terms over a period of time. On the timeline:

- The starting point is labeled with a question mark (?); the end point is marked with \$0.
- An interval stretches across 7 years.
- On the timeline within this interval, several financial terms are labeled:
 - “Interest”
 - “PV of Perpetuity” (present value of perpetuity)
 - “FV of Interest” (future value of interest)
 - “1st PMT” (first payment)
 - “Perpetuity (BGN)”
 - “PMT=\$1,500” (recurring payment of \$1,500)

Some notes about the timeline.

1. The present value at the start of the interest period is unknown.
2. Because the payments are at the beginning of the payment interval, the present value of the perpetuity and the first payment occur at the same time, at the start of the annuity. There is a **7** year gap from the start of the interest period and the present value of annuity/future value of interest/first payment.
3. The **\$1,500** payments for the perpetuity last indefinitely.
4. The present value of the perpetuity becomes the future value of the interest period. Then, the future value of the interest period will be moved back **7** years to the start of the interest

period.

Step 2: Calculate the present value of the perpetuity.

PMT Setting	BGN
N	10,000
PV	?
FV	0
PMT	1,500
I/Y	2.3
P/Y	4
C/Y	2

$$PV = \$263,117.421\dots$$

Step 3: Calculate the present value for the interest period. The present value calculated in the previous step becomes the future value for the interest period:

$$FV = \$263,117.421\dots$$

N	$2 \times 7 = 14$
PV	?
FV	263,117.421...
PMT	0
I/Y	2.3
P/Y	2
C/Y	2

$$PV = \$224,195.65$$

The purchase price of the investment is **\$224,195.65**.

Try It

4) A college invests \$500,000 today in an endowment fund to provide semi-annual scholarships in perpetuity. The first scholarship will be given out 10 years from now. If the endowment fund earns 3.9% compounded monthly, what is the scholarship amount?

Solution

PMT Setting		BGN
<i>N</i>	120	10,000
<i>PV</i>	−500,000	−738,023.537...
<i>FV</i>	738,023.537...	0
<i>PMT</i>	0	14,229.16
<i>I/Y</i>	3.9	3.9
<i>P/Y</i>	12	2
<i>C/Y</i>	12	12

$$PMT = \$14,229.16$$

Section 2.8 Exercises

1. How much money must be invested today to sustain beginning-of-six month payments of \$20,000 in perpetuity at 8.4% compounded semi-annually?

Solution

\$496,190.48

2. What quarterly payments in perpetuity can be received if \$2,400,000 is invested at 4.4% compounded quarterly?

Solution

\$26,400

3. You want to receive \$10,000 every six months in perpetuity from an investment. How much money do you need to invest if the money earns 5.35% compounded quarterly?

Solution

\$371,349.38

4. You invest \$500,000 today in an investment at 5.5% compounded annually. What payment can you receive every six months in perpetuity if the first payment is received today?

Solution

\$13,207.62

5. How much money is required today to fund a perpetuity earning 5.65% annually that needs to pay out \$17,000 at the end of each year?

Solution

\$300,884.96

6. If \$2,500,000 was invested at 4.8% compounded semi-annually, what payment at the end of every six months could be sustained in perpetuity?

Solution

\$60,000

7. The dean of the School of Business and Applied Arts at Red River College in Winnipeg wants to establish a scholarship program for the newly created finance major in its business administration stream. He wants to distribute five \$2,000 scholarships annually starting immediately. How much money must he raise from college supporters if

the perpetuity can earn **3.6%** compounded monthly?

Solution

\$283,224.19

8. Samson just won the grand prize of **\$50** million in the Lotto Max lottery. If he invests the money into a perpetuity fund earning **5.5%** compounded annually, what monthly payment can he expect to receive starting today?

Solution

\$222,589.59

9. An investor states that he would be willing to pay **\$25** for common shares to achieve his desired rate of return of **15%** annually. What are the forecasted annual dividends starting one year from now?

Solution

\$3.75

10. The Coca-Cola Company is forecasted to have dividends of **\$0.50** per share quarterly for the next five years, followed by dividends of **\$0.70** per share quarterly in perpetuity. If an investor desires a **10%** compounded annually rate of return, what amount would she be willing to pay per share?

Solution

\$25.89

11. Indigo's will states that **\$150,000** is to be set aside into a fund that will make annual payments to her grandson starting when he turns **18** years old. If Indigo dies when her grandson is six years old and the fund can earn **4.9%** compounded quarterly, what annual payment will he receive in perpetuity?

Solution

\$12,791.51

12. In **2009**, the Canadian federal government committed **\$1.5** million in annual operating funding for the Canadian Museum for Human Rights in Winnipeg, which opened in **2014**. If the government had funded the museum by setting up an ordinary perpetuity in **2009** that could earn prevailing interest rates of **3%** compounded annually, how much money would have been required?

Solution

\$45,757,082.97

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2.9 REVIEW EXERCISES

Chapter 2 Review Exercises

1. Sangarwe will deposit **\$300** every quarter into an investment annuity earning **4.5%** compounded quarterly for seven years. What is the difference in the amount of money that she will have after seven years if payments are made at the beginning of the quarter instead of at the end?

Solution

\$110.36

2. Canseco wants to have enough money so that he could receive payments of **\$1,500** every month for the next nine-and-a-half years. If the annuity can earn **6.1%** compounded semi-annually, how much less money does he need if he takes his payments at the end of the month instead of at the beginning?

Solution

\$652.42

3. Brianne has a **\$21,000** loan being charged **8.4%** compounded monthly. What are the month-end payments on her loan if the debt will be extinguished in five years?

Solution

\$429.84

4. Consider an investment of **\$225,000** earning **5%** annually. How long could it sustain annual withdrawals of **\$20,000** starting immediately?

Solution

16 years

5. The advertised month-end financing payments on a **\$28,757.72** car are **\$699** for a four-year term. What semi-annual interest rate is being used in the calculation?

Solution

7.9%

6. Kubb Bakery estimates it will need **\$198,000** at a future point to expand its production plant. At the end of each month, the profits of Kubb Bakery average **\$20,000**, of which the owner will commit **70%** toward the expansion. If the savings annuity can earn **7.3%** compounded quarterly, how long will it take to raise the necessary funds?

Solution

1 year, 2 months

7. An investment fund has **\$7,500** in it today and is receiving contributions of **\$795** at the beginning of every quarter. If the fund can earn **3.8%** compounded semi-annually for the first one-and-a-half years, followed by **4.35%** compounded monthly for another one-and-three quarter years, what will be the maturity value of the fund?

Solution

\$19,695.13

8. A **\$17,475** Toyota Matrix is advertised with month-end payments of **\$264.73** for six years. What monthly compounded rate of return (rounded to one decimal) is being charged on the vehicle financing?

Solution

2.9%

9. A variable rate loan has a balance remaining of **\$17,000** after two years of fixed end-of-month payments of **\$655**. If the monthly compounded interest rate on the loan was **5.8%** for the first **10** months followed by **6.05%** for **14** months, what was the initial amount of the loan?

Solution

\$19,894.24

10. Hank has already saved **\$68,000** in his RRSP. Suppose he needs to have **\$220,000** saved by the end of **10** years. What are his monthly payments starting today if the RRSP can earn **8.1%** compounded annually?

Solution**\$394.14**

11. Many consumers carry a balance each month on their credit cards and make minimal payments toward their debt. If a consumer owes **\$5,000** on a credit card being charged **18.3%** compounded daily interest, how long will it take him to pay off his debt with month-end payments of **\$100**?

Solution

8 years

12. You have a loan for **\$20,000** on which you are charged **6%** compounded quarterly. What payment amount at the end of every six months would reduce the loan to **\$15,000** after two years? What is the interest portion of the total payments made?

Solution**\$1799.23, \$2,196.92**

13. Karen is saving **\$1,500** at the end of every six months into an investment that earns **9.4%** compounded monthly for the next **20** years. The maturity value will then be rolled into an investment earning **5.85%** compounded annually, from which she plans on withdrawing **\$23,800** at the beginning of each year. How long will the annuity sustain the withdrawals?

Solution

9 years

14. Being able to start an RRSP with a lump-sum investment can reduce your end-of-month contributions. For any **35**—year term RRSP earning **8.7%** compounded annually, calculate the monthly contribution necessary to have a maturity value of **\$1,000,000** if the starting lump sums are **\$5,000, \$10,000, \$15,000, and \$20,000**. What do you observe from your calculations?

Solution**\$360.92, \$324.06, \$287.19, \$250.31**

15. An annuity needs to pay out **\$1,000** at the end of every quarter for three years. Using an interest rate of **5%** quarterly throughout, what amount of money must be invested today to fund the investment if the first payment is paid out in four years?

Solution**\$9,195.75**

16. Red Deer College wants to set up a scholarship for students in its business programs such that at the end of every year it could distribute a total of **\$50,000**. If the perpetuity fund can earn **4.85%** compounded semiannually, how much money will need to be raised to fund the scholarship?

Solution**\$1,019,577.58**

17. Procter and Gamble shares are valued at **\$61.00** with perpetual year-end dividends of **3.1639%**. What dividend payment would a holder of **750** shares receive in perpetuity assuming the share price and dividend rate remain unchanged?

Solution**\$1447.48**

18. The common shares of The Coca-Cola Company are forecast to pay **\$1.55** per share at the end of the next four years, and then **\$2.05** annually in perpetuity. If the market rate of return on such shares is **2.89%**, what price should an investor be willing to pay today?

Solution**\$69.07**

19. A Canadian college plans to implement a new business major in five years. To support the new program, it wants to offer **15** annual **\$2,500** scholarships at the beginning of each school year in perpetuity. If the scholarship fund can earn **4.65%** compounded annually, what amount of money does the college need to raise today to fund the program?

Solution**\$672,390.21**

20. AVCO Financial is under contract with a national retail chain to purchase its loan contracts on its date of sale. Under a special promotion, the retail chain allows a customer to defer her payments. If AVCO purchases the contract for **\$5,276.83** at its contractual rate of **21%** compounded monthly and the consumer is required to make **30** month-end payments of **\$425**, in how many months will AVCO receive its first payment?

Solution

37 months

21. Jacques Cousteau just won the **\$10** million Powerball lottery. He has been offered the following choices on how to collect his winnings.
- **Option 1:** A one-time lump-sum payment of **\$4, 289, 771.59** today.
 - **Option 2:** Year-end payments of **\$400, 000** for **25** years at **5%** effective with the first payment received in five years.
 - **Option 3:** Annual payments of **\$207, 150** in perpetuity starting today earning **5%** compounded annually.

Which option is the best financial choice?

Solution

PV of Option 1: **\$4, 289, 771.59**, PV of Option 2: **\$4, 638, 049.33**, PV of Option 3: **\$4, 350, 150**; Option #2 is best

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CHAPTER 2 SUMMARY

Key Concepts Summary

- **2.1 Fundamentals of Annuities**

- Understanding what an annuity is.
- The four different types of annuities.

- **2.2 Future Value of Annuities**

- The future value of ordinary annuities.
- The future value of annuities due.
- Variable changes in future value annuity calculations.

- **2.3 Present Value of Annuities**

- The present value of ordinary annuities.
- The present value of annuities due.
- Variable changes in present value annuity calculations.

- **2.4 Calculating the Payment**

- Calculating the annuity payment amount for both ordinary annuities and annuities due.

- **2.5 Calculating the Term of an Annuity**

- Calculating the number of annuity payments for both ordinary annuities and annuities due.
- Calculating the term of an annuity for both ordinary annuities and annuities due.

- **2.6 Calculating the Interest Rate of an Annuity**

- Calculating the interest rate for both ordinary annuities and annuities due.

- **2.7 Deferred Annuities**

- The stages of deferred annuities.
- The four common unknown variables and how to solve for them.

- **2.8 Perpetuities**

- An explanation of perpetuities.
- Ordinary perpetuities and perpetuities due.
- How to solve perpetuity scenarios.

Glossary of Terms

Annuity. A continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation.

Annuity Due. Annuity payments that are each made at the beginning of a payment interval.

Annuity Payment. The dollar amount of the equal periodic payment in an annuity environment.

Deferred Annuity. A financial transaction where annuity payments are delayed until a certain period of time has elapsed.

Down Payment. A portion of the purchase price required up front.

Future Value of an Annuity. The sum of all the future values for all of the annuity payments when they are moved to the end of the last payment interval.

General Annuity. An annuity in which the payment frequency and compounding frequency are unequal.

General Annuity Due. An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.

Ordinary Annuity. Annuity payments that are each made at the end of a payment interval. This is the most common form of an annuity payment.

Ordinary General Annuity. An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs one interval after the beginning of the annuity, while the last payment is on the same date as the end of the annuity.

Ordinary Simple Annuity. An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs one interval after the beginning of the annuity while the last payment is on the same date as the end of the annuity.

Payment Frequency. The number of annuity payments in a complete year.

Payment Interval. The amount of time between each continuous and equal annuity payment.

Period of Deferral. The time segment of a deferred annuity where the single payment earns interest and no contributions are made to the investment.

Perpetuity. A special type of annuity that has fixed, regular payments continuing indefinitely.

Present Value of an Annuity. The sum of all the present values for all of the annuity payments when they are moved to the beginning of the first payment interval.

Simple Annuity. An annuity in which the payment frequency and compounding frequency are equal.

Simple Annuity Due. An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.

Formula & Symbol Hub

Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- N = Total number of annuity payments

Formulas Used

- **Formula 2.1 – Total Number of Payments (Annuity)**

$$n = P/Y \times t$$

- **Formula 2.2 – Future Value of Ordinary Annuity**

$$FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.3 – Future Value of Annuity Due**

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

- **Formula 2.4 – Present Value of Ordinary Annuity**

$$PV = PMT \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

- **Formula 2.5 – Present Value of Annuity Due**

$$PV = PMT \times (1 + i_2) \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$$

- **Formula 2.6 – Present Value of Ordinary Perpetuity**

$$PV = \frac{PMT}{i_2}$$

- **Formula 2.7 – Present Value of Perpetuity Due**

$$PV = (1 + i_2) \times \frac{PMT}{i_2}$$

Calculator

- **Time Value of Money Functions**

- The time value of money buttons are the five buttons located on the third row of your calculator.

Calculator Symbol	Characteristic	Data Entry Requirements
N	The number of payment periods	An integer or decimal number; no negatives
I/Y	The nominal interest rate per year	Percent format without the % sign (i.e., 7% is just 7)
PV	Present value or principal	An integer or decimal number
PMT	Annuity payment amounts	An integer or decimal number
FV	Future value or maturity value	An integer or decimal number

To enter any information into any one of these buttons or variables, called loading the calculator, key in the information first and then press the appropriate button.

- The frequency function is logically placed above the I/Y button and is labeled P/Y . This function addresses payment frequencies and compound interest frequencies. Access the function by pressing **2nd** P/Y to find the following entry fields, through which you can scroll using your arrow buttons.

Calculator Symbol	Characteristic	Data Entry Requirements
P/Y	Annuity payments per year (payment frequency)	A positive, nonzero number only
C/Y	Compounds per year (compounding frequency)	A positive, nonzero number only

To enter any information into one of these fields, scroll to the field on your screen, key in the data, and press **ENTER**. When you enter a value into the P/Y field, the calculator will automatically copy the value into the C/Y field for you. If in fact the C/Y is different, you can change the number manually. To exit the P/Y window, press **2nd** **QUIT**.

- To keying in a question, you must load the calculator with six of the seven variables. To solve for the missing variable, press **CPT** followed by the variable.
- The cash flow sign convention is used for the PV , PMT , and FV buttons. If money leaves you, you must enter it as a negative. If money comes at you, you must enter it as a positive.

• Payment Setting

- The calculator default is for **END** mode, which is the ordinary annuity.
- The annuity type (payment timing) setting can be found on the second shelf above the PMT key. This function works as a toggle.
- To toggle the setting, complete the following sequence:
 1. Press **2nd** **BGN** (the payment button).
 2. Press **2nd** **SET** (the **ENTER** button) to switch payment settings.
 3. Press **2nd** **QUIT** (the **CPT** button) to exit the payment setting menu.
- When the calculator is in annuity due mode, a tiny **BGN** is displayed in the upper right of your calculator.

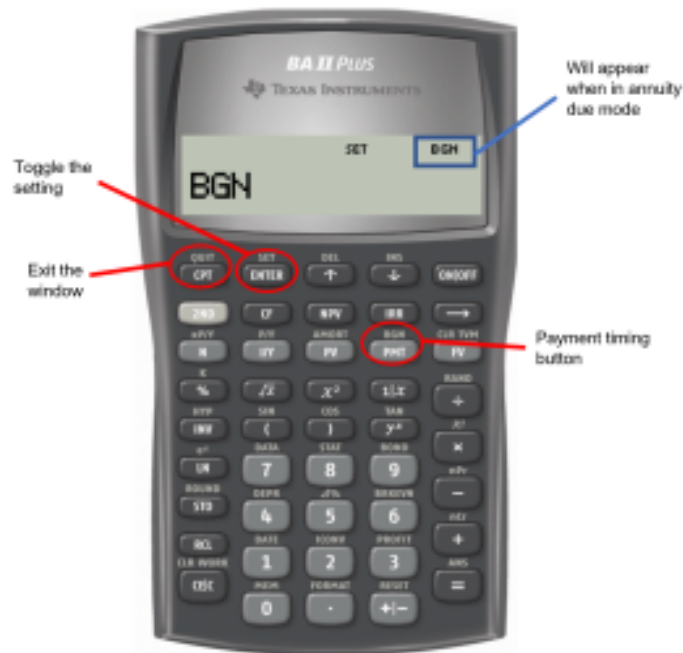


Figure 5.10.1



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=105#oembed-1>

[Ordinary Annuity Calculations \(PV, PMT, FV\)](#) by Joshua Emmanuel [4:31] *Transcript available.*



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=105#oembed-2>

[Annuity Due Calculations Using the BAII Plus](#) by Joshua Emmanuel [2:51] *Transcript Available.*



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=105#oembed-3>

[Annuity Due Calculations Using BAII Plus-Part 2](#) by Joshua Emmanuel [2:05] *Transcript Available*.

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CHAPTER 3: LOANS AND MORTGAGES

Chapter Outline

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[3.1 Amortization Schedules](#)

[3.2 Calculating the Principal and Interest Components of a Loan Payment](#)

[3.3 Calculating the Final Payment](#)

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3.0 INTRODUCTION

Chapter Learning Objectives

By the end of the chapter you should be able to:

- Construct a full or partial amortization schedule for a loan
- Calculate the principal and interest components of any payment
- Calculate the loan balance after any payment
- Calculate the total principal paid and total interest paid for a series of payments
- Calculate the final payment for a loan
- Calculate mortgage payments for the initial term and renewal term
- Calculate amortization periods to reflect changes in payment frequency
- Calculate mortgage balances and amortization periods to reflect prepayments of principal

It would be nice to pay cash for everything that you acquire. However, goods such as housing, automobiles, furniture, and electronics require more money than most Canadians have lying around. If you tried saving up and paying cash for these large purchases, it would take a long time to accumulate enough funds. Take the example of a new home. If you saved \$300 per month at 5% compounded annually, it would take approximately 30 years to save up \$250,000 in cash, and in the meantime, of course, you would still need to pay for somewhere to live.

So it is not necessarily bad to borrow money at some point. However, to borrow wisely you had better understand where your money is going. Would it shock you to learn that in a typical mortgage arrangement most homeowners pay approximately double for their homes? That is, \$250,000 for the home and \$250,000 in interest!

Businesses borrow money for many of the same reasons as consumers. Loans and mortgages in the business world are commonplace. Certain business activities need to be financed upfront. For example, new products

must be researched and developed before a single unit can be sold to consumers. Research and development requires investments that will not be reimbursed until the products turn a profit. All company debts must be accurately recorded onto balance sheets to reflect the balances owing. Income statements must appropriately track interest expenses or earnings. Because businesses can deduct their interest expenses against corporate taxable income and lower their taxes, it is important to see what proportion of their loan payments is going to interest.

When you take out a mortgage for yourself or your business, where does your money go? You need a chart of your loan payments showing how much interest the bank charges and how much is applied against your principal. This chapter takes you through calculating the balance, the principal and interest components of any single payment for a loan. You will now see how to calculate the final payment for a loan. Finally, you will be guided through the largest transactions you personally are likely to make: basic mortgages and renewals.

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3.1 AMORTIZATION SCHEDULES

Learning Objectives

- Construct a full or partial amortization schedule for a loan

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance immediately after a loan payment
- INT = Interest portion of a loan payment or a series of payments
- PRN = Principal portion of a loan payment or a series of payments
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- N = Total number of annuity payments

Introduction

At some point in your life, you will probably take out a loan from a bank, credit union, or some other financial institution. You might need a loan to buy a new car, purchase a home, go back to school, or start a new business. When you borrow money, you have to repay the amount of money you borrowed plus the interest the loan accumulates over the term of the loan. You repay the loan by making regular payments over the term of the loan. Each payment does two things: it pays all of the interest due on the loan at the time the payment is made and the remainder of the payment goes to paying down the loan amount. This process of gradually repaying the loan with periodic payments over the term of the loan is called **amortization**.

What is Amortization?

Amortization is a process by which the principal of a loan is extinguished over the course of an agreed-upon time period through a series of regular payments that go toward both the accruing interest and principal reduction. Two components make up the agreed-upon time component.

1. **Amortization Term.** The amortization term is the length of time for which the interest rate and payment agreement between the borrower and the lender will remain unchanged. Thus, if the agreement is for monthly payments at a 5% fixed rate over five years, it is binding for the entire five years. Or if the agreement is for quarterly payments at a variable rate of prime plus 2% for three years, then interest is calculated on this basis throughout the three years.
2. **Amortization Period.** The amortization period is the length of time it will take for the principal to be reduced to zero. For example, if you agree to pay back your car loan over six years, then after six years you reduce your principal to zero and your amortization period is six years.

In most relatively small purchases, the amortization term and amortization period are identical. For example, a vehicle loan has an agreed-upon interest rate and payments for a fixed term. At the end of the term, the loan is fully repaid. However, larger purchases such as real estate transactions typically involve too much money to be repaid under short time frames. Financial institutions hesitate to agree to amortization terms of much more than five to seven years because of the volatility and fluctuations of interest rates. As a result, a term of five years may be established with an amortization period of 25 years. When the five years elapse, a new term is established as agreed upon between the borrower and lender. The conditions of the new term reflect prevailing interest rates and a payment plan that continues to extinguish the debt within the original amortization period.

Amortization Schedules

An **amortization schedule** shows the payment amount, principal component, interest component, and remaining balance for every payment in the annuity. As the title suggests, it provides a complete understanding of where the money goes, identifying how much of each payment goes to interest and how much goes to principal.

An amortization schedule has five columns:

- **Payment Number** — There is a row for every payment made to repay the loan.
- **Payment** — The periodic payment made to repay the loan. All of the payments are the same (PMT), except for the last payment.
- **Interest Paid** — For each row, the interest paid entry is the amount of interest due on the loan at the time of the corresponding payment.
- **Principal Paid** — For each row, the principal paid entry is the amount of principal repaid at the time of the corresponding payment, after the interest is paid.
- **Balance** — For each row, the balance records how much of the original loan amount remains after the payment is made.

To fill in an amortization schedule, you first need to have all of the details about the loan, including the loan amount (PV), the payment (PMT), the number of payments (N), and the interest rate. If any of these quantities are missing, calculate out the missing value before completing the amortization schedule.

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				Loan Amount ¹
1	PMT^2	INT^3	PRN^4	BAL^5
2	PMT^2	INT^3	PRN^4	BAL^5
\vdots	\vdots	\vdots	\vdots	\vdots
$N-1$	PMT^2			
N	Final Payment ¹⁰	Final INT^9	Final PRN^8	0^7
Totals	Total Amount Paid 12	Total Interest Paid 13	Total Principal Paid 11	

HOW TO

Create an Amortization Schedule

Follow these steps to fill in the amortization schedule.

Step 1: In row **0**, the only entry is the loan amount in the balance column.

Step 2: Fill in the *rounded* loan payment down the payment column, except for the last payment.

Step 3: Calculate the interest paid portion of the payment. The interest is the balance from the previous row times the periodic interest rate:

$$\text{Interest Paid} = \text{Balance from Previous Row} \times i.$$

Note: this calculation assumes that the payment frequency and the compound frequency for the interest rate are the same (i.e. a simple annuity).

Step 4: Calculate the principal paid portion of the payment. The principal paid is the difference between the payment and the interest paid:

$$\text{Principal Paid} = PMT - \text{Interest Paid}.$$

Step 5: Calculate the outstanding balance on the loan after the payment is made. The balance is difference between the balance in the previous row and the principal paid:

$$\text{Balance} = \text{Balance from Previous Row} - \text{Principal Paid}.$$

Step 6: For each payment, repeat steps **2** through **5**, except for the last row.

Step 7: The last balance entry is **0**. Because this is the last payment, the loan must be paid off, which means the balance is reduced to **0**.

Step 8: The final principal paid entry equals the balance entry from row $N - 1$.

Step 9: Calculate the final interest paid portion in the same way as in step **3**.

Step 10: The final payment is the sum of the final interest paid entry and the final principal paid entry:

$$\text{Final Payment} = \text{Final INT} + \text{Final PRN}.$$

Step 11: The total principal paid is the sum of the principal paid column, and is just the loan amount:

$$\text{Total Principal Paid} = \text{Loan Amount.}$$

Step 12: The total amount paid is the sum of the payment column:

$$\text{Total Amount Paid} = (N - 1) \times PMT + \text{Final Payment.}$$

Step 13: The total interest paid is the sum of the interest paid column, and equals the difference between the other two column totals:

$$\text{Total Interest Paid} = \text{Total Amount Paid} - \text{Total Principal Paid.}$$



The manual calculation of the interest paid entry above is based on the assumption that the payment frequency and the compounding frequency are equal. If the payment frequency and the compounding frequency are not equal, an interest conversion is required to convert the interest rate to the equivalent rate with the compounding frequency equal to the payment frequency. However, if you use the TI BAII Plus's built-in amortization worksheet (described below), no interest conversion is required.

1. As you fill in the schedule, round the entries to two decimal places.
2. Make sure that you are using the payment *rounded* to two decimal places in the schedule. In real life, your loan payment is rounded to two decimal places and all of the loan calculations are based on this rounded payment.
3. The final payment must be the exact amount that will result in a 0 balance in the last row. You do not want the last payment to be too large (i.e. an overpayment) because then you are giving the bank more money than necessary. You do not want the last payment to be too small (i.e. an underpayment) because you must have the loan completely paid off with the final payment. Consequently, the last row of the schedule is reverse engineered, working from right to left.

Example 3.1.1

A \$3,000 loan at 8% compounded quarterly is repaid with quarterly payments of \$800. Construct the amortization schedule for the loan.

Solution

Step 1: Calculate the number of payments.

PMT Setting	END
N	?
PV	3,000
FV	0
PMT	−800
I/Y	8
P/Y	4
C/Y	4

$$N = 3.936... \rightarrow 4 \text{ payments}$$

Step 2: Complete the amortization schedule.

Because the payment frequency and the compounding frequency are equal, no interest conversion is required. The calculations for each entry are shown in blue. The periodic interest rate is $i = \frac{8\%}{4} = 2\%$.

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$3,000
1	\$800	\$60 ($3,000 \times 0.02$)	\$740 ($800 - 60$)	\$2,260 ($3,000 - 740$)
2	\$800	\$45.20 ($2,260 \times 0.02$)	\$754.80 ($800 - 45.20$)	\$1,505.20 ($2,260 - 754.80$)
3	\$800	\$30.10 ($1,505.20 \times 0.02$)	\$769.90 ($800 - 30.10$)	\$735.30 ($1,505.20 - 735.30$)
4	\$750.37 ($15.07 + 735.30$)	\$15.07 (735.30×0.02)	\$735.30	\$0
Totals	\$3,150.37 ($3 \times 800 + 735.30$)	\$150.37 ($3150.37 - 3000$)	\$3,000	

Using a Financial Calculator

Although the calculations in an amortization schedule are relatively straightforward, the manual calculations are time-consuming, especially when the schedule has a lot of rows. A financial calculator, such as the TI BAII Plus, has a built-in amortization worksheet that can quickly calculate the entries for each row of the schedule, with the exception of the last row.

Using the TI BAII Plus Calculator to Construct an Amortization Schedule

The amortization worksheet has five variables ($P1$, $P2$, BAL , PRN , INT). You use the up and down arrows to scroll through the amortization worksheet.

- $P1$ is the starting payment number. The calculator works with a single payment or a series of payments.
- $P2$ is the ending payment number. This number is the same as $P1$ when you work with a single payment (i.e. to find the entries for a row of the amortization schedule). When you work with a series of payments later in this chapter, you set it to a number higher than $P1$.
- BAL is the principal balance remaining after the $P2$ payment number. The cash flow sign is correct as indicated on the calculator display.
- PRN is the principal portion of the payments from $P1$ to $P2$ inclusive. Ignore the cash flow sign.
- INT is the interest portion of the payments from $P1$ to $P2$ inclusive. Ignore the cash flow sign.

To use the amortization worksheet to complete an amortization schedule:

1. Solve for any unknown quantities about the loan. You need to know all of the information about the loan first before you can use the amortization worksheet.
2. Enter the values of all seven time value of money variables into the calculator (N , PV , FV , PMT , I/Y , P/Y , C/Y). If you calculated PMT in the first step, you must **re-enter** it **rounded** to two decimals and with the correct cash flow sign. Make sure the payment setting is set to **END**, and obey the cash flow sign convention. Because this is a loan, PV (the loan amount) is positive and **PMT** is negative.
3. Go to the amortization worksheet by pressing **2nd AMORT** (the PV button).

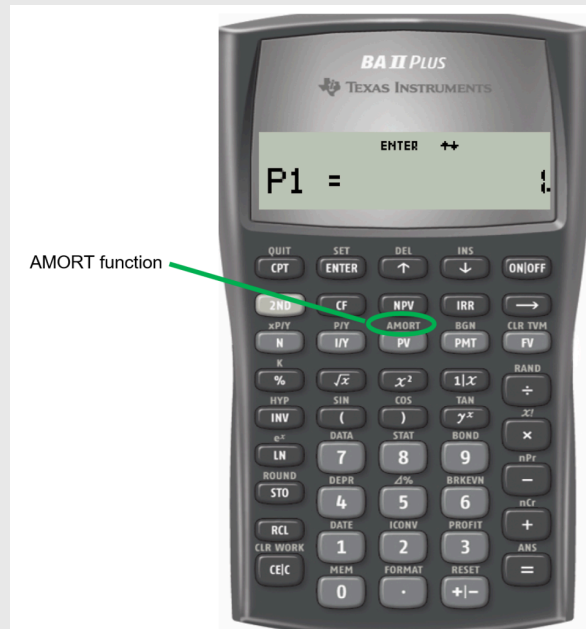


Figure 3.1.1

4. To view the entries for a specific row of the schedule, set $P1$ and $P2$ to the row number. For example, to view the entries for row **5**, set $P1 = 5$ and $P2=5$ [/latex].
 - i. At the $P1$ prompt, enter the row number and press **ENTER**.
 - ii. Press the down arrow.
 - iii. At the $P2$ prompt, enter the row number and press **ENTER**.
 - iv. Press the down arrow.
 - v. The **BAL** entry is the balance entry for the corresponding row.
 - vi. Press the down arrow.
 - vii. The **PRN** entry is the principal paid entry for the corresponding row.
 - viii. Press the down arrow.
 - ix. The **INT** entry is the interest paid entry for the corresponding row.
 - x. Press the down arrow the return to the $P1$ screen.

- Repeat the previous step with a different row number to view the entries for a different row.



Paths to Success

- Because you can enter P/Y and C/Y , there is no need to do an interest conversion if the payment frequency and compounding frequency are different. You simply have to tell the calculator the P/Y and C/Y given in the question, and the calculator will handle any interest conversion internally.
- Make sure to re-enter PMT rounded to 2 decimal places before using the amortization worksheet. If you enter PMT with all of the decimal places, you will not get the correct entries for the amortization schedule.
- The amortization worksheet on the calculator will only work with loans that have payments at the end of the interval (i.e. an ordinary annuity).
- The amortization worksheet gives you the entries for the schedule in reverse order, from right to left: BAL , PRN , and INT .
- As you read the entries off of the amortization worksheet on the calculator and put them in the schedule, round the entries to 2 decimal places.
- For the last row, the amortization worksheet will NOT give you the correct BAL and PRN entries. But, the INT entry for the last row is correct. You will still have to reverse engineer the last row using the steps described above, but you get the INT entry for the last row from the amortization worksheet on the calculator.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=111#oembed-1>

Video: [Amortization Schedule using BAII Plus](#) by Joshua Emmanuel [3:55] *Transcript Available.*

Example 3.1.2

You take out a **\$50,000** business loan at **5%** compounded quarterly. The loan requires semi-annual payments for three years. Construct the amortization schedule for the loan.

Solution

Step 1: Calculate the payment.

PMT Setting	END
N	$2 \times 3 = 6$
PV	50,000
FV	0
PMT	?
I/Y	5
P/Y	2
C/Y	4

$$PMT = \$9,082.24$$

Step 2: Enter the information into the time value of money buttons on the calculator.

Make sure that you re-enter the payment rounded to **2** decimal places.

PMT Setting	END
N	6
PV	50,000
FV	0
PMT	−9,082.24
I/Y	5
P/Y	2
C/Y	4

Step 3: Complete the amortization schedule using the amortization worksheet on the calculator.

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$50,000
1	\$9,082.24	\$1,257.81	\$7,824.43	\$42,175.57
2	\$9,082.24	\$1,060.98	\$8,021.26	\$34,154.31
3	\$9,082.24	\$859.19	\$8,223.05	\$25,931.27
4	\$9,082.24	\$652.33	\$8,429.91	\$17,501.36
5	\$9,082.24	\$440.27	\$8,641.97	\$8,859.39
6	\$9,082.26	\$222.87	\$8,859.39	\$0
Totals	\$54,493.46	\$4,493.46	\$50,000	

- Row 1: In the amortization worksheet, set $P1 = 1$ and $P2 = 1$.
- Row 2: In the amortization worksheet, set $P1 = 2$ and $P2 = 2$.
- Row 3: In the amortization worksheet, set $P1 = 3$ and $P2 = 3$.
- Row 4: In the amortization worksheet, set $P1 = 4$ and $P2 = 4$.
- Row 5: In the amortization worksheet, set $P1 = 5$ and $P2 = 5$.

- *Row 6:*
 - The principal paid (*PRN*) entry is the balance (*BAL*) entry from Row 5.
 - Get the interest paid (*INT*) entry from the calculator by setting $P1 = 6$ and $P2 = 6$.
 - The payment is the sum of the principal paid and interest paid entries:
 $222.87 + 8,859.39 = 9,082.26$.
- *Totals Row:*
 - The principal paid total is the loan amount (\$50,000).
 - The payments total is the sum of the payments:
 $5 \times 9,082.24 + 9,082.26 = 54,493.46$.
 - The interest paid total is the difference in the other two column totals:
 $54,493.46 - 50,000 = 4,493.46$.

Try It

1) A farmer purchased a John Deere combine for **\$369,930**. The equipment dealership sets up a financing plan to allow for end-of-quarter payments of **\$51,000** at **7.8%** compounded monthly. Construct a complete amortization schedule including the totals.

Solution

PMT Setting	END
N	?
PV	369,930
FV	0
PMT	−51,000
I/Y	7.8
P/Y	4
C/Y	12

$N = 7.901... \rightarrow 8 \text{ payments}$

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$369,930
1	\$51,000	\$7,260.63	\$43,739.37	\$326,190.63
2	\$51,000	\$6,402.15	\$44,597.85	\$281,592.78
3	\$51,000	\$5,526.83	\$45,473.17	\$236,119.61
4	\$51,000	\$4,634.33	\$46,365.67	\$189,753.93
5	\$51,000	\$3,724.31	\$47,275.69	\$142,478.24
6	\$51,000	\$2,796.42	\$48,203.58	\$94,274.66
7	\$51,000	\$1,850.33	\$49,149.69	\$45,124.99
8	\$46,010.66	\$885.67	\$45,124.99	\$0
Totals	\$403,010.66	\$33,080.66	\$369,930	

Partial Amortization Schedules

Sometimes, businesses are interested only in creating partial amortization schedules, which are amortization schedules that show only certain payments or a specified range of payments, and not the entire amortization schedule. This may occur for a variety of reasons. For instance, the complete amortization schedule may be too long (imagine weekly payments on a 25-year loan), or maybe you are solely interested in the principal and interest portions during a specific period of time for accounting and tax purposes.

The TI BAII Plus calculator makes it easy to calculate the entries for a specific row of the schedule by setting P1 and P2 equal to the row number you want to view.

Example 3.1.3

You borrowed \$45,000 to purchase a new car. The loan agreement calls for monthly payments of \$900 at 3.5% compounded monthly. Construct a partial amortization schedule showing the details for the first two payments, payment #14, the last two payments, and the totals.

Solution

Step 1: Calculate the number of payments.

PMT Setting	END
N	?
PV	45,000
FV	0
PMT	−900
I/Y	3.5
P/Y	12
C/Y	12

$$N = 54.122... \rightarrow 55 \text{ payments}$$

Step 2: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	55
PV	45,000
FV	0
PMT	−900
I/Y	3.5
P/Y	12
C/Y	12

Step 3: Complete the partial amortization schedule using the amortization worksheet on the calculator.

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$45,000
1	\$900	\$131.25	\$768.75	\$44,231.25
2	\$900	\$129.01	\$770.99	\$43,460.26
14	\$900	\$101.59	\$798.41	\$34,031.06
54	\$900	\$2.94	\$897.06	\$110.51
55	\$110.83	\$0.32	\$110.51	\$0
Totals	\$48,710.83	\$3,710.83	\$45,000	

- Row 1: In the amortization worksheet, set $P1 = 1$ and $P2 = 1$.
- Row 2: In the amortization worksheet, set $P1 = 2$ and $P2 = 2$.
- Row 14: In the amortization worksheet, set $P1 = 14$ and $P2 = 14$.
- Row 54: In the amortization worksheet, set $P1 = 54$ and $P2 = 54$.
- Row 55:

- The principal paid (*PRN*) entry is the balance (*BAL*) entry from Row 54.
- Get the interest paid (*INT*) entry from the calculator by setting $P1 = 55$ and $P2 = 55$.
- The payment is the sum of the principal paid and interest paid entries:
 $0.32 + 110.51 = 110.83$.
- *Totals Row:*
 - The principal paid total is the loan amount (\$45,000).
 - The payments total is the sum of the payments:
 $54 \times 900 + 110.83 = 48,710.83$.
 - The interest paid total is the difference in the other two column totals:
 $48,710.83 - 45,000 = 3,710.83$.

Try It

2) Ron and Natasha had Oasis Leisure and Spa install an in-ground swimming pool for \$51,000. The financing plan through the company allows for end-of-month payments for five years at 6.9% compounded quarterly. Construct a partial amortization schedule showing the last two rows and the totals.

Solution

PMT Setting	END
<i>N</i>	60
<i>PV</i>	51,000
<i>FV</i>	0
<i>PMT</i>	?
<i>I/Y</i>	6.9
<i>P/Y</i>	12
<i>C/Y</i>	4

$$PMT = \$1,006.51$$

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
59	\$1,006.51	\$11.41	\$995.10	\$1,000.98
60	\$1,006.70	\$129.01	\$1,000.98	\$0
Totals	\$60,390.79	\$9,390.79	\$51,000	

Section 3.1 Exercises

1. A \$27,000 loan at 6.8% compounded quarterly is repaid with quarterly payments

over two years. Construct the amortization schedule for the loan.

Solution

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$27,000
1	\$3,638.26	\$459	\$3,179.26	\$23,820.74
2	\$3,638.26	\$404.95	\$3,233.31	\$20,587.43
3	\$3,638.26	\$349.99	\$3,288.27	\$17,299.16
4	\$3,638.26	\$294.09	\$3,344.17	\$13,954.98
5	\$3,638.26	\$237.23	\$3,401.03	\$10,553.96
6	\$3,638.26	\$179.42	\$3,458.84	\$7,095.12
7	\$3,638.26	\$120.62	\$3,517.64	\$3,577.47
8	\$3,638.29	\$60.82	\$3,577.47	\$0
Totals	\$29,106.11	\$2,106.11	\$27,000	

2. A \$192,000 loan at 4.75% compounded semi-annually is repaid with semi-annual payments of \$26,640. Construct the amortization schedule for the loan.

Solution

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$192,000
1	\$26,640	\$4,560	\$22,080	\$169,920
2	\$26,640	\$4,035.60	\$22,604.40	\$147,315.60
3	\$26,640	\$3,498.75	\$23,141.25	\$124,174.35
4	\$26,640	\$2,949.14	\$23,690.86	\$100,483.49
5	\$26,640	\$2,386.48	\$24,253.52	\$76,229.97
6	\$26,640	\$1,810.46	\$24,829.54	\$51,400.43
7	\$26,640	\$1,220.76	\$25,419.24	\$25,981.19
8	\$26,598.24	\$617.05	\$25,981.19	\$0
Totals	\$239,718.24	\$47,718.24	\$192,000	

3. A \$450,000 loan at 7.3% compounded semi-annually is repaid with monthly payments for 25 years. Construct a partial amortization schedule showing the details of payment 100, the last three payments and the totals.

Solution

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
100	\$3,235.66	\$2,262.04	\$973.62	\$376,483.42
298	\$3,235.66	\$57.48	\$3,178.18	\$6,412.91
299	\$3,235.66	\$38.43	\$3,197.23	\$3,215.68
300	\$3,234.95	\$19.27	\$3,215.68	\$0
Totals	\$970.697.29	\$520,697.29	\$450,000	

4. Jennifer purchased a new \$60,000 car for \$5,000 down and monthly payments of

\$750 at **11.2%** compounded monthly. Construct a partial amortization schedule showing the details of payment **30**, the last two payments and the totals.

Solution

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
30	\$750	\$440.16	\$309.84	\$46,849.89
124	\$750	\$8.02	\$741.98	\$116.81
125	\$117.90	\$1.09	\$116.81	\$0
Totals	\$93,117.90	\$38,117.90	\$55,000	

5. To purchase a piece of land, Hillary took out a **\$118,000** loan at **7.95%** compounded semi-annually. The loan agreement required her to make quarterly payments for **15** years. Construct a partial amortization schedule showing the details of the first two payments, payment **28**, the last two payments and the totals.

Solution

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$118,000
1	\$3,368.48	\$2,322.40	\$1,046.08	\$116,953.92
2	\$3,368.48	\$2,301.81	\$1,066.67	\$115,887.24
28	\$3,368.48	\$1,597.93	\$1,770.55	\$79,419.86
59	\$3,368.48	\$128.78	\$3,239.70	\$3,303.30
60	\$3,368.31	\$65.01	\$3,303.30	\$0
Totals	\$202,108.63	\$84,108.63	\$118,000	

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3.2 CALCULATING THE PRINCIPAL AND INTEREST COMPONENTS OF A LOAN PAYMENT

Learning Objectives

- Calculate the principal and interest components of any payment
- Calculate the loan balance after any payment
- Calculate the total principal paid and total interest paid for a series of payments

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance immediately after a loan payment
- INT = Interest portion of a loan payment or a series of payments
- PRN = Principal portion of a loan payment or a series of payments
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency

- N = Total number of annuity payments

Introduction

How much of the principal do you pay off when you make a loan payment? One year ago you purchased your \$250,000 dream home on a 25-year mortgage at a fixed 5% compounded semi-annually interest rate. With monthly contributions of \$1,454.01, or \$17,448.12 in total for the past year, you figure you must have put a serious dent in the balance owing. But you get a rude shock when you inspect your mortgage statement and see that the remaining balance is \$244,806.89, reflecting a principal reduction of only \$5,193.11! The other 70% of your hard-earned money, amounting to \$12,255.01, went solely toward the bank's interest charges.

Many people do not fully understand how their loan payments are portioned out. Over the full course of the 25-year mortgage you will pay \$186,204.46 in interest charges at 5% compounded semi-annually, or approximately 74.5% of the home's price tag. That is a total of \$436,204.46 paid on a \$250,000 home. Term, interest rates, payment amounts, and payment frequency all affect the amount of interest you pay.

These calculations should make it clear that both businesses and consumers need to understand the interest and principal components of loan payments, as well as the remaining balance after each loan payment is made. In the previous section, you already learned how to find the interest, principal, and balance for any loan payment—these are just the entries in the amortization schedule. In this section, we will review how to find these values for any payment, as well as learn how to calculate the total interest paid and total principal paid for a series of payments.

Calculating the Interest Paid, Principal Paid and Balance for a Single Payment

At any point during an amortized loan you can precisely calculate how much any single payment contributes toward principal and interest, and what your outstanding balance is on the loan after a payment is made. This is exactly what each row of the amortization schedule tells you—each row separates the payment into its interest paid and principal paid components, and the balance after the payment is made. Just like in the previous section, you can use the amortization worksheet on your financial calculator to find the interest paid (INT), principal paid (PRN), and balance (BAL) for any payment.

Using the TI BAII Plus Calculator to Find the Interest Paid, Principal Paid, and Balance of a Loan Payment

The amortization worksheet has five variables (P_1 , P_2 , BAL , PRN , INT). You use the up and down arrows to scroll through the amortization worksheet.

- P_1 is the starting payment number. The calculator works with a single payment or a series of payments.
- P_2 is the ending payment number. This number is the same as P_1 when you work with a single payment (i.e. to find the entries for a row of the amortization schedule). When you work with a series of payments later in this section, you set it to a number higher than P_1 .
- BAL is the principal balance remaining after the P_2 payment number. The cash flow sign is correct as indicated on the calculator display.
- PRN is the principal portion of the payments from P_1 to P_2 inclusive. Ignore the cash flow sign.
- INT is the interest portion of the payments from P_1 to P_2 inclusive. Ignore the cash flow sign.

To use the amortization worksheet to find the interest paid, principal paid, and balance of a payment:

1. Solve for any unknown quantities about the loan. You need to know all of the information about the loan first before you can use the amortization worksheet.
2. Enter the values of all seven time value of money variables into the calculator (N , PV , FV , PMT , I/Y , P/Y , C/Y). If you calculated PMT in the first step, you must *re-enter* it *rounded* to two decimals and with the correct cash flow sign. Make sure the payment setting is set to **END**, and obey the cash flow sign convention. Because this is a loan, PV (the loan amount) is positive and PMT is negative.
3. Go to the amortization worksheet by pressing **2nd AMORT** (the PV button).

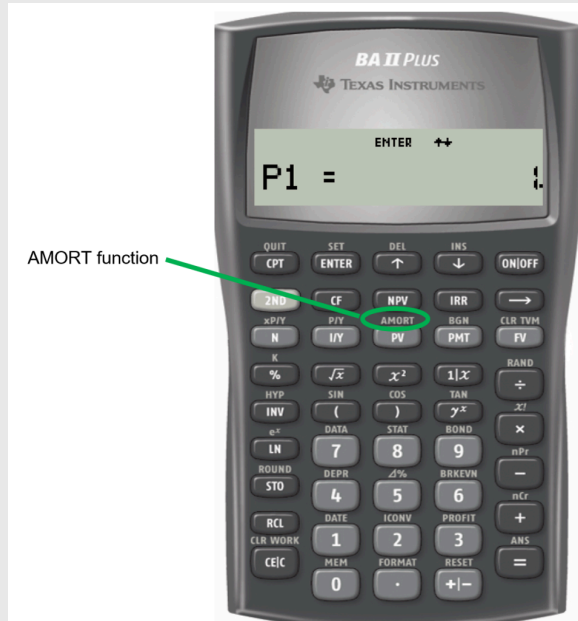


Figure 3.2.1

Image Description

The image shows a Texas Instruments BA II Plus financial calculator. The calculator has a rectangular screen and a keypad with a variety of buttons labeled with financial, numerical, and mathematical symbols.

The screen is displaying the text “P1 = “. Below the screen are the keys to operate the calculator. The key labeled “I/Y” has a green outline around the text “AMORT” and a green arrow pointing toward it, indicating the location of the AMORT function.

4. To view the balance, principal paid, and interest paid for a payment, set P_1 and P_2 to the corresponding payment number. For example, to view the balance, principal paid, and interest paid for payment 5, set $P_1 = 5$ and $P_2 = 5$.
 - i. At the P_1 prompt, enter the payment number and press **ENTER**.
 - ii. Press the down arrow.
 - iii. At the P_2 prompt, enter the payment number and press **ENTER**.
 - iv. Press the down arrow.
 - v. The **BAL** entry is the balance entry for the corresponding row.

- vi. Press the down arrow.
- vii. The ***PRN*** entry is the principal paid entry for the corresponding row.
- viii. Press the down arrow.
- ix. The ***INT*** entry is the interest paid entry for the corresponding row.
- x. Press the down arrow the return to the ***P₁*** screen.

Remember, if you are interested in a *single* payment, you must set ***P₁*** and ***P₂*** to the exact same value. For example, if you want to see the details of the **22nd** payment then both ***P₁* = 22** and ***P₂* = 22**.



As described above, you can find the interest paid and principal paid components of any payment by using the amortization worksheet on your calculator. To calculate these values by hand, use the following formulas.

$$\text{Principal Paid for a Payment} = \text{Payment} - \text{Interest Paid}$$

$$\text{Interest Paid for a Payment} = \text{Balance from the Previous Row} \times i$$

where i is the periodic interest rate for the loan.

But, these formulas require two things: that you know the balance from the previous row and that the payment frequency and the compounding frequency for the loan are equal. Although it is possible to solve both of these problems, such as doing an interest conversion to find the equivalent rate with the compounding frequency equal to the payment frequency, the easiest way to find the interest and principal paid components is to use the amortization worksheet on a financial calculator.

Example 3.2.1

You repaid your student loan of \$20,000 at 3.5% compounded quarterly with month-end payments over five years.

1. What was the outstanding balance on the loan after you made payment 18?
2. How much interest was paid with the 27th payment?
3. How much principal was paid with the 42nd payment?
4. What was the outstanding balance on the loan after three years?

Solution

Step 1: Calculate the payment.

PMT Setting	END
N	$12 \times 5 = 60$
PV	20,000
FV	0
PMT	?
I/Y	3.5
P/Y	12
C/Y	4

$$PMT = \$363.74$$

Step 2: Calculate the balance for payment 18.

To find the balance for payment 18, set $P_1 = 18$ and $P_2 = 18$.

PMT Setting	END
N	60
PV	20,000
FV	0
PMT	−363.74
I/Y	3.5
P/Y	12
C/Y	4
P_1	18
P_2	18

$$BAL = \$14,361.53$$

After 18 payments, the balance on the loan is \$14,361.53.

Step 3: Calculate the interest for payment 27.

To find the interest for payment 27, set $P_1 = 27$ and $P_2 = 27$.

PMT Setting	END
N	60
PV	20,000
FV	0
PMT	−363.74
I/Y	3.5
P/Y	12
C/Y	4
P_1	27
P_2	27

$$INT = \$34.20$$

The interest paid for payment 27 is \$34.20.

Step 4: Calculate the principal for payment 42.

To find the principal for payment 42, set $P_1 = 42$ and $P_2 = 42$.

PMT Setting	END
N	60
PV	20,000
FV	0
PMT	−363.74
I/Y	3.5
P/Y	12
C/Y	4
P_1	42
P_2	42

$$PRN = \$344.21$$

The principal paid for payment 42 is \$344.21.

Step 5: Calculate the balance after three years.

To find the balance after three years, enter the payment number that corresponds to the last payment made in year three. There are 12 payments a year, so the last payment made in year three is 36 (3×12). So, to find the balance after three years, set $P_1 = 36$ and $P_2 = 36$.

PMT Setting	END
N	60
PV	20,000
FV	0
PMT	−363.74
I/Y	3.5
P/Y	12
C/Y	4
P_1	36
P_2	36

$$BAL = \$8,420.49$$

Step 6: Write as a statement.

After three years, the balance on the loan is \$8,420.49.

Note

The calculator thinks in terms of payment numbers, not years. That is, P_1 and P_2 must be payment numbers. In the last part of the previous example, a common mistake is to set $P_1 = 3$ and $P_2 = 3$ to find the balance at the end of year 3. But $P_1 = P_2 = 3$ tells the calculator to find the balance after payment number 3, not after year 3. To find the balance at the end of a particular year, you need the number of the last payment to happen in that year. You can find this payment number by multiplying the payment frequency by the year.

For example, suppose a loan has quarterly payments and you want to find the balance at the end of year 5. The last payment to happen in year five is 20 (4×5). The balance at the end of

year **5** is the same as the balance for payment **20**. In the amortization worksheet you would set $P_1 = 20$ and $P_2 = 20$.

Try It

- 1) A **\$100,000** loan at **4%** compounded semi-annually requires monthly payments of **\$650**.
- How many payments will it take to pay-off the loan?
 - How much principal is paid with the **97th** payment?
 - What is the balance on the loan after the **112th** payment?
 - How much interest is paid with the **175th** payment?
 - What is the balance on the loan at the end of year **10**?

Solution

- Calculate the number of payments.

PMT Setting	END
N	?
PV	100,000
FV	0
PMT	−650
I/Y	4
P/Y	12
C/Y	2

$$N = 215.725... \rightarrow 216 \text{ payments}$$

b. Calculate the principal for payment 97.

PMT Setting	END
N	216
PV	100,000
FV	0
PMT	−650
I/Y	4
P/Y	12
C/Y	2
P_1	97
P_2	97

$$PRN = \$438.48$$

c. Calculate the balance for payment 112.

PMT Setting	END
N	216
PV	100,000
FV	0
PMT	−650
I/Y	4
P/Y	12
C/Y	2
P_1	112
P_2	112

$$BAL = \$56,789.53$$

d. Calculate the interest for payment 175.

PMT Setting	END
N	216
PV	100,000
FV	0
PMT	−650
I/Y	4
P/Y	12
C/Y	2
P_1	175
P_2	175

$$INT = \$82.78$$

e. Calculate the balance after **10** years.

PMT Setting	END
<i>N</i>	216
<i>PV</i>	100,000
<i>FV</i>	0
<i>PMT</i>	−650
<i>I/Y</i>	4
<i>P/Y</i>	12
<i>C/Y</i>	2
<i>P</i> ₁	120
<i>P</i> ₂	120

$$BAL = \$53,048.37$$

Calculating the Total Interest Paid and Total Principal Paid for a Series of Payments

Many times in business, you need to know the principal and interest portions for a series of payments. For example, you might need to know the total interest paid or total principal paid on a loan for a particular year. As above, you can use the amortization worksheet on your financial calculator to find the total principal paid or total interest paid for a series of payments.

Using the TI BAII Plus Calculator to Find the Total Interest Paid or Total Principal Paid for A Series of Loan Payments

To use the amortization worksheet to find the total interest paid or total principal paid:

1. Solve for any unknown quantities about the loan. You need to know all of the information about the loan first before you can use the amortization worksheet.
2. Enter the values of all seven time value of money variables into the calculator (N , PV , FV , PMT , I/Y , P/Y , C/Y). If you calculated PMT in the first step, you must **re-enter** it rounded to two decimals and with the correct cash flow sign. Make sure the payment setting is set to **END**, and obey the cash flow sign convention. Because this is a loan, PV (the loan amount) is positive and PMT is negative.
3. Go to the amortization worksheet by pressing **2nd AMORT** (the PV button).
4. To view the total principal paid or total interest paid for a series of payments, set P_1 to the **first** payment number of the series of payments and set P_2 to the **last** payment number of the series of payments. For example, to view the total principal paid or total interest paid for payments four to seven, set $P_1 = 4$ and $P_2 = 7$. In the outputs from the amortization worksheet:
 - The BAL entry is the balance after payment number P_2 . Note that the BAL entry is only tied to the value of P_2 and does not depend on the value of P_1 .
 - The PRN entry is the total principal paid from payment number P_1 to payment number P_2 .
 - The INT entry is the total interest paid from payment number P_1 to payment number P_2 .



Key Takeaways

The ***PRN*** entry on the amortization worksheet is the sum of the principal paid entries in the amortization schedule starting at payment number P_1 and ending at payment number P_2 . For example, if $P_1 = 4$ and $P_2 = 7$. The ***PRN*** entry tells you the sum of the principal paid column in the amortization schedule starting with payment number 4 and ending with payment number 7.

The ***INT*** entry on the amortization worksheet is the sum of the interest paid entries in the amortization schedule starting at payment number P_1 and ending at payment number P_2 . For example, if $P_1 = 4$ and $P_2 = 7$, the ***INT*** entry tells you the sum of the interest paid column in the amortization schedule starting with payment number 4 and ending with payment number 7.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=113#oembed-1>

Video: [Mortgage Calculations using BAI Plus](#) by Joshua Emmanuel [4:13] *Transcript Available.*

Things to Watch Out For

A common mistake occurs in translating years into payment numbers. You often need to find the total interest paid or total principal paid for a particular year. To do this, you need to set P_1 equal to the number of the first payment that occurs in that year and P_2 equal to the number of the last payment that occurs in that year.

For example, suppose you have monthly payments and you want to know the total interest paid in the fourth year. In error, you might calculate that the fourth year begins with payment **36** and ends with payment **48**, and so enter $P_1 = 36$ and $P_2 = 48$. But the **36th** payment is actually the last payment of the third year. The first payment to occur in year four is the **37th**. So, if you wanted to find the total interest paid in year **4**, $P_1 = 37$ and $P_2 = 48$.

When you need to find the first payment number and last payment number for a particular year, there are two methods you can use to calculate the correct payment numbers.

1. Calculate the payment number at the end of the year in question by multiplying P/Y by the year number: $P/Y \times \text{year number}$. To find the payment number at the start of the year, subtract the payment frequency less one ($P/Y - 1$) from the last payment number of the year. For example, suppose the you have monthly payments and want to find the interest paid for year **4**. The last payment in year **4** is $12 \times 4 = 48$. The first payment in year **4** is $48 - (12 - 1) = 37$.
2. Calculate the payment number at the end of the year in question by multiplying P/Y by the year number: $P/Y \times \text{year number}$. To find the payment number for the first payment in the year, multiply P/Y by the previous year number and then add one to it: $P/Y \times (\text{year} - 1) + 1$. For example, suppose the you have monthly payments and want to find the interest paid for year **4**. The first payment in year **4** is $12 \times 3 + 1 = 37$. The last payment of the fourth year remains at payment **48**.

Example 3.2.2

You borrowed **\$25,000** to renovate your kitchen. The loan agreement calls for quarterly payments for ten years at **3.9%** compounded semi-annually.

1. What as the outstanding balance on the loan after four years?
2. How much interest was paid in year two?
3. How much principal was paid in year five?
4. How much interest was paid in year eight?

Solution

Step 1: Calculate the payment.

PMT Setting	END
N	$4 \times 10 = 40$
PV	25,000
FV	0
PMT	?
I/Y	3.9
P/Y	4
C/Y	2

$$PMT = \$757.10$$

Step 2: Calculate the balance at the end of year four.

To find the balance after four years, enter the payment number that corresponds to the last payment made in year four. There are **4** payments a year, so the last payment made in year four is **16** (4×4). So, to find the balance after four years, set $P_1 = 16$ and $P_2 = 16$.

PMT Setting	END
N	40
PV	25,000
FV	0
PMT	-757.10
I/Y	3.9
P/Y	4
C/Y	2
P_1	16
P_2	16

$$BAL = \$16,140.48$$

After four years, the balance on the loan is \$16,140.48.

Step 3: Calculate the interest paid in year two.

To find the interest paid in year two, set P_1 to first payment number of year two and P_2 to last payment of year two. There are 4 payments a year, so the last payment made in year two is 8 (4×2). The first payment made in year two is 5 ($4 \times 1 + 1$). So, to find the interest paid in year two, set $P_1 = 5$ and $P_2 = 8$.

PMT Setting	END
N	40
PV	25,000
FV	0
PMT	−757.10
I/Y	3.9
P/Y	4
C/Y	2
P_1	5
P_2	8

$$INT = \$857.91$$

The total interest paid in year two is \$857.91.

Step 4: Calculate the principal paid in year five.

To find the principal paid in year five, set P_1 to first payment number of year five and P_2 to last payment of year five. There are 4 payments a year, so the last payment made in year five is 20 (4×5). The first payment made in year five is 17 ($4 \times 4 + 1$). So, to find the principal paid in year five, set $P_1 = 17$ and $P_2 = 20$.

PMT Setting	END
N	40
PV	25,000
FV	0
PMT	-757.10
I/Y	3.9
P/Y	4
C/Y	2
P_1	17
P_2	20

$$PRN = \$2,437.15$$

The total principal paid in year five is \$2,437.15.

Step 5: Calculate the interest paid in year eight.

To find the interest paid in year eight, set P_1 to first payment number of year eight and P_2 to last payment of year eight. There are 4 payments a year, so the last payment made in year eight is **32** (4×8). The first payment made in year eight is **29** ($4 \times 7 + 1$). So, to find the interest paid in year eight, set $P_1 = 29$ and $P_2 = 32$.

PMT Setting	END
N	40
PV	25,000
FV	0
PMT	-757.10
I/Y	3.9
P/Y	4
C/Y	2
P_1	29
P_2	32

$$INT = \$291.84$$

Step 6: Write as a statement.

The total interest paid in year eight is **\$291.84**.

Try It

2) Art Industries just financed a **\$50,000** purchase at **5.9%** effective. It fixes the loan payment at **\$450** per month.

- How many payments will it take to pay-off the loan?
- How much principal is paid in year three?
- How much interest is paid in year ten?

- d. How much principal is paid in year seven?
- e. How much interest is paid in year eleven?

Solution

- a. Calculate the number of payments.

PMT Setting	END
<i>N</i>	?
<i>PV</i>	50,000
<i>FV</i>	0
<i>PMT</i>	−450
<i>I/Y</i>	5.9
<i>P/Y</i>	12
<i>C/Y</i>	1

$$N = 158.969... \rightarrow 159 \text{ payments}$$

- b. Calculate the principal paid in year three.

PMT Setting	END
N	159
PV	50,000
FV	0
PMT	−450
I/Y	5.9
P/Y	12
C/Y	1
P_1	25
P_2	36

$$PRN = \$2,909.70$$

c. Calculate the interest paid in year ten.

PMT Setting	END
N	159
PV	50,000
FV	0
PMT	−450
I/Y	5.9
P/Y	12
C/Y	1
P_1	109
P_2	120

$$INT = \$1,053.70$$

d. Calculate the principal paid in year seven.

PMT Setting	END
N	159
PV	50,000
FV	0
PMT	−450
I/Y	5.9
P/Y	12
C/Y	1
P_1	73
P_2	84

$$PRN = \$3,659.58$$

e. Calculate the interest paid in year eleven.

PMT Setting	END
N	159
PV	50,000
FV	0
PMT	−450
I/Y	5.9
P/Y	12
C/Y	1
P_1	121
P_2	132

$$INT = \$797.27$$

Section 3.2 Exercises

1. A **\$39,000** loan at **6%** compounded semi-annually is repaid with quarterly payments over five years.
 - a. How much interest is paid with the **10th** payment?
 - b. How much principal is paid with the **17th** payment?
 - c. What is the outstanding balance on the loan after the **7th** payment is made?
 - d. How much principal is paid in year **2**?
 - e. How much interest is paid in year **3**?
 - f. What is the outstanding balance on the loan after year **4**?

Solution

- a. **\$340.47**, b. **\$2,138.84**, c. **\$26,639.82**, d. **\$7,326.61**, e. **\$1,303.60**, f. **\$8,748.41**
2. A **\$500,000** loan at **6.25%** effective is repaid with semi-annual payments of **\$35,000**.
 - a. How much principal is paid with the **9th** payment?
 - b. How much interest is paid with the **16th** payment?

- c. What is the outstanding balance on the loan after the **3rd** payment is made?
- d. How much interest is paid in year **4**?
- e. How much principal is paid in year **9**?
- f. What is the outstanding balance on the loan after year **7**?

Solution

- a. **\$24,993.85**, b. **\$4,098.15**, c. **\$439,335.29**, d. **\$22,228.78**, e. **\$64,686.11**, f. **\$163,138.14**
3. A **\$14,000** loan at **6%** compounded monthly is repaid by monthly payments over four years.
- a. What is the size of the monthly payment?
 - b. Calculate the principal portion of the **25th** payment.
 - c. Calculate the interest portion of the **33rd** payment.
 - d. Calculate the total interest paid in the second year.
 - e. Calculate the principal portion of the payments in the third year.
 - f. Calculate the outstanding balance at the end of first year.

Solution

- a. **\$328.79**, b. **\$291.70**, c. **\$25.22**, d. **\$556.26**, e. **\$3,598.25**, f. **\$10,807.68**
4. Quarterly payments are to be made against a **\$47,500** loan at **5.95%** compounded annually with a six-year amortization.
- a. What is the size of the quarterly payment?
 - b. Calculate the principal portion of the sixth payment.
 - c. Calculate the interest portion of the **17th** payment.

- d. Calculate how much the principal will be reduced in the fourth year.
- e. Calculate the total interest paid in the third year.

Solution

- a. **\$2,359.13**, b. **\$1,792.76**, c. **\$257.53**, d. **\$8,109.20**, e. **\$1,782.27**
5. Cathy and Bill just acquired a new Honda Odyssey Touring Edition minivan for **\$60,531.56** under the dealership's purchase financing of **5.65%** compounded annually for eight years.
- a. What are their monthly car payments?
 - b. In the first year, what total amount of interest will they pay?
 - c. In the fourth year, by how much will the principal be reduced?
 - d. What is the balance on the loan at end of sixth year?

Solution

- a. **\$781.07**, b. **\$3,179.72**, c. **\$7,303.28**, d. **\$17,711.66**
6. You took out a business loan of **\$75,000** at **2.9%** compounded quarterly. You repay the loan with monthly payments of **\$1300**.
- a. What is the outstanding balance on the loan after the **33rd** payment is made?
 - b. How much principal is paid with the **41st** payment?
 - c. How much interest is paid with the **29th** payment?
 - d. What is the outstanding balance on the loan after year **2**?
 - e. How much interest is paid in year **3**?
 - f. How much principal is paid in year **4**?

Solution

a. \$36,606.07, b. \$1,232.35, c. \$102.75, d. \$47,381.49, e. \$1,180.56, f.
\$14,842.18

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3.3 CALCULATING THE FINAL PAYMENT

Learning Objectives

- Calculate the final payment for a loan

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance immediately after a loan payment
- INT = Interest portion of a loan payment or a series of payments
- PRN = Principal portion of a loan payment or a series of payments
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- N = Total number of annuity payments

Introduction

If you have ever paid off a loan you may have noticed that your last payment was a slightly different amount than your other payments. Whether you are making monthly insurance premium payments, paying municipal property tax instalments, financing your vehicle, paying your mortgage, receiving monies from an investment annuity, or dealing with any other situation where an annuity is extinguished through equal payments, the last payment typically differs from the rest, by as little as one penny or up to a few dollars. This difference can be much larger if you arbitrarily chose an annuity payment as opposed to determining an accurate payment through time value of money calculations.

Why is it important for this final payment to differ from all of the previous payments? From a consumer perspective, you do not want to pay a cent more toward a debt than you have to. In 2011, the average Canadian is more than \$100,000 in debt across various financial tools such as car loans, consumer debt, and mortgages. Imagine if you overpaid every one of those debts by a dollar. Over the course of your lifetime those overpayments would add up to hundreds or even thousands of dollars.

Calculating the Final Payment

When you calculate out the payment for a loan, the actual payment is rounded to two decimal places. It is rare for a calculated loan payment not to require rounding. The rounding up or down of the payment forms the basis for adjusting the final payment.

Consider the situation where the loan payment is rounded up. Suppose the calculated loan payment is \$999.995. Rounding to two decimals, the actual payment made is \$1,000. So with each \$1,000 loan payment, you are overpaying the debt by $\$1,000 - \$999.995 = \$0.005$. If you make 20 such payments, you end up overpaying the debt by $20 \times \$0.005 = \0.10 . Therefore, when it comes to the final payment you need to compensate for all of the overpayments made, reducing the final payment by \$0.10. Because the principal is slightly smaller at all times as a result of the overpayment, an additional adjustment may be needed because of less interest being calculated.

Now, consider the situation where the loan payment is rounded down. Suppose the calculated loan payment is \$1,000.0025. Rounding to two decimals, the actual payment made is \$1,000. So, with each \$1,000 loan payment, you are underpaying the debt by $\$1,000.0025 - \$1,000 = \$0.0025$. If you make 20 such payments, you end up underpaying by $20 \times \$0.0025 = \0.05 . Therefore, when it comes to the final payment, you need to increase the final payment by this amount. Because the principal is slightly larger at all times as a result of the underpayment, an additional adjustment may be needed because of more interest being calculated.

We have already learned how to calculate the final payment by completing the last row of the amortization schedule. This method gives the following formula for calculating the final payment.

$$\text{Final Payment} = \text{Balance from Second Last Row} + \text{Interest from Last Row}$$

Alternatively, we can calculate the final payment without constructing the last row of the amortization schedule. This method is based on the assumption that all of the payments, including the final payment are the same. The final payment is found by adjusting the regular payment (PMT) by the amount overpaid or underpaid.

$$\text{Final Payment} = PMT - \text{Amount Overpaid}$$

or

$$\text{Final Payment} = PMT + \text{Amount Underpaid}$$

Key Takeaways



An overpayment means that you have paid more through the regular payments than is necessary. The total amount overpaid is subtracted from the payment to find the final payment. Consequently, the final payment is smaller than the other payments.

An underpayment means that you have not paid enough through the regular payments than is necessary. The total amount underpaid is added to the payment to find the final payment. Consequently, the final payment is larger than the other payments.

Using the TI BAII Plus Calculator to Find the Final Payment

To use the amortization worksheet to find the final payment:

1. Solve for any unknown quantities about the loan. You need to know all of the information about the loan first before you can use the amortization worksheet.
2. Enter the values of all seven time value of money variables into the calculator (N , PV , FV , PMT , I/Y , P/Y , C/Y). If you calculated PMT in the first step, you must **re-enter** it rounded to two decimals and with the correct cash flow sign. Make sure the payment setting is set to **END**, and obey the cash flow sign convention. Because this is a loan, PV (the loan amount) is positive and PMT is negative.
3. Go to the amortization worksheet by pressing **2nd AMORT** (the PV button).
4. Enter the payment number for the final payment into P_1 and P_2 .
5. Find the BAL entry. Watch the cash flow sign of the BAL entry to properly interpret what to do with it!
6. Calculate the final payment:

- A *negative* BAL entry indicates an *overpayment*. Then

$$\text{Final Payment} = PMT - \text{Overpayment}$$

- A *positive* BAL entry indicates an *underpayment*. Then

$$\text{Final Payment} = PMT + \text{Underpayment}$$

Example 3.3.1

A \$10,000 loan at 8% compounded quarterly is repaid with month-end payments of \$200. Calculate the final payment.

Solution

Step 1: Calculate the number of payments.

PMT Setting	END
N	?
PV	10,000
FV	0
PMT	-200
I/Y	8
P/Y	12
C/Y	4

$$N = 60.9273... \rightarrow 61 \text{ payments}$$

Step 2: Calculate the balance for payment 61 (the last payment).

To find the balance for payment 61, set $P_1 = 61$ and $P_2 = 61$.

PMT Setting	END
N	61
PV	10,000
FV	0
PMT	−200
I/Y	9
P/Y	12
C/Y	4
P_1	61
P_2	61

$$BAL = -\$14.49$$

Step 3: Calculate the final payment.

Because the BAL entry from the previous step is negative, the **\$14.49** is an overpayment and must be subtracted from the **\$200** payment.

$$\begin{aligned}\text{Final Payment} &= 200 - 14.49 \\ &= \$185.11\end{aligned}$$

Step 4: Write as a statement.

The final payment for the loan is **\$185.11**.

Example 3.3.2

A \$50,000 loan at 5% compounded quarterly is repaid with quarterly payments for 6.5 years. Calculate the final payment.

Solution

Step 1: Calculate the payment.

PMT Setting	END
N	26
PV	50,000
FV	0
PMT	?
I/Y	5
P/Y	4
C/Y	4

$$PMT = \$2,264.36$$

Step 2: Calculate the balance for payment 26 (the last payment).

To find the balance for payment 26, set $P_1 = 26$ and $P_2 = 26$. Remember to re-enter the payment rounded to two decimal places.

PMT Setting	END
N	26
PV	50,000
FV	0
PMT	−2,264.36
I/Y	5
P/Y	4
C/Y	4
P_1	26
P_2	26

$$BAL = \$0.13$$

Step 3: Calculate the final payment.

Because the BAL entry from the previous step is positive, the **\$0.13** is an underpayment and must be added to the **\$2,264.36** payment.

$$\begin{aligned}\text{Final Payment} &= 2,264.36 + 0.13 \\ &= \$2,264.49\end{aligned}$$

Step 4: Write as a statement.

The final payment for the loan is **\$2,264.49**.

Try It

1) Semi-annual payments are made against a **\$97,500** loan at **7.5%** compounded semi-annually with a 10-year amortization. Calculate the final payment.

Solution

PMT Setting	END
<i>N</i>	20
<i>PV</i>	97,500
<i>FV</i>	0
<i>PMT</i>	?
<i>I/Y</i>	7.5
<i>P/Y</i>	2
<i>C/Y</i>	2

$$PMT = \$7,016.30$$

PMT Setting	END
N	20
PV	97,500
FV	0
PMT	-7,016.30
I/Y	7.5
P/Y	2
C/Y	2
P_1	20
P_2	20

$$BAL = \$0.13$$

$$\begin{aligned}\text{Final Payment} &= 7,016.30 + 0.13 \\ &= \$7,016.43\end{aligned}$$

Try It

2) A \$65,000 loan at 3.5% compounded quarterly is repaid with \$600 monthly payments. Calculate the final payment.

Solution

PMT Setting	END
N	?
PV	65,000
FV	0
PMT	−600
I/Y	3.5
P/Y	12
C/Y	4

$$N = 130.309... \rightarrow 131 \text{ payments}$$

PMT Setting	END
N	131
PV	65,000
FV	0
PMT	−600
I/Y	3.5
P/Y	12
C/Y	4
P_1	131
P_2	131

$$BAL = -\$413.98$$

$$\begin{aligned} \text{Final Payment} &= 600 - 413.98 \\ &= \$186.02 \end{aligned}$$

Section 3.3 Exercises

1. A \$15,000 loan at 10% compounded quarterly is repaid with quarterly payments for three years. Calculate the final payment.

Solution

\$1,462.27

2. A \$85,000 loan at 6.75% compounded monthly is repaid with monthly payments for seven years. Calculate the final payment.

Solution

\$1,273.03

3. A \$32,000 loan at 8.25% effective is repaid with annual payments for 10 years. Calculate the final payment.

Solution

\$4,822.82

4. A \$250,000 loan at 5.9% compounded semi-annually is repaid with monthly payments of \$2,400. Calculate the final payment.

Solution

\$1,265.95

5. A \$28,250 loan at 9% compounded quarterly is repaid by monthly payments over five years. Calculate the final payment.

Solution

\$585.50

6. You took out a \$90,000 home renovation loan at 2.7% compounded semi-annually. You make quarterly payments of \$3,000 to repay the loan. Calculate your final payment.

Solution

\$1,864.93

7. Stuart and Shelley just purchased a new **\$65,871.88** Nissan Titan Crew Cab SL at **8.99%** compounded monthly for a seven-year term. What is the amount of the final payment?

Solution

\$1,059.89

8. Wile E. Coyote owes the ACME Corporation **\$75,000** for various purchased goods. Wile agrees to make **\$1,000** payments at the end of every month at **10%** compounded quarterly until the debt is repaid in full. What is the amount of the final payment?

Solution

\$512.66

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3.4 MORTGAGES

Learning Objectives

- Calculate mortgage payments for the initial term and renewal term
- Calculate amortization periods to reflect changes in payment frequency
- Calculate mortgage balances and amortization periods to reflect prepayments of principal

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance immediately after a loan payment
- INT = Interest portion of a loan payment or a series of payments
- PRN = Principal portion of a loan payment or a series of payments
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- N = Total number of annuity payments

Introduction

A **mortgage** is a special type of loan that is collaterally secured by real estate. In essence, the loan has a lien against the property, which is the right to seize the property for the debt to be satisfied. An individual or business taking out a mortgage is obliged to pay back the amount of the loan with interest based on a predetermined contract. The lender, a bank, or some other financial institution, has a claim on the real estate property in the event that the mortgage goes into default, which means that it is not paid as per the agreement. In these instances, financial institutions will pursue foreclosure of the property, which allows for the tenants to be evicted and the property to be sold. The proceeds of the sale are then used to pay off the mortgage. A mortgage always involves two parties. The individual or business that borrows the money is referred to as the **mortgagor**, and the financial institution that lends the money is referred to as the **mortgagee**.

Types of Mortgages

In the mortgage contract, the two parties can agree to either a fixed interest rate or a variable interest rate. Either way, the mortgage always forms an ordinary annuity because interest is not payable in advance.

- Under a ***fixed interest rate***, the principal is repaid through a number of equal payments that cover both the interest and principal components of the loan. The interest portion is highest at the beginning and gradually declines over the amortization period of the mortgage. In Canada, fixed interest rates are either annually or semi-annually compounded, with the latter being the prevailing choice.
- In a ***variable interest rate*** mortgage, the principal is repaid through an agreed-upon number of unequal payments that fluctuate with changes in borrowing rates. The principal and interest portions of the payment vary as interest rates fluctuate, meaning that the interest portion can rise at any point with any increase in rates. When rates change, a common practice in many financial institutions is to change the variable interest rate as of the first day of the next month. If the rate change does not coincide with a mortgage payment date, then the interest portion is calculated in a way similar to the procedure for a demand loan, where the exact number of days at the different rates must be determined. In Canada, variable interest rates are usually compounded monthly.

The mortgage agreement can be open or closed. An **open mortgage** has very few rules and it allows the mortgagor to pay off the debt in full or make additional prepayments at any given point in any amount without penalty. A **closed mortgage** has many rules that determine how the mortgage is to be paid. It does not allow the mortgager to pay off the debt in full until the loan matures. As a marketing tool, most closed mortgages have “top-up” options that allow the mortgagor to make additional payments (such as an additional 20% per year) against the mortgage without penalty. Any payments exceeding the maximums or early payment of the mortgage are penalized heavily, with a three-month minimum interest charge that could be increased up to a

measure called the interest rate differential, which effectively assesses the bank's loss and charges the mortgagor this full amount.

Calculating Mortgage Payments and Outstanding Balance

Previously, we learned how to calculate the payment for a loan and how to find the remaining balance on a loan after any payment is made. Because a mortgage is a special type of loan, we can use these same techniques on a mortgage.

Example 3.4.1

A new home is purchased for \$408,726 with a \$50,000 down payment and a 25-year mortgage for the outstanding balance. The mortgage is repaid with monthly payments. The interest rate on the mortgage is fixed at 5.29% compounded semi-annually for a five year term.

1. What is the size of the monthly payments?
2. What is the balance on the mortgage at the end of the five year term?

Solution

Step 1: Calculate the loan amount.

$$\begin{aligned}\text{Loan Amount} &= 408,726 - 50,000 \\ &= \$358,726\end{aligned}$$

Step 2: Calculate the payment.

PMT Setting	END
N	$12 \times 25 = 300$
PV	358,726
FV	0
PMT	?
I/Y	5.29
P/Y	12
C/Y	2

$$PMT = \$2,145.98$$

The monthly mortgage payments are \$2,145.98.

Step 3: Calculate the balance after five years.

To find the balance after five years, enter the payment number that corresponds to the last payment made in year five. There are **12** payments a year, so the last payment made in year five is **60** (5×12). So, to find the balance after five years, set $P_1 = 60$ and $P_2 = 60$.

PMT Setting	END
N	300
PV	358,726
FV	0
PMT	-2,145.98
I/Y	5.29
P/Y	12
C/Y	2
P_1	60
P_2	60

$$BAL = \$318,927.89$$

Step 4: Write as a statement.

At the end of the five year term, the balance is **\$318,927.89**.

Key Takeaways



In the previous example, the payment is calculated based on the entire **25** year amortization period, and not the five year term. This is true for any mortgage. The payments are always calculated based on the assumption that the current interest rate will last for the entire amortization period, which determines the length of time over which the loan is repaid. The amortization period forms the basis for calculating the payment. The term has no effect on the payment calculation. The term dictates only the time frame during which the current mortgage arrangement (interest rate, payment frequency, type, and so on) remains in effect.

Renewing the Mortgage

When the term of a mortgage expires, the balance remaining becomes due in full. Typically the balance owing is still quite substantial, so the mortgage must be renewed. As discussed earlier, this means that the mortgagor assumes another mortgage, not necessarily with the same financial institution, and the amortization term is typically reduced by the length of the first term. The length of the second term of the mortgage then depends on the choice of the mortgagor. Other variables such as payment frequency and the interest rate may or may not change.

