

Business and Financial Mathematics

BUSINESS AND FINANCIAL MATHEMATICS

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Fanshawe College Pressbooks
London Ontario



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Collaborators

This project was a collaboration between the author and the team in the OER Design Studio at Fanshawe. The following staff and students were involved in the creation of this project:

- Melanie Mitchell Sparkes, *Instructional Designer*
- Shauna Roch, *Project Lead*

ABOUT THIS BOOK

Business and Financial Mathematics introduces students to the ideas, concepts, and applications of the mathematics of business and finance. Topics include the mathematics of merchandising, cost-volume-profit analysis, simple and compound interest, annuities, loans, bonds, and business investment decisions. The book is written at an introductory level, designed for students studying business, finance, or other related areas. The text emphasizes the understanding and application of technology, in the form of a financial calculator, to solve problems, but some knowledge of algebra is required.

Although the text focuses on concepts and applications, every effort has been made to provide both information essential to understanding a topic and sound methodological development. Generally accepted terminology and notation for each topic is used throughout without becoming overly focused on technical details.

In place of most manual calculations, especially those related to compound interest and annuities, the text utilizes the Texas Instrument BAII Plus financial calculator. Because of the prevalence and use of technology like financial calculators, it is important for students to understand how to leverage a financial calculator's capabilities. Throughout the text, information is provided on how to use the BAII Plus calculator to perform calculations and the solutions to examples illustrate the required information need to solve problems using the calculator.

The text is organized into eight chapters, and then divided into sub-chapters by concept or topic. The chapters are as follows:

Chapter 1: Mathematics of Merchandising	Chapter 1 covers the calculations involved in the merchandising chain, including discounts, markup, and markdown.
Chapter 2: Cost-Volume-Profit Analysis	Chapter 2 examines the concepts associated with break-even analysis through revenue and cost functions, the contribution margin, and the contribution rate.
Chapter 3: Simple Interest and its Applications	Chapter 3 introduces simple interest and the application of simple interest to short-term GICs, treasury bills and commercial papers, and demand loans.
Chapter 4: Compound Interest	Chapter 4 extends the concept of interest to compound interest, including the calculation of future value, present value, the interest rate, and term.
Chapter 5: Annuities	Chapter 5 explores the calculation of future value, present value, payment, interest rate and term to both ordinary annuities and annuities due, as well as deferred annuities and perpetuities.
Chapter 6: Loans and Mortgages	Chapter 6 extends annuity calculations to loans and mortgages, focusing on the amortization schedule, calculations associated with mortgage renewal, and calculations related to a mortgage's pre-payment options.
Chapter 7: Bonds and Sinking Funds	Chapter 7 covers the concepts of bonds, including calculating the purchase price of a bond, the quotation of a bond, bond schedules, and sinking funds.
Chapter 8: Business Investment Decisions	Chapter 8 examines the use of net present value and internal rate of return in making a business decision.

For the Student

Each sub-chapter in this text begins with a list of relevant learning objectives and concludes with a concept review that highlights the key topics in the sub-chapter. Where appropriate, videos are included to review, enhance, and extend the material covered in the text. At the end of each chapter, a series of exercises are included to check retention and assess understanding.

Accessibility Statement

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CHANGES FROM ADAPTED SOURCES

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The following is a summary of changes that were made in this version:

Overall	• Book reformatted for Pressbooks where applicable. • Chapters were moved, combined, or separated as required. • Sub-chapters were moved, combined, or separated as required. • Latex code updated throughout. • Added learning objectives at the start of every sub-chapter. • Revised and updated chapter and sub-chapter titles throughout. • Revised, rewrote, and updated content throughout. • Replaced most by-hand solutions with financial calculator solutions. • Added additional images and videos. • Added exercises at the end of each sub-chapter, with unnecessary exercises removed. • Added answers to each question in exercise section at end of each sub-chapter. • Removed Concept Checks and H5P activities. • Removed Chapter Exercises Solutions and Test Bank chapters from end of book.
Chapter 1: Mathematics of Merchandising	• Removed some content, images, and examples as required. • Added some additional content, examples, and exercises. • Added sub-chapter on invoices from Accounting Applications chapter.
Chapter 2: Cost-Volume-Profit Analysis	• Removed some content, images, and examples as required. • Added some additional content, examples, and exercises. • Separated and reorganized content in sections on Cost-Volume-Net Income Analysis and Break-Even Analysis into several different sub-chapters.
Chapter 3: Simple Interest and Its Applications	• Removed some content, images, and examples as required. • Added some additional content, examples, and exercises. • Added calculator demonstration videos.
Chapter 4: Compound Interest	• Removed some content, images, and examples as required. • Added some additional content, examples, and exercises. • Replaced most manual calculations with calculator-only solutions. • Added calculator demonstration videos.
Chapter 5: Annuities	• Removed some content, images, and examples as required. • Added some additional content, examples, and exercises. • Replaced most manual calculations with calculator-only solutions. • Added calculator demonstration videos. • Added sub-chapters on deferred annuities and perpetuities from Applications of Annuities chapter.

Chapter 6: Loans and Mortgages	• Removed some content, images, and examples as required. • Added some additional content, examples, and exercises. • Replaced most manual calculations with calculator-only solutions. • Added calculator demonstration videos. • Added content on mortgage pre-payment options.
Chapter 7: Bonds and Sinking Funds	• Removed some content, images, and examples as required. • Added some additional content, examples, and exercises. • Replaced most manual calculations with calculator-only solutions. • Added calculator demonstration videos. • Separated section on Determining the Value of a Bond into several sub-chapters. • Added content on quoted price of a bond.
Chapter 8: Business Investment Decisions	• Removed some content, images, and examples as required. • Added some additional examples and exercises. • Replaced most manual calculations with calculator-only solutions. • Added calculator demonstration videos.

MathJax

All mathematical notation was written in LaTeX to allow for MathJax functionality.

Please note that “some screen readers support MathML, MathJax’s internal format. Screen readers like ChromeVox, JAWS (on IE), and TextHelp support MathJax directly (most only version 2); other screen readers are supported by the assistive-mml extension as of version 3.0.1.” (MathJax Consortium, 2021). It is important to also note that the quality of screen reader support varies greatly with the software you are using and the various settings enabled. For more information on MathJax and screen reader support please visit the [MathJax Consortium Accessibility Features page](#).

PART I

MATHEMATICS OF MERCHANDISING

When you buy a new cell phone, it is very important that the right price is set. The price that the cell phone is sold for needs to do several things, including:

- Be seen as fair by you the buyer.
- Pay for the costs (plastics, battery, buttons, circuit boards, headset) and expenses (employees, factory, electricity, distribution) of making the cell phone.
- Allow the seller's business to make some extra money as profit so that it can grow its business further.

If your business is selling a product, you will need to pay close attention to price adjustments because they affect profitability. Various discounts, like putting items on sale, may increase sales but lower the amount of profit per transaction. How do you know where to set the balance to maximize profit overall?

As a student in a business program, consider this chapter essential to the success of any business. Whether your pricing strategy is high or low, the company must ensure that it can still pay its bills as a minimum requirement. And that requires careful juggling of many factors. If it fails to manage its pricing properly, the company will go bankrupt!

This chapter will make you a smarter business professional and a wiser consumer. You shop retail almost every day and regularly purchase goods and services. If you understand how product pricing works, you can make sense of “deals.” You can easily explain why the same product sells for two different prices at two different stores.

You must learn the language of marketers to perform merchandising mathematics involving product costs, expenses, prices, markups, markdowns, and ultimately profitability. Once the study of the various pricing components is complete, we will see how the various pieces of the pricing puzzle fit together into a cohesive merchandising environment.

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1.1 DISCOUNTS

LEARNING OBJECTIVES

- Solve problems involving a single trade discount rate or a series of trade discount rates.
- Calculate a single discount rate that is equivalent to a series of discounts.

You mutter in exasperation, “Why can’t they just set one price and stick with it?” Your mind boggles at all the competing discounts you encounter at the mall in your search for that perfect Batman toy for your nephew. Walmart is running their Rollback promotion and is offering a Batmobile for 25% off, regularly priced at \$49.99. Toys R’ Us has an outlet in the parking lot where the regular price for the same toy is \$59.99, but all Batman products are being cleared out at 40% off. You head over to The Bay for a warehouse clearance event that has the same toy priced at \$64.99 but at 35% off. It is also Bay Days, which means you can scratch and win a further 10% to 20% off the sale price.

The **cost** of a product is the amount of money required to obtain the merchandise. If you are a consumer, the ticketed price tag on the product is your cost. If you are a reseller (also known as a *middleman* or *intermediary*), what you pay to your supplier for the product is your cost. If you are a manufacturer, then your cost equals all of the labor, materials, and production expenditures that went into creating the product.

A **discount** is a reduction in the price of a product. As a consumer, you are bombarded with discounts all the time. Retailers use various terms for discounts, including *sales* or *clearance*. If your business purchases a product from a supplier, any discount it receives lowers how much the business pays to acquire the product. When a business buys products, the price paid is the cost to the business. Therefore, a lower price means a lower cost.

If your business is the one selling the product, any discount offered lowers the selling price and reduces revenue per sale. Because the revenue must cover all costs and expenses associated with the product, the lower price

means that the business reduces profits per sale. In business, it is common practice to express a discount as a percentage off the regular price.

The distribution of goods or products that are ultimately sold to consumers forms a chain, called a **merchandising chain** or **distribution chain**. For example, consider a manufacturer such as Kellogg Canada Inc. The distribution chain to get products from the manufacturer, in this case Kellogg Canada, to the consumer is illustrated in the top part of the image below. To distribute its products to various regions across Canada, Kellogg Canada uses various regional wholesalers. Each wholesaler then resells the product to retailers in its local trade area. Although some retailers (such as the Real Canadian Superstore) are very large, and Kellogg Canada distributes directly to these organizations, bypassing the wholesaler as represented by the blue arrow. Finally, consumers shop at these retailers and acquire Kellogg products.