For example, assume a mortgage is initially taken out with a 25-year amortization period and a five-year term. After five years, the mortgage becomes due in full. Unable to pay it, the mortgagor renews the mortgage for the remaining 20-year amortization period, and also opts for a three-year term in assuming the new mortgage.

When those three years are over, the mortgagor renews the mortgage for the remaining 17-year amortization period and again makes another term decision. This process repeats until the debt is ultimately paid off.

HOW TO

Calculate the New Payment When a Mortgage is Renewed

When the mortgage is renewed for another term, the interest rate or payment frequency may change. Consequently, the payments on the mortgage will also change when the mortgage is renewed and the new term goes into affect. To find the new payment when the mortgage is renewed, calculate the payment using the remaining balance at the end of the previous term and the remaining time from the amortization period.

The steps to find the new payment upon renewal are:

Step 1: Calculate the original payment for the mortgage before the mortgage is renewed.

Step 2: Calculate the remaining balance at the end of the previous term.

Step 3: Calculate the new payment for the mortgage using the balance from the previous step as the loan amount (the present value), the remaining time on the amortization period, and incorporating any changes that occur in the new term, such as a new interest rate or change in payment frequency.

Example 3.4.2

A \$389,000 mortgage is repaid with monthly payments and amortized over 20 years. The interest rate for the first three year term is 4.9% compounded semi-annually. At the end of the three year term the mortgage is renewed at 5.85% compounded semi-annually. What is the size of the monthly payment when the mortgage is renewed?

Solution

Step 1: Calculate the payment for the first term.

PMT Setting	END
N	$12 \times 20 = 240$
PV	389,000
FV	0
PMT	?
I/Y	4.9
P/Y	12
C/Y	2

$$PMT = \$2,535.26$$

The monthly mortgage payments are \$2,535.26.

Step 2: Calculate the balance at the end of the three year term.

To find the balance after three years, enter the payment number that corresponds to the last payment made in year three. There are **12** payments a year, so the last payment made in year three is **36** (3×12). So, to find the balance after three years, set $P_1 = 36$ and $P_2 = 36$.

PMT Setting	END
N	240
PV	389,000
FV	0
PMT	-2,535.26
I/Y	4.9
P/Y	12
C/Y	2
P_1	36
P_2	36

$$BAL = \$351,770.37$$

At the end of the three year term, the balance is **\$351,770.37**.

Step 3: Calculate the payments when the mortgage is renewed.

The interest rate changes to **5.85%** compounded semi-annually. The time used to find the new payments is based on the *remaining* amortization period. The original amortization period is **20** years and **3** years have past to get to the time of renewal, so the remaining amortization period is **20 - 3 = 17** years. The new payments are calculated using the new interest rate, **5.85%** compounded semi-annually, the **17** years remaining on the amortization period, and the remaining balance, **\$351,770.37**, from the end of the previous term.

PMT Setting	END
N	$12 \times 17 = 204$
PV	351,770.37
FV	0
PMT	?
I/Y	5.85
P/Y	12
C/Y	2

$$PMT = \$2,711.92$$

Step 4: Write as a statement.

The monthly payments when the mortgage is renewed are **\$2,711.92**.

Try It

1) A **\$628,200** house is purchased for **\$100,000** down and monthly payments for **30** years. The interest rate for the first four year term is **6.49%** compounded semi-annually.

1. Calculate the monthly payments.
2. What is the balance on the mortgage at the end of the four year term?
3. If the mortgage is renewed at **6.19%** compounded semi-annually, what is the size of the new monthly payments?

Solution

1. Calculate the monthly payments.

PMT Setting	END
N	$12 \times 30 = 360$
PV	528,200
FV	0
PMT	?
I/Y	6.49
P/Y	12
C/Y	2

$$PMT = \$3,305.29$$

2. Calculate the balance at the end of the four year term.

PMT Setting	END
N	360
PV	528,200
FV	0
PMT	-3,305.29
I/Y	6.49
P/Y	12
C/Y	2
P_1	48
P_2	48

$$BAL = \$501,665.54$$

3. Calculate the new payment upon renewal.

PMT Setting	END
N	$12 \times 26 = 312$
PV	501,665.54
FV	0
PMT	?
I/Y	6.19
P/Y	12
C/Y	2

$$PMT = \$3,123.63$$

Changing the Payment Frequency

Although monthly mortgage payments are the most common payment frequency, mortgages can be paid off with semi-monthly, bi-weekly, or weekly payments. By making more frequency payments, the mortgagor can reduce the amortization period and pay-off the mortgage faster.

To determine how much the amortization period is shortened by changing the payment frequency, calculate the amortization period for the new payment frequency and then calculate the difference in the two amortization periods.

Example 3.4.3

A \$280,000 mortgage at 5.3% compounded semi-annually is repaid with monthly payments of

\$1678 amortized over 25 years. How much shorter is the amortization period if weekly payments of \$400 are made instead of the monthly payments?

Solution

Step 1: Calculate the amortization period for the weekly payments.

PMT Setting	END
N	?
PV	280,000
FV	0
PMT	-400
I/Y	5.3
P/Y	52
C/Y	2

$$N = 1,211.952... \rightarrow 1,212 \text{ payments}$$

Convert (rounded UP) N to years and months.

$$\begin{aligned}
 \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\
 &= \frac{1,212}{52} \\
 &= 23.307... \\
 &\rightarrow 23 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.307... \times 12 \\
 &= 3.692... \\
 &\rightarrow 4 \text{ months}
 \end{aligned}$$

The amortization period for the weekly payments is 23 years, 4 months.

Step 2: Calculate the difference in the amortization periods.

$$\begin{aligned}
 \text{Difference} &= 25 \text{ years} - (23 \text{ years and } 4 \text{ months}) \\
 &= 1 \text{ year and } 8 \text{ months}
 \end{aligned}$$

Step 3: Write as a statement.

With the weekly payments, the amortization period is shortened by **1** year and **8** months.

Try It

A **\$300,000** mortgage at **4.7%** compounded semi-annually is repaid with monthly payments of **\$1,925** for **20** years. How much shorter is the amortization period if bi-weekly payments of **\$1,000** are made instead of the monthly payments?

Solution

1. Calculate the amortization period for the bi-weekly payments.

PMT Setting	END
N	?
PV	300,000
FV	0
PMT	-1,000
I/Y	4.7
P/Y	26
C/Y	2

$$N = 430.370... \rightarrow 431 \text{ payments}$$

$$\begin{aligned}
 \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\
 &= \frac{431}{26} \\
 &= 16.576... \\
 &\rightarrow 16 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.5769... \times 12 \\
 &= 6.923... \\
 &\rightarrow 7 \text{ months}
 \end{aligned}$$

2. Calculate the difference in the amortization periods.

$$\begin{aligned}
 \text{Difference} &= 20 \text{ years} - (16 \text{ years and } 7 \text{ months}) \\
 &= 3 \text{ year and } 5 \text{ months}
 \end{aligned}$$

With the bi-weekly payments, the amortization period is shortened by 3 year and 5 months.

Lump-Sum Payments

As noted above, closed mortgages allow the mortgagor to make extra payments, in addition to the regular periodic payments. Any extra payments applied to the mortgage are applied directly to the principal because the interest is paid by the regular periodic payment. Consequently, these extra payments allow the mortgagor to repay the mortgage sooner. What affect do these extra payments have on the amortization period? How much is the amortization period shortened by when extra payments are made?

One of these pre-payment options is to make a lump-sum payment. A lump-sum payment is simply a single amount of money applied to the mortgage. For example, you might make a \$10,000 lump-sum payment to your mortgage at the end of the sixth year. This means that at the end of the sixth year of the amortization period, you pay \$10,000 to the mortgage principal. What affect does this have on the amount of time it takes to re-pay the mortgage?

HOW TO

Apply a Lump Sum Payment

To calculate how much the amortization period is shortened by when a lump-sum payment is applied:

Step 1: Find any missing information about the mortgage. For example, you might have to calculate out the payment or the number of payments, depending on what information is missing.

Step 2: Calculate the balance on the mortgage at the time the lump-sum payment is made.

Step 3: Subtract the lump-sum payment from the balance found in Step 2. This amount becomes the remaining balance on the mortgage.

Step 4: Calculate the number of regular periodic payments required to pay-off the balance found in step 3.

Step 5: Calculate the difference between the number of payments without the lump-sum payment applied and the number of payments with the lump-sum payment included.

Step 6: Convert the difference found in step 5 to years and months to find how much the amortization period is shortened by with the lump-sum payment included.

Things to Watch Out For

The lump sum payment is NOT counted as one of the payments in step 4. The number of payments refers to the number of regularly scheduled periodic payments.

Example 3.4.4

A \$400,000 mortgage at 3.5% compounded semi-annually is repaid with monthly payments for 25 years. How much is the amortization period shortened by if a \$15,000 lump-sum payment is made at the end of year five?

Solution

Step 1: Calculate the monthly payments.

PMT Setting	END
N	$12 \times 25 = 300$
PV	400,000
FV	0
PMT	?
I/Y	3.5
P/Y	12
C/Y	2

$$PMT = \$1,997.08$$

The monthly mortgage payments are \$1,997.08.

Step 2: Calculate the balance at the end of five years (the time of the lump-sum payment).

To find the balance after five years, enter the payment number that corresponds to the last payment made in year five. There are 12 payments a year, so the last payment made in year five is 60 (5×12). So, to find the balance after five years, set $P_1 = 60$ and $P_2 = 60$.

PMT Setting	END
N	300
PV	400,000
FV	0
PMT	-1,997.08
I/Y	3.5
P/Y	12
C/Y	2
P_1	60
P_2	60

$$BAL = \$345,120.01$$

After five years, the balance is \$345,120.01.

Step 3: Subtract the lump-sum payment from the balance found in the previous step.

$$\begin{aligned}\text{New Balance} &= 345,120.01 - 15,000 \\ &= \$330,120.01\end{aligned}$$

Step 4: Calculate the number of payments needed to re-pay the new balance found in the previous step.

PMT Setting	END
N	?
PV	330,120.01
FV	0
PMT	-1,997.08
I/Y	3.5
P/Y	12
C/Y	2

$$N = 225.263... \rightarrow 226 \text{ payments}$$

Step 5: Calculate the difference between the number of payments without the lump-sum payment and the number of payments with the lump-sum payment.

There are **300** payments without the lump-sum payment applied. With the lump-sum payment applied, there are **60** payments before the lump-sum payment and **226** payment after. So with the lump-sum payment, there are a total of $226 + 60 = 286$ payments.

$$\begin{aligned} \text{Difference} &= 300 - 286 \\ &= 14 \text{ payments} \end{aligned}$$

With the lump-sum payment applied, it takes **14** fewer payments to pay-off the mortgage.

Step 6: Convert the difference found in the previous step to years and months.

$$\begin{aligned} \text{Number of Years} &= \frac{14}{12} \\ &= 1.1666... \\ &\rightarrow 1 \text{ year} \end{aligned}$$

$$\begin{aligned} \text{Number of Months} &= 0.16666... \times 12 \\ &= 2 \text{ months} \end{aligned}$$

Step 7: Write as a statement.

The amortization period is shortened by **1** year and **2** months.

Try It

2) A **\$300,000** mortgage at **2.8%** compounded semi-annually is repaid with monthly payments for **20** years. How much is the amortization period shortened by if a **\$10,000** lump-sum payment is made at the end of year four?

Solution

1. Calculate the monthly payments.

PMT Setting	END
N	240
PV	300,000
FV	0
PMT	?
I/Y	2.8
P/Y	12
C/Y	2

$$PMT = \$1,631.51$$

2. Calculate the balance at the end of four years.

PMT Setting	END
N	240
PV	300,000
FV	0
PMT	-1,631.51
I/Y	2.8
P/Y	12
C/Y	2
P1	48
P2	48

$$BAL = \$252,555.63$$

3. Subtract the lump-sum payment from the balance.

$$\begin{aligned}\text{New Balance} &= 252,555.63 - 10,000 \\ &= \$242,555.63\end{aligned}$$

4. Calculate the number of payments needed to re-pay the new balance found in the previous step.

PMT Setting	END
N	?
PV	242,555.63
FV	0
PMT	-1,631.51
I/Y	2.8
P/Y	12
C/Y	2

$$N = 182.530... \rightarrow 183 \text{ payments}$$

5. Calculate the difference between the number of payments and convert the difference to years and months.

$$\begin{aligned} \text{Difference} &= 240 - (48 + 183) \\ &= 9 \text{ payments} \end{aligned}$$

$$\begin{aligned} \text{Number of Years} &= \frac{9}{12} \\ &= 0.75 \\ &\rightarrow 0 \text{ years} \end{aligned}$$

$$\begin{aligned} \text{Number of Months} &= 0.75 \times 12 \\ &= 9 \text{ months} \end{aligned}$$

The amortization period is shortened by 9 months.

Increasing the Amount of the Periodic Payment

Another one of the pre-payment options is to increase the periodic payment amount. For example, you might decide to increase your periodic payment by \$10 at the end of the sixth year. This means that starting with the first payment in year seven, each of the periodic payments will be \$10 higher. Because the interest was already paid by the original periodic payment amount, this extra \$10 gets applied to the mortgage principal. What affect does this have on the amount of time it takes to re-pay the mortgage?

HOW TO

Increasing the Periodic Payment

To calculate how much the amortization period is shortened by when the amount of the periodic payment is increased:

Step 1: Find any missing information about the mortgage. For example, you might have

to calculate out the payment or the number of payments, depending on what information is missing.

Step 2: Calculate the balance on the mortgage at the time the payment is increased.

Step 3: Calculate the new payment by adding the payment increase to the original payment amount.

Step 4: Calculate the number of new periodic payments required to pay-off the balance found in step 2.

Step 5: Calculate the difference between the number of payments without the payment increase applied and the number of payments with the payment increase included.

Step 6: Convert the difference found in step 5 to years and months to find how much the amortization period is shortened by with the payment increase included.

Example 3.4.5

A \$375,000 mortgage at 3.1% compounded semi-annually is repaid with monthly payments for 20 years. How much is the amortization period shortened by if the payments are increased by \$200 at the end of year eight?

Solution

Step 1: Calculate the monthly payments.

PMT Setting	END
N	$12 \times 20 = 240$
PV	375,000
FV	0
PMT	?
I/Y	3.1
P/Y	12
C/Y	2

$$PMT = \$2,094.82$$

The monthly mortgage payments are \$2,094.82.

Step 2: Calculate the balance at the end of eight years (the time of the payment increase).

To find the balance after eight years, enter the payment number that corresponds to the last payment made in year eight. There are **12** payments a year, so the last payment made in year eight is **96** (8×12). So, to find the balance after eight years, set $P_1 = 96$ and $P_2 = 96$.

PMT Setting	END
N	240
PV	375,000
FV	0
PMT	-2,094.82
I/Y	3.1
P/Y	12
C/Y	2
P_1	96
P_2	96

$$BAL = \$251,916.89$$

After eight years, the balance is \$251,916.89.

Step 3: Add the payment increase to the payment to find the new payment amount.

$$\begin{aligned}\text{New Payment} &= 2,094.82 + 200 \\ &= \$2,294.82\end{aligned}$$

Step 4: Calculate the number of payments needed to re-pay the balance found in Step 2 with the new payment found in Step 3.

PMT Setting	END
N	?
PV	251,916.89
FV	0
PMT	-2,294.82
I/Y	3.1
P/Y	12
C/Y	2

$$N = 129.108... \rightarrow 130 \text{ payments}$$

Step 5: Calculate the difference between the number of payments without the payment increase and the number of payments with the payment increase.

There are **240** payments without the payment increase. With the payment increase, there are **96** payments before the payment increase and **130** payment after. So with the payment increase, there are a total of $96 + 130 = 226$ payments.

$$\begin{aligned} \text{Difference} &= 240 - 226 \\ &= 14 \text{ payments} \end{aligned}$$

With the payment increase, it takes **14** fewer payments to pay-off the mortgage.

Step 6: Convert the difference found in the previous step to years and months.

$$\begin{aligned} \text{Number of Years} &= \frac{14}{12} \\ &= 1.1666... \\ &\rightarrow 1 \text{ year} \end{aligned}$$

$$\begin{aligned} \text{Number of Months} &= 0.16666... \times 12 \\ &= 2 \text{ months} \end{aligned}$$

Step 7: Write as a statement.

The amortization period is shortened by **1** year and **2** months.

Try It

3) A \$500,000 mortgage at 4.3% compounded semi-annually is repaid with monthly payments for 25 years. How much is the amortization period shortened by if the payments are increased by \$75 at the end of year three?

Solution

1. Calculate the monthly payments.

PMT Setting	END
N	300
PV	500,000
FV	0
PMT	?
I/Y	4.3
P/Y	12
C/Y	2

$$PMT = \$2,712.04$$

2. Calculate the balance at the end of three years.

PMT Setting	END
N	300
PV	500,000
FV	0
PMT	-2,712.04
I/Y	4.3
P/Y	12
C/Y	2
P_1	36
P_2	36

$$BAL = \$464,114.56$$

3. Calculate the new payment.

$$\begin{aligned}\text{New Payment} &= 2,712.04 + 75 \\ &= \$2,787.04\end{aligned}$$

4. Calculate the number of payments needed to re-pay the balance found in Step 2.

PMT Setting	END
N	?
PV	464,114.56
FV	0
PMT	-2,787.04
I/Y	4.3
P/Y	12
C/Y	2

$$N = 252.4769... \rightarrow 253 \text{ payments}$$

5. Calculate the difference between the number of payments and convert the difference to years and months.

$$\begin{aligned} \text{Difference} &= 300 - (36 + 253) \\ &= 11 \text{ payments} \end{aligned}$$

$$\begin{aligned} \text{Number of Years} &= \frac{11}{12} \\ &= 0.9166... \\ &\rightarrow 0 \text{ years} \end{aligned}$$

$$\begin{aligned} \text{Number of Months} &= 0.9166... \times 12 \\ &= 11 \text{ months} \end{aligned}$$

The amortization period is shortened by **11** months.

Section 3.4 Exercises

1. A **\$434,693** mortgage is repaid with monthly payments over **30** years. The interest rate for the first three year term is **4.5%** compounded semi-annually. If the mortgage is renewed at **5.25%** compounded semi-annually, what is the size of the monthly payments upon renewal?

Solution

\$2,370.32

2. A **\$318,222** mortgage is repaid with bi-weekly payments over **25** years. The interest rate for the first two year term is **3%** compounded semi-annually. If the mortgage is renewed at **6.8%** compounded semi-annually, what is the size of the bi-weekly

payments upon renewal?

Solution

\$985.74

3. Five years ago, Asia purchased her **\$322,000** home in Edmonton with a **25**—year amortization. In her first five-year term, she made monthly payments and was charged **4.89%** compounded semi-annually. She will renew the mortgage on the same amortization timeline for a three year term at **5.49%** compounded semi-annually with monthly payments.
 - a. Calculate the balance remaining after the first term.
 - b. What is the new mortgage payment amount for the second term?
 - c. What is the balance at the end of the second term?

Solution

a. **\$284,498.75**; b. **\$1,945.52**; c. **\$258,813.11**

4. Luke took out a **\$350,000** mortgage at **3.7%** compounded semi-annually and repaid the mortgage with monthly payments of **\$1,700**.
 - a. How long will it take Luke to repay the mortgage?
 - b. How much would the amortization period be shortened by if Luke made weekly payments of **\$425** instead of the monthly payments?

Solution

a. **27** years, **2** months; b. **3** years, **5** months

5. Sally took out a **\$475,000** mortgage at **2.9%** compounded semi-annually. She repaid the mortgage with monthly payments for **25** years. How much would the amortization period be shortened by if Sally made bi-weekly payments of **\$1,100** instead of the monthly payments?

Solution

2 years, **4** months

6. Sam received a **\$280,000** mortgage at **4.5%** compounded semi-annually. She repaid the mortgage with monthly payments for **20** years. How much would the amortization period be shortened by if Sam made a **\$12,000** lump-sum payment at the end of year two?

Solution

1 year, 2 months

7. A **\$420,000** mortgage at **3.47%** compounded semi-annually is repaid with monthly payments for **30** years. How much would the amortization period be shortened by if a lump-sum payment of **\$20,000** is made at the end of year six?

Solution

1 year, 11 months

8. Jesse received a **\$570,000** mortgage at **2.9%** compounded semi-annually. He repaid the mortgage with monthly payments for **25** years. How much would the amortization period be shortened by if Jesse increases his monthly payments by **\$175** at the end of year five?

Solution

1 year, 7 months

9. A **\$190,000** mortgage at **3.2%** compounded semi-annually is repaid with monthly payments for **15** years. How much would the amortization period be shortened by if the monthly payments are increased by **10%** at the end of year three?

Solution

1 year, 3 months

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3.5 REVIEW EXERCISES

Chapter 3 Review Exercises

1. Annanya took out a \$42,500 loan at 6.6% compounded monthly with monthly payments over the six-year amortization period. Calculate the total principal and interest portions for the third year.

Solution

$$PRN = \$6,810.95, INT = \$1,786.45$$

2. Two years ago, Sumandeep borrowed \$20,000 at 9.45% compounded monthly. She has been paying end-of-month payments since, and the last payment will be today. Calculate the amount of the final payment.

Solution

$$\$917.82$$

3. Hogwild Industries borrowed \$75,000 to purchase some new equipment. The terms of the ordinary loan require quarterly payments for three years with an interest rate of 7.1% compounded semi-annually. Calculate the total interest and principal portions for the third year.

Solution

$$PRN = \$26,763.03, INT = \$1,187.52$$

4. Dr. Strong of Island Lakes Dental Centre acquired a new Panoramic X-ray machine for his practice. The \$7,400 for the machine, borrowed at 8.8% compounded annually, is to be repaid in four end-of-quarter instalments. Develop a complete amortization schedule and total the interest paid.

Solution

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$7,400
1	\$1,949.59	\$157.69	\$1,791.90	\$5,608.10
2	\$1,949.59	\$119.50	\$1,830.09	\$3,778.01
3	\$1,949.59	\$80.51	\$1,869.08	\$1,908.93
4	\$1,949.61	\$40.68	\$1,908.93	\$0
Totals	\$7,798.38	\$398.38	\$7,400	

5. Dr. Miller acquired a new centrifuge machine from Liaoyang Longda Pharmaceutical Machinery Company (LLPMC) for his medical practice. He is to pay off the **\$60,341** through **20** month-end payments. LLPMC has set the interest rate on the loan at **9.5%** compounded quarterly. Develop a partial amortization schedule for the third to sixth payments.

Solution

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
3	\$3,272.05	\$429.84	\$2,842.21	\$51,880.64
4	\$3,272.05	\$407.51	\$2,864.54	\$49,016.11
5	\$3,272.05	\$385.01	\$2,887.04	\$46,129.07
6	\$3,272.05	\$362.33	\$2,909.72	\$43,219.35

6. Kerry, who is a pharmacist, just became a new franchisee for Shoppers Drug Mart. As part of her franchising agreement, her operation is to assume a **\$1.2** million mortgage to be financed over the next **15** years. She is to make payments after every six months. Head office will charge her a rate of **14.25%** compounded annually. Determine the amount of her mortgage payment.

Solution

\$95,615.95

7. Alibaba took out a 25-year amortization **\$273,875** mortgage five years ago at **4.85%** compounded semi-annually and has been making monthly payments. He will renew the mortgage for a three-year term today at an interest rate of **6.1%** compounded semi-annually on the same amortization schedule. What are his new monthly mortgage payments?

Solution

\$1,735.84

8. Monthly payments are to be made against an **\$850,000** loan at **7.15%** compounded annually with a 15-year amortization.
- What is the size of the monthly payment?
 - Calculate the principal portion of the **100th** payment.
 - Calculate the interest portion of the **50th** payment.
 - Calculate how much the principal will be reduced in the second year.
 - Calculate the total interest paid in the fifth year.

Solution

a. **\$7,604.85**; b. **\$4,771.37**; c. **\$4,026.56**; d. **\$35,827.23**; e. **\$47,183.46**

9. An investment annuity of **\$100,000** earning **4.5%** compounded quarterly is to make payments at the end of every three months with a 10-year amortization.
- What is the size of the quarterly payment?
 - Calculate the principal portion of the **20th** payment.
 - Calculate the interest portion of the **33rd** payment.
 - Calculate how much the principal will be reduced in the second year.
 - Calculate the total interest paid in the seventh year.

Solution

- a. \$3, 118.35; b. \$2, 465.45; c. \$166.96; d. \$8, 480.07; e. \$1, 866.95
10. Four years ago, Katrina became a landlord and opened her new four-unit apartment housing unit with an initial mortgage at **6.83%** compounded semi-annually in the amount of **\$971, 000** less a **\$100, 000** down payment. She amortized over **30** years and opted for monthly payments. Upon renewing her mortgage today, she is taking a two-year term at **5.1%** compounded semi-annually while continuing with monthly payments and the original amortization timeline.
- a. Calculate the interest and principal portions in her first term.
 - b. What is the balance remaining after the first term?
 - c. What is the new mortgage payment amount in the second term?
 - d. What is the balance remaining after the second term?

Solution

- [illegible]

Solution

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
32	\$3,000	\$55.70	\$2,944.30	\$1,575.34
33	\$1,594.75	\$19.41	\$1,575.34	\$0
Totals	\$97,594.75	\$17,594.75	\$80,000	

- c. You purchased a \$400,000 house. You paid 10% as a down payment and took out a mortgage for the balance at 4.5% compounded semi-annually. You repaid the mortgage with monthly payments for 25 years. How much is the amortization period shortened by if a lump-sum payment of \$15,000 is made at the end of the fourth year?

Solution

1 year, 6 months

- d. A \$525,000 mortgage at 3.9% compounded semi-annually is repaid with monthly payments for 25 years. How much is the amortization period shortened by if the monthly payments are increased by \$200 at the end of year three?

Solution

2 years, 3 months

- e. A \$600,000 mortgage at 2.75% compounded semi-annually is repaid with monthly payments of \$2,445 for 30 years. How much is the amortization period shortened by if mortgage is repaid with semi-monthly payments of \$1,300 instead of the monthly payments?

Solution

2 years, 8 months

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CHAPTER 3 SUMMARY

Key Concepts Summary

- **3.1 Amortization Schedules**

- The concept of amortization.
- The development of an amortization schedule.
- The development of a partial amortization schedule.

- **3.2 Calculating the Principal and Interest Components of a Loan**

- How to calculate interest and principal components for a single payment.
- How to calculate interest and principal components for a series of payments.
- Variable changes in future value annuity calculations.

- **3.3 Calculating the Final Payment**

- Understanding why the final payment is different than all the other payments.
- How to calculate the exact amount of the final payment.

- **3.4 Mortgages**

- The language and concepts involved in mortgages.
- How to calculate the mortgage payment.
- The procedure involved in renewing a mortgage.
- The procedure involved with prepayment options.

Glossary of Terms

Amortization. A process by which the principal of a loan is extinguished over the course of an agreed-upon time period through a series of regular payments that go toward both the accruing interest and principal reduction.

Amortization Period. The length of time it will take for the principal of a loan to be reduced to zero.

Amortization Schedule. A table that shows the payment amount, principal component, interest component, and remaining balance for every payment in the annuity.

Amortization Term. The length of time for which the interest rate and payment agreement between the borrower and the lender will remain unchanged.

Closed Mortgage. A type of mortgage that has many strict rules and does not allow the mortgager to pay off the debt in full until the loan matures. Early payment incurs substantial penalties.

Foreclosure. A process that allows a financial institution to evict tenants from a mortgaged property and put the property up for sale. The proceeds of the sale are then used to pay off the mortgage.

Mortgage. A special type of loan that is collaterally secured by real estate property.

Mortgagee. The financial institution that lends the money for a mortgage.

Mortgagor. The individual or business that borrows the money for a mortgage.

Open Mortgage. A type of mortgage that has very few rules and allows the mortgagor to pay off the debt in full or with additional prepayments at any given point without penalty.

Partial Amortization Schedule. Amortization schedules that show only a specified range of payments and not the entire annuity.

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance immediately after a loan payment
- INT = Interest portion of a loan payment or a series of payments
- PRN = Principal portion of a loan payment or a series of payments
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- N = Total number of annuity payments

Calculator

- **Amortization Worksheet**
- **AMORT** is located on the 2nd shelf above the PV button.

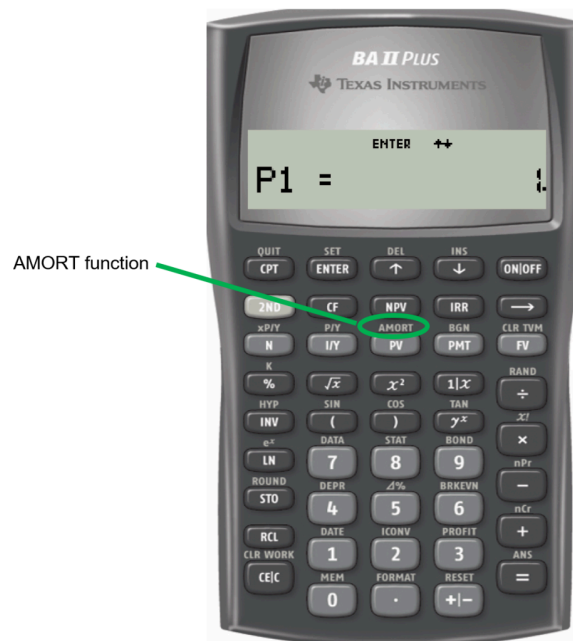


Figure 3.6.1

- There are five variables in the amortization worksheet:
 - $P1$ is the starting payment number. The calculator can work with a single payment or a series of payments.
 - $P2$ is the ending payment number. This number is the same as $P1$ when you are concerned with just a single payment. When working with a series of payments, you can set it to a higher number.
 - BAL is the principal balance remaining after the payment number entered into the $P2$ variable. The cash flow sign is correct as indicated on the calculator display.
 - PRN is the principal portion of the payments from $P1$ to $P2$ inclusive. Ignore the cash flow sign.
 - INT is the interest portion of the payments from $P1$ to $P2$ inclusive. Ignore the cash flow sign.
- To use the amortization worksheet:
 - Enter all seven of the time value of money variables accurately (N , I/Y , PV , PMT , FV , P/Y , and C/Y). If PMT was computed, you must re-enter it with only two decimals while retaining the correct cash flow sign convention.
 - Press **2nd AMORT**.
 - Enter a value for $P1$, press **ENTER** and then the down arrow.
 - Enter a value for $P2$, press **ENTER** and then the down arrow. Note that the higher the numbers entered in $P1$ or $P2$, the longer it will take the calculator to compute the outputs. It is possible that the calculator will go blank and take a few moments before displaying the

outputs.

- Press the down arrow to scroll through *BAL*, *PRN*, and *INT* to read the output.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=121#oembed-1>

Video: [Amortization Schedule using BAII Plus](#) by Joshua Emmanuel [3:55] *Transcript Available.*



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=121#oembed-2>

Video: [Mortgage Calculations using BAII Plus](#) by Joshua Emmanuel [4:13] *Transcript Available.*

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CHAPTER 4: BONDS AND SINKING FUNDS

Chapter Outline

[4.0 Introduction](#)

[4.1 Bond Terminology](#)

[4.2 Calculating the Purchase Price of a Bond on an Interest Payment Date](#)

[4.3 Calculating the Yield Rate](#)

[4.4 Bond Schedules](#)

[4.5 Quotation of Bonds](#)

[4.6 Sinking Funds](#)

[4.7 Review Exercises](#)

[Chapter 4 Summary](#)

4.0 INTRODUCTION

Chapter Learning Objectives

By the end of the chapter you should be able to:

- Define terminology used in bonds and bond transactions
- Calculate the purchase price of a bond on an interest payment date
- Calculate the yield to maturity of a bond on an interest payment date
- Calculate the investor's yield when a bond is sold before maturity
- Construct an amortization of the premium bond schedule
- Construct an accumulation of the discount bond schedule
- Calculate the purchase price of a bond on a non-interest payment date
- Calculate the quoted price of a bond
- Construct a sinking fund schedule

One of the most powerful ways for an institution to raise large amounts of capital is by issuing marketable bonds. A company or a government will essentially borrow the required financing from investors such as you and me. Individually we cannot provide the needed funds, but the bond issuer can raise a large sum of money from us collectively. One of the largest bond issuers in Canada is the Government of Canada. This money is primarily used to fund Canada's national debt.

On the consumer side, marketable bonds represent another investment opportunity you can include in your RRSP portfolio. They feature regular interest payments and a chance to grow your savings through trading on the bond market. This chapter begins with the basics of how bonds operate. Then you will learn how to calculate the purchase price of a bond and work with bond interest rates. Bond issuance usually comes with a sinking fund provision. Finally, we will explore the financial and accounting responsibilities associated with amortization of bond premiums and accrual of bond discounts.

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4.1 BOND TERMINOLOGY

Learning Objectives

- Define terminology used in bonds and bond transactions

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance
- BV = Book value
- FV = Face value of bond
- PMT = Annuity payment amount
- N = Number of annuity payments
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency

Formulas Used

- Formula 4.1 – **Bond Payment**

$$\text{Payment} = \text{Face Value} \times \text{periodic bond rate}$$

Introduction

A marketable **bond** is a debt that is secured by a specific corporate asset, that establishes the issuer's responsibility toward a creditor for paying interest at regular intervals, and for repaying the principal at a fixed later date. A **debenture** is the same as a marketable bond, except that the debt is not secured by any specific corporate asset. Mathematically, the calculations are identical for these two financial tools, which this textbook refers to as *bonds* for simplicity.

Bond Terminology

- **Issue Date.** The **bond issue date** is the date that the bond is issued and available for purchase by creditors. Interest accrues from this date.
- **Face Value.** Also called the *par value* or *denomination of the bond*, the **face value** is the principal amount of the debt. It is what the investor lent to the bond-issuing corporation. The amount, usually a multiple of \$100, is found in small denominations up to \$10, 000 for individual investors and larger denominations up to \$50, 000 or more for corporate investors.
- **Coupon Rate.** Also known as the *bond rate* or *nominal rate*, the **coupon rate** is the nominal interest rate paid on the face value of the bond. The coupon rate is fixed for the life of the bond. Most commonly the interest is calculated semi-annually and payable at the end of every six-month period over the entire life of the bond, starting from the issue date. All coupon rates used in this textbook are assumed to be semi-annually compounded, unless stated otherwise.
- **Yield Rate.** The **yield rate**, or *market rate*, is the prevailing nominal rate of interest in the open bond market. Because bonds are actively traded, this rate fluctuates based on economic and financial conditions. On the issue date, the market rate determines the coupon rate that is tied to the bond. Market rates are usually compounded semi-annually, as will be assumed in this textbook unless otherwise stated. Therefore, marketable bonds form ordinary simple annuities, because the interest payments and the market rate are both compounded semi-annually, and the payments occur at the end of the interval.

- **Redemption Value.** Also called the *redemption price* or *maturity value*, the **redemption value** is the amount the bond issuer will pay to the bondholder upon maturity of the bond. The redemption price normally equals the face value of the bond, in which case the bond is said to be “redeemable at par” because interest on the bond has already been paid in full periodically throughout the term, leaving only the principal in the account. In some instances a bond issuer may in fact redeem the bond at a premium, which is a price greater than the face value. The redemption price is then stated as a percentage of the face value, such as 103%. For introductory purposes, this textbook sticks to the most common situation, where the redemption price equals the face value.
- **Maturity Date.** Also known as the *redemption date* or *due date*, the **maturity date** is the day upon which the redemption price will be paid to the bondholder (along with the final interest payment), thereby extinguishing the debt.
- **Selling Date.** The date that a bond is actively traded and sold to another investor through the bond market is known as the **selling date**. In the timeline, the selling date can appear anywhere on the timeline between the issue date and maturity date, and it may occur more than once as the bond is sold by one investor after another.
- **Purchase Price.** The **purchase price** is the price the bond holder pays to purchase the bond. Although the redemption value and the periodic interest payments remain fixed throughout the lifetime of the bond, the purchase price fluctuates depending on various market conditions, such as the current yield rate.

Example 4.1.1

A \$10,000 bond was issued on March 1, 2019 with a 7% coupon and 16 years to maturity. The bond was purchased on July 13, 2023 when the yield to maturity was 5%.

- Face Value (or Redemption Value) = \$10,000
- Issue Date = March 1, 2019
- Purchase Date (or Selling Date) = July 23, 2023
- Maturity Date (or Redemption Date) = March 1, 2035
- Coupon Rate = 7% compounded semi-annually
- Yield Rate = 5% compounded semi-annually

Things to Watch Out For

Unless otherwise stated, the coupon rate and the yield rate are compounded semi-annually and the coupon payments are made every six months.

In this chapter we only deal with bonds that are redeemed for their face value at the time of maturity. So, the redemption value of the bond will equal its face value.

Premium, Discount, and At Par Bonds

The price at which a bond is purchased in the market may not be the face value of the bond. In fact, the price of a bond fluctuates with the market rate over time. There are three scenarios relating the purchase price of the bond to its face value.

- ***Bonds purchased at par.*** The purchase price of the bond *equals* its face value. This happens when the coupon rate equals the yield rate.
- ***Bonds purchased at premium.*** The purchase price of the bond is *higher* than its face value. This happens when the coupon rate is greater than the yield rate. In such cases, the bond is providing a higher rate of return (through the coupons) than an investment in the market (earning the lower yield rate). Consequently, the bond will be in demand and will sell at a higher price than its face value. The difference between the purchase price and the face value is called the *premium*.
- ***Bonds purchased at discount.*** The purchase price of the bond is *lower* than its face value. This happens when the coupon rate is less than the yield rate. In such cases, the bond is providing a lower rate of return (through the coupons) than an investment in the market (earning the higher yield rate). Consequently, the bond will not be in demand and will sell at a lower price than its face value. The difference between the face value and the purchase price is called the *discount*.

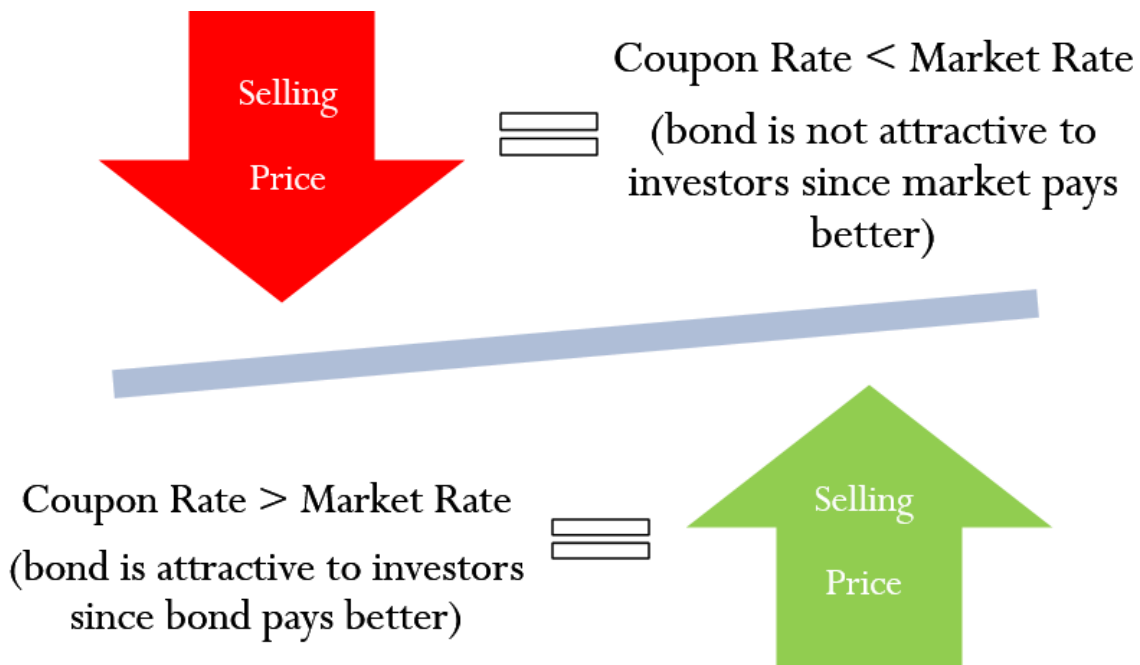


Figure 4.1.1

Image Description

The image illustrates the relationship between coupon rates, market rates, and selling prices of bonds using arrows and text to explain the concept.

At the top, there is a large red downward-pointing arrow with the text "Selling Price" written inside it. To the right of this arrow, there is text that reads "Coupon Rate < Market Rate (bond is not attractive to investors since market pays better)". An equals sign is displayed between the red arrow and the text.

Below this, a tilted blue bar is shown representing a balance or scale.

At the bottom, there is a large green upward-pointing arrow with the text "Selling Price" written inside it. To the left of this arrow, there is text that reads "Coupon Rate > Market Rate (bond is attractive to investors since bond pays better)". An equals sign is displayed between the green arrow and the text.

Example 4.1.2

Identify each of the following bonds as premium, discount, or at par.

- a. A \$5,000 bond with a 3% coupon is purchased when there are five years to maturity and the yield to maturity is 4.5%.
- b. A \$12,000 bond with a 4% coupon is purchased when there are ten years to maturity and the yield to maturity is 4%.
- c. A \$7,000 bond with a 6% coupon is purchased when there are twelve years to maturity and the yield to maturity is 3.7%.

Solution

- a. Discount because the coupon rate (3%) is less than the yield rate (4.5%)
- b. At par because the coupon rate and yield rate are equal.
- c. Premium because the coupon rate (6%) is higher than the yield rate (3.7%).

Calculating the Bond Payment

The **bond payment**, or coupon payment, is the payment the bond holder received semi-annually throughout the investment. This interest is not converted to the principal of the bond, and therefore does not compound. Instead, the bond payment is directly paid to the bond holder. The amount of the bond payment depends only on the face value of the bond and the coupon rate.

4.1 Bond Payment

$$\text{Bond Payment} = FV \times b$$

- **Bond Payment:** is the payment the bond holder received semi-annually throughout the investment.
- **FV** is the Face Value: of the bond when the bond is redeemed at maturity.
- **b** is Periodic Coupon Rate: where $b = \frac{\text{coupon rate}}{\text{number of coupons per year}}$

Example 4.1.3

A \$5,000 bond has a 3% coupon. Calculate the bond payment.

Solution

$$\begin{aligned} \text{Bond Payment} &= FV \times b \\ &= 5,000 \times \frac{0.03}{2} \\ &= \$75 \end{aligned}$$



Key Takeaways

Remember, the bond payments and coupon rate are assumed to be semi-annual, unless stated otherwise.

For the bond in the previous example, the bond holder will receive a **\$75** payment at the end of every six months over the life time of the bond.

Try It

1) A **\$12,000** bond has a **7%** coupon. Calculate the bond payment.

Solution

$$\begin{aligned}\text{Bond Payment} &= FV \times b \\ &= 12,000 \times \frac{0.07}{2} \\ &= \$420\end{aligned}$$

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4.2 CALCULATING THE PURCHASE PRICE OF A BOND ON AN INTEREST PAYMENT DATE

Learning Objectives

- Calculate the purchase price of a bond on an interest payment date

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance
- BV = Book value
- FV = Face value of bond
- PMT = Annuity payment amount
- N = Number of annuity payments
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency

Formulas Used

- **Formula 4.1 – Bond Payment**

$$\text{Payment} = \text{Face Value} \times \text{periodic bond rate}$$

- **Formula 4.2 – Purchase Price on an Interest Payment Date**

$$\text{Purchase Price} = \text{Present Value of Face Value} + \text{Present Value of Bond Payments}$$

- **Formula 4.3 – Premium (Bonds)**

$$\text{Premium} = \text{Purchase Price} - \text{Face Value}$$

- **Formula 4.4 – Discount (Bonds)**

$$\text{Discount} = \text{Face Value} - \text{Purchase Price}$$

Introduction

Exactly what is your marketable bond worth? When you inspect the financial section of your local newspaper, you pay particular attention to the quotes on bonds, into which you invested a portion of your RRSP portfolio. Ten years ago, when prevailing bond rates were 5.5%, you purchased 10 Government of Canada \$5,000 face value bonds with a 5% coupon and 20 years remaining to maturity. At the time you paid \$4,699.02 each for them. Today's prevailing bond rates have dropped to 3.35% but are expected to rise in the near future. Should you hold on to those bonds? Should you cash them in by selling them in the bond market?

Marketable bonds and debentures are nonredeemable, which means the only way to cash these bonds in before the maturity date is to sell them to another investor. Therefore, the key mathematical calculation is what to pay for the bond. The selling date, maturity date, coupon rate, redemption price, and market rate together determine the bond's purchase price.

On the bond's issue date, the market rate determines the coupon rate, so these two rates are identical. As a result, the price of the bond equals its face value. After the bond is issued, interest starts to accrue on it, and the market rate begins to fluctuate based on market conditions. This changes the price of the bond.

Because the bond pays interest semi-annually, two days of the year are defined as the interest payment dates. For

now, we are interested in finding out the purchase price of a bond if the bond is purchased on one of these two interest payment dates. Because the bond sells on its interest payment date, the most recent interest payment date is the selling date.

When you purchase a bond, you are buying two promises: to be paid the face value of the bond at maturity and to be paid the periodic bond payments. The purchase price takes both of these promises into account. The purchase price of the bond on an interest payment date is the sum of the present value of all of the remaining payments on the bond on this date and the present value of the face value of the bond on this date, using the yield rate to calculate the present values.

4.2 Purchase Price on an Interest Payment Date

$$\text{Purchase Price} = PV \text{ of Payments} + PV \text{ of Face Value}$$

Purchase Price: is the calculated purchase price on the interest payment date.

PV of Payments: is the sum of the present value of the remaining payments.

PV of Face Value: is the present value of the bond's face value as of the payment date.

Key Takeaways



The bond payment issued on the date of purchase (which in this case is an interest payment date) goes to the seller of the bond, and not the purchaser. The purchaser only starts to receive payments on the first interest payment date after the purchase of the bond.

The price the seller sells the bond for equals the price the purchaser pays for the bond.

Similar to T-bills and commercial papers, the time for the purchase price calculation is based on the remaining time to maturity (i.e. the time from the interest payment date to the maturity date) at the time of purchase.

Recall from the previous section that a bond can be bought at a premium, discount, or a par.

- *Bonds purchased at par.* The purchase price of the bond *equals* its face value. This happens when the coupon rate is the same as the yield rate.
- *Bonds purchased at premium.* The purchase price of the bond is *higher* than its face value. The difference between the purchase price and the face value is called the **premium**.

4.3 Premium (Bonds)

$$\text{Premium} = \text{Purchase Price} - \text{Face Value}$$

- *Bonds purchased at discount.* The purchase price of the bond is **lower** than its face value. The difference between the face value and the purchase price is called the **discount**.

4.4 Discount (Bonds)

$$\text{Discount} = \text{Face Value} - \text{Purchase Price}$$

You can use the financial calculator to find the purchase price of the bond. After all, the purchase price is just the sum of two present value calculations. You can calculate each present value separately and then add the results together. Alternatively, the calculator can do both calculations simultaneously and output the purchase price.

Using the TI BAII Plus Calculator to Find the Purchase Price of a Bond on an Interest Payment Date

- Set the calculator to the **END** payment setting.
- Enter values for the known variables (FV , PMT , N , I/Y , P/Y and C/Y), paying close attention to the cash flow sign convention for PMT and FV .
 - Because you are purchasing the bond, you will become the bond holder and receive the payments and the face value. So PMT (the bond payment) and FV (the face value) are both entered with a positive cash flow sign.
- After all of the known quantities are loaded into the calculator, press CPT and then PV to solve for the purchase price.

Things to Watch Out For

There are two interest rates associated with a bond: the yield rate and the coupon rate. They are NOT interchangeable.

- The coupon rate is only used to calculate the bond payment. The coupon rate is NEVER used as the value for I/Y .
- The yield rate is the prevailing interest rate in the market. In the purchase price calculation, the yield rate is the interest rate entered for I/Y .

Example 4.2.1

A \$50,000 bond with a 10.15% coupon is purchased when there are 10.5 years remaining to maturity. If the yield to maturity at the time of purchase is 4.31%, what is the purchase price? What is the amount of premium or discount?

Solution

Step 1: The given information is

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual.

$$FV = \$50,000$$

$$P/Y = 2$$

$$I/Y = 4.31\%$$

$$C/Y = 2$$

$$t = 10.5 \text{ years}$$

$$\text{Coupon Rate} = 10.15\%$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 50,000 \times \frac{0.1015}{2} \\ &= \$2,537.50 \end{aligned}$$

Step 3: Calculate the purchase price.

PMT Setting	END
N	$2 \times 10.5 = 21$
PV	?
FV	50,000
PMT	2,537.50
I/Y	4.31
P/Y	2
C/Y	2

$$PV = \$74,452.86$$

Step 4: Calculate the premium or discount. Because the purchase price is greater than the face value, this is a premium bond.

$$\begin{aligned}
 \text{Premium} &= \text{Purchase Price} - \text{Face Value} \\
 &= 74,452.86 - 50,000 \\
 &= \$24,452.86
 \end{aligned}$$

Step 5: Write as a statement.

The purchase price of the bond is **\$74,452.86** and the premium is **\$24,452.86**.

Example 4.2.2

A **\$25,000** bond with a **8.92%** coupon is purchased when there are **22.5** years remaining to maturity. If the yield to maturity at the time of purchase is **9.46%**, what is the purchase price? What is the amount of premium or discount?

Solution

Step 1: The given information is

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual.

$$FV = \$25,000$$

$$P/Y = 2$$

$$I/Y = 9.46\%$$

$$C/Y = 2$$

$$t = 22.5 \text{ years}$$

$$\text{Coupon Rate} = 8.92\%$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 25,000 \times \frac{0.0892}{2} \\ &= \$1,115 \end{aligned}$$

Step 3: Calculate the purchase price.

PMT Setting	END
N	$2 \times 22.5 = 45$
PV	?
FV	25,000
PMT	1,115
I/Y	9.46
P/Y	2
C/Y	2

$$PV = \$23,751.28$$

Step 4: Calculate the premium or discount.

Because the purchase price is less than the face value, this is a discount bond.

$$\begin{aligned}
 \text{Discount} &= \text{Face Value} - \text{Purchase Price} \\
 &= 25,000 - 23,751.28 \\
 &= \$1,248.72
 \end{aligned}$$

Step 5: Write as a statement.

The purchase price of the bond is **\$23,751.28** and the discount is **\$1,248.72**.

Try It

1) A **\$7,000** bond with a coupon rate of **5%** is redeemable in ten years. Calculate the purchase price and the premium or discount if the yield rate is **7%**.

Solution

PMT Setting	END
<i>N</i>	$2 \times 10 = 20$
<i>PV</i>	?
<i>FV</i>	7,000
<i>PMT</i>	175
<i>I/Y</i>	7
<i>P/Y</i>	2
<i>C/Y</i>	2

$$PV = \$6,005.13$$

$$\begin{aligned}
 \text{Discount} &= 7,000 - 6,005.13 \\
 &= \$994.87
 \end{aligned}$$

Example 4.2.3

An investor purchased a \$10,000 bond with a coupon rate of 6.5% when there was 15 years to maturity and the yield rate was 4%. Four years later, the investor decided to sell the bond when the yield to maturity was 5.2%.

1. At what price did the investor purchase the bond?
2. At what price did the investor sell the bond?
3. What was the investor's gain or loss on the sale of the bond?

Solution

Step 1: Calculate the investor's purchase price.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 10,000 \times \frac{0.065}{2} \\
 &= \$325
 \end{aligned}$$

PMT Setting	END
N	$2 \times 15 = 30$
PV	?
FV	10,000
PMT	325
I/Y	4
P/Y	2
C/Y	2

$$PV = \$12,799.56$$

The investor paid \$12,799.56 for the bond.

Step 2: Calculate the selling price of the bond.

The selling price of the bond for the investor equals the purchase price of the bond for the person the investor is selling the bond to. The purchase price is based on the time to maturity and the yield rate at the time the bond is sold. The investor purchased the bond when there was 15 years to maturity and then sold the bond four years later. So at the time of the sale there was 11 (15 – 4) years to maturity.

PMT Setting	END
N	$2 \times 11 = 22$
PV	?
FV	10,000
PMT	325
I/Y	5.2
P/Y	2
C/Y	2

$$PV = \$11,078.66$$

The investor sold the bond for \$11,078.66.

Step 3: Calculate the gain or loss on the sale.

Because the investor sold the bond for less than they paid for it, the investor realized a loss on the sale.

$$\begin{aligned}\text{Loss} &= 12,799.56 - 11,078.66 \\ &= \$1,720.90\end{aligned}$$

Step 4: Write as a statement.

The investor's lost \$1,720.90 on the sale of the bond.

Try It

2) You purchased a **\$3,000** bond paying a **2.5%** coupon rate when there was eight years to maturity and the yield rate was **5.7%**. After two years, you decided to sell the bond when the yield to maturity was **4.9%**.

- What price did you pay for the bond?
- What price did you sell the bond for?
- What was your gain or loss on the sale of the bond?

Solution

- Calculate the purchase price of the bond.

PMT Setting	END
<i>N</i>	$2 \times 8 = 16$
<i>PV</i>	?
<i>FV</i>	3,000
<i>PMT</i>	37.50
<i>I/Y</i>	5.7
<i>P/Y</i>	2
<i>C/Y</i>	2

$$PV = \$2,390.09$$

- Calculate the selling price of the bond.

PMT Setting	END
N	$2 \times 6 = 12$
PV	?
FV	3,000
PMT	37.50
I/Y	4.9
P/Y	2
C/Y	2

$$PV = \$2,629.60$$

c. Calculate the gain on the sale of the bond.

$$\begin{aligned}\text{Gain} &= 2,629.60 - 2,390.09 \\ &= \$239.51\end{aligned}$$

Section 4.2 Exercises

1. A \$1,000 bond with a 7.16% coupon is redeemable in 27 years.
 - a. Calculate the purchase price if the yield to maturity is 6.3%.
 - b. Calculate the premium or discount.

Solution

- a. **\$1,110.93**; b. Premium=**\$110.93**
2. A **\$5,000** bond with a **7.64%** coupon is redeemable in **6.5** years.
- a. Calculate the purchase price if the yield to maturity is **3.96%**.
- b. Calculate the premium or discount.

Solution

- a. **\$6,045.43**; b. Premium=**\$1,045.53**
3. With **17** years until maturity, Julio purchased an **\$8,000** Government of Saskatchewan bond with a coupon rate of **5.55%**. The market yield was **8.65%**.
- a. Calculate the purchase price of the bond.
- b. Calculate the premium or discount.

Solution

- a. **\$5,777.23**; b. Discount=**\$2,222.77**
4. A **\$5,000** with **14** years to maturity has a coupon rate of **4.42%** and a yield of **4.07%**.
- a. Calculate the purchase price of the bond.
- b. Calculate the premium or discount.

Solution

- a. **\$5,185.37**; b. Premium=**\$185.37**
5. A **\$25,000** bond has a coupon rate of **5.12%**. James purchased the bond when there was seven years to maturity and the yield rate was **4.18%**. Five years later, James sold the bond when the yield to maturity was **3.87%**.
- a. Calculate the price James paid for the bond.
- b. Calculate the price James sold the bond for.

- c. Calculate James' gain or loss on the sale of the bond.

Solution

a. \$26,413.52; b. \$25,595.90; c. Loss=\$817.62

6. William purchased a \$18,000 bond with a coupon rate of 11.25%, twelve years to maturity and a yield rate of 8.3%. After holding the bond for seven years, William sold the bond when the yield rate was 5.7%.

- At what price did William purchase the bond?
- At what price did William sell the bond?
- What was William's gain or loss on the sale of the bond?

Solution

a. \$21,986.61; b. \$22,293.64; c. Gain=\$307.03

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4.3 CALCULATING THE YIELD RATE

Learning Objectives

- Calculate the yield to maturity of a bond on an interest payment date
- Calculate the investor's yield when a bond is sold before maturity

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance
- BV = Book value
- FV = Face value of bond
- PMT = Annuity payment amount
- N = Number of annuity payments
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency

Formulas Used

- Formula 4.1 – **Bond Payment**

$$\text{Payment} = \text{Face Value} \times \text{periodic bond rate}$$

- Formula 4.2 – **Purchase Price on an Interest Payment Date**

$$\text{Purchase Price} = \text{Present Value of Face Value} + \text{Present Value of Bond Payments}$$

- Formula 4.3 – **Premium (Bonds)**

$$\text{Premium} = \text{Purchase Price} - \text{Face Value}$$

- Formula 4.4 – **Discount (Bonds)**

$$\text{Discount} = \text{Face Value} - \text{Purchase Price}$$

Introduction

The **yield to maturity** is a bond's overall rate of return when purchased at a market price and held until maturity. It includes both the semi-annual interest that the bondholders earn on their investment along with the gain or loss resulting from the difference between the market price on the selling date and the redemption price. This yield to maturity is exactly equal to the market rate of return on the date of purchase. For example, if you purchase a bond today with a yield rate of 5.5% and you hold onto the bond until maturity, you will realize a yield to maturity of 5.5%.

You can use the financial calculator to find the yield rate for a bond. To calculate the yield rate, you need to know the purchase price of the bond, along with the other information about the bond.

Using the TI BAII Plus Calculator to Find the Yield Rate for a Bond

- Set the calculator to the **END** payment setting.
- Enter values for the known variables (PV , FV , PMT , N , P/Y and C/Y), paying close attention to the cash flow sign convention for PV , PMT and FV .
 - Because you are paying out the purchase price, PV (the purchase price) is entered with a negative cash flow sign.
 - Because you are the bond holder, you receive the payments and the face value. So PMT (the bond payment) and FV (the face value) are both entered with a positive cash flow sign.
- After all of the known quantities are loaded into the calculator, press CPT and then I/Y to solve for the yield rate.

Key Takeaways



The calculated yield rate is the market rate of return that the bond holder earns if the bond is purchased at the given purchase price and held until maturity. It represents the rate of return the bond holder will receive on their investment, provided they hold the bond until maturity. If the bond holder decides to sell the bond at any point before the maturity date, they will realize a different rate of return on the bond.

To calculate a yield rate, the purchase date must be on an interest payment date.

Example 4.3.1

Suppose a \$10,000 face value bond is purchased for \$7,688.52 with 20 years until maturity and has a coupon rate of 4% semi-annually. What yield will the bondholder realize if she holds onto it until maturity?

Solution

Step 1: The given information is

Because no other information is given, the frequency of the payments and the compounding frequency for the yield rate are assumed to be semi-annual.

$$PV = \$7,688.52$$

$$FV = \$10,000$$

$$P/Y = 2$$

$$C/Y = 2$$

$$t = 20 \text{ years}$$

$$\text{Coupon Rate} = 4\%$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 10,000 \times \frac{0.04}{2} \\ &= \$200 \end{aligned}$$

Step 3: Calculate the yield rate.

PMT Setting	END
N	$2 \times 20 = 40$
PV	$-7,688.52$
FV	10,000
PMT	200
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 6\%$$

Step 4: Write as a statement.

The yield rate is 6% compounded semi-annually.

In order to calculate the yield rate, you need to know the purchase price of the bond. As in the above example, you could be told the purchase price directly. An alternative way to be given information about the purchase price is through a market quotation.

The **market quotation** of a bond is the purchase price expressed as a percentage of the face value. For example, if a bond has a market quotation of 97.5, it means that the bond's purchase price is 97.5% of its face value. If the face value of the bond is \$5,000, the purchase price is $0.975 \times 5,000 = \$4,875$.

$$\text{Purchase Price} = \text{Face Value} \times \text{Market Quotation}$$

Things to Watch Out For

Although market quotations are percentages, they generally do not include a percent sign when written. It is common practice to see phrases like “The bond is purchased at 102.3.” or “The

bond has a market quotation of **102.3**.” This does NOT mean that the bond is purchased for **\$102.30**. In this type of phrasing, it is understood that the “**102.3**” is a percent and that the purchase price of the bond is **102.3%** of the face value of the bond.

Example 4.3.2

A \$5,000 bond has a 7% coupon. The bond is purchased at **103.7** when there are **10** years to maturity. What is the yield rate on the bond?

Solution

Step 1: The given information is

Because no other information is given, the frequency of the payments and the compounding frequencies are assumed to be semi-annual.

$$FV = \$5,000$$

$$P/Y = 2$$

$$C/Y = 2$$

$$t = 10 \text{ years}$$

$$\text{Coupon Rate} = 7\%$$

$$\text{Market Quotation} = 103.7\%$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 5,000 \times \frac{0.07}{2} \\ &= \$175 \end{aligned}$$

Step 3: Calculate the purchase price (PV).

$$\begin{aligned}
 PV &= FV \times \text{market quotation} \\
 &= 5,000 \times 1.037 \\
 &= \$5,185
 \end{aligned}$$

Step 4: Calculate the yield rate.

PMT Setting	END
N	$2 \times 10 = 20$
PV	$-5,185$
FV	$5,000$
PMT	175
I/Y	$?$
P/Y	2
C/Y	2

$$I/Y = 6.49\%$$

Step 5: Write as a statement.

The yield rate is **6.49%** compounded semi-annually.

Try It

1) A **\$13,000** bond with a coupon rate of **4.5%** is redeemable in **15** years. The bond is purchased today at **95.4**. Calculate the yield rate on the bond.

Solution

PMT Setting	END
N	$2 \times 15 = 30$
PV	$-12,402$
FV	13000
PMT	292.50
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 4.94\%$$

The Investor's Yield

The yield-to-maturity calculation requires that the bond is held until its maturity date, at which point the future value redemption price is known and equal to its face value. However, this condition may not hold. The bond holder may sell the bond at any point before the maturity date. If the bond is sold, the future value is based on the prevailing bond rate at that time, and the price is generally not equal to its face value. What then is the bond's yield for the investor? To find the investor's yield, calculate the rate of return (I/Y) with the present value set to the purchase price, the future value set to the selling price, the payment set to the bond payment, and time based on the length of time the investor held the bond.

You can use the financial calculator to find the investor's yield. To calculate the investor's yield, you need to know the purchase price of the bond, the selling price of the bond, and the other information about the bond.

Using the TI BAII Plus Calculator to Find the Investor's Yield

- Set the calculator to the **END** payment setting.
- Enter values for the known variables (PV , FV , PMT , N , P/Y and C/Y), paying close attention to the cash flow sign convention for PV , PMT and FV .
 - Because you paid out the purchase price, PV (the purchase price) is entered with a negative cash flow sign.
 - Because you received the bond payments as the bond holder, PMT (the bond payment) is entered with a positive cash flow sign.
 - Because you are selling the bond, you will receive the selling price. So FV (the selling price) is entered with a positive cash flow sign.
- After all of the known quantities are loaded into the calculator, press CPT and then I/Y to solve for the investor's yield.

Key Takeaways



1. To calculate the investor's yield, the purchase date and the selling date must be on an interest payment dates.
2. For an investor's yield calculation, the time is based on the length of time between the purchase date and the selling date. The time to maturity is not involved in an investor's yield

calculation.

3. In an investor's yield calculation, the present value is the purchase price of the bond and the future value is the selling price of the bond. The face value is not involved in an investor's yield calculation (except to find the bond payment).

Example 4.3.3

A \$1,000 face value bond with a 7% coupon and 12 years to maturity was purchased for \$1,084.68. It sold seven years later for \$920.87. What rate of return did the bond holder realize on the sale?

Solution

Step 1: The given information, related to the sale, is

Because no other information is given, the frequency of the payments and the compounding frequencies are assumed to be semi-annual.

$$PV = \$1,084.68$$

$$FV = \$920.87$$

$$P/Y = 2$$

$$C/Y = 2$$

$$t = 7 \text{ years}$$

$$\text{Coupon Rate} = 7\%$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 1,000 \times \frac{0.07}{2} \\ &= \$35 \end{aligned}$$

Step 3: Calculate the bond holder's rate of return on the sale.

PMT Setting	END
N	$2 \times 7 = 14$
PV	$-1,084.68$
FV	920.87
PMT	35
I/Y	$?$
P/Y	2
C/Y	2

$$I/Y = 4.6\%$$

Step 4: Write as a statement.

The bond holder's rate of return on the sale is **4.6%** compounded semi-annually.

Example 4.3.4

Sara purchased a **\$25,000** bond paying **2.8%** when the yield rate was **3.7%** and there was **10** years to maturity. After three years, Sara decides to sell the bond when the yield rate was **4.5%**.

- What price did Sara pay to purchase the bond?
- At what price did Sara sell the bond?
- What rate of return did Sara realize on sale of the bond?

Solution

Step 1: Calculate the Sara's purchase price.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 25,000 \times \frac{0.028}{2} \\
 &= \$350
 \end{aligned}$$

PMT Setting	END
N	$2 \times 10 = 20$
PV	?
FV	25,000
PMT	350
I/Y	3.7
P/Y	2
C/Y	2

$$PV = \$23,133.56$$

Sara paid **\$23,133.56** for the bond.

Step 2: Calculate the selling price of the bond.

The selling price of the bond for the investor equals the purchase price of the bond for the person the investor is selling the bond to. The purchase price is based on the time to maturity and the yield rate at the time the bond is sold. The investor purchased the bond when there was **10** years to maturity and then sold the bond three years later. So at the time of the sale there was **7 (10 – 3)** years to maturity.

PMT Setting	END
N	$2 \times 7 = 14$
PV	?
FV	25,000
PMT	350
I/Y	4.5
P/Y	2
C/Y	2

$$PV = \$22,472.11$$

Sara sold the bond for \$22,472.11.

Step 3: Calculate Sara's rate of return on the sale.

PMT Setting	END
N	$2 \times 3 = 6$
PV	-23,133.56
FV	22,472.11
PMT	350
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 2.10\%$$

Step 4: Write as a statement.

Sara's rate of return on the sale is 2.10% compounded semi-annually.

Try It

2) John purchased a **\$12,000** bond with a **4.7%** coupon for **\$13,750.09** when there was **17** years to maturity and the yield rate was **3.55%**. After **11** years, John sold the bond when the yield rate was **5.3%**.

- At what price did John sell the bond?
- What rate of return did John realize on the sale of the bond?

Solution

- Calculate the sale price of the bond.

PMT Setting	END
N	$2 \times 6 = 12$
PV	?
FV	12,000
PMT	282
I/Y	5.3
P/Y	2
C/Y	2

$$PV = \$11,634.05$$

- Calculate the rate of return on the sale.

PMT Setting	END
N	$2 \times 11 = 22$
PV	$-13,750.09$
FV	$11,634.05$
PMT	282
I/Y	$?$
P/Y	2
C/Y	2

$$I/Y = 2.9\%$$

Section 4.3 Exercises

1. A \$1,000 bond with a 6% coupon is purchased for \$1,546.16 when there was 25 year to maturity. Calculate the yield to maturity.

Solution

2.91%

2. A \$5,000 bond paying 5.75% is purchased for 76.4822 when there was 18.5 years to maturity. Calculate the yield to maturity.

Solution

8.25%

3. A \$25,000 bond has a 7.25% coupon rate. The bond was purchased for \$23,123.77 and sold six years later for \$21,440.13. Calculate the investor's rate of return on the sale price.

Solution

6.8363%

4. A \$100,000 bond with a 8.65% coupon rate is purchased for \$106,590. The bond is sold 11.5 years later for \$109,687.75. Calculate the investor's rate of return on the sale price.

Solution

8.271%

5. A \$500,000 bond has a 9% coupon rate and is redeemable in 15 years. The bond is purchased at 105.72. Calculate the yield rate.

Solution

8.33%

6. A \$495,000 face value Province of Ontario bond issued with a coupon rate of 7.5% is sold for \$714,557.14. If 20 years remain until maturity, what was the yield to maturity?

Solution

4.2%

7. A \$100,000 face value bond is issued with a 5% coupon and 22 years until maturity. If it sells for \$76,566.889.5 years later, what was the yield to maturity?

Solution

8%

8. A \$50,000 face value Government of Canada bond is purchased for \$48,336.48 and sold nine years later for \$51,223.23. If the coupon rate is 4.985%, calculate the investor's yield.

Solution

5.6742%

9. Great-West Life (GWL) issued a \$100,000 face value bond carrying a coupon rate of

6.14% and **20** years to maturity. Eight years later, GWL decides to buy back some of its outstanding bonds when current market rates are **7.29%**. If the bond was held from the issue date until the selling date, what yield did the bondholder realize on her investment?

Solution

5.2091%

10. A **\$50,000** face value bond with a **5.1%** coupon is purchase **8.5** years before maturity for **\$53,336.24**. After **1.5** years, the bond is sold for **\$53,109.69**.
- Calculate the yield to maturity on the purchase date.
 - Calculate the yield to maturity on the selling date.
 - Calculate the investor's yield.

Solution

a. **4.16%**; b. **4.07%**; c. **4.5041%**

11. A **\$75,000** bond carrying a **4.08%** coupon is purchased **19.5** years before maturity when the yield rate was **4.2%**. After **3.5** years, the bond is sold when yield rate is **4.05%**.
- Calculate the purchase price of the bond along with the amount of the premium or discount.
 - Calculate the selling price of the bond along with the amount of the premium or discount.
 - Calculate the rate of return when the bond was sold.

Solution

a. Purchase Price=**\$73,809.92**, Discount=**\$1,190.08**; b. Purchase Price=**\$75,263.06**, Premium=**\$263.06**; c. **4.6701%**

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4.4 BOND SCHEDULES

Learning Objectives

- Construct an amortization of the premium bond schedule
- Construct an accumulation of the discount bond schedule

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance
- BV = Book value
- FV = Face value of bond
- PMT = Annuity payment amount
- N = Number of annuity payments
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency

Formulas Used

- Formula 4.1 – **Bond Payment**

$$\text{Payment} = \text{Face Value} \times \text{periodic bond rate}$$

- Formula 4.2 – **Purchase Price on an Interest Payment Date**

$$\text{Purchase Price} = \text{Present Value of Face Value} + \text{Present Value of Bond Payments}$$

- Formula 4.3 – **Premium (Bonds)**

$$\text{Premium} = \text{Purchase Price} - \text{Face Value}$$

- Formula 4.4 – **Discount (Bonds)**

$$\text{Discount} = \text{Face Value} - \text{Purchase Price}$$

Introduction

Recall that the purchase price of a bond does not necessarily equal its face value. The bond may be purchased for more than its face value, resulting in a premium, or the bond may be purchased for less than its face value, resulting in a discount. When an investor purchases a bond, the investor receives the bond payments as well as the bond's face value at maturity. What are the tax implications associated with owning a bond? For example, the bond payments are taxable income for the investor. But, what are the tax implications of the premium or discount that is realized when the bond is redeemed at maturity?

The **book value of a bond** is the face value of the bond plus or minus any unamortized premium or discount. The initial book value of a bond is its purchase price. A common practice to deal with the tax implications of the premium or discount is to make adjustments to the book value of the bond over the holding period of the bond. The adjustments to the book value involve amortizing small parts of the premium or discount during each payment interval until the book value of the bond equals the bond's face value.

Amortization of the Premium

When a bond is purchased for more than its face value, the result is a premium. For example, suppose an investor pays \$1,050 for a \$1,000 face value bond. If the investor holds onto the bond until maturity,

only the redemption price of \$1,000 is returned. The bond premium of \$50 represents a capital loss for the investor because the premium is not recovered at maturity. A **capital loss** is the amount by which the current value of an asset falls short of the original purchase price. What does the investor do with this capital loss? One option is for the investor to apply the capital loss against their income at the time of redemption, which results in a lower taxable amount at the time of redemption. But, commonly accepted practices allow the investor to amortize the capital loss over the period of time that the bond is held, and not just in the period during which the capital loss actually occurs (at maturity). This allows the investor to spread the capital loss over each payment interval, claiming a little bit of the capital loss against each bond payment in order to lower the taxable amount for each payment interval. This process is called the **amortization of the premium**.

The amortization of the premium schedule records the bond payment, the amount of the premium that is claimed with each payment, the book value, and the remaining premium to be amortized. The amortization of the premium schedule shows how much premium is claimed against each payment and the resulting book value.

An amortization of the premium schedule has six columns:

- **Payment Number** — There is a row for every bond payment during the holding period of the bond.
- **Bond Payment** — The periodic bond payment (PMT). All of the payments are the same, including the last payment.
- **Interest on Book Value at Yield** — For each row, the interest on book value entry is the interest portion of the current bond payment based on the book value from the previous row.
- **Amortized Premium** — For each row, the amortized premium entry is the amount of the premium that is claimed as a capital loss in that payment interval.
- **Book Value** — For each row, the book value is the face value of the bond plus the remaining unamortized premium.
- **Remaining Premium to be Amortized** — For each row, the remaining premium is the amount of premium that is left to claim against the payments.

To fill in an amortization of the premium schedule, you first need to have all of the details about the bond, including the face value (FV), the purchase price (PV), the bond payment (PMT), the number of payments (N), and the yield rate. If any of these quantities are missing, calculate out the missing value before completing the amortization of the premium schedule.

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				Purchase Price ¹	Premium ¹
1	PMT^2	$INT_{\text{on } BV}^3$	AP^4	BV^5	RP^6
2	PMT^2	$INT_{\text{on } BV}^3$	AP^4	BV^5	RP^6
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$N - 1$	PMT^2	$INT_{\text{on } BV}^3$	AP^4	BV^5	RP^6
N	PMT^2	$INT_{\text{on } BV}^3$	AP^4	Face Value ⁸	0 ⁹
Totals	Total Payments ¹⁰	Total Interest ¹²	Total Amortized Premium ¹¹		D

HOW TO

Fill In the Amortization of the Premium Schedule

Follow these steps to fill in the amortization of the premium schedule.

Step 1: In row 0, the only entries are in the book value and remaining premium to be amortized columns. The initial book value is the purchase price of the bond and the initial remaining premium is the premium.

Step 2: Each entry in the bond payment column is the bond payment. All of the payments in this column are the same, including the last payment.

Step 3: Calculate the interest on the book value at yield. The interest is the book value from the previous row times the periodic yield rate:

$$\text{Interest on Book Value} = \text{Book Value from Previous Row} \times i$$

Note: this calculation uses the periodic yield rate, not the periodic bond rate.

Step 4: Calculate the amortized premium. The amortized premium is the difference between the payment and the interest on the book value:

$$\text{Amortized Premium} = PMT - \text{Interest on Book Value}.$$

Step 5: Calculate the new book value. The book value is the difference between the book value in the previous row and the amortized premium:

$$\text{Book Value} = \text{Book Value from Previous Row} - \text{Amortized Premium}$$

Step 6: Calculate the remaining premium to be amortized. The remaining premium is the difference between the remaining premium from the previous row and the amortized premium:

$$\text{Remaining Premium} = \text{Remaining Premium from Previous Row} - \text{Amortized Premium}$$

Step 7: For each payment, repeat steps 2 through 6, including for the last row.

Step 8: The last book value is the face value of the bond. If the calculations are done correctly, this will automatically happen.

Step 9: The last remaining premium to be amortized is 0. If the calculations are done correctly, this will automatically happen.

Step 10: The total payments is the sum of the payment column:

$$\text{Total Payments} = N \times PMT.$$

Step 11: The total amortized premium is the sum of the amortized premium column, and is just the premium:

$$\text{Total Amortized Premium} = \text{Premium}.$$

Step 12: The total interest is the sum of the interest on book value column, and equals the difference between the other two column totals:

$$\text{Total Interest} = \text{Total Payments} - \text{Total Amortized Premium}.$$



Key Takeaways

The manual calculation of the interest on the book value entry above is based on the assumption that the payment frequency and the compounding frequency are equal. For bonds this is generally not an issue because most bonds have semi-annual payments and a yield rate that compounds semi-annually. If the payment frequency and the compounding frequency are not equal, an interest conversion would be required to convert the interest rate to the equivalent rate with the compounding frequency equal to the payment frequency. However, if you use the TI BAII Plus's built-in amortization worksheet (described below), no interest conversion is required.

As you fill in the schedule, round the entries to two decimal places.

The amortized premium column represents how much of the premium is claimed as a capital loss against the payment for that payment interval.

The amortization of the premium schedule presented here assumes the bond is purchased on an interest payment date. Bonds that are not purchased on an interest payment date have added complications that will not be addressed here.

Example 4.4.1

A \$2,000 bond with a coupon rate of 8% is redeemable in two years. The bond was purchased when the yield rate was 5%. Construct the amortization of the premium schedule.

Solution

Step 1: The given information is

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual.

$$FV = \$2,000$$

$$P/Y = 2$$

$$I/Y = 5\%$$

$$C/Y = 2$$

$$t = 2 \text{ years}$$

$$\text{Coupon Rate} = 8\%$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 2,000 \times \frac{0.08}{2} \\ &= \$80 \end{aligned}$$

Step 3: Calculate the purchase price.

PMT Setting	END
N	$2 \times 2 = 4$
PV	?
FV	2,000
PMT	80
I/Y	5
P/Y	2
C/Y	2

$$PV = \$2,112.86$$

The purchase price of the bond is \$2,112.86.

Step 4: Calculate the premium.

$$\begin{aligned}
 \text{Premium} &= \text{Purchase Price} - \text{Face Value} \\
 &= 2,112.86 - 2,000 \\
 &= \$112.86
 \end{aligned}$$

The premium is \$112.86.

Step 5: Complete the amortization of premium schedule.

Because the payment frequency and the compounding frequency are equal, no interest conversion is required. The calculations for each entry are shown in blue. The periodic yield

rate is $i = \frac{5\%}{2} = 2.5\%$.

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$2,112.86	\$112.86
1	\$80	\$52.82 (2,112.86 × 0.025)	\$27.18 (80 – 52.82)	\$2,085.68 (2,112.86 – 27.18)	\$85.68 (112.86 – 27.18)
2	\$80	\$52.14 (2,085.68 × 0.025)	\$27.86 (80 – 52.14)	\$2,057.82 (2,085.68 – 27.86)	\$57.82 (85.68 – 27.86)
3	\$80	\$51.45 (2,057.82 × 0.025)	\$28.55 (80 – 51.45)	\$2,029.27 (2,057.82 – 28.55)	\$29.27 (57.82 – 28.55)
4	\$80	\$50.73 (2,029.27 × 0.025)	\$29.27 (80 – 50.73)	\$2,000	\$0 (29.27 – 29.27)
Totals	\$320 (4 × 80)	\$207.14 (320 – 112.86)	\$112.86		

Things to Watch Out For

In the previous example, the result of the amortization of the premium schedule is that the investor ends up paying taxes only on **\$207.14** (the total of the interest on the book value at

yield column) instead of **\$320** (the payments total). That is, by spreading the **\$112.86** capital loss over the holding period of the bond, the total taxable amount is reduced from **\$320** down to **\$207.14**.

Sometimes there is a minor difference in the face value and the remaining premium to be amortized in the last row, where the last entries in these columns are off by a cent or two. This is a normal occurrence that happens because of the rounding of the entries in the schedule.

Although the calculations in an amortization of premium schedule are relatively straightforward, the manual calculations are time-consuming, especially when the schedule has a lot of rows. The amortization worksheet on a financial calculator, such as the TI BAII Plus, can be used to quickly calculate the entries for each row of the schedule.

Using the TI BAII Plus Calculator to Construct an Amortization of the Premium Schedule

To use the amortization worksheet to complete an amortization of the premium schedule:

1. Solve for any unknown quantities about the bond. You need to know all of the information about the bond first before you can use the amortization worksheet.
2. Enter all the value of all seven time value of money variables into the calculator (N , PV , FV , PMT , I/Y , P/Y , C/Y). If you calculated the purchase price (PV) in the first step, you must *re-enter* it rounded to two decimals and with the correct cash flow sign. Make sure the payment setting is set to **END**, and obey the cash flow sign convention. Because this is a bond, PV (the purchase price) is negative, FV (the face value) is positive, and PMT (the bond payment) is positive.
3. Go to the amortization worksheet by pressing **2nd AMORT** (the PV button).

4. To view the entries for a specific row of the schedule, set P_1 and P_2 to the row number. For example, to view the entries for row 5, set $P_1 = 5$ and $P_2 = 5$:
 - i. At the P_1 prompt, enter the row number and press **ENTER**.
 - ii. Press the down arrow.
 - iii. At the P_2 prompt, enter the row number and press **ENTER**.
 - iv. Press the down arrow.
 - v. The **BAL** entry is the book value entry for the corresponding row.
 - vi. Press the down arrow.
 - vii. The **PRN** entry is the amortized premium entry for the corresponding row.
 - viii. Press the down arrow.
 - ix. The **INT** entry is the interest on book value at yield entry for the corresponding row.
 - x. Press the down arrow the return to the P_1 screen.
5. Repeat the previous step with a different row number to view the entries for a different row.