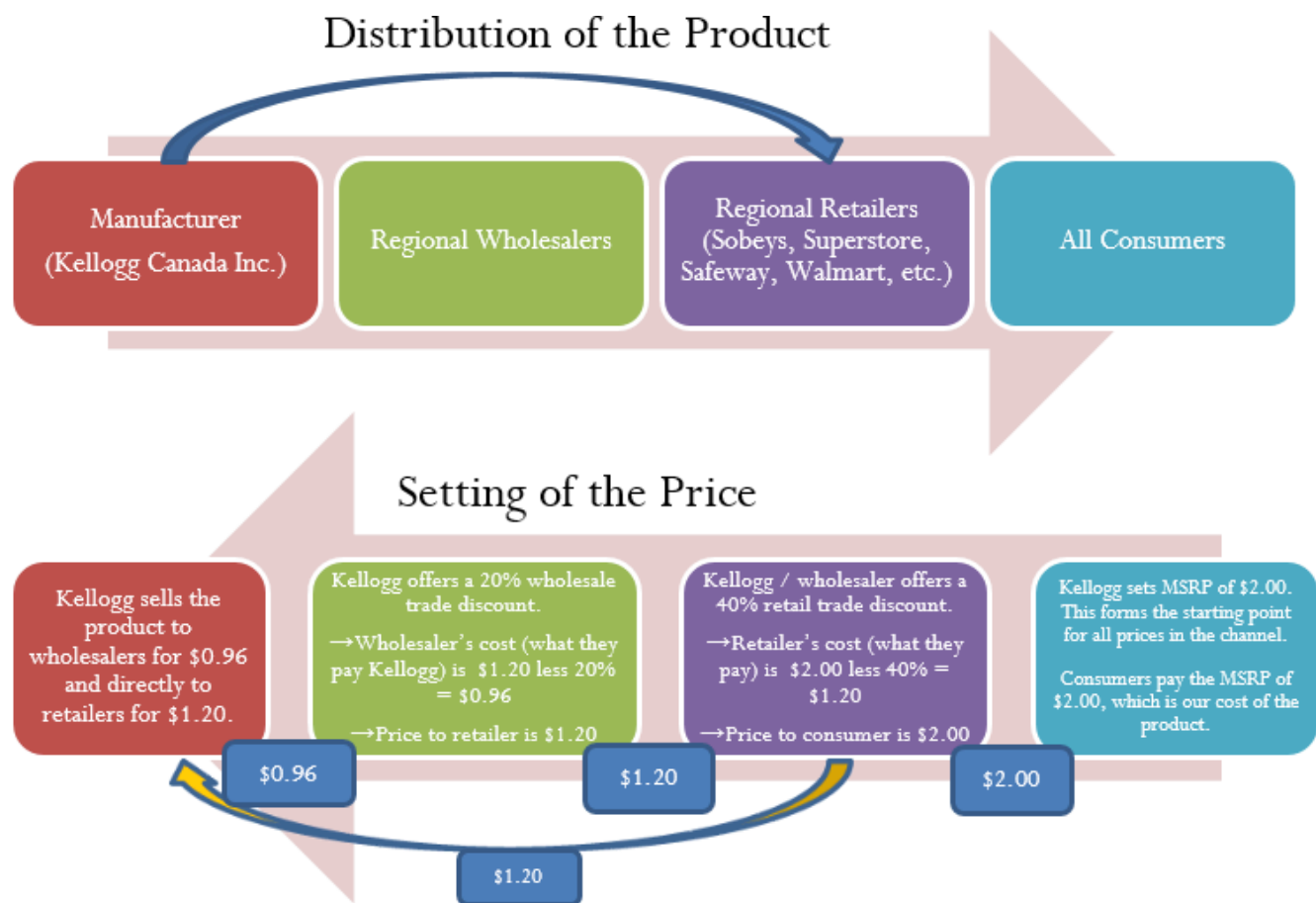


Figure 1.1.1

The relationship of distribution to pricing is illustrated in the bottom part of the image above. Kellogg Canada sets a **list price** or **manufacturer's suggested retail price**, known as the MSRP. This is a recommended retail price based on consumer market research. Because grocery retailers commonly carry thousands or tens of

thousands of products, the MSRP helps the retailer to determine the retail price at which the product should be listed. In this case, assume a \$2.00 MSRP for a particular product, which is the price consumers will pay for the product.

The retailer must pay something less than \$2.00 to make money when selling the product. Kellogg Canada understands its distributors and calculates that to be profitable most retailers must pay approximately 40% less than the MSRP. Therefore, Kellogg Canada offers a 40% discount. If the retailer purchases directly from Kellogg, as illustrated by the yellow arrow, the price paid by the retailer to acquire the product is \$2.00 less 40%, or \$1.20. Smaller retailers acquire the product from a wholesaler for the same price. Thus, the retailer's cost equals the wholesaler's price (or Kellogg Canada's price if the retailer purchases it directly from Kellogg).

The wholesaler's price is \$1.20. Again, Kellogg Canada, knowing that the wholesaler must pay something less than \$1.20 to be profitable, offers an additional 20% discount exclusively to the wholesaler. So the price paid by the wholesaler to acquire the product from Kellogg Canada is \$1.20 less 20%, or \$0.96. This \$0.96 forms Kellogg Canada's price to the wholesaler, which equals the wholesaler's cost.

In summary, this discussion illustrates two key pricing concepts:

- Companies higher up in the distribution chain pay lower prices than those farther down the chain. Companies receive discounts off the MSRP based on their level in the distribution chain. This may result in multiple discounts, such as a wholesaler receiving both the retailer's discount and an additional discount for being a wholesaler.
- One organization's price becomes the next organization's cost (assuming the typical distribution chain structure):

Manufacturer's Price = Wholesaler's Cost

Wholesaler's Price = Retailer's Cost

Retailer's Price = Consumer's Cost

Types of Discounts

Businesses or consumers are offered numerous types of discounts, of which five of the most common are trade, quantity, loyalty, sale, and seasonal.

- **Trade Discount.** A **trade discount** is a discount offered to businesses only based on the type of business and its position in the distribution chain, such as a retailer, wholesaler, or any other member of the chain that resells the product. Consumers are ineligible for trade discounts. In the above example, two trade discounts are offered—a 40% retail trade discount and a second 20% wholesale trade discount.

Typically, a business that is higher up in the distribution chain receives a combination of these trade discounts.

- **Quantity Discount.** A **quantity discount**, or volume discount, is a discount for purchasing larger quantities of a certain product.
- **Loyalty Discount.** A **loyalty discount** is a discount that a seller gives to a purchaser for repeat business. As a consumer, you see this regularly in marketing programs or with credit cards that offer cash back programs.
- **Sale Discount.** A **sale discount** is a temporary lowering of the price from a product's regular selling price. Businesses put items on sale for a variety of reasons, such as selling excess stock or attracting shoppers.
- **Seasonal Discount.** A **seasonal discount** offered to consumers and businesses for purchasing products out of season. At the business level, manufacturers tend to offer seasonal discounts to encourage retailers, wholesalers, or distributors to purchase products before they are in season.

Single Discounts

The price of a product after applying a single discount to the list price is called the **net price**. Figuring out the price after applying a single discount is called a net price calculation. When a business calculates the net price of a product, it is interested in what you still have to pay, not in what has been removed.

Given the list price (L) of an item and a single discount (d), the net price (N) of the item is:

$$N = L \times (1 - d)$$

where

- **N is the net price.** The net price is the price of the item after the discount is removed from the list price. It is a dollar amount representing what price remains after you have applied the discount.
- **L is the list price.** The list price is the normal or regular dollar price of the item before any discounts. It is the MSRP or any dollar amount before you remove the discount.
- **d is the discount rate.** The discount rate represents the percentage (in decimal format) of the list price that is deducted.

NOTES

1. In the net price formula, you take 1 and subtract the discount rate to determine the rate owing. For example, if you are eligible for a 20% discount, then you must pay 80% of the list price.
2. The net price formula above requires the discount be given as percentage (i.e. 20% discount). Sometimes a discount is expressed as a dollar amount (i.e. take \$5 off).

The **discount amount** (D) is the actual dollar amount subtracted from the list price to get the net price.

The discount amount is

$$D = d \times L$$

where

- **D is the discount amount.**
- **d is the discount rate.** The discount rate represents the percentage (in decimal format) of the list price that is deducted.
- **L is the list price.** The list price is the normal or regular dollar price of the item before any discounts.

An alternative formula for the discount amount is

$$D = L - N$$

where

- **D is the discount amount.**
- **L is the list price.** The list price is the normal or regular dollar price of the item before any discounts.
- **N is the net price.** The net price is the price of the item after the discount is removed from the list price.

NOTES

1. The last formula, $D = L - N$, can be rearrange to express the net price in terms of D and L : $N = L - D$.
2. When working with single discounts, you are not always solving for the net price. Sometimes you must calculate the discount percent or the list price. At other times you know information about the discount amount but need to solve for list price, net price, or the discount rate. Ultimately, the formula that you use will depend on what information you have and what you are trying to find.
3. Many of the pricing problems take multiple steps that combine various formulas, so you need to work through the problem systematically. In any pricing problem, you must understand which variables are provided and match them up to the known formulas. To get to your end goal, you must look for formulas in which you know all but one variable. In these cases, solving for variables will move you forward toward solving the overall pricing problem.

EXAMPLE

A manufacturer that sells jeans directly to its retailers uses market research to find out it needs to offer a 25% trade discount. In doing so, the retailers will then be able to price the product at a list price of \$59.99. What price should retailers pay for the jeans?

Solution:

Step 1: The given information is

$$L = \$59.99$$

$$d = 25\%$$

Step 2: Solve for the net price.

$$\begin{aligned}
 N &= L \times (1 - d) \\
 &= 59.99 \times (1 - 0.25) \\
 &= 59.99 \times 0.75 \\
 &= \$44.99
 \end{aligned}$$

The manufacturer should sell the jeans to the retailers for \$44.99.

NOTES

1. Until you arrive at the final answer, avoid rounding any interim number, unless you have some special reason to do so.
2. Round all dollar amounts to the nearest cent.
3. Round all percentages to four decimals when in decimal format and two decimals when in percent format (i.e. 0.0523 or 5.23%).

EXAMPLE

A retailer pays a net price of \$27.50 for a winter jacket after receiving a retail trade discount of 45%. What was the list price of the jacket?

Solution:

Step 1: The given information is

$$\begin{aligned}
 N &= \$27.50 \\
 d &= 45\%
 \end{aligned}$$

Step 2: Solve for the list price.

$$\begin{aligned} N &= L(1-d) \\ 27.50 &= L(1-0.45) \\ 27.50 &= L(0.55) \\ \frac{27.50}{0.55} &= L \\ 50 &= L \end{aligned}$$

The list price of the jacket was \$50.

EXAMPLE

You are shopping at a local sporting goods store for a new environmentally friendly water bottle. The price tag reads \$14.75, which is \$10.24 off the regular price. Determine the discount rate applied.

Solution:

Step 1: The given information is

$$N = \$14.75$$

$$D = \$10.24$$

Step 2: Solve for the list price.

$$\begin{aligned} N &= L - D \\ 14.75 &= L - 10.24 \\ 14.75 + 10.24 &= L \\ 24.99 &= L \end{aligned}$$

Step 3: Solve for the discount rate.

$$\begin{aligned} D &= d \times L \\ 10.24 &= d \times 24.99 \\ \frac{10.24}{24.99} &= d \\ 0.4098 &= d \\ 40.98\% &= d \end{aligned}$$

The discount rate was 40.98%.

TRY IT

A retailer sells shoes for \$150. During a sale, a discount of 30% is offered. What is the net price of the shoes?

Click to see Solution

$$\begin{aligned} N &= L \times (1 - d) \\ &= 150 \times (1 - 0.3) \\ &= 150 \times 0.7 \\ &= \$105 \end{aligned}$$

TRY IT

A new model of cell phone is on sale for \$750 after a discount of 15%. What was the list price of the cell phone?

Click to see Solution

$$\begin{aligned}
 N &= L \times (1 - d) \\
 750 &= L \times (1 - 0.15) \\
 750 &= L \times 0.85 \\
 \frac{750}{0.85} &= L \\
 \$882.35 &= L
 \end{aligned}$$

TRY IT

A local food distributor lists a case of canned soup for \$68.50. The distributor sells the case to a local grocery retailer for \$52.35. What discount rate did the distributor offer the retailer?

Click to see Solution

$$\begin{aligned}
 N &= L \times (1 - d) \\
 52.35 &= 68.50 \times (1 - d) \\
 \frac{52.35}{68.50} &= 1 - d \\
 0.76423... &= 1 - d \\
 d &= 1 - 0.76423... \\
 d &= 0.2358 \\
 d &= 23.58\%
 \end{aligned}$$

Multiple Discounts

You are driving down the street when you see a large sign at Old Navy that says, “Big sale, take an additional 25% off already reduced prices!” In other words, products on sale (the first discount) are being reduced by an additional 25% (the second discount).

Businesses commonly receive more than one discount when they make a purchase. Consider a transaction in which a business receives a 30% trade discount as well as a 10% volume discount. First, you have to understand that this is **NOT** a $30\% + 10\% = 40\%$ discount. The second discount is always applied to the net price **after** the

first discount is applied. Therefore, the second discount has a smaller base upon which it is calculated. If there are more than two discounts, you deduct each subsequent discount from continually smaller bases.

Given the list price of an item and a series of discounts, the net price of the item is

$$N = L \times (1 - d_1) \times (1 - d_2) \times \cdots \times (1 - d_n)$$

where

- **N is the net price.** The net price is the price of the item after all of the discounts have been deducted.
- **L is the list price.** The list price is the normal or regular dollar price of the item before any discounts.
- **d_1 is the first discount rate, d_2 is the second discount rate, and so on to d_n the n th discount rate.** The discount rate represents the percentage (in decimal format) of the list price that is deducted. When there is more than one discount, you must extend beyond the $N = L \times (1 - d)$ formula by multiplying another discount expression. These discounts are represented by the same d symbol. However, each discount receives a subscript to make its symbol unique. Therefore, the first discount receives the symbol of d_1 , the second discount receives the symbol d_2 , and so on. Recall that the symbol n represents the number of pieces of data (a count), so you can expand or contract this formula to the exact number of discounts being offered.

It is often difficult to understand exactly how much of a discount is being received when multiple discounts are involved. Often it is convenient to summarize the multiple discount percentages into a single percentage. This makes it easier to calculate the net price and aids in understanding the discount benefit. Simplifying multiple percent discounts into a single percent discount is called finding the **single equivalent discount**. Whether you apply the multiple discounts or just the single equivalent discount, you arrive at the same net price.

To convert multiple discount percentages into a single equivalent discount percent

$$d_e = 1 - [(1 - d_1) \times (1 - d_2) \times \cdots (1 - d_n)]$$

where

- **d_e is the single equivalent discount rate.** That is, d_e is equal to the series of multiple discounts. Recall that taking $(1 - d)$ calculates what you pay. Therefore, if you take 1, which represents the entire amount, and reduce it by what you pay, the rate left over must be what you did not pay. In other words, it is the discount rate.
- **d_1 is the first discount rate, d_2 is the second discount rate, and so on to d_n the n th discount rate.** Because there are multiple discounts, each discount receives a numerical subscript to give it a unique identifier. You can expand or contract the formula to the exact number of discounts being

offered.

NOTES

1. As noted above, the single equivalent discount rate is NOT equal to the sum of the discounts. That is, if you have a series of 30%, 10% and 5% discounts, the single equivalent discount is actually 40.15%, and NOT 45%.
2. The order of the discounts **does not** matter in determining the net price. Remember from the rules of BEDMAS that you can complete multiplication in any order. In other words, it does not matter if you take 30% first followed by 10% or 10% first followed by 30%.
3. Notice in the single equivalent discount formula that the list price and the net price are not involved in the calculation of the single equivalent discount.

EXAMPLE

A retail dealership purchases some snowmobiles to stock in its stores. Examining the merchandising terms of the manufacturer, the dealership notices that it would be eligible to receive a 35% trade discount, a 15% volume discount, and a 3% loyalty discount. Because it is June and snowmobiles are out of season, the manufacturer offers a seasonal discount of 12% for purchases made before June 30.

1. If the list price for the snowmobiles is \$12,399 and the dealership purchases this item on June 15, what price would the dealership pay?
2. What single discount is equivalent to the series of four discounts offered?

Solution:

Step 1: The given information is

$$\begin{array}{l} L = \$12,399 \\ d_1 = 35\% \\ d_2 = 15\% \\ d_3 = 3\% \\ d_4 = 12\% \end{array}$$

Step 2: Solve for the net price.

$$N = L \times (1 - d_1) \times (1 - d_2) \times (1 - d_3) \times (1 - d_4) \\ = 12,399 \times (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12) \\ = 12,399 \times 0.65 \times 0.85 \times 0.97 \times 0.88 \\ = \$5,847.54$$

Step 3: Solve for the single equivalent discount rate.

$$d_e = 1 - \left[(1 - d_1) \times (1 - d_2) \times (1 - d_3) \times (1 - d_4) \right] \\ = 1 - \left[(1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12) \right] \\ = 1 - (0.65 \times 0.85 \times 0.97 \times 0.88) \\ = 1 - 0.471614 \\ = 0.528386 \\ = 52.8386\%$$

The net price of the snowmobiles was \$5,847.54. The single equivalent discount rate was 52.84%.

NOTE

In the previous example, because you know the list price (\$12,399) and the net price (\$5,847.54), you could solve for the single equivalent discount by solving for d in $N = L \times (1 - d)$:

$$\begin{aligned} N &= L \times (1 - d) \\ 5,847.54 &= 12,399 \times (1 - d) \\ \frac{5,847.54}{12,399} &= 1 - d \\ 0.47161384 &= 1 - d \\ d &= 0.528386 \end{aligned}$$

EXAMPLE

An advertisement claims that at 60% off, you are saving \$18. However, today there is an additional 20% off. What price should you pay for this item? What percent savings does this represent?

Solution:

Step 1: The given information is

$$\begin{aligned} D_1 &= \$18 \\ d_1 &= 60\% \\ d_2 &= 20\% \end{aligned}$$

Step 2: Solve for the list price.

$$\begin{aligned} D_1 &= d_1 \times L \\ \$18 &= 0.6 \times L \\ L &= \frac{18}{0.6} \\ L &= \$30 \end{aligned}$$

Step 3: Solve for the net price.

$$\begin{aligned} N &= L \times (1-d_1) \times (1-d_2) \\ &= \$30 \times (1-0.6) \times (1-0.2) \\ &= \$30 \times 0.4 \times 0.8 \\ &= \$9.60 \end{aligned}$$

Step 4: Solve for the single equivalent discount rate.

$$\begin{aligned} d_e &= 1 - \left[(1-d_1) \times (1-d_2) \right] \\ &= 1 - \left[(1-0.6) \times (1-0.2) \right] \\ &= 1 - (0.4 \times 0.8) \\ &= 1 - 0.32 \\ &= 0.68 \\ &= 68\% \end{aligned}$$

You should pay \$9.60 for the item, which represents a 68% savings.

EXAMPLE

An online store sells a pair of shoes for \$73.50 less a 15% discount. A department store sells the same shoes for \$87.25 less a 20% discount.

1. What is the net price of the shoes at the online store?
2. What additional discount does the department store have to offer to match the price offered by the online store?