Paths to Success

- On the amortization worksheet, **BAL** is the book value entry, **PRN** is the amortized premium entry, and **INT** is the interest on book value at yield entry.
- You cannot get the entries for the last column, remaining premium to be amortized, from the amortization worksheet on the calculator. This entry will still need to be calculated manually.
- Make sure to re-enter **PV** rounded to 2 decimal places before using the amortization worksheet. If you enter **PV** with all of the decimal places, you will not get the correct entries for the amortization of the premium schedule.
- As you read the entries off of the amortization worksheet on the calculator and put them in the

schedule, round the entries to 2 decimal places.

Example 4.4.2

A \$5,000 bond has a 4% coupon and is redeemable in two years. It was purchased to yield 3%. Construct the amortization of the premium schedule.

Solution

Step 1: Calculate the purchase price.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 5,000 \times \frac{0.04}{2} \\ &= \$100 \end{aligned}$$

PMT Setting	END
N	$2 \times 2 = 4$
PV	?
FV	5,000
PMT	100
I/Y	3
P/Y	2
C/Y	2

$$PV = \$5,096.36$$

Step 2: Calculate the premium.

$$\begin{aligned} \text{Premium} &= \text{Purchase Price} - \text{Face Value} \\ &= 5,096.36 - 5,000 \\ &= \$96.36 \end{aligned}$$

Step 3: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	4
PV	-5,096.36
FV	5,000
PMT	100
I/Y	3
P/Y	2
C/Y	2

Step 4: Complete the amortization of the premium schedule using the amortization worksheet on the calculator.

Remember, the calculator will not tell you the entries for the last column (remaining premium to be amortized), so you will need to complete this column manually.

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$5,096.36	\$96.36
1	\$100	\$76.45	\$23.55	\$5,072.81	\$72.81
2	\$100	\$76.09	\$23.91	\$5,048.90	\$48.90
3	\$100	\$75.73	\$24.27	\$5,024.63	\$24.63
4	\$100	\$75.37	\$24.63	\$5,000	\$0
Totals	\$400	\$303.63	\$96.36		

- **Row 1:** In the amortization worksheet, set $P_1 = 1$ and $P_2 = 1$. The entry for the last column (remaining premium to be amortized) is $96.36 - 23.55$.
- **Row 2:** In the amortization worksheet, set $P_1 = 2$ and $P_2 = 2$. The entry for the last column (remaining premium to be amortized) is $72.81 - 23.91$.

- **Row 3:** In the amortization worksheet, set $P_1 = 3$ and $P_2 = 3$. The entry for the last column (remaining premium to be amortized) is $48.90 - 24.27$.
- **Row 4:** In the amortization worksheet, set $P_1 = 4$ and $P_2 = 4$. The entry for the last column (remaining premium to be amortized) is $24.63 - 24.63$.
- **Totals Row:**
 - The amortized premium total is the premium (\$96.36).
 - The payments total is the sum of the payments: $4 \times 100 = 400$.
 - The interest on book value total is the difference in the other two column totals: $400 - 96.36 = 303.63$.

Try It

1) A \$10,000 bond has a coupon rate of 6.5%. The bond was purchased when there was three years to maturity and the yield rate was 5%. Construct the amortization of the premium schedule.

Solution

PMT Setting	END
N	$2 \times 3 = 6$
PV	?
FV	10,000
PMT	325
I/Y	5
P/Y	2
C/Y	2

$$PV = \$10,413.11$$

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$10,413.11	\$413.11
1	\$325	\$260.33	\$64.67	\$10,348.44	\$348.44
2	\$325	\$258.71	\$66.29	\$10,282.15	\$282.15
3	\$325	\$257.05	\$67.95	\$10,214.20	\$214.20
4	\$325	\$255.36	\$69.64	\$10,144.56	\$144.56
5	\$325	\$253.61	\$71.39	\$10,073.17	\$73.17
6	\$325	\$251.83	\$73.17	\$10,000	\$0
Totals	\$1,950	\$303.63	\$413.11		

Accumulation of the Discount

When a bond is purchased for less than its face value, the result is a discount. For example, suppose an investor pays \$950 for a \$1,000 face value bond. If the investor holds onto the bond until maturity, the investor receives the redemption price of \$1,000. The bond discount of \$50 represents a capital gain for the investor. A **capital gain** is the amount by which the current value of an asset exceeds the original purchase price. What does the investor do with this capital gain? One option is for the investor to claim all of the capital gain as taxable income at the time of redemption, which results in a higher taxable amount at the time of redemption. But, commonly accepted practices allow the investor to distribute the capital gain over the period of time that the bond is held, and not just in the period during which the capital gain actually occurs (at maturity). This allows the investor to spread the capital gain over each payment interval, claiming a little bit of the capital gain along with each bond payment. This results in a higher taxable amount for each payment interval, but results in lower taxes overall because the capital gain is not claimed as a single lump-sum taxable expense on the maturity date. This process is called the **accumulation of the discount**.

The accumulation of the discount schedule records the bond payment, the amount of the discount that is claimed with each payment, the book value, and the remaining discount to be accumulated. The accumulation of the discount schedule shows how much discount is claimed with each payment and the resulting book value.

An accumulation of the discount schedule has six columns:

- **Payment Number** — There is a row for every bond payment during the holding period of the bond.
- **Bond Payment** — The periodic bond payment (PMT). All of the payments are the same, including the last payment.
- **Interest on Book Value at Yield** — For each row, the interest on book value entry is the interest portion of the current bond payment based on the book value from the previous row.
- **Accumulated Discount** — For each row, the accumulated discount entry is the amount of the discount that is claimed as a capital gain in that payment interval.
- **Book Value** — For each row, the book value is the face value of the bond minus the remaining unaccumulated discount.
- **Remaining Discount to be Accumulated** — For each row, the remaining discount is the amount of discount that is left to claim with the payments.

To fill in an accumulation of the discount schedule, you first need to have all of the details about the bond, including the face value (FV), the purchase price (PV), the bond payment (PMT), the number of payments (N), and the yield rate. If any of these quantities are missing, calculate out the missing value before completing the accumulation of the discount schedule.

Total Accumulated Discount¹¹

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				Purchase Price ¹	Discount ¹
1	PMT^2	$INT_{BV^3}^{\text{on}}$	AD^4	BV^5	RD^6
2	PMT^2	$INT_{BV^3}^{\text{on}}$	AD^4	BV^5	RD^6
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$N - 1$	PMT^2	$INT_{BV^3}^{\text{on}}$	AD^4	BV^5	RD^6
N	PMT^2	$INT_{BV^3}^{\text{on}}$	AD^4	Face Value ⁸	0 ⁹
Totals	Total Payments ¹⁰	Interest ¹²			

HOW TO

Fill In the Accumulation of the Discount Schedule

Step 1: In row **0**, the only entries are in the book value and remaining discount to be accumulated columns. The initial book value is the purchase price of the bond and the initial remaining discount is the discount.

Step 2: Each entry in the bond payment column is the bond payment. All of the payments in this column are the same, including the last payment.

Step 3: Calculate the interest on the book value at yield. The interest is the book value from the previous row times the periodic yield rate:

$$\text{Interest on Book Value} = \text{Book Value from Previous Row} \times i$$

Note: this calculation uses the periodic yield rate, not the periodic bond rate.

Step 4: Calculate the accumulated discount. The accumulated discount is the difference between the interest on the book value and the payment:

$$\text{Accumulated Discount} = \text{Interest on Book Value} - PMT.$$

Step 5: Calculate the new book value. The book value is sum of the book value in the previous row and the accumulated discount:

$$\text{Book Value} = \text{Book Value from Previous Row} + \text{Accumulated Discount}$$

Step 6: Calculate the remaining discount to be accumulated. The remaining discount is the difference between the remaining discount from the previous row and the accumulated discount:

$$\begin{aligned} \text{Remaining Discount} &= \text{Remaining Discount from Previous Row} \\ &\quad - \text{Accumulated Discount} \end{aligned}$$

Step 7: For each payment, repeat Steps **2** through **6**, including for the last row.

Step 8: The last book value is the face value of the bond. If the calculations are done correctly, this will automatically happen.

Step 9: The last remaining discount to be accumulated is **0**. If the calculations are done correctly, this will automatically happen.

Step 10: The total payments is the sum of the payment column:

$$\text{Total Payments} = N \times PMT.$$

Step 11: The total accumulated discount is the sum of the accumulated discount column, and is just the discount:

$$\text{Total Accumulated Discount} = \text{Discount}.$$

Step 12: The total interest is the sum of the interest on book value column, and equals the sum of the other two column totals:

$$\text{Total Interest} = \text{Total Payments} + \text{Total Accumulated Discount}.$$



Paths to Success

The manual calculation of the interest on the book value entry above is based on the assumption that the payment frequency and the compounding frequency are equal. For bonds this is generally not an issue because most bonds have semi-annual payments and a yield rate that compounds semi-annually. If the payment frequency and the compounding frequency are not equal, an interest conversion would be required to convert the interest rate to the equivalent rate with the compounding frequency equal to the payment frequency. However, if you use the TI BAII Plus's built-in amortization worksheet (described below), no interest conversion is required.

As you fill in the schedule, round the entries to two decimal places.

The accumulated discount column represents how much of the discount is claimed as a capital gain with the payment for that payment interval.

The accumulation of the discount schedule presented here assumes the bond is purchased on an interest payment date. Bonds that are not purchased on an interest payment date have added complications that will not be addressed here.

Example 4.4.3

A \$3,000 bond with a coupon rate of 5% is redeemable in two years. The bond was purchased when the yield rate was 8%. Construct the accumulation of the discount schedule.

Solution

Step 1: The given information is

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual.

$$FV = \$3,000$$

$$P/Y = 2$$

$$I/Y = 8\%$$

$$C/Y = 2$$

$$t = 2 \text{ years}$$

$$\text{Coupon Rate} = 5\%$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 3,000 \times \frac{0.05}{2} \\ &= \$75 \end{aligned}$$

Step 3: Calculate the purchase price.

PMT Setting	END
N	$2 \times 2 = 4$
PV	?
FV	3,000
PMT	75
I/Y	8
P/Y	2
C/Y	2

$$PV = \$2,836.65$$

The purchase price of the bond is \$2,836.65.

Step 4: Calculate the discount.

$$\begin{aligned}
 \text{Discount} &= \text{Face Value} - \text{Purchase Price} \\
 &= 3,000 - 2,836.65 \\
 &= \$163.35
 \end{aligned}$$

The discount is \$163.35.

Step 5: Complete the accumulation of discount schedule.

Because the payment frequency and the compounding frequency are equal, no interest conversion is required. The calculations for each entry are shown in blue. The periodic yield

$$\text{rate is } i = \frac{8\%}{2} = 4\%.$$

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Premium to be Amortized
0				\$2,836.65	\$163.35
1	\$75	\$113.47 (2,836.65 × 0.04)	\$38.47 (113.47 – 75)	\$2,875.12 (2,836.65 + 38.47)	\$124.88 (163.35 – 38.47)
2	\$75	\$115 (2,875.12 × 0.04)	\$40 (115 – 75)	\$2,915.12 (2,915.12 + 40)	\$84.88 (124.88 – 40)
3	\$75	\$116.60 (2,915.12 × 0.04)	\$41.60 (116.60 – 75)	\$2,956.72 (2,915.12 + 41.60)	\$43.28 (84.88 – 41.60)
4	\$75	\$118.27 (2,956.72 × 0.04)	\$43.27 (118.27 – 75)	\$2,999.99 (2,956.72 + 43.27)	\$0.01 (43.28 – 43.27)
Totals	\$300 (4 × 75)	\$463.34 (300 + 163.34)	\$163.34		



Key Takeaways

In the previous example, there is a minor difference of **\$0.01** in the final book value and the final remaining discount to be accumulated in row **4** and the totals in the interest on the book value and accumulated discount totals. This sometimes happens, and is caused by rounding the entries at each step.

In the previous example, the result of the accumulation of discount schedule is that the investor ends up paying taxes on **\$463.34** (the total of the interest on the book value at yield column)

instead of **\$300** (the payments total). That is, by spreading the **\$163.35** capital gain over the holding period of the bond, the total taxable amount is increased from **\$300** to **\$463.34**. Because the capital gain is spread out over the payments, this results in a lower tax amount overall.

Although the calculations in an accumulation of the discount schedule are relatively straightforward, the manual calculations are time-consuming, especially when the schedule has a lot of rows. The amortization worksheet on a financial calculator, such as the TI BAII Plus, can be used to quickly calculate the entries for each row of the schedule.

Using the TI BAII Plus Calculator to Construct an Accumulation of the Discount Schedule

To use the amortization worksheet to complete an accumulation of the discount schedule:

1. Solve for any unknown quantities about the bond. You need to know all of the information about the bond first before you can use the amortization worksheet.
2. Enter all the value of all seven time value of money variables into the calculator (N , PV , FV , PMT , I/Y , P/Y , C/Y). If you calculated the purchase price (PV) in the first step, you must re-enter it rounded to two decimals and with the correct cash flow sign. Make sure the payment setting is set to **END**, and obey the cash flow sign convention. Because this is a bond, PV (the purchase price) is negative, FV (the face value) is positive, and PMT (the bond payment) is positive.
3. Go to the amortization worksheet by pressing **2nd AMORT** (the PV button).
4. To view the entries for a specific row of the schedule, set P_1 and P_2 to the row number. For example, to view the entries for row **5**, set $P_1 = 5$ and $P_2 = 5$:
 - i. At the P_1 prompt, enter the row number and press **ENTER**.

- ii. Press the down arrow.
 - iii. At the P_2 prompt, enter the row number and press **ENTER**.
 - iv. Press the down arrow.
 - v. The BAL entry is the book value entry for the corresponding row.
 - vi. Press the down arrow.
 - vii. The PRN entry is the accumulated discount entry for the corresponding row.
 - viii. Press the down arrow.
 - ix. The INT entry is the interest on book value at yield entry for the corresponding row.
 - x. Press the down arrow the return to the P_1 screen.
5. Repeat the previous step with a different row number to view the entries for a different row.



- On the amortization worksheet, BAL is the book value entry, PRN is the accumulation of discount entry, and INT is the interest on book value at yield entry.
- You cannot get the entries for the last column, remaining discount to be accumulated, from the amortization worksheet on the calculator. This entry will still need to be calculated manually.
- Make sure to re-enter PV rounded to 2 decimal places before using the amortization worksheet. If you enter PV with all of the decimal places, you will not get the correct entries for the accumulation of the discount schedule.
- As you read the entries off of the amortization worksheet on the calculator and put them in the schedule, round the entries to 2 decimal places.

Example 4.4.4

A \$7,000 bond has a 3% coupon and is redeemable in two years. It was purchased to yield 5%. Construct the accumulation of the discount schedule.

Solution

Step 1: Calculate the purchase price.

$$\begin{aligned}PMT &= FV \times \frac{\text{coupon rate}}{2} \\&= 7,000 \times \frac{0.03}{2} \\&= \$105\end{aligned}$$

PMT Setting	END
N	$2 \times 2 = 4$
PV	?
FV	7,000
PMT	105
I/Y	5
P/Y	2
C/Y	2

$$PV = \$6,736.66$$

Step 2: Calculate the discount.

$$\begin{aligned}\text{Premium} &= \text{Face Value} - \text{Purchase Price} \\&= 7,000 - 6,736.66 \\&= \$263.34\end{aligned}$$

Step 3: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	4
PV	−6,736.66
FV	7,000
PMT	105
I/Y	5
P/Y	2
C/Y	2

Step 4: Complete the accumulation of the discount schedule using the amortization worksheet on the calculator.

Remember, the calculator will not tell you the entries for the last column (remaining discount to be accumulated), so you will need to complete this column manually.

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$6,736.66	\$263.34
1	\$105	\$168.42	\$63.42	\$6,800.08	\$199.92
2	\$105	\$170	\$65	\$6,865.08	\$134.92
3	\$105	\$171.63	\$66.63	\$6,931.71	\$68.29
4	\$105	\$173.29	\$68.29	\$7,000	\$0
Totals	\$420	\$683.34	\$263.34		

- **Row 1:** In the amortization worksheet, set $P_1 = 1$ and $P_2 = 1$. The entry for the last column (remaining discount to be accumulated) is $263.34 - 63.42$.
- **Row 2:** In the amortization worksheet, set $P_1 = 2$ and $P_2 = 2$. The entry for the last column (remaining discount to be accumulated) is $199.92 - 65$.
- **Row 3:** In the amortization worksheet, set $P_1 = 3$ and $P_2 = 3$. The entry for the last

column (remaining discount to be accumulated) is $134.92 - 66.63$.

- **Row 4:** In the amortization worksheet, set $P_1 = 4$ and $P_2 = 4$. The entry for the last column (remaining discount to be accumulated) is $68.29 - 68.29$.
- **Totals Row:**
 - The accumulated discount total is the discount (\$263.34).
 - The payments total is the sum of the payments: $4 \times 105 = 420$.
 - The interest on book value total is the sum of other two column totals:
 $420 + 263.34 = 683.34$.

Try It

1) A \$5,000 bond has a coupon rate of 3.6%. The bond was purchased when there was 2.5 years to maturity and the yield rate was 6%. Construct the accumulation of the discount schedule.

Solution

PMT Setting	END
N	$2 \times 2.5 = 5$
PV	?
FV	5,000
PMT	90
I/Y	6
P/Y	2
C/Y	2

$$PV = \$4,725.22$$

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$4,725.22	\$274.78
1	\$90	\$141.76	\$51.76	\$4,776.98	\$223.02
2	\$90	\$143.31	\$53.31	\$4,830.29	\$169.71
3	\$90	\$144.91	\$54.91	\$4,885.19	\$114.80
4	\$90	\$146.56	\$56.56	\$4,941.75	\$58.25
5	\$90	\$148.25	\$58.25	\$5,000	\$0
Totals	\$450	\$703.33	\$253.33		

Section 4.4 Exercises

1. A \$20,000 bond has a 5% coupon rate. The bond was purchased when there are three years to maturity and the yield rate was 6.75%. Construct the appropriate bond schedule for the bond.

Solution

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$19,063.66	\$936.34
1	\$500	\$643.40	\$143.40	\$19,207.06	\$792.94
2	\$500	\$648.24	\$148.24	\$19,355.30	\$644.70
3	\$500	\$653.24	\$153.24	\$19,508.54	\$491.46
4	\$500	\$658.41	\$158.41	\$19,666.95	\$333.05
5	\$500	\$663.76	\$163.76	\$19,830.71	\$169.29
6	\$500	\$669.29	\$169.29	\$20,000	\$0
Totals	\$3,000	\$3,936.34	\$936.34		

2. A \$50,000 bond with an 8% coupon is redeemable in two years. The bond was purchased when the yield to maturity was 5%. Construct the appropriate bond scheduled for the bond.

Solution

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$52,821.48	\$2,821.48
1	\$2,000	\$1,320.54	\$679.46	\$52,142.02	\$2,142.02
2	\$2,000	\$1,303.55	\$696.45	\$51,445.57	\$1,445.57
3	\$2,000	\$1,286.14	\$713.86	\$50,731.71	\$731.71
4	\$2,000	\$1,268.29	\$731.71	\$50,000	\$0
Totals	\$8,000	\$5,178.52	\$2,821.48		

3. Three years before maturity, a \$55,000 face value bond carrying a 5.5% coupon is acquired when posted market rates are 4.77% compounded semi-annually. Construct

the appropriate bond schedule for the bond.

Solution

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$56,110.02	\$1,110.02
1	\$1,512.50	\$1,338.22	\$174.28	\$55,935.74	\$935.74
2	\$1,512.50	\$1,334.07	\$178.43	\$55,757.31	\$757.31
3	\$1,512.50	\$1,329.81	\$182.69	\$55,574.62	\$574.62
4	\$1,512.50	\$1,325.45	\$187.05	\$55,387.58	\$387.58
5	\$1,512.50	\$1,320.99	\$191.51	\$55,196.07	\$196.07
6	\$1,512.50	\$1,316.43	\$196.07	\$55,000	\$0
Totals	\$9,075	\$57,964.98	\$1,110.02		

4. When the current bond market is yielding **5.89%** compounded semi-annually, Jennifer purchases a **\$10,000** face value bond carrying a **4.2%** coupon with three years until maturity for her RRSP. Construct the appropriate bond schedule for the bond.

Solution

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$9,541.41	\$458.59
1	\$210	\$280.99	\$70.99	\$9,612.40	\$387.60
2	\$210	\$283.09	\$73.09	\$9,685.49	\$314.52
3	\$210	\$285.24	\$75.24	\$9,760.73	\$239.27
4	\$210	\$287.45	\$77.45	\$9,838.19	\$161.82
5	\$210	\$289.73	\$79.73	\$9,917.92	\$82.08
6	\$210	\$292.08	\$82.08	\$10,000	\$0
Totals	\$1,260	\$1,718.59	\$458.59		

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4.5 QUOTATION OF BONDS

Learning Objectives

- Calculate the purchase price of a bond on a non-interest payment date
- Calculate the quoted price of a bond

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance
- BV = Book value
- FV = Face value of bond
- PMT = Annuity payment amount
- N = Number of annuity payments
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency

Formulas Used

- Formula 4.1 – **Bond Payment**

$$\text{Payment} = \text{Face Value} \times \text{periodic bond rate}$$

- Formula 4.2 – **Purchase Price on an Interest Payment Date**

$$\text{Purchase Price} = \text{Present Value of Face Value} + \text{Present Value of Bond Payments}$$

- Formula 4.3 – **Premium (Bonds)**

$$\text{Premium} = \text{Purchase Price} - \text{Face Value}$$

- Formula 4.4 – **Discount (Bonds)**

$$\text{Discount} = \text{Face Value} - \text{Purchase Price}$$

- Formula 4.5 – **Quoted Price (Bonds)**

$$\text{Flat Price} - \text{Accrued Interest}$$

- Formula 4.6 – **Quoted Price as a Percentage of Face Value**

$$\frac{\text{Quoted Price}}{\text{Face Value}} \times 100\%$$

Calculating the Purchase Price of a Bond on a Non-Interest Payment Date

In a previous section, we learned how to calculate the purchase price of a bond when the bond is purchased on an interest payment date. But, bonds can be bought or sold on any day of the year. How do we calculate the purchase price when a bond is purchased on a non-interest payment date?

For each day that elapses after an interest payment date, interest for the next payment starts to accrue. When the bond is sold in between interest payment dates, the interest, called the **accrued interest**, that accrues on the bond since the interest payment date before the purchase date up to but not including the purchase date is paid to the seller. This accrued interest is included in the purchase price paid to the seller. That is, the buyer has to pay the seller the purchase price of the bond on the previous interest payment date, called the **market**

price, plus the accrued interest. The market price plus the accrued interest is called the **flat price**, and is the actual amount the buyer pays the seller on the purchase date.

Why does the buyer have to pay the seller the accrued interest? Suppose a bond makes semi-annual interest payments of \$50 and the buyer acquires the bond two months after the last interest payment date. Because the seller held the bond for two months of the six-month payment interval, it is fair and reasonable for the seller to receive the interest earned during those two months. However, the next interest payment occurs four months after the sale of the bond, at which time the buyer, who now owns the bond, will receive the full \$50 interest payment for the full six month payment interval. Thus, at the time of buying the bond, the buyer has to pay the seller the bond's market price plus the portion of the next interest payment that legally belongs to the seller. In this example, an interest amount representing two of the six months needs to be paid.

Arriving at the bond's purchase price in between interest payment dates is a little complex because the flat price is increasing according to a compound interest formula, but in practice the accrued interest on the bond is increasing according to a simple interest formula. If this seems peculiar to you, you would be right, but that is just how bonds happen to work!

HOW TO

Find the Flat Price of a Bond

Follow these steps to find the flat price, the actual purchase price of a bond between interest payment dates.

Step 1: Identify four important dates:

- Purchase date.
- Maturity date.
- Interest payment date immediately before the purchase date.
- Interest payment date immediately after the purchase date.

Step 2: Calculate the market price on the interest payment date immediately before the purchase date. The market price is the purchase price if the bond is actually purchased on the interest payment date immediately before the purchase date. This calculation is the same purchase price calculation we learned in a previous section.

Step 3: Calculate the flat price on the purchase date by adding the accrued interest to the

market price. This is a compound interest only calculation that moves the market price from the payment date immediately before the purchase date to the purchase date. In this calculation, you calculate the future value with the present value set to the market price from the previous step, the payment is 0, the interest rate is the yield rate, and N is

$$N = \frac{\text{Number of days from previous interest payment date to purchase date}}{\text{Number of days in between interest payment dates}}$$

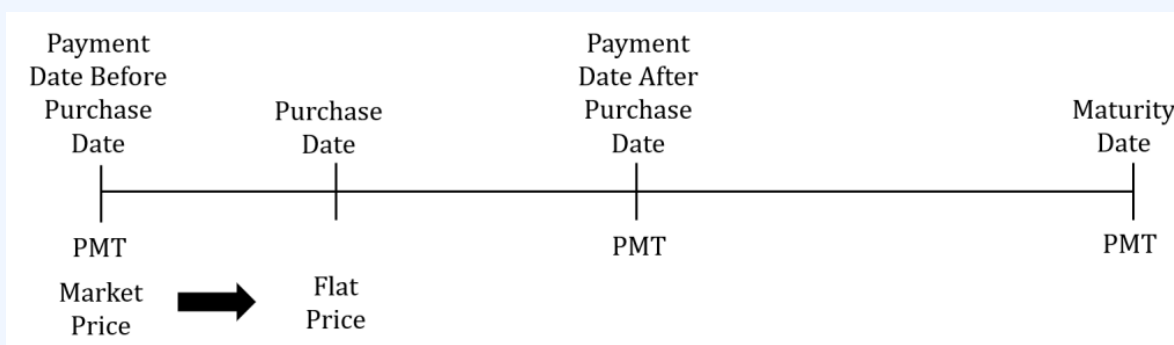


Figure 4.5.1

Image Description

The image is a horizontal timeline illustrating the sequence of events related to bond payments and prices. The timeline has five primary markers:

- **Payment Date Before Purchase Date:** The first marker on the left is labeled “Payment Date Before Purchase Date”.
- **Purchase Date:** The second marker is labeled “Purchase Date”.
- **Payment Date After Purchase Date:** The third marker is labeled “Payment Date After Purchase Date”.
- **Maturity Date:** The final marker on the right is labeled “Maturity Date”.
- **PMT:** The text “PMT” is positioned beneath each of these markers to denote payment events.
- **Market Price and Flat Price:** Between the payment date before purchase date and the purchase date, there’s a bold arrow labeled “Market Price” pointing rightward to the text “Flat Price”.



Key Takeaways

The market price is the purchase price on the payment date before the purchase price. That is, if the actual purchase date was this payment date, you would pay the market price to buy the bond.

Two things happen on the maturity date: the investor receives the face value of the bond and the last bond payment. Consequently, the maturity date is a payment date. Because the payments happen on the same two dates of every year, the month and day of the maturity date tells you one of the payment dates. The other payment date happens six months later (or six months earlier depending on how you think about it) on the same day. For example, suppose the maturity date is January 1, 2040. This means that January 1 is one of the two payment dates. The other payment date happens six months later, in this case July 1.

Example 4.5.1

A \$1,000 bond with a 7% coupon has a maturity date of February 1, 2035. The bond was purchased on May 20, 2023 when the yield to maturity was 5%. Calculate the purchase price.

Solution

Step 1: Identify the four dates.

Because February 1 is the maturity date, February 1 is one of the two payment dates. The other payment date is six months later, in this case August 1.

- Purchase date: May 20, 2023

- Maturity date: February 1, 2035
- Payment date before purchase date: February 1, 2023
- Payment date after purchase date: August 1, 2023

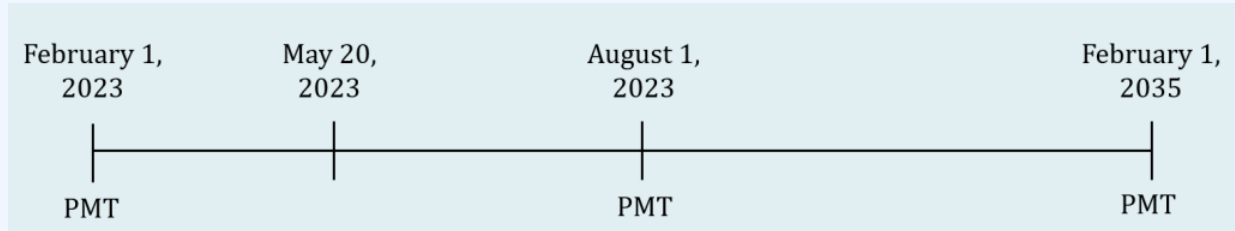


Figure 4.5.2

Image Description

The image shows a horizontal timeline with the following dates:

- February 1, 2023
- May 20, 2023
- August 1, 2023
- February 1, 2035

Each date except May 20, 2023 has a label “PMT” beneath it, indicating a significant event or milestone occurring on these dates.

Step 2: Calculate the market price on February 1, 2023.

The time for this calculation is the number of years between February 1, 2023 (the payment date before the purchase date) and February 1, 2035 (the maturity date), which is **12** years.

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$\begin{aligned}
 FV &= \$1,000 \\
 P/Y &= 2 \\
 I/Y &= 5\% \\
 C/Y &= 2 \\
 t &= 12 \text{ years} \\
 \text{Coupon Rate} &= 7\%
 \end{aligned}$$

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 1,000 \times \frac{0.07}{2} \\
 &= \$35
 \end{aligned}$$

PMT Setting	END
N	$2 \times 12 = 24$
PV	?
FV	1,000
PMT	35
I/Y	5
P/Y	2
C/Y	2

$$PV = \$1,178.85$$

The market price on February 1, 2023 is \$1,178.85.

Step 3: Calculate the flat price on May 20, 2023.

This is a compound interest calculation that moves the market price from the previous step forward in time to the purchase date. The value of N for this compound interest calculation is

$$N = \frac{\text{Number of days from February 1, 2023 to May 20, 2023}}{\text{Number of days from February 1, 2023 to August 1, 2023}}$$

$$= \frac{108}{181}$$

N	$\frac{108}{181}$
PV	1,178.85
FV	?
PMT	0
I/Y	5
P/Y	2
C/Y	2

$$FV = \$1,196.35$$

Step 4: Write as a statement.

The flat price on May 20, 2023 is **\$1,196.35**. This is the purchase price of the bond on May 20, 2023.



Paths to Success

In the above example, the market price of **\$1,178.85** is the purchase price if you purchased the bond on February 1, 2023.

To calculate the number of days between the dates in Step 3, use the **DATE** function on the financial calculator.

In step 3 above, you will need to convert the fraction $\frac{108}{181}$ to a decimal in order to enter it into your calculator.

Remember to keep *all* of the decimals when you enter this number into N to avoid any possible round-off error.

The flat price calculation in step 3 is a compound interest only calculation, which means the payment is 0. This is a future value calculation because the market price is moving forward in time to get to the purchase date. The interest rate for this calculation is the yield rate.

Using the TI BAII Plus Calculator to Count the Number of Days Between Dates

To count the number of days between two dates:

1. Press **2nd DATE** (the **1** button) to enter the date worksheet.
2. At the **DT1** screen, enter the first date. Dates are entered in the form mm.ddyy. For example, enter May 19, 2023 as **05.1923**. After entering the date press **ENTER**.
3. Press the down arrow.
4. At the **DT2** screen, enter the second date. Dates are entered in the form mm.ddyy. For example, enter August 7, 2023 as **08.0723**. After entering the date press **ENTER**.
5. Press the down arrow.
6. At the **DBD** screen press **CPT** to calculate the number of days between the two entered dates. For example, there are **80** days between May 19, 2023 and August 7, 2023.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=139#oembed-1>

Example 4.5.2

A 25-year, \$2,000 bond with a 3% coupon was issued on April 1, 2018. The bond was purchased on January 27, 2023 when the yield to maturity was 8%. Calculate the purchase price.

Solution

Step 1: Identify the four dates.

The bond was issued on April 1, 2018 and reaches maturity 25 years later. So, the maturity date is April 1, 2043. Because April 1 is the maturity date, April 1 is one of the two payment dates. The other payment date is six months later, in this case October 1.

- Purchase date: January 27, 2023
- Maturity date: April 1, 2043
- Payment date before purchase date: October 1, 2022
- Payment date after purchase date: April 1, 2023

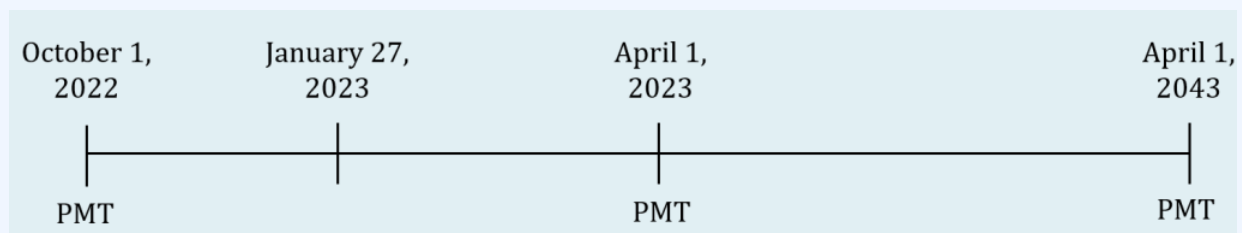


Figure 4.5.3

Image Description

The image shows a horizontal timeline with the following dates:

- October 1, 2022
- January 27, 2023
- April 1, 2023
- April 1, 2043

Each date except January 27, 2023 has a label “PMT” beneath it, indicating a payment occurring on these dates.

Step 2: Calculate the market price on October 1, 2022.

The time for this calculation is the number of years between October 1, 2022 (the payment date before the purchase date) and April 1, 2043 (the maturity date), which is **20.5** years.

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$FV = \$2,000$$

$$P/Y = 2$$

$$I/Y = 8\%$$

$$C/Y = 2$$

$$t = 20.5 \text{ years}$$

$$\text{Coupon Rate} = 3\%$$

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 2,000 \times \frac{0.03}{2} \\ &= \$30 \end{aligned}$$

PMT Setting	END
N	$2 \times 20.5 = 41$
PV	?
FV	2,000
PMT	30
I/Y	8
P/Y	2
C/Y	2

$$PV = \$1,000.35$$

The market price on October 1, 2022 is \$1,000.35.

Step 3: Calculate the flat price on January 27, 2023.

This is a compound interest calculation that moves the market price from the previous step forward in time to the purchase date. The value of N for this compound interest calculation is

$$N = \frac{\text{Number of days from October 1, 2022 to January 27, 2023}}{\text{Number of days from October 1, 2022 to April 1, 2023}}$$

$$= \frac{118}{182}$$

N	$\frac{118}{182}$
PV	1,000.35
FV	?
PMT	0
I/Y	8
P/Y	2
C/Y	2

$$FV = \$1,026.11$$

Step 4: Write as a statement.

The flat price on January 27, 2023 is \$1,026.11. This is the purchase price of the bond on January 27, 2023.

Try It

1) A \$5,000 bond with a 4.7% coupon has a maturity date of December 1, 2040. The bond was purchased on September 19, 2023 when the yield rate was 5.9%. Calculate the purchase price.

Solution

The four dates are

- Purchase date: September 19, 2023
- Maturity date: December 1, 2040
- Payment date before purchase date: June 1, 2023
- Payment date after purchase date: December 1, 2023

Calculate the market price on June 1, 2023.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 5,000 \times \frac{0.047}{2} \\
 &= \$117.50
 \end{aligned}$$

PMT Setting	END
N	$2 \times 17.5 = 25$
PV	?
FV	5,000
PMT	117.50
I/Y	5.9
P/Y	2
C/Y	2

$$PV = \$4,350.65$$

Calculate the flat price on September 19, 2023.

N	$\frac{110}{183}$
PV	4,350.65
FV	?
PMT	0
I/Y	5.9
P/Y	2
C/Y	2

$$FV = \$4,427.35$$

Quotation of Bonds

The flat price we calculated in the previous examples is the actual price a buyer would pay to purchase those bonds on the given purchase date. The flat price is the market price (on the payment date before the purchase date) plus any interest that has accrued since the previous payment. However, two different bonds purchased on the same day can have very different amounts of accrued interest, depending on where the purchase date is in relation to each bond's payment date. This can make it very difficult for an investor to directly compare the price of the bonds.

To make a meaningful comparison of bond prices, the flat prices are adjusted for their differing amount of accrued interest. The **quoted price** of bond is the flat price excluding the accrued coupon interest.

4.5 Quoted Price (Bonds)

$$\text{Quoted Price} = \text{Flat Price} - \text{Accrued Interest}$$

$$\text{Accrued Interest} = PMT \times \frac{\# \text{ days from previous interest pmt date to purchase date}}{\# \text{ days in between interest pmt dates}}$$

NOTE

The fraction used to find the accrued interest is the same fraction calculated in Step 3 of the flat price calculations above.

Quoted prices for bonds are normally given as a percent of their face value. This is the **market quotation** of the bond. Market quotations allow investors to easily compare bond prices.

4.6 Market Quotation as a Percentage of Face Value

$$\text{Percent of Face Value} = \frac{\text{Quoted Price}}{\text{Face Value}} \times 100\%$$

Example 4.5.3

A \$1,000 bond with a 7% coupon has a maturity date of February 1, 2035. The bond was purchased on May 20, 2023 when the yield to maturity was 5%. Calculate the quoted price as a percentage of face value.

Solution

The flat price of the bond on May 20, 2023 is \$1,196.35 (found in the first example above).

Step 1: Calculate the accrued interest.

$$\begin{aligned}\text{Accrued Interest} &= PMT \times \frac{\text{Number of days from February 1, 2023 to May 20, 2023}}{\text{Number of days from February 1, 2023 to August 1, 2023}} \\ &= 35 \times \frac{108}{181} \\ &= \$20.88\end{aligned}$$

Step 2: Calculate the quoted price.

$$\begin{aligned}\text{Quoted Price} &= \text{Flat Price} - \text{Accrued Interest} \\ &= 1,196.35 - 20.88 \\ &= \$1,175.47\end{aligned}$$

Step 3: Calculate the quoted price as a percentage of face value.

$$\begin{aligned}\text{Percent of Face Value} &= \frac{\text{Quoted Price}}{\text{Face Value}} \times 100\% \\ &= \frac{1,175.47}{1,000} \times 100\% \\ &= 117.55\%\end{aligned}$$

Step 4: Write as a statement.

The quoted price as a percentage of face value is 117.55%.

Example 4.5.4

A \$8,000 bond with a 4% coupon has a maturity date of November 1, 2037. The bond was purchased on February 5, 2023 when the yield to maturity was 7.5%. Calculate the quoted price as a percentage of face value.

Solution

Step 1: Identify the four dates.

Because November 1 is the maturity date, November 1 is one of the two payment dates. The other payment date is six months later, in this case May 1.

- Purchase date: February 5, 2023
- Maturity date: November 1, 2037
- Payment date before purchase date: November 1, 2022
- Payment date after purchase date: May 1, 2023

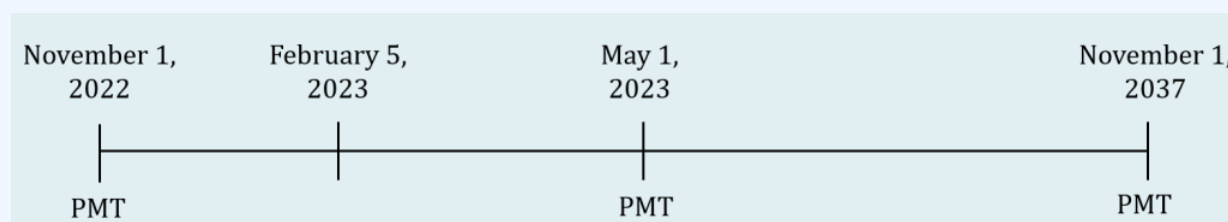


Figure 4.5.4

Image Description

The image is a timeline with the following details:

- The timeline begins on November 1, 2022.
- There is a mark on February 5, 2023.
- Another mark is on May 1, 2023.
- The timeline ends on November 1, 2037.

Each of the marks on the timeline except the second one (February 5, 2023) is labeled “PMT”.

Step 2: Calculate the market price on November 1, 2022.

The time for this calculation is the number of years between November 1, 2022 (the payment date before the purchase date) and November 1, 2037 (the maturity date), which is 15 years.

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$FV = \$8,000$$

$$P/Y = 2$$

$$I/Y = 7.5\%$$

$$C/Y = 2$$

$$t = 15 \text{ years}$$

$$\text{Coupon Rate} = 4\%$$

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 8,000 \times \frac{0.04}{2} \\ &= \$160 \end{aligned}$$

PMT Setting	END
N	$2 \times 15 = 30$
PV	?
FV	8,000
PMT	160
I/Y	7.5
P/Y	2
C/Y	2

$$PV = \$5,503.91$$

The market price on November 1, 2022 is \$5,503.91.

Step 3: Calculate the flat price on February 5, 2023.

This is a compound interest calculation that moves the market price from the previous step forward in time to the purchase date. The value of N for this compound interest calculation is

$$N = \frac{\text{Number of days from November 1, 2022 to February 5, 2023}}{\text{Number of days from November 1, 2022 to May 1, 2023}}$$

$$= \frac{96}{181}$$

N	$\frac{96}{181}$
PV	5,503.91
FV	?
PMT	0
I/Y	7.5
P/Y	2
C/Y	2

$$FV = \$5,612.43$$

The flat price on February 5, 2023 is \$5,612.43.

Step 4: Calculate the accrued interest.

$$\begin{aligned} \text{Accrued Interest} &= PMT \times \frac{\text{Number of days from November 1, 2022 to February 5, 2023}}{\text{Number of days from November 1, 2022 to May 1, 2023}} \\ &= 160 \times \frac{96}{181} \\ &= \$84.86 \end{aligned}$$

Step 5: Calculate the quoted price.

$$\begin{aligned} \text{Quoted Price} &= \text{Flat Price} - \text{Accrued Interest} \\ &= 5,612.43 - 84.86 \\ &= \$5,527.57 \end{aligned}$$

Step 6: Calculate the quoted price as a percentage of face value.

$$\begin{aligned}
 \text{Percent of Face Value} &= \frac{\text{Quoted Price}}{\text{Face Value}} \times 100\% \\
 &= \frac{5,527.57}{8,000} \times 100\% \\
 &= 69.09\%
 \end{aligned}$$

Step 7: Write as a statement.

The quoted price as a percentage of face value is **69.09%**.

Try It

2) A 15—year, \$7,000 bond with a 2.9% coupon was issued on September 1, 2015. The bond was purchased on April 15, 2023 when the yield to maturity was 4.1%. Calculate the quoted price as a percentage of face value.

Solution

The four dates are

- Purchase date: April 15, 2023
- Maturity date: September 1, 2030
- Payment date before purchase date: March 1, 2023
- Payment date after purchase date: September 1, 2023

Calculate the market price on March 1, 2023.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 7,000 \times \frac{0.029}{2} \\
 &= \$101.50
 \end{aligned}$$

PMT Setting	END
N	$2 \times 7.5 = 15$
PV	?
FV	7,000
PMT	101.50
I/Y	4.1
P/Y	2
C/Y	2

$$PV = \$6,462.34$$

Calculate the flat price on April 15, 2023.

N	$\frac{45}{184}$
PV	6,462.34
FV	?
PMT	0
I/Y	4.1
P/Y	2
C/Y	2

$$FV = \$6,494.50$$

Calculate the accrued interest.

$$\begin{aligned}
 \text{Accrued Interest} &= PMT \times \frac{\text{Number of days from March 1, 2023 to April 15, 2023}}{\text{Number of days from March 1, 2023 to September 1, 2023}} \\
 &= 101.50 \times \frac{45}{184} \\
 &= \$24.82
 \end{aligned}$$

Calculate the quoted price.

$$\begin{aligned}\text{Quoted Price} &= \text{Flat Price} - \text{Accrued Interest} \\ &= 6,494.50 - 24.82 \\ &= \$6,469.68\end{aligned}$$

Calculate the quoted price as a percentage of face value.

$$\begin{aligned}\text{Percent of Face Value} &= \frac{\text{Quoted Price}}{\text{Face Value}} \times 100\% \\ &= \frac{6,469.68}{7,000} \times 100\% \\ &= 92.42\%\end{aligned}$$

Section 4.5 Exercises

1. A **\$50,000** bond with a **6.33%** coupon rate is redeemable on January 1, 2038. The bond was purchased on May 2, 2023 when the yield rate was **4.19%**.
 - a. Calculate the flat price on the purchase date.
 - b. Calculate the quoted price on the purchase date.
 - c. Calculate the quoted price as a percentage of face value on the purchase date.

Solution

- a. **\$62,689.98**; b. **\$61,632.07**; c. **123.26%**
2. A **30**—year, **\$2,500** bond with a **10.4%** coupon rate was issued on November 1, 2006. The bond was purchased on September 13, 2023 when the yield to maturity was

12.39%.

- a. Calculate the flat price on the purchase date.
- b. Calculate the quoted price on the purchase date.
- c. Calculate the quoted price as a percentage of face value on the purchase date.

Solution

a. \$2,275.89; b. \$2,180.51; c. 87.22%

3. A \$15,000 bond with a 9.75% coupon was purchased on January 2, 2023 when the yield rate was 10.1%. The bond matures on April 1, 2040.
 - a. Calculate the flat price on the purchase date.
 - b. Calculate the quoted price on the purchase date.
 - c. Calculate the quoted price as a percentage of face value on the purchase date.

Solution

a. \$14,944.40; b. \$14,570.74; c. 97.14%

4. A 20-year, \$10,000 bond was issued on September 1, 2019 with a 2% coupon. The bond was purchased on November 17, 2023 when the yield rate was 5.25%.
 - a. Calculate the flat price on the purchase date.
 - b. Calculate the quoted price on the purchase date.
 - c. Calculate the quoted price as a percentage of face value on the purchase date.

Solution

a. \$6,582.90; b. \$6,540.59; c. 130.81%

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4.6 SINKING FUNDS

Learning Objectives

- Construct a sinking fund schedule

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance
- BV = Book value
- FV = Face value of bond
- PMT = Annuity payment amount
- N = Number of annuity payments
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency

Formulas Used

- Formula 4.1 – **Bond Payment**

$$\text{Payment} = \text{Face Value} \times \text{periodic bond rate}$$

- Formula 4.2 – **Purchase Price on an Interest Payment Date**

$$\text{Purchase Price} = \text{Present Value of Face Value} + \text{Present Value of Bond Payments}$$

- Formula 4.3 – **Premium (Bonds)**

$$\text{Premium} = \text{Purchase Price} - \text{Face Value}$$

- Formula 4.4 – **Discount (Bonds)**

$$\text{Discount} = \text{Face Value} - \text{Purchase Price}$$

- Formula 4.5 – **Quoted Price (Bonds)**

$$\text{Flat Price} - \text{Accrued Interest}$$

- Formula 4.6 – **Quoted Price as a Percentage of Face Value**

$$\frac{\text{Quoted Price}}{\text{Face Value}} \times 100\%$$

- Formula 4.7 – **Periodic Cost of Debt**

$$\text{Sinking Fund Payment} + \text{Periodic Interest Payment}$$

- Formula 4.8 – **Book Value**

$$\text{Loan Amount} - \text{Balance}$$

Introduction

A **sinking fund** is a special account into which an investor, either an individual or a business, makes annuity payments so that sufficient funds are on hand by a specified date to meet a future savings goal or debt

obligation. In its simplest terms, a sinking fund is a financial savings place. As the definition indicates, a sinking fund has is used for one of two main purposes:

- **Capital Savings.** When your goal is to acquire some form of a capital asset by the end of the fund, you have a capital savings sinking fund. What is a capital asset? It is any tangible property that is not easily converted into cash. For example, saving up to buy a home, car, warehouse, or even new production machinery are all capital savings.
- **Debt Retirement.** When your goal is to pay off some form of debt by the end of the fund, you have a debt retirement sinking fund. Perhaps as a consumer, if you were able to get a 0% interest plan with no payments for one year, you might want to make monthly payments into your own savings account so that you would have the needed funds to pay off your purchase when it comes due. Businesses usually set up sinking funds for the retirement of stocks, bonds, and debentures.

Sinking Funds and Debt Retirement

Whether the sinking fund is for capital savings or debt retirement, the mathematical calculations and procedures are identical. However, this section will focus on using sinking funds for debt retirement. Now, why discuss sinking funds in the chapter about bonds? Many bonds carry a sinking fund provision. Once the bond has been issued, the company must start regular contributions to a sinking fund because large sums of money have been borrowed over a long time frame, and investors need assurance that the bond issuer will be able to repay its debt upon bond maturity.

To provide further assurance to bondholders, the sinking fund is typically managed by a neutral third party rather than the bond-issuing company. This third-party company ensures the integrity of the fund, working toward the debt retirement in a systematic manner according to the provisions of the sinking fund. Investors much prefer bonds or debentures that are backed by sinking funds and third-party management because they are less likely to default.

Sinking funds are an alternative way to pay off a loan or debt. A sinking fund is used to accumulate the principal only owed on a debt so that the principal of the debt can be repaid in its entirety on the maturity date. For example, sinking funds are used to accumulate the face value of bonds so that money is available to pay the face value at maturity. Sinking funds are NOT used to pay the interest due on the debt. For example, sinking funds are not used to pay the periodic bond payments.

In the case of bonds or debentures, sinking funds are most commonly set up as ordinary simple annuities that match the timing of the bond interest payments. Thus, when a bond issuer makes a bond interest payment to its bondholders, it also makes an annuity payment to its sinking fund. In other applications, any type of annuity is possible, whether ordinary or due, general or simple.

When a sinking fund is used to retire a debt, there are two interest rates associated with the debt.

- ***Interest rate for the loan or debt.*** Because sinking funds are used to pay off loans, bonds, or other debts, there is an interest rate for calculating the amount of interest charged on the loan. For example, if the sinking fund is used to accumulate the face value of a bond, the interest rate on the debt is the coupon rate of the bond. This interest rate is only for calculating the interest due on the loan or debt, and does not have anything to do with the money deposited into the sinking fund.
- ***Interest rate for the sinking fund.*** Because a sinking fund is an interest-earning account, there is an interest rate for calculating the interest earned by the money in the sinking fund.

When a sinking fund is established to retire a debt, there are two different periodic costs or expenses made in relation to the debt.

- ***Periodic interest payments on the debt.*** The interest rate for the loan or debt is used to calculate the periodic interest payment. For example, if a sinking fund is used to accumulate the face value of a bond, the bond issuer still has to make the periodic bond payments to the bond holders. These periodic interest payments are not directly related to the sinking fund, but are a cost associated with the debt.
- ***Periodic payments made to the sinking fund.*** When a periodic interest payment is made on the debt, a payment is made into the sinking fund with the goal of accumulating the loan amount. The interest rate for the sinking fund is used to calculate the periodic annuity payment for the sinking fund. For example, if a sinking fund is used to accumulate the face value of a bond, the bond issuer makes a deposit into the sinking fund at the same time they make the bond payment. Because these periodic sinking fund payments are made as a result of a debt, they are also costs associated with the debt.

The periodic cost of a debt retired with a sinking fund is the total amount paid each payment interval as a result of the debt.

4.7 Periodic Cost of Debt

$$\text{Periodic Cost of the Debt} = \text{Periodic Interest Payment} + \text{Periodic Sinking Fund Payment}$$

The payments made into a sinking fund form an annuity, and are calculated the same as any other annuity

payment with the future value of the sinking fund set to the loan amount and using the interest rate associated with the sinking fund. Because the goal of the sinking fund is to accumulate *at least* the required amount, sinking fund payments are always rounded UP to the next cent. Consequently, all of the payments made to a sinking fund, including the last payment, are the same.

Example 4.6.1

A bank issued a \$10,000,000 face value bond carrying a 5.1% coupon and 30 years until maturity. The bank set up a sinking earning 4.5% to accumulate the face value of the bond.

- Calculate the sinking fund payment.
- Calculate the periodic expense of the debt.

Solution

Step 1: The given information for the sinking fund is

Because no other information is given, the frequency of the payments (for both the bond and the sinking fund) and the compounding frequencies (for the coupon rate and the sinking fund rate) are assumed to be semi-annual.

$$FV = \$10,000,000$$

$$I/Y = 4.5\%$$

$$P/Y = 2$$

$$C/Y = 2$$

$$t = 30 \text{ years}$$

Step 2: Calculate the sinking fund payment.

PMT Setting	END
N	$2 \times 30 = 60$
PV	0
FV	10,000,000
PMT	?
I/Y	4.5
P/Y	2
C/Y	2

$$PMT = \$80,353.2748... \rightarrow \$80,353.28$$

The sinking fund payment is **\$80,353.28**. Remember, sinking fund payments always get rounded UP to the next cent.

Step 3: Calculate the bond payment.

$$\begin{aligned}
 \text{Bond Payment} &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 10,000,000 \times \frac{0.051}{2} \\
 &= \$255,000
 \end{aligned}$$

The bond payments are **\$255,000**.

Step 4: Calculate the periodic cost of the debt.

$$\begin{aligned}
 \text{Periodic Cost of the Debt} &= \text{Periodic Interest Payment} + \text{Periodic Sinking Fund Payment} \\
 &= 255,000 + 80,353.28 \\
 &= \$335,353.28
 \end{aligned}$$

Step 5: Write as a statement.

The periodic cost of the debt is **\$335,353.28**. This means that every six months the bank must pay out a total of **\$335,353.28** because of the debt. Of this amount, **\$255,000** goes to paying the bond payments and **\$80,353.28** goes to the sinking fund to accumulate the **\$10,000,000** face value of the bonds.

Key Takeaways



The goal of a sinking fund is to accumulate the loan amount so that the loan amount can be paid off in one lump-sum payment at the end of the term. So, the loan amount becomes the future value of the sinking fund.

Sinking fund payments always get rounded UP to the next cent. This ensures that the final amount in the sinking fund will be at or over the loan amount. Rounding the payment up guarantees that the balance in the sinking fund at the end of the term will always be at or over the loan amount.

Try It

1) A company issued bonds worth **\$200,000** to raise money to build an expansion to its factory. The bonds have a coupon rate of **3.9%** compounded semi-annually and ten years to maturity. The company established a sinking fund earning **2.7%** compounded semi-annually to accumulate the face value of the bonds.

- a. Calculate the sinking fund payment.
- b. Calculate the periodic expense of the debt.

Solution

- a. Calculate the sinking fund payment.

PMT Setting	END
N	$2 \times 10 = 20$
PV	0
FV	200,000
PMT	?
I/Y	2.7
P/Y	2
C/Y	2

$$PMT = \$8,777.63$$

b. Calculate the periodic expense of the debt.

$$\begin{aligned}
 \text{Bond Payment} &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 200,000 \times \frac{0.039}{2} \\
 &= \$3,900
 \end{aligned}$$

Write as a statement.

$$\begin{aligned}
 \text{Periodic Cost of the Debt} &= \text{Periodic Interest Payment} + \text{Periodic Sinking Fund Payment} \\
 &= 3,900 + 8,777.63 \\
 &= \$12,677.63
 \end{aligned}$$

Sinking Fund Schedules

When a company takes out a loan or issues bonds, these are debts to the company. Through a sinking fund the company saves up money to extinguish that debt. The **book value of the debt** is the difference between the principal amount owing on the debt (i.e. the loan amount or face value of the bond) and the accumulated balance in the sinking fund at any point in time. For example, if the company issued \$10 million in bonds and has accumulated \$1 million in its sinking fund, the book value of the debt is \$9 million.

4.8 Book Value

$$\text{Book Value} = \text{Principal} - \text{Fund Balance}$$

A sinking fund schedule is a table that records the sinking fund contribution, the interest earned by the fund, the increase in the fund, the accumulated balance for every payment, and the current book value of the debt. A sinking fund schedule is very similar to an amortization schedule except that the balance increases instead of decreases and the interest is earned instead of being paid.

A sinking fund schedule has six columns:

- **Payment Number.** There is a row for every payment into the sinking fund.
- **Payment.** The sinking fund payment (PMT). Because the sinking fund payment is rounded up, all of the payments are the same, including the last payment.
- **Interest.** The interest earned by the fund at the end of each payment interval.
- **Increase.** The total amount added to the fund with each payment interval.
- **Balance.** The current amount accumulated in the fund for each payment interval.
- **Book Value.** The book value of the debt.

To fill in a sinking fund schedule, you first need to have all of the details about the fund, including the loan amount (FV), the sinking fund payment (PMT), the number of payments (N), and the sinking fund's interest rate. If any of these quantities are missing, calculate out the missing value before completing the sinking fund schedule.

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				0	Loan Amount ¹
1	PMT^2	INT^3	INC^4	BAL^5	BV^6
2	PMT^2	INT^3	INC^4	BAL^5	BV^6
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$N - 1$	PMT^2	INT^3	INC^4	BAL^5	BV^6
N	PMT^2	INT^3	INC^4	BAL^5	BV^6
Totals	Total Payments ⁸	Total Interest ¹⁰	Total Increase ⁹		

HOW TO

Fill In a Sinking Fund Schedule

Follow these steps to fill in a sinking fund schedule.

Step 1: In row 0, the only entries are in the balance and book value columns. The initial balance is 0 and the initial book value is the loan amount (the future value of the sinking fund).

Step 2: Each entry in the payment column is the sinking fund payment. If you have to calculate out the payment, remember to round the payment up to the next cent. All of the payments in this column are the same, including the last payment.

Step 3: Calculate the interest. The interest is the balance from the previous row times the periodic interest rate:

$$\text{Interest} = \text{Balance from Previous Row} \times i$$

Note: this calculation uses the periodic sinking fund rate, not the periodic interest rate associated with the loan.

Step 4: Calculate the increase. The increase is the sum of the payment and the interest:

$$\text{Increase} = PMT + \text{Interest}.$$

Step 5: Calculate the new balance. The balance is the sum of the balance in the previous row and the increase:

$$\text{Balance} = \text{Balance from Previous Row} + \text{Increase}$$

Step 6: Calculate the new book value. The book value is the difference between the book value from the previous row and the increase:

$$\text{Book Value} = \text{Book Value from Previous Row} - \text{Increase}.$$

Step 7: For each payment, repeat steps 2 through 6, including for the last row.

Step 8: The total payments is the sum of the payment column:

$$\text{Total Payments} = N \times PMT.$$

Step 9: The total increase is the sum of the increase column, and is the last balance entry:

$$\text{Total Increase} = \text{Balance in Last Row}.$$

Step 10: The total interest is the sum of the interest column, and equals the difference between the other two column totals:

$$\text{Total Interest} = \text{Total Payments} - \text{Total Increase}.$$



The manual calculation of the interest entry above is based on the assumption that the payment frequency and the compounding frequency are equal. If the payment frequency and the compounding frequency are not equal, an interest conversion would be required to convert the interest rate to the equivalent rate with the compounding frequency equal to the payment frequency. However, if you use the TI BAII Plus's built-in amortization worksheet (described below), no interest conversion is required.

As you fill in the schedule, round the entries to two decimal places.

The sinking fund schedule presented here assumes the payments are made at the end of the payment interval. That is, the sinking fund schedule presented above is for an ordinary annuity. If the sinking fund is an annuity

due (payments at the beginning), the calculations are the same except for the interest column, where the interest is based on both the balance from the previous row and the payment.

Example 4.6.2

A company has to repay a \$20,000 loan in two years. The company establishes a sinking fund earning 4% compounded semi-annually and makes end-of-six-month payments into the fund to accumulate the loan amount. Construct the sinking fund schedule.

Solution

Step 1: The given information is

$$FV = \$20,000$$

$$I/Y = 4\%$$

$$P/Y = 2$$

$$C/Y = 2$$

$$t = 2 \text{ years}$$

Step 2: Calculate the sinking fund payment.

PMT Setting	END
N	$2 \times 2 = 4$
PV	0
FV	20,000
PMT	?
I/Y	4
P/Y	2
C/Y	2

$$PMT = \$4,852.48$$

The sinking fund payment is \$4,852.48.

Step 3: Complete the sinking fund schedule.

Because the payment frequency and the compounding frequency are equal, no interest conversion is required. The calculations for each entry are shown in blue. The periodic

interest rate is $i = \frac{4\%}{2} = 2\%$.

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				\$0	\$20,000
1	\$4,852.48	\$0 (0×0.02)	\$4,852.48 ($4,852.48 + 0$)	\$4,852.48 ($0 + 4,852.48$)	\$15,147.52 ($20,000 - 4,852.48$)
2	\$4,852.48	\$97.05 (4582.48×0.02)	\$4,949.53 ($4852.48 + 97.05$)	\$9,802.01 ($4852.48 + 4949.53$)	\$10,197.99 ($15,147.52 - 4949.53$)
3	\$4,852.48	\$196.04 (9802.01×0.02)	\$5,048.52 ($4852.48 + 196.04$)	\$14,850.5 ($9802.01 + 5048.52$)	\$5,149.47 ($10,197.99 - 5048.52$)
4	\$4,852.48	\$297.01 ($14,850.53 \times 0.02$)	\$5,149.49 ($4852.48 + 297.01$)	\$20,000.02 ($14,850.53 + 5149.49$)	-\$0.02 ($5149.47 - 5149.49$)
Totals	\$19,409.92 (4×4852.48)	\$590.10 ($20,000.02 - 19,409.92$)	\$20,000.02		

Things to Watch Out For

In the previous example, the final balance is slightly more than the required **\$20,000** because the sinking fund payment was rounded up to the next cent. By rounding the payment up, we ensure that the sinking fund has at least **\$20,000** at the end of the term.

Although the calculations in a sinking fund schedule are relatively straightforward, the manual calculations are time-consuming, especially when the schedule has a lot of rows. The amortization worksheet on a financial calculator, such as the TI BAII Plus, can be used to quickly calculate the entries for each row of the schedule.

Using the TI BAII Plus Calculator to Construct a Sinking Fund Schedule

To use the amortization worksheet to complete a sinking fund schedule:

1. Solve for any unknown quantities about the sinking fund. You need to know all of the information about the sinking fund first before you can use the amortization worksheet.
2. Enter all the value of all seven time value of money variables into the calculator (N , PV , FV , PMT , I/Y , P/Y , C/Y). If you calculated the payment in the first step, you must *re-enter* it rounded up to the next cent and with the correct cash flow sign. Make sure the payment setting is set to **END**, and obey the cash flow sign convention. Because PMT (the sinking fund payment) is paid out to the fund, PMT is negative. At the end of the term, FV is received, so FV is positive.
3. Go to the amortization worksheet by pressing **2nd AMORT** (the PV button).
4. To view the entries for a specific row of the schedule, set P_1 and P_2 to the row number. For example, to view the entries for row 5, set $P_1 = 5$ and $P_2 = 5$.
 - i. At the P_1 prompt, enter the row number and press **ENTER**.
 - ii. Press the down arrow.
 - iii. At the P_2 prompt, enter the row number and press **ENTER**.
 - iv. Press the down arrow.
 - v. The BAL entry is the balance entry for the corresponding row.
 - vi. Press the down arrow.
 - vii. The PRN entry is the increase entry for the corresponding row.
 - viii. Press the down arrow.

- ix. The ***INT*** entry is the interest entry for the corresponding row.
 - x. Press the down arrow the return to the ***P₁*** screen.
5. Repeat the previous step with a different row number to view the entries for a different row.

Key Takeaways



On the amortization worksheet, ***BAL*** is the balance entry, ***PRN*** is the increase entry, and ***INT*** is the interest entry.

You cannot get the entries for the last column, the book value, from the amortization worksheet on the calculator. This entry will still need to be calculated manually. You can find the book value for any row by subtracting the balance for the row from the loan amount:

$$\text{Book Value} = \text{Principal} - \text{Fund Balance}$$

Make sure to re-enter ***PMT*** rounded up to the next cent before using the amortization worksheet. Otherwise, you will not get the correct entries for the sinking fund schedule.

As you read the entries off of the amortization worksheet on the calculator and put them in the schedule, round the entries to **2** decimal places.

Example 4.6.3

A company set up a sinking fund to accumulated the **\$30,000** they need to repay a loan. The sinking fund earns **3.5%** compounded semi-annually. The company made semi-annual deposits into the sinking fund for **2.5** years. Construct the sinking fund schedule.

Solution

Step 1: Calculate the sinking fund deposit.

PMT Setting	END
N	$2 \times 2.5 = 5$
PV	0
FV	30,000
PMT	?
I/Y	3.5
P/Y	2
C/Y	2

$$PMT = \$5,793.65$$

Step 2: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	5
PV	0
FV	30,000
PMT	-5,793.65
I/Y	3
P/Y	2
C/Y	2

Step 4: Complete the sinking fund schedule using the amortization worksheet on the calculator.

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				\$0	\$30,000
1	\$5,793.65	\$0	\$5,793.65	\$5,793.65	\$24,206.35
2	\$5,793.65	\$101.39	\$5,895.04	\$11,688.69	\$18,311.31
3	\$5,793.65	\$204.55	\$5,998.20	\$17,686.89	\$12,313.11
4	\$5,793.65	\$309.52	\$6,103.17	\$23,790.06	\$6,209.94
5	\$5,793.65	\$416.33	\$6,209.98	\$30,000.04	-\$0.04
Totals	\$28,968.25	\$1,031.79	\$30,000.04		

- **Row 1:** In the amortization worksheet, set $P_1 = 1$ and $P_2 = 1$. The entry for the last column (the book value) is $30,000 - 5,793.65$.
- **Row 2:** In the amortization worksheet, set $P_1 = 2$ and $P_2 = 2$. The entry for the last column (the book value) is $30,000 - 11,688.69$.
- **Row 3:** In the amortization worksheet, set $P_1 = 3$ and $P_2 = 3$. The entry for the last column (the book value) is $30,000 - 17,686.89$.

- **Row 4:** In the amortization worksheet, set $P_1 = 4$ and $P_2 = 4$. The entry for the last column (the book value) is $30,000 - 23,790.06$.
- **Row 5:** In the amortization worksheet, set $P_1 = 5$ and $P_2 = 5$. The entry for the last column (the book value) is $30,000 - 30,000.04$.
- **Totals Row:**
 - The increase total is the last balance entry ($\$30,000.04$).
 - The payments total is the sum of the payments: $5 \times 5,793.65 = 28,968.25$.
 - The interest total is the difference in the other two column totals:
 $30,000.04 - 28,968.25 = 1,031.79$.

Try It

2) A **\$10,000** loan was repaid using a sinking fund that was earning **4.5%** compounded semi-annually. Deposits were made every six months into the fund for three years to accumulate the loan amount. Construct the sinking fund schedule.

Solution

PMT Setting	END
N	$2 \times 3 = 6$
PV	0
FV	10,000
PMT	?
I/Y	4.5
P/Y	2
C/Y	2

$$PMT = \$1,575.35$$

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				\$0	\$10,000
1	\$1,575.35	\$0	\$1,575.35	\$1,575.35	\$8,424.65
2	\$1,575.35	\$35.45	\$1,610.80	\$3,186.15	\$6,813.85
3	\$1,575.35	\$71.69	\$1,647.04	\$4,833.18	\$5,166.82
4	\$1,575.35	\$108.75	\$1,684.10	\$6,517.28	\$3,482.72
5	\$1,575.35	\$146.64	\$1,721.99	\$8,239.27	\$1,760.73
6	\$1,575.35	\$185.38	\$1,760.73	\$10,000	\$0
Totals	\$9,452.10	\$547.90	\$10,000		

Other Sinking Fund Calculations

Similar to working with loans, the amortization worksheet on the financial calculator can be applied to sinking funds to find a partial sinking fund schedule, to find the total interest or the total increase for a series of payments, to find the balance in the fund after any payment, or to find the book value after any payment.

Example 4.6.4

A company took out a **\$25,000** loan and establishes a sinking fund earning **2.7%** compounded semi-annually to accumulate the loan amount. The company makes semi-annual payments into the fund for ten years. Construct a partial sinking fund schedule showing the details of payment #7, the last two payments and the totals.

Solution

Step 1: Calculate the sinking fund deposit.

PMT Setting	END
N	$2 \times 10 = 20$
PV	0
FV	25,000
PMT	?
I/Y	2.7
P/Y	2
C/Y	2

$$PMT = \$1,097.21$$

Step 2: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	20
PV	0
FV	25,000
PMT	-1,097.21
I/Y	2.7
P/Y	2
C/Y	2

Step 3: Complete the partial sinking fund schedule using the amortization worksheet on the calculator.

Payment Number	Payment	Interest	Increase	Balance	Book Value
7	\$1,097.21	\$91.93	\$1,189.14	\$7,988.62	\$17,011.38
19	\$1,097.21	\$299.54	\$1,396.75	\$23,584.57	\$1,415.43
20	\$1,097.21	\$318.39	\$1,415.60	\$25,000.17	-\$0.17
Totals	\$21,944.20	\$3,055.97	\$25,000.17		

- **Row 7:** In the amortization worksheet, set $P_1 = 7$ and $P_2 = 7$.
- **Row 19:** In the amortization worksheet, set $P_1 = 19$ and $P_2 = 19$.
- **Row 20:** In the amortization worksheet, set $P_1 = 20$ and $P_2 = 20$.
- **Book Value for each Row:** Book Value = 25,000 – Balance
- **Totals Row:**
 - The increase total is the last balance entry (\$25,000.17).
 - The payments total is the sum of the payments: $20 \times 1,097.21 = 21,944.20$.
 - The interest total is the difference in the other two column totals:
 $25,000.17 - 21,944.20 = 3,055.97$.

Using the TI BAII Plus Calculator for Other Sinking Fund Calculations

To use the amortization worksheet to find the total interest, the total increase or the balance for a sinking fund:

1. Solve for any unknown quantities about the sinking fund. You need to know all of the information about the sinking fund first before you can use the amortization worksheet.
2. Enter all the value of all seven time value of money variables into the calculator (N , PV , FV , PMT , I/Y , P/Y , C/Y). If you calculated the payment in the first step, you must *re-enter* it rounded up to the next cent and with the correct cash flow sign. Make sure the payment setting is set to **END**, and obey the cash flow sign convention. Because PMT (the sinking fund payment) is paid out to the fund, PMT is negative. At the end of the term, FV is received, so FV is positive.
3. Go to the amortization worksheet by pressing **2nd AMORT** (the PV button).
4. To find the balance in the fund after a payment, set P_1 and P_2 equal to the payment number. The BAL entry is the balance after payment number P_2 . Note that the BAL entry is only tied to the value of P_2 and does not depend on the value of P_1 .
5. To view the total increase or the total interest for a series of payments, set P_1 to the *first* payment number of the series of payments and set P_2 to the *last* payment number of the series of payments. For example, to view the total increase or the total interest for payments four to seven, set $P_1 = 4$ and $P_2 = 7$. In the outputs from the amortization worksheet:
 - The PRN entry is the total increase from payment number P_1 to payment number P_2 .
 - The INT entry is the total interest from payment number P_1 to payment number P_2 .



Key Takeaways

The *PRN* entry on the amortization worksheet is the sum of the increase entries in the sinking fund schedule starting at payment number P_1 and ending at payment number P_2 . For example, if $P_1 = 4$ and $P_2 = 7$, the *PRN* entry tells you the sum of the increase column in the sinking fund schedule starting with payment number 4 and ending with payment number 7.

The *INT* entry on the amortization worksheet is the sum of the interest entries in the sinking fund schedule starting at payment number P_1 and ending at payment number P_2 . For example, if $P_1 = 4$ and $P_2 = 7$, the *INT* entry tells you the sum of the interest column in the sinking fund schedule starting with payment number 4 and ending with payment number 7.

The calculator thinks in terms of payment numbers, not years. That is, P_1 and P_2 must be payment numbers.

Example 4.6.5

A company issued \$500,000 worth of bonds with 15 years to maturity. The company set up a sinking fund earning 4.7% compounded semi-annually to accumulate the face value of the bonds and made semi-annual payments into the sinking fund.

- What is the sinking fund payment?
- What was the balance in the fund after payment 18?

- c. By how much did the sinking fund increase with payment **9**?
- d. How much interest was paid in year **10**?
- e. What was the book value in the fund at the end of year **8**?

Solution

Step 1: Calculate the payment.

PMT Setting	END
N	$12 \times 15 = 30$
PV	0
FV	500,000
PMT	?
I/Y	4.7
P/Y	2
C/Y	2

$$PMT = \$11,663.61$$

Step 2: Calculate the balance for payment 18.

To find the balance for payment **18**, set $P_1 = 18$ and $P_2 = 18$.

PMT Setting	END
N	30
PV	0
FV	500,000
PMT	-11,663.61
I/Y	4.7
P/Y	2
C/Y	2
P_1	18
P_2	18

$$BAL = \$257,632.82$$

After 18 payments, the balance in the fund is \$257,632.82.

Step 3: Calculate the increase for payment 9.

To find the interest for payment 9, set $P_1 = 9$ and $P_2 = 9$.

PMT Setting	END
N	30
PV	0
FV	500,000
PMT	-11,663.61
I/Y	4.7
P/Y	2
C/Y	2
P_1	9
P_2	9

$$PRN = \$14,045.45$$

The increase for payment 9 is \$14,045.45.

Step 4: Calculate the interest paid for year ten.

To find the interest paid in year two, set P_1 to first payment number of year ten and P_2 to last payment of year ten. There are 2 payments a year, so the last payment made in year two is 20 (2×10). The first payment made in year ten is 19 ($2 \times 9 + 1$). So, to find the interest paid in year ten, set $P_1 = 19$ and $P_2 = 20$.

PMT Setting	END
N	30
PV	0
FV	500,000
PMT	-11,663.61
I/Y	4.7
P/Y	2
C/Y	2
P_1	10
P_2	20

$$INT = \$12,525.11$$

The interest paid in year ten is **\$12,525.11**.

Step 5: Calculate the book value after eight years.

To find the book value, calculate the balance after eight years and then subtract the balance from **\$500,000**. To find the balance after eight years, enter the payment number that corresponds to the last payment made in year eight. There are **2** payments a year, so the last payment made in year eight is **16** (8×2). So, to find the balance after eight years, set $P_1 = 16$ and $P_2 = 16$.

PMT Setting	END
N	30
PV	0
FV	500,000
PMT	-11,663.61
I/Y	4.7
P/Y	2
C/Y	2
P_1	16
P_2	16

$$BAL = \$223,407.95$$

$$\begin{aligned}\text{Book Value} &= 500,000 - 223,407.95 \\ &= \$276,592.05\end{aligned}$$

Step 6: Write as a statement.

After eight years, the book value in the fund is **\$276,592.05**.

Things to Watch Out For

A common mistake occurs in translating years into payment numbers. You often need to find the total interest or the total increase in the sinking fund for a particular year. To do this, you need to set P_1 equal to the number of the first payment that occurs in that year and P_2 equal to the number of the last payment that occurs in that year.

For example, suppose you have monthly payments and you want to know the total interest in the fourth year. In error, you might calculate that the fourth year begins with payment **36** and

ends with payment 48, and so enter $P_1 = 36$ and $P_2 = 48$. But the 36^{th} payment is actually the last payment of the third year. The first payment to occur in year four is the 37^{th} . So, if you wanted to find the total interest in year 4, $P_1 = 37$ and $P_2 = 48$.

When you need to find the first payment number and last payment number for a particular year, there are two methods you can use to calculate the correct payment numbers.

1. Calculate the payment number at the end of the year in question by multiplying P/Y by the year number: $P/Y \times \text{year number}$. To find the payment number at the start of the year, subtract the payment frequency less one ($P/Y - 1$) from the last payment number of the year. For example, suppose the you have monthly payments and want to find the interest paid for year 4. The last payment in year 4 is $12 \times 4 = 48$. The first payment in year 4 is $48 - (12 - 1) = 37$.
2. Calculate the payment number at the end of the year in question by multiplying P/Y by the year number: $P/Y \times \text{year number}$. To find the payment number for the first payment in the year, multiply P/Y by the previous year number and then add one to it: $P/Y \times (\text{year} - 1) + 1$. For example, suppose the you have monthly payments and want to find the interest paid for year 4. The first payment in year 4 is $12 \times 3 + 1 = 37$. The last payment of the fourth year remains at payment 48.

Try It

3) A company sold \$100,000 worth of bonds and set up a sinking fund earning 4% compounded semi-annually to retire the bonds in nine years. The company made semi-annual deposits into the sinking fund.

- a. What is the sinking fund payment?
- b. How much interest did the sinking fund earn with the 10th payment?

- c. By how much did the sinking fund increase with the 7th payment?
- d. What is the balance on the loan after eight years?
- e. How much interest did the sinking fund earn in year three?
- f. By how much did the sinking fund increase in year six?
- g. What is the book value in the fund after year five?

Solution

- a. Calculate the payment.

PMT Setting	END
N	$2 \times 9 = 18$
PV	0
FV	100,000
PMT	?
I/Y	4
P/Y	2
C/Y	2

$$PMT = \$4,670.22$$

- b. Calculate the interest for the 10th payment.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	-4,670.22
I/Y	4
P/Y	2
C/Y	2
P_1	10
P_2	10

$$INT = \$911.13$$

c. Calculate the increase for the 7th payment.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	-4,670.22
I/Y	4
P/Y	2
C/Y	2
P_1	7
P_2	7

$$PRN = \$5,259.43$$

d. Calculate the balance at the end of year 8.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	-4,670.22
I/Y	4
P/Y	2
C/Y	2
P_1	16
P_2	16

$$BAL = \$87,049.56$$

e. Calculate the interest for year 3.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	-4,670.22
I/Y	4
P/Y	2
C/Y	2
P_1	5
P_2	6

$$INT = \$871.06$$

f. Calculate the increase for year **6**.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	−4,670.22
I/Y	4
P/Y	2
C/Y	2
P_1	11
P_2	12

$$PRN = \$11,499.80$$

g. Calculate the book value after year **5**.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	-4,670.22
I/Y	4
P/Y	2
C/Y	2
P_1	10
P_2	10

$$BAL = \$51,137.61$$

$$\begin{aligned}\text{Book Value} &= 100,000 - 51,137.61 \\ &= \$48,862.39\end{aligned}$$

Section 4.6 Exercises

1. A company set up a sinking fund earning **6.2%** effective to accumulate **\$10,000** to repay a loan. The company makes quarterly payments into the sinking fund for two years. Construct the sinking fund schedule for the loan.

Solution

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				\$0	\$10,000
1	\$1,185.21	\$0	\$1,185.21	\$1,185.21	\$8,814.79
2	\$1,185.21	\$17.96	\$1,203.17	\$2,388.38	\$7,611.62
3	\$1,185.21	\$36.19	\$1,221.40	\$3,609.78	\$6,390.22
4	\$1,185.21	\$54.70	\$1,239.91	\$4,849.68	\$5,150.32
5	\$1,185.21	\$73.48	\$1,258.69	\$6,108.38	\$3,891.62
6	\$1,185.21	\$92.55	\$1,277.76	\$7,386.14	\$2,613.86
7	\$1,185.21	\$111.92	\$1,297.13	\$8,683.27	\$1,316.73
8	\$1,185.21	\$131.57	\$1,316.78	\$10,000.05	—\$0.05
Totals	\$9,481.68	\$518.37	\$10,000.05		

2. A company issued \$5,000,000 worth of bonds and set up a sinking fund earning 7% compounded semi-annually to retire the bonds in six years. The company made semi-annual payments into the fund. Construct a sinking fund schedule showing the details of third year, the last two payments and the totals.

Solution

Payment Number	Payment	Interest	Increase	Balance	Book Value
5	\$342,419.75	\$50,514.79	\$392,934.54	\$1,836,214.23	\$3,163,785.77
6	\$342,419.75	\$64,267.50	\$406,687.25	\$2,242,901.47	\$3,757,098.53
11	\$342,419.75	\$140,597.13	\$483,016.88	\$4,500,077.59	\$49,922.41
12	\$342,419.75	\$157,502.72	\$499,922.47	\$5,000,000.05	—\$0.05
Totals	\$4,109,037	\$890,963.05	\$5,000,000.05		

3. A company borrowed \$800,000 to expand their factory. They establish a sinking fund earning 5% compounded quarterly to accumulate the loan amount in 8 years. The company made monthly payments into the sinking fund. Construct a sinking fund schedule showing the details of payment 37, the last two payments and the totals.

Solution

Payment Number	Payment	Interest	Increase	Balance	Book Value
37	\$6,800.52	\$1,093.21	\$7,893.73	\$271,355.37	\$528,644.63
95	\$6,800.52	\$3,236.08	\$10,036.60	\$789,922.04	\$10,077.96
96	\$6,800.52	\$3,277.72	\$10,078.24	\$800,000.29	−\$0.29
Totals	\$652,849.92	\$890,963.05	\$5,000,000.05		

4. A \$500,000 bond with a 6.5% coupon is redeemable in ten years. The bond comes with a sinking fund that requires deposits timed to match the bond payments. The sinking fund earns 4% compounded semi-annually.
- What is the sinking fund payment?
 - What is the periodic expense of the debt?
 - What is the balance in the fund after two years?
 - How much interest did the fund earn with the 17th payment?
 - By how much did the fund increase with the 8th payment?
 - How much interest did the fund earn in year nine?
 - By how much did the fund increase in year six?
 - What is the book value at the end of year seven?