Solution:

Step 1: The given information for the online store is

$$L = \$73.50$$

$$d = 15\%$$

Step 2: Solve for the net price at the online store.

$$\begin{aligned} N &= L \times (1-d) \\ &= 73.50 \times (1-0.15) \\ &= 73.50 \times 0.85 \\ &= \$62.48 \end{aligned}$$

Step 3: The department store needs to have the same net price as the online store: \$62.48. The given information for the department store is

$$L = \$87.25$$

$$N = \$62.48$$

$$d_1 = 20\%$$

Step 4: Solve for the second discount rate.

$$\begin{aligned} N &= L \times (1-d_1) \times (1-d_2) \\ 62.48 &= 87.25 \times (1-0.2) \times (1-d_2) \\ 62.48 &= 69.80 \times (1-d_2) \\ \frac{62.48}{69.80} &= 1-d_2 \\ 0.89512894 &= 1-d_2 \\ d_2 &= 0.10487106 \\ &= 10.49\% \end{aligned}$$

The net price at the online store is \$62.48. The department store needs to offer an additional discount of 10.49% to match the net price of the online store.

TRY IT

A retailer purchases coats for a net price of \$75 after discounts of 20% and 5%. What was the list price of the coats?

Click to see Solution

$$\begin{array}{l} N = L \times (1-d_1) \times (1-d_2) \\ 75 = L \times (1-0.2) \times (1-0.05) \\ 75 = L \times 0.8 \times 0.95 \\ 75 = L \times 0.76 \\ \frac{75}{0.76} = L \\ \$98.68 = L \end{array}$$

TRY IT

What single discount is equivalent to the series of discounts 40%, 18% and 3%?

Click to see Solution

$$\begin{array}{l} d_e = 1 - \left[(1-d_1) \times (1-d_2) \times (1-d_3) \right] \\ = 1 - \left[(1-0.4) \times (1-0.18) \times (1-0.03) \right] \\ = 1 - (0.6 \times 0.82 \times 0.97) \\ = 1 - 0.47724 \\ = 0.52276 \\ = 52.276\% \end{array}$$

TRY IT

An electronics store sells a tablet for \$799 less discounts of 30% and 8%. An online store sells the same tablet for \$849 less a 25% discount. What additional discount does the online store need to have to match the price offered by the electronics store?

Click to see Solution

$$\begin{aligned}
 N &= L \times (1 - d_1) \times (1 - d_2) \\
 &= 799 \times (1 - 0.3) \times (1 - 0.08) \\
 &= 799 \times 0.7 \times 0.92 \\
 &= \$514.56
 \end{aligned}$$

$$\begin{aligned}
 N &= L \times (1 - d_1) \times (1 - d_2) \\
 &= 849 \times (1 - 0.25) \times (1 - d_2) \\
 &= 849 \times 0.75 \times (1 - d_2) \\
 &= 636.75 \times (1 - d_2) \\
 &= \frac{514.56}{636.75} \\
 &= 1 - d_2 \\
 d_2 &= 0.191896 \\
 &\approx 19.19\%
 \end{aligned}$$

Exercises

1. What is the net price of an item with a list price of \$980 after a discount of 42%? What is the total discount amount?

Click to see Answer

net price=\$568.40, discount amount=\$411.60

2. What is the list price of an item with a net price of \$366.05 and discounts of 18%, 4%, and 7%? What is

the total discount amount? What single discount rate is equivalent to the series of discounts given?

Click to see Answer

list price=\$500, discount amount=\$133.95, single discount rate=26.7904%

3. A wholesaler of stereos normally qualifies for a 35% trade discount on all electronic products purchased from its manufacturer. If the list price of a stereo is \$399.95, what net price will the wholesaler pay?

Click to see Answer

\$259.97

4. Mary is shopping at the mall where she sees a sign that reads, "Everything in the store is 30% off, including sale items!" She wanders in and finds a blouse on the clearance rack. A sign on the clearance rack states, "All clearance items are 50% off." If the blouse is normally priced at \$69.49, what price should Mary pay for it?

Click to see Answer

\$24.32

5. A distributor sells some shoes directly to a retailer. The retailer pays \$16.31 for a pair of shoes that has a list price of \$23.98. What trade discount percent is the distributor offering to its retailers?

Click to see Answer

31.99%

6. A retailer purchases supplies for its head office. If the retailer pays \$16.99 for a box of paper and was eligible for a 15% volume discount, what was the original list price for the box of paper?

Click to see Answer

\$19.99

7. Mountain Equipment Co-op has purchased a college backpack for \$29 after discounts of 30%, 8%, and 13%. What is the list price for the backpack? What single discount is equivalent to the three discounts?

Click to see Answer

list price=\$51.76, single discount rate=43.97%

8. Walmart purchased the latest CD recorded by Selena Gomez. It received a total discount of \$10.08 off the list price for the CD, which represents a discount percent of 42%.
 - a. What was the list price?
 - b. What was the net price paid for the CD?

Click to see Answer

a. \$24; b. \$13.92

9. Best Buy just acquired an HP Pavilion computer for its electronics department. The net price on the computer is \$260.40 and Best Buy receives discounts of 40% and 38%.
 - a. What single discount is equivalent to the two discounts?
 - b. What is the list price?
 - c. What is the total discount amount?

Click to see Answer

a. 62.8%; b. \$700; c. \$439.60

10. TELUS retails a Samsung cellphone at the list price of \$399.99. TELUS can purchase the phone from its supplier and receive a 20% trade discount along with a 5% volume discount.
 - a. What is the single equivalent percent discount?
 - b. What net price does TELUS pay for the phone?
 - c. How much of a discount in dollars does this represent?

Click to see Answer

a. 24%; b. \$303.99; c. \$96

11. A wholesaler offers the following discounts: 10% seasonal discount for all purchases made between March 1 and May 1, 15% cumulative quantity discount whenever more than 5,000 units are purchased in any month, 5% loyalty discount for customers who have made regular purchases every month for at least one year, and a 33% trade discount to any retailer. Ed's Retail Superstore makes a purchase of 200 watches, list price of \$10, from the wholesaler on April 29. This month alone, Ed's has ordered more than 5,000 watches. However, Ed's has purchased from the wholesaler for only the past six months. Determine the total price that Ed's should pay for the watches.

Click to see Answer

\$1,026

12. Sk8 is examining an invoice. The list price of a skateboard is \$109.00, and the invoice states it received a trade discount of 15% and quantity discount of 10% as well as a loyalty discount. However, the amount of the loyalty discount is unspecified.
- If Sk8 paid \$80.88 for the skateboard, what is the loyalty discount percent?
 - If the loyalty discount is applied after all other discounts, what amount of loyalty dollars does Sk8 save per skateboard?

Click to see Answer

a. 3%; b. \$2.50

13. Currently, a student can qualify for up to six different tuition discounts at a local college based on such factors as financial need or corporate sponsorships. Mary Watson just applied to the college and qualifies for all six discounts: 20%, 15%, 23%, 5%, 3%, and 1%.
- She is confused and wants the college to tell her what single discount percent she is receiving. What should the college tell her?
 - If her total list tuition comes to \$6,435.00, how much should she pay?

Click to see Answer

a. 52.23%; b. \$3,073.82

14. Sumandeep is very loyal to her local hairstylist. Because she is loyal, her hairstylist gives her three different discounts: 10%, 5%, and 5%. These discounts amount to \$14.08 in savings.
- What was the list price her hairstylist charged her?
 - What amount did she pay her hairstylist?
 - If her hairstylist increases prices by 5%, what are the list price, net price, and total discount amount?

Click to see Answer

a. \$74.99; b. \$60.91; c. list price=\$78.74, net price=\$63.96, discount amount=\$14.78

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1.2 INVOICES, TERMS OF PAYMENTS, AND CASH DISCOUNTS

LEARNING OBJECTIVES

- Solve problems involving cash discounts for full and partial payments.

In the world of business, most purchases are not paid for in cash. Instead, businesses tend to work through an invoicing system in which they send out bills to their clients on a regular basis. In accounting, this means that the purchase is placed into accounts receivables until such time as a cheque arrives and the purchase is converted to cash. Invoices provide detailed transaction information, listing the amount owed and also indicating the terms on which payment is expected. Companies may offer what are known as cash discounts as incentives for early invoice payment. The rationale for these discounts is simple—a sale is not a sale until you have the cash in hand. The longer an invoice remains in accounts receivable, the less likely that it will be paid. Thus, it might turn into bad debt, which the creditor cannot collect at all.

Invoicing is less common in consumer purchases because of the sheer volume of transactions involved and the higher risk of nonpayment. Imagine purchasing items at Walmart and receiving an invoice to pay your bill next month instead of paying cash. It is hard to fathom the number of invoices Walmart would have to distribute monthly. How much would it cost to collect those debts? How many of those invoices would go unpaid? That is why consumer purchases typically do not involve invoicing.

Invoicing does commonly occur at a consumer level on credit card transactions, along with many services where the business may not be able to assess the exact amount of the bill at the time of the transaction or until the service is delivered. Two examples illustrate this point:

- Think of your MasterCard bill. You are able to make purchases, say, from March 9 to April 8. Then on April 9 the company sends out a statement saying you have until April 29 to pay your bill. If you do not,

interest and late payment penalties are involved.

- You have a dental visit for a regular cleaning. Before charging you, the dentist's office needs to determine how much your insurance will cover. It may not find out the answer for a day or two, so it sends you an invoice at a later date once it hears from the insurance company. The invoice terms indicate that payment is due upon receipt and you will incur late penalties if payment is not forthcoming.

This section explores the most common aspects of invoicing, including terms of payments and cash discounts. You will learn to calculate the amount required to pay invoices and how to reduce the outstanding balance of an invoice if a partial payment is received.

Invoice Terms

You must know how to read a business invoice. A typical invoice includes the following information:

- **Invoicing Company.** The invoice must identify who is sending and issuing the invoice.
- **Invoice Date.** The date on which the invoice was printed, along with the invoice tracking number. When an invoice is paid, the cheque must reference the invoice number so that the invoicing company can identify which invoice to credit the payment to.
- **Transaction Details.** The details of the transaction might include number of units, unit prices, and any discounts for which the item is eligible.
- **Invoice Total.** The total amount owing is indicated, including any taxes or additional charges.
- **Terms of Payment.** The terms of payment include any cash discounts and due dates. The date of commencement is determined from this part of the invoice.
- **Late Penalty.** A penalty, if any, for late payments is indicated on the invoice. Whether or not a company enforces these late penalties is up to the invoice-issuing organization.