Solution

a. \$20,578.40; b. \$36,828.40; c. \$84,816.10; d. \$7,671.33; e. \$23,638.11; f. \$15,907.66; g. \$50,671.61; h. \$171,281.91

5. A **\$125,000** face value bond is issued with **15** years to maturity and a coupon rate of **5%**. A sinking fund is established to accumulate the face value of the bond. The sinking fund earns **7.4%** compounded semi-annual and the payments are time to match the bond payments.

- What is the sinking fund payment?
- What is the periodic expense of the debt?
- What is the balance in the fund half way through the term?
- How much interest did the fund earn with the **20th** payment?
- By how much did the fund increase with the **10th** payment?
- How much interest did the fund earn in year seven?
- By how much did the fund increase in year three?
- What is the book value at the end of year twelve?

Solution

a. \$2,342.79; b. \$5,467.79; c. \$45,878.92; d. \$2,329.50; e. \$3,248.94; f. \$2,694.64; g. \$5,518.73; h. \$36,885.21

6. A company borrowed **\$750,000** at **3.8%** compounded quarterly which required them to make quarterly interest payments. The company set up a sinking fund earning **4.9%** compounded quarterly to retire the debt in eight years. The company makes quarterly payments into the sinking fund at the same time they make the quarterly interest payments on the loan.

- What is the sinking fund payment?
- What is the periodic expense of the debt?

- c. What is the balance in the fund at the end of year five?
- d. How much interest did the fund earn with the **13**th payment?
- e. By how much did the fund increase with the **25**th payment?
- f. How much interest did the fund earn in year four?
- g. By how much did the fund increase in year six?
- h. What is the book value at the end of year two?

Solution

a. \$19,284.57; b. \$33,534.57; c. \$434,053.97; d. \$3,033.85; e. \$25,829.55; f. \$13,789.23; g. \$100,229.96; h. \$588,944.27

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4.7 REVIEW EXERCISES

Chapter 4 Review Exercises

1. A \$100,000 face value bond has a 5.03% coupon. The bond is purchased when there were 13.5 years to maturity and the yield to maturity was 4.29%.
 - a. What is the purchase price of the bond?
 - b. What is the premium or discount?

Solution

a. \$107,523.91; b. Premium=\$7,523.91

2. Greg purchased a \$5,000 bond with a 4.3% coupon when there was 12 years to maturity and the yield to maturity was 5.7%. Four years later, Greg sold the bond when the yield rate was 3.9%.
 - a. What did Greg pay to purchase the bond?
 - b. What is the premium or discount?
 - c. At what price did Greg sell the bond?
 - d. What was Greg's gain or loss on the sale of the bond?

Solution

a. \$4,397.56; b. Discount=\$602.44; c. \$5,136.32; d. Gain=\$738.76

3. A 25-year, \$50,000 bond with a 5.75% coupon was issued on June 1, 2008. The bond

was purchased on November 21, 2019 when the yield to maturity was **3.85%**.

- a. Calculate the flat price on the purchase date.
- b. Calculate the quoted price as a percentage of face value.

Solution

a. **\$61,302.61**; b. **119.89%**

4. A **\$35,000** bond with a **7%** coupon is purchased for **132.92** when there was **12** years to maturity. Calculate the yield rate.

Solution

3.6%

5. An investor purchases a **\$55,000** bond with a **4%** coupon for **\$33,227.95** when there was **10.5** years to maturity. After **4** years, the investor sold the bond for **\$60,231.63**.

- a. Calculate the yield to maturity on the purchase date.
- b. Calculate the yield to maturity on the selling date.
- c. Calculate the investor's rate of return when the bond was sold.

Solution

a. **10.24%** b. **2.41%**; c. **20.66%**

6. A **\$17,000** bond with a **3.8%** coupon has a maturity date of October 1, 2035. The bond is purchased on May 30, 2023 when the yield to maturity was **4.19%**.

- a. Calculate the flat price on the purchase date.
- b. Calculate the quoted price as a percentage of face value.

Solution

a. **\$16,469.68**; b. **96.27%**

7. A **\$6,500** bond with a **6.7%** coupon is purchased three years before maturity when

the yield rate was 8%. Construct the appropriate bond schedule for the bond.

Solution

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$6,278.52	\$221.48
1	\$217.75	\$251.14	\$33.39	\$6,311.91	\$188.09
2	\$217.75	\$252.48	\$34.73	\$6,346.64	\$153.36
3	\$217.75	\$253.87	\$36.12	\$6,382.75	\$117.25
4	\$217.75	\$255.31	\$37.56	\$6,420.31	\$79.69
5	\$217.75	\$256.81	\$39.06	\$6,459.38	\$40.62
6	\$217.75	\$258.38	\$40.63	\$6,500	\$0
Totals	\$1,306.50	\$1,527.98	\$221.48		

8. A \$4,000 bond with a 5.2% coupon is purchase two years before maturity when the yield rate was 3.7%. Construct the appropriate bond schedule for the bond.

Solution

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$4,114.65	\$114.65
1	\$104	\$76.12	\$27.88	\$4,086.77	\$86.77
2	\$104	\$75.61	\$28.39	\$4,058.38	\$58.38
3	\$104	\$75.08	\$28.91	\$4,029.46	\$29.46
4	\$104	\$74.54	\$29.46	\$4,000	\$0
Totals	\$416	\$301.35	\$114.65		

9. A **\$62,000** face value bond carrying an **8.88%** coupon is purchased on July 15, 2023. The bond matures on November 1, 2037. At the time of purchase, the market rate on the bond was **4.44%**.
- Calculate the flat price on the purchase date.
 - Calculate the quoted price.

Solution

a. **\$92,021.62**; b. **\$90,899.55**

10. A company with a **\$150,000** debt establishes a sinking fund earning **3.89%** compounded quarterly to retire the debt in full in ten years. The company makes quarterly payments into the sinking fund. Construct a partial sinking fund schedule showing the details of the payments in year three, the last two payments and the totals.

Solution

Payment Number	Payment	Interest	Increase	Balance	Book Value
9	\$3,085.78	\$248.41	\$3,334.19	\$28,877.23	\$121,122.77
10	\$3,085.78	\$280.83	\$3,366.61	\$32,243.84	\$117,756.16
11	\$3,085.78	\$313.57	\$3,399.35	\$35,643.19	\$114,356.81
12	\$3,085.78	\$346.63	\$3,432.41	\$39,075.60	\$110,924.40
39	\$3,085.78	\$1,371.64	\$4,457.42	\$145,499.62	\$4,500.38
40	\$3,085.78	\$1,414.98	\$4,500.76	\$150,000.38	—\$0.38
Totals	\$123,431.20	\$26,569.18	\$150,000.38		

11. A **\$300,000** face value bond with a **4%** coupon is issued with **15** years to maturity. A sinking fund earning **6.35%** compounded semi-annually is set-up to accumulate the face value of the bonds.
- Calculate sinking fund payment.

- b. What is the periodic expense of the debt?
- c. How much interest does the fund earn with the 17^{th} payment?
- d. What is the book value of the fund after the 9^{th} payment?
- e. What is the balance in the fund after three years?
- f. By how much does the fund increase with the 23^{rd} payment?
- g. On which payment does fund reach the half way point to the \$300,000?
- h. By how much does the fund increase in the 10^{th} year?
- i. How much interest does the fund earn in the 12^{th} year?
- j. What is the book value of the fund after the 6^{th} year?

Solution

- a. \$6,129.04; b. \$12,129.04; c. \$3,977.07; d. \$237,288.64; e. \$39,819.74; f. \$12,190.77; g. 19; h. \$21,857.64; i. \$12,510.51; j. \$212,146.63
12. A company borrowed \$400,000 and set up a sinking fund earning 3.7% compounded semi-annually to retire the debt in seven years. The company made monthly deposits into the fund and rounded the payment up to the next dollar.
- a. Calculate sinking fund payment.
 - b. What is the balance in the fund at the half way point?
 - c. How much interest does the fund earn in year five?
 - d. Calculate the amount by which the sinking fund increased in year two.
 - e. What is the book value in the fund at the end of year three?
 - f. What is the final balance in the fund?
 - g. What is the total increase in the fund?

h. What is the total amount of interest accumulated by the fund?

Solution

a. \$4,184; b. \$187,214.25; c. \$8,918.61; d. \$52,968.39; e. \$241,023.63; f. \$400,060.56; g. \$400,060.56; i. \$48,604.56;

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CHAPTER 4 SUMMARY

Key Concepts

- **4.1 Bond Terminology**

- Bond definition, characteristics, and key terminology.

- **4.2 Calculating the Purchase Price of a Bond on an Interest Payment Date**

- Calculating a marketable bond price when the selling date is the interest payment date.
- Calculating premiums and discounts.

- **4.3 Calculating the Yield Rate**

- Calculating the yield on the bond to maturity.
- Calculating the yield realized by an investor if a bond is purchased and sold before maturity.

- **4.4 Bond Schedules**

- How to amortize a bond premium or accrue a bond discount.

- **4.5 Quotation of Bonds**

- Calculating a marketable bond price when the selling date is a noninterest payment date.
- Calculating the quoted price of a bond.

- **4.6 Sinking Funds**

- Sinking funds and their purposes.
- How to construct complete ordinary sinking fund schedules.

- How to construct partial ordinary sinking fund schedules.
- Other sinking fund calculations.

Glossary of Terms

Accrued Interest. The amount of interest that the bond has earned but has not yet paid out since the previous interest payment date.

Bond. A debt that is secured by a specific corporate asset and that establishes the issuer's responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date.

Book Value of the Debt. The difference between the principal amount owing on the bond and the accumulated balance in the sinking fund at any point in time.

Capital Gain. The amount by which the current value of an asset exceeds the original purchase price.

Capital Loss. The amount by which the current value of an asset falls short of the original purchase price.

Coupon Rate. Also known as the bond rate or nominal rate, this is the nominal interest rate paid on the face value of the bond.

Debenture. A debt that is not secured by a specific corporate asset and that establishes the issuer's responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date.

Discount. The amount by which a bond's selling price falls short of its face value.

Face Value. Also called the par value or denomination of the bond, this is the principal amount of the debt that the investor lent to the bond-issuing corporation.

Issue Date. The date that a bond is issued and available for purchase by creditors.

Market Quotation. The purchase price expressed as a percentage of face value.

Market Rate. This is the prevailing nominal rate of interest in the open bond market.

Maturity Date. Also known as the redemption date or due date, this is the day upon which the redemption price will be paid to the bondholder (along with the final interest payment), thereby extinguishing the debt.

Periodic Cost of the Debt. The periodic total of the interest payments and the sinking fund payments.

Premium. The amount by which a bond's selling price exceeds its face value.

Purchase Price. Also known as the cash price or flat price, this is the amount of money an investor must directly pay out to acquire the bond. It represents the total of the market price and the accrued interest.

Quoted Price. This is the actual value of the bond excluding any accrued interest.

Redemption Price. Also called the redemption value or maturity value, this is the amount the bond issuer will pay to the bondholder upon maturity of the bond.

Selling Date. This is the date that a bond is actively traded and sold to another investor through the bond markets.

Sinking Fund. A special account into which an investor, whether an individual or a business, makes annuity payments such that sufficient funds will be on hand by a specified date to meet a future savings goal or debt obligation.

Sinking Fund Schedule. A table that shows the sinking fund contribution, interest earned, and the accumulated balance for every payment in the annuity.

Yield to Maturity. A bond's overall rate of return when purchased at a market price and held until maturity. It includes both the semi-annual interest that the bondholders earn on their investment along with the gain or loss resulting from the difference between the market price and the redemption price.

Formula & Symbol Hub

Symbols Used

- BAL = Principal balance
- BV = Book value
- FV = Face value of bond
- PMT = Annuity payment amount
- N = Number of annuity payments
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency

Formulas Used

- Formula 4.1 – **Bond Payment**

$$\text{Payment} = \text{Face Value} \times \text{periodic bond rate}$$

- Formula 4.2 – **Purchase Price on an Interest Payment Date**

$$\text{Purchase Price} = \text{Present Value of Face Value} + \text{Present Value of Bond Payments}$$

- Formula 4.3 – **Premium (Bonds)**

$$\text{Premium} = \text{Purchase Price} - \text{Face Value}$$

- Formula 4.4 – **Discount (Bonds)**

$$\text{Discount} = \text{Face Value} - \text{Purchase Price}$$

- Formula 4.5 – **Quoted Price (Bonds)**

$$\text{Flat Price} - \text{Accrued Interest}$$

- Formula 4.6 – **Quoted Price as a Percentage of Face Value**

$$\frac{\text{Quoted Price}}{\text{Face Value}} \times 100\%$$

- Formula 4.7 – **Periodic Cost of Debt**

Sinking Fund Payment + Periodic Interest Payment

- Formula 4.8 – **Book Value**

$$\text{Loan Amount} - \text{Balance}$$

Calculator

• Date Function

- **2nd DATE** to access.
- Enter two of the three variables (*DT1*, *DT2*, *DBD*) by pressing **ENTER** after each input and using the up arrow and down arrow to scroll through the display. The variables are:
 - *DT1* = The starting date of the transaction
 - *DT2* = The ending date of the transaction
 - *DBD* = The days between the dates, counting the first day but not the last, which is the time period of the transaction.
 - *ACT/360* = A setting for determining how the calculator determines the *DBD*. In Canada, you should maintain this setting on *ACT*, which is the actual number of days. In other countries, such as the United States, they treat each year as having 360 days (the 360 setting) and each month as having 30 days. If you need to toggle this setting, press **2nd SET**.
- Enter all dates in the format of MM.DDYY, where MM is the numerical month, DD is the day, and YY is the last two digits of the year. DD and YY must always be entered with both digits.
- Press **CPT** on the unknown (when it is on the screen display) to compute the answer.

• Amortization Worksheet for Bond Schedules

- To use the amortization worksheet for a bond schedule:
 - Enter all seven of the time value of money variables accurately (*N*, *I/Y*, *PV*, *PMT*, *FV*,

P/Y , and C/Y). Because money is received by the bond holder, PMT and FV are positive. Because the purchase price is paid out by the bond holder, PV is negative.

- Press **2nd AMORT**.
- Enter a value for $P1$, press **ENTER** and then the down arrow.
- Enter a value for $P2$, press **ENTER** and then the down arrow. Note that the higher the numbers entered in $P1$ or $P2$, the longer it will take the calculator to compute the outputs. It is possible that the calculator will go blank and take a few moments before displaying the outputs.
- Press the down arrow to scroll through BAL , PRN , and INT to read the output. The BAL number is the book value entry of the bond schedule. The PRN number is the amortized premium or accumulated discount entry of the bond schedule. The INT number is the interest on the book value at yield entry of the bond schedule.

• Amortization Worksheet for Sinking Fund Schedules

- To use the amortization worksheet for a sinking fund schedule:
 - Enter all seven of the time value of money variables accurately (N , I/Y , PV , PMT , FV , P/Y , and C/Y). If PMT was computed, you must re-enter it rounded up to the next cent while retaining the correct cash flow sign convention. Because money is invested into the account, PMT is negative. Because the money is withdrawn from the fund, FV is positive.
 - Press **2nd AMORT**.
 - Enter a value for $P1$, press **ENTER** and then the down arrow.
 - Enter a value for $P2$, press **ENTER** and then the down arrow. Note that the higher the numbers entered in $P1$ or $P2$, the longer it will take the calculator to compute the outputs. It is possible that the calculator will go blank and take a few moments before displaying the outputs.
 - Press the down arrow to scroll through BAL , PRN , and INT to read the output. The BAL number is the balance entry of the sinking fund schedule. The PRN number is the increase entry of the sinking fund schedule. The INT number is the interest entry of the sinking fund schedule.

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CHAPTER 5: BUSINESS INVESTMENT DECISIONS

Chapter Outline

[5.0 Introduction](#)

[5.1 Net Present Value](#)

[5.2 Internal Rate of Return](#)

[5.3 Review Exercises](#)

[Chapter 5 Summary](#)

5.0 INTRODUCTION

Chapter Learning Objectives

By the end of the chapter you should be able to:

- Calculate the net present value of an investment
- Evaluate and compare investment decisions using the net present value
- Calculate the internal rate of return of an investment
- Evaluate and compare investment decisions using the internal rate of return

You must make financial decisions throughout your professional career and everyday personal life. Some of these decisions are easy. To make the best choice you need a little intuition and simple calculation. Other decisions are very challenging, confronting you with a great array of competing options, each of which is bundled with numbers projected from now until ... whenever. If you tried just to wing it in these latter scenarios, you could make a catastrophic mistake.

For example, suppose you are the production manager for a company that soon needs to replace a critical machine that is nearing the end of its useful life. At your invitation, salespeople from three competing companies have paid you a visit this week. Each of them showed you an impressive replacement machine. The three machines appear to be equal in design and performance, but each carries a different price tag. Each machine also differs widely in operating costs such as power consumption, consumables, and labour costs. The maintenance requirements also follow different timetables and have different costs. In all cases, you can either purchase the machines or lease them through the supplier's leasing plan.

These machines are not cheap. Because the one you select represents such a significant investment on your company's part, good financing for it is an integral part of the decision. Will the money to pay for it come from a bank loan, a bond issuance, or by issuing some common shares? Perhaps the finance department

can withdraw some money from your organization's current investments. Each of these funding sources is associated with a different interest rate.

Your company relies on you to choose the best machine at the lowest possible cost. From a strictly financial perspective, and assuming that all machines are equally productive, which of the three machines is your best choice?

This business investment decision should not be guesswork. In previous chapters you have already learned many of the fundamental financial skills required to make effective monetary decisions. In this chapter you will amalgamate and apply your existing skills in dealing with compound interest, ordinary annuities, annuities due, leases, loans, and much more. Using a concept known as cash flows, there are two different mechanisms you will learn about to help make these types of business investment decisions: net present value and internal rate of return. Each of these techniques allow you to analyze a decision-making scenario, determine if a particular project is profitable, and select the best option from several choices.

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5.1 NET PRESENT VALUE

Learning Objectives

- Calculate the net present value of an investment
- Evaluate and compare investment decisions using the net present value

Formula & Symbol Hub

Symbols Used

- NPV = net present value
- IRR = internal rate of return

Formulas Used

- **Formula 5.1 – Net Present Value**

$$NPV = (\text{Sum of } PV \text{ of All Future Cash Flows}) - (\text{Initial Investment})$$

or

$$NPV = (\text{Sum of } PV \text{ of Cash Inflows}) - (\text{Sum of } PV \text{ of Cash Outflows})$$

Introduction

If you decide to pursue a course of action, what is the net benefit or cost of making that decision? How can you tell systematically which choice is financially superior?

Suppose that your marketing department thinks it has stumbled onto the next hot product to hit the market. The estimated research and development costs are \$10 million today with an expected \$5 million revitalization cost in five years. The product hits shelves in two years. Estimated year-end profits are \$4 million in the first three years and \$3 million for the subsequent two years. At that time, the rapid pace of technology will end the life cycle of the new product. This last year will earn an estimated \$1 million in profits. The funds needed for this project will be obtained at 12% compounded annually. Does this project make financial sense? Your co-worker argues that the decision is clear: You would spend \$15 million to earn \$19 million, so the project should go forward. What do you think?

Your co-worker's analysis summed the nominal numbers. This clearly violates the fundamental concept of time value of money—for example, you know that the projected profit of \$1 million in the last year of the product is not worth \$1 million today. To get an accurate picture of the potential benefit of this product, we need a technique that takes into account the time value of money.

What Is Involved in Making Monetary Business Decisions?

Investment decisions must consider both the type of decision being made along with the source of funding, which influences the interest rate used in financial calculations.

What type of decision is begin made?

Although there is no such thing as a “standard” or “normal” decision, many business decisions seem to fall into the same three categories.

- ***Making One Choice from Multiple Options.*** When you face a variety of options that will solve your problem, which option is financially the best?
- ***Deciding Whether to Pursue One Course of Action.*** When you face only one particular course of action, do you do it or not?
- ***Making Multiple Choices under Constraints.*** If you have limited resources and many different opportunities from which you can choose more than one, how do you select which opportunities to pursue and which to let pass by?

What monetary source is being used and what does it cost?

The structure of many decisions follows the old adage that you have to spend money to make money. Many courses of action require an investment up front (costs) to receive the reward (profits) in the future. This means you need to tap a source of money before you can go forward with any course of action. Individuals and corporations can access many of the same monetary sources, but businesses have additional options that are not available to individuals.

- **Personal Funding Sources.** You have limited options for obtaining money. Typically, you can either use debt financing such as a loan from a bank or a line of credit, or you can withdraw money from an investment fund or savings account.
 - If you borrow, the financial cost of the project is the interest rate that is being charged on the borrowed funds. For example, if you fund the project using a loan at 8.5% compounded annually, the loan's interest rate becomes the nominal rate you should use in the required time value of money calculations.
 - If you withdraw money from an investment, the cost of this project is the interest rate you forego on those funds. For example, if you take the money out of a GIC earning 4% compounded annually, you should use the GIC's interest rate in the decision calculations.
- **Business Funding Sources.** Businesses have substantially more choices than individuals when obtaining financing. In addition to debt financing and current investments, businesses also can issue bonds or use equity financing, such as offering company shares. Bartering with business partners is also possible. The same principle applies, though: whatever interest rate the funding source involves carries forward into any decision-making calculations.

Rate of return and cost of capital

The **rate of return** is the required rate of return that a business expects to receive from investing in the project. That is, the rate of return is what the business would earn by investing the same amount into a similar investment with the same risk. The rate of return is also known as the **cost of capital**, which is the cost of investing. Unless otherwise specified, the cost of capital is an interest rate that compounds annually.

As mentioned above, if the business does not have the funds on hand to invest in the project, some alternative funding source is required to carry out the project. In these cases the source of funding you use determines the cost of capital or interest rate in these calculations. If you use multiple sources to gather the needed funds, your interest rate is a weighted average of all the debt and equity financing rates. If you use a single source then the weighted average is simply the one source's rate. This weighted interest rate is the cost of capital.

Cash Flows

A **cash flow** is a movement of money into or out of a particular project. Therefore, any time you incur a cost or receive a benefit on a particular project, you have a cash flow.

- When money is received, such as revenue or profit, you have a cash in-flow. Cash in-flows are recorded as positive.
- When money is paid out, such as an expense or a cost, you have a cash out-flow. Cash out-flows are recorded as negative.

As you work with these continuous cash flows, where to put an amount on the timeline depends on whether it is a cost or a benefit.

- **Costs.** Unless otherwise noted, capital costs are spent at the beginning of any time segment. Thus, if there are capital costs in year two, you can assume you incur them at the beginning of year two, which is the same as the end of year one.
- **Benefits.** Revenues, net profits (product sales minus product costs), or other benefits are realized at the end of any time segment. Thus, if there are net profits in year two, you can assume you realize them at the end of year two.

Net Present Value

The **net present value** (*NPV*) method calculates the difference between all benefits and costs for any given project in **today's dollars**. This decision-making technique calculates the value of all future cash in-flows and cash out-flows brought back to today (which serves as the focal date), using the rate of return or cost of capital as the interest rate. The word *net* means that the technique considers the invested money, all costs or expenses, along with all rewards such as profits in its calculations.

5.1 Net Present Value

$$NPV = \text{Sum of the } PV \text{ of Future Cash Flow} \\ - \text{Initial Investment}$$

- **NPV** is the Net Present Value: which is the value of all future cash in-flows and out-flows in today's dollars
- **Sum of the PV of Future Cash Flow** represents the total present value of cash flow
- **Initial Investment** represents the original invested amount, which is subtracted from the cash flow calculation.

HOW TO

Calculate the Sum of the Present Value of Future Cash Flows

To calculate the sum of the present value of the future cash flows:

Step 1: Using today, or year 0, as the focal date, find the present value of all **future** cash flows, inflows and outflows, where the interest rate is the cost of capital.

Step 2: Add up the present values of the cash inflows and subtract the present values of the cash outflows.

The initial investment is money spent today, and so is already at the focal date. Because most projects require advance funding, this amount represents a cash outflow today, against which you must net the present value of all future cash flows.

An alternative way to think of the net present value is that **NPV** is the sum of the present value of the cash inflows minus the sum of the present value of the cash outflows. Because the initial investment is a cash outflow, the initial investment is included in the second sum. As above, today is used as the focal date and the present value of each cash flow is found at the focal date.

$$NPV = \text{Sum of the } PV \text{ of Cash Inflows} - \text{Sum of the } PV \text{ of Cash Outflows}$$

The net present value nets out all costs and benefits. Here is how to interpret the **NPV**:

- **NPV = 0.** The project repays the amounts of its investments plus the required cost of capital. In other words, the project breaks even. The business should accept the project.
- **NPV > 0.** The project provides returns that exceed its investments and the cost of capital. The project makes financial sense because it will add value to the company. The business should accept the project.

- $NPV < 0$. The project does not provide enough return to pay for its investments and the cost of capital. The project does not make financial sense because it will lose money for the business. The business should reject the project.

Of course, businesses often have many nonfinancial reasons for pursuing or not pursuing certain courses of action. These reasons are beyond the scope of this textbook.

Observe how NPV relates to each of the three common types of business decision categories.

- ***Making One Choice from Multiple Options.*** You should choose the project with the highest NPV .
- ***Choosing Whether to Pursue One Course of Action.*** You should pursue the project if the NPV is greater than or equal to \$0.
- ***Making Multiple Choices under Constraints.*** You should choose the projects that maximize the total NPV .

Example 5.1.1

A local business is considering purchasing a new machine. The machine costs \$80,000. The machine will reduce costs by \$25,000 in year 1, \$17,000 in year 2 and \$10,000 in year 3. At the end of year 3, the business can sell the machine for \$20,000. The business's cost of capital is 8%.

- Calculate the net present value for the machine.
- Should the business purchase the machine?

Solution

Step 1: Calculate the net cash flow for each year.

Record the cash flows in a table.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−80,000	−80,000
1	25,000	0	25,000
2	17,000	0	17,000
3	10,000 20,000	0	30,000

Step 2: Calculate the PV for each future net cash flow.

	Year 1	Year 2	Year 3
N	1	2	3
PV	23,148.15	14,574.76	23,814.97
FV	−25,000	−17,000	−30,000
PMT	0	0	0
I/Y	8	8	8
P/Y	1	1	1
C/Y	1	1	1

Step 3: Calculate the NPV .

$$\begin{aligned}
 NPV &= \text{Sum of the } PV \text{ of future cash flows} - \text{Initial Investment} \\
 &= 23,148.15 + 14,574.76 + 23,814.97 - 80,000 \\
 &= -\$18,462.12
 \end{aligned}$$

Step 4: Write as a statement.

The NPV for the machine is $-\$18,462.12$.

Because the $NPV < 0$, the business should not purchase the machine.



Create a table to record the cash flows for each time period. The table should record the cash inflows, the cash outflows, and the net cash flow for each time period. Alternatively, use a timeline to illustrate the cash flows for each time period.

Because cash flows can be inflows or outflows, write all numbers with their cash flow sign conventions in the table to avoid any confusion. That is, cash inflows are recorded as positive and cash outflows are recorded as negative. This helps you keep track of the different flows and minimizes the possibility of incorrectly netting out all present values.

Although the net present value can be found by finding the present value of all the future cash flows, as illustrated in the above example, this can be a cumbersome calculation when there are numerous cash flows over an extended period of time, even with the aid of a financial calculator. Most financial calculators include a cash flow worksheet and a built-in function to calculate the net present value.

Using the TI BAII Plus Cash Flow Worksheet to Calculate Net Present Value

The photo illustrates the buttons used in the cash flow function of your BAII Plus calculator. Access the cash flow function by pressing the **CF** button on the keypad. Immediately clear the memory using **2nd CLR WORK** to delete any previously entered data. Use the up arrow and down arrow to scroll through the various lines. You must strictly obey the cash flow sign convention and press **ENTER** after keying in any data.

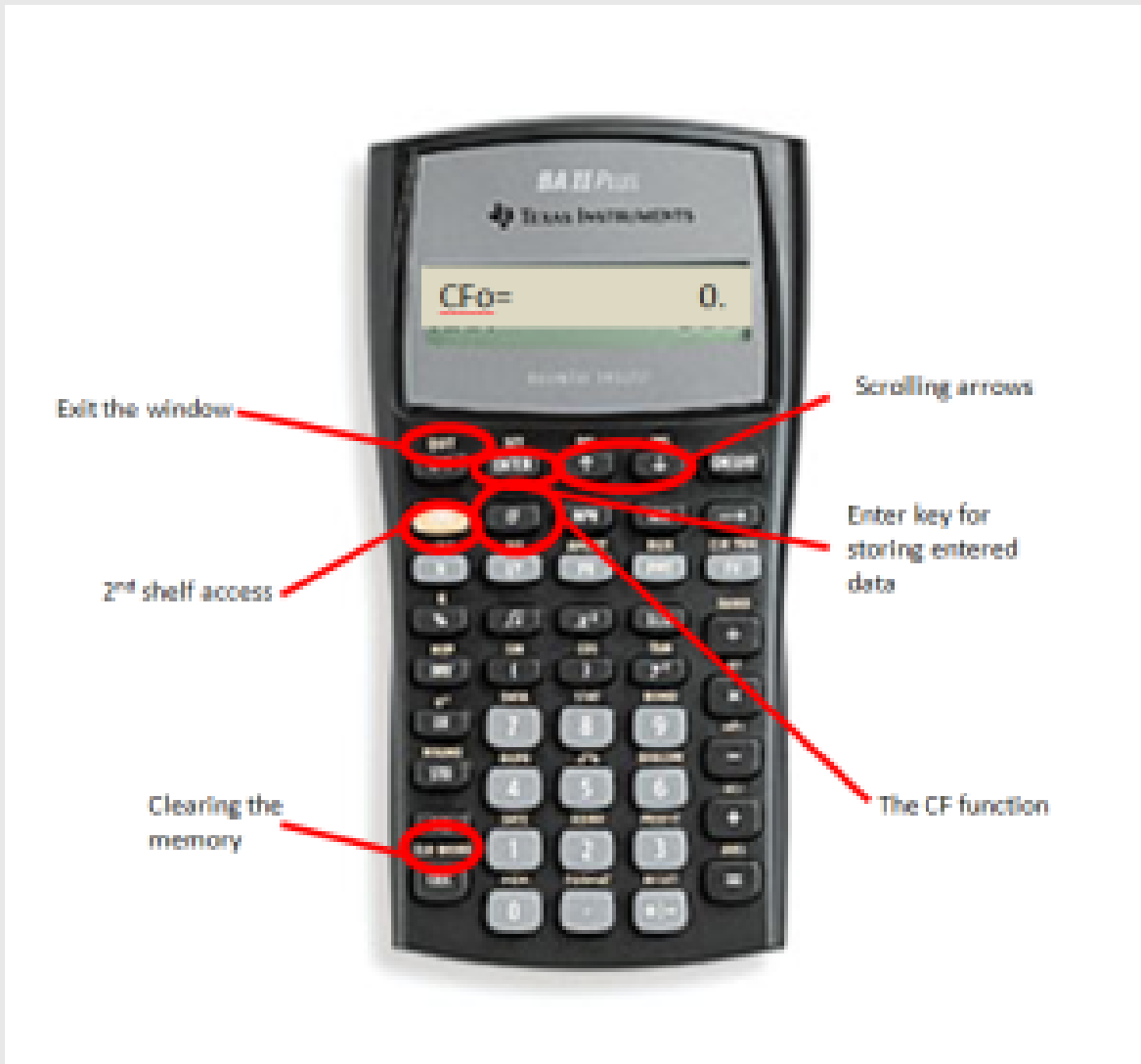


Figure 5.1.1

Image Description

Image of a BA II Plus calculator by Texas Instruments.

The image shows a Texas Instruments BA II Plus calculator with labeled buttons/functions:

- The screen at the top displays ' $CFO =$ ' and ' $0.$ '.
- Immediately below the screen, the following buttons are circled and labeled from left to right:
 - First button: 'Exit the window.'
 - Second button: ' 2^{nd} shelf access.'

- Third button: '**ENTER** key for storing entered data.'
- Fourth button: 'Scrolling arrows.'
- Further down on the next row of buttons, the following button is circled and labeled:
 - Second button: 'The **CF** function.'
- Below these buttons, the following button is circled and labeled:
 - In the left column, near the bottom: 'Clearing the memory.'

Here are the various lines appearing in the cash flow function:

- **CF₀** = any cash flow today (i.e. the initial investment).
- **CXX** = a particular cash flow, where **XX** is the cash flow number starting with **01**. You must key in the cash flows in order from the first time segment to the last. You cannot skip a time segment even if it has a value of zero. Each time segment is a placeholder on the timeline.
- **FX** = the frequency of a particular cash flow, where **XX** is the cash flow number. It is how many times *in a row* the corresponding cash flow amount occurs. This allows you to enter recurring amounts instead of keying them in separately. By default, the calculator sets this variable to **1**.

Once you have entered all cash flows, you must access other functions to generate the output. For net present value, press **NPV** on the keypad as illustrated in the photo.

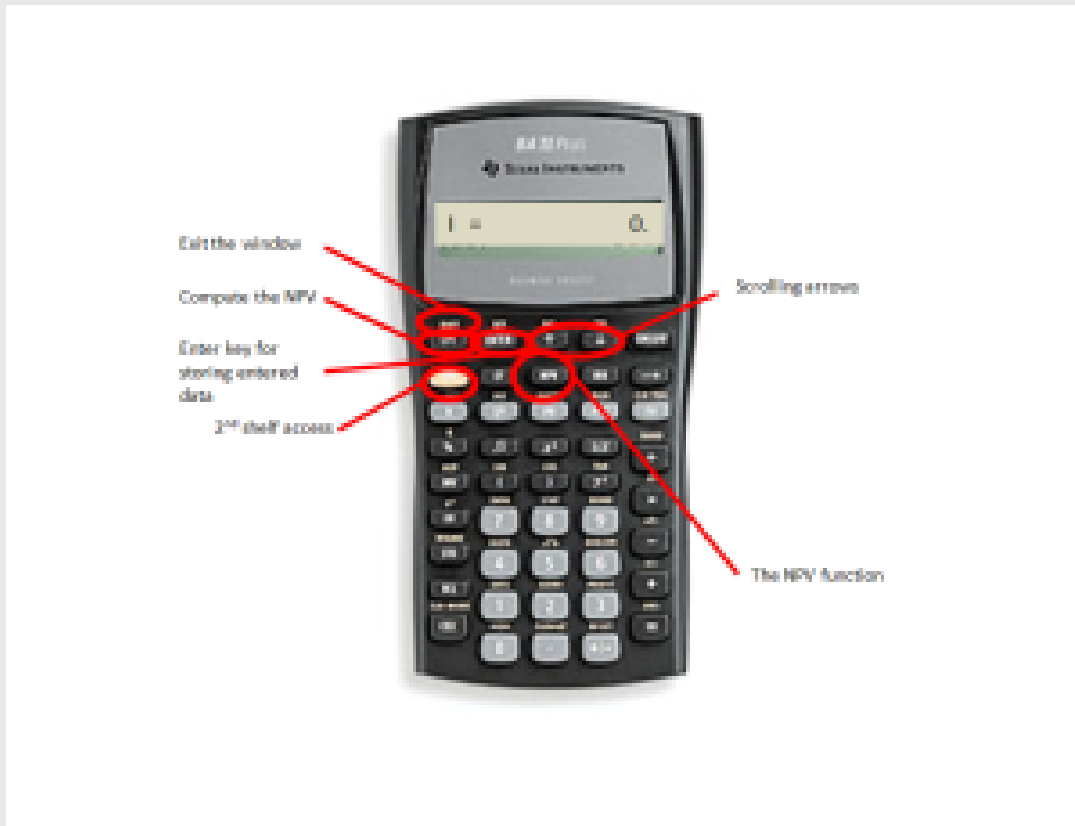


Figure 5.1.2

Image Description

This image shows a Texas Instruments BA II Plus financial calculator. Several buttons are circled and annotated with their functions:

- Button to the right of the screen is labeled “Exit the window.”
- Button below this is labeled “Scrolling arrows.”
- Button under first two is labeled “Compute the *NPV*.”
- Button to the left of *NPV* is labeled “Enter key for storing entered data.”
- Top left button is labeled “*2nd* shelf access.”
- The “*NPV* Function” is highlighted next to a button on the rightmost side, in the middle row of buttons.

In the *NPV* menu:

- *I* = the matching periodic interest rate for the interval of each time segment. If the timeline is drawn with yearly intervals, then this must be the annual cost of capital. If the timeline is

drawn with semi-annual intervals, this needs to be the matching semi-annual periodic cost of capital.

- **NPV** = net present value. Press **CPT** to calculate this amount. The output nets all future cash flows against any initial investment.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=152#oembed-1>

Video: [Cash Flows 1: Net Present Value \(NPV\) and IRR Calculations](#) by Joshua Emmanuel [2:52] *Transcript Available.*

Key Takeaways



For each period, only one cash flow can be entered in the financial calculator. If more than one cash flow exists for a certain period, enter the NET cash flow for that period. The net cash flow for a period is the sum of the cash inflows minus the sum of the cash outflows.

When using the cash flow worksheet on a financial calculator, you must enter a cash flow for EVERY period. If there is no cash flow for a certain period, a **0** is recorded as the net cash flow for that period. Enter **0** for **C0** on the calculator for that period and **1** for **F0**.

Example 5.1.2

A local business is considering purchasing a new machine. The machine costs \$80,000. The machine will reduce costs by \$25,000 in year 1, \$17,000 in year 2 and \$10,000 in year 3. At the end of year 3, the business can sell the machine for \$20,000. The business's cost of capital is 8%.

Find the net present value.

Solution

Step 1: Calculate the net cash flow for each year.

Record the cash flows in a table.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−80,000	−80,000
1	25,000	0	25,000
2	17,000	0	17,000
3	10,000 20,000	0	30,000

Step 2: Enter the net cash flow data into the cash flow worksheet on the financial calculator.

1. Press **CF**.
2. Press **2nd CLR WORK**. (This step clears any previously recorded data from the **CF** worksheet.)
3. At the **CF0** screen, enter −80,000 and press **ENTER**. (This step enters the −80,000 net cash flow for year 0.)
4. Press the down arrow.

5. At the **C01** screen, enter **25,000** and press **ENTER**. (This step enters the **25,000** net cash flow for year **1**.)
6. Press the down arrow.
7. At the **F01** screen, enter **1** and press **ENTER**. (This records the frequency of the **25,000** net cash flow from year **1**.)
8. Press the down arrow.
9. At the **C02** screen, enter **17,000** and press **ENTER**. (This step enters the **17,000** net cash flow for year **2**.)
10. Press the down arrow.
11. At the **F02** screen, enter **1** and press **ENTER**. (This records the frequency of the **17,000** net cash flow from year **2**.)
12. Press the down arrow.
13. At the **C03** screen, enter **30,000** and press **ENTER**. (This step enters the **30,000** net cash flow for year **3**.)
14. Press the down arrow.
15. At the **F03** screen, enter **1** and press **ENTER**. (This records the frequency of the **30,000** net cash flow from year **3**.)
16. Press **2nd QUIT**. (This step exits the **CF** worksheet.)

Step 3: Calculate the *NPV* on the financial calculator.

<i>I</i>	8
<i>NPV</i>	−\$18,462.12

1. Press ***NPV***.
2. At the ***I*** screen, enter **8** and press **ENTER**. (This step enters the **8%** cost of capital.)

3. Press the down arrow.
4. At the *NPV* screen, press *CPT*.

Try It

1) A company wants to develop an app for its business over the course of five years. The development requires an investment of **\$900,000** today and another investment of **\$200,000** one year from now. Starting in year three, the company forecasts that the app will generate profits of **\$150,000** every year for three years. At the end of the fifth year, the company plans to sell the app to a competitor for **\$1,250,000**. The company's cost of capital is **7%**.

- a. Calculate the *NPV* for this project.
- b. Is this a sound investment decision?

Solution

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−900,000	−900,000
1	0	−200,000	−200,000
2	0	0	0
3	150,000	0	150,000
4	150,000	0	150,000
5	150,000 1,250,000	0	1,400,000

<i>I</i>	7
<i>NPV</i>	\$148,143.73

Because the $NPV = \$148,143.73 > 0$, this is a sound investment decision.

Example 5.1.3

Lethbridge Community College is considering purchasing new industrial photocopier equipment. The college has narrowed down the choices to two comparable machines from Xerox and Canon.

- The Xerox machine can be acquired for **\$81,900** today and is expected to have a life span of four years. The machine is expected to reduce labour and materials costs by **\$31,000** each year. A **\$10,000** maintenance package is expected to be purchased at the end of year two. After the four years, the machine can be sold for **\$16,000**.
- The Canon machine retails for **\$74,800** and is expected to have a life span of four years. The machine is forecasted to save **\$26,000** in labour and materials each year. A **\$4,000** maintenance procedure is needed at the end of both years two and three. The salvage value of the machine is estimated at **\$11,750**.

The cost of capital is **11%**. Which machine should be purchased, and how much money will it save over the alternative?

Solution

Step 1: Calculate the *NPV* for the Xerox machine.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−81,900	−81,900
1	31,000	0	31,000
2	31,000	−10,000	21,000
3	31,000	0	31,000
4	31,000 16,000	0	47,000

<i>I</i>	11
<i>NPV</i>	\$16,699.29

Step 2: Calculate the *NPV* for the Canon machine.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−74,800	−74,800
1	26,000	0	26,000
2	26,000	−4,000	22,000
3	26,000	−4,000	22,000
4	26,000 11,750	0	37,750

<i>I</i>	11
<i>NPV</i>	\$7,432.42

Step 3: Write as a statement.

The college should pick the Xerox machine because it has the higher *NPV*. By choosing the Xerox machine, the college will save $\$16,699.29 - 7,432.42 = \$9,266.87$.

Try It

2) The operations team at an engineering firm is considering investing in two different projects, but they only have the funds to invest in one project.

- Project 1 requires an initial investment of \$650,000 and another investment of \$250,000 at the end of year 2. The project will generate annual profits of \$200,000 starting in year 2 until the end of the project in year 5. At the end of the project, the firm can sell the project for \$500,000.
- Project 2 requires an initial investment of \$800,000 and another investment of \$200,000 at the end of year 3. The project will generate annual profits of \$150,000 starting in year 1 until the end of the project in year 5. At the end of the project, the firm can sell the project for \$600,000.

The firm's cost of capital is 9%. Which project should the firm pick?

Solution

Project 1:

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−650,000	−650,000
1	0	0	0
2	200,000	−250,000	−50,000
3	200,000	0	200,000
4	200,000	0	200,000
5	200,000 500,000	0	700,000

<i>I</i>	9
<i>NPV</i>	\$58,989.71

Project 2:

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−800,000	−800,000
1	150,000	0	150,000
2	150,000	0	150,000
3	150,000	−200,000	−50,000
4	150,000	0	150,000
5	150,000 600,000	0	750,000

<i>I</i>	9
<i>NPV</i>	\$18,969.83

The firm should select Project **1** because it has the higher *NPV*.

Section 5.1 Exercises

1. A new diamond deposit has been found in northern Alberta. Your researchers have determined that it will cost **\$2.5** million to purchase the land and prepare it for mining. At the end of both the first and second years, another **\$1** million investment will be required to establish the mining operations. Starting at the end of the second year, the deposit is expected to earn net profits of **\$3** million, which will be sustained for three years before the deposit is depleted. If the cost of capital is **16%**, should your company

pursue this venture? Provide calculations to support your decision.

Solution

$NPV = \$1,703,103.15$; pursue project

2. A company can pursue only one of two available projects, both of which require a **\$3** million investment today. In project A, another investment of **\$2** million is required at the end of the first year, and three years of **\$2.5** million in year-end profits will start in the second year. In project B, another investment of **\$2** million is required at the end of the third year, and four years of **\$1.7** million in year-end profits will start in the first year. If the cost of capital is **12%**, which project should be chosen? How much better is your chosen alternative?

Solution

NPV of Project A = $\$575,516.22$; NPV of Project B = $\$739,933.36$; Project B

3. A new marketing project requires initial research and development costs of **\$550,000** today with further investments of **\$100,000** and **\$75,000** in the third and fifth year, respectively. Initially the project will lose **\$50,000** in its first year and then earn profits of **\$200,000** for the next four years followed by profits of **\$140,000** in the last two years. At the end of the project, its capital goods can be sold for an estimated **\$50,000**. If the cost of capital is **13%**, should the marketing project be pursued? Show calculations to support your recommendation.

Solution

$NPV = -\$29,798.58$; do not pursue project

4. One of your company's clients has proposed a contract offering an estimated **\$150,000** in net profits for the next six years; however, your company would be required to invest **\$585,000** today to acquire the needed resources for the project. Determine whether the project should be accepted if the cost of capital is
 - a. **10%**
 - b. **13%**
 - c. **16%**

Solution

a. $NPV = \$68,289.10$; accept project b. $NPV = \$14,632.47$; accept project c. $NPV = -\$32,289.61$; do not accept project.

5. To replace an aging production machine, Johnston Distributors is considering the purchase of a new robotic machine for **\$950,000**. The machine is expected to have a service life of eight years and an estimated residual value of **\$75,000**. In each of the first four years, the machine is expected to produce production efficiencies and labour savings of **\$250,000**. In the last four years, the annual savings are expected to rise to **\$300,000**. Hydro costs for the machine are estimated at **\$25,000** starting at the end of the first year and increasing by **5%** each year thereafter. Maintenance costs are estimated at the end of the second, fourth, and sixth years in the amounts of **\$40,000**, **\$60,000**, and **\$80,000**, respectively. If the cost of capital is **16%**, determine if the new robotic machine should be purchased. Provide calculations to support your answer.

Solution

$NPV = \$15,496.12$; purchase machine

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5.2 INTERNAL RATE OF RETURN

Learning Objectives

- Calculate the internal rate of return of an investment
- Evaluate and compare investment decisions using the internal rate of return

Formula & Symbol Hub

Symbols Used

- NPV = net present value
- IRR = internal rate of return

Formulas Used

- **Formula 5.1 – Net Present Value**

$$NPV = (\text{Sum of } PV \text{ of All Future Cash Flows}) - (\text{Initial Investment})$$

or

$$NPV = (\text{Sum of } PV \text{ of Cash Inflows}) - (\text{Sum of } PV \text{ of Cash Outflows})$$

Introduction

The net present value presented in the previous section provides an exact monetary magnitude of a project. However, the common mindset in business focuses on profitability as a percentage and not as a dollar amount. Thus, decisions are based on the **internal rate of return** for a project, or *IRR* for short. The *IRR* is the annual percentage rate of return on the investment so that the present value of all cash inflows equals the present value of all cash outflows. That is, the *IRR* is the rate of return required to make $NPV = 0$.

To interpret the *IRR* and how the *IRR* can be used to make a decision about a project, consider the relationship between the value of NPV and the *IRR*.

- $NPV = 0$. By definition, the cost of capital and the *IRR* are the same value.
- $NPV > 0$. Because the *IRR* is the rate of return that makes $NPV = 0$, this means that a higher rate of return must be used in order for the $NPV > 0$. In other words, the *IRR* is greater than the cost of capital.
- $NPV < 0$. Because the *IRR* is the rate of return that makes $NPV = 0$, this means that a lower rate of return must be used in order for the $NPV < 0$. In other words, the *IRR* is less than the cost of capital.

In summary, here is how to interpret the *IRR* and use it to make a decision about the viability of a project.

- $IRR = \text{cost of capital}$. The project breaks even. This is the minimum financial condition to pursue the project. The business should accept the project.
- $IRR > \text{cost of capital}$. The project is profitable because the project makes enough money to cover the costs. The business should accept the project.
- $IRR < \text{cost of capital}$. The project is not profitable because the project does not make enough money to cover the costs. The business should reject the project.

Solving for the internal rate of return is difficult without the use of a financial calculator or other technology. Similar to calculating the NPV in the previous section, the *IRR* can be calculated by using the cash flow worksheet on the financial calculator and a built-in function to calculate the internal rate of return.

Using the TI BAII Plus Cash Flow Worksheet to Calculate the Internal Rate of Return

Enter the cash flow data into the cash flow worksheet, as discussed in the previous section. Once you have entered all cash flows, you must access other functions to generate the output. For internal rate of return, press *IRR* on the keypad, as illustrated in the photo, followed by the *CPT* button to perform the calculation. The output is a percentage in percent format. To exit the window, press **2nd QUIT**.

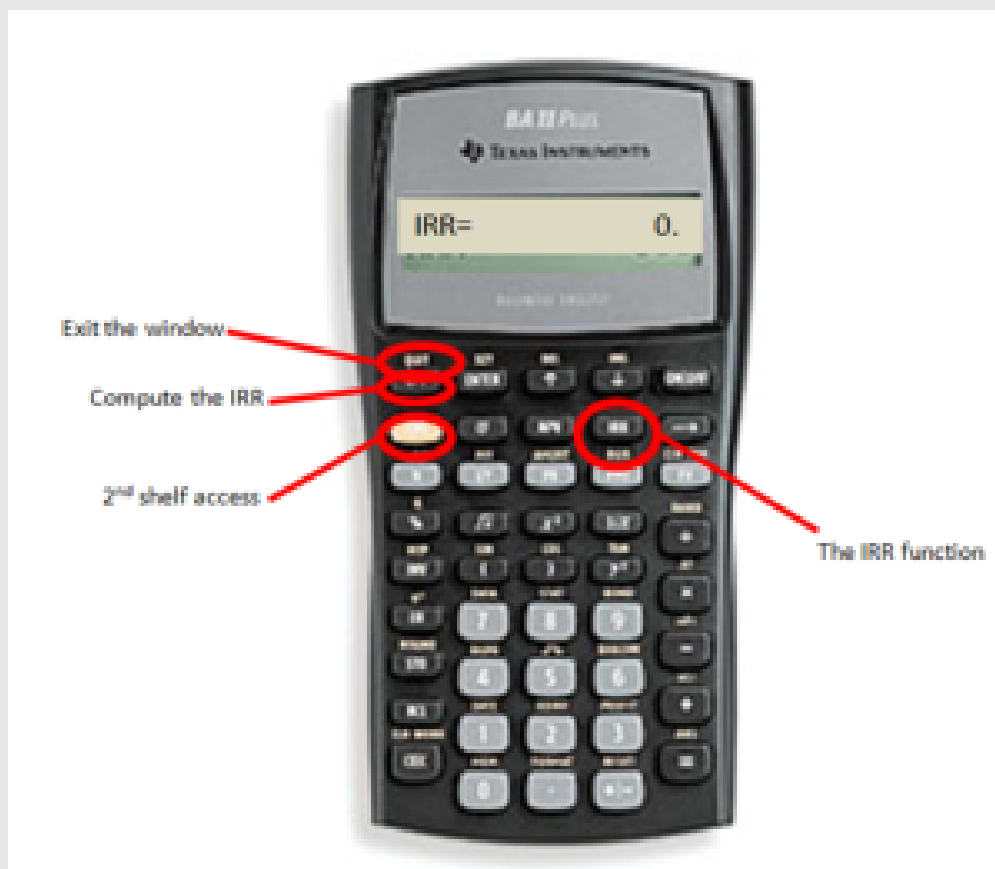


Figure 5.2.1

Image Description

This image displays a Texas Instruments BA II Plus calculator. It highlights specific buttons labeled with red circles and annotations demonstrating the functions of those buttons:

- The button labeled “Exit the window” is the $CE|C$ button located near the top left of the numeric keypad.
- The button labeled “Compute the IRR ” is the CPT button positioned below the $CE|C$ button and slightly to the right.
- The button labeled “ 2^{nd} shelf access” is the 2^{nd} button located to the right of the CPT button.
- The “ IRR function” is highlighted as being located on an oval button directly below the screen on the calculator (next to the 2^{nd} button).

The display screen at the top of the calculator shows “ $IRR = 0.$ ”



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=155#oembed-1>

Video: [Cash Flows 1: Net Present Value \(NPV\) and IRR Calculations](#) by Joshua Emmanuel [2:52] *Transcript Available.*



Key Takeaways

Create a table to record the cash flows for each time period. The table should record the cash inflows, the cash outflows, and the net cash flow for each time period. Alternatively, use a timeline to illustrate the cash flows for each time period.

Because cash flows can be inflows or outflows, write all numbers with their cash flow sign conventions in the table to avoid confusion. That is, cash inflows are recorded as positive and cash outflows are recorded as negative. This helps you keep track of the different flows and minimizes the possibility of incorrectly netting out all present values.

For each period, only one cash flow can be entered in the financial calculator. If more than one cash flow exists for a certain period, enter the NET cash flow for that period. The net cash flow for a period is the sum of the cash inflows minus the sum of the cash outflows.

When using the cash flow worksheet on a financial calculator, you must enter a cash flow for EVERY period. If there is no cash flow for a certain period, a **0** is recorded as the net cash flow for that period. Enter **0** for **C0** on the calculator for that period and **1** for **F0**.

Example 5.2.1

Tim Hortons has purchased the lease on a three-year onsite concession space in the cafeteria at a local college for **\$750,000**. The franchise is expected to earn **\$400,000**, **\$500,000**, and **\$600,000** in profits per year for the first three years, respectively.

- a. Calculate the internal rate of return for this investment.
- b. If the cost of capital is 20%, did Tim Hortons make a smart financial decision?

Solution

Step 1: Record the cash flows in a table.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−750,000	−750,000
1	400,000	400,000	400,000
2	500,000	500,000	500,000
3	600,000	600,000	600,000

Step 2: Calculate the IRR .

IRR	40.92%
-------	--------

1. Press IRR .
2. At the IRR screen, press CPT .

Step 3: Write as a statement.

Because the $IRR = 40.92\% > 20\%$, this was a smart decision.

Try It

- 1) A research project requires an investment of \$350,000 today and another investment of

\$200,000 one year from today. Starting from the third year, the project will reduce costs by \$250,000 every year for three years.

- Calculate the internal rate of return for the project.
- If the cost of capital is 12%, is the project financially viable?

Solution

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−350,000	−350,000
1	0	−200,000	−200,000
2	0	0	0
3	250,000	0	250,000
4	250,000	0	250,000
5	250,000	0	250,000

<i>IRR</i>	8.95%
------------	-------

Because the $IRR = 8.95\% < 12\%$, the project is not financially viable.

Like the net present value, the internal rate of return can be used to choose one project from several different projects. Relative to their initial investment, projects with higher *IRRs* are more profitable. Consequently, *IRR* can be used to rank projects. When making a choice from several options, the option with the highest *IRR* should be accepted because it will be the most profitable.

Example 5.2.3

A company is consider two different investments.

- Investment A requires an initial investment of \$300,000 today and will provide returns of \$100,000 in year two, \$150,000 in year three, and \$400,000 in year five.
- Investment B requires an initial investment of \$500,000 today and will provide returns of \$350,000 in year one, \$200,000 in year four, and \$75,000 in year five.

Calculate the internal rate of return for each investment. Which investment should the company select?

Solution

Step 1: Calculate the *IRR* for Investment A.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−300,000	−300,000
1	0	0	0
2	100,000	0	100,000
3	150,000	0	150,000
4	0	0	0
5	400,000	0	400,000

<i>IRR</i>	21.76%
------------	--------

Step 2: Calculate the *IRR* for Investment B.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−500,000	−500,000
1	350,000	0	350,000
2	0	0	0
3	0	0	0
4	200,000	0	200,000
5	75,000	0	75,000

<i>IRR</i>	10.14%
------------	--------

Step 3: Write as a statement.

The company should select Investment A because it has the higher *IRR*.

Try It

2) A company is considering two different projects to deal with an equipment issue. The company can either upgrade the equipment (Project A) or they can retrofit the equipment (Project B).

- Project A requires an initial investment of **\$300,000** in new equipment that will result in increased revenue of **\$90,000** each year for five years.
- Project B requires an initial investment of **\$125,000** to retrofit existing equipment that will result in increased revenue of **\$30,000** each year for five years.

Calculate the internal rate of return for each project. Which project should the company select?

Solution**Project A:**

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−300,000	−300,000
1	90,000	0	90,000
2	90,000	0	90,000
3	90,000	0	90,000
4	90,000	0	90,000
5	90,000	0	90,000

<i>IRR</i>	15.24%
------------	--------

Project B:

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	−125,000	−125,000
1	30,000	0	30,000
2	30,000	0	30,000
3	30,000	0	30,000
4	30,000	0	30,000
5	30,000	0	30,000

<i>IRR</i>	6.40%
------------	-------

The company should select Project A because it has the higher *IRR*.



Paths to Success

As mentioned above, the internal rate of return can be used to rank projects. However, a couple of things need to be considered about using the internal rate of return in this way.

1. ***The Cost of Capital is Ignored.*** The internal rate of return does not factor in the cost of capital in its computations. Recall that the fundamental concept of time value of money requires all money to be on the same date using an appropriate rate of interest—the cost of capital—before any decision can be made. Therefore, if you have not factored in the cost of capital, then your analysis is incomplete.
2. ***The Magnitude of the Decision is Ignored.*** It can easily happen that an alternative has a high *IRR* but a low *NPV*. For example, using a cost of capital of 10% consider two alternatives. Alternative A invests \$1 and one year later returns \$1.50. The *IRR* is 50%, while the *NPV* is \$0.36. Alternative B invests \$1,000 and one year later returns \$1,250. The *IRR* is 25%, while the *NPV* is \$136.36. If choosing between these two options based on the *IRR*, you select Alternative A, resulting in a net present value that is \$136 lower than for Alternative B.

Section 5.2 Exercises

1. A project requires an investment of \$225,000 today and is projected to have annual profits of \$34,000 for nine years. The capital assets can be sold at the end of the ninth year for \$23,500. Calculate the *IRR* for this project. If capital could be acquired at a cost of 12%, should the project be pursued?

Solution

$IRR = 7.91\%$; do not pursue project

2. A new diamond deposit has been found in northern Alberta. Your researchers have

determined that it will cost **\$4.5** million to purchase the land and prepare it for mining. Starting in the second year, the lode is expected to earn net profits of **\$3** million, which will be sustained for three years before the deposit is depleted. Calculate the IRR on the diamond mine. If the cost of capital is **21%**, should the deposit be acquired?

Solution

$IRR = 26.78\%$; pursue project

3. Schick is considering launching only one of two new products to compete with the Gillette Fusion ProGlide Power razor.
 - Product A requires a **\$10** million research and development investment with forecasted profits are expected to be **\$3** million in years two to four, **\$1.5** million in years five to seven, and **\$0.5** million in years eight to ten.
 - Product B requires a **\$6.5** million research and development investment with forecasted profits of **\$2.75** million in years one to two, **\$1** million in years three to five, and **\$0.35** million in years six to ten.

Calculate the IRR for each product. Which product should Schick launch?

Solution

IRR of Product A = 10.59%; IRR of Product B = 16.89%; launch Product B

4. The Dragons (from CBC's *Dragon's Den*) are considering a venture from an Albertan entrepreneur. His proposal requires the Dragons to invest **\$500,000** immediately. Based on market conditions, the product is estimated to have year-end profits of **\$100,000**, **\$200,000**, **\$300,000**, **\$200,000**, and finally **\$100,000**. The Dragons will invest only if the internal rate of return exceeds their cost of capital of **20%**. Should they invest?

Solution

$IRR = 22.78\%$; pursue project

5. A Western Canadian producer is considering purchasing a five-year product licence from Sunkist Growers Inc. The product licence will cost **\$1,500,000**, and the required equipment will cost **\$1** million. Equipment upgrades will be **\$100,000** at the end of the

second and fourth years. Expected profits are \$1 million in each of the first three years and \$750,000 in the last two years. The producer will purchase product licences only if the IRR is greater than 25%. Should the product licence be purchased?

Solution

$IRR = 22.91\%$; do not pursue project

6. Consider the following two projects, of which only one can be selected:

- Project A. Immediate investment of \$225,000, profits in years four to six of \$160,000.
- Project B. Immediate investment of \$112,500, profits in years one to three of \$55,000 followed by profits of \$20,000 in years four to six.

Calculate the IRR for each project. Which project should be selected?

Solution

IRR of Project A = 16.54%; IRR of Project B = 32%; select Project B

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5.3 REVIEW EXERCISES

Chapter 5 Review Exercises

1. Jeremy is thinking of starting up a candle manufacturing business. The initial outlay for equipment, moulds, and other required production equipment is **\$15,000**. Working part time on this hobby business, Jeremy estimates that he will lose **\$2,000** in the first year, break even in the second year, and earn annual profits of **\$5,000**, **\$10,000**, and **\$15,000** in subsequent years. After the five years, he hopes to sell the business to an investor for **\$17,500**. If his cost of capital is **8.25%** compounded annually, should he pursue this venture? Provide net present value calculations to support your answer.

Solution

NPV=**\$16,241.50**; pursue venture

2. A company pursues all projects that exceed its **15%** cost of capital. Project “Affinity” is available to the company with an initial investment of **\$655,000**. Forecasted annual profits are expected to be **\$200,000** for the first three years and **\$160,000** for the subsequent three years. Based on the *IRR*, should the company pursue Project “Affinity”?

Solution

IRR=**17.47%**; pursue project

3. Coca-Cola is considering two new flavours—cherry and vanilla—for expansion of one of its product lines. Based on historical and competitive information, the cherry flavour isn’t quite as popular as the vanilla flavour, but it will endure in the market longer. Both options require an investment of **\$2.25** million in equipment and modifications. The cherry flavour is expected to have annual profits of **\$750,000** for the first two years

and \$500,000 for five more years. The vanilla flavour is expected to initially have annual profits of \$950,000 for two years, \$650,000 for two years, and \$250,000 for the last three years. The cost of capital is 16%. Calculate the *NPV* for each flavour. Which flavour should Coca-Cola launch?

Solution

NPV of Cherry = \$170,590.69; *NPV* of Vanilla = \$360,483.17; launch Vanilla

4. A saleswoman informs the head librarian that investing in her new automated library product is a steal at \$103,000. Before purchasing, the head librarian investigates the benefits of the automated product, which is expected to have a useful life of seven years and a 15% cost of capital. Each year, \$24,000 in labour savings and reduced book shrinkage is projected. However, the automated products require regular maintenance of \$1,000 in the first four years and \$2,000 in the last three years. Electricity bills will also rise by \$1,500 each year in the first three years and \$2,100 each year in the last four years. Calculate the *NPV* for this product. Is the saleswoman correct? Should the automated system be purchased?

Solution

NPV = −\$15,982.73; do not pursue automation

5. Hershiser has been researching different methods of management leadership that are designed to increase employee motivation and performance. If he pursues a new leadership style, it will require an investment of \$165,000 to take the training. A new executive assistant will be hired at a cost of \$55,000 annually with expected raises of 5% each year. The techniques and tactics from the style are expected to have an impact on employees for five years before needing to be refreshed and updated with more modern techniques. Increased motivation and performance will produce net profits of \$100,000 in the first year, rising by 10% in each subsequent year.
- Calculate the internal rate of return.
 - If the cost of capital is 10%, should Hershiser take the management leadership course?

Solution

$IRR = 22.36\%$; take course

6. Mariners Inc. in Halifax has the option to purchase only one of two available offshore fishing rights. Both rights are available for purchase on a four-year contract before they have to be renewed.
- Fishing right #1 costs **\$600,000** to acquire, and **\$64,000** is needed in new equipment to harvest the fish in the area. For each of the first two years, **\$21,000** will be spent on equipment maintenance and replacement followed by **\$33,000** in both of the last two years. Annual profits are expected at **\$300,000**.
 - Fishing right #2 costs **\$400,000** to acquire, and **\$82,000** is needed in new equipment, which requires annual costs of **\$30,000** for each of the first two years and **\$41,000** for both of the last two years. Annual profits are expected to be **\$250,000**.

Calculate the net present value for both projects if the cost of capital is **14%**. What decision do you recommend?

Solution

NPV of #1 = \$133,721.11; NPV of #2 = \$145,079.14; choose fishing right #2.

7. The following projects are available, but only one can be chosen. The cost of capital in all cases is **16%**.
- Project A requires an initial investment of **\$180,000** followed by profits of **\$30,000** in years one to four, **\$40,000** in years five to seven, and **\$50,000** in years eight to ten with a residual value of **\$100,000** in year ten.
 - Project B requires an initial investment of **\$335,000** followed by profits of **\$65,000** in years one to three, **\$85,000** in years four to six, and **\$110,000** in years seven to eight with a residual value of **\$60,000** in year ten.
 - Project C requires an initial investment of **\$372,000** followed by profits of

\$150,000 in years four to nine and a residual value of \$70,000 in the tenth year.

- a. If the cost of capital is 16%, calculate the net present value for each project. Which project should be selected?
- b. If the cost of capital is 11%, calculate the net present value for each project. What decision should be selected?

Solution

a. NPV of A = \$15,962.27; NPV of B = \$19,359.83;
 NPV of C = -\$2,034; choose Project B
 b. NPV of A = \$71,533.63;
 NPV of B = \$97,566.83; NPV of C = \$116,652.84; choose Project C

8. The Banff Gondola is looking to increase its operational capacity through modifications to its gondola equipment. If it invests \$2.3 million it can upgrade the motors, strengthen the necessary supports, and add an additional cable car to its line. The expected life of the project is eight years before the equipment will need to be replaced. Annual increased maintenance costs are \$50,000 at the end of every year for four years and then \$100,000 at the end of every year during the last four years. Because of construction and down times, Banff Gondola is forecasting a loss of \$415,000 in the first year. Afterwards, it projects that the equipment will result in increased profits of \$850,000 per year. Calculate the internal rate of return for the project.

Solution

$IRR = 16.07\%$

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CHAPTER 5 SUMMARY

Key Concepts

- **5.1 Net Present Value**

- The characteristics of making decisions including decision types, monetary sources, and interest rates.
- Making decisions through net present value.

- **5.2 Internal Rate of Return**

- Using internal rate of return to choose whether to pursue one course of action.

Glossary of Terms

Cash Flow. A movement of money into or out of a particular project.

Cost of Capital. A weighted average of all of the debt and equity financing rates used to provide needed funds for a project.

Internal Rate of Return (IRR). The annual rate of return on the investment being made such that the net present value of all cash flows in a particular project equals zero.

Net Present Value (NPV). The difference in today's dollars between all benefits and costs for any given project.

Formula & Symbol Hub

Symbols Used

- NPV = net present value
- IRR = internal rate of return

Formulas Used

- Formula 5.1 – **Net Present Value**

$$NPV = (\text{Sum of } PV \text{ of All Future Cash Flows}) - (\text{Initial Investment})$$

or

$$NPV = (\text{Sum of } PV \text{ of Cash Inflows}) - (\text{Sum of } PV \text{ of Cash Outflows})$$

Calculator

• Cash Flow Worksheet

- Access the cash flow function by pressing CF on the keypad.
- Always clear the memory using **2nd CLR WORK** so that any previously entered data is deleted.
- Use the up arrow and down arrow to scroll through the various lines.
- Strictly adhere to the cash flow sign convention when using this function and press **ENTER** after keying in the data.
- To exit the window, press **2nd QUIT**.
- The various lines are summarized below:
 - CF_0 = any cash flow today.
 - CXX = a particular cash flow, where XX is one of a series of cash flow numbers starting with 01. You must key in cash flows in order from the first time segment to the last. You cannot skip a time segment, even if it has a value of zero, because each time segment is a placeholder on the timeline.

- FXX = the frequency of a particular cash flow, where XX is the cash flow number. It is how many times *in a row* the corresponding cash flow amount occurs. This allows you to enter recurring amounts together instead of keying them in separately. By default, the calculator sets this variable to 1.

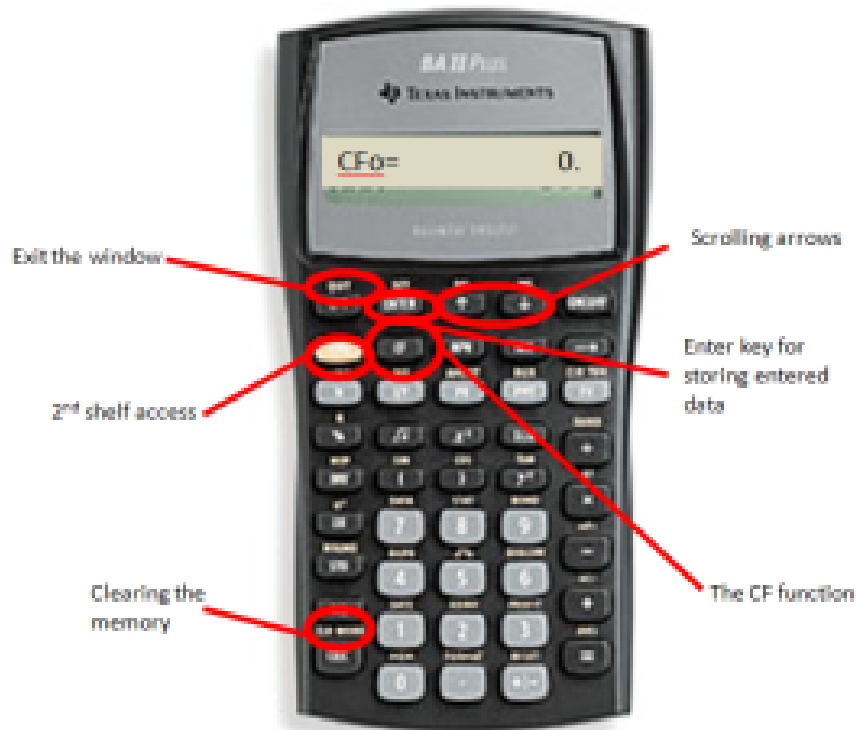


Figure 5.4.1

• Net Present Value

- Use this function after you have entered all cash flows.
- Press NPV on the keypad to access the function.
- Use the up arrow and down arrow to scroll through the window.
- To exit the window, press **2nd QUIT**.
- This window has two lines:
 - I = the matching periodic interest rate for the interval of each time segment.
 - NPV = the net present value. Press CPT to calculate this amount.

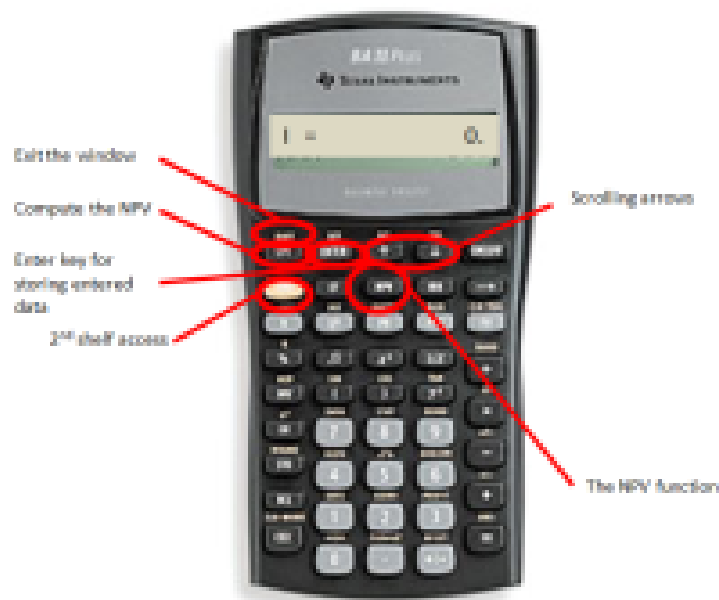


Figure 5.4.2

• Internal Rate of Return

- Use this function after you have entered all cash flows.
- Press *IRR* on the keypad to access the function.
- Press *CPT* button to perform the calculation. The output is in percent format.
- To exit the window, press **2nd QUIT**.

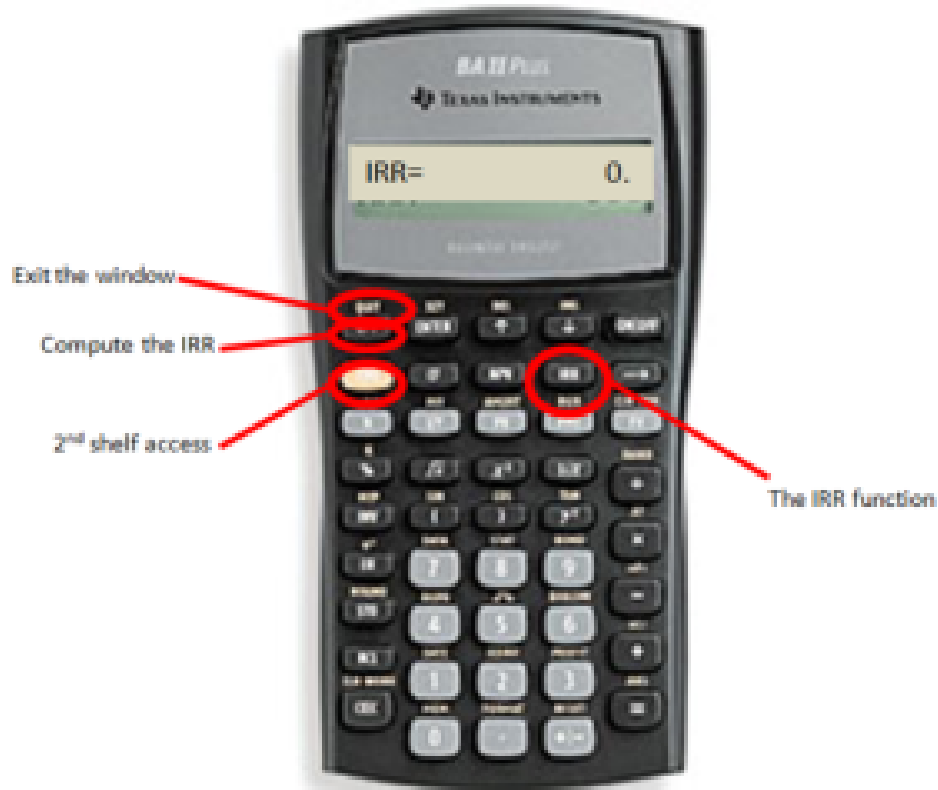


Figure 5.4.3



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[Cash Flows 1: Net Present Value \(NPV\) and IRR Calculations](#) by Joshua Emmanuel [2:52] (transcript available).

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CHAPTER 6: DESCRIPTIVE STATISTICS

Chapter Outline

[6.0 Introduction](#)

[6.1 Descriptive Statistics](#)

[6.2 Frequency Distribution and Histograms](#)

[6.3 Descriptive Measures](#)

[6.4 Skewness](#)

[6.5 Measures of Location](#)

[6.6 Graphical Representation](#)

[6.7 Review Exercises](#)

[Chapter 6 Summary](#)

6.0 INTRODUCTION

Chapter Learning Objectives

By the end of the chapter, you should be able to:

- Explain the difference between population and sample.
- Differentiate quantitative and continuous variables.
- Display data using a histogram.
- Analyze and interpret data presented in a graph.
- Calculate the mean, median, mode, range, variance, standard deviation, Standard Error of the Means, and Coefficient of Variation for any population or sample.
- Identify symmetric distribution and skewed sets.
- Identify the shape of a set of data.
- Recognize, describe, calculate, and interpret the measures of location of data: quartiles and percentiles.
- Contrast how you can represent data. Know what data sets and purposes would incline you to use one over another.

Statistics has become the universal language of the sciences, and data analysis can lead to powerful results. As scientists, researchers, and managers working in the natural resources sector, we all rely on statistical analysis to help us answer the questions that arise in the populations we manage. For example:

- Has there been a significant change in the mean saw timber volume in the red pine stands?
- Has there been an increase in the number of invasive species found in the Great Lakes?
- What proportion of white-tail deer in New Hampshire have weights below the limit considered healthy?
- Did fertilizer A, B, or C have an effect on the corn yield?

These are typical questions that require statistical analysis for the answers. In order to answer these questions,

a good random sample must be collected from the population of interest. We then use descriptive statistics to organize and summarize our sample data. The next step is inferential statistics, which allows us to use our sample statistics and extend the results to the population while measuring the reliability of the result. However, before we begin exploring different types of statistical methods, a brief review of descriptive statistics is needed.

Statistics is the science of collecting, organizing, summarizing, analyzing, and interpreting information.

Good statistics come from good samples and are used to draw conclusions or answer questions about a population. We use sample statistics to estimate population parameters (the truth). So, let's begin there...

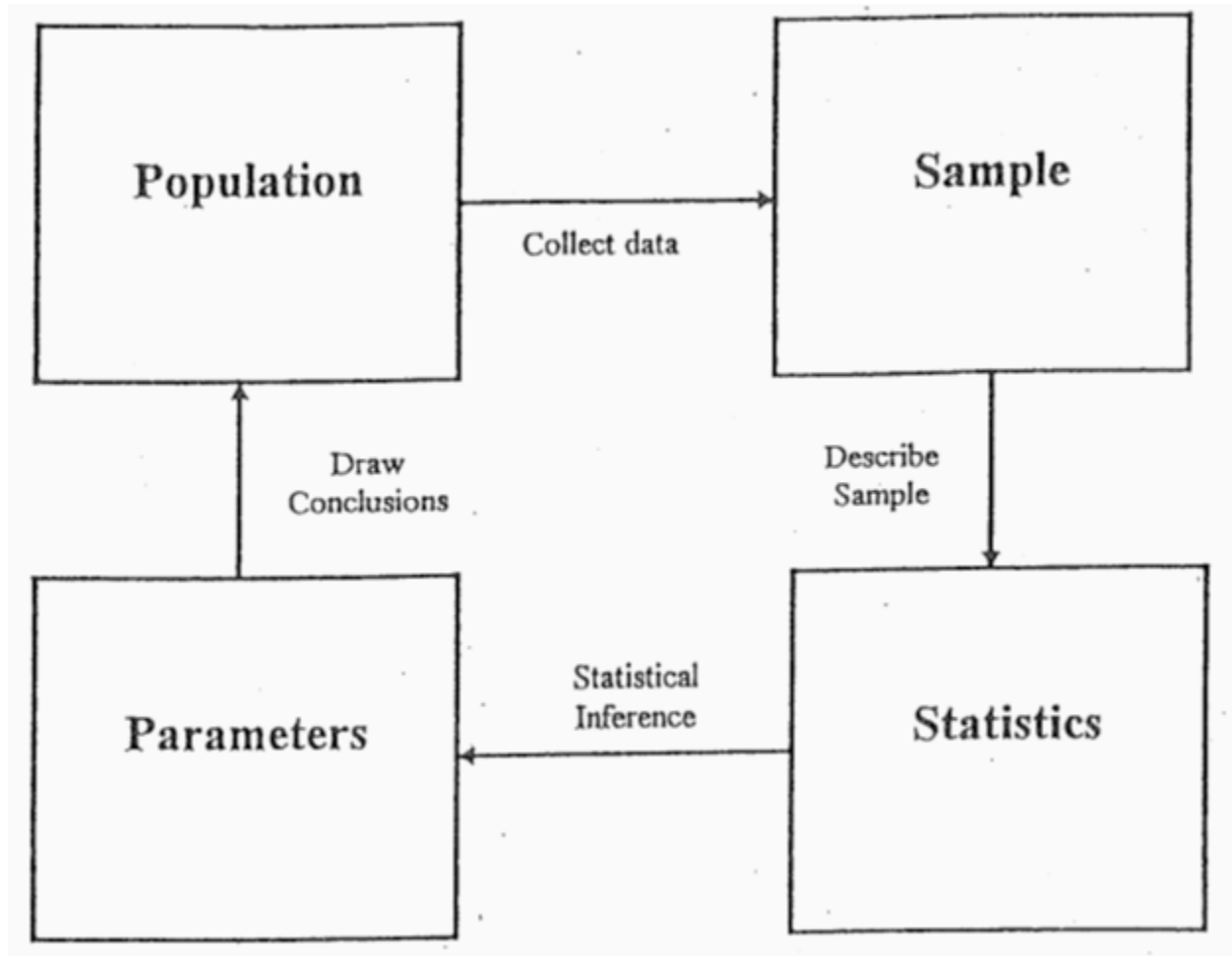


Figure 6.0.1 Using sample statistics to estimate population parameters.

Image Description

This image is a flowchart depicting the relationship between four concepts: Population, Sample, Statistics, and Parameters. Each concept is represented as a labelled box.

- **Population** box is placed at the top-left corner.

- **Sample** box is placed at the top-right corner.
- **Parameters** box is placed at the bottom-left corner.
- **Statistics** box is placed at the bottom-right corner.

Arrows are used to show the flow between these concepts:

- An arrow labelled “Collect data” points from **Population** to **Sample**.
 - An arrow labelled “Describe Sample” points from **Sample** to **Statistics**.
 - An arrow labeled “Statistical Inference” points from **Statistics** to **Parameters**.
 - An arrow labelled “Draw Conclusions” points from **Parameters** to **Population**.
-

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6.1 DESCRIPTIVE STATISTICS

Learning Objectives

- Explain the difference between population and sample.
- Differentiate quantitative and continuous variables.

Formula & Symbol Hub

Symbols Used

- μ = Population Mean
- \bar{x} = Sample Mean
- σ^2 = Population Variance
- s^2 = Sample Variance

A **population** is the group to be studied, and population data is a collection of all elements in the population. For example:

- All the fish in Long Lake.
- All the lakes in the Adirondack Park.
- All the grizzly bears in Yellowstone National Park.