Dates of Commencement

All invoice terms are affected by what is known as the **date of commencement**, which is the first day from which all due dates stated on the invoice will stem. The date of commencement is determined in one of three ways, as illustrated below.

- **Ordinary Dating.** In **ordinary invoice dating**, or just invoice dating for short, the date of commencement is the same date as the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on March 19. This is the default manner in which most companies issue their invoices, so if an invoice does not specify any other date of commencement you can safely

assume it is using ordinary dating.

- **End-of-Month Dating (EOM).** **End-of-month invoice dating** applies when the terms of payment include the wording “end-of-month” or the abbreviation “EOM” appears after the terms of payment. In end-of-month dating, the date of commencement is the last day of the same month as indicated by the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on the last day of March, or March 31. Many companies use this method of dating to simplify and standardize all of their due dates, in that if the date of commencement is the same for all invoices, then any terms of payment will also share the same dates.
- **Receipt-of-Goods Dating (ROM).** **Receipt-of-goods invoice dating** applies when the terms of payment include the wording “receipt-of-goods” or the abbreviation “ROG” appears after the terms of payment. In receipt-of-goods dating, the date of commencement is the day on which the customer physically receives the goods. Therefore, if the invoice is printed on March 19 but the goods are not physically received until April 6, then all terms of payment commence on April 6. Companies with long shipping times involved in product distribution or long lead times in production commonly use this method of dating.

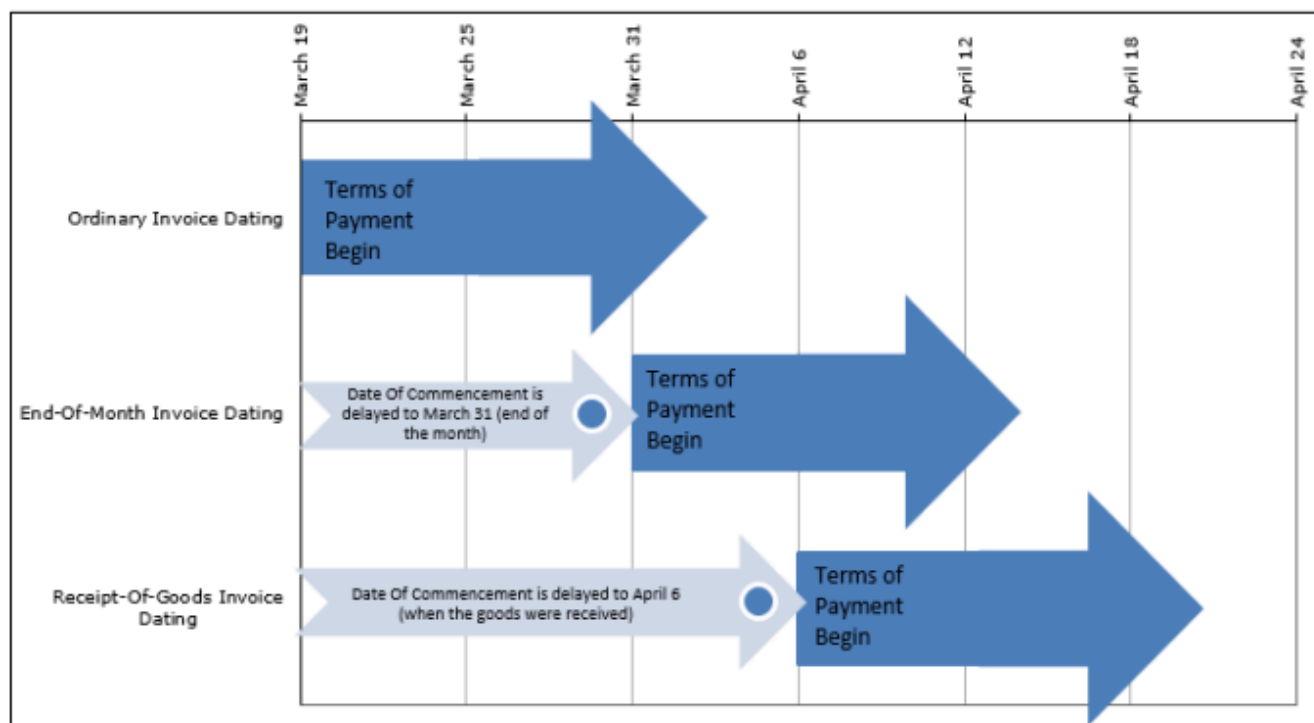


Figure 1.2.1

NOTE

1. In this section, we will focus on ordinary dating only.
2. If an invoice does not specifically state EOM or ROM, it is assumed to use ordinary dating.

Terms of Payment

The most common format for the terms of paying an invoice are

3/10, n /30

where

- **3 is the cash discount.** A cash discount is the percentage of the balance owing on an invoice that can be deducted for payment received either in full or in part during the discount period. In this case, 3% is deducted from the total invoice amount.
- **10 is the discount period.** The discount period is the number of days from the date of commencement in which the customer is eligible to take advantage of the cash discount. For this example, there is a period of 10 days during which a payment received in full or any portion thereof is eligible for the 3% cash discount.
- **n /30 is the credit period.** The credit period is the number of interest-free days from the date of commencement that the customer has to pay the invoice in full before the creditor applies any penalties to the invoice.

EXAMPLE

An invoice dated May 7 has payment terms **5/10, n /30**.

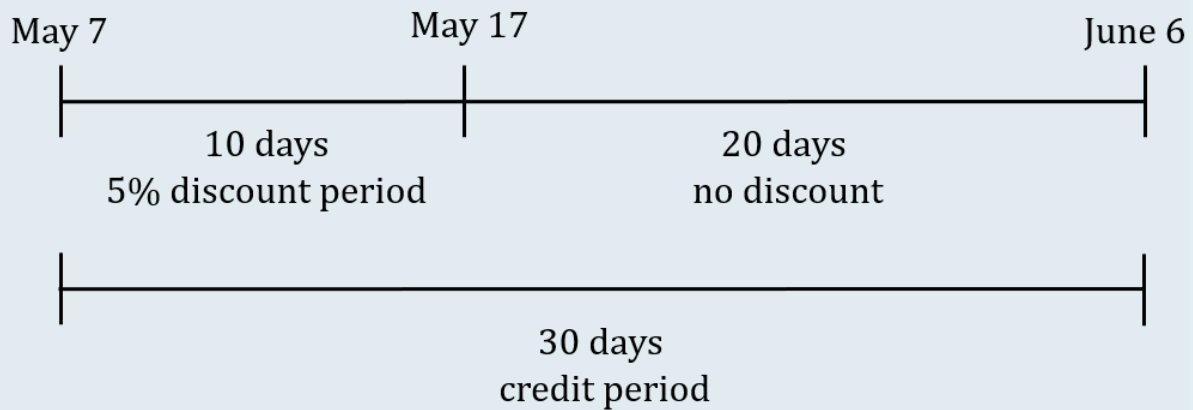


Figure 1.2.2

- Using ordinary dating, the terms of payment begin on May 7.
- For the first 10 days, there is a 5% discount period. Any payments made during this period receive a 5% discount. This discount period ends on May 17 (10 days after the invoice date).
- After May 17, there is no discount period. Any payments made during this period receive no discount.
- The terms of payment end 45 days after the date of the invoice on June 6.

EXAMPLE

An invoice dated May 7 has payment terms $5/10, 2/25, n/45$.

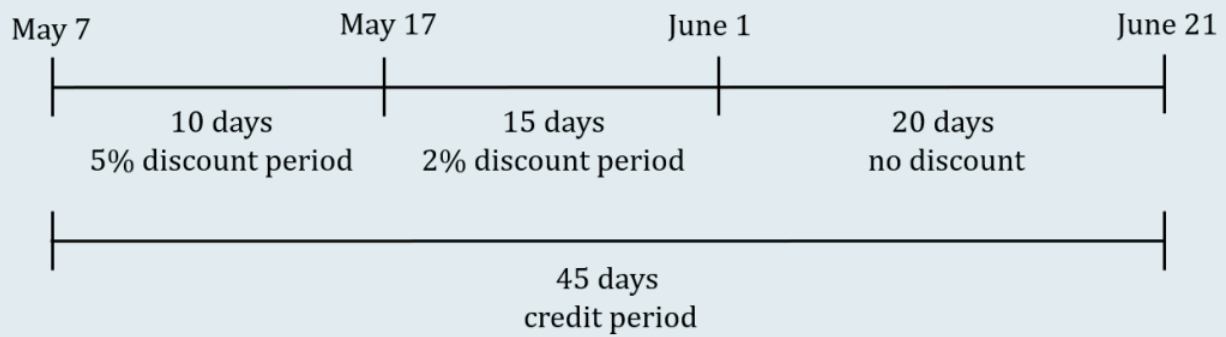


Figure 1.2.3

- Using ordinary dating, the terms of payment begin on May 7.
- For the first 10 days, there is a 5% discount period. Any payments made during this period receive a 5% discount. This discount period ends on May 17 (10 days after the invoice date). During this period, only the larger discount of 5% is applied.
- For the next 15 days, there is a 2% discount period. Any payments made during this period receive a 2% discount. This discount period ends on June 1 (25 days after the invoice date).
- After June 1, there is no discount period. Any payments made during this period receive no discount.
- The terms of payment end 45 days after the date of the invoice on June 6.

NOTE

When there are multiple discount periods, as in the above example, only one discount applies at a time. The second discount period begins after the first discount period.

Some common business practices are implemented across most industries. Note that you must always check with the invoicing organization to ensure it is applying these practices.

- **No Net Figure.** If no net figure is stated in the terms of payment, you should assume that the credit period is 20 days after the last discount period. If there are no discount periods, then the credit period is

20 days from the date of commencement. For example, **3/10, 2/15** means that the credit period ends 20 days after the second discount period of 15 days. Hence, the credit period is 35 days after the date of commencement.

- **No Cash Discount.** Not every invoice receives a cash discount. If no terms indicate a cash discount, then the invoicing company seeks full payment only. For example, **n/30** means that no cash discount applies and the credit period ends 30 days from the date of commencement.
- **Nonbusiness Days.** Most businesses operate Monday through Friday and are closed on weekends and holidays. As such, any date falling on a nonworking day is moved to the next business day. For example, if an invoice is ordinary dated December 21 and lists a cash discount of **2/10**, the end of the discount period falls on January 1. Because this is New Year's Day, the discount period extends to January 2. In this textbook, you should apply this practice to any of the five known statutory holidays (New Year's Day, Good Friday or Easter Monday in Quebec only, Canada Day, Labor Day, and Christmas Day).

Types of Payments

When businesses pay invoices, three situations can occur:

1. **Full Payment.** A full payment means that the company wants to pay its invoice in full and reduce its balance owing to zero dollars. This is the most common practice.
2. **Partial Payment.** A partial payment means that the company wants to lower its balance owing but will not reduce that balance to zero. A company will generally employ this method when it wants to either lower its accounts payable or demonstrate good faith in paying its invoices. Partial payment may also occur if the company wants to take advantage of a cash discount but lacks the funds to clear the invoice in its entirety.
3. **Late Payment.** A late payment occurs when the company pays its invoice either in full or partially after the credit period has elapsed. Late payments occur for a variety of reasons, but the most common are either insufficient funds to pay the invoice or a simple administrative oversight.

Full Payments

The good news is that a cash discount is just another type of discount similar to what you encountered in the previous section, and you do the calculations using the exact same formula.

$$\text{Amount Paid} = \text{Amount Credited} \times (1 - d)$$

where

- **Amount Credited** is the amount deducted from the invoice balance as a result of the payment. It is what the payment is worth once the cash discount has been factored into the calculation.
- **Amount Paid** is the actual payment amount made that must be converted into its value toward the invoice total. Because the payment represents an amount after the discount has been removed, you need to find out what it is worth before the cash discount was removed.
- ***d* is the cash discount rate.** Any payment received within any discount period, whether in full or partial, is eligible for the specified cash discount indicated in the terms of payment.

EXAMPLE

An invoice for \$5,000 is dated May 7 with payment terms **5/10, 2/25, n/45**.

1. What amount paid on May 15 will clear the invoice?
2. What amount paid on May 31 will clear the invoice?
3. What amount paid on June 10 will clear the invoice?

Solution:

The timeline for this invoice is

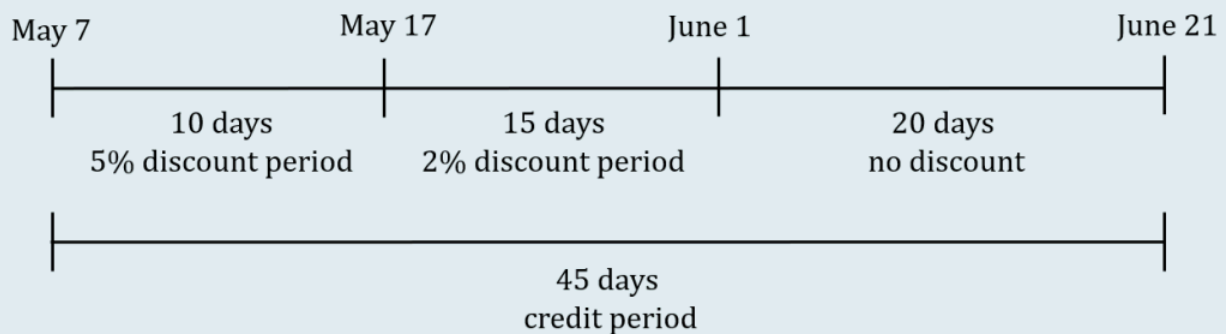


Figure 12.4

Step 1: May 15 is during the first discount period, so the payment qualifies for the 5% discount.

$$\begin{aligned}
 \text{Amount Paid} &= \text{Amount Credited} \times (1 - d) \\
 &= 5,000 \times (1 - 0.05) \\
 &= 5,000 \times 0.95 \\
 &= \$4,750
 \end{aligned}$$

On May 15, a payment of \$4,750 will clear the invoice.

Step 2: May 31 is during the second discount period, so the payment qualifies for the 2% discount.

$$\begin{aligned}
 \text{Amount Paid} &= \text{Amount Credited} \times (1 - d) \\
 &= 5,000 \times (1 - 0.02) \\
 &= 5,000 \times 0.98 \\
 &= \$4,900
 \end{aligned}$$

On May 31, a payment of \$4,900 will clear the invoice.

Step 3: A payment on June 10 occurs during the no discount period. So a payment of \$5,000 will clear the invoice on June 10.

NOTE

If there is one area of invoicing that causes the most confusion, it is assigning information to the net price and invoice amount variables. These are commonly assigned backwards. Remember these rules so that you always get the correct answer.

1. The invoice amount or any reference to the balance owing on an invoice is *always* a list price.

$$\text{List Price} = \text{Invoice Amount} = \text{Amount Credited}$$

2. The payment of an invoice, whether in full, partial, or late, is *always* a net price.

$$\text{Net Price} = \text{Payment Amount} = \text{Amount Paid}$$

TRY IT

You receive an invoice for \$35,545.50 dated August 14. The terms of payment are listed as **3/10, 1/20, n/30**.

1. What amount paid on August 24 will clear the invoice?
2. What amount paid on September 3 will clear the invoice?
3. What amount paid on September 13 will clear the invoice?

Click to see Solution

1. August 24 is the last day of the first discount period.

$$\begin{aligned}
 \text{Amount Paid} &= \text{Amount Credited} \times (1 - d) \\
 &= 35,545.50 \times (1 - 0.03) \\
 &= 35,545.50 \times 0.97 \\
 &= \$34,479.14
 \end{aligned}$$

On August 24, a payment of \$34,479.14 will clear the invoice.

2. September 3 is the last day of the second discount period.

$$\begin{aligned}
 \text{Amount Paid} &= \text{Amount Credited} \times (1 - d) \\
 &= 35,545.50 \times (1 - 0.01) \\
 &= 35,545.50 \times 0.99 \\
 &= \$35,190.05
 \end{aligned}$$

On September 3, a payment of \$35,190.05 will clear the invoice.

3. September 13 is the last day of the credit period when there is no discount. A payment of \$35,545.50 will clear the invoice on September 13.

Who cares about 1% or 2% discounts when paying bills? While these percentages may not sound like a lot, remember that these discounts occur over a very short time frame. For example, assume you just received an invoice for \$102.04 with terms of **2/10, n/30**. When taking advantage of the 2% cash discount, you must pay the bill 20 days early, resulting in a \$100 payment. That is a \$2.04 savings over the course of 20 days.

To understand the significance of that discount, imagine that you had \$100 sitting in a savings account at your bank. Your savings account must have a balance of \$102.04 twenty days later. This requires your savings account to earn an annual interest rate of 44.56%! Therefore, a 20-day 2% discount is the same thing as earning interest at 44.56%.

Partial Payments

Suppose you receive a \$21,000 invoice with payment terms $2/15, 1/25, n/60$. You do not have sufficient funds to take full advantage of the 2% discount period. Not wanting to lose out entirely, you decide to submit a partial payment of \$10,000 before the first discount period elapses. What is the balance remaining on the invoice after you make this \$10,000 payment?

Unless you pay attention to invoicing concepts, it is easy to get confused. Perhaps you think that \$10,000 should be removed from the \$21,000 invoice total, thereby leaving a balance owing of \$11,000. Or maybe you think the payment should receive the discount of 2%, which would be \$200. Are you credited with \$9,800 off of your balance? Or maybe \$10,200 off of your balance? In all of these scenarios, you would be committing a serious mistake and miscalculating your balance owing. Let's look at the correct way of handling this payment.

Recall that

$$\text{Amount Paid} = \text{Amount Credited} \times (1 - d)$$

Rearranging this formula

$$\text{Amount Credited} = \frac{\text{Amount Paid}}{1 - d}$$

where

- **Amount Credited** is the amount deducted from the invoice balance as a result of the payment. It is what the payment is worth once the cash discount has been factored into the calculation.
- **Amount Paid** is the actual payment amount made that must be converted into its value toward the invoice total. Because the payment represents an amount after the discount has been removed, you need to find out what it is worth before the cash discount was removed.
- **d is the cash discount rate.** Any payment received within any discount period, whether in full or partial, is eligible for the specified cash discount indicated in the terms of payment.

EXAMPLE

You receive a \$21,000 invoice with payment terms **2/15, 1/25, n/60**. You submit a partial payment of \$10,000 during the first discount period. What is the balance remaining on the invoice after you make this \$10,000 payment?

Solution:

The timeline for this invoice is

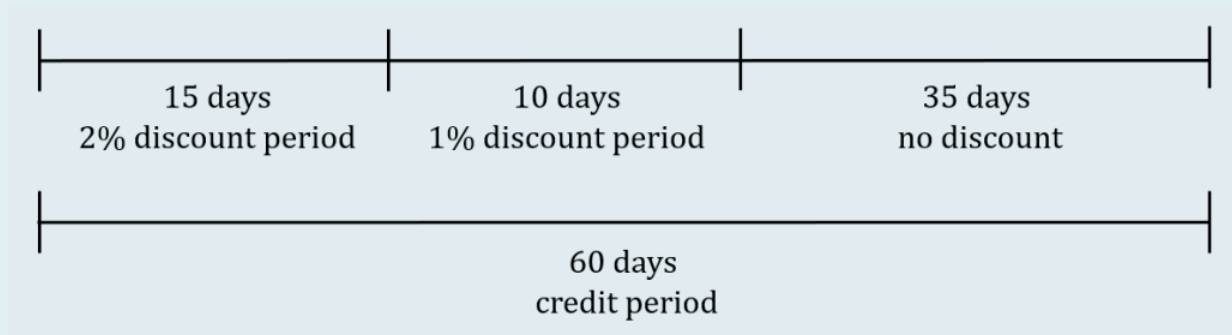


Figure 1.2.5

Step 1: The given information is

$$\begin{aligned}\text{Amount Paid} &= \$10,000 \\ d &= 2\%\end{aligned}$$

Step 2: Calculate the amount credited.

$$\begin{aligned}\text{Amount Credited} &= \frac{\text{Amount Paid}}{1 - d} \\ &= \frac{10,000}{1 - 0.02} \\ &= \frac{10,000}{0.98} \\ &= \$10,204.08\end{aligned}$$

When the \$10,000 payment is made during the first discount period, \$10,204.08 will be credited against the invoice.