A **sample** is a subset of data drawn from the population of interest. For example:

- 100 fish randomly sampled from Long Lake.
- 25 lakes randomly selected from the Adirondack Park.
- 60 grizzly bears with a home range in Yellowstone National Park.

Populations are characterized by descriptive measures called parameters. Inferences about parameters are based on sample statistics. For example, the population mean (μ) is estimated by the sample mean (\bar{x}). The population variance (σ^2) is estimated by the sample variance (s^2).

Variables are the characteristics we are interested in. For example:

- The length of fish in Long Lake.
- The pH of lakes in the Adirondack Park.
- The weight of grizzly bears in Yellowstone National Park.

Variables are divided into two major groups: qualitative and **quantitative**. Qualitative variables have values that are attributes or categories. Mathematical operations cannot be applied to qualitative variables. Examples of qualitative variables are gender, race, and petal colour. Quantitative variables have values that are typically numeric, such as measurements. Mathematical operations can be applied to these data. Examples of quantitative variables are age, height, and length.

Quantitative variables can be broken down further into two more categories: **discrete** and **continuous** variables. Discrete variables have a finite or countable number of possible values. Think of discrete variables as “hens.” Hens can lay 1 egg, or 2 eggs, or 13 eggs... There is a limited, definable number of values that the variable could take on.



Continuous variables have an infinite number of possible values. Think of continuous variables as “cows.” Cows can give 4.6713245 gallons of milk, or 7.0918754 gallons of milk, or 13.272698 gallons of milk ... There are an almost infinite number of values that a continuous variable could take on.



Try It

Is the variable qualitative or quantitative?

Species	Weight	Diameter	Zip Code
---------	--------	----------	----------

Solution

Qualitative	Quantitative	Quantitative	Qualitative
-------------	--------------	--------------	-------------

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6.2 FREQUENCY DISTRIBUTION AND HISTOGRAMS

Learning Objectives

- Create and interpret frequency tables.
- Display data using a histogram.
- Analyze and interpret data presented in a graph.

Frequency

Twenty students were asked how many hours they worked per day. Their responses, in hours, are recorded in the table below:

5	6	3	3	2	4	7	5	2	3
5	6	5	4	4	3	5	2	5	3

The following table lists the different data values in ascending order and their frequencies.

Frequency Table of Student Work Hours

Data Value	Frequency
2	3
3	5
4	3
5	6
6	2
7	1

A **frequency** is the number of times a value of the data occurs. According to the table, there are three students who work two hours, five students who work three hours, and so on. The sum of the values in the frequency column is 20, which is the total number of students included in the sample.

A **relative frequency** is the ratio (fraction or proportion) of the number of times a value of the data occurs in the set of all outcomes to the total number of outcomes. To find the relative frequencies divide each frequency by the total number of students in the sample—in this case 20. Relative frequencies can be written as fractions, percents, or decimals. The sum of the values in the relative frequency column is 1 or 100%.

Frequency Table of Student Work Hours with Relative Frequencies

Data Value	Frequency	Relative Frequency
2	3	$\frac{3}{20} = 0.15$
3	5	$\frac{5}{20} = 0.25$
4	3	$\frac{3}{20} = 0.15$
5	6	$\frac{6}{20} = 0.30$
6	2	$\frac{2}{20} = 0.10$
7	1	$\frac{1}{20} = 0.05$

Cumulative frequency is the accumulation of the previous frequencies. To find the cumulative frequencies, add all of the previous frequencies to the frequency for the current row, as shown in the table below. The last entry of the cumulative frequency column is the number of observations in the data.

Frequency Table of Student Work Hours with Relative and Cumulative Frequencies

Data Value	Frequency	Relative Frequency	Cumulative Frequency
2	3	0.15	3
3	5	0.25	$3 + 5 = 8$
4	3	0.15	$8 + 3 = 11$
5	6	0.30	$11 + 6 = 17$
6	2	0.10	$17 + 2 = 19$
7	1	0.05	$19 + 1 = 20$

Cumulative relative frequency is the accumulation of the previous relative frequencies. To find the cumulative relative frequencies, add all the previous relative frequencies to the relative frequency for the current row, as shown in the table below. The last entry of the cumulative relative frequency column is 1 or 100%, indicating that 100% of the data has been accumulated.

Frequency Table of Student Work Hours with Relative, Cumulative and Cumulative Relative Frequencies

Data Value	Frequency	Relative Frequency	Cumulative Frequency	Cumulative Relative Frequency
2	3	0.15	3	0.15
3	5	0.25	8	$0.15 + 0.25 = 0.40$
4	3	0.15	11	$0.40 + 0.15 = 0.55$
5	6	0.30	17	$0.55 + 0.30 = 0.85$
6	2	0.10	19	$0.85 + 0.10 = 0.95$
7	1	0.05	20	$0.95 + 0.05 = 1.00$

NOTE

Because of rounding of the relative frequencies, the relative frequency column may not always sum to 1 or 100%, and the last entry in the cumulative relative frequency column may not be 1 or 100%. However, they each should be close to 1 or 100%. If all of the decimals are kept in the calculations, the relative frequency column will sum to 1 or 100% and the last cumulative relative frequency will be 1 or 100%.



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[mathematicsfinance/?p=1208#oembed-1](https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=1208#oembed-1)

Watch this video: [Frequency Distributions](#) by Joshua Emmanuel [8:40] (transcript available).

Histograms

A **histogram** is a visual display of a frequency chart. It consists of contiguous, vertical boxes with both a horizontal axis and a vertical axis. The horizontal axis is labelled with the classes or categories from the frequency chart. The vertical axis is labelled either **frequency** or **relative frequency** (or percent frequency or probability). The graph will have the same shape as either label on the vertical axis, but the scale on the vertical axis will be different. The histogram gives us the shape of the data, the center of the data, and the spread of the data.

Recall that the frequency is the number of times an observation falls into that particular class, and the relative frequency is the frequency for the class divided by the total number of data values in the sample. For example, if three students in Mr. Ahab's English class of 40 students received from 90% to 100%, then the frequency

of the 90% to 100% class is 3, and the relative frequency is $\frac{3}{40} = 0.075$. So, 7.5% of the students received between 90% and 100%.

To construct a histogram, first decide how many bars or intervals, also called classes, represent the data. For clarity, many histograms consist of 5 to 15 bars or classes, but the number of bars is determined by the person constructing the histogram. Choose a starting point for the first class that is less than the smallest data value. A convenient starting point is a lower value carried out to one more decimal place than the value with the most decimal places. For example, if the value with the most decimal places is 6.1 and this is the smallest value, a convenient starting point is 6.05. We say that 6.05 has more precision. If the value with the most decimal places is 2.23 and the lowest value is 1.5, a convenient starting point is 1.495. If the value with the most decimal places is 3.234 and the lowest value is 1.0, a convenient starting point is 0.9995. If all the data happen to be integers and the smallest value is 2, then a convenient starting point is 1.5. Also, when the starting point and other boundaries are carried to one additional decimal place, no data value will fall on a boundary.

A rule of thumb is to use a histogram when the data set consists of 100 values or more.



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Video: [Histograms | Applying mathematical reasoning | Pre-Algebra | Khan Academy](#) by [Khan Academy](#) [6:8] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*

Example 6.2.1

The following data are the heights (in inches to the nearest half inch) of 100 male semiprofessional soccer players. The heights are **continuous** data because height is a measurement.

60	64	64.5	66	66.5	67	67.5	69	70	71
60.5	64	64.5	66	66.5	67	67.5	69	70	71
61	64	64.5	66.5	66.5	67	67.5	69	70	72
61	64	66	66.5	67	67	68	69	70	72
61.5	64	66	66.5	67	67	68	69	70	72
63.5	64.5	66	66.5	67	67.5	69	69.5	70	72.5
63.5	64.5	66	66.5	67	67.5	69	69.5	70.5	72.5
63.5	64.5	66	66.5	67	67.5	69	69.5	70.5	73
64	64.5	66	66.5	67	67.5	69	69.5	70.5	73.5
64	64.5	66	66.5	67	67.5	69	69.5	71	74

The smallest data value is 60. Because the data with the most decimal places has one decimal (for instance, 61.5), we want our starting point to have two decimal places. Because the numbers 0.5, 0.05, 0.005, etc. are convenient numbers, use 0.05 and subtract it from 60, the smallest value, for the convenient starting point. Then the starting point is, then, $60 - 0.05 = 59.95$. The largest value is 74, so $74 + 0.05 = 74.05$ is the ending value.

Next, calculate the width of each bar or class interval. To calculate this width, subtract the starting point from the ending value and divide by the number of classes (you must decide how many classes you want). Suppose we want to have eight classes.

$$\text{Class Width} = \frac{74.05 - 59.95}{8} = 1.76$$

We will round up to two and make each bar or class interval two units wide. Rounding up to two is one way to prevent a value from falling on a boundary. Rounding to the next number is often necessary even if it goes against the standard rules of rounding. For this example, using 1.76 as the width would also work.

The boundaries for the classes are:

$$\begin{aligned}
 &59.95 \\
 &59.95 + 2 = 61.95 \\
 &61.95 + 2 = 63.95 \\
 &63.95 + 2 = 65.95 \\
 &65.95 + 2 = 67.95 \\
 &67.95 + 2 = 69.95 \\
 &69.95 + 2 = 71.95 \\
 &71.95 + 2 = 73.95 \\
 &73.95 + 2 = 75.95
 \end{aligned}$$

The heights 60 through 61.5 inches are in the first class 59.95–61.95. The heights that are 63.5 are in the second class 61.95–63.95. The heights that are 64 through 64.5 are in the third class 63.95–65.95. The heights 66 through 67.5 are in the fourth class 65.95–67.95. The heights 68 through 69.5 are in the fifth class 67.95–69.95. The heights 70 through 71 are in the sixth class 69.95–71.95. The heights 72 through 73.5 are in the seventh class 71.95–73.95. The height 74 is in the last class 73.95–75.95.

The following histogram displays the heights on the x -axis and relative frequency on the y -axis.

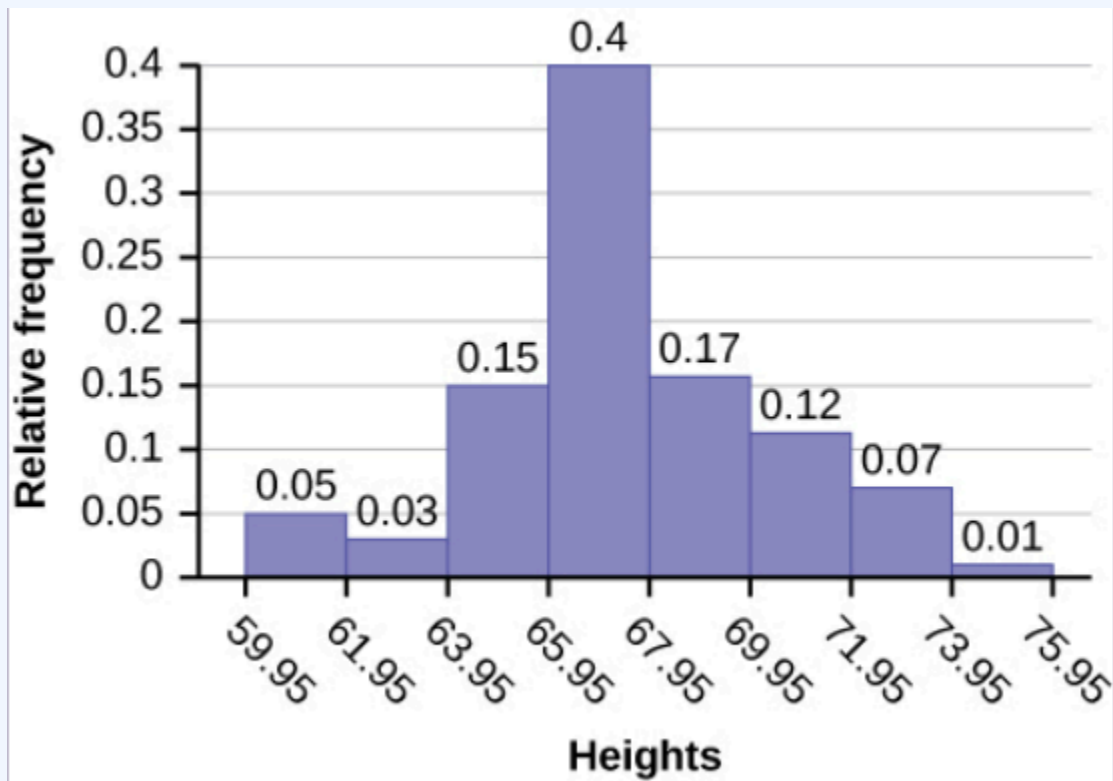


Image Description

The image is a histogram displaying the relative frequency of heights. The x-axis is labeled “Heights” with values ranging from 59.95 to 75.95. The y-axis is labeled “Relative frequency” with values ranging from 0 to 0.4. Each bar represents an interval of heights and their corresponding relative frequency values.

Height Interval	Relative Frequency
59.95	0.05
61.95	0.03
63.95	0.15
65.95	0.4
67.95	0.17
69.95	0.12
71.95	0.07
73.95	0.01

The highest bar is between 65.95 and 67.95, with a relative frequency of 0.4, indicating that this height range occurs most frequently in the data set. Other height intervals show varying lower frequencies.

Note

A guideline that is followed by some for the width of a bar or class interval is to take the square root of the number of data values and then round to the nearest whole number, if necessary. For example, if there are 150 values of data, take the square root of 150 and round to 12 bars or classes.

Try It

The following data are the shoe sizes of 50 male students. The sizes are continuous data since shoe size is measured. Construct a histogram and calculate the width of each bar or class interval. Suppose you choose six bars.

9	9	9.5	9.5	10	10	10	10	10	10
10.5	10.5	10.5	10.5	10.5	10.5	10.5	10.5	11	11
11	11	11	11	11	11	11	11	11	11
11	11.5	11.5	11.5	11.5	11.5	11.5	11.5	12	12
12	12	12	12	12	12.5	12.5	12.5	12.5	14

Solution

- Smallest value: **9**
- Largest value: **14**
- Convenient starting value: $9 - 0.05 = 8.95$
- Convenient ending value: $14 + 0.05 = 14.05$
- Class width: $\frac{14.05 - 8.95}{6} = 0.85$

The calculations suggest using **0.85** as the width of each bar or class interval. You can also use an interval with a width equal to one.

Example 6.2.2

The following data are the number of books bought by 50 part-time college students at ABC College. The number of books is **discrete data** because books are counted.

1	1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2	2
2	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	4	4	4
4	4	4	5	5	5	5	5	6	6

Eleven students buy one book. Ten students buy two books. Sixteen students buy three books. Six students buy four books. Five students buy five books. Two students buy six books.

Because the data are integers, subtract 0.5 from 1, the smallest data value and add 0.5 to 6, the largest data value, to get the starting and ending points. Then the starting point is 0.5 and the ending value is 6.5.

Next, calculate the width of each bar or class interval. If the data are discrete and there are not too many different values, a width that places the data values in the middle of the bar or class interval is the most convenient. Because the data consist of the numbers 1, 2, 3, 4, 5, 6 and the starting point is 0.5, a width of one places the 1 in the middle of the interval from 0.5 to 1.5, the 2 in the middle of the interval from 1.5 to 2.5, the 3 in the middle of the interval from 2.5 to 3.5, the 4 in the middle of the interval from _____ to _____, the 5 in the middle of the interval from _____ to _____, and the _____ in the middle of the interval from _____ to _____.

Solution:

- 3.5 to 4.5
- 4.5 to 5.5
- 6
- 5.5 to 6.5

Calculate the number of bars as follows:

$$\frac{6.5 - 0.5}{\text{number of bars}} = 1$$

where 1 is the width of a bar. Therefore, the number of bars is 6.

The following histogram displays the number of books on the x -axis and the frequency on the y -axis.

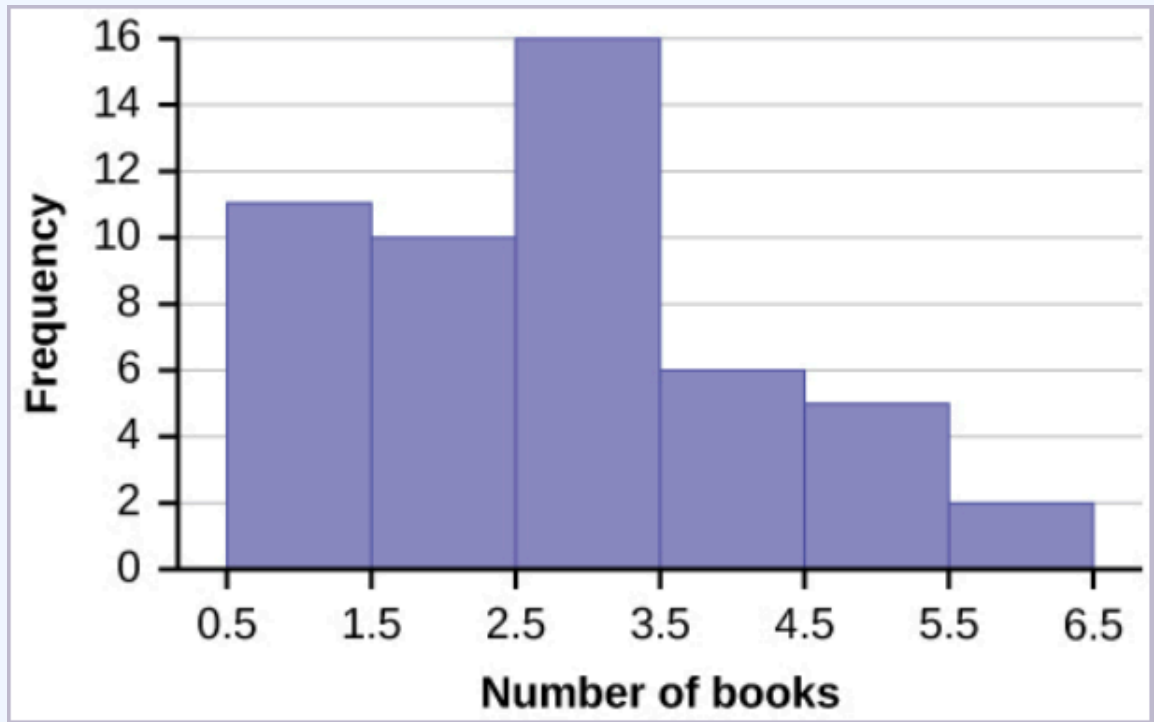


Image Description

The image is a histogram displaying the frequency of books read, divided into different number ranges. The x-axis is labelled "Number of books" and ranges from 0.5 to 6.5, with increments of 1. The y-axis is labelled "Frequency" and ranges from 0 to 16, with increments of 2.

Bars in the histogram represent the following data:

- From 0.5 to 1.5: Frequency is 10.
- From 1.5 to 2.5: Frequency is 12.
- From 2.5 to 3.5: Frequency is 16.
- From 3.5 to 4.5: Frequency is 5.
- From 4.5 to 5.5: Frequency is 3.
- From 5.5 to 6.5: Frequency is 2.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=1208#oembed-3>

[Video: How to Construct a Histogram in Excel using built-in Data Analysis](#) by [Joshua Emmanuel](#) [1:59] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*

Try It

The following data are the number of sports played by 50 student-athletes. The number of sports is discrete data because sports are counted.

1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
2	2	3	3	3	3	3	3	3	3

Fill in the blanks for the following sentence. Because the data consist of the numbers 1, 2, 3, and the starting point is 0.5, a width of one places the 1 in the middle of the interval 0.5 to _____, the 2 in the middle of the interval from _____ to _____, and the 3 in the middle of the interval from _____ to _____.

Solution

- 1.5

- 1.5 to 2.5
- 2.5 to 3.5

Example 6.2.3

Using this data set, construct a histogram.

Number of Hours My Classmates Spent Playing Video Games on Weekends				
9.95	10	2.25	16.75	0
19.5	22.5	7.5	15	12.75
5.5	11	10	20.75	17.5
23	21.9	24	23.75	18
20	15	22.9	18.8	20.5

**Image Description**

The image is a histogram titled “Hours Spent Playing Video Games on Weekends.” It displays data on the number of students and the hours they spend playing video games during weekends. The x-axis represents the “Number of hours,” ranging from 0 to 25. The y-axis represents the “Number of students,” ranging from 0 to 10. The bars indicate the following data:

Number of hours	Number of students
0-5	2
5-10	3
10-15	4
15-20	6
20-25	9

Some values in this data set fall on boundaries for the class intervals. A value is counted in a class interval if it falls on the left boundary but not if it falls on the right boundary. Different researchers may set up histograms for the same data in different ways. There is more than one correct way to set up a histogram.

Attribution

“[2.2 Histograms, Frequency Polygons, and Time Series Graphs](#)” & “[1.4 Frequency, Frequency Tables, and Levels of Measurement](#)” in [Introduction to Statistics](#) by Valerie Watts is licensed under a [Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License](#), except where otherwise noted.

6.3 DESCRIPTIVE MEASURES

Learning Objectives

- Calculate the mean, median, mode, range, variance, standard deviation, Standard Error of the Means, and Coefficient of Variation for any population or sample.
- Identify symmetric distribution and skewed sets.

Formula & Symbol Hub

Symbols Used

- μ = Population mean
- σ^2 = Population variance
- σ = Population standard deviation
- \bar{x} = Sample Mean
- s^2 = Sample variance
- s = Sample standard deviation
- $s_{\bar{x}}$ = Standard error of the means

Formulas Used

- Formula 6.1 – **Population Mean**

$$\mu = \frac{\sum x_i}{N}$$

- Formula 6.2 – **Sample mean**

$$\bar{x} = \frac{\sum x_i}{n}$$

- Formula 6.3 – **Population Variance**

$$\sigma^2 = \frac{\sum (x_i - \mu)^2}{N}$$

- Formula 6.4 – **Sample Variance**

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1} = \frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n - 1}$$

- Formula 6.5 – **Population Standard Deviation**

$$\sigma = \sqrt{\sigma^2}$$

- Formula 6.6 – **Sample Standard Deviation**

$$s = \sqrt{s^2}$$

- Formula 6.7 – **Standard Error of the Means**

$$s_{\bar{x}} = \sqrt{\frac{s^2}{n}} = \frac{s}{\sqrt{n}}$$

- Formula 6.8 – **Coefficient of Variation (Population)**

$$CV = \frac{\sigma}{\mu} \times 100$$

- Formula 6.9 – **Coefficient of Variation (Sample)**

$$CV = \frac{s}{\bar{x}} \times 100$$

Descriptive measures of populations are called parameters and are typically written using Greek letters. The population mean is μ (mu). The population variance is σ^2 (sigma squared), and population standard deviation is σ (sigma).

Descriptive measures of samples are called statistics and are typically written using Roman letters. The sample mean is \bar{x} (x-bar). The sample variance is s^2 and the sample standard deviation is s . Sample statistics are used to estimate unknown population parameters.

In this section, we will examine descriptive statistics in terms of **measures of center** and **measures of dispersion**. These descriptive statistics help us to identify the center and spread of the data.

Measures of Center

Mean

The arithmetic mean of a variable, often called the average, is computed by adding up all the values and dividing by the total number of values.

The population mean is represented by the Greek letter μ (mu). The sample mean is represented by \bar{x} (x-bar). The sample mean is usually the best, unbiased estimate of the population mean. However, the mean is influenced by extreme values (outliers) and may not be the best measure of center with strongly **skewed data**. The following equations compute the population mean and sample mean.

6.1 Population Mean

$$\mu = \frac{\sum x_i}{N}$$

6.2 Sample Mean

$$\bar{x} = \frac{\sum x_i}{n}$$

x_i is an element in the data set.

μ and \bar{x} are the population and sample means: the averages of all values in their respective data sets.

N and n are the number of elements in the data set: N is used for calculations of the population mean while n is used for calculations of the sample mean.

Try It

Find the mean for the following sample data set: 6.4, 5.2, 7.9, 3.4

Solution

$$\bar{x} = \frac{6.4 + 5.2 + 7.9 + 3.4}{4} = 5.725$$

Median

The median of a variable is the middle value of the data set when the data are sorted in order from least to greatest. It splits the data into two equal halves, with 50% of the data below the median and 50% above the median. The median is resistant to the influence of outliers and may be a better measure of the center with strongly skewed data.

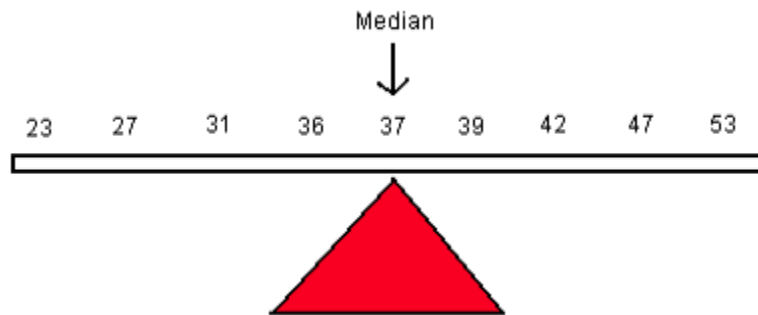


Figure 6.2.1 The median value of a data set.

Image Description

The image is a visual representation of a set of numbers balanced on a red triangle, indicating the median value. The numbers, in ascending order from left to right, are 23, 27, 31, 36, 37, 39, 42, 47, and 53. Above the number 37, there is an arrow pointing down with the label “Median.”

The calculation of the median depends on the number of observations in the data set.

To calculate the median with an odd number of values (n is odd), first sort the data from smallest to largest.

Example 6.2.1

23, 27, 29, 31, 35, 39, 40, 42, 44, 47, 51

The median is **39**. It is the middle value that separates the lower **50%** of the data from the upper **50%** of the data.

To calculate the median with an even number of values (n is even), first sort the data from smallest to largest and take the average of the two middle values.

Example 6.2.2

23, 27, 29, 31, 35, 39, 40, 42, 44, 47

$$M = \frac{35 + 39}{2} = 37$$

Mode

The mode is the most frequently occurring value and is commonly used with qualitative data as the values are categorical. Categorical data cannot be added, subtracted, multiplied or divided, so the mean and median cannot be computed. The mode is less commonly used with quantitative data as a measure of center. Sometimes, each value occurs only once, and the mode will not be meaningful.



Understanding the relationship between the mean and median is important. It gives us insight into the distribution of the variable. For example, if the distribution is skewed right (positively skewed), the mean will increase to account for the few larger observations that pull the distribution to the right. The median will be less affected by these extremely large values, so in this situation, the mean will be larger than the median. In a symmetric distribution, the mean, median, and mode will all be similar in value. If the distribution is skewed left (negatively skewed), the mean will decrease to account for the few smaller observations that pull the distribution to the left. Again, the median will be less affected by these extremely small observations, and in this situation, the mean will be less than the median.

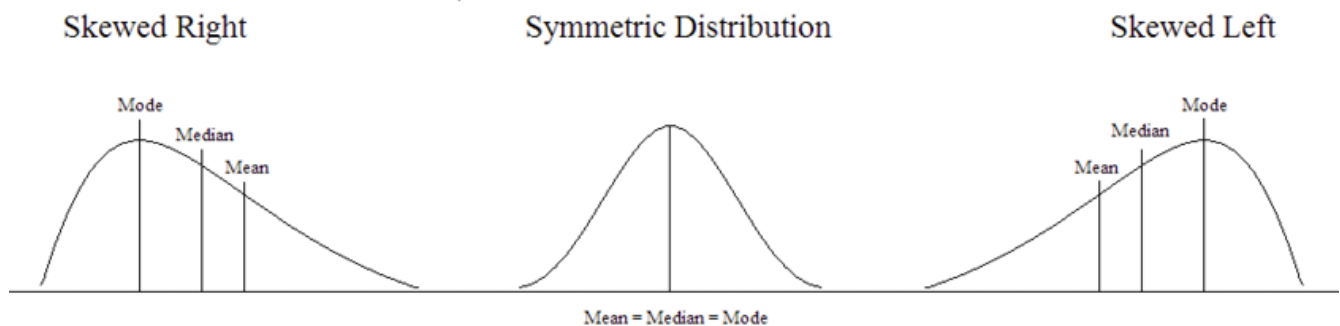


Figure 6.2.2 Illustration of skewed and symmetric distributions.

Image Description

The image contains three graphs illustrating different types of data distributions: skewed right, symmetric distribution, and skewed left.

Skewed Right: The graph on the left is labelled “Skewed Right.” It shows a distribution where the bulk of the data is concentrated on the left side, with a long tail stretching to the right. Three vertical lines are marked on the graph, indicating the mode (highest point), median, and mean (from left to right). The right tail skews the mean to the right.

Symmetric Distribution: The middle graph is labelled “Symmetric Distribution.” It depicts a bell-shaped distribution where the data is evenly distributed around the center. A single vertical line is present at the center of the graph, indicating that the mean, median, and mode are all equal. Below the graph is the notation, “Mean = Median = Mode”.

Skewed Left: The graph on the right is labelled “Skewed Left.” This graph shows a distribution where the majority of the data is concentrated on the right side, with a long tail extending to the left. Three vertical lines are marked, showing the mean, median, and mode (from left to right). The left tail skews the mean to the left.

Measures of Dispersion

Measures of center look at the average or middle values of a data set. Measures of dispersion look at the spread or variation of the data. Variation refers to the amount of values that vary among themselves. Values in a data set that are relatively close to each other have lower measures of variation. Values that are spread farther apart have higher measures of variation.

Examine the two histograms below. Both groups have the same mean weight, but the values of Group A are more spread out compared to the values in Group B. Both groups have an average weight of **267 lb.** but the weights of Group A are more variable.

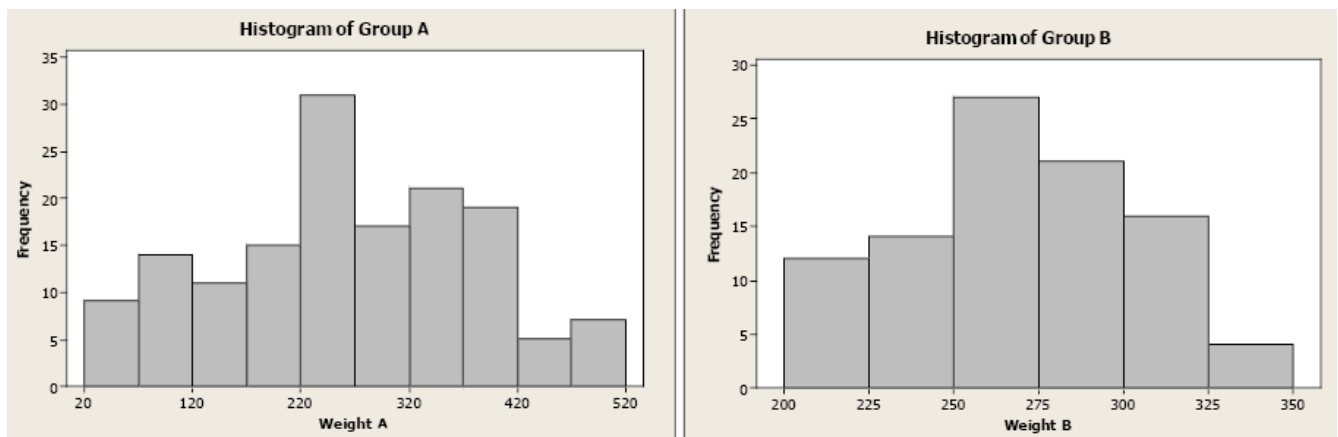


Figure 6.2.3 Histograms of Group A and Group B.

Image Description

This image contains two histograms placed side by side.

On the left side, there is a histogram titled “Histogram of Group A.”

- The x-axis is labeled “Weight A” and ranges from 20 to 520.
- The y-axis is labelled “Frequency” and ranges from 0 to 35.
- The histogram shows the distribution of weights in Group A, with the tallest bar reaching a frequency of just above 30 in the weight range of 220 — 320. The frequencies vary between 5 and 30 across different weight intervals.

On the right side, there is a histogram titled “Histogram of Group B .”

- The x-axis is labeled “Weight B ” and ranges from 200 to 350.
- The y-axis is labelled “Frequency” and ranges from 0 to 30.
- The histogram shows the distribution of weights in Group B , with the tallest bar reaching a frequency of just above 25 in the weight range of 250 – 275. The frequencies vary between 5 and 30 across different weight intervals.

Range

The range of a variable is the largest value minus the smallest value. It is the simplest measure and uses only these two values in a quantitative data set.

Try It

Find the range for the given data set.

12, 29, 32, 34, 38, 49, 57

Solution

$$\text{Range} = 57 - 12 = 45$$

Variance

The variance uses the difference between each value and its arithmetic mean. The differences are squared to deal with positive and negative differences. The sample variance (s^2) is an unbiased estimator of the population variance (σ^2), with $n - 1$ degrees of freedom.

Degrees of freedom: In general, the degrees of freedom for an estimate is equal to the number of values minus the number of parameters estimated en route to the estimate in question.

The sample variance is unbiased due to the difference in the denominator. If we used n in the denominator instead of $n - 1$, we would consistently underestimate the true population variance. To correct this bias, the denominator is modified to $n - 1$.

6.3 Population Variance

$$\sigma^2 = \frac{\sum (x_i - \mu)^2}{N}$$

6.4 Sample Variance

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1} = \frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n - 1}$$

x_i is an element in the data set.

σ^2 and s^2 are the population and sample variances: the spreads of values in their data sets.

N and n are the number of elements in a data set: again, N is used for the population data set and n for the sample data set.

μ and \bar{x} are the population and sample means: these values can be calculated with formulas 6.1 and 6.2 respectively

Try It

Compute the variance of the sample data: 3, 5, 7. The sample mean is 5.

Solution

$$s^2 = \frac{(3 - 5)^2 + (5 - 5)^2 + (7 - 5)^2}{3 - 1} = 4$$

Standard Deviation

The standard deviation is the square root of the variance (both population and sample). While the sample variance is the positive, unbiased estimator for the population variance, the units for the variance are squared. The standard deviation is a common method for numerically describing the distribution of a variable.

6.5 Population Standard Deviation

$$\sigma = \sqrt{\sigma^2}$$

6.6 Sample Standard Deviation

$$s = \sqrt{s^2}$$

σ (sigma) and s are the population and sample standard deviations.

σ^2 and s^2 are the population and sample variances: these values can be calculated with formulas 6.3 and 6.4 respectively.

Try It

Compute the standard deviation of the sample data: 3, 5, 7 with a sample mean of 5.

Solution

$$s = \sqrt{\frac{(3 - 5)^2 + (5 - 5)^2 + (7 - 5)^2}{3 - 1}} = \sqrt{4} = 2$$

Standard Error of the Means

Commonly, we use the sample mean \bar{x} to estimate the population mean μ . For example, if we want to estimate the heights of eighty-year-old cherry trees, we can proceed as follows:

- Randomly select 100 trees
- Compute the sample mean of the 100 heights
- Use that as our estimate

We want to use this sample mean to estimate the true but unknown population mean. But our sample of 100 trees is just one of many possible samples (of the same size) that could have been randomly selected. Imagine if we take a series of different random samples from the same population and all the same size:

- Sample 1—we compute sample mean \bar{x}
- Sample 2—we compute sample mean \bar{x}
- Sample 3—we compute sample mean \bar{x}
- Etc.

Each time we sample, we may get a different result as we are using a different subset of data to compute the sample mean. This shows us that the sample mean is a random variable!

The sample mean (\bar{x}) is a random variable with its own probability distribution called the sampling distribution of the sample mean. The distribution of the sample mean will have a mean equal to μ and a standard deviation equal to $\frac{s}{\sqrt{n}}$.

The standard error $\frac{s}{\sqrt{n}}$ is the standard deviation of all possible sample means.

In reality, we would only take one sample, but we need to understand and quantify the sample-to-sample variability that occurs in the sampling process.

The standard error is the standard deviation of the sample means and can be expressed in different ways.

6.7 Standard Error of the Means

$$s_{\bar{x}} = \sqrt{\frac{s^2}{n}} = \frac{s}{\sqrt{n}}$$

$s_{\bar{x}}$ is the standard error.

n is the number of values in the sample data set.

s^2 is the sample variance calculated by formula 6.4

s is the sample standard deviation calculated by formula 6.6

Example 6.2.3

Describe the distribution of the sample mean.

A population of fish has weights that are normally distributed with $\mu = 8$ lb. and $s = 2.6$ lb. If you take a sample of size $n = 6$, the sample mean will have a normal distribution with a mean of 8 and a standard deviation (standard error) of $\frac{2.6}{\sqrt{6}} = 1.061$ lb.

If you increase the sample size to 10, the sample mean will be normally distributed with a mean of 8 lb. and a standard deviation (standard error) of $\frac{2.6}{\sqrt{10}} = 0.822$ lb.

Notice how the standard error decreases as the sample size increases.

The **Central Limit Theorem** (CLT) states that the sampling distribution of the sample means will approach a normal distribution as the sample size increases. If we do not have a normal distribution, or know nothing about our distribution of our random variable, the CLT tells us that the distribution of the \bar{x} 's will become normal as n increases. How large does n have to be? A general rule of thumb tells us that $n \geq 30$.

The Central Limit Theorem tells us that regardless of the shape of our population, the sampling distribution of the sample mean will be normal as the sample size increases.

Coefficient of Variation

To compare standard deviations between different populations or samples is difficult because the standard deviation depends on units of measure. The coefficient of variation expresses the standard deviation as a percentage of the sample or population mean. It is a unitless measure.

6.8 Coefficient of Variation (Population)

$$CV = \frac{\sigma}{\mu} \times 100$$

6.9 Coefficient of Variation (Sample)

$$CV = \frac{s}{\bar{x}} \times 100$$

CV is the coefficient of variation.

σ and s are the population and sample standard deviations, calculated by formulas 6.5 and 6.6 respectively.

μ and \bar{x} are the population and sample means, calculated by formulas 6.1 and 6.2 respectively.

Example 6.2.4

Fisheries biologists were studying the length and weight of Pacific salmon. They took a random sample and computed the mean and standard deviation for length and weight (given below). While the standard deviations are similar, the differences in units between lengths and weights make it difficult to compare the variability. Computing the coefficient of variation for each variable allows the biologists to determine which variable has the greater standard deviation.

	Sample Mean	Sample Standard Deviation
Length	63 cm	19.97 cm
Weight	37.6 kg	19.39 kg
	$CV_L = \frac{19.97}{53.0} \times 100 = 31.5\%$	$CV_W = \frac{19.39}{37.6} \times 100 = 51.6\%$

There is greater variability in Pacific salmon weight compared to length.

Variability

Variability is described in many different ways. Standard deviation measures point-to-point variability within a sample, i.e., variation among individual sampling units. Coefficient of variation also measures point to point variability but on a relative basis (relative to the mean), and is not influenced by measurement units. Standard error measures the sample-to-sample variability, i.e. variation among repeated samples in the sampling process. Typically, we only have one sample and standard error allows us to quantify the uncertainty in our sampling process.

Basic Statistics Example using Excel and Minitab Software

Consider the following tally from 11 sample plots on Heiburg Forest, where X_i is the number of downed logs per acre. Compute basic statistics for the sample plots.

ID	X_i	X_i^2	$(X_i - \bar{X})$	$(X_i - \bar{X})^2$	Order
1	25	625	-7.27	52.8529	4
2	35	1225	2.73	7.4529	6
3	55	3025	22.73	516.6529	10
4	15	225	-17.25	298.2529	2
5	40	1600	7.73	59.7529	8
6	25	625	-7.27	52.8529	5
7	55	3025	22.73	516.6529	11
8	35	1225	2.73	7.4529	7
9	45	2025	12.73	162.0529	9
10	5	25	-27.27	743.6529	1
11	20	400	-12.27	150.1819	3
Sum	355	14025	0.0	2568.1519	
	$\sum_{i=1}^n X_i$	$\sum_{i=1}^n X_i^2$	$\sum_{i=1}^n (X_i - \bar{X})$	$\sum_{i=1}^n (X_i - \bar{X})^2$	

1. Sample mean:

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n} = \frac{355}{11} = 32.27$$

2. Median = 35

3. Variance:

$$\begin{aligned}
 S^2 &= \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n - 1} = \frac{2568.1519}{11 - 1} = 256.82 \\
 &= \frac{\sum_{i=1}^n X_i^2 - \frac{\left(\sum_{i=1}^n X_i\right)^2}{n}}{n - 1} = \frac{14025 - \frac{(355)^2}{11}}{11 - 1} = 256.82
 \end{aligned}$$

4. Standard Deviation:

$$S = \sqrt{S^2} = \sqrt{256.82} = 16.0256$$

5. Range: $55 - 5 = 50$

6. Coefficient of variation:

$$CV = \frac{S}{X} \cdot 100 = \frac{16.0256}{32.27} \cdot 100 = 49.66\%$$

7. Standard Error of the Mean:

$$\begin{aligned} S_{\bar{X}} &= \sqrt{\frac{S^2}{n}} = \sqrt{\frac{256.82}{11}} = 4.8319 \\ &= \frac{S}{\sqrt{n}} = \frac{16.0256}{\sqrt{11}} = 4.8319 \end{aligned}$$

Attribution

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6.4 SKEWNESS

Learning Objectives

- Identify the shape of a set of data.

Symmetrical Distribution

Consider the following data set:

4	5	6	6	6	7	7	7
7	7	7	8	8	8	9	10

This data set can be represented by the following histogram. Each interval has width one, and each value is located in the middle of an interval.

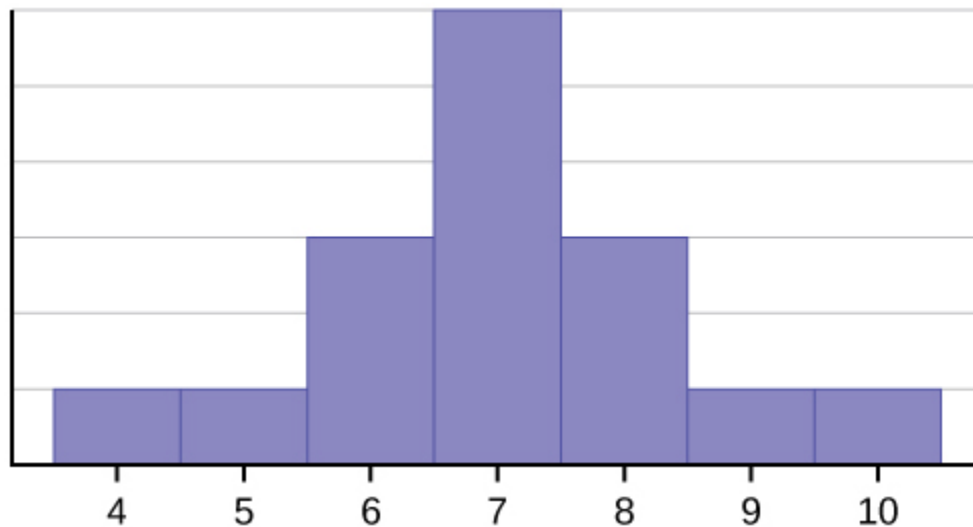


Figure 6.3.1 Histogram: symmetrical distribution of data

Image Description

A histogram with data values ranging from 4 to 10 along the x-axis. The y-axis represents the frequency of occurrences within each bin, though specific frequency values are not labelled. The histogram has the following distribution:

- Values at 4 and 10 have lower frequencies.
- Frequencies increase for values at 6 and 8.
- The highest frequency occurs at the value 7, forming a peak in the center.
- The distribution appears roughly symmetrical, centred around the value 7.

The histogram above displays a **symmetrical** distribution of data. A distribution is symmetrical if a vertical line can be drawn at some point in the histogram so that the shapes to the left and the right of the vertical line are mirror images. For the above data set, the mean, the median, and the mode are each seven. In a perfectly symmetrical distribution, the mean and the median are the same. This example has one mode, and the mode is the same as the mean and median. In a symmetrical distribution that has multiple modes, the modes would be different from the mean and median.

Left Skewed Distribution

Consider the following data set:

4	5	6	6	6	7	7	7	7	8
---	---	---	---	---	---	---	---	---	---

This data set can be represented by the following histogram. Each interval has width one, and each value is located in the middle of an interval.

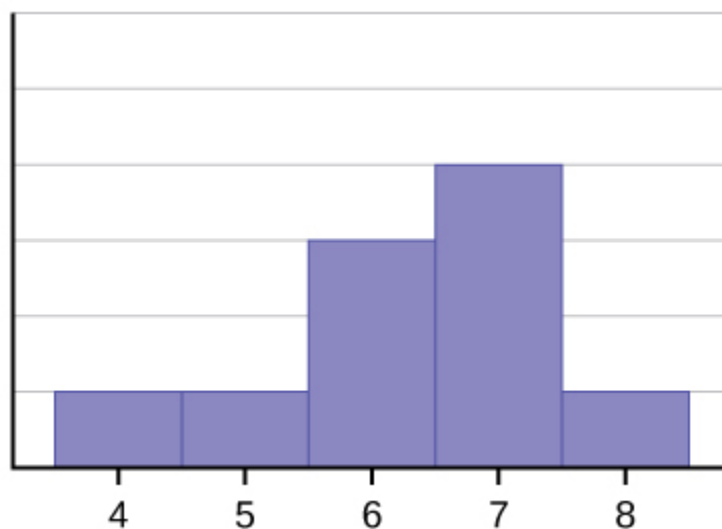


Figure 6.3.2 Histogram: skewed to the left

Image Description

A histogram displaying data values from 4 to 8 on the x-axis. The y-axis indicates frequency, though specific frequency values are not shown. The histogram has the following characteristics:

- Lower frequencies are observed at values 4, 5, and 8.
- Frequencies increase at values 6 and 7, with the highest frequency at 7.
- The distribution is skewed slightly towards the higher end, with a peak at 7.

The histogram above is not symmetrical. The right-hand side seems “chopped off” compared to the left side. A distribution of this type is called *skewed to the left* because it is pulled out to the left. The mean of this data is 6.3, the median is 6.5, and the mode is 7. *Notice that the mean is less than the median, and they are both less than the mode.* The mean and the median both reflect the skewing, but the mean reflects it more so.

Right Skewed Distribution

Consider the following data set:

6	7	7	7	7	8	8	8	9	10
---	---	---	---	---	---	---	---	---	----

This data set can be represented by the following histogram. Each interval has width one, and each value is located in the middle of an interval.

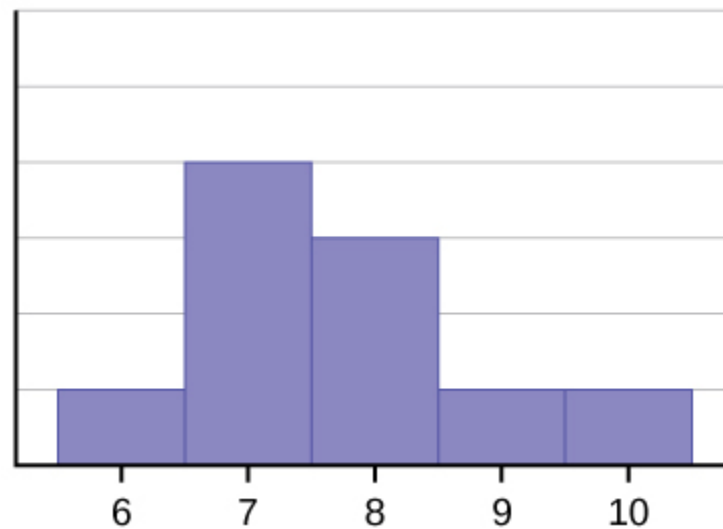


Figure 6.3.2 Histogram: skewed to the right

Image Description

A histogram with data values ranging from 6 to 10 on the x-axis. The y-axis represents frequency, although specific values are not labelled. The histogram's distribution shows:

- Lower frequencies at values 9 and 10.
- Higher frequencies at values 7 and 8, with the highest peak at 7.
- The distribution is slightly skewed to the right, with most data clustered around 7 and 8.

The histogram above is also not symmetrical. In this case, the data is **skewed to the right**. The mean for this data is 7.7, the median is 7.5, and the mode is 7. Of the three statistics, *the mean is the largest, while the mode is the smallest*. Again, the mean reflects the skewing the most.



- If the distribution of the data is symmetrical, **mean** = **median** = **mode** (assuming there is only one mode). If there are multiple modes in a symmetric distribution, the modes would be different from the mean and the median, but the mean and median would still be equal.
- If the distribution of the data is skewed to the left, **mean** < **median** < **mode**.
- If the distribution of the data is skewed to the right, **mean** > **median** > **mode**.



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Video: [Elementary Business Statistics | Skewness and the Mean, Median, and Mode](#) by Janux [3:58] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*

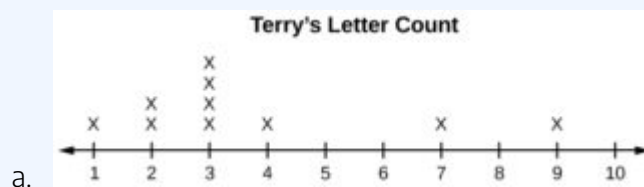
Example 6.4.1

Statistics are used to compare and sometimes identify authors. The following list shows a simple random sample that compares the letter counts for three authors.

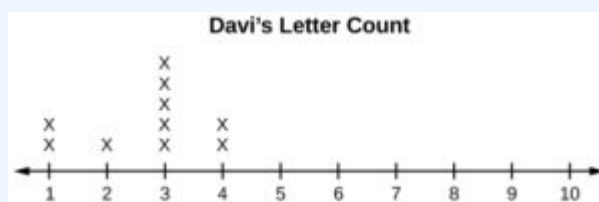
Terry									
7	9	3	3	3	4	1	3	2	2
Davis									
3	3	3	4	1	4	3	2	3	1
Maris									
2	3	4	4	4	6	6	6	8	3

1. Make a dot plot for the three authors and compare the shapes.
2. Calculate the mean for each.
3. Calculate the median for each.
4. Describe any pattern you notice between the shape and the measures of center.

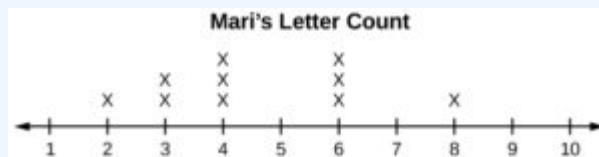
Solution:



Terry's distribution has a right (positive) skew.



Davis' distribution has a left (negative) skew



Maris' distribution is symmetrically shaped.

- b. Terry's mean is 3.7, Davis' mean is 2.7, Maris' mean is 4.6.
- c. Terry's median is three, Davis' median is three. Maris' median is four.
- d. It appears that the median is always closest to the high point (the mode), while the mean tends to be farther out on the tail. In a symmetrical distribution, the mean and the median are both centrally located close to the high point of the distribution.

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6.5 MEASURES OF LOCATION

Learning Objectives

- Recognize, describe, calculate, and interpret the measures of location of data: quartiles and percentiles.

Formula & Symbol Hub

Symbols Used

- Q_1 = First Quartile
- Q_2 = Second Quartile or median
- Q_3 = Third Quartile

Formulas Used

- Formula 6.10 – **Interquartile Range**

$$IQR = Q_3 - Q_1$$

The common measures of location are **quartiles** and **percentiles**. Previously, we learned that the **median** is a number that measures the “center” of the data. But the median can also be thought of as a measure of location

because the median is the “middle value” of a set of data. The median is a number that separates ordered data into halves. Half of the values in the data are the same number or smaller than the median and half of the values in the data are the same number or larger.

For example, consider the following data, already ordered from smallest to largest:

1	1	2	2	4	6	6.8
7.2	8	8.3	9	10	10	11.5

Because there are 14 observations, the median is between the seventh value, 6.8, and the eighth value, 7.2. To find the median, add the two values together and divide by two:

$$\frac{6.8 + 7.2}{2} = 7$$

The median is seven. We can see that half (or 50%) of the values are less than seven and half (or 50%) of the values are larger than seven.

The median is an example of both a quartile and a percentile. The median is also the second quartile, Q_2 , and the 50th percentile, P_{50} .

Quartiles

Quartiles are numbers that separate the data into quarters (four parts). Like the median, quartiles may or may not be an actual value in the set of data. To find the quartiles, order the data (from smallest to largest) and then find the median or second quartile. The first quartile, Q_1 , is the middle value of the lower half of the data and the third quartile, Q_3 , is the middle value of the upper half of the data. To get the idea, consider the same (ordered) data set used above:

1	1	2	2	4	6	6.8
7.2	8	8.3	9	10	10	11.5

The median or **second quartile** is seven. The lower half of the data are:

1	1	2	2	4	6	6.8
---	---	---	---	---	---	-----

The middle value of the lower half of the data is 2. The number 2, which is part of the data, is the **first**

quartile, Q_1 . One-fourth (or 25%) of the entire sets of values are the same as or less than 2 and three-fourths (or 75%) of the values are more than two.

The upper half of the data are:

7.2	8	8.3	9	10	10	11.5
-----	---	-----	---	----	----	------

The middle value of the upper half of the data is 9. The **third quartile, Q_3** , is 9. Three-fourths (or 75%) of the values are less than 9. One-fourth (or 25%) of the values in the data set are greater than or equal to 9.

The **interquartile range** is a number that indicates the spread of the middle half or the middle 50% of the data. It is the difference between the third quartile (Q_3) and the first quartile (Q_1).

6.10 Interquartile Range

$$IQR = Q_3 - Q_1$$

The **IQR** can help to determine potential *outliers*. A value is suspected to be a potential outlier if it is less than $1.5 \times IQR$ below the first quartile or more than $1.5 \times IQR$ above the third quartile. Potential outliers always require further investigation.

Note

A potential outlier is a data point that is significantly different from the other data points. These special data points may be errors, some kind of abnormality, or they may be a key to understanding the data.



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Video: “[Median, Quartiles and Interquartile Range : ExamSolutions](#)” by [ExamSolutions](#) [12:36] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*

Calculating Quartiles in Excel

To find quartiles in Excel, use the `quartile.exc(array, quartile number)` function.

- For array, enter the array or cell range containing the data.
- For quartile number, enter the quartile (1, 2 or 3) being calculated.

The output from the `quartile.exc` function is the value of the corresponding quartile. For example, `quartile.exc(array,1)` returns the value of the first quartile where 25% of the observations in the data are (strictly) less than the value of the first quartile.

Visit the [Microsoft page](#) for more information about the `quartile.exc` function.

Note

We are using the `quartile.exc` function, and not the `quartile.inc` function, because we want the percent of the observations in the data to be strictly less than the value of the quartile.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=1303#oembed-2>

Video: “[How To Find Quartiles and Construct a Boxplot in Excel](#)” by [Joshua Emmanuel](#) [4:13] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*

Example 6.5.1

For the following 13 real estate prices, calculate the three quartiles and the *IQR*. Determine if any prices are potential outliers. The prices are in dollars.

389,950	230,500	158,000	479,000	639,000	114,950	5,500,000
387,000	659,000	529,000	575,000	488,800	1,095,000	

Solution:

Enter the data into an Excel spreadsheet. For this example, suppose we entered the data in column A from cell A1 to A13.

For the first quartile Q_1 :

Function	quartile.exc	Answer
Field 1	A1:A13	\$308,750
Field 2	1	

For the second quartile Q_2 :

Function	quartile.exc	Answer
Field 1	A1:A13	\$488,800
Field 2	2	

For the third quartile Q_3 :

Function	quartile.exc	Answer
Field 1	A1:A13	\$649,000
Field 2	3	

For the IQR: $IQR = 649,000 - 308,750 = \$340,250$

To determine if there are any outliers:

$$1.5 \times IQR = 1.5 \times 340,250 = 510,375$$

$$Q_1 - 1.5 \times IQR = 308,750 - 510,375 = -201,625$$

$$Q_3 + 1.5 \times IQR = 649,000 + 510,375 = 1,159,375$$

No house price is less than $-\$201,625$. However, $\$5,500,000$ is more than $\$1,159,375$. Therefore, $\$5,500,000$ is a potential outlier.

Note

Quartiles have the same units as the data. In this case, the data is measured in dollars, so the quartiles are also in dollars.

Try It

For the following 11 salaries, calculate the three quartiles and the *IQR*. Are any of the salaries outliers? The salaries are in dollars.

33,000	72,000	54,000
64,500	68,500	120,000
28,000	69,000	40,500
54,000	42,000	

Solution

Enter the data into an Excel spreadsheet. For this example, suppose we entered the data in column A from cell A1 to A11.

For the first quartile Q_1 :

Function	quartile.exc	Answer
Field 1	A1:A11	\$40,500
Field 2	1	

For the second quartile Q_2 :

Function	quartile.exc	Answer
Field 1	A1:A11	\$54,000
Field 2	2	

For the third quartile Q_3 :

Function	quartile.exc	Answer
Field 1	A1:A11	\$69,000
Field 2	3	

For the IQR: $IQR = 69,000 - 40,500 = \$28,500$

To determine if there are any outliers:

$$1.5 \times IQR = 1.5 \times 28,500 = 42,750$$

$$Q_1 - 1.5 \times IQR = 40,500 - 42,750 = -2,250$$

$$Q_3 + 1.5 \times IQR = 69,000 + 42,750 = 111,750$$

No salary is less than $-\$2,250$. However, $\$120,000$ is more than $\$111,750$, so $\$120,000$ is a potential outlier.

Try It

Find the interquartile range for the following two data sets and compare them.

Test Scores for Class A									
69	96	81	79	65	76	83	99	89	67
90	77	85	98	66	91	77	69	80	94

Test Scores for Class B									
90	72	80	92	90	97	92	75	79	68
70	80	99	95	78	73	71	68	95	100

Solution

Enter the data into an Excel spreadsheet. For this example, suppose we entered the data for Class A into column A from cell A1 to A20 and the data for Class B into column B from cell B1 to B20.

Class A

For the first quartile Q_1 :

Function	quartile.exc	Answer
Field 1	A1:A20	70.75
Field 2	1	

For the third quartile Q_3 :

Function	quartile.exc	Answer
Field 1	A1:A20	90.75
Field 2	3	

For the IQR: $IQR = 90.75 - 70.75 = 20$

Class B

For the first quartile Q_1 :

Function	quartile.exc	Answer
Field 1	B1:B20	72.25
Field 2	1	

For the third quartile Q_3 :

Function	quartile.exc	Answer
Field 1	B1:B20	94.25
Field 2	3	

For the IQR: $IQR = 94.25 - 72.25 = 22$

The data for Class B has a larger IQR , so the scores between Q_3 and Q_1 (the middle 50% of the data) for the data for Class B are more spread out and not clustered about the median.

Percentiles

Percentiles are numbers that separate the (ordered) data into hundredths (100 parts). Like quartiles, percentiles may or may not be part of the data. The n th percentile, P_n , is the value where $n\%$ of the observations in the data are less than the value of the n th percentile. To score in the 90th percentile of an exam does not mean, necessarily, that you received 90% on a test. The 90th percentile means that 90% of test scores are less than your score and 10% of the test scores are the same or greater than your test score.

Quartiles are special percentiles. The first quartile, Q_1 , is the same as the 25th percentile and the third quartile, Q_3 , is the same as the 75th percentile. The median is the 50th percentile.

Percentiles are useful for comparing values. For this reason, universities and colleges use percentiles extensively. One instance in which colleges and universities use percentiles is when SAT results are used to determine a minimum testing score that will be used as an acceptance factor. For example, suppose Duke accepts SAT scores at or above the 75th percentile. That translates into an SAT score of at least 1220.

Percentiles are mostly used with very large data sets. Therefore, if you were to say that 90% of the test scores are less (and not the same or less) than your score, it would be acceptable because removing one particular data value is not significant.

Calculating Percentiles in Excel

To find the k th percentiles in Excel, use the `percentile.exc(array, percent)` function.

- For array, enter the array or cell range containing the data.
- For percent, enter the percentile (as a decimal) being calculated. For example, if we are calculating the 60th percentile, we would enter 0.6 for the percent in the `percentile.exc` function.

The output from the `percentile.exc` function is the value of the corresponding percentile. For example, `percentile.exc(array, 0.6)` returns the value of the 60th percentile where 60% of the observations in the data are (strictly) less than the value of the 60th percentile.

Visit the [Microsoft page](#) for more information about the `percentile.exc` function.

Note

We are using the `percentile.exc` function, and not the `percentile.inc` function, because we want the percent of the observations in the data to be strictly less than the value of the percentile.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=1303#oembed-3>

Video: “[Percentiles – How to calculate Percentiles, Quartiles, ...](#)” by [Joshua Emmanuel](#) [3:44] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*

Example 6.5.2

Listed are twenty-nine ages (in years) for trees found in the Saint Louis Botanical Garden.

18	21	22	25	26	27	29	30	31	33
36	37	41	42	47	52	55	57	58	62
64	67	69	71	72	73	74	76	77	

1. Find the 70th percentile.
2. Find the 83rd percentile.

Solution:

Enter the data into an Excel spreadsheet. For this example, suppose we entered the data in column A from cell A1 to A29.

For the 70th percentile P_{70} :

Function	percentile.exc	Answer
Field 1	A1:A29	64 years
Field 2	0.7	

For the 83rd percentile P_{83} :

Function	percentile.exc	Answer
Field 1	A1:A29	71.9 years
Field 2	0.83	

Note

Percentiles have the same units as the data. In this case, the data is measured in years, so the percentiles are also in years.

Try It

Listed are 29 ages (in years) for Academy Award winning best actors.

18	21	22	25	26	27	29	30	31	33
36	37	41	42	47	52	55	57	58	62
64	67	69	71	72	73	74	76	77	

Calculate the 20th percentile and the 55th percentile.

Solution

Enter the data into an Excel spreadsheet. For this example, suppose we entered the data in column A from cell A1 to A29.

For the 20th percentile P_{20} :

Function	percentile.exc	Answer
Field 1	A1:A29	27 years
Field 2	0.2	

For the 55th percentile P_{55} :

Function	percentile.exc	Answer
Field 1	A1:A29	53.5 years
Field 2	0.55	

Interpreting Percentiles and Quartiles

A percentile indicates the relative standing of a data value when data are sorted into numerical order from smallest to largest. Percentages of data values are less than the value of the n th percentile. For example, 15% of the data values are less than the value of the 15th percentile. Note that low percentiles always correspond to lower data values, and high percentiles always correspond to higher data values.

A percentile may or may not correspond to a value judgment about whether it is “good” or “bad.” The interpretation of whether a certain percentile is “good” or “bad” depends on the context of the situation to which the data applies. In some situations, a low percentile would be considered “good,” but in other contexts a high percentile might be considered “good”. In many situations, there is no value judgment that applies.

Understanding how to interpret percentiles or quartiles properly is important not only when describing data but also when calculating probabilities in later chapters of this text. When writing the interpretation of a percentile or quartile in the context of the given data, the sentence should contain the following information:

- Information about the context of the situation being considered,
- The data value (value of the variable) that represents the percentile/quartile.
- The percent of individuals or items with data values below the percentile/quartile.

Example 6.5.3

On a timed math test, the first quartile for the time it took to finish the exam was 35 minutes. Interpret the first quartile in the context of this situation.

Solution

- Interpretation: 25% of students finished the exam in less than 35 minutes.
- In this context, a low percentile could be considered good, as finishing more quickly on a timed exam is desirable. (If you take too long, you might not be able to finish.)

Try It

For the 100-meter dash, the third quartile for times for finishing the race was 11.5 seconds. Interpret the third quartile in the context of the situation.

Solution

- Interpretation: 75% of runners finished the race in less than 11.5 seconds.
- In this context, a lower percentile is good because finishing a race more quickly is desirable.

Example 6.5.4

On a 20 question math test, the 70th percentile for the number of correct answers was 16. Interpret the 70th percentile in the context of this situation.

Solution:

- Interpretation: 70% of students answered less than 16 questions correctly.

Try It

On a 60-point written assignment, the 80th percentile for the number of points earned was 49. Interpret the 80th percentile in the context of this situation.

Solution

- Interpretation: 80% of students earned less than 49 points.

Example 6.5.5

At a community college, it was found that the 30th percentile of credit units that students are enrolled for is 7 units. Interpret the 30th percentile in the context of this situation.

Solution:

- Interpretation: 30% of students are enrolled in less than 7 credit units.
- In this context, there is no “good” or “bad” value judgment associated with a higher or lower percentile. Students attend community college for varied reasons and needs, and their course load varies according to their needs.

Try It

During a season, the 40th percentile for points scored per player in a game is 8. Interpret the 40th percentile in the context of this situation.

Solution

- Interpretation: 40% of players scored fewer than 8 points.

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6.6 GRAPHICAL REPRESENTATION

Learning Objectives

- Contrast how you can represent data. Know what data sets and purposes would incline you to use one over another.

Data organization and summarization can be done graphically as well as numerically. Tables and graphs allow for a quick overview of the information collected and support the presentation of the data used in the project. While there are a multitude of available graphics, this chapter will focus on a specific few commonly used tools.

Pie Charts

Pie charts are an excellent visual tool, allowing the reader to quickly see the relationship between categories. It is important to label each category clearly, and adding the frequency or relative frequency is often helpful. However, too many categories can be confusing. Be careful of putting too much information in a pie chart. The first pie chart gives a clear idea of the representation of fish types relative to the whole sample. The second pie chart is more difficult to interpret, with too many categories. It is important to select the best graphic when presenting the information to the reader.

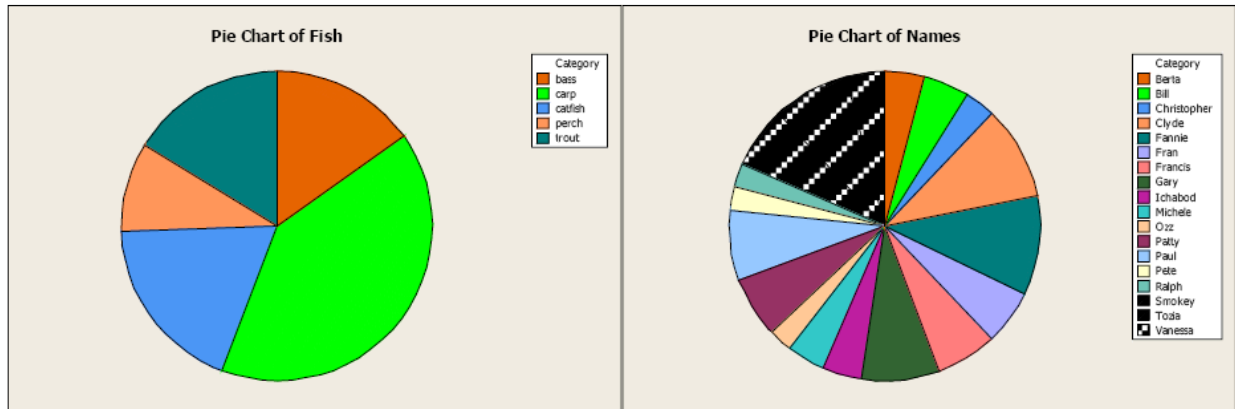


Figure 6.4.1 Comparison of pie charts.

Image Description

There are two adjacent pie charts in this image:

1. The pie chart on the left is titled “Pie Chart of Fish” and contains segments of different colours representing categories such as bass, carp, catfish, perch, and trout. The colours for each category are depicted in a legend situated to the right of the pie chart.
2. The pie chart on the right is titled “Pie Chart of Names” and contains a larger variety of segments, each representing names like Barta, Bill, Christopher, Clyde, and others. Each segment is coloured differently, with the corresponding colours listed in a legend to the right of the chart.

Both charts are set against a light-coloured background with clear labels and legends for reference.

Bar Charts and Histograms

Bar charts graphically describe the distribution of a qualitative variable (fish type), while histograms describe the distribution of a quantitative variable discrete or continuous variables (bear weight).

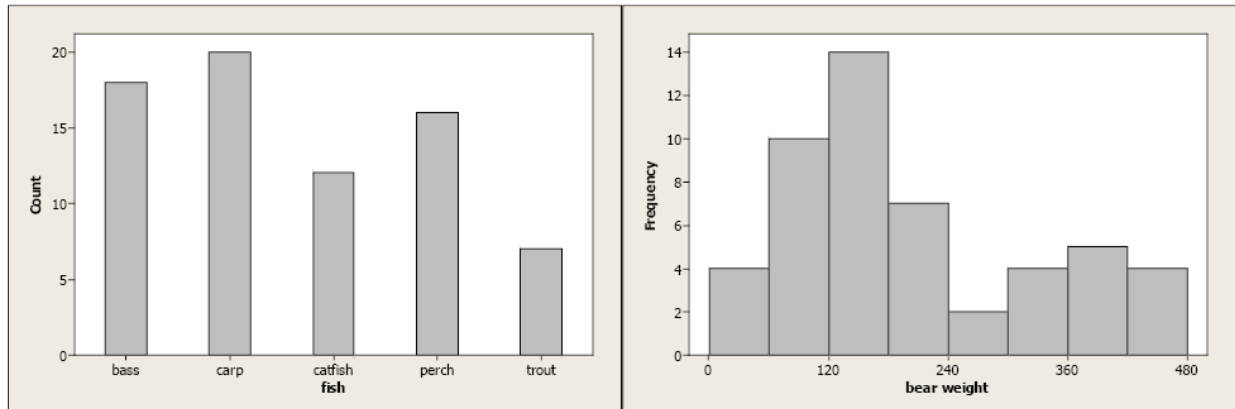


Figure 6.4.2 Comparison of a bar chart for qualitative data and a histogram for quantitative data.

Image Description

There are two bar charts side by side, each within its own beige background area.

The left chart depicts the count of different types of fish. The x-axis is labelled “fish,” and the y-axis is labelled “Count.”

- Bass: approximately 18
- Carp: 20
- Catfish: approximately 11
- Perch: approximately 15
- Trout: approximately 8

The right chart shows the frequency distribution of bear weights. The x-axis is labelled “bear weight,” with increments from 0 to 480, and the y-axis is labelled “Frequency.”

- 0: approximately 4
- 120: approximately 11
- 240: 15
- 360: approximately 5
- 480: approximately 4

In both cases, the bars’ equal width and the y-axis are clearly defined. With qualitative data, each category is represented by a specific bar. With continuous data, lower and upper-class limits must be defined with equal class widths. There should be no gaps between classes, and each observation should fall into one class.

Boxplots

Boxplots use the 5-number summary (minimum and maximum values with the three quartiles) to illustrate the center, spread, and distribution of your data. When paired with histograms, they give an excellent description, both numerically and graphically, of the data.

With symmetric data, the distribution is bell-shaped and somewhat symmetric. In the boxplot, we see that $Q1$ and $Q3$ are approximately equidistant from the median, as are the minimum and maximum values. Also, both whiskers (lines extending from the boxes) are approximately equal in length.

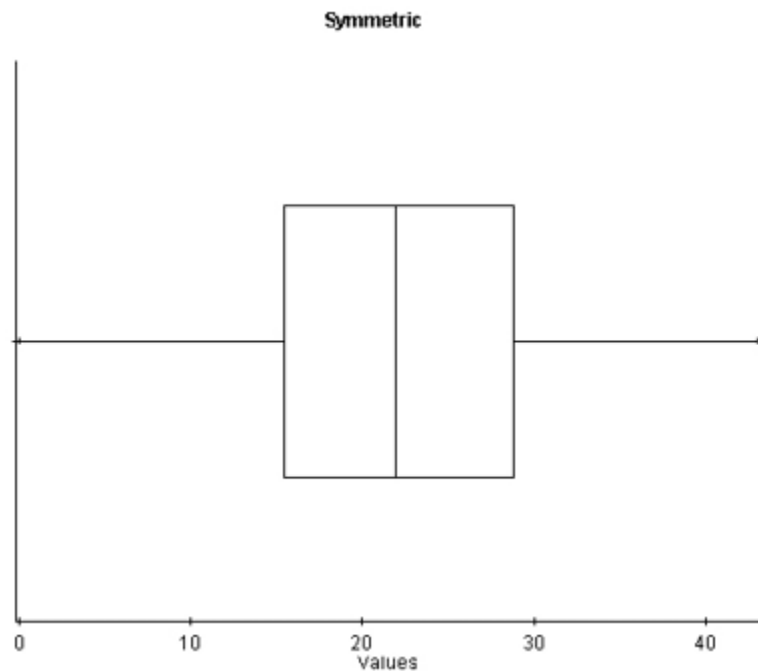


Figure 6.4.3 A Boxplot of normal distribution.

Image Description

The image is a box plot labelled “Symmetric” displayed on a two-dimensional grid.

The x-axis is labelled “Values” and ranges from 0 to 40 in increments of 10. The y-axis does not have specific labels or notches but provides the vertical alignment.

In the middle of the plot, there is a rectangular box that extends from a value slightly above 10 to a value slightly below 30 on the x-axis. The median is represented by a vertical line inside the box at around 20.

On either side of the box, horizontal lines (whiskers) extend outward. The left whisker runs from the start of the box to a value slightly above 0 on the x-axis, and the right whisker runs from the end of the box to a value slightly below 40 on the x-axis.

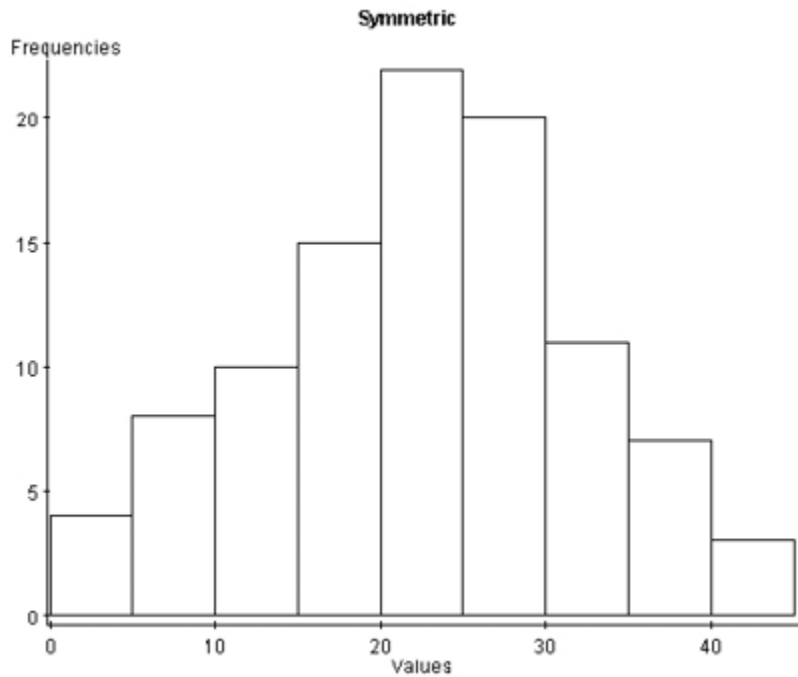


Figure 6.4.4 A Histogram of normal distribution.

Image Description

The image depicts a histogram labelled “Symmetric.” The x-axis is labelled “Values” and ranges from 0 to 40 in increments of 10. The y-axis is labeled “Frequencies” and ranges from 0 to 25 in increments of 5.

- The bar representing the range 0 – 5 has a frequency of about 5.
- The bar representing the range 5 – 10 has a frequency about 10.
- The bar representing the range 10 – 15 has a frequency slightly above 10.
- The bar representing the range 15 – 20 has a frequency between 15 and 20.
- The bar representing the range 20 – 25 has the highest frequency, slightly above 20.
- The bar representing the range 25 – 30 has a frequency about 18.
- The bar representing the range 30 – 35 has a frequency slightly above 10.
- The bar representing the range 35 – 40 has a frequency of about 10.
- The bar representing the range 40 – 45 has a frequency of about 5.

With skewed left distributions, we see that the histogram looks “pulled” to the left. In the boxplot, $Q1$ is farther away from the median as are the minimum values, and the left whisker is longer than the right whisker.

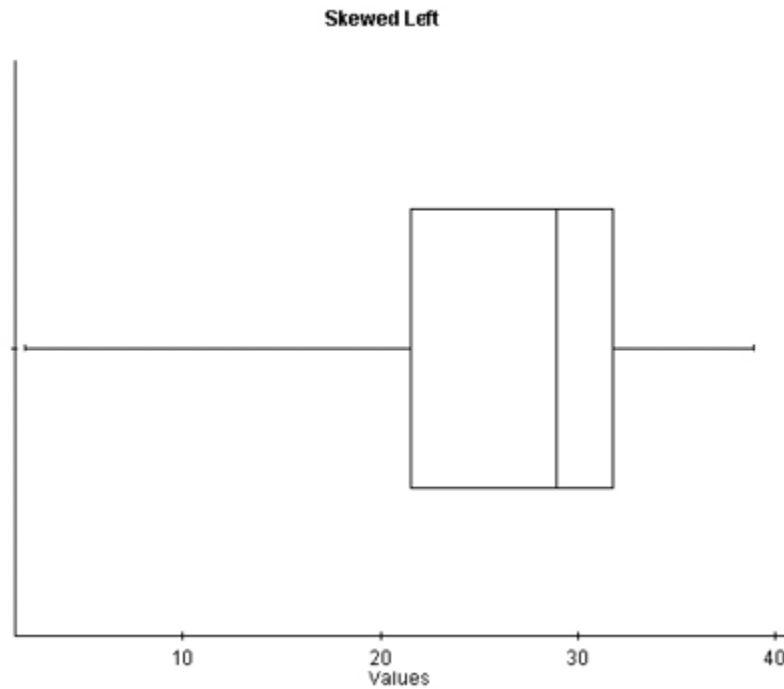


Figure 6.4.5 A Boxplot of skewed left distribution.

Image Description

The image is a box plot labeled “Skewed Left.” It consists of a rectangular box with horizontal lines extending from either side. The main elements are:

- A horizontal axis labeled “Values” with tick marks and numbers ranging from 10 to 40, in increments of 10.
- The box spans from approximately 20 to 32 on the horizontal axis.
- A vertical line inside the box indicating the median, positioned closer to the right edge of the box around 30.
- The left whisker extends from the left side of the box at approximately 20, to just before the 10 mark on the horizontal axis.
- The right whisker extends from the right side of the box at approximately 32, ending roughly around 36 on the horizontal axis.

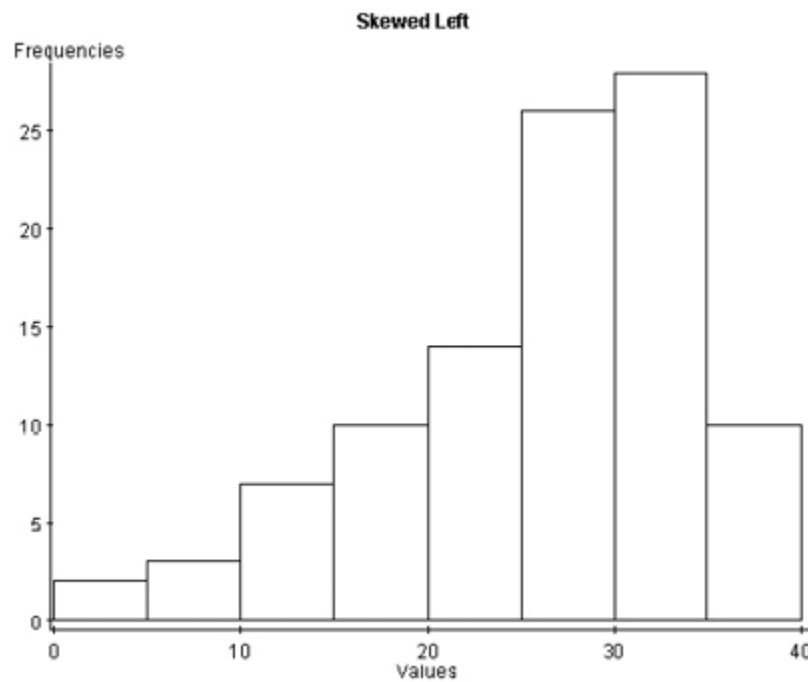


Figure 6.4.6 A Histogram of skewed left distribution.

Image Description

The image is a histogram titled “Skewed Left”. The horizontal axis is labeled “Values” and the vertical axis is labeled “Frequencies”. The vertical axis ranges from 0 to 30 in increments of 5, while the horizontal axis ranges from 0 to 40 in increments of 10.

The histogram consists of several bars indicating the frequency of different ranges of values:

- The first bar (0 — 5) has a frequency of about 2.
- The second bar (5 — 10) has a frequency of about 4.
- The third bar (10 — 15) has a frequency of about 7.
- The fourth bar (15 — 20) has a frequency of about 10.
- The fifth bar (20 — 25) has a frequency of about 15.
- The sixth bar (25 — 30) has a frequency of about 20.
- The seventh bar (30 — 35) has the highest frequency, about 27.
- The eighth bar (35 — 40) has a frequency of about 12.

With skewed right distributions, we see that the histogram looks “pulled” to the right. In the boxplot, $Q3$ is farther away from the median, as is the maximum value, and the right whisker is longer than the left whisker.

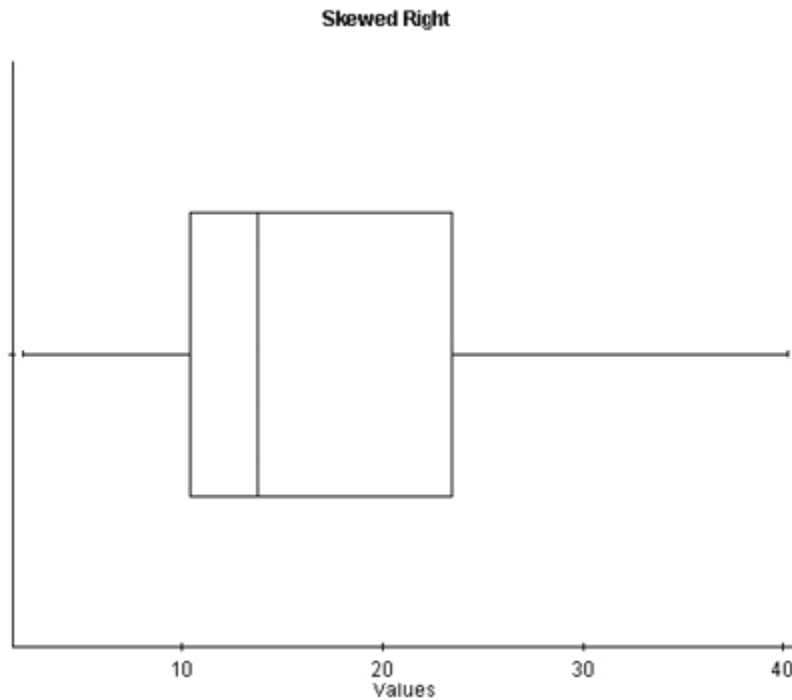


Figure 6.4.7 A Boxplot of skewed right distribution.

Image Description

This image depicts a box plot labeled “Skewed Right.” The horizontal axis is labeled “Values” and ranges from 0 to 40. The box plot includes a rectangular box with its left edge around 10 and its right edge around 23. A vertical line inside the box, representing the median, is closer to the left edge of the box. The whiskers extend from the edges of the box to approximately 5 on the left side and 37 on the right side, indicating a right-skewed distribution.

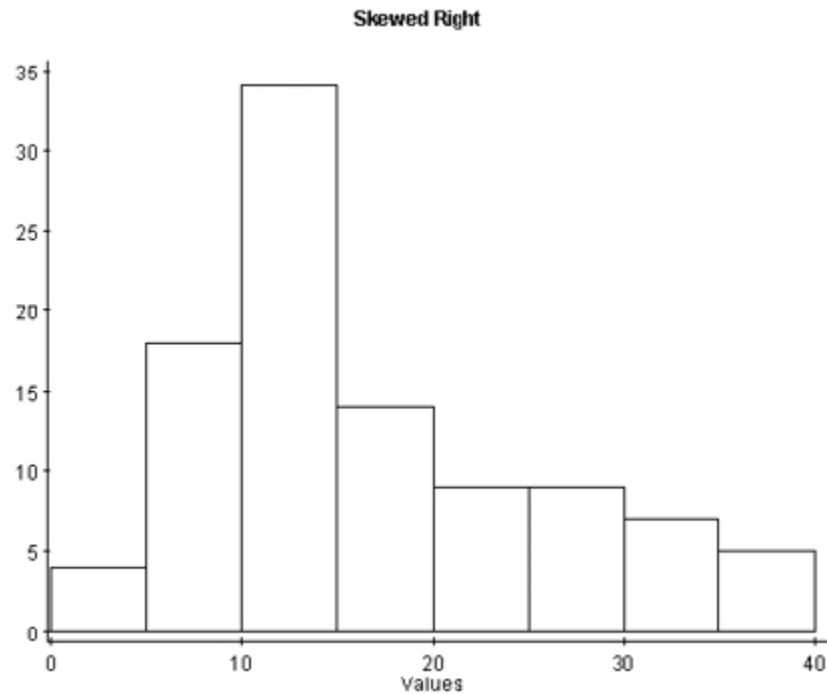


Figure 6.4.8 A Histogram of skewed right distribution.

Image Description

This image is a histogram titled “Skewed Right.” The horizontal axis is labeled “Values” and ranges from 0 to 40. The vertical axis is unlabeled but ranges from 0 to 35, with tick marks in intervals of 5.

The bars represent the frequency of different ranges of values:

- The first bar, ranging from 0 to approximately 5, has a height of about 5.
- The second bar, ranging from approximately 5 to 10, has a height of about 20.
- The third bar, ranging from approximately 10 to 15, has the highest height of about 35.
- The fourth bar, ranging from approximately 15 to 20, has a height of about 15.
- The fifth bar, ranging from approximately 20 to 25, has a height of about 10.
- The sixth bar, ranging from approximately 25 to 30, has a height of about 10.
- The seventh bar, ranging from approximately 30 to 35, has a height of about 7.
- The eighth bar, ranging from approximately 35 to 40, has a height of about 5.

The distribution of the data is skewed to the right, with a higher concentration of values on the lower end and a tail extending towards higher values.

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6.7 REVIEW EXERCISES

Chapter 6 Review Exercises

1. Consider the following population data:

2, 4, 8, 4, 6,

2, 7, 8, 4, 3,

8, 9, 4, 3, 5

- a) Calculate the mean.
- b) Calculate the median.
- c) Calculate the mode.
- d) Calculate the range
- e) calculate the standard deviation.
- f) Describe the skewness of the data.

Solution

a) **5.13**

b) **4**

c) **4**

d) **7**

e) **2.28**

f) right skewed

2. A sample of monthly credit card charges (in dollars) is shown below:

236, 1710, 1351, 825, 7450,

316, 4135, 1333, 1584, 387

- a) Calculate the mean.
- b) Calculate the median.
- c) Calculate the mode.
- d) Calculate range
- e) calculate the standard deviation.
- f) Describe the skewness of the data.

Solution

a) **\$1, 932.70**

b) **\$1, 342**

c) no mode.

D) **\$7, 214**

e) **\$2, 242.73**

f) right skewed.

3. People visiting video rental stores often rent more than one DVD at a time. The probability distribution for DVD rentals per customer at Video To Go is given in the following table. There is a five-video limit per customer at this store, so nobody ever rents more than five DVDs.

x	$P(x)$
0	0.03
1	0.50
2	0.24
3	
4	0.07
5	0.04

- Describe the random variable x .
- Find the probability that a customer rents three DVDs.
- Find the probability that a customer rents at least four DVDs.
- Find the probability that a customer rents at most two DVDs.

Solution

- The number of DVDs a given customer is currently renting.
- $1 - 0.03 - 0.50 - 0.24 - 0.07 - 0.04 = 0.09$
- $0.07 + 0.04 = 0.11$
- $0.03 + 0.50 + 0.24 = 0.77$

4. The following are the grades for 40 students on an accounting test:

75 89 66 52 90 68 83 84 77 60
 98 47 87 65 87 49 65 72 73 81
 53 77 41 88 74 97 85 76 74 63
 69 72 81 87 76 58 63 70 72 55

- Calculate the range for this data.
- Calculate class width (round up to the next whole number) for constructing frequency distribution table.
- Start the minimum number 41 as the lower limit of the first class. What are the upper limits for the frequency distribution table?
- Construct a frequency distribution table with classes, frequency, relative frequencies,

cumulative frequencies and cumulative relative frequencies.

(e) Construct the histogram corresponding to the table in part (d).

(f) Calculate mean of this data.

(g) Calculate the mode for this data.

(h) Calculate the median for this data.

(i) Describe skewness of this data

Solution

a. 57

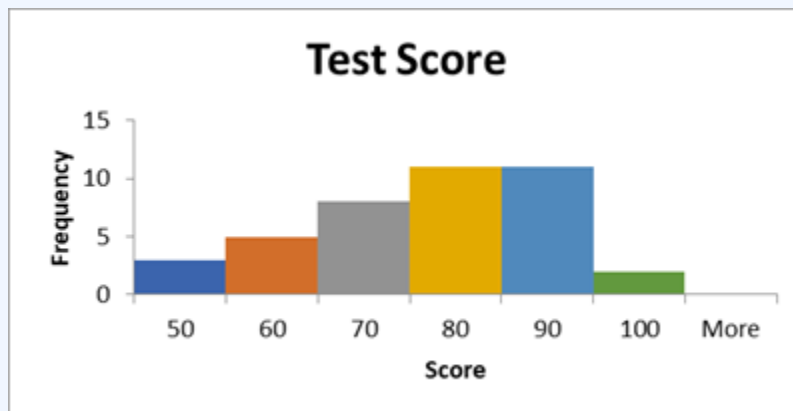
b. 10

c. 50, 60, 70, 80, 100

d.

Lower Class Limit	Upper Class Limit	Frequency	Relative Frequency	Cumulative Frequency	Relative Cumulative Frequency
41	50	3	8%	3	0.075
51	60	5	13%	8	20%
61	70	8	20%	16	40%
71	80	11	28%	27	68%
81	90	11	28%	38	95%
91	100	2	5%	40	100%

e.



- f. Mean: 72.48
- g. Median: 73.5
- h. Mode: 87, 72
- i. Skew to the left

5. Consider the following sample data:

2 7 8 4 3
8 9 4 3 5

- (a) Calculate the mean.
- (b) Calculate the median.
- (c) Calculate the mode.
- (d) Calculate the range.
- (e) Calculate the standard deviation.
- (f) Describe the skewness of the data.

Solution

- a. 5.3
- b. 4.5
- c. 8, 4, 3
- d. 7
- e. 2.50
- f. positively skewed since the mean is greater than the median.

6. A sample of monthly credit card charges (in dollars) is shown below:

316 4135 133 1584 387

- (a) Calculate the mean monthly credit card charge.
- (b) Calculate the median monthly credit card charge.
- (c) Calculate the mode.
- (d) Calculate the standard deviation.
- (e) Describe the skewness of the data.

Solution

- a. \$1311
- b. \$387
- c. No mode
- d. \$1679.36
- e. Skew to the right

7. A sample of students who graduated with student loan debt is shown below. The data is in thousands of dollars.

12.2 2 11.5 17.8 4

- (a) Calculate the mean loan debt.
- (b) What is the median loan debt?
- (c) What is the mode?
- (d) Describe the skewness of this data.
- (e) Calculate the standard deviation?

Solution

- a. \$9,500
- b. \$11,500
- c. No mode
- d. negatively skewed since the mean is smaller than the median
- e. \$6,455.23

8. The following data show the annual salary, in thousands of dollars, for superintendents in 20 school districts in Rochester, New York.

187 175 165 162 172
 184 172 208 172 175
 174 202 215 182 170
 185 197 164 156 183

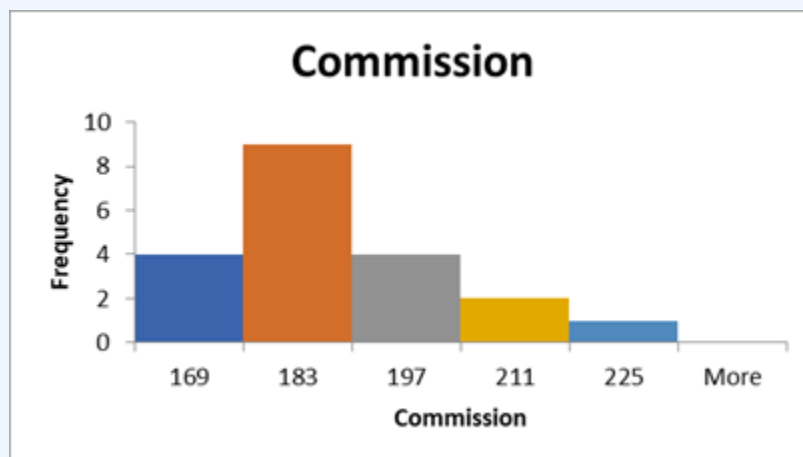
- (a) Construct a frequency distribution table for this data.
- (b) Construct the histogram corresponding to the table in part (a).
- (c) Based on the histogram, what measure of central tendency best describes this data?

Solution

a.

Upper Class Limit	Frequency	Relative Frequency	Cumulative Frequency	Relative Cumulative Frequency
169	4	20%	4	0.2
183	9	45%	13	65%
197	4	20%	17	85%
211	2	10%	19	95%
225	1	5%	20	100%

b.



c. Skewed to the right. Median is the best.

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CHAPTER 6 SUMMARY

Key Concepts Summary

- **6.1 Descriptive Statistics**

- Differentiating Population and Sample
- Differentiating Quantitative and Continuous variables

- **6.2 Frequency Distribution and Histograms**

- Creating and interpreting frequency tables
- Displaying, analyzing, and interpreting data in histograms

- **6.3 Descriptive Measures**

- Calculating the Measures of Center and Measures of Dispersion for sets of data

- **6.4 Skewness**

- Identifying the shape of a data set

- **6.5 Measures of Location**

- Recognize, describe, calculate, and interpret the measures of location of data: quartiles and percentiles.

- **6.6 Graphical Representation**

- Using graphs and charts to visualize data

Glossary of Terms

Central Limit Theorem. A theorem stating that as a sample size increases the sample mean will grow closer to normal.

Continuous Variable. A variable with an infinite number of possible values.

Frequency or relative frequency. (or percent frequency or probability).

Histogram. A visual display of a frequency chart.

Measures of Center. Measurements used to determine the center of data.

Measures of Dispersion. Measurements used to determine the spread of data.

Median. A number that measures the “center” of the data.

Normal Distribution. A “bell-shaped” symmetric distribution of data that can be used for estimating probabilities and proportions within a population.

Percentiles. Numbers that separate the (ordered) data into hundredths.

Population. A group to be studied.

Quantitative variable. A variable with a finite and countable number of possible values.

Quartiles. Numbers that separate the data into quarters.

Sample. A selection from the population.

Skew. A term to describe a curve that does not meet the criteria of a normal distribution.

Variable. The characteristics to be studied from the population and/or sample.

Formula & Symbol Hub

Symbols Used

- μ = Population Mean
- \bar{x} = Sample Mean
- σ^2 = Population Variance
- s^2 = Sample Variance
- σ = Population standard deviation
- s = Sample standard deviation
- $s_{\bar{x}}$ = Standard error of the means
- z = a Z-score

Formulas Used

- Formula 6.1 – **Population Mean**

$$\mu = \frac{\sum x_i}{N}$$

- Formula 6.2 – **Sample mean**

$$\bar{x} = \frac{\sum x_i}{n}$$

- Formula 6.3 – **Population Variance**

$$\sigma^2 = \frac{\sum (x_i - \mu)^2}{N}$$

- Formula 6.4 – **Sample Variance**

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1} = \frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n - 1}$$

- **Formula 6.5 – Population Standard Deviation**

$$\sigma = \sqrt{\sigma^2}$$

- **Formula 6.6 – Sample Standard Deviation**

$$s = \sqrt{s^2}$$

- **Formula 6.7 – Standard Error of the Means**

$$s_{\bar{x}} = \sqrt{\frac{s^2}{n}} = \frac{s}{\sqrt{n}}$$

- **Formula 6.8 – Coefficient of Variation (Population)**

$$CV = \frac{\sigma}{\mu} \times 100$$

- **Formula 6.9 – Coefficient of Variation (Sample)**

$$CV = \frac{s}{\bar{x}} \times 100$$

- **Formula 6.10 – Interquartile Range**

$$IQR = Q_3 - Q_1$$

CHAPTER 7: DISCRETE RANDOM VARIABLES

Chapter Outline

[7.0 Introduction to Discrete Random Variables](#)

[7.1 Probability Distribution of a Discrete Random Variable](#)

[7.2 Expected Value and Standard Deviation for a Discrete Probability Distribution](#)

[Chapter 7 Summary](#)

7.0 INTRODUCTION

Chapter Learning Objectives

By the end of the chapter, you should be able to:

- Recognize, understand, and construct discrete probability distributions.
- Calculate and interpret the expected value of a probability distribution.
- Calculate the standard deviation for a probability distribution.

A student takes a ten-question, true-false quiz. Because the student had such a busy schedule, he or she could not study and guess randomly at each answer. What is the probability of the student passing the test with at least a 70%?

Small companies might be interested in the number of long-distance phone calls their employees make during the peak time of the day. Suppose the average is 20 calls. What is the probability that the employees make more than 20 long-distance phone calls during the peak time?

These two examples illustrate two different types of probability problems involving **discrete random variables**. Recall that discrete data are data that you can count. A **random variable** describes the outcomes of a statistical experiment in words. The values of a random variable can vary with each repetition of an experiment.



You can use probability and discrete random variables to calculate the likelihood of lightning striking the ground five times during a half-hour thunderstorm. Photo by [Leszek Leszczynski](#), CC BY 2.0.

Random Variables

Upper case letters such as X or Y denote a random variable. Lowercase letters like x or y denote the value of a random variable. If X is a random variable, then X is written in words, and x is given as a number.

For example, let X be the number of heads you get when you toss three fair coins. The sample space for the toss of three fair coins is $TTT, THH, HTH, HHT, HTT, THT, TTH, HHH$. Then, $x = 0, 1, 2, 3$. X is in words and x is a number. Notice that for this example, the x values are countable outcomes. Because you can count the possible values that X can take on, and the outcomes are random (the x values are $0, 1, 2, 3$), X is a discrete random variable.

A **random variable** describes a characteristic of interest in a population being studied. Common notation for variables are upper case Latin letters X, Y, Z, \dots and common notation for a specific value from the domain (set of all possible values of a variable) are lower case Latin letters x, y , and z . For example, if X is the number of children in a family, then x represents a specific integer $0, 1, 2, 3, \dots$. Variables in statistics differ from variables in intermediate algebra in the two following ways:

- The domain of the random variable is not necessarily a numerical set. The domain may be expressed in words. For example, if X is hair colour, then the domain is {black, blond, gray, green, orange}.
- We can tell what specific value x the random variable X takes only after performing the experiment.

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7.1 PROBABILITY DISTRIBUTION OF A DISCRETE RANDOM VARIABLE

Learning Objectives

- Recognize, understand, and construct discrete probability distributions.

A **random variable** describes the outcomes of a statistical experiment in words. The values of a random variable can vary with each repetition of an experiment.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=1043#oembed-1>

Video: “[Random Variables and Probability Distributions](#)” by [Dr Nic’s Maths and Stats](#) [4:39] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*

The **probability distribution** for a random variable lists all the possible values of the random variable and the probability the random variable takes on each value. The probability distribution for a random variable describes how probabilities are distributed over the values of the random variable. A probability distribution can be a table, with a column for the values of the random variable and another column for the corresponding

probability, or a graph, like a histogram with the values of the random variable on the horizontal axis and the probabilities on the vertical axis.

In a probability distribution, each probability is between 0 and 1, inclusive. Because all possible values of the random variable are included in the probability distribution, the sum of the probabilities is 1.

Example 7.1.1

A child psychologist is interested in the number of times a newborn baby's crying wakes its mother after midnight. For a random sample of 50 mothers, the following information was obtained. Let X be the number of times per week a newborn baby's crying wakes its mother after midnight. For this example, the values of the random variable are $x = 0, 1, 2, 3, 4, 5$.

In the table, the left column contains all of the possible values of the random variable and the right column, $P(x)$, is the probability that X takes on the corresponding value x . For example, in the first row, the value of the random variable is 0, and the probability the random variable is 0 is $\frac{2}{50}$.

In the context of this example, that means that the probability a newborn baby's crying wakes its mother 0 times per week is $\frac{2}{50}$.

x	$P(x)$
0	$\frac{2}{50}$
1	$\frac{11}{50}$
2	$\frac{23}{50}$
3	$\frac{9}{50}$
4	$\frac{4}{50}$
5	$\frac{1}{50}$

Because X can only take on the values 0, 1, 2, 3, 4, and 5, X is a discrete random variable. Note that each probability is between 0 and 1 and the sum of the probabilities is 1:

$$\frac{2}{50} + \frac{11}{50} + \frac{23}{50} + \frac{9}{50} + \frac{4}{50} + \frac{1}{50} = 1$$

Try It

Suppose Nancy has classes three days a week. She attends classes three days a week 80% of the time, two days a week 15% of the time, one day a week 4% of the time, and no days 1% of the time. Suppose one week is randomly selected.

1. Let X be the number of days Nancy -----.
2. X takes on what values?
3. Suppose one week is randomly chosen. Construct a probability distribution table like the one in the example above. The table should have two columns labelled x and $P(x)$. What does the $P(x)$ column sum to?

Solution

1. Let X be the number of days Nancy attends class per week.
2. 0, 1, 2, and 3.

3.

x	$P(x)$
0	0.01
1	0.04
2	0.15
3	0.80

The $P(x)$ column sums to 1.

Example 7.1.2

Jeremiah has basketball practice two days a week. Ninety percent of the time, he attends both practices. Eight percent of the time, he attends one practice. Two percent of the time, he does not attend either practice. What is X , and what values does it take on?

Solution:

X is the number of days Jeremiah attends basketball practice per week. X takes on the values 0, 1, and 2.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=1043#oembed-2>

Video: “[Constructing a probability distribution for random variable | Khan Academy](#)” by [Khan Academy](#) [6:47] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*

Key Takeaways



A probability distribution for a random variable describes how the probabilities are distributed over the random variable—in other words; the probability distribution describes the probability that the random variable takes on a specific value. A probability distribution includes all possible values the random variable can take on and the corresponding probability. Each probability is between 0 and 1, inclusive, and the sum of the probabilities is 1.

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7.2 EXPECTED VALUE AND STANDARD DEVIATION FOR A DISCRETE PROBABILITY DISTRIBUTION

Learning Objectives

- Calculate and interpret the expected value of a probability distribution.
- Calculate the standard deviation for a probability distribution.

Formula & Symbol Hub

Symbols Used

- σ = standard deviation
- $P(x)$ = probability of x
- μ = mean or expected value
- \sum = summation symbol

Formulas Used

- Formula 7.1 – **Expected Value**

$$E(x) = \sum (x \times P(x))$$

- Formula 7.2 – **Standard Deviation of a Probability Distribution**

$$\sigma = \sqrt{\sum ((x - \mu)^2 \times P(x))}$$

Expected Value of a Probability Distribution

The **expected value** is often referred to as the “long-term” average or mean. That is, over the long term of repeatedly doing an experiment, you would expect this average.

Suppose you toss a coin and record the result. What is the probability that the result is heads? If you flip a coin two times, does the probability tell you that these flips will result in one heads and one tail? You might toss a fair coin ten times and record nine heads. Probability does not describe the short-term results of an experiment. Probability gives information about what can be expected in the long term. To demonstrate this, Karl Pearson once tossed a fair coin 24,000 times! He recorded the results of each toss, obtaining heads 12,012 times. In his experiment, Pearson illustrated the Law of Large Numbers.

The Law of Large Numbers states that, as the number of trials in a probability experiment increases, the difference between the theoretical probability of an event and the relative frequency approaches zero—the theoretical probability and the relative frequency get closer and closer together. When evaluating the long-term results of statistical experiments, we often want to know the “average” outcome. This “long-term average” is known as the **mean** or **expected value** of the experiment and is denoted by μ or $E(x)$. In other words, after conducting many trials of an experiment, you would expect this average value.

The **expected value**, denoted by μ or $E(x)$, is a weighted average where each value of the random variable is weighted by the value’s corresponding probability.

7.1 Expected Value

$$E(x) = \sum (x \times P(x))$$

$E(x)$ is the **expected value**: A weighted average where each value of the random variable is weighted by the value's corresponding probability.

\sum is the **summation symbol**: Indicates that the expression to its right is repeatedly added.

x is a **random variable**: A variable that holds a random value.

$P(x)$ is the **probability**: The probability of the random value occurring.

Example 7.2.1

A men's soccer team plays soccer zero one or two days a week. The probability that they play zero days is 0.2, the probability that they play one day is 0.5, and the probability that they play two days is 0.3. Find the long-term average or expected value of the number of days per week the men's soccer team plays soccer.

Solution:

First, let the random variable X be the number of days the men's soccer team plays soccer per week. X takes on the values 0, 1, 2. The table below shows the probability distribution for X , and includes an additional column $x \times P(x)$ that we will use to calculate the expected value. In this new column, we will multiply each x value by its corresponding probability.

x	$P(x)$	$x \times P(x)$
0	0.2	$0 \times 0.2 = 0$
1	0.5	$1 \times 0.5 = 0.5$
2	0.3	$2 \times 0.3 = 0.6$

Add the last column to find the long-term average or expected value:

$$\begin{aligned}
 E(x) &= (0 \times 0.2) + (1 \times 0.5) + (2 \times 0.3) \\
 &= 0 + 0.5 + 0.6 \\
 &= 1.1
 \end{aligned}$$

The expected value is 1.1. The men's soccer team would, on average, expect to play soccer 1.1 days per week. The number 1.1 is the long-term average or expected value if the men's soccer team plays soccer week after week after week.

NOTE

The expected value does not represent a value that the random variable takes on. The expected value is an average. In this case, the expected value of 1.1 is the average number of times the team plays per week. To understand what this means, imagine that each week, you recorded the number of times the soccer team played that week. You do this repeatedly for many, many, many weeks. Then you calculate the mean of the numbers you recorded (using the techniques we learned previously)—the mean of these numbers equals 1.1, the expected value. The number of trials must be very, very large in order for the mean of the values recorded from the trials to equal the expected value calculated using the expected value formula.

Standard Deviation of a Probability Distribution

Like data, probability distributions have standard deviations. The **standard deviation**, denoted σ , of a probability distribution for a random variable X describes the spread or variability of the probability distribution. The standard deviation is the standard deviation you expect when doing an experiment over and over.

7.2 Standard Deviation of a Probability Distribution

$$\sigma = \sqrt{\sum ((x - \mu)^2 \times P(x))}$$

σ is the standard deviation: A measure of how much the values tend to deviate from the mean or expected value of the distribution.

\sum is the summation symbol: Indicates that the expression to which it applies is repeatedly added.

x is a random variable: Represents a possible value within the distribution.

μ is the mean: The expected value (average) of the distribution.

$P(x)$ is the probability: The probability or likelihood of x having its current value within the distribution.

To calculate the standard deviation of a probability distribution, find each deviation from its expected value, square it, multiply it by its probability, add the products, and take the square root.

Example 7.2.2

Let X be the number of times per week a newborn baby's crying wakes its mother after midnight. The probability distribution for X is:

x	$P(x)$
0	0.04
1	0.22
2	0.46
3	0.18
4	0.08
5	0.02

Find the expected value and standard deviation of the number of times a newborn baby's crying wakes its mother after midnight.

Solution:

For the expected value:

x	$P(x)$	$x \times P(x)$
0	0.04	0
1	0.22	0.22
2	0.46	0.92
3	0.18	0.54
4	0.08	0.32
5	0.02	0.1

$$\begin{aligned}\mu &= 0 + 0.22 + 0.92 + 0.54 + 0.32 + 0.1 \\ &= 2.1\end{aligned}$$

On average, a newborn wakes its mother after midnight 2.1 times per week.

For the standard deviation: For each value x , multiply the square of its deviation by its probability (each deviation has the format $x - \mu$).

x	$P(x)$	$(x - \mu)^2 \times P(x)$
0	0.04	$(0 - 2.1)^2 \times 0.04 = 0.1764$
1	0.22	$(1 - 2.1)^2 \times 0.22 = 0.2662$
2	0.46	$(2 - 2.1)^2 \times 0.46 = 0.0046$
3	0.18	$(3 - 2.1)^2 \times 0.18 = 0.1458$
4	0.08	$(4 - 2.1)^2 \times 0.08 = 0.2888$
5	0.02	$(5 - 2.1)^2 \times 0.02 = 0.1682$
	Sum	1.05

Add the values in the third column of the table and then take the square root of this sum:

$$\begin{aligned}\sigma &= \sqrt{1.05} \\ &= 1.024...\end{aligned}$$

Try It

A hospital researcher is interested in the number of times the average post-op patient will ring the nurse during a 12-hour shift. Let X be the number of times a post-op patient rings for the nurse. For a random sample of 50 patients, the following information was obtained. What is the expected value? What is the standard deviation?

x	$P(x)$
0	0.08
1	0.16
2	0.32
3	0.28
4	0.12
5	0.04

Solution

For the expected value:

x	$P(x)$	$x \times P(x)$
0	0.08	0
1	0.16	0.16
2	0.32	0.64
3	0.28	0.84
4	0.12	0.48
5	0.04	0.2

$$\begin{aligned}\mu &= 0 + 0.16 + 0.64 + 0.84 + 0.48 + 0.2 \\ &= 2.32\end{aligned}$$

For the standard deviation:

x	$P(x)$	$(x - \mu)^2 \times P(x)$
0	0.08	0.430592
1	0.16	0.278784
2	0.32	0.032768
3	0.28	0.129472
4	0.12	0.338688
5	0.04	0.287296

$$\sigma = \sqrt{0.430592 + 0.27878784 + 0.032768 + 0.129472 + 0.338688 + 0.287296} \\ = 1.22....$$

Example 7.2.3

Suppose you play a game of chance in which five numbers are chosen from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. A computer randomly selects five numbers from zero to nine with replacement. You pay \$2 to play and could profit \$100,000 if you match all five numbers in order (you get your \$2 back plus \$100,000). Over the long term, what is your expected profit of playing the game?

Solution:

To do this problem, set up an expected value table for the amount of money you can profit. Let X be the amount of money you profit. The values of x are not 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Because you are interested in your profit (or loss), the values of x are \$100,000 and -\$2 dollars.

To win, you must get all five numbers correct, in order. The probability of choosing one correct number is

$\frac{1}{10} = 0.1$ because there are ten numbers. You may choose a number more than once. The probability of choosing all five numbers correctly and in order is

$$\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = 0.00001$$

Therefore, the probability of winning is 0.00001 and the probability of losing is $1 - 0.00001 = 0.99999$.

The expected value is as follows:

x	$P(x)$	$x \times P(x)$
-2	0.99999	-1.99998
100,000	0.00001	1

$$\begin{aligned} E(x) &= -1.99998 + 1 \\ &= -0.99998 \end{aligned}$$

Because -0.99998 is about -1, you would, on average, expect to lose approximately \$1 for each game you play. However, each time you play, you either lose \$2 or profit \$100,000. The \$1 is the average (or expected) LOSS per game after playing this game over and over.

Try It

You are playing a game of chance in which four cards are drawn from a standard deck of 52 cards. You guess the suit of each card before it is drawn. The cards are replaced in the deck on each draw. You pay \$1 to play. If you guess the right suit every time, you get your money back and \$256. What is your expected profit from playing the game over the long term?

Solution

Let X be the amount of money you profit. The values of x are $-\$1$ (for a loss) and $\$256$ (for a win). The probability of winning (guessing the correct suit on each draw) is

$$\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = 0.0039$$

The probability of losing is

$$1 - 0.0039 = 0.9961$$

The expected value is as follows:

x	$P(x)$	$x \times P(x)$
-1	0.9961	-0.9961
256	0.0039	0.9984

$$\begin{aligned} E(x) &= -0.9961 + 0.9984 \\ &= 0.0023 \end{aligned}$$

Playing the game over and over again means you would average \$0.0023 in profit per game.

Example 7.2.4

Suppose you play a game with a biased coin where the probability of heads is $\frac{2}{3}$. You play each game by tossing the coin once. If you toss a head, you pay \$6. If you toss a tail, you win \$10. If you play this game many times, will you come out ahead?

1. Define a random variable X .
2. Construct the probability distribution for X .
3. What is the expected value? Do you come out ahead?

Solution:

1. Let X be the amount of profit per game. The values of x are $-\$6$ (for a loss) and $\$10$ (for a win).

2.

x	$P(x)$
10	$\frac{1}{3}$
-6	$\frac{2}{3}$

3.

x	$P(x)$	$x \times P(x)$
10	$\frac{1}{3}$	$\frac{10}{3}$
-6	$\frac{2}{3}$	-4

$$\begin{aligned} E(x) &= \frac{10}{3} + (-4) \\ &= -0.67 \end{aligned}$$

On average, you lose \$0.67 each time you play the game, so you do not come out ahead.

Try It

Suppose you play a game with a spinner that has three colours on it: red, green, and blue. The probability of landing on red is 40% and the probability of landing on green is 20%. You play a

game by spinning the spinner once. If you land on red, you pay \$10. If you land on blue, you do not pay or win anything. If you land on green, you win \$10. What is the expected value of this game? Do you come out ahead?

Solution

Let X be the amount won in a game. The values of x are $-\$10$ (for red), 0 (for blue) and $\$10$ (for a green).

x	$P(x)$	$x \times P(x)$
-10	0.4	-4
0	0.4	0
10	0.2	2

$$\begin{aligned} E(x) &= -4 + 0 + 2 \\ &= -2 \end{aligned}$$

On average, you lose \$2 per game. So you do not come out ahead.

Try It

On May 11, 2013 at 9:30 PM, the probability that moderate seismic activity (one moderate earthquake) would occur in the next 48 hours in Japan was about 1.08%. You bet that a moderate earthquake will occur in Japan during this period. If you win the bet, you win \$100. If you lose the bet, you pay \$10. Let X be the amount of profit from a bet. Find the mean and standard deviation of X .

Solution

x	$P(x)$	$x \times P(x)$	$(x - \mu)^2 \times P(x)$
100	0.0108	1.08	127.8726
-10	0.9892	-9.892	1.3961

$$\begin{aligned}\mu &= 1.08 + (-9.892) \\ &= -8.812\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{127.7826 + 1.3961} \\ &= 11.3696....\end{aligned}$$



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<https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=1045#oembed-1>

Video: “[Mean \(expected value\) of a discrete random variable | AP Statistics | Khan Academy](#)” by [Khan Academy](#) [4:32] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*



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Video: “[Mean, Standard Deviation, Variance on BA II Plus Calculator | Probability Distribution \$\bar{x}\$, SD, \$\sigma^2\$](#) ” by [Joshua Emmanuel](#) [2:21] (Transcript Available)



Key Takeaways

The expected value, or mean, of a discrete random variable predicts the long-term results of a statistical experiment that has been repeated many times. The standard deviation of a probability distribution is used to measure the variability of possible outcomes.

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7.3 REVIEW EXERCISES

Chapter 7 Review Exercises

1. A company has created a table to evaluate how long new hires remain with the company. Complete the following table using the data provided.

x	$P(x)$
0	0.12
1	0.18
2	0.30
3	0.15
4	
5	0.10
6	0.05

- a) $P(x = 4)$
b) $P(x \geq 4)$
c) On average, how long would you expect a new hire to stay with the company?
d) What does the column $P(x)$ sum to?

Solution

- a) 0.10
b) 0

c) Two years

d) 1

2. A baker is deciding how many batches of muffins to make to sell in his bakery. He wants to make enough to sell to everyone and no fewer. Through observation, the baker has established a probability distribution.

x	$P(x)$
1	0.15
2	0.35
3	0.40
4	0.10

a) Define the random variable x .

b) What is the probability the baker will sell more than one batch?

c) What is the probability the baker will sell exactly one batch?

d) On average, how many batches should the baker make?

Solution

a) The number of batches the baker makes.

b) $0.35 + 0.40 + 0.10 = 0.85$

c) 15

3. Ellen has music practice three days a week. She practices for all of the three days 85 of the time, two days 8 of the time, one day 4 of the time, and no days 3 of the time. One week is selected at random.

a) Define the random variable x .

b) Construct a probability distribution table for the data.

Solution

- a) The amount of the three days which Ellen practices.,
b)

x	$P(x)$
0	0.03
1	0.04
2	0.08
3	0.85

4. We know that for a probability distribution function to be discrete, it must have two characteristics. One is that the sum of the probabilities is one. What is the other characteristic?

Solution

Each probability is between zero and one.

5. Javier volunteers in community events each month. He does not do more than five events in a month. He attends exactly five events **35** of the time, four events **25** of the time, three events **20** of the time, two events **10** of the time, one event **5** of the time, and no events **5** of the time.

- a) Define the random variable x .
b) What values does x take on?
c) Construct a probability distribution table with the data.
d) Find the probability that Javier volunteers for less than three events each month.
e) Find the probability that Javier volunteers for at least one event each month.

Solution

- a) The number of events Javier volunteers for each month
b) Discrete values between **0** and **5**,

c)

x	$P(x)$
0	0.05
1	0.05
2	0.10
3	0.25
4	0.35

d) $0.05 + 0.05 + 0.10 = 0.20$

e) $1 - 0.05 = 0.95$

6. Suppose that the probability distribution function for the number of years it takes to earn a Bachelor of Science (B.S.) degree is given in the following table.

x	$P(x)$
3	0.05
4	0.40
5	0.30
6	0.15
7	0.10

a) Define the random variable x .b) What does it mean that the values zero, one, and two are not included for x in the PDF?**Solution**

a) The number of years it takes to earn a B.S. degree

b) Nobody sampled has acquired a B.S. in **0, 1, or 2** years.

7. Complete the expected value table.

x	$P(x)$	$x * P(x)$
0	0.2	
1	0.2	
2	0.4	
3	0.2	

Solution

$$x * P(x)$$

$$0(0.2) = 0$$

$$1(0.2) = 0.2$$

$$2(0.4) = 0.8$$

$$3(0.2) = 0.6$$

8. Find the expected value from the expected value table.

x	$P(x)$	$x * P(x)$
2	0.1	0.2
4	0.3	1.2
6	0.4	2.4
8	0.2	1.6

Solution

$$0.2 + 1.2 + 2.4 + 1.6 = 5.4$$

9. Find the standard deviation

x	$P(x)$	$x * P(x)$	$(x - \mu)^2 * P(x)$
2	0.1	$2(0.1) = 0.2$	$(2 - 5.4)^2 * 0.1 = 1.156$
4	0.3	$4(0.3) = 1.2$	$(4 - 5.4)^2 * 0.3 = 0.588$
6	0.4	$6(0.4) = 2.4$	$(6 - 5.4)^2 * 0.4 = 0.144$
8	0.2	$8(0.2) = 1.6$	$(8 - 5.4)^2 * 0.2 = 1.352$

Solution

$$\frac{1.156 + 0.588 + 0.144 + 1.352}{4} = 0.81$$

10. Identify the mistake in the probability distribution table

x	$P(x)$	$x * P(x)$
1	0.15	0.15
2	0.25	0.50
3	0.30	0.90
4	0.20	0.80
5	0.15	0.75

Solution

The values of $P(x)$ do not sum to 1

11. A physics professor wants to know what percent of physics majors will spend the next several years doing post-graduate research. He has the following probability distribution.

x	$P(x)$
1	0.35
2	0.20
3	0.15
4	
5	0.10
6	0.05

- Define the random variable x .
- Define $P(x)$, or the probability of x .
- Find the probability that a physics major will do post-graduate research for four years.
- Find the probability that a physics major will do post-graduate research for at most three years.
- On average, how many years would you expect a physics major to spend doing post graduate research?

Solution

- The amount of years a student will spend doing post-graduate research.
- The probability that a student will spend x amount of years doing post-graduate research
- $1 - 0.35 - 0.20 - 0.15 - 0.05 = 0.15$
- $1 - 0.35 - 0.20 - 0.15 - 0.10 - 0.05 = 0.15$
- $$1(0.35) + 2(0.20) + 3(0.15) + 4(0.15) + 5(0.10) + 6(0.05) = 0.35 + 0.40 + 0.45 + 0.60 + 0.50 + 0.30 = 2.6 \text{ years}$$

12. A venture capitalist, willing to invest \$1,000,000, has three investments to choose from. The first investment, a software company, has a 10 chance of returning \$5,000,000 profit, a

30 chance of returning \$1,000,000 profit, and a **60** chance of losing the million dollars. The second company, a hardware company, has a **20** chance of returning \$3,000,000 profit, a **40** chance of returning \$1,000,000 profit, and a **40** chance of losing the million dollars. The third company, a biotech firm, has a **10** chance of returning \$6,000,000 profit, a **70** of no profit or loss, and a **20** chance of losing the million dollars.

- Construct a probability distribution function for each investment.
- Find the expected value for each investment.
- Which is the safest investment? Why do you think so?
- Which is the riskiest investment? Why do you think so?
- Which investment has the highest expected return, on average?

Solution

a)

x	$P(x)$	$x * P(x)$
5,000,000	0.10	500,000
1,000,000	0.30	300,000
0	0.60	$[\text{latex}0][\text{/latex}]$

x	$P(x)$	$x * P(x)$
3,000,000	0.20	600,000
1,000,000	0.40	400,000
0	0.40	0

x	$P(x)$	$x * P(x)$
6,000,000	0.10	600,000
1,000,000	0.70	700,000
0	0.20	0

- b) Expected value of **800,000** for the software company, **1,000,000** for the hardware

company, and **1,300,000** for the biotech firm

- c) The safest investment would be in the biotech firm. Justifications include there being the lowest probability for losing the investment, and the highest potential for profit.
- d) The riskiest investment would be in the software company. Justifications include there being the highest possibility for losing the investment, and the lowest potential for profit.
- e) On average, the biotech firm has the highest expected return.

13. Suppose that **20,000** married adults in the United States were randomly surveyed as to the number of children they have. The results are compiled and are used as theoretical probabilities. Let x be the number of children married people have.

x	$P(x)$	$x * P(x)$
0	0.10	
1	0.20	
2	0.30	
3		
4	0.10	
5	0.05	
6(or more)	0.05	

- a) Find the probability that a married adult has three children.
- b) In words, what does the expected value in this example represent?
- c) Find the expected value.
- d) Is it more likely that a married adult will have two to three children or four to six children? How do you know?

Solution

a) $1 - 0.05 - 0.05 - 0.10 - 0.30 - 0.20 - 0.10 = 0.20$

b) The average amount of children that the survey participants had.

c)

$$0(0.10) + 1(0.20) + 2(0.30) + 3(0.20) + 4(0.10) + 5(0.05) + 6(0.05) = 2.25$$

d) The probability of a married adult having two to three children is $0.30 + 0.20$ according to this table. The probability of a married adult having four to six children is $0.10 + 0.05 + 0.05 = 0.20$ according to this table. Therefore, it is more probable that a married adult will have two to three children.

14. Suppose that the PDF for the number of years it takes to earn a Bachelor of Science (B.S.) degree is given as in following table. On average, how many years do you expect it to take for an individual to earn a B.S.?

x	$P(x)$
3	0.05
4	0.40
5	0.30
6	0.15
7	0.10

Solution

$$3(0.05) + 4(0.40) + 5(0.30) + 6(0.15) + 7(0.10) = 4.85$$

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CHAPTER 7 SUMMARY

Key Concepts Summary

- **7.1 Probability Distribution of a Discrete Random Variable**
 - Recognizing, understanding, and constructing discrete probability functions
- **7.2 Expected Value and Standard Deviation for a Discrete Probability Distribution**
 - Calculating and interpreting the expected values of probability distribution functions
 - Calculating the standard deviation for a probability distribution

Glossary of Terms

Discrete Random Variable. A random variable expressed in countable terms (i.e. the number of phone calls made in a week).

Expected Value. The average value expected after long-term repetition of an experiment.

Law of Large Numbers. A rule stating that as the number of trials of a probability experiment increases, the difference between the expected value and observed value should decrease.

Probability Distribution. A listing of all the possible values of a random variable, and the likelihood that they will occur.

Random Variable. A description of the data being observed that may vary between repetitions of an experiment.

Formula & Symbol Hub

Symbols Used

- σ = standard deviation
- $P(x)$ = probability of x
- μ = mean or expected value
- \sum = summation symbol

Formulas Used

- Formula 7.1 – **Expected Value**

$$E(x) = \sum (x \times P(x))$$

- Formula 7.2 – **Standard Deviation of a Probability Distribution**

$$\sigma = \sqrt{\sum ((x - \mu)^2 \times P(x))}$$

CHAPTER 8: CORRELATION AND SIMPLE LINEAR REGRESSION

Chapter Outline

[8.0 Introduction](#)

[8.1 Linear Correlation](#)

[8.2 Linear Regression](#)

[8.3 Review Exercises](#)

[Chapter 8 Summary](#)

8.0 INTRODUCTION

Chapter Learning Objectives

By the end of the chapter, you should be able to:

- Calculate a line of best fit for correlated variables using linear regression.
- Calculate the Coefficient of determination.

In many studies, we measure more than one variable for each individual. For example, we measure precipitation and plant growth, or number of young with nesting habitat, or soil erosion and volume of water. We collect pairs of data and instead of examining each variable separately (univariate data), we want to find ways to describe **bivariate data**, in which two variables are measured on each subject in our sample. Given such data, we begin by determining if there is a relationship between these two variables. As the values of one variable change, do we see corresponding changes in the other variable?

We can describe the relationship between these two variables graphically and numerically. We begin by considering the concept of **correlation**.

Correlation is defined as the statistical association between two variables.

A correlation exists between two variables when one of them is related to the other in some way. A scatterplot is the best place to start. A scatterplot (or scatter diagram) is a graph of the paired (x, y) sample data with a horizontal x -axis and a vertical y -axis. Each individual (x, y) pair is plotted as a single point.

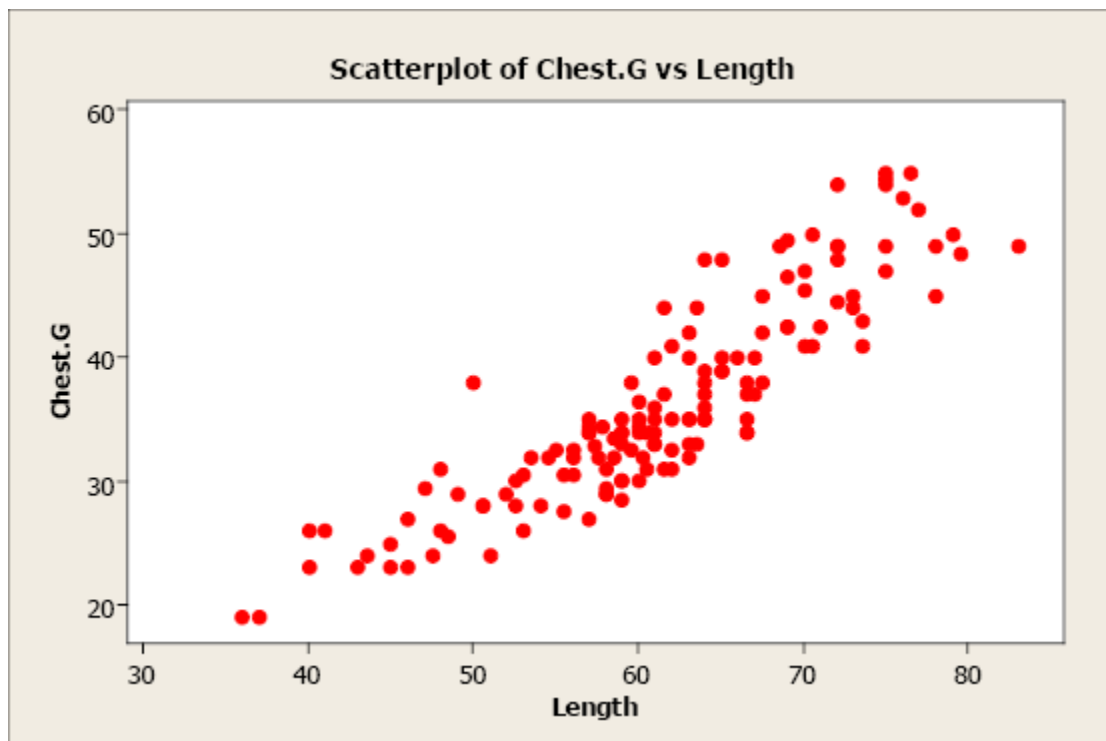


Figure 8.0.1 Scatterplot of chest girth versus length.

Image Description

The image is a scatterplot titled “Scatterplot of Chest.G vs Length.” It shows a positive correlation between two variables: Chest.G (on the y-axis) and Length (on the x-axis). The x-axis is labeled “Length” and ranges from 30 to 80, while the y-axis is labeled “Chest.G” and ranges from 20 to 60. The plot contains numerous red dots representing individual data points, which are dispersed in such a way that they generally trend upwards from the lower-left corner to the upper-right corner of the plot, indicating that as Length increases, Chest.G also tends to increase.

In this example, we plot bear chest girth (y) against bear length (x). When examining a scatterplot, we should study the overall pattern of the plotted points. In this example, we see that the value for chest girth does tend to increase as the value of length increases. We can see an upward slope and a straight-line pattern in the plotted data points.

Key Takeaways



A scatterplot can identify several different types of relationships between two variables.

- A relationship has no correlation when the points on a scatterplot do not show any pattern.
- A relationship is non-linear when the points on a scatterplot follow a pattern but not a straight line.
- A relationship is linear when the points on a scatterplot follow a somewhat straight line pattern. This is the relationship that we will examine.

Linear relationships can be either positive or negative. Positive relationships have points that incline upwards to the right. As x values increase, y values increase. As x values decrease, y values decrease. For example, when studying plants, height typically increases as diameter increases.

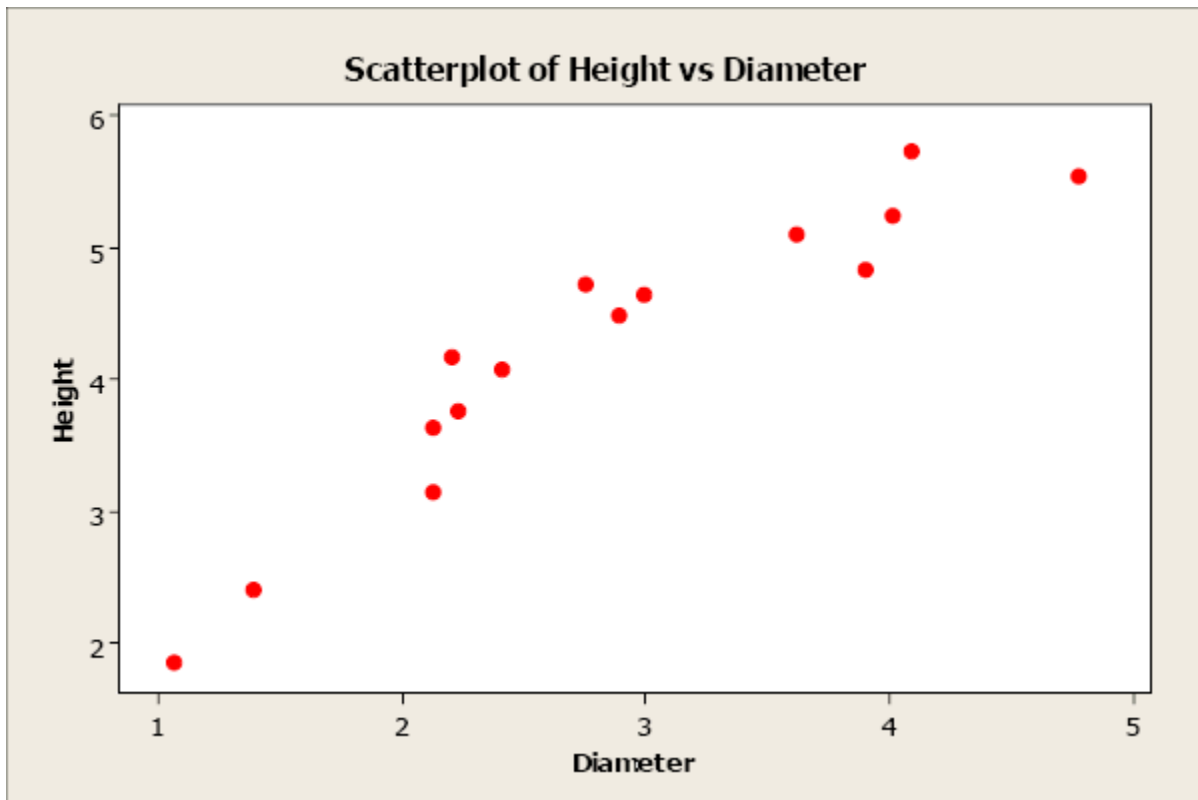


Figure 8.0.2 Scatterplot of height versus diameter.

Image Description

A scatter plot showing the relationship between height and diameter. The x-axis is labeled “Diameter” and ranges from 1 to 5. The y-axis is labeled “Height” and ranges from 2 to 6. There are fourteen red points plotted, indicating a positive correlation between height and diameter.

Negative relationships have points that decline downward to the right. As x values increase, y values decrease. As x values decrease, y values increase. For example, as wind speed increases, wind chill temperature decreases.

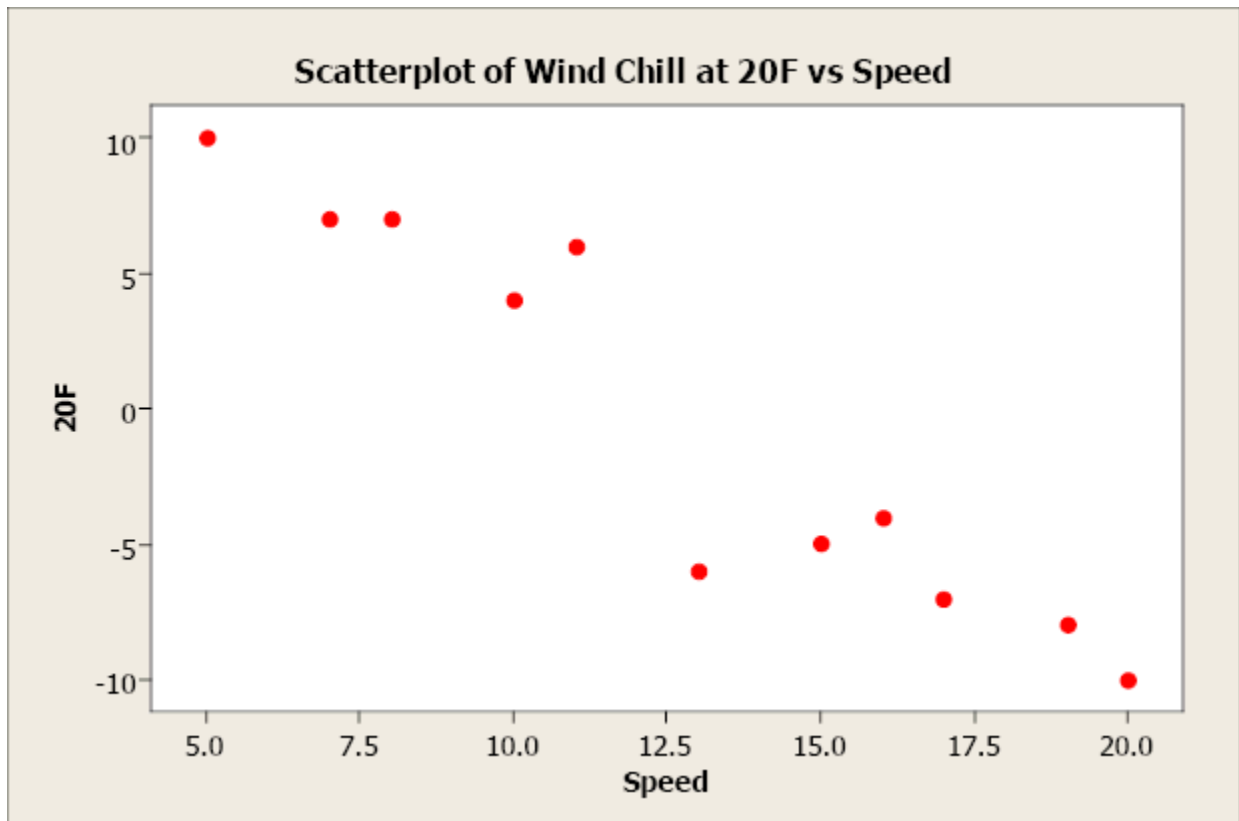


Figure 8.0.3 Scatterplot of temperature versus wind speed.

Image Description

This image is a scatterplot graph titled “Scatterplot of Wind Chill at $20^{\circ} F$ vs Speed.” The graph depicts data points representing wind chill temperatures at various wind speeds, measured at 20 degrees Fahrenheit.

The x-axis is labeled “Speed” and ranges from 5.0 to 20.0 . The y-axis is labeled “ $20^{\circ} F$ ” and ranges from -10 to 10 .

Red dots are plotted on the graph to represent the data points. As the speed increases from left to right, the wind chill temperature decreases. At a speed of approximately 5.0 , the wind chill is close to 10.0 . At a speed of approximately 20.0 , the wind chill is close to -10.0 .

Non-linear relationships have an apparent pattern, just not linear. For example, as age increases height increases up to a point then levels off after reaching a maximum height.

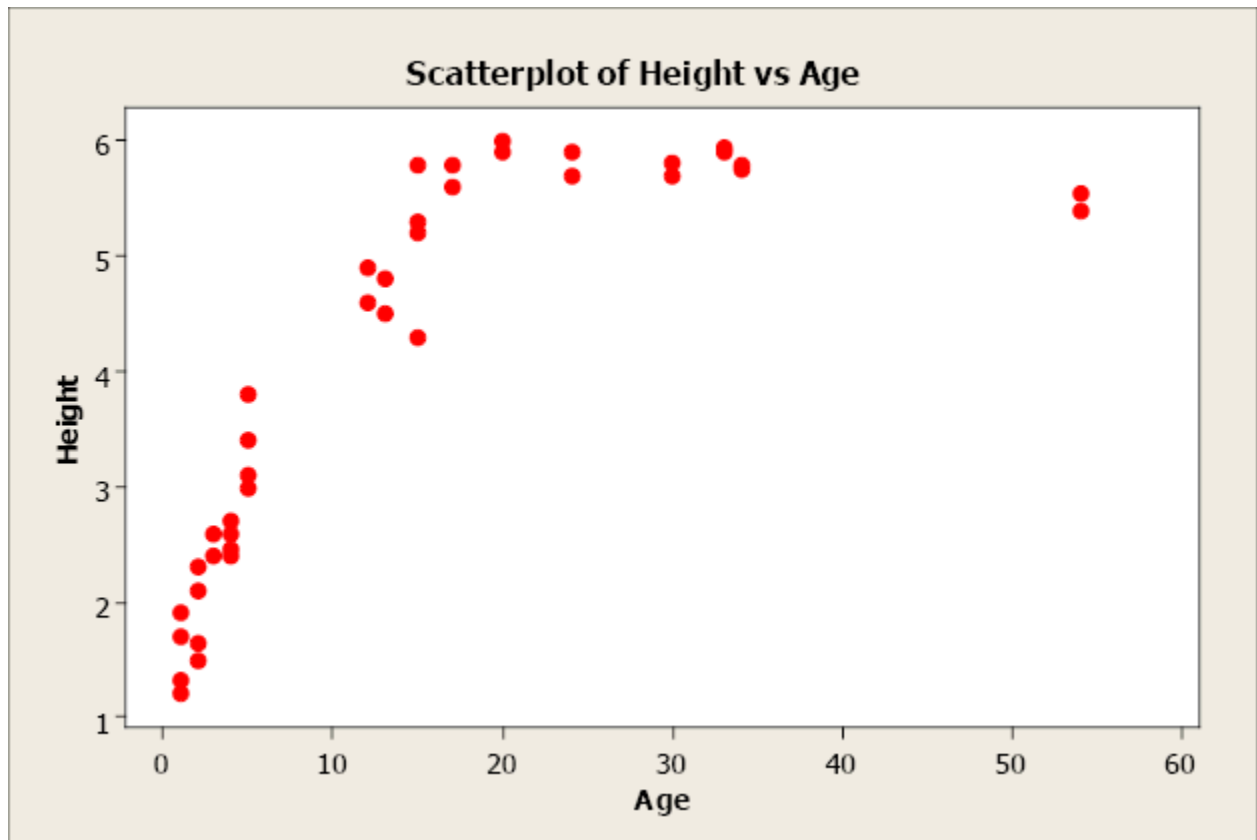


Figure 8.0.4 Scatterplot of height versus age.

Image Description

This image is a scatterplot titled “Scatterplot of Height vs Age”. The plot displays individual data points (marked by red dots) representing height on the y-axis and age on the x-axis. The x-axis is labeled “Age” and ranges from 0 to 60, while the y-axis is labeled “Height” and ranges from 1 to 6.

Most data points cluster between ages 0 and 10 with heights ranging from just over 1 to nearly 4. As age increases to between 10 and 20, the height of the data points also increases, with the majority reaching heights between 4 and 6. Beyond the age of 20, the height remains relatively constant around the 6 mark, with a few data points extending up to age 50.

When two variables have no relationship, there is no straight-line relationship or non-linear relationship. When one variable changes, it does not influence the other variable.

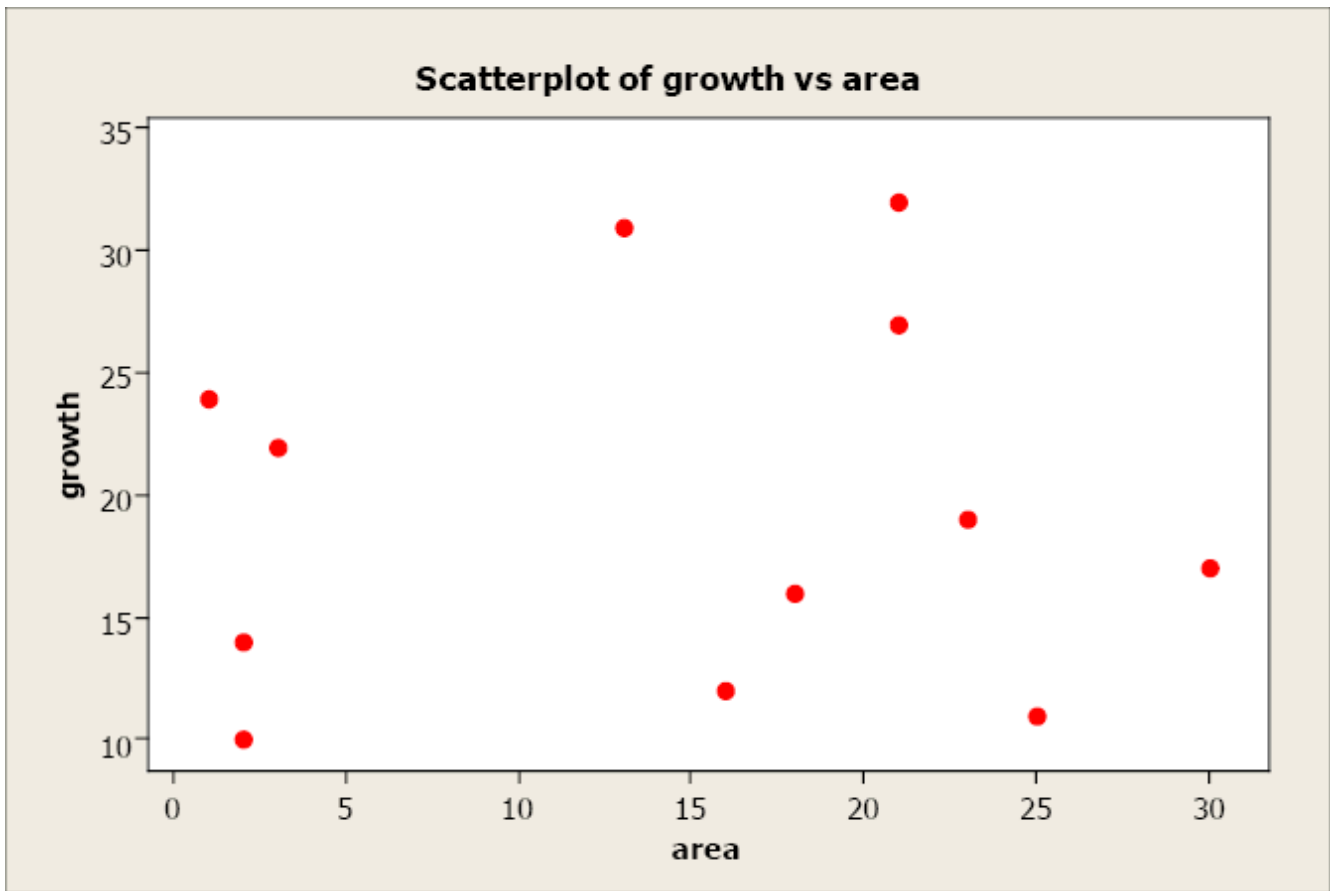


Figure 8.0.5 Scatterplot of growth versus area.

Image Description

The image is a scatterplot chart titled “Scatterplot of growth vs area.” The x-axis is labeled “area” with values ranging from 0 to 30, and the y-axis is labeled “growth” with values ranging from 10 to 35. There are ten red dots plotted at various coordinates, representing different values of growth vs area:

- Approximately (2, 25)
- Approximately (4, 15)
- Approximately (5, 23)
- Approximately (10, 12)
- Approximately (15, 30)
- Approximately (20, 21)
- Approximately (20, 15)
- Approximately (22, 27)
- Approximately (25, 32)
- Approximately (30, 18)

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8.1 LINEAR CORRELATION

Learning Objectives

- Calculate the linear correlation coefficient for any given sets and use it to determine strong/weak positive/negative or no correlation between the sets

Formula & Symbol Hub

Symbols Used

- \bar{x} = mean of set x
- S_x = standard deviation of set x
- r = linear correlation coefficient

Formulas Used

- **Formula 8.1 – Linear Correlation Coefficient**

$$r = \frac{\sum \frac{(x_i - \bar{x})}{S_x} \frac{(y_i - \bar{y})}{S_y}}{n - 1}$$

Linear Correlation Coefficient

Because visual examinations are largely subjective, we need a more precise and objective measure to define the **correlation** between the two variables. To quantify the strength and direction of the relationship between two variables, we use the linear correlation coefficient:

8.1 Linear Correlation Coefficient

$$r = \frac{\sum \frac{(x_i - \bar{x})}{S_x} \frac{(y_i - \bar{y})}{S_y}}{n - 1}$$

r is the Linear Correlation Coefficient.

x_i and y_i are values from data sets x and y .

\bar{x} and \bar{y} are the sample means of data sets x and y .

S_x and S_y are the sample standard deviations of data sets x and y .

n is the sample size of sets x and y .

An alternate computation of the correlation coefficient is:

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$$

$$\text{where } S_{xx} = \sum x^2 - \frac{(\sum x)^2}{n}$$

$$S_{xy} = \sum xy - \frac{(\sum x)(\sum y)}{n}$$

$$S_{yy} = \sum y^2 - \frac{(\sum y)^2}{n}$$

The linear correlation coefficient is also referred to as Pearson's product moment correlation coefficient in honor of Karl Pearson, who originally developed it. This statistic numerically describes how strong the straight-line or linear relationship is between the two variables and the direction, positive or negative.

The properties of r :

- It is always between -1 and $+1$.
- It is a unitless measure so r would be the same value whether you measures the two variables in pounds and inches or in grams and centimeters.
- Positive values of r are associated with positive relationships.
- Negative values of r are associated with negative relationships.

Examples of Positive Correlation

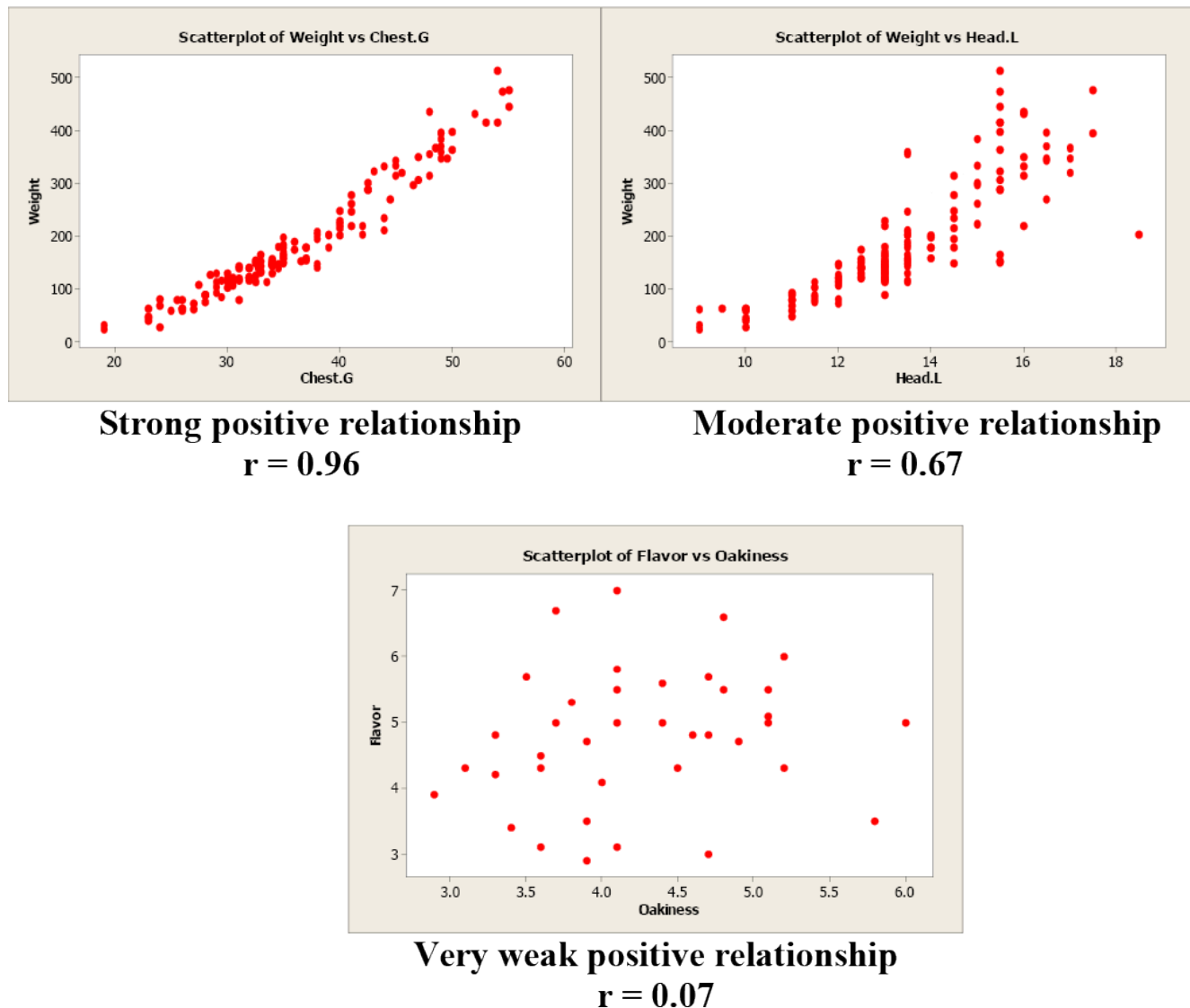


Figure 8.1.1 Examples of positive correlation

Image Description

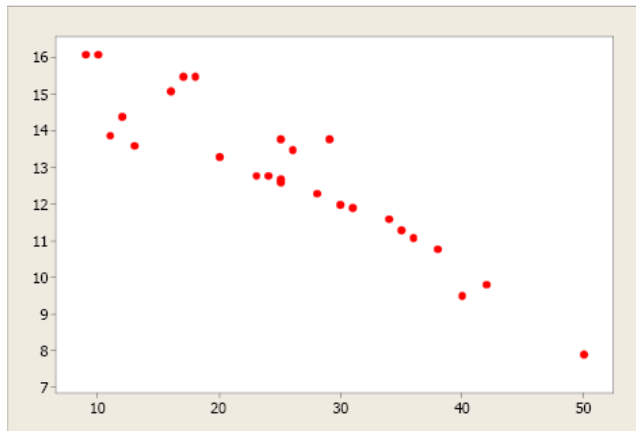
The image contains three scatterplots that depict relationships between different variables, each with associated correlation coefficient values (r) displayed below the scatterplots.

Top-left scatterplot: Title "Scatterplot of Weight vs Chest.G". The x-axis is labeled "Chest.G" and ranges from 20 to 60. The y-axis is labeled "Weight" and ranges from 0 to 500. There are red dots forming a clear upward linear pattern, indicating a strong positive relationship. Below the scatterplot, the text reads: "Strong positive relationship $r = 0.96$ ".

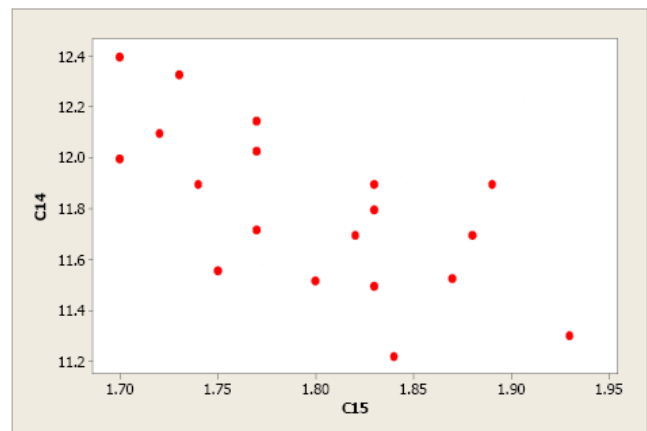
Top-right scatterplot: Title “Scatterplot of Weight vs Head.L”. The x-axis is labeled “Head.L” and ranges from 10 to 18. The y-axis is labeled “Weight” and ranges from 0 to 500. There are red dots forming a moderately upward pattern, indicating a moderate positive relationship. Below the scatterplot, the text reads: “Moderate positive relationship $r = 0.67$ ”.

Bottom scatterplot: Title “Scatterplot of Flavor vs Oakiness”. The x-axis is labeled “Oakiness” and ranges from 3.0 to 6.0. The y-axis is labeled “Flavor” and ranges from 3 to 7. There are red dots scattered without any distinct pattern, indicating a very weak positive relationship. Below the scatterplot, the text reads: “Very weak positive relationship $r = 0.07$ ”.

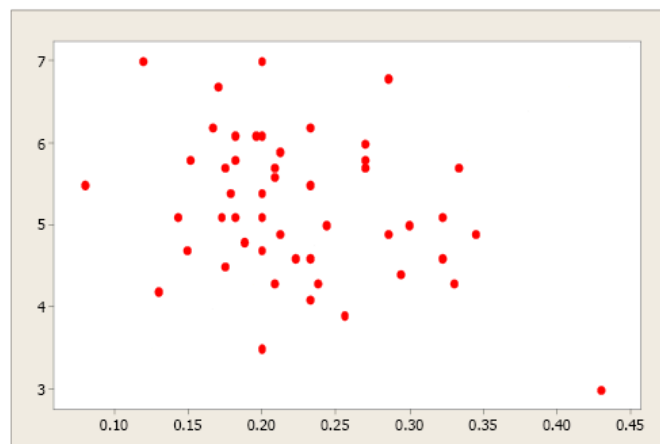
Examples of Negative Correlation



Very strong negative relationship
 $r = -0.93$



Moderately strong negative relationship
 $r = -0.67$



Very weak negative relationship
 $r = -0.13$

Figure 8.1.2 Examples of negative correlation.

Image Description

This image contains three scatterplots demonstrating relationships between paired data points, each featuring red dots representing individual data points.

The top left scatterplot titled “Very strong negative relationship, $r = -0.93$ ” shows a clear downward trend, indicating a very strong negative correlation. Data points are tightly clustered along a descending line.

The top right scatterplot titled “Moderately strong negative relationship, $r = -0.67$ ” shows a moderate downward trend with a moderate amount of dispersion around the trend line, indicating a moderately strong negative correlation.

The bottom center scatterplot titled “Very weak negative relationship, $r = -0.13$ ” shows a very weak downward trend with considerable scatter around the trend line, indicating a very weak negative correlation.

Correlation is not causation!!! Just because two variables are correlated does not mean that one variable causes another variable to change.

Examine these next two scatterplots. Both of these data sets have an $r = 0.01$, but they are very different. Plot 1 shows little linear relationship between x and y variables. Plot 2 shows a strong non-linear relationship. Pearson’s linear correlation coefficient only measures the strength and direction of a linear relationship. Ignoring the scatterplot could result in a serious mistake when describing the relationship between two variables.

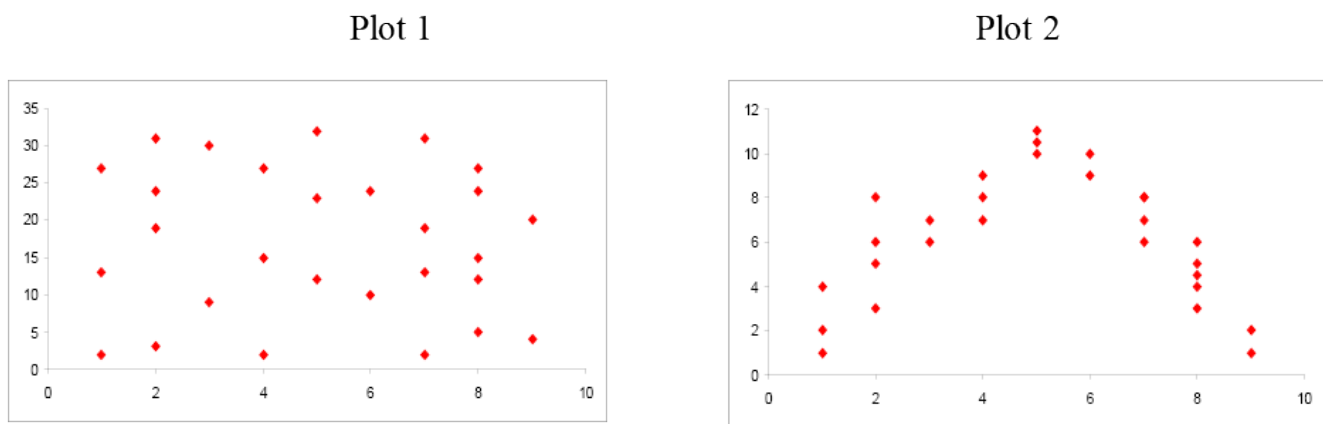


Figure 8.1.3 Comparison of scatterplots.

Image Description

This image contains two scatter plots side by side.

Plot 1: This scatter plot features red diamond-shaped markers, dispersed randomly across the plot area. The X-axis ranges from 0 to 10, and the Y-axis ranges from 0 to 35.

Plot 2: This scatter plot also features red diamond-shaped markers, but they are arranged in a parabolic shape. The X-axis ranges from 0 to 10, and the Y-axis ranges from 0 to 12.

When you investigate the relationship between two variables, always begin with a scatterplot. This graph allows you to look for patterns (both linear and non-linear). The next step is to quantitatively describe the strength and direction of the linear relationship using r . Once you have established that a linear relationship exists, you can take the next step in model building.

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8.2 LINEAR REGRESSION

Learning Objectives

- Calculate a line of best fit for correlated variables using linear regression.
- Calculate the Coefficient of determination

Formula & Symbol Hub

Symbols Used

- \bar{x} = mean of set x
- S_x = standard deviation of set x
- b_0 = the y intercept of a line
- b_1 = the slope of a line
- x = the predictor variable in linear regression
- \hat{y} = the estimate of the mean in linear regression

Formulas Used

- **Formula 8.2 – Simple Linear Regression**

$$\hat{y} = b_0 + b_1 x$$

Simple Linear Regression

Once we have identified two variables that are correlated, we would like to model this relationship. We want to use one variable as a **predictor** or explanatory variable to explain the other variable, the **response** or dependent variable. In order to do this, we need a good relationship between our two variables. The model can then be used to predict changes in our response variable. A strong relationship between the predictor variable and the response variable leads to a good model.

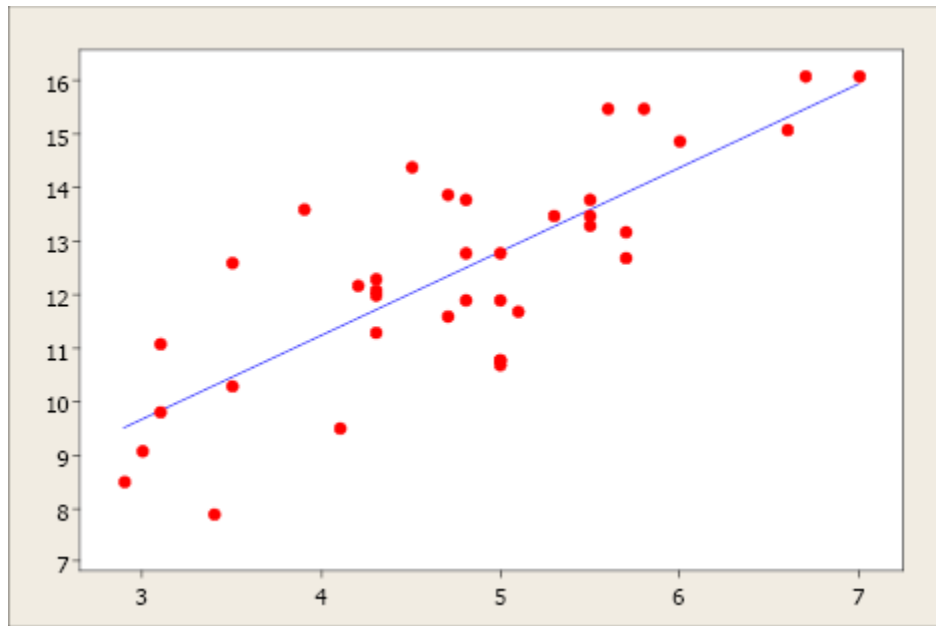


Figure 8.2.1 Scatterplot with regression model.