Step 3: Calculate the balance on the invoice.

$$\begin{aligned}\text{Balance} &= 21,000 - 10,204.08 \\ &= \$10,795.92\end{aligned}$$

The balance on the invoice is \$10,795.92.

NOTE

In the case where a payment falls within a cash discount period, the amount credited toward an invoice total is always larger than the actual payment amount. If the payment does not fall within any discount period, then the amount credited toward an invoice total is equal to the actual payment amount (because there is no cash discount).

To help you understand why a partial payment works in this manner, assume an invoice is received in the amount of \$103.09 and the customer pays the invoice in full during a 3% cash discount period. What amount is paid? The answer is

Amount Paid = $103.09 \times (1 - 0.03) = 100$. Therefore, any payment of \$100 made during a 3% cash discount period is always equivalent to a pre-discount invoice credit of \$103.09. If the balance owing is more than \$103.09, that does not change the fact that the \$100 payment during the discount period is worth \$103.09 toward the invoice balance.

EXAMPLE

Heri just received an invoice from his supplier for \$68,435.27 and is dated June 5 with terms of **2.5/10, 1/25, n/45**. Heri sent two partial payments in the amounts of \$20,000 on June 15 and \$30,000 on June 29. He wants to clear his invoice by making a final payment on July 18. What is the amount of the final payment?

Solution:

The timeline for the invoice is

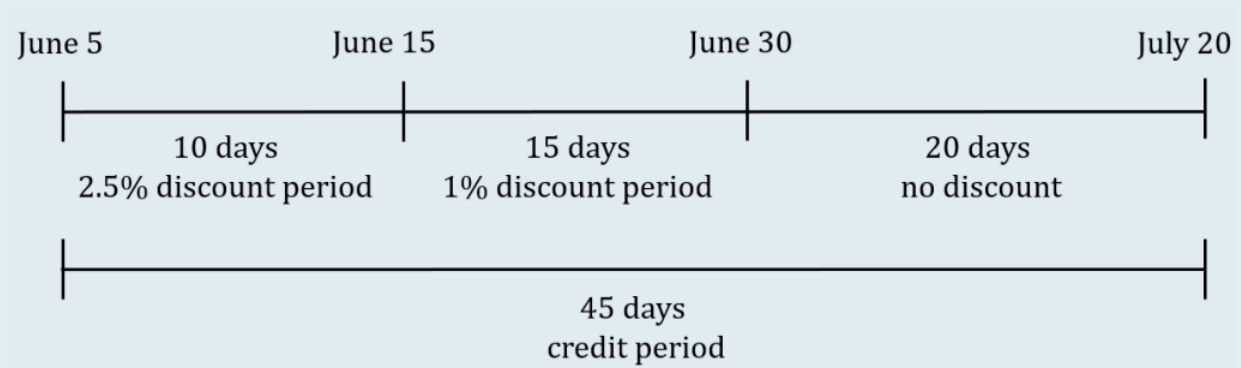


Figure 1.2.6

Step 1: Calculate the amount credited for the \$20,000 payment. The \$20,000 payment on June 15 occurs during the first discount period.

$$\begin{aligned}
 \text{Amount Credited} &= \frac{\text{Amount Paid}}{1 - d} \\
 &= \frac{20,000}{1 - 0.025} \\
 &= \frac{20,000}{0.975} \\
 &= \$20,512.82
 \end{aligned}$$

Step 2: Calculate the amount credited for the \$30,000 payment. The \$30,000 payment on June 29 occurs during the second discount period.

$$\begin{aligned}
 \text{Amount Credited} &= \frac{\text{Amount Paid}}{1 - d} \\
 &= \frac{30,000}{1 - 0.01} \\
 &= \frac{30,000}{0.99} \\
 &= \$30,303.03
 \end{aligned}$$

Step 3: Calculate the balance on the invoice after the two payment are made.

$$\begin{aligned}
 \text{Balance} &= 68,435.27 - 20,512.82 - 30,303.03 \\
 &= \$17,619.42
 \end{aligned}$$

The first two payments receive a credit of \$20,512.82 and \$30,303.03 toward the invoice total, respectively. This leaves a balance on the invoice of \$17,619.42. Because the final payment is made during the credit period, but not within any cash discount period, the final payment is the exact amount of the balance owing and equals \$17,619.42.

TRY IT

An invoice of \$4,700 has payment terms $4/15, n/45$. A payment of \$2,500 was made during the discount period. What payment made after the discount period will clear the invoice?

Click to see Solution

$$\begin{aligned}\text{Amount Credited} &= \frac{\text{Amount Paid}}{1 - d} \\ &= \frac{2,500}{1 - 0.04} \\ &= \frac{2,500}{0.96} \\ &= \$2,604.17\end{aligned}$$

$$\begin{aligned}\text{Balance} &= 4,700 - 2,604.17 \\ &= \$2,905.83\end{aligned}$$

A payment of \$2,905.83 after the discount period will clear the invoice.

TRY IT

An invoice for \$50,000 is dated April 7 with payment terms **3.5/10, 2/30, n/60**. A payment of \$15,000 is made on April 15 and another payment of \$9,000 is made on May 6. What amount must be paid on June 6 to clear the invoice? What is the total amount paid to clear the invoice?

Click to see Solution

1. Calculate the amount credited on April 15.

$$\begin{aligned}
 \text{Amount Credited} &= \frac{\text{Amount Paid}}{1 - d} \\
 &= \frac{15,000}{1 - 0.035} \\
 &= \frac{15,000}{0.965} \\
 &= \$15,544.04
 \end{aligned}$$

2. Calculate the amount credited on May 6.

$$\begin{aligned}
 \text{Amount Credited} &= \frac{\text{Amount Paid}}{1 - d} \\
 &= \frac{9,000}{1 - 0.02} \\
 &= \frac{9,000}{0.98} \\
 &= \$9,183.67
 \end{aligned}$$

3. Calculate the payment on June 6.

$$\begin{aligned}
 \text{Payment} &= 50,000 - 15,544.04 - 9,183.67 \\
 &= \$25,317.29
 \end{aligned}$$

4. Calculate the total amount paid.

$$\begin{aligned}\text{Total Paid} &= 15,000 + 9,000 + 25,317.29 \\ &= \$49,317.29\end{aligned}$$

Exercises

1. An invoice of \$136,294.57 is dated January 14 with terms $2/10, n/30$. What payment on January 22 will clear the invoice?

Click to see Answer

\$133,568.68

2. An invoice of \$57,775 is dated June 16 with terms $3/15, 1.5/25, n/45$. A payment of \$20,000 is made on July 2 and a payment of \$15,000 is made on July 10. Calculate the remaining balance on the invoice.

Click to see Answer

\$21,928.01

3. A company received an invoice dated July 13 in the amount of \$206,731.75. The terms are $2/10, 1/20, n/30$. If the full payment is made on August 1, what amount is paid?

Click to see Answer

\$204,664.43

4. Time Bomb Traders Inc. received an invoice dated March 27 in the amount of \$36,448.50 with terms $1.5/15, 0.5/30, n/45$.

- What amount is the full payment if the payment is made on April 4?
- What amount is the full payment if the payment is made on April 20?
- What amount is the full payment if the payment is made on May 1?

Click to see Answer

a. \$37,003.55; b. \$36,631.66; c. \$36,448.50

5. An invoice for \$23,694.50 dated April 13 has payment terms $3/15, 2/25, n/60$. A \$10,000 payment was made on April 13 and a payment of \$5,000 was made on May 3. The final payment to clear the invoice was made on May 31.
- Calculate the final payment.
 - What was the total amount paid to clear the invoice?

Click to see Answer

a. \$8,283.18; b. \$23,283.18

6. Family Foods received an invoice dated July 2 from Kraft Canada in the amount of \$13,002.96 with terms of $2/15, 1/30, n/45$. Payments of \$4,000 are made on July 17 and August 1. What payment amount is required to pay the remaining balance on the invoice on August 10?

Click to see Answer

\$4,880.92

7. An invoice dated November 6 in the amount of \$38,993.65 with terms $2.5/10, 1/25, n/45$ is received by your company. You make a partial payment of \$15,000 on November 15 and a partial payment of \$20,000 on November 30. You make a final payment on December 10.
- Calculate the final payment.
 - What was the total amount paid to clear the invoice?

Click to see Answer

a. \$3,407.01; b. \$38,407.01

8. An invoice for \$100,000 has payment terms $3.5/10, n/30$. What payment must be made during the discount period to reduce the balance on the invoice to \$60,000?

Click to see Answer

\$38,600

9. An invoice for \$75,000 has payment terms $2/15, n/30$. What payment must be made during the

discount period to reduce the balance on the invoice by \$20,000?

Click to see Answer

\$19,600

10. You receive an invoice dated August 20 for your office supplies totaling \$10,235.97. The terms of payment on invoice are $3.75/15, 1.5/30, n/45$. You make three equal payments of \$2,000 on August 25, September 4 and September 19. What payment on September 29 will settle the invoice?

Click to see Answer

\$4,049.67

11. An invoice for \$100,000 dated March 2 has terms of $4/10, 3/20, 2/30, 1/40, n/60$. If four equal payments are made on March 10, March 17, March 31, and April 5 resulting in full payment of the invoice, calculate the amount of each payment.