Image Description

The image is a scatter plot with red dots representing data points. The x-axis ranges from 2 to 8, and the y-axis ranges from 7 to 17. A blue line representing a trend line is drawn diagonally from the bottom-left to the top-right, indicating a positive linear relationship between the variables on the x-axis and y-axis. The data points are scattered around the trend line, showing some variance but generally following the upward trend.

A simple linear regression model is a mathematical equation that allows us to predict a response for a given predictor value.

Our model will take the form of

8.2 Simple Linear Regression

$$\hat{y} = b_0 + b_1 x$$

b_0 is the y-intercept.

b_1 is the slope.

x is the predictor variable.

\hat{y} is the estimate of the mean value of the response variable for any value of the predictor variable.

The y-intercept is the predicted value for the response (y) when $x = 0$. The slope describes the change in y for each one unit change in x . Let's look at this example to clarify the interpretation of the slope and intercept.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=500#oembed-1>

Video: “[BAII Plus – Correlation and regression coefficients](#)” by [Joshua Emmanuel](#) [4:53] is licensed under the [Standard YouTube License](#). Transcript and closed captions available on YouTube.

Example 8.2.1

A hydrologist creates a model to predict the volume flow for a stream at a bridge crossing with a predictor variable of daily rainfall in inches.

$\hat{y} = 1.6 + 29x$. The y -intercept of **1.6** can be interpreted this way: On a day with no rainfall, there will be **1.6** gal. of water/min. flowing in the stream at that bridge crossing. The slope tells us that if it rained one inch that day the flow in the stream would increase by an additional **29** gal./min. If it rained **2** inches that day, the flow would increase by an additional **58** gal./min.

Try It

What would be the average stream flow if it rained **0.45** inches that day?

Solution

$$\hat{y} = 1.6 + 29x = 1.6 + 29(0.45) = 14.65 \text{ gal./min.}$$



Paths to Success

The Least-Squares Regression Line (shortcut equations)

The equation is given by $\hat{y} = b_0 + b_1 x$

where $b_1 = r \left(\frac{S_y}{S_x} \right)$ is the slope and $b_0 = \hat{y} - b_1 \hat{x}$ is the y -intercept of the regression line.

An alternate computational equation for slope is:

$$b_1 = \frac{\sum xy - \frac{(\sum x)(\sum y)}{n}}{\sum x^2 - \frac{(\sum x)^2}{n}} = \frac{S_{xy}}{S_{xx}}$$

This simple model is the **line of best fit** for our sample data. The regression line does not go through every point; instead it balances the difference between all data points and the straight-line model. The difference between the observed data value and the predicted value (the value on the straight line) is the error or **residual**. The criterion to determine the line that best describes the relation between two variables is based on the residuals.

$$\text{Residual} = \text{Observed} - \text{Predicted}$$

For example, if you wanted to predict the chest girth of a black bear given its weight, you could use the following model.

$$\text{Chest girth} = 13.2 + 0.43 \text{ weight}$$

The predicted chest girth of a bear that weighed 120 lb. is 64.8 in.

$$\text{Chest girth} = 13.2 + 0.43(120) = 64.8 \text{ in.}$$

But a measured bear chest girth (observed value) for a bear that weighed 120 lb. was actually 62.1 in.

The residual would be $62.1 - 64.8 = -2.7$ in.

A negative residual indicates that the model is over-predicting. A positive residual indicates that the model is under-predicting. In this instance, the model over-predicted the chest girth of a bear that actually weighed 120 lb.

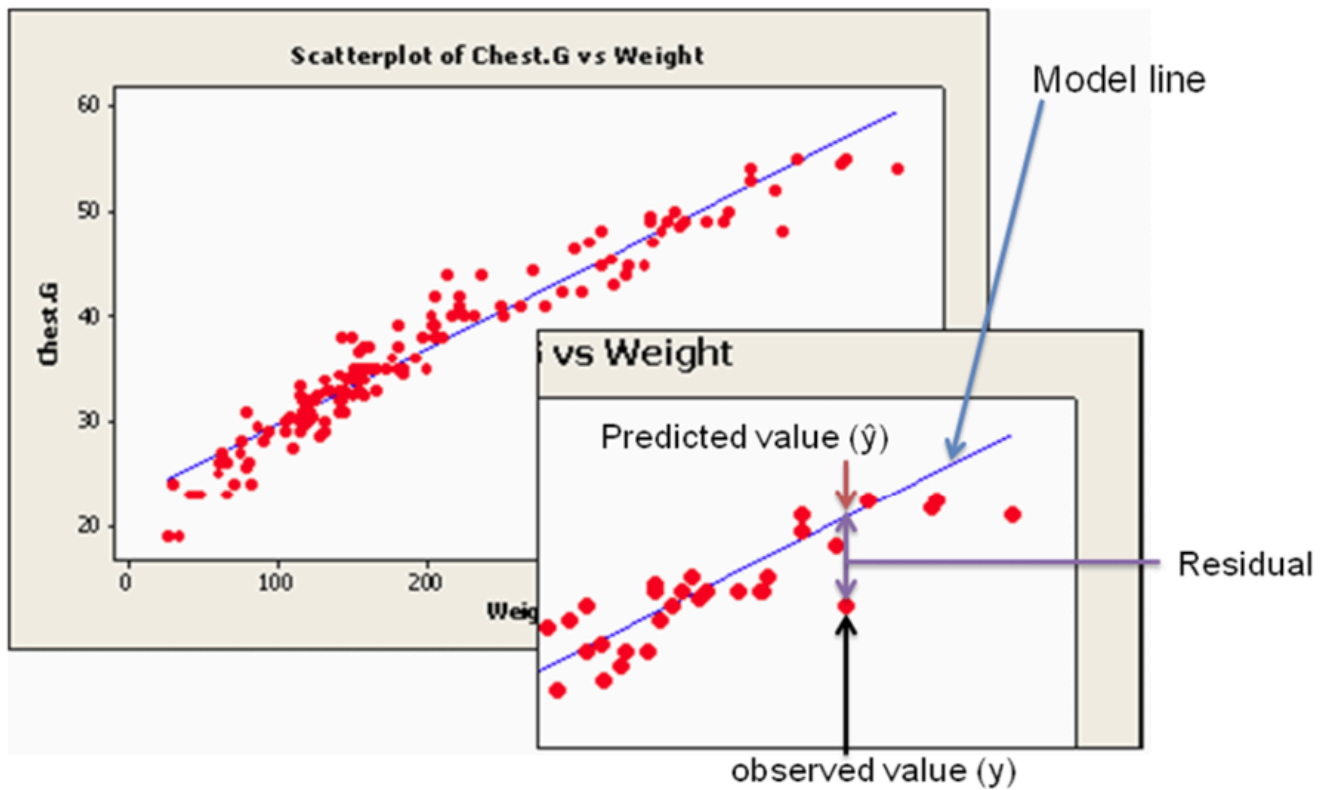


Figure 8.2.2 Scatterplot with regression model illustrating a residual value.

Image Description

The image features two main graphs depicting a scatter plot of chest circumference (Chest.G) versus weight. Data points are illustrated with red dots.

In the larger main graph, titled “Scatterplot of Chest.G vs Weight,” a blue model line plots a trend through the data points, showing a positive linear relationship.

The inset graph focuses on a specific data point. The graph shows two primary values for the highlighted data point: the observed value (y) marked with a black arrow from the x-axis to the red dot and the predicted value (\hat{y}) represented by where the blue model line intersects the same x-coordinate. The vertical distance between the red dot (observed value) and the blue line (predicted value) is labeled as “Residual.”

The labels indicate the slope of the blue line as “Model line” and the vertical line between observed and predicted values is noted as “Residual.”

This random error (residual) takes into account all unpredictable and unknown factors that are not included in the model. An ordinary least squares regression line minimizes the sum of the squared errors between the

observed and predicted values to create a best fitting line. The differences between the observed and predicted values are squared to deal with the positive and negative differences.

Coefficient of Determination

After we fit our regression line (compute b_0 and b_1), we usually wish to know how well the model fits our data. To determine this, we need to think back to the idea of analysis of variance. In ANOVA, we partitioned the variation using sums of squares so we could identify a treatment effect opposed to random variation that occurred in our data. The idea is the same for regression. We want to partition the total variability into two parts: the variation due to the regression and the variation due to random error. And we are again going to compute sums of squares to help us do this.

Suppose the total variability in the sample measurements about the sample mean is denoted by $\sum (y_i - \bar{y})^2$, called the sums of squares of total variability about the mean (SST). The squared difference between the predicted value \hat{y} and the sample mean is denoted by $\sum (\hat{y}_i - \bar{y})^2$, called the sums of squares due to regression (SSR). The SSR represents the variability explained by the regression line. Finally, the variability which cannot be explained by the regression line is called the sums of squares due to error (SSE) and is denoted by $\sum (y_i - \hat{y})^2$. SSE is actually the squared residual.

SST	=SSR	+SSE
$\sum (y_i - \bar{y})^2$	$= \sum (\hat{y}_i - \bar{y})^2$	$+ \sum (y_i - \hat{y})^2$

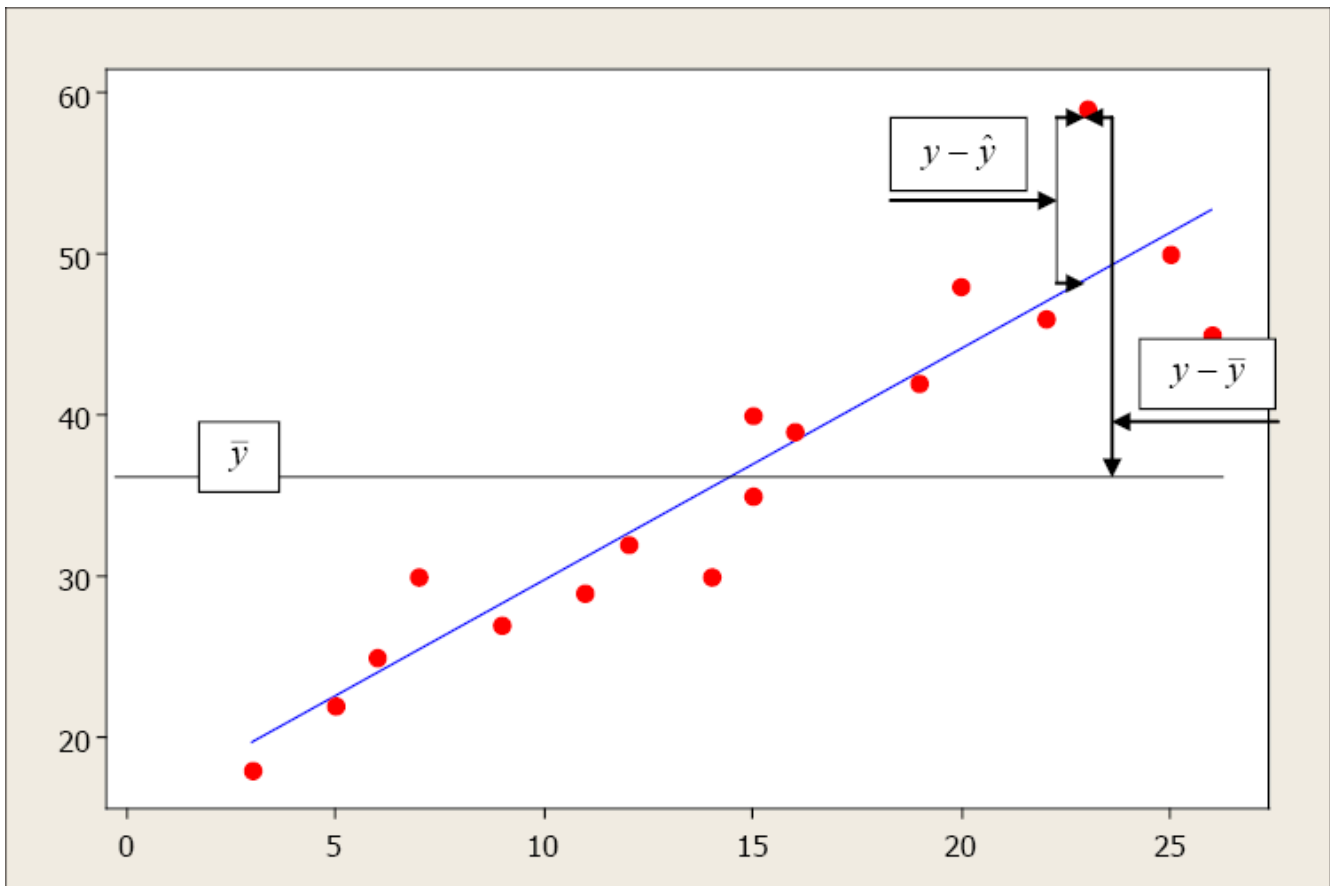


Figure 8.2.3 An illustration of the relationship between the mean of the y 's and the predicted and observed value of a specific y .

Image Description

This image is a scatter plot with red data points displaying a positive correlation. It includes a blue line representing the line of best fit. The x-axis ranges from 0 to 25, while the y-axis ranges from 0 to 60. There are two horizontal gray lines: one at approximately $y = 20$, representing the mean of y (\bar{y}), and another at approximately $y = 37$, marking a specific value on the line of best fit and near the data points.

Annotations in the plot explain statistical concepts:

- One labeled box near the left points to the mean of y (\bar{y}).
- A box at the upper end near the right side indicates the residual ($y - \hat{y}$) with an arrow pointing downward.
- Another labeled box below that shows the difference between actual value and mean ($y - \bar{y}$), also with an arrow pointing downward.

Arrows and labels help visually explain the relationships between actual data points, the mean, and the line of best fit.

The sums of squares and mean sums of squares (just like ANOVA) are typically presented in the regression analysis of variance table. The ratio of the mean sums of squares for the regression (MSR) and mean sums of squares for error (MSE) form an f-test statistic used to test the regression model.

The relationship between these sums of square is defined as

$$\text{Total Variation} = \text{Explained Variation} + \text{Unexplained Variation}$$

The larger the explained variation, the better the model is at prediction. The larger the unexplained variation, the worse the model is at prediction. A quantitative measure of the explanatory power of a model is R^2 , the Coefficient of Determination:

$$R^2 = \frac{\text{Explained Variation}}{\text{Total Variation}}$$

The Coefficient of Determination measures the percent variation in the response variable (y) that is explained by the model.

- Values range from 0 to 1.
- An R^2 close to zero indicates a model with very little explanatory power.
- An R^2 close to one indicates a model with more explanatory power.

The Coefficient of Determination and the linear correlation coefficient are related mathematically.

$$R^2 = r^2$$

However, they have two very different meanings: r is a measure of the strength and direction of a linear relationship between two variables; R^2 describes the percent variation in y that is explained by the model.

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8.3 REVIEW EXERCISES

Chapter 8 Review Exercises

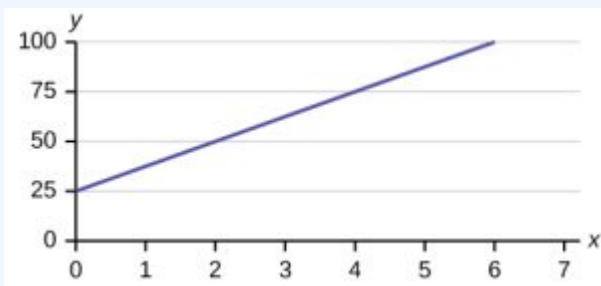
1. A vacation resort rents SCUBA equipment to certified divers. The resort charges an up-front fee of \$25 and another fee of \$12.50 an hour.

- What are the dependent and independent variables?
- Find the equation that expresses the total fee in terms of the number of hours the equipment is rented.
- Graph the equation from 2.

Solution

a. dependent variable: fee amount; independent variable: time

b. $y = 25 + 12.50x$



2. Which of the following equations are linear?

a. $y = 6x + 8$

b. $y + 7 = 3x$

c. $y - x = 8x^2$

d. $4y = 8$

Solution

a. linear

b. linear

c. not linear

d. not linear

4. The table below contains real data for the first two decades of AIDS reporting. Use the columns “year” and “# AIDS cases diagnosed.” Why is “year” the independent variable and “# AIDS cases diagnosed.” the dependent variable (instead of the reverse)?

Adults and Adolescents only, United States

Year	# AIDS cases diagnosed	# AIDS deaths
Pre-1981	91	29
1981	319	121
1982	1,170	453
1983	3,076	1,482
1984	6,240	3,466
1985	11,776	6,878
1986	19,032	11,987
1987	28,564	16,162
1988	35,447	20,868
1989	42,674	27,591
1990	48,634	31,335
1991	59,660	36,560
1992	78,530	41,055
1993	78,834	44,730
1994	71,874	49,095
1995	68,505	49,456
1996	59,347	38,510
1997	47,149	20,736
1998	38,393	19,005
1999	25,174	18,454
2000	25,522	17,347
2001	25,643	17,402
2002	26,464	16,371
Total	802,118	489,093

Solution

The number of cases depends on the year. Therefore, the year becomes the independent variable, and the number of flu cases is the dependent variable.

5. A specialty cleaning company charges an equipment fee and an hourly labour fee. A linear equation that expresses the total amount of the fee the company charges for each session is $y = 50 + 100x$.

- What are the independent and dependent variables?
- What is the y -intercept and what is the slope? Interpret them using complete sentences.

Solution

- independent: hour; dependent: fee.
- The y -intercept is 50 ($a = 50$). At the start of the cleaning, the company charges a one-time fee of \$50 (this is when $x = 0$). The slope is 100 ($b = 100$). For each session, the company charges \$100 for each hour they clean.

6. Due to erosion, a river shoreline is losing several thousand pounds of soil each year. A linear equation that expresses the total amount of soil lost per year is $y = 12,000x$.

- What are the independent and dependent variables?
- How many pounds of soil does the shoreline lose in a year?
- What is the y -intercept? Interpret its meaning.

Solution

- independent: years; dependent lost soil.
- 12,000 pounds of soil
- y -intercept is zero means when the time (years) is zero, the loss of soil is zero.

7. The price of a single issue of stock can fluctuate throughout the day. A linear equation that represents the price of stock for Shipment Express is $y = 15 - 1.5x$ where x is the number of hours passed in an eight-hour day of trading.

- What are the slope and y -intercept? Interpret their meaning.

- b. If you owned this stock, would you want a positive or negative slope? Why?

Solution

- a. The slope is -1.5 ($b = -1.5$). This means the stock is losing value at a rate of \$1.50 per hour. The y-intercept is \$15 ($a = 15$). This means the price of stock before the trading day was \$15.
- b. A positive slope so that the stock gains value.

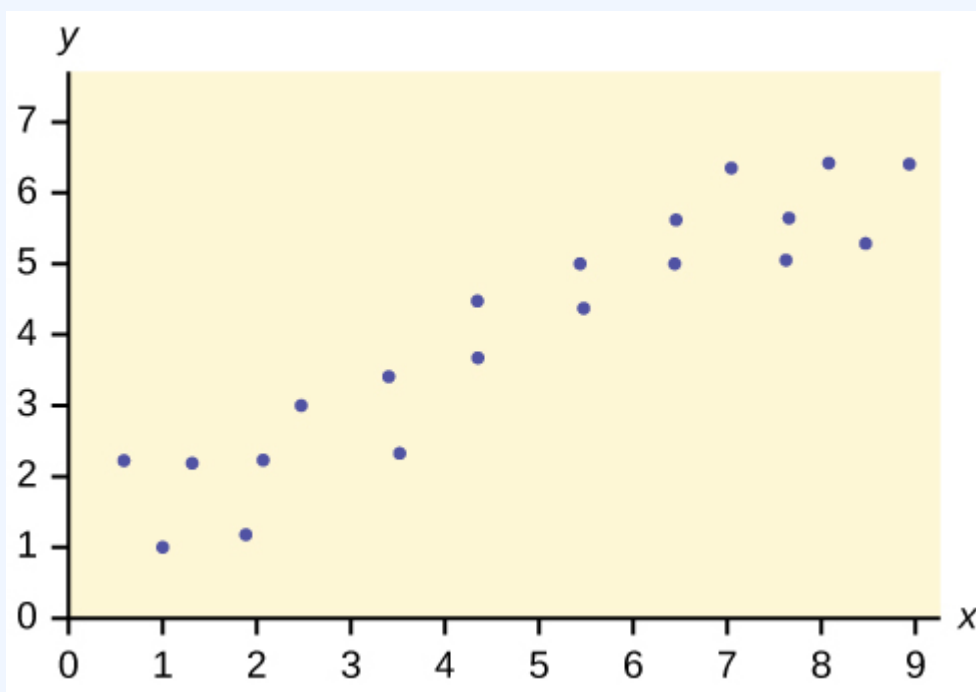
8. For each of the following situations, state the independent variable and the dependent variable.

- a. A study is done to determine if elderly drivers are involved in more motor vehicle fatalities than other drivers. The number of fatalities per 100,000 drivers is compared to the age of drivers.
- b. A study is done to determine if the weekly grocery bill changes based on the number of family members.
- c. Insurance companies base life insurance premiums partially on the age of the applicant.
- d. Utility bills vary according to power consumption.
- e. A study is done to determine if a higher education reduces the crime rate in a population.

Solution

- a. Independent Variable: Age of drivers
Dependent Variable: Number of fatalities per 100,000 drivers
- b. Independent Variable: Number of family members
Dependent Variable: Weekly grocery bill
- c. Independent Variable: Age of the applicant
Dependent Variable: Life insurance premiums
- d. Independent Variable: Power consumption
Dependent Variable: Utility bills
- e. Independent Variable: Level of education in the population
Dependent Variable: Crime rate in the population

9. Does the scatter plot appear linear? Strong, moderate, or weak? Positive or negative?

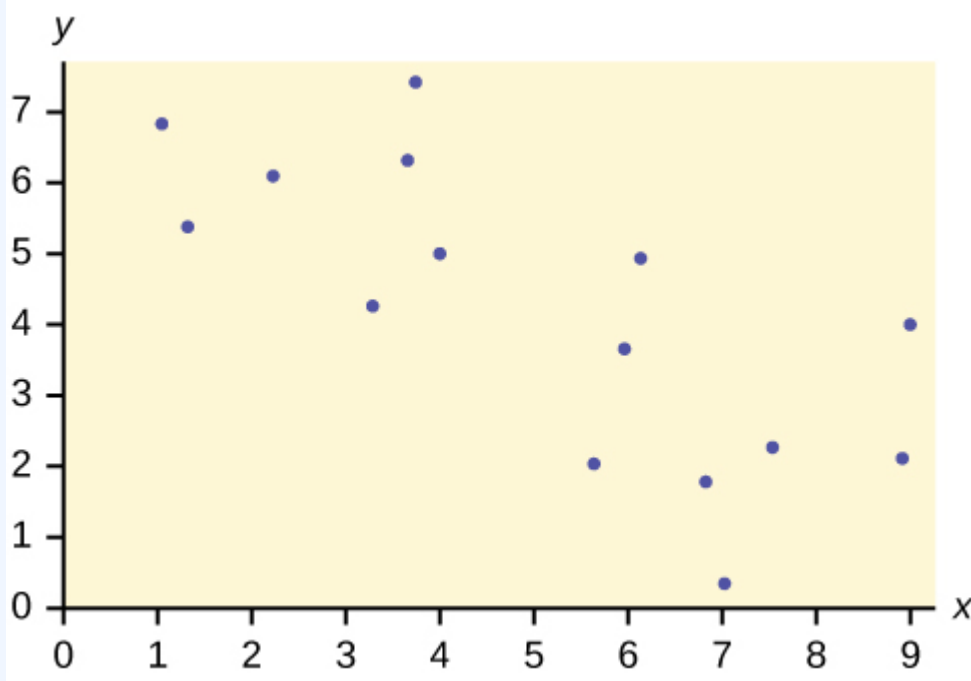
**Image Description**

This image is a scatter plot with the x-axis labelled as “x” and the y-axis labelled as “y.” Both axes have values ranging from 0 to 9. The background of the plot area is light yellow. There are 20 blue dots plotted, representing data points, that display a positive linear trend. As the values of x increase, the values of y tend to increase accordingly.

Solution

The plot is not linear, but it is strongly positively correlated.

10. Does the scatter plot appear linear? Strong, moderate, or weak? Positive or negative?

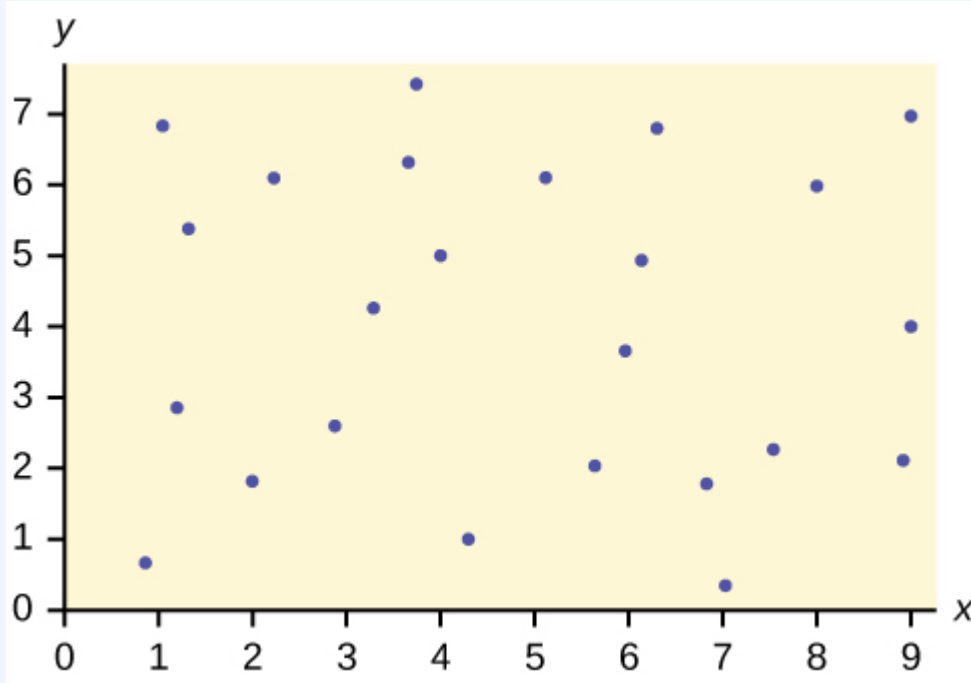
**Image Description**

The image is a scatter plot with a yellow background. The x-axis is labelled “x” and runs from 0 to 9, while the y-axis is labelled “y” and runs from 0 to 7. The plot features several blue dots representing data points scattered throughout the graph. The distribution of points suggests a general negative correlation between the x and y variables, as the y-values tend to decrease as the x-values increase.

Solution

The data appears to be linear with a moderate negative correlation.

11. Does the scatter plot appear linear? Strong, moderate, or weak? Positive or negative?

**Image Description**

The image is a scatter plot on a yellow background with dots that represent data points. The x-axis is labelled from 0 to 9, while the y-axis is labelled from 0 to 7. The data points are scattered across the plot area with no clear pattern or trend.

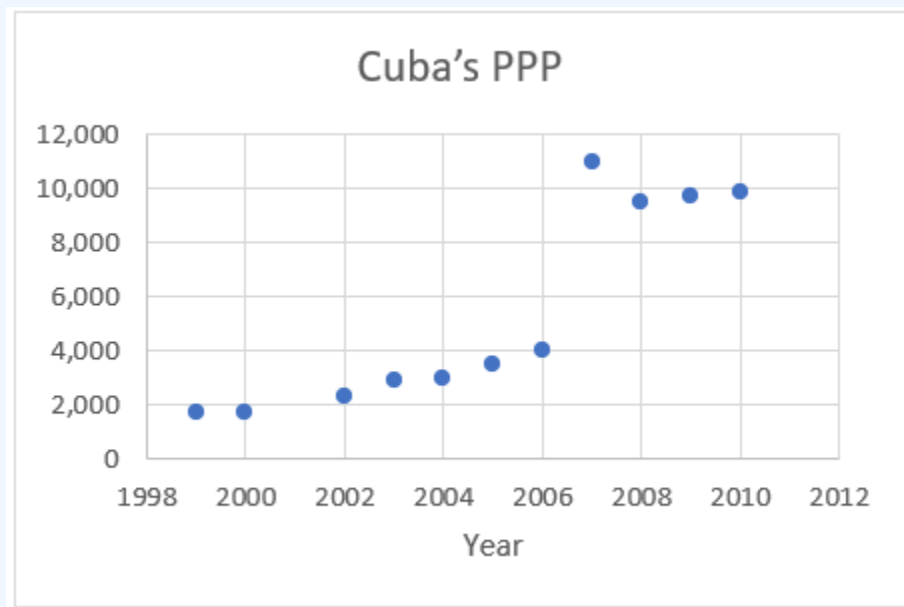
Solution

Not linear but moderately negatively correlated.

12. The Gross Domestic Product Purchasing Power Parity is an indication of a country's currency value compared to another country. The table below shows the GDP PPP of Cuba as compared to US dollars. Construct a scatter plot of the data.

Year	Cuba's PPP	Year	Cuba's PPP
1999	1,700	2006	4,000
2000	1,700	2007	11,000
2002	2,300	2008	9,500
2003	2,900	2009	9,700
2004	3,000	2010	9,900
2005	3,500		

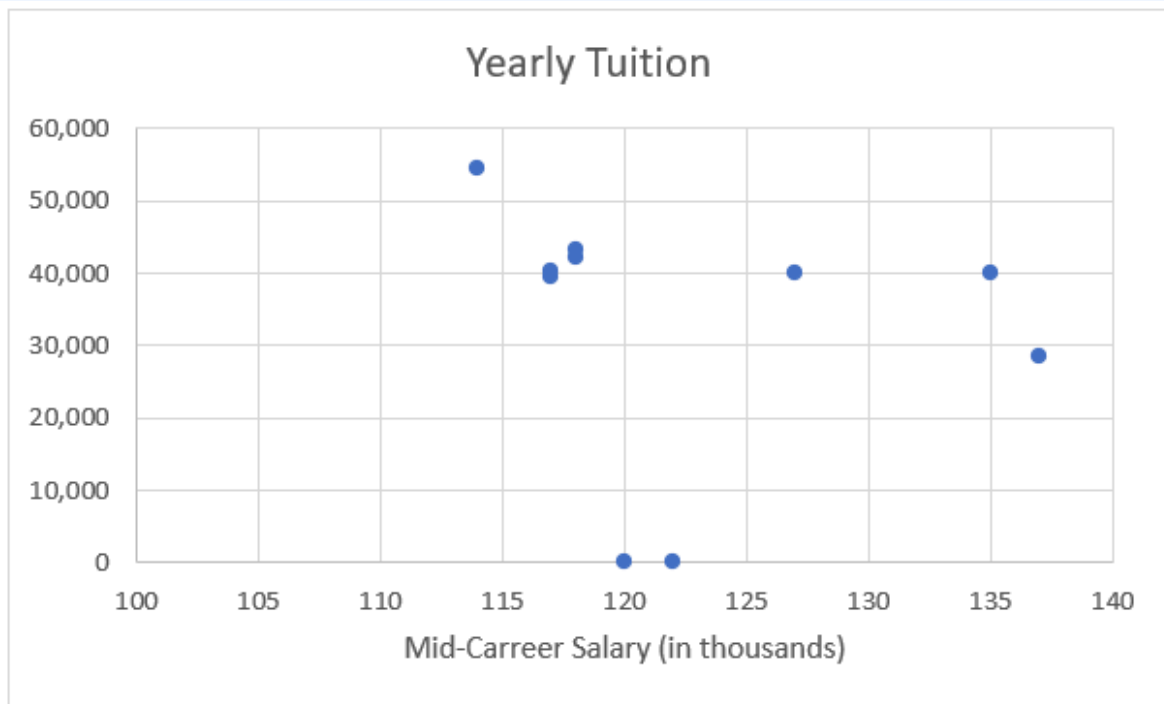
Solution



13. Does the higher cost of tuition translate into higher-paying jobs? The table lists the top ten colleges based on mid-career salary and the associated yearly tuition costs. Construct a scatter plot of the data.

School	Mid-Career Salary (in thousands)	Yearly Tuition
Princeton	137	28,540
Harvey Mudd	135	40,133
CalTech	127	39,900
US Naval Academy	122	0
West Point	120	0
MIT	118	42,050
Lehigh University	118	43,220
NYU-Poly	117	39,565
Babson College	117	40,400
Stanford	114	54,506

Solution



14. A random sample of ten professional athletes produced the following data where x is the

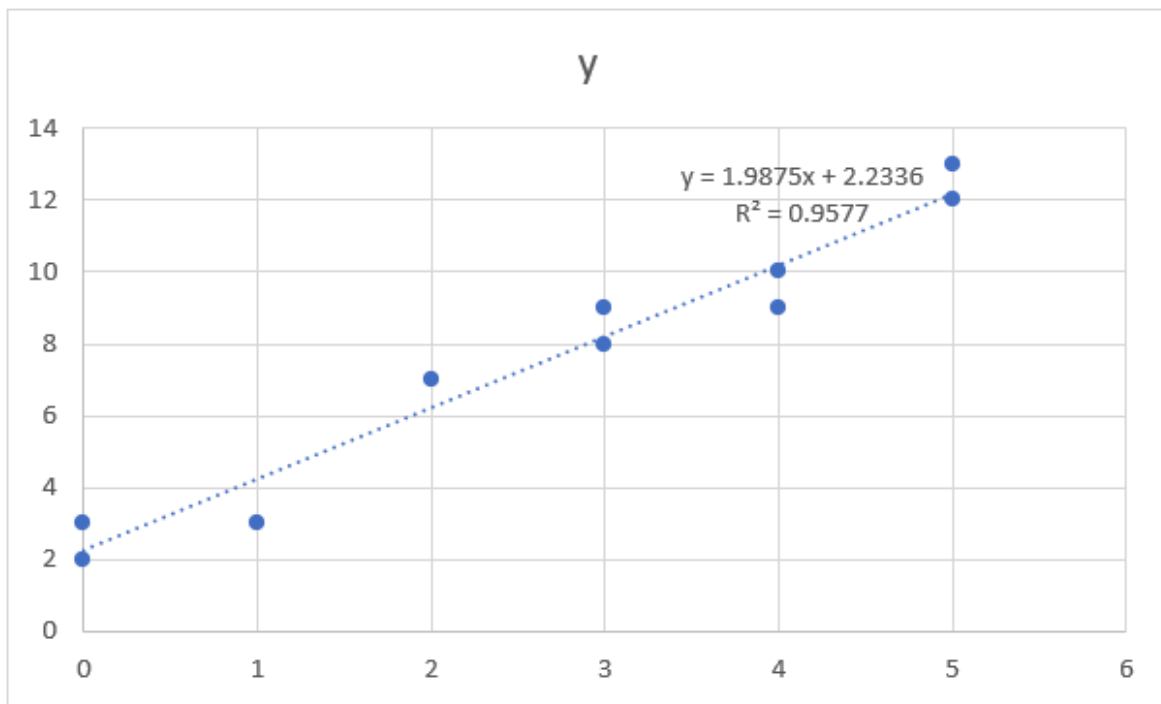
number of endorsements the player has and y is the amount of money made (in millions of dollars).

x	y	x	y
0	2	5	12
3	8	4	9
2	7	3	9
1	3	0	3
5	13	4	10

- Draw a scatter plot of the data.
- Use regression to find the equation for the line of best fit.
- What is the slope of the line of best fit? What does it represent?
- What is the y -intercept of the line of best fit?

Solution

a.



b. $\hat{y} = 2.23 + 1.99x$

c. The slope is 1.99 ($b = 1.99$). It means that for every endorsement deal a professional player gets, he gets an average of another \$1.99 million in pay each year.

d. The y-intercept is 2.23.

15. What does an r value of zero mean?

Solution

It means that there is no correlation between the data sets.

16. What is the process through which we can calculate a line that goes through a scatter plot with a linear pattern?

Solution

The process through a line that goes through a scatter plot with a linear pattern is linear regression.

17. Explain what it means when a correlation has an r^2 of 0.72.

Solution

It means that 72% of the variation in the dependent variable (y) can be explained by the variation in the independent variable (x).

18. An electronics retailer used regression to find a simple model to predict sales growth in the first quarter of the new year (January through March). The model is good for 90 days, where x is the day. The model can be written as $\hat{y} = 101.32 + 2.48x$ where \hat{y} is in thousands of dollars.

- What would you predict the sales to be on day 60?
- What would you predict the sales to be on day 90?

Solution

- \$250,120
- \$324,520

19. A landscaping company is hired to mow the grass for several large properties. The total area of the properties combined is 1,345 acres. The rate at which one person can mow is $\hat{y} = 1350 - 1.2x$ where x is the number of hours and \hat{y} represents the number of acres left to mow.

- How many acres will be left to mow after 20 hours of work?
- How many acres will be left to mow after 100 hours of work?
- How many hours will it take to mow all of the lawns?

Solution

- 1,326 acres
- 1,230 acres
- 1,125 hours, or when $x = 1,125$

20. The table below contains real data for the first two decades of AIDS reporting.

Adults and Adolescents only, United States

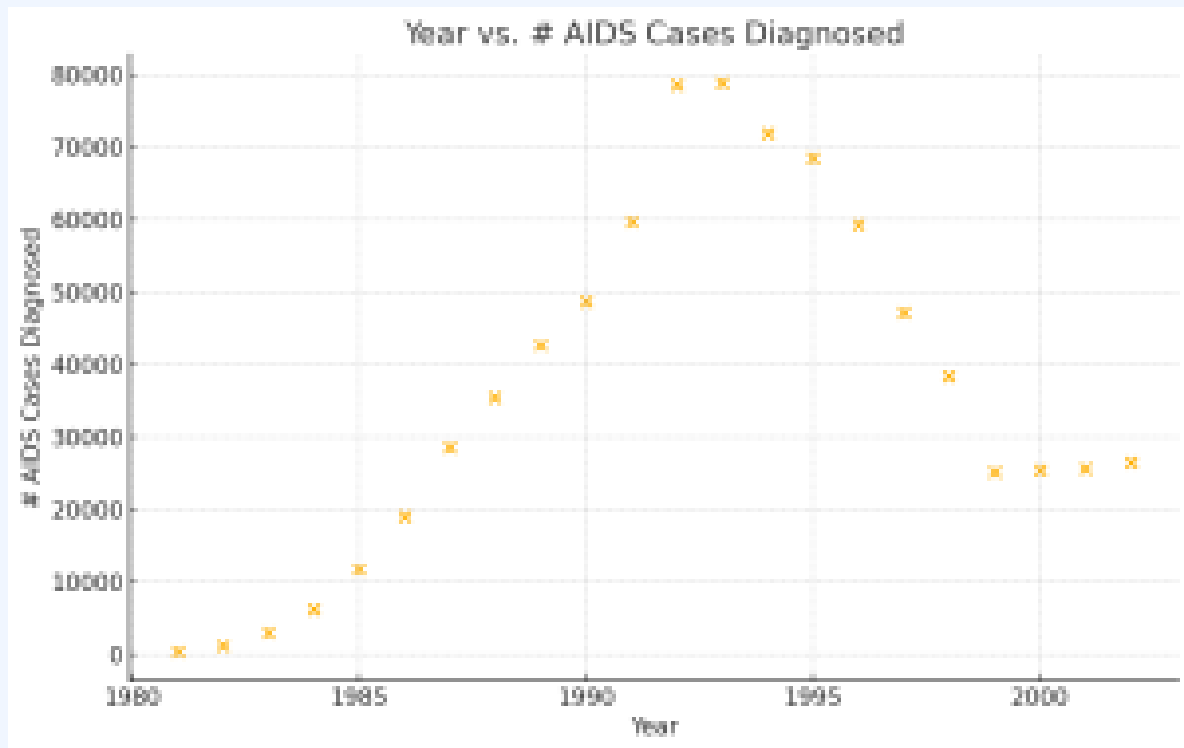
Year	# AIDS cases diagnosed	# AIDS deaths
Pre-1981	91	29
1981	319	121
1982	1,170	453
1983	3,076	1,482
1984	6,240	3,466
1985	11,776	6,878
1986	19,032	11,987
1987	28,564	16,162
1988	35,447	20,868
1989	42,674	27,591
1990	48,634	31,335
1991	59,660	36,560
1992	78,530	41,055
1993	78,834	44,730
1994	71,874	49,095
1995	68,505	49,456
1996	59,347	38,510
1997	47,149	20,736
1998	38,393	19,005
1999	25,174	18,454
2000	25,522	17,347
2001	25,643	17,402
2002	26,464	16,371
Total	802,118	489,093

- a. Graph “year” versus “# AIDS cases diagnosed” (plot the scatter plot). Do not include pre-1981 data.

- b. Calculate the correlation coefficient.
- c. Interpret the correlation coefficient.
- d. Find the linear regression equation.
- e. Interpret the slope of the linear regression equation.
- f. What is the predicted number of diagnosed cases for the year 1985?
- g. What is the predicted number of diagnosed cases for the year 1970? Why doesn't this answer make sense?
- h. Calculate the coefficient of determination.
- i. Interpret the coefficient of determination.
- j. Calculate the standard error of the estimate.
- k. Interpret the standard error of the estimate.

Solution

a.



b. 0.45

c. There is a moderate positive correlation between the year and the number of AIDS cases diagnosed, indicating that, on average, AIDS cases tended to increase with time over this period, although the relationship is not very strong.

d. $y = 1749.78x - 3448225.05$

e. or each additional year, the number of diagnosed AIDS cases is expected to increase by approximately 1,750 cases, according to the model.

f. 25,082

g. -1,164. This prediction doesn't make sense because the value is negative, which isn't possible for case counts. This shows that the linear model isn't suitable for predicting years far outside the observed range.

h. 0.20

i. Approximately 20% of the variance in the number of AIDS cases diagnosed can be explained by the year, indicating a weak fit of the linear model to the data.

j. 21,869.27

k. The standard deviation of the residuals (errors) is around 21,869 cases, indicating that the model's predictions can differ from the actual number of cases by a substantial margin, reflecting the variability in the data that the model doesn't capture.

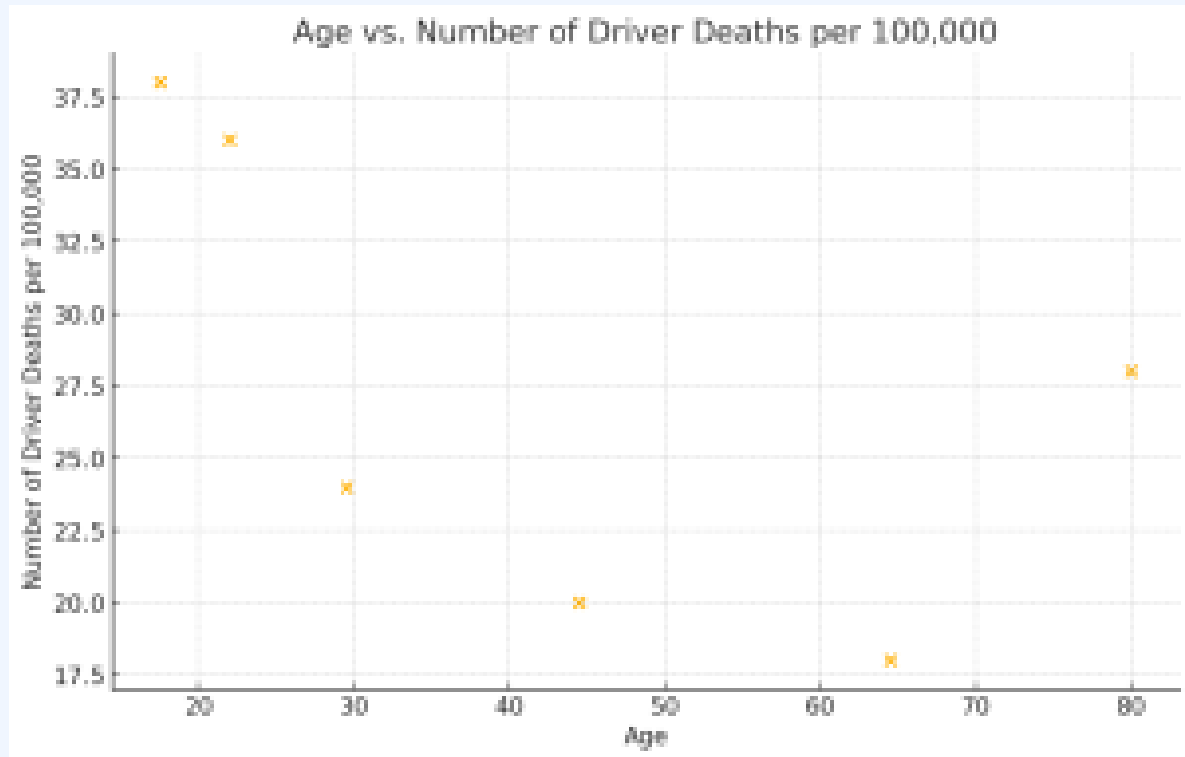
21. Recently, the annual number of driver deaths per 100,000 for the selected age groups was as follows:

Age	Number of Driver Deaths per 100,000
17.5	38
22	36
29.5	24
44.5	20
64.5	18
80	28

- Using “ages” as the independent variable and “Number of driver deaths per 100,000” as the dependent variable, make a scatter plot of the data.
- Calculate the least squares (best-fit) line.
- Interpret the slope of the least squares line.
- Predict the number of deaths of people aged 40.
- Find the correlation coefficient.
- Interpret the correlation coefficient.
- Find the coefficient of determination.
- Interpret the coefficient of determination.
- Find the standard error of the estimate.
- Interpret the standard error of the estimate.

Solution

a.



b. $y = -0.19x + 35.58$

c. The slope of -0.19 indicates that, on average, the number of driver deaths per 100,000 decreases by about 0.19 for each additional year of age.

d. Approximately 27.91 deaths per 100,000.

e. -0.58

f. There is a moderate negative correlation between age and the number of driver deaths per 100,000, meaning that as age increases, the number of driver deaths tends to decrease to some extent.

g. 0.33

h. About 33% of the variance in the number of driver deaths per 100,000 can be explained by age, which indicates a moderate fit of the linear model to the data.

i. 6.15

j. The standard deviation of the residuals (errors) is approximately 6.15 deaths per 100,000,

which means that predictions based on this model can deviate from actual values by about 6.15 on average, reflecting the spread of the data around the regression line.

22. The table below shows the life expectancy for an individual born in the United States in certain years.

Year of Birth	Life Expectancy
1930	59.7
1940	62.9
1950	70.2
1965	69.7
1973	71.4
1982	74.5
1987	75
1992	75.7
2010	78.7

- Decide which variable should be the independent variable and which should be the dependent variable.
- Draw a scatter plot of the ordered pairs.
- Find the correlation coefficient
- Interpret the correlation coefficient.
- Find the linear regression equation.
- Interpret the slope of the linear regression equation.
- What is the estimated life expectancy for someone born in 1950? Why doesn't this value match the life expectancy given in the table for 1950?
- What is the estimated life expectancy for someone born in 1982?

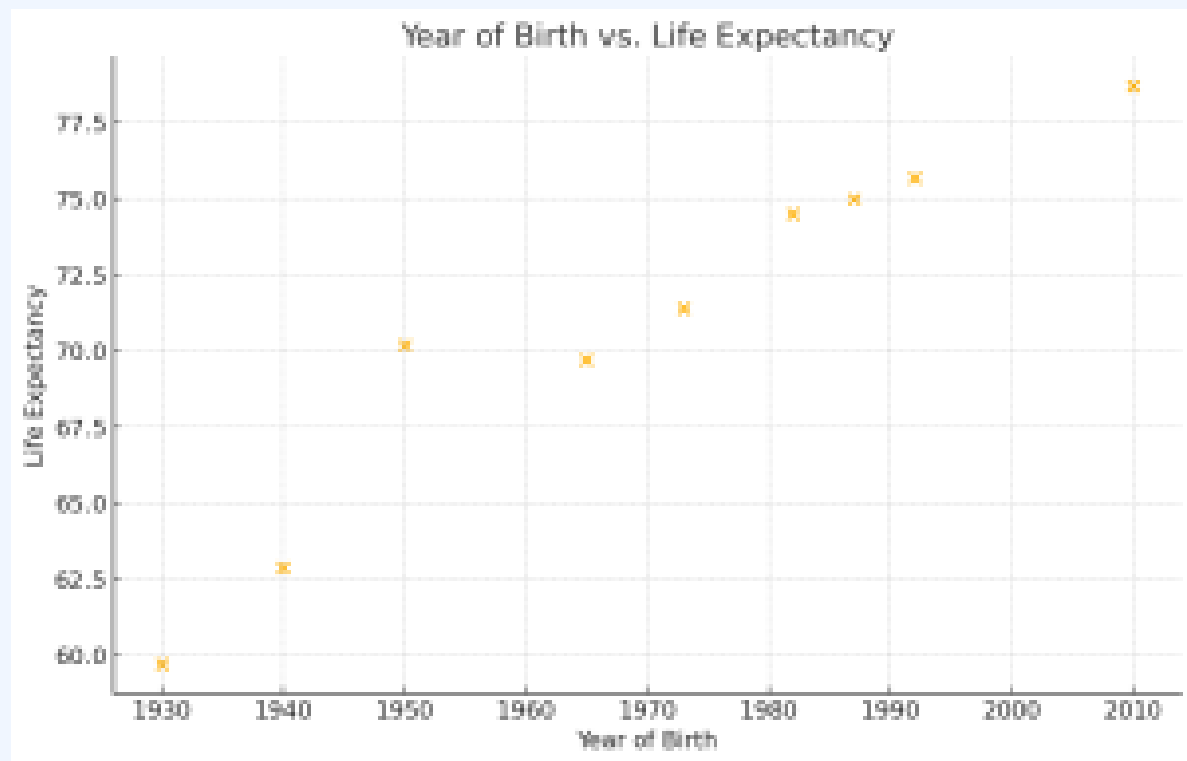
- i. Using the regression equation, find the estimated life expectancy for someone born in 1850. Is this an accurate estimate for that year? Explain why or why not.
- j. Calculate the coefficient of determination.
- k. Interpret the coefficient of determination.
- l. Calculate the standard error of the estimate.
- m. Interpret the standard error of the estimate.

Solution

a. Independent Variable: Year of Birth

Dependent Variable: Life Expectancy

b.



c. 0.96

d. There is a strong positive correlation between the year of birth and life expectancy, suggesting that life expectancy has generally increased over time.

e. $y = 0.23x - 377.24$ $y = 0.23x - 377.24$

f. The slope of 0.23 means that for each additional year, the life expectancy increases by approximately 0.23 years.

g. 66.34 years. This estimate does not match the actual life expectancy given in the table (70.2 years) because the linear regression equation provides an average trend, not exact values for each year.

h. 73.62 years

i. 43.59 years. This estimate is not accurate for that year because the regression model is based on data from 1930 onward. Extrapolating back to 1850 goes beyond the data's range, where the trend may not apply.

j. 0.92

k. Approximately 92% of the variance in life expectancy can be explained by the year of birth, indicating a very strong fit of the model to the data.

l. 1.61

m. The standard deviation of the residuals (errors) is approximately 1.61 years, indicating that predicted life expectancy values typically differ from actual values by about 1.61 years.

23. The maximum discount value of the Entertainment® card for the “Fine Dining” section, Edition ten, for various pages is given in the table below.

Page number	Maximum value (\$)
4	16
14	19
25	15
32	17
43	19
57	15
72	16
85	15
90	17

- Decide which variable should be the independent variable and which should be the dependent variable.
- Draw a scatter plot of the ordered pairs.
- Find the correlation coefficient.
- Interpret the correlation coefficient.
- Find the linear regression equation.
- Interpret the slope of the linear regression equation.
- What is the estimated maximum value for restaurants on page 10?
- What is the estimated maximum value for restaurants on page 70?
- Using the regression equation, find the estimated maximum value for restaurants on page 200. Is this an accurate estimate for that page? Explain why or why not.
- Calculate the coefficient of determination.
- Interpret the coefficient of determination.
- Calculate the standard error of the estimate.

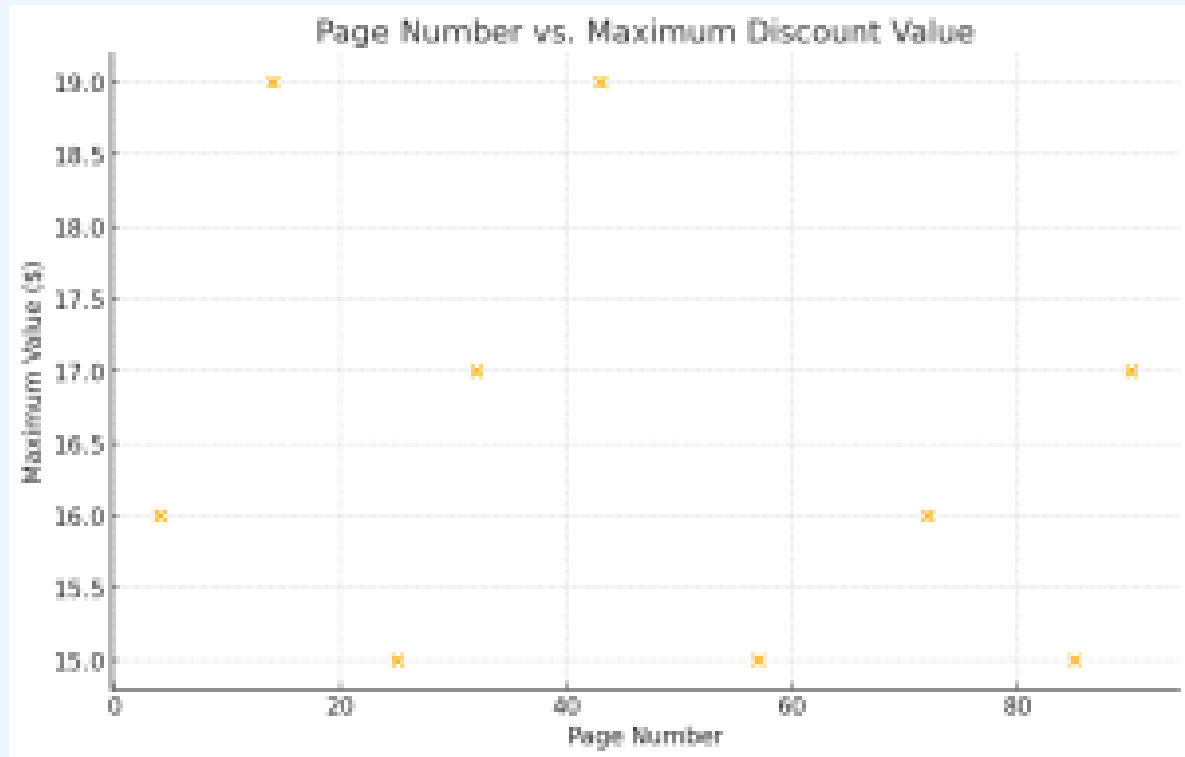
m. Interpret the standard error of the estimate.

Solution

a. Independent Variable: Page Number

Dependent Variable: Maximum Discount Value

b.



c. -0.28

d. There is a weak negative correlation between page number and maximum discount value, suggesting a slight tendency for the maximum value to decrease as the page number increases, though the relationship is weak.

e. $y = -0.01x + 17.22$

f. The slope of -0.01 indicates that, on average, the maximum discount value decreases by about 1 cent for each additional page number.

g. Approximately \$17.08

h. Approximately \$16.23

i. Approximately \$14.39. This estimate may not be accurate for page 200 because it is far outside the range of observed page numbers, and the trend seen in the available data may not hold at such a high page number.

j. 0.08

k. Only about 8% of the variance in the maximum discount value can be explained by the page number, indicating a very weak fit of the linear model to the data.

l. 1.44

m. The standard deviation of the residuals is about \$1.44, suggesting that predictions from this model may differ from actual values by around \$1.44, which is relatively high given the small range of discount values.

24. The table below gives the gold medal times for every other Summer Olympics for the women's 100-meter freestyle (swimming).

Year	Time (seconds)
1912	82.2
1924	72.4
1932	66.8
1952	66.8
1960	61.2
1968	60.0
1976	55.65
1984	55.92
1992	54.64
2000	53.8
2008	53.1

- Decide which variable should be the independent variable and which should be the dependent variable.
- Draw a scatter plot of the ordered pairs.

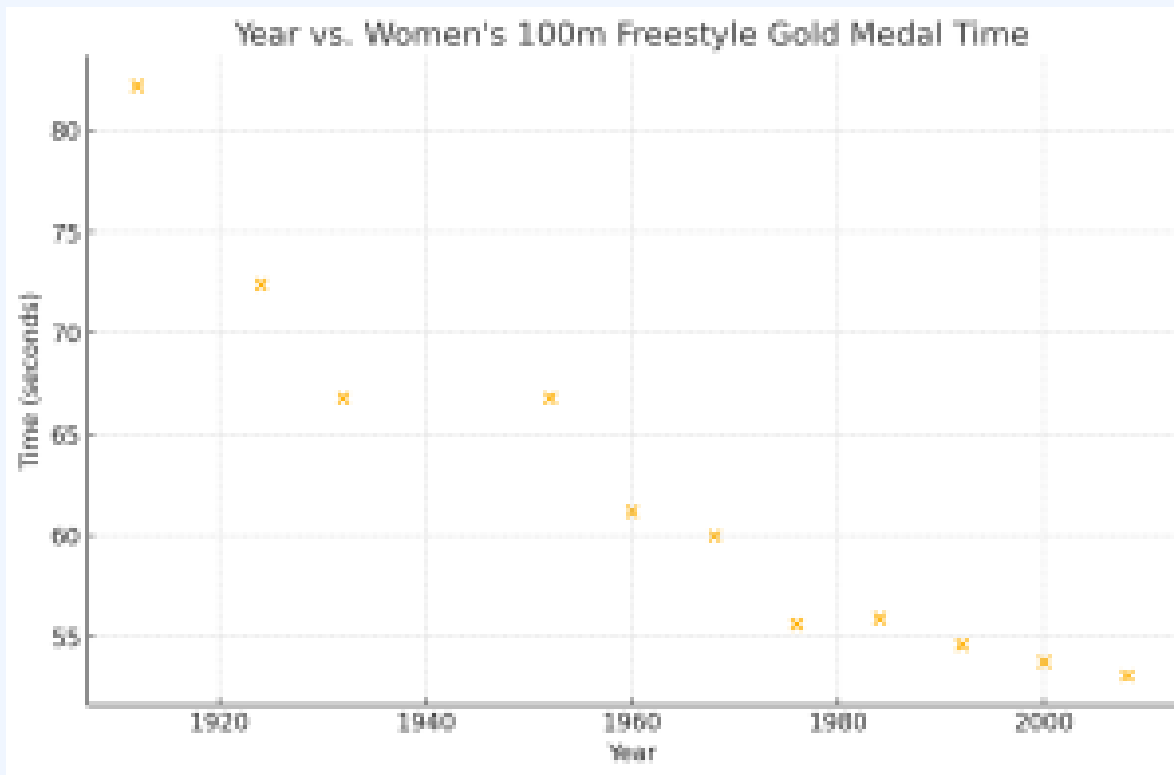
- c. Find the correlation coefficient.
- d. Interpret the correlation coefficient.
- e. Find the linear regression equation.
- f. Interpret the slope of the linear regression equation.
- g. What is the estimated gold medal time for 1932?
- h. What is the estimated gold medal time for 1984?
- i. Calculate the coefficient of determination.
- j. Interpret the coefficient of determination.
- k. Calculate the standard error of the estimate.
- l. Interpret the standard error of the estimate.

Solution

a. Independent Variable: Year

Dependent Variable: Time (in seconds)

b.



c. -0.95

d. There is a strong negative correlation between the year and the gold medal time, indicating that as time progressed, the gold medal time generally decreased.

e. $y = -0.28x + 603.43$

f. The slope of -0.28 suggests that, on average, the gold medal time decreased by approximately 0.28 seconds for each additional year.

g. Approximately 70.97 seconds

h. Approximately 56.63 seconds

i. 0.90

j. About 90% of the variance in gold medal times can be explained by the year, indicating that the model provides a strong fit to the data.

k. 2.70 seconds

l. The standard deviation of the residuals is approximately 2.70 seconds, suggesting that

predictions from this model typically differ from actual values by about 2.70 seconds, reflecting a reasonably good fit of the linear trend to the data.

25. The height (sidewalk to roof) of notable tall buildings in America is compared to the number of stories of the building (beginning at street level).

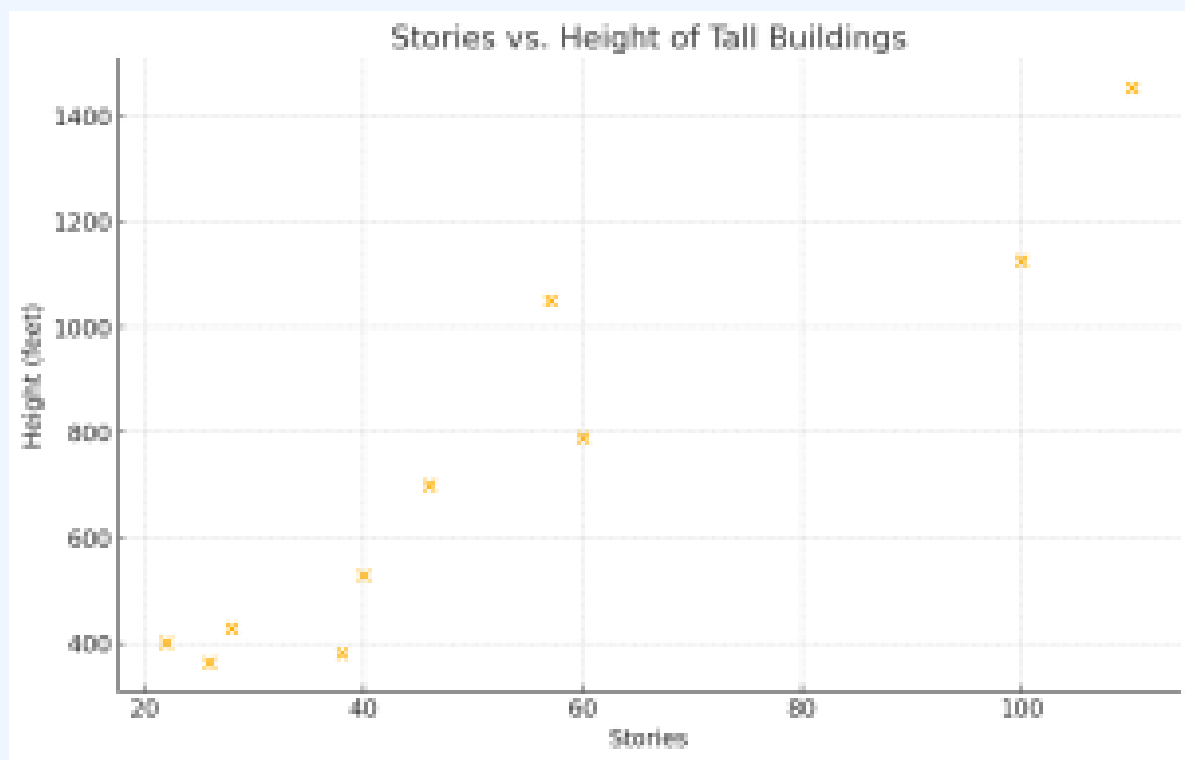
Height (in feet)	Stories
1,050	57
428	28
362	26
529	40
790	60
401	22
380	38
1,454	110
1,127	100
700	46

- Using “stories” as the independent variable and “height” as the dependent variable, draw a scatter plot of the ordered pairs.
- Find the correlation coefficient.
- Interpret the correlation coefficient.
- Find the linear regression equation.
- Interpret the slope of the linear regression equation.
- What is the estimated height for a 32-story building?
- What is the estimated height for a 94-story building?
- Using the regression equation, find the estimated height for a 6-story building. Is this an accurate estimate for the height of a 6-story building? Explain why or why not.

- i. Calculate the coefficient of determination.
- j. Interpret the coefficient of determination.
- k. Calculate the standard error of the estimate.
- l. Interpret the standard error of the estimate.

Solution

a.



- b. 0.94
- c. There is a strong positive correlation between the number of stories and building height, indicating that taller buildings generally have more stories.
- d. $y = 11.76x + 102.43$
- e. The slope of 11.76 suggests that, on average, each additional story adds approximately 11.76 feet to the building's height.
- f. Approximately 478.70 feet

g. Approximately 1207.72 feet

h. Approximately 172.98 feet. This estimate might not be accurate for a 6-story building as it lies outside the typical range of the dataset (mainly taller buildings), so the trend may not apply well for such a small number of stories.

i. 0.89

j. Approximately 89% of the variance in building height can be explained by the number of stories, indicating that the linear model provides a strong fit to the data.

k. 118.75 feet

l. The standard deviation of the residuals is approximately 118.75 feet, indicating that predictions from this model can typically differ from actual values by around 118.75 feet, which suggests some variability in the height-to-story relationship across the buildings.

26. The following table shows data on average per capita wine consumption and heart disease rate in a random sample of 10 countries.

Yearly wine consumption in litres	2.5	3.9	2.9	2.4	2.9	0.8	9.1	2.7	0.8	0.7
Death from heart diseases	221	167	131	191	220	297	71	172	211	300

- Decide which variable should be the independent variable and which should be the dependent variable.
- Draw a scatter plot of the ordered pairs.
- Find the correlation coefficient.
- Interpret the correlation coefficient.
- Find the linear regression equation.
- Interpret the slope of the linear regression equation.
- Calculate the coefficient of determination.
- Interpret the coefficient of determination.

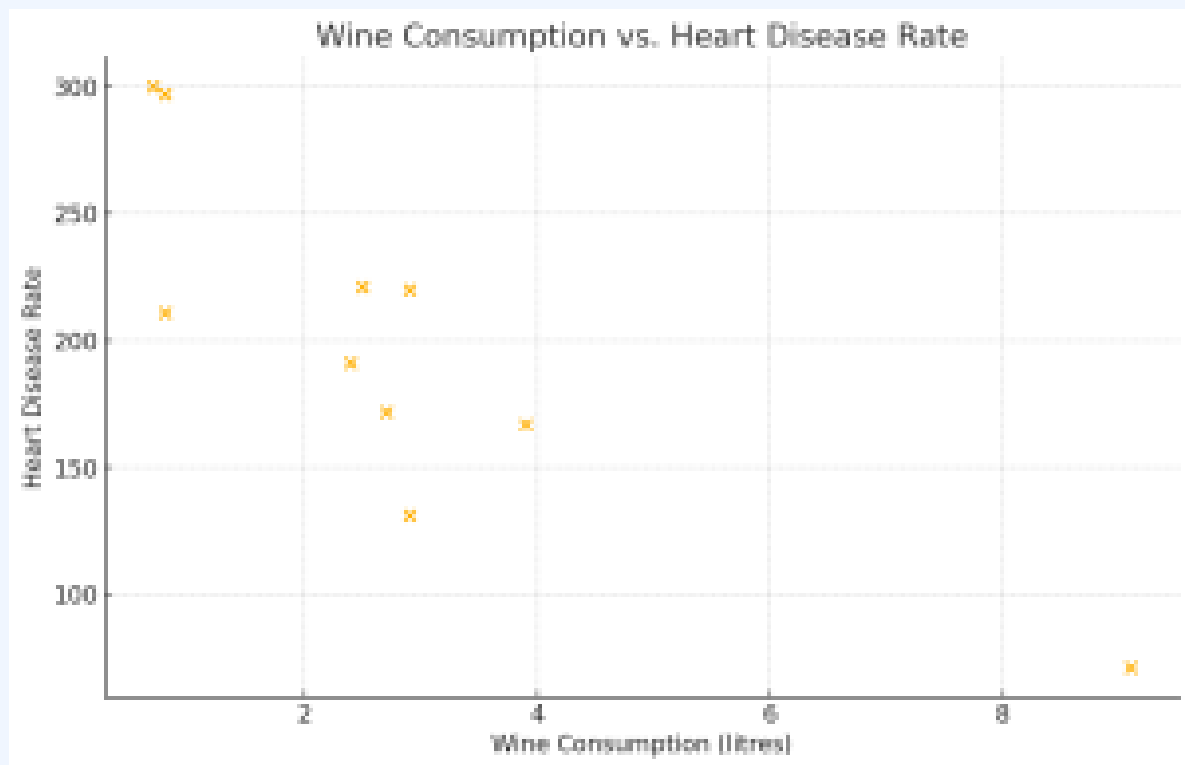
- i. Calculate the standard error of the estimate.
- j. Interpret the standard error of the estimate.

Solution

a. Independent Variable: Yearly Wine Consumption (litres)

Dependent Variable: Heart Disease Rate

b.



- c. -0.84
- d. There is a strong negative correlation between wine consumption and heart disease rate, suggesting that, as average wine consumption increases, the heart disease rate tends to decrease.
- e. $y = -23.88x + 266.63$
- f. The slope of -23.88 suggests that, on average, for each additional litre of wine consumed per capita, the heart disease rate decreases by approximately 23.88 deaths per 100,000.
- g. 0.70

h. About 70% of the variance in heart disease rates can be explained by wine consumption, indicating that the linear model provides a reasonably strong fit to the data.

i. 36.28

j. The standard deviation of the residuals (errors) is approximately 36.28 deaths per 100,000, meaning that predictions from this model typically differ from actual values by about 36.28, reflecting a moderate level of variability around the regression line.

27. The following table consists of one student athlete's time (in minutes) to swim 2000 yards and the student's heart rate (beats per minute) after swimming on a random sample of 10 days:

Swim Time	Heart Rate
34.12	144
35.72	152
34.72	124
34.05	140
34.13	152
35.73	146
36.17	128
35.57	136
35.37	144
35.57	148

- Decide which variable should be the independent variable and which should be the dependent variable.
- Draw a scatter plot of the ordered pairs.
- Find the correlation coefficient.
- Interpret the correlation coefficient.
- Find the linear regression equation.
- Interpret the slope of the linear regression equation.

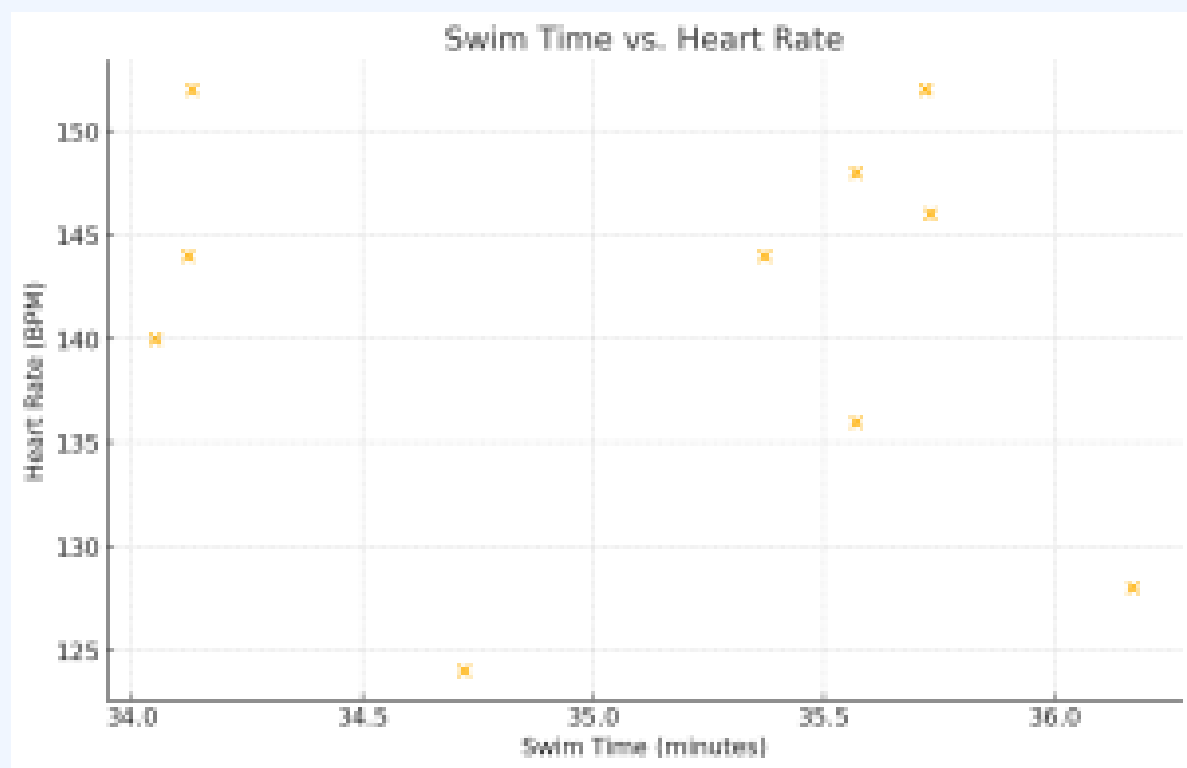
- g. What is the estimated heart rate for a swim time of 34.75 minutes?
- h. Calculate the coefficient of determination.
- i. Interpret the coefficient of determination.
- j. Calculate the standard error of the estimate.
- k. Interpret the standard error of the estimate.

Solution

a. Independent Variable: Swim Time (minutes)

Dependent Variable: Heart Rate (BPM)

b.



c. -0.12

d. There is a very weak negative correlation between swim time and heart rate, suggesting almost no relationship between the two variables.

e. $y = -1.49x + 193.88$

f. The slope of -1.49 suggests that, on average, each additional minute of swim time slightly decreases the heart rate by about 1.49 BPM, though this effect is minimal.

g. Approximately 141.95 BPM

h. 0.02

i. Only about 2% of the variance in heart rate can be explained by swim time, indicating that the linear model does not effectively describe the data.

j. 8.97 BPM

k. The standard deviation of the residuals (errors) is approximately 8.97 BPM, indicating that predictions from this model typically differ from actual values by around 8.97 BPM, reflecting the weak relationship and poor predictive accuracy of the model.

28. The table below gives percent of workers who are paid hourly rates for the years 1979 to 1992.

Year	Percent of workers paid hourly rates
1979	61.2
1980	60.7
1981	61.3
1982	61.3
1983	61.8
1984	61.7
1985	61.8
1986	62.0
1987	62.7
1990	62.8
1992	62.9

- a. Decide which variable should be the independent variable and which should be the dependent variable.

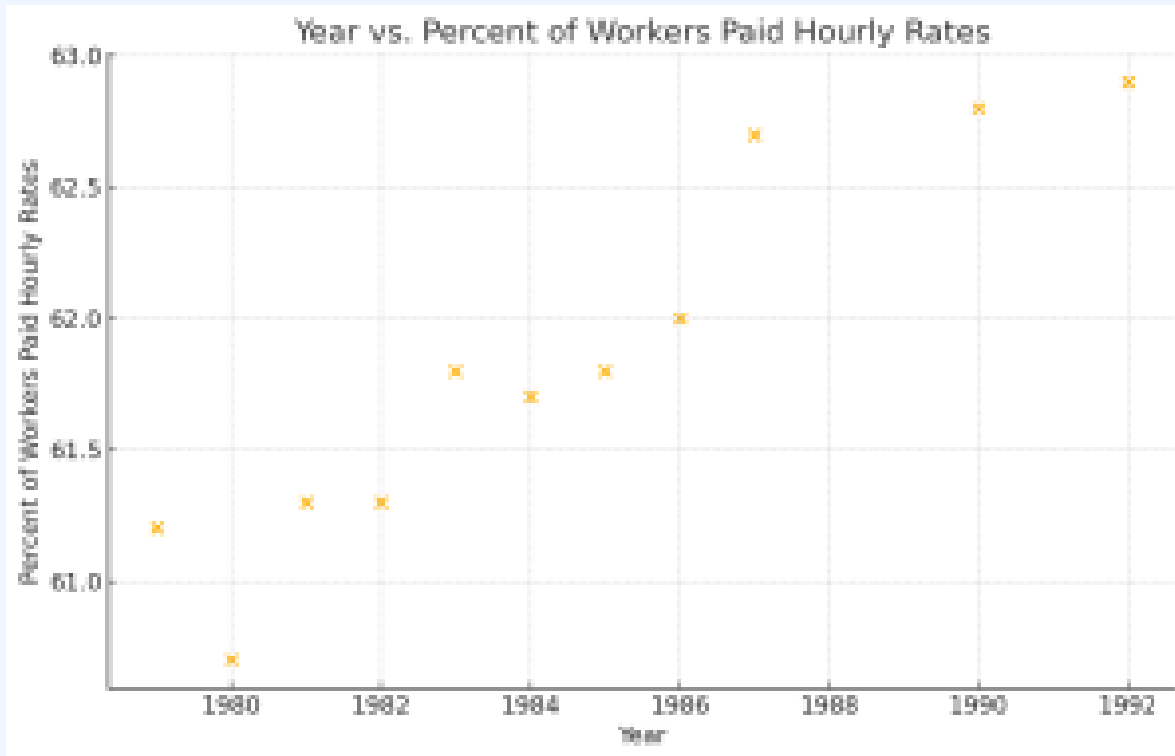
- b. Draw a scatter plot of the ordered pairs.
- c. Find the correlation coefficient.
- d. Interpret the correlation coefficient.
- e. Find the linear regression equation.
- f. Interpret the slope of the linear regression equation.
- g. What is the estimated percentage of workers paid hourly rates in 1988?
- h. Calculate the coefficient of determination.
- i. Interpret the coefficient of determination.
- j. Calculate the standard error of the estimate.
- k. Interpret the standard error of the estimate.

Solution

a. Independent Variable: Year

Dependent Variable: Percent of Workers Paid Hourly Rates

b.



c. 0.94

d. There is a strong positive correlation between the year and the percentage of workers paid hourly rates, suggesting that this percentage has generally increased over time.

e. $y = 0.17x - 266.89$

f. The slope of 0.17 indicates that, on average, the percentage of workers paid hourly rates increased by about 0.17% each year.

g. Approximately 62.42%

h. 0.89

i. About 89% of the variance in the percentage of workers paid hourly rates can be explained by the year, indicating a very strong fit of the linear model to the data.

j. 0.22%

k. The standard deviation of the residuals is approximately 0.22%, meaning that predictions from this model typically differ from actual values by around 0.22%, showing a high degree of accuracy in the model's predictions.

29. The table below shows the average heights for American boys in 1990.

Age (years)	Height (cm)
birth	50.8
2	83.8
3	91.4
5	106.6
7	119.3
10	137.1
14	157.5

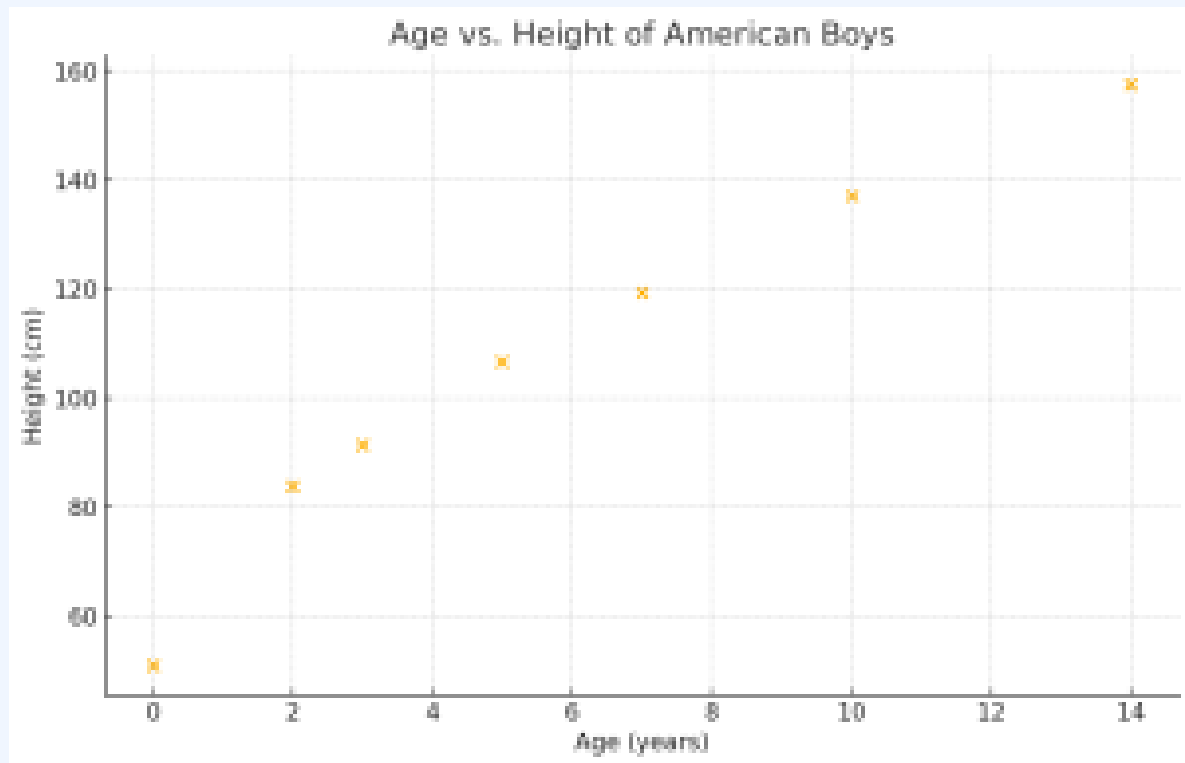
- Decide which variable should be the independent variable and which should be the dependent variable.
- Draw a scatter plot of the ordered pairs.
- Find the correlation coefficient.
- Interpret the correlation coefficient.
- Find the linear regression equation.
- Interpret the slope of the linear regression equation.
- What is the estimated average height for a one-year-old?
- Using the regression equation, find the estimated average height for a 62-year-old man. Do you think that your answer is reasonable? Explain why or why not.
- Calculate the coefficient of determination.
- Interpret the coefficient of determination.
- Calculate the standard error of the estimate.
- Interpret the standard error of the estimate.

Solution

a. Independent Variable: Age (years)

Dependent Variable: Height (cm)

b.



c. 0.98

d. There is a very strong positive correlation between age and height, indicating that as age increases, height generally increases as well.

e. $y = 7.09x + 65.09$

f. The slope of 7.09 indicates that, on average, height increases by about 7.09 cm for each additional year of age.

g. Approximately 72.18 cm

h. Approximately 504.97 cm. This prediction is not reasonable as it suggests an implausible height. The model is only accurate for the age range of the data (birth to 14 years), where growth patterns are different from those in adulthood. Extrapolating far beyond this range leads to unrealistic results.

i. 0.95

j. About 95% of the variance in height can be explained by age, indicating an excellent fit of the linear model to the data within the observed range.

k. 7.13 cm

l. The standard deviation of the residuals is approximately 7.13 cm, meaning predictions from this model typically differ from actual values by around 7.13 cm within the age range of the data.

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CHAPTER 8 SUMMARY

Key Concepts Summary

- **8.1 Linear Correlation**

- Calculating the linear correlation coefficients for any given sets and using it to determine strong/weak positive/negative or no correlation between the sets

- **8.2 Linear Regression**

- Calculating a line of best fit for correlated variables using linear regression.
- Calculating the Coefficient of determination

Glossary of Terms

Bivariate Data. Two variables measured on each subject.

Correlation. The statistical association between two variables

Line of Best Fit. The balancing of the differences between the observed data to create a linear model that can be used to estimate new data points.

Predictor Variable. A random variable chosen to predict the changes in another variable.

Residual. The difference between the data predicted by the line of best fit and the actual observed data.

Response Variable. The random variable that is expected to change in response to changes in the predictor variable.

Formula & Symbol Hub

Symbols Used

- \bar{x} = mean of set x
- S_x = standard deviation of set x
- r = linear correlation coefficient
- b_0 = the y intercept of a line
- b_1 = the slope of a line
- x = the predictor variable in linear regression
- \hat{y} = the estimate of the mean in linear regression

Formulas Used

- **Formula 8.1 – Linear Correlation Coefficient**

$$r = \frac{\sum \frac{(x_i - \bar{x})}{S_x} \frac{(y_i - \bar{y})}{S_y}}{n - 1}$$

- **Formula 8.2 – Simple Linear Regression**

$$\hat{y} = b_0 + b_1 x$$

CHAPTER 9: RISK & RETURN

Chapter Outline

[9.0 Introduction](#)

[9.1 What is Risk?](#)

[9.2 Expected Return and Standard Deviation of a Single Security](#)

[9.3 Expected Return and Standard Deviation of a Portfolio](#)

[9.4 Interpreting Expected Return and Standard Deviation](#)

[9.5 Beta](#)

[9.6 Important Implications of the CAPM/SML](#)

[9.7 Review Exercises](#)

[Chapter 9 Summary](#)

[Solutions to CH 9 Exercises](#)

9.0 INTRODUCTION

Chapter Learning Objectives

By the end of the chapter, you should be able to:

- Define the concept of risk and explain how both the probability and magnitude of outcomes impact the degree of risk.
 - Identify sources of risk and differentiate between general economic risk factors and firm-specific risk factors.
 - Explain the concepts of probability distributions, expected return and standard deviation for a single security.
 - Calculate and interpret the expected return and standard deviation of a single security given a probability distribution.
 - Explain the concept of correlation.
 - Explain the concept of expected return and standard deviation for portfolios.
 - Calculate and interpret the expected return and standard deviation for two-stock portfolios.
 - Explain/diagram the concept and implications of portfolio diversification.
 - Differentiate between firm-specific (diversifiable) risk, market (non-diversifiable) risk, and total risk.
 - Identify when each risk type of risk measurement is appropriate.
 - Calculate and interpret beta.
 - Explain, calculate, and interpret the Capital Asset Pricing Model and Security Market Line.
 - Identify potential concerns regarding the viability of the Capital Asset Pricing Model and the Security Market Line.
-

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9.1 WHAT IS RISK?

Learning Objectives

- Evaluate the interaction of probability and the severity of outcomes to determine the overall risk in various financial scenarios, such as coin flips and investment opportunities.
- Identify and categorize different sources of risk, including general economic conditions, company-specific factors, and financial risk factors, and explain their potential impacts on investment outcomes.
- Implement diversification strategies to minimize the impact of company-specific factors on investment portfolios, understanding the benefits and limitations of diversification in risk management.

Risk refers to the possibility of an unfavourable event occurring. The higher the risk, the greater the probability of an unfavourable event or the more unfavourable the event could be. The interaction of the probability of the unfavourable event and the degree of negativity associated with the event is critical to determining the risk.

Example 9.1.1

Imagine that you are going to participate in a coin flip. It will cost you **\$1.00** to participate. If you flip a head, you get **\$1.01**. If you flip a tail, you get **\$0.99**. Even though there is a fairly high

probability of the unfavourable event (**50%** chance of tails), the outcome is so minor (you lose **\$0.01**) that this would be a low-risk event.

Now, consider a slightly different coin flip. This time, instead of flipping the coin once, you will flip it **3** times. If you get **3** heads, you receive **\$10,000**. If you flip **3** tails, you owe **\$10,000**. Anything else you get your **\$1.00** back. Even though the probability of the bad outcome is much smaller (there is only a **12.5%** chance of flipping **3** straight tails), this is a much riskier event due to the bad outcome being substantially worse.

However, most people do not consider flying a commercial airliner a high-risk event (even though the worst-case scenario is quite severe) because of the extremely low probability of a fatal crash (less than 1 in 10 million). It is not just the probability or the degree of an unfavourable outcome but the combination of the two that matters for rational risk analysis.

In finance terms, our “unfavourable event” refers to earning less than expected. Any time we have a chance to earn less than expected on an investment opportunity, we are exposed to risk. Note that this is a more strict definition than defining risk as the possibility of losing money. We want to be careful to think of risk as the possibility of earning less than expected instead of losing money.

Sources of Risk

General Economic Conditions

- Inflation
- Recession
- Interest Rates

Company Specific Factors

Business Risk Factors

- Volatility of Sales
- Volatility of Input Prices
- Strikes and Other Labour Situations

- Product Lines
- Fixed vs. Variable Costs

Financial Risk Factors

- Degree of Debt Financing

Diversification

Diversification refers to the concept that by holding a number of different securities (ideally not just stocks) from a spectrum of industries, we can negate the impact of company-specific factors on our returns. We will come back to this issue (one of the most important financial concepts) in more detail later in this chapter.

Key Takeaways



Note that the above list is a sample of broad factors and not a specific list. For example, consider what happens when we have a significant increase in oil and gasoline prices. One immediate impact is inflation. The higher energy prices are, by definition, inflation in energy, but it goes beyond that. Now, it costs more for firms to distribute their products to suppliers, which is likely to cause inflation to spill over to other areas. As we search for alternative energy sources (like ethanol), we may see corn prices rising. Since corn is used to feed cattle, this could lead to an increase in beef costs as well. Also, if consumers are now spending more to fill up their gas tanks and more to buy a variety of food products, there would be less money available to spend on entertainment and other goods/services. This could lead to a recessionary environment (could, not will, because there are always so many influences on the economy that this is just one of many

factors impacting economic growth). The point here is not the specific impacts of higher oil/gasoline prices but that many economic risk factors may have more complex interactions than are apparent at the surface.

One other thought on risk to remember as we move through this chapter. Throughout the chapter, we will treat risk and potential returns as primarily objective and measurable. However, in practice, one of the biggest challenges of risk management is trying to figure out what bad outcomes are and how likely they are. In practice, the results of our models are only as good as the inputs we put into them.

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9.2 EXPECTED RETURN AND STANDARD DEVIATION OF A SINGLE SECURITY

Learning Objectives

- Apply the formula for expected return to determine the anticipated rate of return for a single security based on the probability distribution of possible outcomes.
- Use the standard deviation formula to measure the variability of returns for a single security, interpreting the implications of higher or lower standard deviation on investment risk.
- Analyze probability distributions to understand the likelihood of various states of nature and their corresponding returns, enhancing the ability to predict and manage financial risks.

Formula & Symbol Hub

Symbols Used

- $\sum_{i=1}^n$ = repeatedly sum the expression from i to n
- \bar{k} = expected return of a stock
- P_i = the probability of the i th possible outcome (state of nature)
- k_i = the return under the i th outcome (state of nature)

- P_n = the probability of the nth possible outcome (state of nature)
- k_n = the return under the nth outcome (state of nature)
- σ = standard deviation

Formulas Used

- Formula 9.1 – **Expected Return (Single Security)**

$$\bar{k} = \sum_{i=1}^n P_i k_i$$

OR

$$\bar{k} = P_1 k_1 + P_2 k_2 + \dots + P_n k_n$$

- Formula 9.2 – **Standard Deviation (Single Security)**

$$\sigma = \sqrt{\sum_{i=1}^n P_i (k_i - \bar{k})^2}$$

OR

$$\sigma = \sqrt{P_1 (k_1 - \bar{k})^2 + P_2 (k_2 - \bar{k})^2 + \dots + P_n (k_n - \bar{k})^2}$$

Expected Return

The **expected return** of a security is based on the probability distribution of returns. Before we get into the details of the expected return, let's briefly introduce the concept of a probability distribution. A probability distribution is a representation of possible outcomes (states of nature) that may occur and the likelihood (probability) of each outcome. If you think of a coin flip, the probability distribution has two possible outcomes (heads or tails) and each outcome has a 50% chance of occurring (technically, this is not true as even in [a "fair" coin flip, the side that starts up has about a 51% chance of occurring](#)). When dealing with financial securities, the number of possible outcomes is nearly infinite and it is not possible to know the exact outcomes or probabilities of those outcomes. Therefore, we are really only approximating the true probability distribution.



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Specifically, the **expected return** is the probability of a specific state of nature occurring times the return under that state of nature summed across all possible states of nature.

9.1 Expected Return (Single Security)

$$\bar{k} = \sum_{i=1}^n P_i k_i$$

OR

$$\bar{k} = P_1 k_1 + P_2 k_2 + \dots + P_n k_n$$

\bar{k} represents the expected return of the stock

$\sum_{i=1}^n$ means “sum the following expression repeatedly from i to n ”

P_i represents the probability of the i th possible outcome (state of nature)

k_i represents the return under the i th outcome (state of nature)

P_n represents the probability of the n th possible outcome (state of nature)

k_n represents the return under the n th outcome (state of nature)

Example 9.2.1

After researching Stock A we have determined that there are **3** possible outcomes for the next year (**3** states of nature). The first possibility is the economy enters a recession causing the stock to have a return of **−15%**. The probability of this occurring is **20%**. The second possibility is that the economy goes smoothly, but does not experience rapid growth causing the stock to rise and offer a **10%** return. The probability of this occurring is **50%**. The third possibility is that the economy booms, causing the stock to provide a **35%** rate of return. The probability of the economy booming is **30%** (note that the probabilities must sum to **1.0** and the states of nature should be mutually exclusive).

State of Nature	Probability	Return
Recession	0.20	−15%
Normal	0.50	10%
Boom	0.30	35%

What is the expected rate of return?

Solution

Substitute probabilities and returns into the Expected Return formula (Formula 6.1) and solve:

$$\begin{aligned}\bar{k} &= \sum_{i=1}^n P_i k_i \\ \bar{k} &= (.20)(-15\%) + (.50)(10\%) + (.30)(35\%) \\ \bar{k} &= -3\% + 5\% + 10.5\% \\ \bar{k} &= 12.5\%\end{aligned}$$

The expected return is **12.5%**.



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Standard Deviation

The **standard deviation** measures the variability of possible returns and is represented by the lower-case Greek symbol sigma (σ). The smaller the standard deviation, the more likely we are going to earn something “close” to our expected return. The greater the standard deviation, the greater the chance that we may earn something far more (good) or far less (bad) than our expected return. The formula for this is (remember that \bar{k} is our symbol for expected return):

9.2 Standard Deviation (Single Security)

$$\sigma = \sqrt{\sum_{i=1}^n P_i \left(k_i - \bar{k} \right)^2}$$

OR

$$\sigma = \sqrt{P_1 \left(k_1 - \bar{k} \right)^2 + P_2 \left(k_2 - \bar{k} \right)^2 + \dots + P_n \left(k_n - \bar{k} \right)^2}$$

σ (sigma) represents the standard deviation

P_i represents the probability of the i th outcome (state of nature)

k_i represents the return under the i th outcome (state of nature)

\bar{k} represent the expected return for the stock

P_n represents the probability of the n th outcome (state of nature)

k_n represents the return under the n th outcome (state of nature)

Things to Watch Out For

- It is easy to get confused with decimals and percentages. The best way to do these calculations is to always leave the weights as decimals and the returns as a regular number. For instance, if you have a probability of **0.10** and a return of **15%**, you would put the probability into your calculator as **0.10** and the return as **15**.
- Be careful with your order of operations.
 1. Do $(k_1 - \bar{k})$ first
 2. Then square that
 3. Then multiply by P_1
 4. Repeat for all n states of nature
 5. Add them up
 6. Finally, take the square root

Example 9.2.2

Using the data and calculated expected return from Example 9.2.1 above, calculate the standard deviation for stock A?

Solution

Substitute probability, return, and expected return into the formula for standard deviation (Formula 9.2) and solve:

$$\begin{aligned}\sigma &= \sqrt{\sum_{i=1}^n P_i (k_i - \bar{k})^2} \\ \sigma &= \sqrt{0.2(-15 - 12.5)^2 + 0.5(10 - 12.5)^2 + 0.3(35 - 12.5)^2} \\ \sigma &= \sqrt{0.2(756.25) + 0.5(6.25) + 0.3(506.25)} \\ \sigma &= \sqrt{151.25 + 3.13 + 151.88} \\ \sigma &= \sqrt{306.25} \\ \sigma &= 17.50\%\end{aligned}$$

The standard deviation for Stock A is **17.50%**.



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9.3 EXPECTED RETURN AND STANDARD DEVIATION OF A PORTFOLIO

Learning Objectives

- Use the expected return formula for a portfolio to determine the weighted average return of multiple stocks within a portfolio.
- Apply the standard deviation formula for a two-stock portfolio to assess the overall risk, considering the weights, individual standard deviations, and correlations between the stocks.
- Evaluate how the correlation between stocks affects the overall risk of a portfolio, demonstrating the benefits of diversification in reducing portfolio volatility.

Formula & Symbol Hub

Symbols Used

- $\sum_{i=1}^n$ = repeatedly sum the expression from i to n
- \bar{k} = expected return of a stock
- σ = standard deviation
- W_n = the weight (proportion of portfolio) of stock n

- \bar{k}_n = expected return for stock n
- $corr_{x,y}$ = the correlation between the returns of stocks x and y

Formulas Used

- Formula 9.3 – **Expected Return (Portfolio)**

$$\bar{k}_p = \sum_{i=1}^n W_i \bar{k}_i$$

OR

$$\bar{k}_p = W_1 \bar{k}_1 + W_2 \bar{k}_2 + \dots + W_n \bar{k}_n$$

- Formula 9.4 – **Standard Deviation (Two-Stock Portfolio)**

$$\sigma_p = \sqrt{W_1^2 \sigma_1^2 + W_2^2 \sigma_2^2 + 2W_1 W_2 \sigma_1 \sigma_2 corr_{1,2}}$$

Expected Return

The expected return for a portfolio is simply the weighted average of each stock held in the portfolio. The formula here is

9.3 Expected Return (Portfolio)

$$\bar{k}_p = \sum_{i=1}^n W_i \bar{k}_i$$

OR

$$\bar{k}_p = W_1 \bar{k}_1 + W_2 \bar{k}_2 + \dots + W_n \bar{k}_n$$

\bar{k}_p represents the expected return for the portfolio

W_1 represents the weight (proportion of portfolio) of stock 1

\bar{k}_1 represents the expected return for stock 1

W_n represents the weight (proportion of portfolio) of stock n

\bar{k}_n represents the expected return for stock n

Example 9.3.1

What is the expected return of a portfolio made up of **60%** Stock A and **40%** Stock B when the expected return for Stock A is **10%** and the expected return for Stock B is **20%**?

Solution

Substitute weights and expected returns for stocks in the portfolio into the portfolio expected return formula (Formula 6.3) and solve:

$$\begin{aligned}\bar{k}_p &= \sum_{i=1}^n W_i \bar{k}_i \\ \bar{k}_p &= (.60)(10\%) + (.40)(20\%) \\ \bar{k}_p &= 6\% + 8\% \\ \bar{k}_p &= 14\%\end{aligned}$$



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Standard Deviation

The standard deviation of a portfolio becomes more complicated. It depends not only on the standard deviation and weightings of each stock, but also on the correlation between pairs of stocks.

Correlation

The correlation between a pair of stocks measures how closely the returns for each stock are related. A negative correlation means that the price of one stock tends to fall while the other rises (prices/returns are inversely related). A positive correlation means that the price of one stock tends to rise while the other rises (prices/returns are positively related). Correlations can range from -1.0 to 1.0 . Correlations for real-world variables are almost never at the extremes (perfect positive correlation, no correlation, or perfect negative correlation).

See the [Observed Betas, Correlations, and Standard Deviations Table in Appendix B](#) for some standard deviations and correlations from actual companies over the past five years.

Observations from the Table

- The last two rows/columns are for an aggregate US bond fund and the S&P 500 ETF. The purpose of these is to provide a proxy for the US bond market and the US stock market.
- The vast majority (65 of 66) of correlations between pairs of stocks are positive. This is because all stocks are impacted by general economic factors.
- The average correlation across pairs is a low, positive value (0.31 in our sample). While general economic factors cause a tendency for stock to move together, firm-specific factors push correlations towards zero.
- Stocks in similar industries tend to have higher correlations than the average stock (0.50 vs. 0.31).
- Each stock has a positive correlation with the overall stock market.
- The stock and bond market have a negative (although essentially zero) correlation during this time-period.
- The bond market has a much lower standard deviation than the stock market, but also generated much lower returns during this time-period.
- Most individual stocks have a standard deviation that is higher than the overall stock market. This is because the stock market represents a diversified portfolio which has eliminated most of the **firm-specific risk**. The exception is Pepsi in our sample which is essentially the same standard deviation.
- The historical annualized returns are not the same as the expected returns. It is unrealistic to expect 30%+ annual returns for Deere, Boston Beer or Amazon going forward (which is not the same as

saying that these stocks can't generate those returns). Also, it is unreasonable to expect people to buy stock in Molson Coors and Exxon with the anticipation of losing nearly 12% or 8% each year.

- While standard deviations will vary depending on overall market conditions, large cap stocks over this 5—year window had standard deviations of approximately 15 — 42%. Note that the standard deviations will be bigger or smaller depending on both the stock and the time period and that Boston Beer's 42.5% is noticeably larger than the next closest company.

Standard Deviation for a Two-Stock Portfolio

In order to calculate the standard deviation of a two-stock portfolio, we will use the following formula:

9.4 Standard Deviation (Two-Stock Portfolio)

$$\sigma_p = \sqrt{W_1^2 \sigma_1^2 + W_2^2 \sigma_2^2 + 2W_1 W_2 \sigma_1 \sigma_2 \text{corr}_{1,2}}$$

σ_p represents the standard deviation of the portfolio

W_1 represents the weight (proportion of portfolio) of stock 1

σ_1 represent the standard deviation of stock 1

W_2 represents the weight (proportion of portfolio) of stock 2

σ_2 represent the standard deviation of stock 2

$\text{corr}_{1,2}$ represents the correlation between the returns of stocks 1 and 2

Example 9.3.2

Consider a two-stock portfolio in which **60%** of your money is invested in stock A and **40%** of your money is invested in stock B. Stock A has a standard deviation of **50%** and stock B has a standard deviation of **70%**. The correlation between the returns for stock A and stock B are **0.30**. You want to find the standard deviation of this portfolio.

Solution

Substitute values into Formula 6.4 for calculating the standard deviation of a two-stock portfolio and solve:

$$\begin{aligned}\sigma_p &= \sqrt{W_1^2 \sigma_1^2 + W_2^2 \sigma_2^2 + 2W_1 W_2 \sigma_1 \sigma_2 \text{corr}_{1,2}} \\ \sigma_p &= \sqrt{(0.6)^2 (50)^2 + (0.4)^2 (70)^2 + 2(0.6)(0.4)(50)(70)(0.3)} \\ \sigma_p &= \sqrt{(0.36)(2500) + (0.16)(4900) + (504)} \\ \sigma_p &= \sqrt{900 + 784 + 504} \\ \sigma_p &= \sqrt{2188} \\ \sigma_p &= 46.78\%\end{aligned}$$

Note that in Example 9.3.2, the standard deviation of the portfolio is less than the standard deviation of either stock separately. This illustrates the concept of diversification. As long as the correlation is less than **1.0** (which it will be for any two stocks), the risk of the portfolio is less than the weighted average risk of the two securities which make up the portfolio (and sometimes — like here — even less than the lowest risk stock in the portfolio). While we will not cover the process of calculating the expected return and standard deviation for larger portfolios in this class, in practice, most portfolios hold far more securities.



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9.4 INTERPRETING EXPECTED RETURN AND STANDARD DEVIATION

Learning Objectives

- Assess the expected return and risk of different investment opportunities using the concept of expected return, standard deviation, and risk aversion to make informed investment decisions.
- Distinguish between diversifiable (firm-specific) risk and non-diversifiable (market) risk, explaining how diversification can reduce overall portfolio risk.
- Compare different investment options by analyzing the trade-offs between expected return and associated risks, considering factors such as individual risk aversion and market conditions.

The expected return gives us an idea of how much we will make on the investment. Remember that it is not how much we will make but how much we would make on average if we could repeat the holding period an infinite number of times. Think of a situation where you are asked to pick a number between 1 and 10. If you select the correct number you get \$100 and if not you get nothing. Any time that you try this, you will either receive \$100 (if you are lucky) or \$0. However, if you could repeat the exercise 100,000 times, you would find that you would make almost exactly \$10 per time. It is critical to know expected values when selecting investments. For instance, if someone offered you the opportunity to do this exercise for \$9 per pick, and you could pick as often as you wanted, it would be an excellent opportunity. Alternatively, if you were offered the same thing for \$11, you would (hopefully) walk away. However, it is just as important to understand that the expected return is only an average return and not the return you will receive in any particular instance. This example also illustrates why it is so important to focus on the process and not the result. If you had the opportunity to play this game once for a cost of \$2 and you lost, it still was a smart decision to play and you should do it again if you got the chance. If you had the opportunity to play this game once for a cost of \$20 and you won, it was still a bad decision to play and you should pass if offered the opportunity to try again.

Example 9.4.1

Now consider a similar exercise — pick a number between **1** and **5**. If you select the correct number, you get **\$50** and if not, you get nothing. The expected value for this exercise is the same as the previous exercise (**\$10**). So, imagine that you are offered the opportunity to participate in either one (pick from **10** numbers or pick from **5** numbers) for **\$8**. You can only play once. Most people will now choose to pick from **5** numbers. Why? Because it has less risk (a lower standard deviation) and offers the same expected return. This is the concept of **risk aversion**. As a side note, if you still picked the **10**—number game, it is likely because the stakes are small (the entertainment value of the gamble outweighs the financial aspect). As the stakes increase, the vast majority of people will choose the 5-number game.

Example 9.4.2

Let's put this in financial terms. Consider two stocks.

- Stock A has an expected return of **10%** and a standard deviation of **25%**. Stock B has an expected return of **10%** and a standard deviation of **30%**. Which should you choose and why? (Answer to follow ...think about it first)

Now consider two other stocks.

- Stock C has an expected return of **7%** and a standard deviation of **20%**. Stock D has an expected return of **9%** and a standard deviation of **28%**. Which should you choose and why? (Again, spend some time thinking before reading the next paragraph).

In the first example, you should choose stock A, and everyone else should choose stock A. Stock B offers us no additional compensation (expected return) to entice us to take the higher risk (standard deviation). Therefore, it is irrational in a risk-averse framework to invest in stock B. In the

second example, you could choose stock C or stock D and someone else may make the same choice or the opposite choice. Here, the choice is based on your individual level of risk aversion. Stock D is riskier, but it also compensates us for that risk with a higher expected return. Is the compensation enough? That depends on the individual. For those that are less risk-averse, they require less additional compensation to take on the extra risk so they will likely take stock D. For those that are more risk averse, they will take stock C because the extra compensation is not enough for them to take the extra risk.

Take a few moments and think of what factors impact your level of risk aversion. Typically, we find that age, personality, number of dependents, wealth, income, variability of income, past life experiences, and other factors all combine to influence one's level of risk aversion. One last thought — remember that taking stock D does not mean you will earn a higher return, just that you will earn a higher return on average. If you always earned a higher return from stock D, then it wouldn't be riskier. Visually, you can think of it along the lines of the diagram below. At low levels of risk, the range of returns will be close to the expected return. At high levels of risk, the range of returns will be higher. The expected return increases with risk, but so too does the range of potential returns.

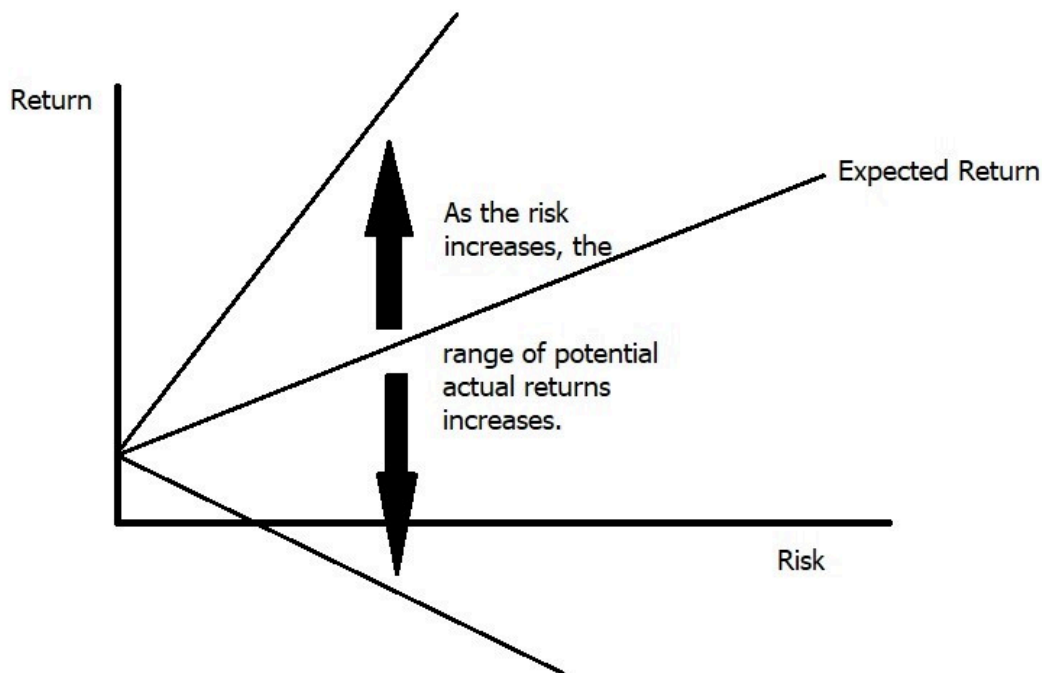


Figure 9.4.1 Range of Potential Actual Returns. This graph shows Return plotted on the y-axis and Risk plotted on the x-axis, with a line representing the expected return between two other lines representing the range of potential returns. As the risk increases, the range of potential actual returns increases.

Diversifiable and Non-Diversifiable Risk

Remember earlier, we discussed the possibility of lowering our firm-specific risk by holding a number of stocks from a wide range of industries? This concept of diversification allows us to greatly reduce our risk by holding a portfolio. **Diversifiable risk** refers to risk factors that are isolated to one particular firm or industry.

For example, if a drug manufacturer gets hit with a lawsuit related to one of the drugs it produces, that is likely to have an isolated impact. It will not have any effect on the stock prices of auto manufacturers, grocery retailers, banks, etc. Once we have enough stocks in our portfolio, bad news for any one of them will have a small impact on our overall portfolio as long as that bad news is contained to that one firm (in other words, as long as it comes from a diversifiable risk factor). By holding approximately 25 — 50 stocks, we can eliminate a large portion of our diversifiable risk (Statman, 1987). While a portfolio of 10 stocks has much less risk than a portfolio of 5 stocks, a portfolio of 100 stocks offers very little risk reduction compared to a much smaller 50-stock portfolio (assuming our stocks are from a variety of different industries and have other differing characteristics).

Does this mean that we have eliminated all of our risk when we hold a 50-stock (or even 500-stock) portfolio? No, we are still subject to general economic risk that affects all securities. This leftover risk is referred to as **Non-diversifiable risk** (or **market/systematic risk**). Examples of non-diversifiable risks include political events (such as wars), energy price shocks, changes in interest rates, recessions, etc. Any risk factor that impacts virtually all stocks is referred to as a non-diversifiable risk factor because it will impact our portfolio regardless of how well we have diversified our investments. Another name for non-diversifiable risk is **market risk** because these sources of risk tend to affect the entire market as opposed to an individual security or industry. Note: Market risk does not refer to risk impacting a specific industry. Instead, it refers to risk impacting the broad economy (most stocks). If a risk factor only impacts one or two industries without carrying over to the broader market, it is classified as firm-specific.

The following graph (Figure 9.4.2) does a good job of illustrating the concept of diversification and firm-specific vs. market risk.

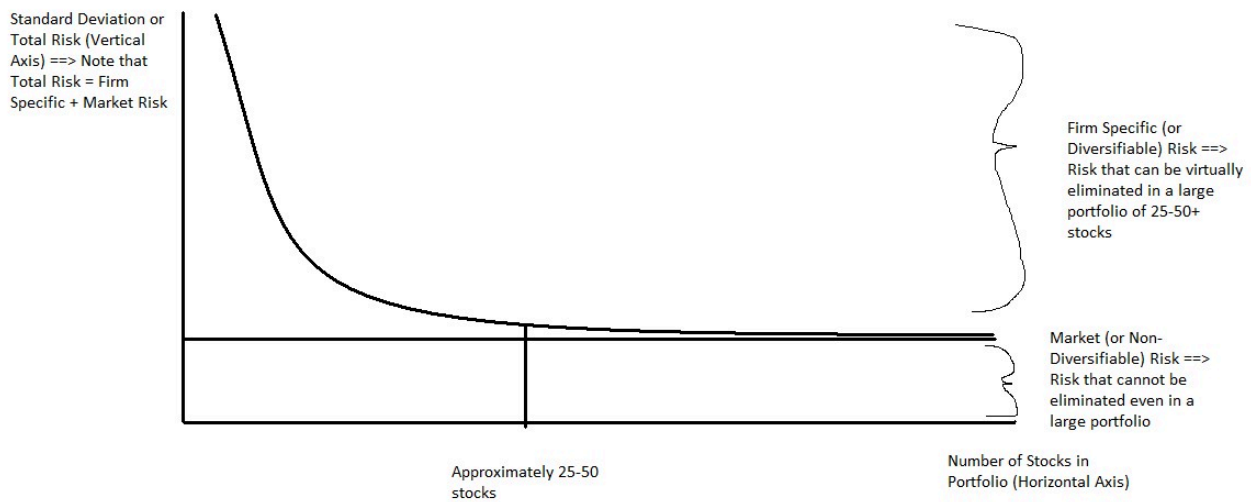


Figure 9.4.2 Firm-Specific Risk vs Market Risk. On this graph, Standard Deviation (Total Risk) is plotted on the y-axis, and Number of Stocks in Portfolio is plotted on the x-axis. The curve slopes sharply downward at first, before levelling out at around a 15-50 stock portfolio size. This shows that firm specific risk can be virtually eliminated by a portfolio around this size, whereas the remaining Market Risk cannot be eliminated no matter how large the portfolio may be.

Given the relative ease with which investors can virtually eliminate their firm-specific (diversifiable) risk, for most investors the level of non-diversifiable (market) risk associated with an investment becomes more important. It is important to note that while market risk impacts all stocks, it does not impact all stocks equally. Therefore, we need a tool to measure how sensitive a stock is to the overall market. This tool is known as BETA. Standard deviation measures total risk (diversifiable risk + market risk) for a security, while beta measures the degree of market (non-diversifiable) risk. We won't introduce a risk measurement for diversifiable risk.



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9.5 BETA

Learning Objectives

- Apply the Beta formula to determine the market risk of a stock by using the standard deviation of the stock, the correlation between the stock and the market, and the standard deviation of the market.
- Utilize the Security Market Line formula to calculate the required return for a stock based on its Beta, the risk-free rate of interest, and the expected return on the market.
- Differentiate between standard deviation and Beta as risk measurements, and determine the appropriate context for using each when evaluating the risk of individual securities and diversified portfolios.

Formula & Symbol Hub

Symbols Used

- β_A = the Beta of Stock A
- σ_A = the standard deviation of stock A
- $\text{corr}_{A,MKT}$ = the correlation between stock A and the overall market
- σ_{MKT} = the standard deviation of the overall market
- k_A = the required return for stock A,

- k_{RF} = the risk-free rate of interest (often approximated by the yield on 10-year Treasury bond)
- \bar{k}_m = the expected return on the market (often approximated by the S&P 500)

Formulas Used

- Formula 9.5 – **Beta**

$$\beta_A = \frac{(\sigma_A)(\text{corr}_{A,MKT})}{\sigma_{MKT}}$$

- Formula 9.6 – **Required Return (SML)**

$$k_A = k_{RF} + \beta_A(\bar{k}_m - k_{RF})$$

In addition to serving as a measure of market risk, **Beta** tells us how a particular stock moves in relation to the rest of the stock market as a whole.

9.5 Beta

$$\beta_A = \frac{(\sigma_A)(\text{corr}_{A,MKT})}{\sigma_{MKT}}$$

β_A represents the Beta of Stock A

σ_A represents the standard deviation of stock A

$\text{corr}_{A,MKT}$ represents the correlation between stock A and the overall market

σ_{MKT} represents the standard deviation of the overall market

Example 9.5.1

Stock A has a standard deviation of **60%** while the overall stock market has a standard deviation of **25%**. Assuming that the correlation between Stock A and the overall market is **0.30**, what is the beta of Stock A?

Solution

Substitute values into the Beta formula (Formula 9.5.1) and solve:

$$\beta_A = \frac{(\sigma_A)(\text{corr}_{A,MKT})}{\sigma_{MKT}}$$

$$\beta_A = \frac{(60)(.30)}{25}$$

$$\beta_A = \frac{18}{25}$$

$$\beta_A = 0.72$$

What is the Market?

The market refers to a portfolio of all investment assets (stocks, bonds, gold, art, etc.). However, in more practical terms, the market usually refers to the stock market and can be measured by a **market index** (such as the S&P 500 or Dow Jones Industrial Average).

How do we Interpret Beta?

Most betas range between 0.35 and 1.8 (there are many outside this range, but the majority of stocks fall in this range – review [Observed Correlations, Returns, Standard Deviations and Betas Table in Appendix B](#) and note that all the stocks other than Wal-Mart fall in this window.)

Table 9.5.1 Interpreting Beta

$\beta > 1$	Beta greater than 1.0 implies greater than average risk
$\beta = 1$	Beta equal to 1.0 implies average risk
$\beta < 1$	Beta less than 1.0 implies less than average risk

Go back to our example where we calculated the beta for stock A. By itself, stock A is much riskier than the overall market as determined by its standard deviation. However, when we consider it as part of an overall portfolio its risk is much lower (less than average) due to the fact that it has a relatively low correlation to the overall market. The riskiness of stock A depends on whether we plan to use it as a stand alone investment or as part of a portfolio.

Standard Deviation vs. Beta

At this point, we have introduced two risk measurements. The first is standard deviation and the second is beta. In some cases, these two risk measurements will tell a different story. For instance, stock A may have a standard deviation of 30% and a beta of 0.8 while stock B may have a standard deviation of 25% and a beta of 1.3. Which stock is riskier? The answer is depends on the specific situation. Because each risk measurement is measuring a different type of risk (standard deviation measures total risk while beta measures market risk), we need to think of situations where each is appropriate.

Single Security and/or Poorly Diversified Portfolio

If you are going to place your entire investment into a single security or a poorly diversified portfolio, then standard deviation is the appropriate risk measurement. In this situation, you have not diversified away the majority of the firm-specific risk, so you need to include it in your analysis. Standard deviation does this because it includes both sources (market and firm-specific) of risk.

Adding a Security to a Well-Diversified Portfolio

If you own a well-diversified portfolio and you are planning to add a single security to that portfolio, then the firm-specific risk of the security you are adding is not relevant. The reason it is not relevant is because it will be one of many stocks in the large portfolio and the firm-specific risk will be diversified away. What matters is how that stock moves with the overall market. Since we measure this market risk with beta, our appropriate risk measurement here is beta.

Choosing Between 2 (or more) Well-Diversified Portfolios

If you are choosing between two or more portfolios that are each well-diversified, then you can use either standard deviation or beta as your risk measurement. The reason for this is that at this point, the firm-specific risk is already diversified away so that your total risk and market risk should be essentially the same. Thus, whichever portfolio has the higher standard deviation should also have the higher beta (if not, you know the portfolios are not well-diversified).

Beta and Required Return: Capital Asset Pricing Model (CAPM)

During the mid-1960's and early-1970's some finance professors developed the **Capital Asset Pricing Model (CAPM)**. (Sharpe, 1964; Lintner, 1965; Black, 1972) One of the key components of this model is the **Security Market Line (SML)** which states that the **required rate of return** for a stock is dependent on the beta of that stock. While technically, the SML is a subset of the larger model (CAPM), in practice the two terms are typically used interchangeably. Thus, think of them as the same basic model.

9.6 Required Return (SML)

$$k_A = k_{RF} + \beta_A (\bar{k}_m - k_{RF})$$

k_A is the required return for stock A

k_{RF} is the **risk-free rate** of interest (often approximated by the yield on 10-year Treasury bond)

β_A is the beta for stock A

\bar{k}_m is the expected return on the market (often approximated by the S&P 500)

Example 9.5.2

Stock B has a beta of **1.4**. The expected return on the market is **11%** and the Treasury bond rate is **5%**. Calculate the required return for stock B.

Solution

Substitute the given values into Formula 9.6 for required return and solve:

$$\begin{aligned} k_B &= k_{RF} + \beta_B (\bar{k}_m - k_{RF}) \\ k_B &= 5\% + (1.4) (11\% - 5\%) \\ k_B &= 5\% + (1.4) (6\%) \\ k_B &= 5\% + 8.4\% \\ k_B &= 13.4\% \end{aligned}$$



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/mathematicsfinance/?p=258#oembed-2>

Video: “[Beta and the SML Part One](#)” by [Kevin Bracker](#) [7:43] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*



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Video: “[Beta and the Security Market Line Two](#)” by [Kevin Bracker](#) [7:30] is licensed under the [Standard YouTube License](#). *Transcript and closed captions available on YouTube.*

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9.6 IMPORTANT IMPLICATIONS OF THE CAPM/SML

Learning Objectives

- Describe how the Security Market Line (SML) predicts that high beta stocks should earn higher returns than low beta stocks and discuss the role of beta as the primary factor influencing differences in stock returns.
- Evaluate how rising interest rates and increased investor risk aversion affect required returns and stock prices, using the concepts of the risk premium and equilibrium conditions.
- Critically analyze the empirical findings and limitations of the Capital Asset Pricing Model (CAPM) and the Security Market Line (SML), and explore alternative models that have been proposed to explain stock returns.

Consider the following:

- According to the SML, high beta stocks should, on average, earn higher returns than low beta stocks.
- According to the SML, the only factor that should cause consistent differences in returns across stocks is beta.
- When interest rates rise, required returns should increase and (all else equal) cause stock prices to decline.
- When investors become more risk-averse, the **risk premium** ($k_m - k_{RF}$) should increase which will increase required returns and (all else equal) cause stock prices to decline.

Under **equilibrium** conditions, the required return should equal the expected return. Let's consider this for a minute. In our example above, we estimated that the required return for stock B should be **13.4%**. What would happen if the expected return \bar{k} for this stock was **16%**?



For practice in understanding this concept, think through what would happen if the required return was 13.4% and the expected return was 9%. Also, think about what types of things may push us out of equilibrium and why this is relevant for explaining stock price movements.

Empirical Findings of the SML

While the CAPM/SML met with a lot of early success and became the standard for estimating required returns in the field of finance very quickly, it ran into problems in the 1990s and is deemed less reliable at the moment. (Fama and French, 1992) There have been some alternative models ([Fama-French 3-Factor model](#), [Carhart 4-Factor model](#), [Fama-French 5-Factor model](#), and the [q-Factor model](#)) introduced since then. However, there has not been a new standard-bearer to take its place. As of right now, the SML is still commonly used in practice. However, there is a growing focus on the alternatives. My view is that it is critical for you to know and

understand the basic premise of the CAPM and SML (that role of market risk in explaining returns) and to be aware of some of the problems (listed below).

- While market returns play a major role in explaining the returns of individual stocks, Beta doesn't do a very good job of explaining future returns. In other words, after controlling for other factors, it is not clear that high beta stocks actually outperform.
- Small firms seem to earn higher returns than can be explained by beta.
- Firms with a low **market-to-book ratio (MV/BV)** tend to earn higher returns than can be explained by beta.
- Firms that have been top performers in the past 6-12 months tend to earn higher returns in the following 6-12 months than can be explained by beta.

Again, the above flaws do not mean that the CAPM/SML are useless. They provide a simplified framework for understanding how market risk relates to returns. However, recognize that they are not perfect models and the process of understanding stock returns is more complex than the SML indicates. As our knowledge increases, better models will likely evolve.

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9.7 REVIEW EXERCISES

Chapter 8 Review Exercises

1. What is Risk? What two factors are important in determining the degree of risk?

Solution

In Finance terms, risk refers to the possibility of earning less than expected on a particular investment over a given time period. For instance, if I make an investment and anticipate earning a 5% rate of return over the next year, I am exposed to risk if there is any chance that I earn less than that 5%. The two factors that determine the degree of risk are (A) the extent to which I can earn less than expected and (B) the likelihood that I will earn less than expected. For instance, if my expected return is 5% and the worst possible outcome is that I earn 4%, there is not a lot of risk involved. On the other hand, if there is a chance that I will lose 60%, my risk is much higher. Also, if I will make 5% 99.5% of the time and lose 50% only 0.5% of the time, my risk is also not very great because the chance of a bad outcome is very small. A high risk investment means that there is both a significant chance of earning less than expected and the amount less that I can earn is relatively large.

2. "As long as we can't lose any money, we have a risk-free investment." Discuss this comment.

Solution

This is a false comment. You can avoid losing money and still earn less than expected. Very small rates of return over long periods of time are still "bad" outcomes and the chance of that happening means you are exposed to risk even if you can't lose money.

3. Both investing and gambling can be defined as "undertaking risk in order to earn a profit." Explain how these two activities are different and why society generally takes a

more favorable view of investing compared to gambling.

Solution

While both activities use risk in an attempt to earn a profit, there are several important differences. First, the risk-return profile is different. Most forms of gambling have a negative expected return (the “house edge”). Even games where you might argue that some people have a positive expected return (sports betting, poker, etc.), the average individual will lose money due to the fixed cost of paying the gambling institution offering the wager. On the other hand, investing typically offers a positive rate of return. While stocks and bonds have both had periods of negative returns, the long-run rate of return on both is significantly positive.

Another difference is that investing provides capital to a variety of businesses. Without stock and bond markets, there would be far fewer jobs and product innovation as the stock and bond markets provide the capital for businesses to operate. Gambling on the other hand tends to transfer wealth instead of create it.

Note that the above discussion does not imply that gambling is wrong and investing is right. Instead, the point is to illustrate that the combination of positive expected returns to investors and providing capital to businesses makes investing a net positive to society while gambling tends to transfer wealth from gamblers to gambling institutions and is thus neutral in creating wealth.

4. Explain the concept of correlation? If two securities have a high positive correlation what does that mean? Give an example of two securities that might be highly correlated. If two securities have a low positive correlation what does that mean? Give an example of two securities that might have a low positive correlation. If two securities are negatively correlated what does that mean? Give an example of two securities that might be negatively correlated?

Solution

Correlation measures the extent to which two variables (returns) move together. If two securities have a high positive correlation (say somewhere between **0.5** and **0.99**) that means that they have a strong tendency to move in the same direction. When one security increases in value, it is likely that the other will as well. When one decreases in value, it is likely that the other will as well. However, since the correlation is less than **1.0**, they will not ALWAYS move in the same direction. Two securities with high positive correlations could be any two securities in the same industry. Industry pressures will lead

the securities to move in a similar direction most of the time. However, firm specific factors will keep the correlation less than **1.0**.

If two securities have a low positive correlation (**0.01** to **0.49**) that means that they have a weak tendency to move in the same direction. While they move in the same direction (up or down) more frequently than not, it is not uncommon for one to be up while the other is down. Two securities with low positive correlations could be any two companies without a specific connection (for instance, Amazon and Exxon). Both will be exposed to the same general economic factors (leading to a positive correlation), but the different industry and firm specific issues will keep the correlations low. In general, most stocks should have low positive correlations.

If two securities have a negative correlation, that means that they tend to move in opposite directions (the closer to **-1.0**, the stronger the inverse relationship). Due to general economic factors, few stocks will have negative correlations (at least as an expectation). One possible example might be oil producers and trucking firms. Since an increase in oil prices will improve the value of assets and profits for the oil firms and lower profits for trucking firms, we might see negative correlations. However, the ability of trucking firms to pass along higher prices and periods with low oil price volatility will mean that even here we may not see that significant of an inverse relationship.

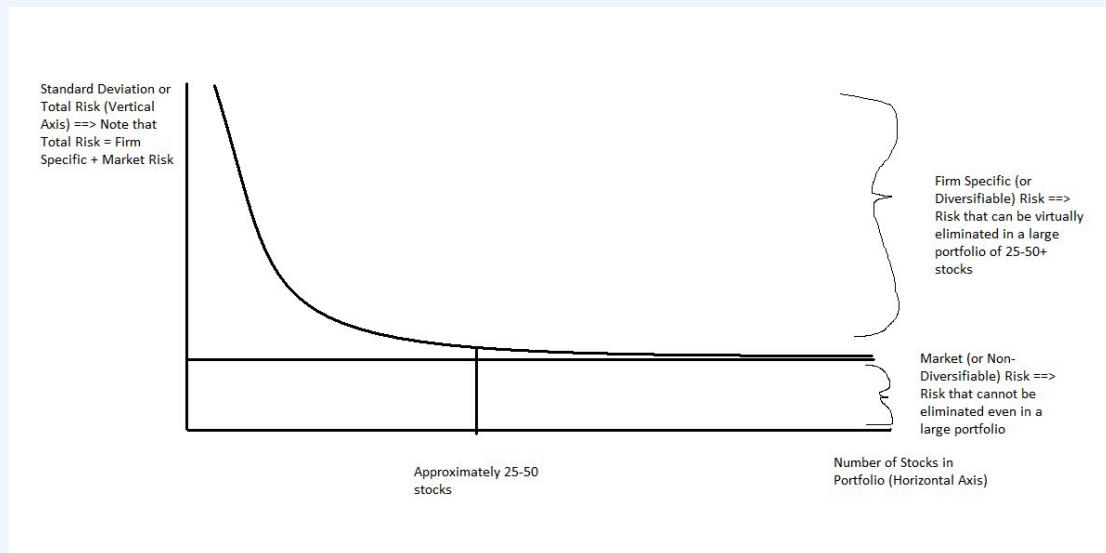
5. What is diversifiable risk? What is market or non-diversifiable risk? Give an example of each.

Solution

Diversifiable (Firm Specific) risk is risk that can be virtually eliminated by holding a large portfolio of securities from a variety of industries. An example of a diversifiable risk would be the success or failure of a new product introduction by a firm. For instance, if Ford's new auto line is poorly received and results in a bad year or two for the automaker, this will not impact the other stocks in my portfolio. Thus, as long as I own several other stocks in a variety of industries, my overall portfolio will not be hurt much by Ford's problems.

Non-Diversifiable (Market or General Economic) risk is risk that can NOT be eliminated by holding a large portfolio of securities from a variety of industries. An example of a non-diversifiable risk would be a sudden surge in inflation. This would result in most stocks declining and I couldn't eliminate this risk regardless of how many stocks were in my portfolio.

The diagram below indicates how we can diversify away most of our firm specific risk in a large portfolio (25-50+ stocks), but market risk remains.



6. Standard deviation measures what kind of risk? When is this important? When is it not important?

Solution

Standard Deviation measures total risk. Total risk is both firm specific and market risk combined. Standard deviation is important when I am considering putting all of my money into a single security or a non-diversified portfolio. The reason total risk needs to be considered in these situations is that we have not diversified away our firm-specific risk so that it is still relevant. On the other hand, if we are adding a security to a well-diversified portfolio, the standard deviation of that stock is not important. Because we own a well-diversified portfolio, the firm specific risk of the stock will be unimportant and we only need to focus on the market risk.

7. Beta (β) measures what kind of risk? When is this important? When is it not important?

Solution

Beta is a measure of market risk. While every firm has market risk and it can not be eliminated in a portfolio, some firms are more sensitive to market (general economic) factors than others. Therefore, the market risk of some stocks will be higher than the market risk of others. A portfolio of stocks with high market risk can still be diversified (so that firm specific issues are not relevant) and yet increase at a rate faster than the

market or decrease at a rate faster than the market. When we are adding a security to a well-diversified portfolio, we no longer need to be concerned about firm-specific risk, but only about market risk and how it will affect our portfolio's sensitivity to general economic/market forces. When we are considering investing all of our money into a single stock, beta is not relevant because we need to focus on the total risk instead of just the market risk factors.

8. Explain why we can use either beta or standard deviation when comparing two well-diversified portfolios. Is this true for any two portfolios?

Solution

When we are comparing two well-diversified portfolios, either beta or standard deviation will work. That is because since both portfolios have essentially zero firm-specific risk, the total risk of the portfolio will effectively equal the market risk of the portfolio. Whichever portfolio has more market risk (higher beta) will also have more total risk (higher standard deviation).

The table below illustrates when we should focus on standard deviation (σ) as a measure of risk and when we should focus on beta (β) as the appropriate measure of risk.

	Evaluating the risk of a single security in isolation or comparing two poorly diversified portfolios	Evaluating the risk of a security to add to a well-diversified portfolio	Comparing the risk level of two well-diversified portfolios
σ	X		X
β		X	X

9. If I had a stock with a beta of **1.25** and I thought the stock market was going to climb by **10%** over the next **2** months, how much should I expect my stock to move? How about if I thought the stock market was going to drop by **5%**? What would happen under each of the previous two cases if my beta was **0.6** instead of **1.25**?

Solution

The stock should increase by **12.5%** $\Rightarrow (1.25 \times 10\%)$. While in reality it will likely increase by either more or less than **12.5%**, my best guess today would be for an increase of **12.5%**. If it increased by more than **12.5%**, it would be because firm specific factors turned out positive and if it increased by less than **12.5%** (or dropped), it would be because firm specific factor turned out to be negative. The **12.5%** is based on

the firm's sensitivity to market factors. If I thought the stock market was going to decline by 5%, I would expect my stock to decrease in value by $6.25\% \Rightarrow (1.25 \times -5\%)$. For a stock with a beta of 0.6, it should increase by 6% if the market climbs by 10% and fall by 3% if the market declines by 5%.

10. According to the SML, if we purchase only securities with a high beta, we should (on average) earn higher returns. True or False? Explain your answer.

Solution

True. High beta stocks imply higher than average risk. Higher than average risk implies higher than average expected returns. We can also see this by looking at the security market line. If we have a beta of more than one, the required return will be greater than the expected return on the market. Note that this relationship is based on expected returns and not actual returns. The nature of risk is that sometimes we will lose money and with high risk we may lose a lot. While our expected returns are greater than the average return on the market, our realized returns may be higher or lower. Over long periods of time with well diversified portfolios, we should earn higher than average returns if beta is a good measurement of risk.

11. Stock A has an expected return of 10% and Stock B has an expected return of 20%. This means that if we buy Stock B, we will be wealthier at the end of the year than if we bought stock A. True or False and explain.

Solution

False. Expected return does not mean realized return. While ON AVERAGE, we would make more money from stock B over the course of the next year, we don't get to repeat next year 10,000 times to reach our average. Instead, we only get one shot. Its possible that stock B will earn quite a bit less than its expected return and stock A will earn quite a bit more.

12. Due to a recent news announcement, the expected return on XYZ Corp. just went from 13% to 18%. Assuming that the stock was in equilibrium prior to the announcement and the announcement did not affect the required return, explain what will happen to XYZ's stock price (and expected return) in the immediate future to bring the stock back into equilibrium. How long should this process take?

Solution

In equilibrium, expected return should equal required return. If this is not true, the price

will adjust until we move back into equilibrium. In this example, the expected return moves to **18%**, while the required return stays at **13%**. This means investors anticipate earning an **18%** return on average while a **13%** return is enough to compensate them for the risk they are taking. Since they can earn **5%** more than they need to compensate them for the risk, they will buy the stock. As a large number of investors start buying the stock, the price will increase. Since the expected cash flows associated with the stock are not changing, the higher price means that there will be a lower expected return (see a more detailed explanation of this below). The price will continue to go up until the expected return falls to **13%**. At that point, there is no reason for people to keep buying the stock and pushing up the price because they are only earning the required return. At this new price, the stock is worth buying, but at any higher price it would not be. Thus, we are back in equilibrium.

This process should happen instantaneously if markets are efficient. People will see the discrepancy between expected return and required return and start buying immediately. With everyone wanting to buy at the same time, the price will respond immediately.

A lot of people struggle with the idea that an increasing price leads to a lower expected return. At first glance, it seems counter-intuitive as if we see the stock price going up we might think that it is a “good” stock and expect it to continue to go up more in the future. However, if you think of a stock as the present value of all expected cash flows AND we increase the price while holding cash flows constant, you can see that you are paying more for the same set of cash flows. Since you are paying more for the same set of cash flows, you should expect to get a lower rate of return on your investment.

13. Find the expected return and standard deviation of each stock.

Stock A

Probability	Return
0.20	−30%
0.40	15%
0.40	30%

Stock B	
Probability	Return
0.30	−5%
0.40	10%
0.30	20%

If you were going to put all of your money into one of these two stocks, which should you pick?

Solution

$$\text{Exp Ret A} = 0.20(-30\%) + 0.40(15\%) + 0.40(30\%) = -6\% + 6\% + 12\% = 12\%$$

$$\text{Exp Ret B} = 0.30(-5\%) + 0.40(10\%) + 0.30(20\%) = -1.5\% + 4\% + 6\% = 8.5\%$$

$$\text{St. Dev. A} = \sqrt{0.20(-30\% - 12\%)^2 + 0.40(15\% - 12\%)^2 + 0.40(30\% - 12\%)^2}$$

$$\text{St. Dev. A} = \sqrt{352.8 + 3.6 + 129.6}$$

$$\text{St. Dev. A} = 22.05\%$$

$$\text{St. Dev. B} = \sqrt{0.30(-5\% - 8.5\%)^2 + 0.40(10\% - 8.5\%)^2 + 0.30(20\% - 8.5\%)^2}$$

$$\text{St. Dev. B} = \sqrt{54.68 + 0.90 + 39.68}$$

$$\text{St. Dev. B} = 9.76\%$$

If you are going to put all of your money into one stock, you could choose either one. People that are more risk-averse will choose Stock B while people that are less risk-averse will choose Stock A. Note that both choices are consistent with risk aversion because the stock with the higher risk (Stock A) also has a higher expected return.

14. Solve the following:

- Find the expected return and standard deviation of each stock

Probability	Return of Stock C	Return of Stock D
0.30	−10%	25%
0.50	15%	10%
0.20	40%	0%

- b. Calculate the expected return and standard deviation of a portfolio made up of **50%** stock C and **50%** stock D if the correlation is **−0.75**.
- c. Would you prefer to put your money in stock C, stock D or the **50/50** portfolio? Explain.

15. **Solution**

a. $\text{Exp Ret C} = 0.30(-10\%) + 0.50(15\%) + 0.20(40\%) = -3\% + 7.5\% + 8\% = 12.5\%$

$\text{Exp Ret D} = 0.30(25\%) + 0.50(10\%) + 0.20(0\%) = 7.5\% + 5\% + 0\% = 12.5\%$

$\text{St.Dev. C} = \sqrt{0.30(-10\% - 12.5\%)^2 + 0.50(15\% - 12.5\%)^2 + 0.20(40\% - 12.5\%)^2}$

$\text{St.Dev. C} = \sqrt{151.88 + 3.13 + 151.25}$

$\text{St.Dev. C} = 17.50\%$

$\text{St.Dev. D} = \sqrt{0.30(25\% - 12.5\%)^2 + 0.50(10\% - 12.5\%)^2 + 0.20(0\% - 12.5\%)^2}$

$\text{St.Dev. D} = \sqrt{46.88 + 3.13 + 31.25}$

$\text{St.Dev. D} = 9.01\%$

b. $\text{Exp Ret Port} = 0.5(12.5\%) + 0.5(12.5\%) = 12.5\%$

$\text{St.Dev. Port} = \sqrt{(0.5)^2(17.5\%)^2 + (0.5)^2(9.01\%)^2 + 2(0.5)(0.5)(17.5\%)(9.01\%)(-0.75)}$

$\text{St.Dev. Port} = \sqrt{76.56 + 20.30 + -59.13}$

$\text{St.Dev. Port} = 6.14\%$

- c. We should choose the portfolio because all three choices have the same expected return, but the portfolio has the lowest risk. All risk-averse investors will choose the portfolio over either stock individually in this situation.

16. Assume you had two stocks. Stock A had an expected return of **20%** and a standard

deviation of **25%**. Stock B had an expected return of **15%** and a standard deviation of **20%**. You want to create a portfolio made up of **65%** Stock A and **35%** Stock B. Find the expected return and standard deviation of this portfolio under the following conditions.

- a. Correlation between stock A and B is **1.0**
- b. **3**b. Correlation between stock A and B is **0.5**
- c. **3**c. Correlation between stock A and B is **0.0**
- d. **3**d. Correlation between stock A and B is **−0.5**
- e. **3**e. Correlation between stock A and B is **−1.0**

Solution

a. **Exp Ret Port** = $0.65(20\%) + 0.35(15\%) = 18.25\%$

$$\text{St.Dev. Port} = \sqrt{(0.65)^2(25\%)^2 + (0.35)^2(20\%)^2 + 2(0.65)(0.35)(25\%)(20\%)(1.0)}$$

$$\text{St.Dev. Port} = \sqrt{264.06 + 49 + 227.5}$$

$$\text{St. Dev. Port} = 23.25\%$$

b. **Exp Ret Port** = $0.65(20\%) + 0.35(15\%) = 18.25\%$

$$\text{St.Dev. Port} = \sqrt{(0.65)^2(25\%)^2 + (0.35)^2(20\%)^2 + 2(0.65)(0.35)(25\%)(20\%)(0.5)}$$

$$\text{St.Dev. Port} = \sqrt{264.06 + 49 + 113.75}$$

$$\text{St.Dev. Port} = 20.66\%$$

c. **Exp Ret Port** = $0.65(20\%) + 0.35(15\%) = 18.25\%$

$$\text{St. Dev. Port} = \sqrt{(0.65)^2(25\%)^2 + (0.35)^2(20\%)^2 + 2(0.65)(0.35)(25\%)(20\%)(0)}$$

$$\text{St.Dev. Port} = \sqrt{264.06 + 49 + 0}$$

$$\text{St.Dev. Port} = 17.69\%$$

d. **Exp Ret Port** = $0.65(20\%) + 0.35(15\%) = 18.25\%$

$$\text{St. Dev. Port} = \sqrt{(0.65)^2(25\%)^2 + (0.35)^2(20\%)^2 + 2(0.65)(0.35)(25\%)(20\%)(-0.5)}$$

$$\text{St. Dev. Port} = \sqrt{264.06 + 49 + -113.75}$$

$$\text{St. Dev. Port} = 14.12\%$$

$$e. \text{Exp Ret Port} = 0.65(20\%) + 0.35(15\%) = 18.25\%$$

$$\text{St.Dev. Port} = \sqrt{(0.65)^2(25\%)^2 + (0.35)^2(20\%)^2 + 2(0.65)(0.35)(25\%)(20\%)(-1.0)}$$

$$\text{St. Dev. Port} = \sqrt{264.06 + 49 + -227.5}$$

$$\text{St. Dev. Port} = 9.25\%$$

Note that the expected return is not a function of the correlation while the standard deviation declines as the correlation drops. This is the “free lunch” of diversification – as long as we invest in securities with a correlation of less than one, we will lower our risk (st. deviation will be less than the weighted average of each stock’s st. deviation) without lowering our expected return (the exp. return will be the weighted average of each stock’s expected return).

17. The stock of Ralph’s Restaurants has a standard deviation of **70%** and has a correlation with the market of **0.40**. The expected return for the market is **13%** and it has a standard deviation of **20%**. Currently the risk-free rate of return is **5%**.
- What is the beta for Ralph’s Restaurants?
 - What is the required return for Ralph’s Restaurants?
 - What is the expected return for Ralph’s restaurants in equilibrium?

Solution

$$a. \beta = \frac{(70)(0.40)}{20} = 1.40$$

$$b. \text{Req. Return} = 5\% + 1.40(13\% - 5\%) = 5\% + 11.2\% = 16.2\%$$

- c. In equilibrium, expected return equals required return. Since the required return is **16.2%**, the expected return should be **16.2%** as well.

18. We are purchasing a stock that just paid a dividend (D_0) of **\$1.50**. The growth rate in dividends for this stock is **4%** and it has a beta of **1.3**. The expected return on the market is **12%** and the current Treasury rate is **7%**. How much should we pay for this stock?

Solution

First, we know that the stock price should be found by using the following formula

$$P_0 = \frac{D_1}{k-g}$$

Now, we need to find each of the three inputs.

g is given at 4%.

$$D_1 = D_0(1 + g), \text{ so } D_1 = 1.50(1.04) = \$1.56.$$

$$k = k_{rf} + \beta (\bar{k}_m - k_{rf})$$

$$k = 7\% + 1.3 (12\% - 7\%)$$

$$k = 13.5\%$$

Once we get all the inputs, we plug them back into the pricing model

$$P_0 = \frac{1.56}{0.135 - 0.04}$$

$$P_0 = \$16.42$$

The most we should pay for the stock today is **\$16.42**.

Attribution

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CHAPTER 9 SUMMARY

Key Concepts Summary

The relationship between risk and return is an essential element of financial analysis. Because investors tend to behave in a risk-averse manner, we anticipate that higher risk investments should have higher expected returns. In this chapter, we formalize that analysis by introducing measures of risk (standard deviation and beta) and expected return. Standard deviation captures the dispersion of possible returns (total risk) and is best used when evaluating individual investments in isolation or poorly diversified portfolios. Beta captures the risk of securities relative to the overall market (market risk) and is best used when evaluating individual investments within a portfolio. A portfolio represents a collection of securities held together and is an essential tool for diversifying away firm-specific risk. The lower the correlation between a pair of securities, the more potential diversification benefits there are. This is reminiscent of the “don’t put all your eggs in one basket” cliché. While diversification can virtually eliminate the impact of firm-specific risk, it cannot eliminate market risk. Therefore, investors should be compensated for the degree of market risk that they are exposed to in their investments. The Security Market Line (SML) formalizes the risk-return tradeoff with respect to beta, the risk-free rate, and the market risk premium. It hypothesizes that higher beta stocks should generate higher returns and beta should be the only factor that systematically differentiates returns. Unfortunately, the SML has not held up well to empirical testing which has led to the introduction of newer attempts to formalize the relationship between risk and return. While the specific nature of risk and return is not finalized, it is safe to say that, in general, higher risk should be rewarded with higher expected returns. However, it is important to remember that expected returns are not the same as realized returns due to the nature of risk.

Glossary of Terms

Beta. A measure of a stock's volatility in relation to the overall market. It indicates the stock's sensitivity to market movements.

Capital Asset Pricing Model (CAPM). A model used to determine the expected return on an asset based on its beta, the risk-free rate, and the expected market return.

Correlation. A statistical measure that describes the extent to which two securities move in relation to each other.

Diversifiable Risk (Firm-Specific Risk). The portion of an investment's risk that can be eliminated through diversification by holding a portfolio of various assets. It is specific to a company or industry.

Diversification. An investment strategy that involves holding a variety of assets to reduce the overall risk of the portfolio. It aims to mitigate the impact of poor performance in any single investment.

Equilibrium. A state in which the required return equals the expected return, leading to stable prices as the market is in balance.

Expected Return. The average return an investor anticipates earning from an investment, based on the probabilities of different outcomes.

Firm-Specific Risk. Risk factors that affect a particular company or industry, such as business operations, management decisions, and specific events.

Market Index. A statistical measure that tracks the performance of a group of stocks, representing a portion of the market. Examples include the S&P 500 and the Dow Jones Industrial Average.

Market Risk. The overall risk that affects the entire market or a large segment of it, influenced by economic, political, and social factors.

Market Risk Premium. The additional return expected from holding a risky market portfolio instead of risk-free assets. It is the difference between the expected return on the market and the risk-free rate.

Market-to-Book Ratio. A financial ratio comparing the market value of a company's stock to its book value. A low ratio can indicate undervaluation, while a high ratio can indicate overvaluation.

Non-Diversifiable Risk (Market/Systematic Risk). The risk inherent to the entire market or market segment that cannot be eliminated through diversification. It is associated with factors that affect all securities, such as economic events.

Required Return. The minimum return an investor expects to achieve by investing in a particular asset, given its risk level.

Risk refers to the possibility of an unfavourable event occurring.

Risk Aversion. The preference for lower risk and the desire to avoid uncertainty in investment decisions. Risk-averse investors prefer investments with lower risk, even if they offer lower returns.

Risk-Free Rate. The theoretical rate of return of an investment with zero risk, often approximated by the yield on a 10-year Treasury bond.

Risk Premium. The return in excess of the risk-free rate that investors require as compensation for the additional risk of an investment.

Security Market Line (SML). A graphical representation of the Capital Asset Pricing Model (CAPM) that shows the relationship between the expected return of a security and its beta with the market.

Standard Deviation. A measure of the dispersion of returns for a given security or market index. It quantifies the amount of variation or volatility from the average return.

Formula & Symbol Hub

Symbols Used

- β_A = the Beta of Stock A
- $\text{corr}_{x,y}$ = the correlation between the returns of stocks x and y
- $\text{corr}_{A,MKT}$ = the correlation between stock A and the overall market
- \bar{k} = expected return of a stock

- \bar{k}_n = expected return for stock n
- \bar{k}_m = the expected return on the market (often approximated by the S&P 500)
- k_A = the required return for stock A
- k_{RF} = the risk-free rate of interest (often approximated by the yield on 10-year Treasury bond)
- k_i = the return under the i th outcome (state of nature)
- k_n = the return under the n th outcome (state of nature)
- P_i = the probability of the i th possible outcome (state of nature)
- P_n = the probability of the n th possible outcome (state of nature)
- $\sum_{i=1}^n$ = repeatedly sum the expression from i to n
- σ = standard deviation
- σ_A = the standard deviation of stock A
- σ_{MKT} = the standard deviation of the overall market
- W_n = the weight (proportion of portfolio) of stock n

Formulas Used

- **Formula 9.1 – Expected Return (Single Security)**

$$\bar{k} = \sum_{i=1}^n P_i k_i$$

OR

$$\bar{k} = P_1 k_1 + P_2 k_2 + \dots + P_n k_n$$

- **Formula 9.2 – Standard Deviation (Single Security)**

$$\sigma = \sqrt{\sum_{i=1}^n P_i (k_i - \bar{k})^2}$$

OR

$$\sigma = \sqrt{P_1 (k_1 - \bar{k})^2 + P_2 (k_2 - \bar{k})^2 + \dots + P_n (k_n - \bar{k})^2}$$

- **Formula 9.3 – Expected Return (Portfolio)**

$$\bar{k}_p = \sum_{i=1}^n W_i \bar{k}_i$$

OR

$$\bar{k}_p = W_1 \bar{k}_1 + W_2 \bar{k}_2 + \dots + W_n \bar{k}_n$$

- **Formula 9.4 – Standard Deviation (Two-Stock Portfolio)**

$$\sigma_p = \sqrt{W_1^2 \sigma_1^2 + W_2^2 \sigma_2^2 + 2W_1 W_2 \sigma_1 \sigma_2 \text{corr}_{1,2}}$$

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APPENDIX 1: SOFTWARE SOLUTIONS

Descriptive Statistics: Data

Variable	N	N*	Mean	SE Mean	StDev	Variance	CoefVar	Minimum	Q1
Dara	11	0	32.27	4.83	16.03	256.82	29.66	5.00	20.00

Variable	Median	Q3	Maximum	IQR
Data	35.00	45.00	55.00	25.00

Excel

Open up Excel and enter the data in the first column of the spreadsheet. Select DATA > Data Analysis > Descriptive Statistics. For the input range, select data in column A. Check “Labels in First Row” and “Summary Statistics”. Also, check “Output Range” and select a location for output.

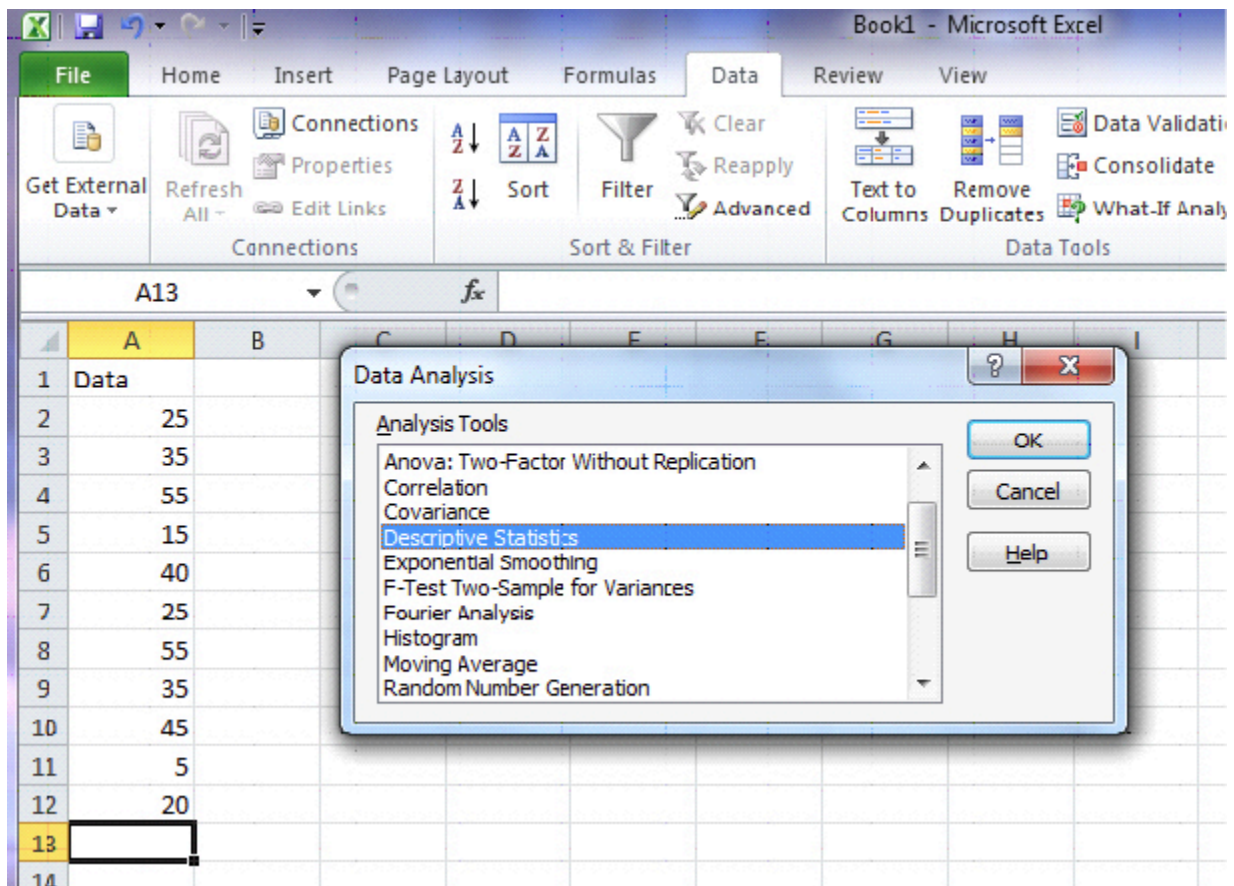


Image Description

The image depicts a Microsoft Excel 2010 interface with the “Data” tab active. There is an open “Data Analysis” dialog box at the center, displaying a list of analysis tools. “Descriptive Statistics” is selected from the list. Visible Ribbon Tabs and Buttons:

- The active “Data” tab shows options like:
 - Get External Data
 - Connections
 - Sort & Filter
 - Data Tools
 - Outline
- Buttons under “Get External Data”: Fetch Data, Refresh All
- Buttons under “Connections”: Connections, Properties, Edit Links
- Buttons under “Sort & Filter”: Sort A to Z, Sort Z to A, Advanced
- Buttons under “Data Tools”: Text to Columns, Remove Duplicates, Data Validation, Consolidate, What-If Analysis

Open “Data Analysis” Dialog Box:

Options

Anova: Two-Factor Without Replication

Correlation

Covariance

Descriptive Statistics

Exponential Smoothing

F-Test Two-Sample for Variances

Fourier Analysis

Histogram

Moving Average

Random Number Generation

- Buttons within the dialog box: OK, Cancel, Help

Excel Sheet:

- Sheet name: “Book1 – Microsoft Excel”
- Selected cell: *A13*
- Worksheet data in column *A*:
 - *A1 : Data*
 - *A2 : 25*
 - *A3 : 25*
 - *A4 : 30*
 - *A5 : 55*
 - *A6 : 15*
 - *A7 : 35*
 - *A8 : 65*
 - *A9 : 35*
 - *A10 : 25*
 - *A11 : 55*
 - *A12 : 20*

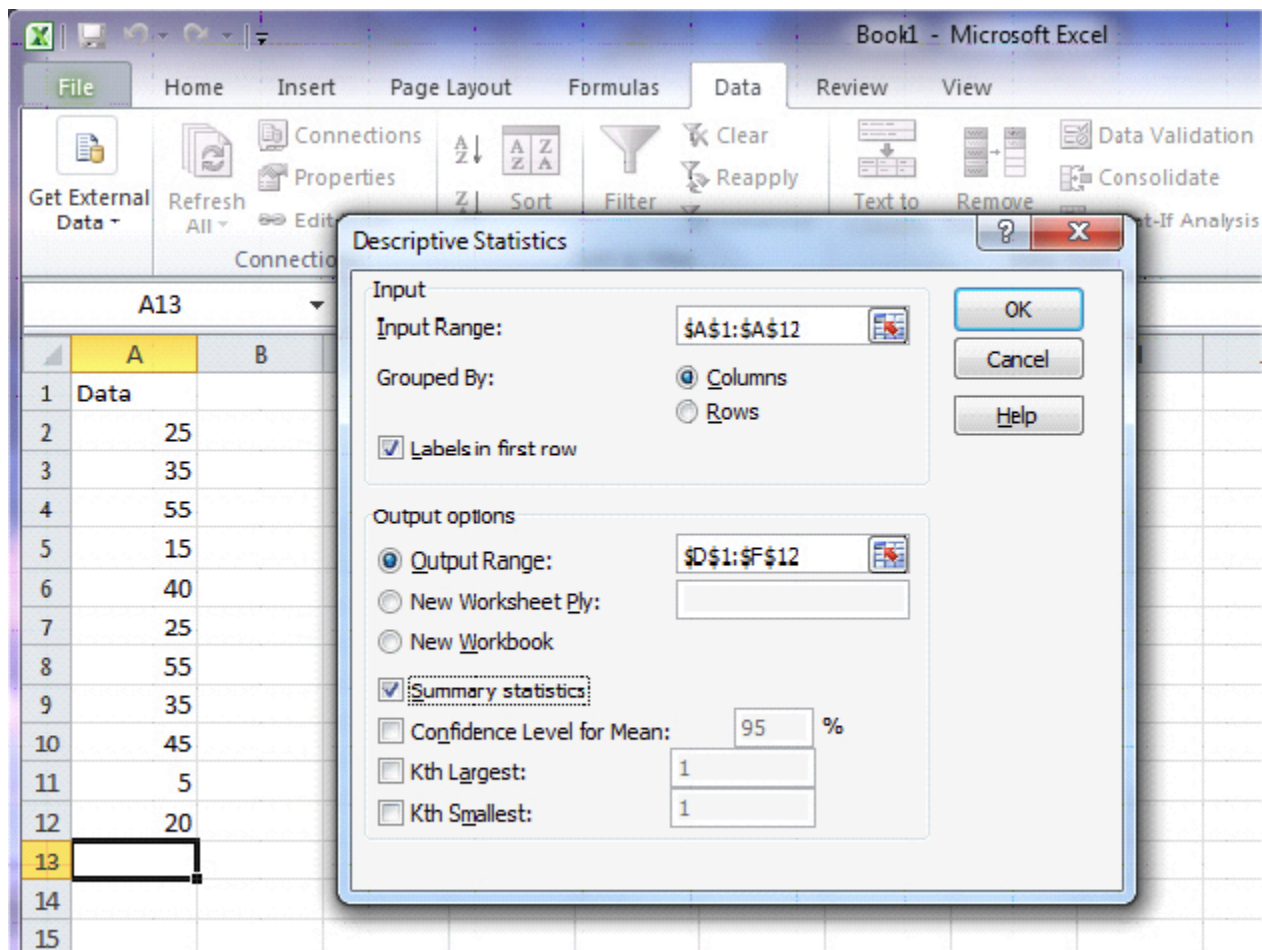


Image Description

The image shows a screenshot of a Microsoft Excel application. In the background, a worksheet titled “Book1” is open, with column headers labeled “A” and “B”. The worksheet contains a column of numerical data under column “A,” specifically labeled “Data” in cell A1, with values 25, 35, 45, 50, 55, 65, 75, 85, 95, and 105 in cells A2 through A11. Cell A12 is empty, and cell A13 is highlighted. Column B is empty.

In the foreground, a “Descriptive Statistics” dialog box is open. The dialog box is divided into two main sections:

1. Input:

- Input Range: ‘\$A\$1 : \$A\$12’
- Grouped By: Columns (selected)
- A checkbox labeled “Labels in first row” is checked.

2. Output options:

- Output Range: ‘\$D\$1 : \$F\$12’

- Radio buttons for ‘Output Range’ (selected), ‘New Worksheet Ply’, and ‘New Workbook’.
- A checkbox for “Summary statistics” is checked.
- Fields with labels for “Confidence Level for Mean:” set to **95**, and fields for “ K^{th} Largest” and “ K^{th} Smallest” both set to **1**.

On the right side of the dialog box, there are three buttons: ‘OK’, ‘Cancel’, and ‘Help’. The Excel Ribbon is visible at the top with various tabs like File, Home, Insert, Page Layout, Formulas, Data, Review, and View. The “Data” tab is currently selected, displaying options such as Connections, Sort, Filter, Data Tools, and Outline.

Data	
Mean	32.27273
Standard Error	4.831884
Median	35
Mode	25
Standard Deviation	16.02555
Sample Variance	256.8182
Kurtosis	−0.73643
Skewness	−0.05982
Range	50
Minimum	5
Maximum	55
Sum	355
Count	11

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VERSIONING HISTORY

This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.1. If the edits involve a number of changes, the version number increases to the next full number.

The files posted alongside this book always reflect the most recent version.

Version	Date	Change	Affected Web Page
1.0	7 November 2024	First Publication	N/A