Click to see Answer

\$24,371.79

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1.3 MARKUP

LEARNING OBJECTIVES

- Calculate the selling price, cost, expenses, or profit of an item.
- Calculate the amount of markup of an item.
- Calculate the rates of markup on an item.

As you wait in line to purchase your Iced Caramel Macchiato at Starbucks, you look at the pricing menu and think that \$4.99 seems like an awful lot of money for a frozen coffee beverage. Clearly, the coffee itself does not cost anywhere near that much. But then gazing around the café, you notice the carefully applied color scheme, the comfortable seating, the high-end machinery behind the counter, and a seemingly well-trained barista who answers customer questions knowledgeably. Where did the money to pay for all of this come from? You smile as you realize your \$4.99 pays not just for the macchiato, but for everything else that comes with it.

The process of taking a product's cost and increasing it by some amount to arrive at a selling price is called **markup**. This process is critical to business success because every business must ensure that it does not lose money when it makes a sale. From the consumer perspective, the concept of markup helps you make sense of the prices that businesses charge for their products or services. This in turn helps you to judge how reasonable some prices are (and hopefully to find better deals).

The Selling Price

Before you learn to calculate markup, you first have to understand the various components of a selling price. When your business acquires merchandise for resale, this is a monetary outlay representing a cost. When you then resell the product, the price you charge must recover more than just the product cost. You must also recover all the selling and operating expenses associated with the product. Ultimately, you also need to make some money, or profit, as a result of the whole process.

Most people think that marking up a product must be a fairly complex process. The relationship between the three components of cost (C), expenses (E), and profits (P) and the selling price (S) is given by the following formula

$$S = C + E + P$$

where

- **S is the selling price.** Once you calculate what the business paid for the product (cost), the bills it needs to cover (expenses), and how much money it needs to earn (profit), you arrive at a selling price by summing the three components.
- **C is the cost.** The cost is the amount of money that the business must pay to purchase or manufacture the product. If manufactured, the cost represents all costs incurred to make the product. If purchased, this number results from applying an appropriate discount, as we learned in a previous section. There is a list price from which the business will deduct discounts to arrive at the net price. The net price paid for the product equals the cost of the product. If a business purchases or manufactures a product for \$10 then it must sell the product for at least \$10. Otherwise, it fails to recover what was paid to acquire or make the product in the first place—a path to sheer disaster!
- **E is the expenses.** Expenses are the financial outlays involved in selling the product. Beyond just purchasing the product, the business has many more bills to pay, including wages, taxes, leases, equipment, electronics, insurance, utilities, fixtures, décor, and many more.
- **P is the profit.** Profit is the amount of money that remains after a business pays all of its costs and expenses. A business needs to add an amount above its costs and expenses to allow it to grow. If it adds too much profit, though, the product's price will be too high, in which case the customer may refuse to purchase it. If it adds too little profit, the product's price may be too low, in which case the customer may perceive the product as shoddy and once again refuse to purchase it. Many businesses set general guidelines on how much profit to add to various products.

NOTES

1. The expenses must be recovered and may be calculated as:
 - A fixed dollar amount per unit.
 - A percentage of the product cost. For example, if a business forecasts total

merchandise costs of \$100,000 for the coming year and total business expenses of \$50,000, then it may set a general guideline of adding 50% ($\$50,000 \div \$100,000$) to the cost of a product to cover expenses.

- A percentage of the product selling price based on a forecast of future sales. For example, if a business forecasts total sales of \$250,000 and total business expenses of \$50,000, then it may set a general guideline of adding 20% ($\$50,000 \div \$250,000$) of the selling price to the cost of a product to cover expenses.

2. As with expenses, the profit may be expressed as:

- A fixed dollar amount per unit.
- A percentage of the product cost.
- A percentage of the selling price.

EXAMPLE

Suppose a business pays \$75 to acquire a product. Through analyzing its finances, the business estimates expenses at \$25 per unit and it figures it can add \$50 in profit. Calculate the selling price.

Solution:

Step 1: The given information is

$$C = \$75$$

$$E = \$25$$

$$P = \$50$$

Step 2: Calculate the selling price per unit.

$$\begin{aligned} S &= C + E + P \\ S &= \$75 + \$25 + \$50 \\ S &= \$150 \end{aligned}$$

The selling price per unit is \$150.

NOTE

The most common mistake in working with pricing components occurs in identifying and labeling the information correctly. It is critical to identify and label information correctly. You have to pay attention to details such as whether you are expressing the expenses in dollar format or as a percentage of either cost or selling price. Systematically work your way through the information provided piece by piece to ensure that you do not miss an important detail.

EXAMPLE

Mary's Boutique purchases a dress for resale at a cost of \$23.67. The owner determines that each dress must contribute \$5.42 to the expenses of the store. The owner also wants this dress to earn \$6.90 toward profit. What is the regular selling price for the dress?

Solution:

Step 1: The given information is

$$C = \$23.67$$

$$E = \$5.42$$

$$P = \$6.90$$

Step 2: Calculate the selling price per unit.

$$\begin{array}{l} S \\ \hline S = C + E + P \\ S = 23.67 + 5.42 + 6.90 \\ S = 35.99 \end{array}$$

The selling price of the dress is \$35.99.

EXAMPLE

John's Discount Store just completed a financial analysis. The company purchases a product for \$19.99 less a trade discount of 45%. The company determined that expenses average 20% of the product cost and profit averages 15% of the product cost. What will be the regular selling price for the product?

Solution:

Step 1: The given information is

$$L = \$19.99$$

$$d = 45\%$$

$$E = 20\% \text{ of cost}$$

$$= 0.2 \times C$$

$$P = 15\% \text{ of cost}$$

$$= 0.15 \times C$$

Step 2: Calculate the cost per unit. The cost equals the net price.

$$\begin{aligned} C &= L \times (1-d) \\ &= 19.99 \times (1-0.45) \\ &= 19.99 \times 0.55 \\ &= \$10.99 \end{aligned}$$

Step 3: Calculate the expenses.

$$E = 0.2 \times C$$

$$= 0.2 \times 10.99$$

$$= \$2.20$$

Step 4: Calculate the profit.

$$P = 0.15 \times C$$

$$= 0.15 \times 10.99$$

$$= \$1.65$$

Step 5: Calculate the selling price per unit.

$$\begin{aligned} S &= C + E + P \\ &= 10.99 + 2.20 + 1.65 \\ &= \$14.84 \end{aligned}$$

The selling price of the product is \$14.84.

EXAMPLE

Based on last year's results, Benthall Appliance learned that its expenses average 30% of the regular selling price. It wants a 25% profit based on the selling price. If Benthall Appliance purchases a fridge for \$1,200, what is the regular unit selling price?

Solution:

Step 1: The given information is

$$\begin{array}{l} C = \$1,200 \\ E = 30\% \text{ of selling price} \\ P = 0.3 \times S \\ P = 25\% \text{ of selling price} \\ P = 0.25 \times S \end{array}$$

Step 2: Calculate the selling price per unit.

$$\begin{array}{l} S = C + E + P \\ S = 1,200 + 0.3 \times S + 0.25 \times S \\ S = 1,200 + (0.3 + 0.25) \times S \\ S = 1,200 + 0.55 \times S \\ S - 0.55 \times S = 1,200 \\ (1 - 0.55) \times S = 1,200 \\ 0.45 \times S = 1,200 \\ S = \frac{1,200}{0.45} \\ S = \$2,666.67 \end{array}$$

The selling price of the fridge is \$2,666.67.

EXAMPLE

If a company knows that its profits are 15% of the selling price and expenses are 30% of cost, what is the cost of an MP3 player that has a regular selling price of \$39.99?

Solution:

Step 1: The given information is

$$\begin{aligned} S &= \$39.99 \\ E &= 30\% \text{ of cost} \\ &= 0.3 \times C \\ P &= 15\% \text{ of selling price} \\ &= 0.15 \times S \end{aligned}$$

Step 2: Calculate the profit.

$$\begin{aligned} P &= 0.15 \times S \\ &= 0.15 \times 39.99 \\ &= \$6.00 \end{aligned}$$

Step 3: Calculate the cost per unit.

$$\begin{aligned} S &= C + E + P \\ 39.99 &= C + 0.3 \times C + 6 \\ &= (1 + 0.3) \times C + 6 \\ 39.99 &= 1.3 \times C + 6 \\ 33.99 &= 1.3 \times C \\ \frac{33.99}{1.3} &= C \\ C &= \$26.15 \end{aligned}$$

The cost of the MP3 player is \$26.15.

TRY IT

A company purchases a product to sell in its stores for \$50. The company has expenses that are

27% of the selling price and profit that is 13% of the selling price. What is the selling price of the product?

Click to see Solution

$$\begin{array}{l} S = C + E + P \\ S = 50 + 0.27S + 0.13S \\ S = 50 + (0.27 + 0.13)S \\ S = 50 + 0.4S \\ S - 0.4S = 50 \\ 0.6S = 50 \\ S = \frac{50}{0.6} \\ S = \$83.33 \end{array}$$

TRY IT

A retailer sells a pair of shoes for \$75. The retailer's expenses are 40% of cost and profit is 25% of cost. What is the cost of the shoes?

Click to see Solution

$$\begin{array}{l} S = C + E + P \\ 75 = C + 0.4C + 0.25C \\ 75 = (1 + 0.4 + 0.25)C \\ 75 = 1.65C \\ C = \frac{75}{1.65} \\ C = \$45.45 \end{array}$$

Markup

Most companies sell more than one product, each of which has different price components with varying costs, expenses, and profits. Can you imagine trying to compare 50 different products, each with three different components? You would have to juggle 150 numbers! To make merchandising decisions more manageable and

comparable, many companies combine expenses and profit together into a single quantity, called the **markup** (M).

One of the most basic ways a business simplifies its merchandising is by combining the dollar amounts of its expenses and profits together into the markup. The amount of markup is

$$M = E + P$$

where

- **M is the markup amount.** Markup is taking the cost of a product and converting it into a selling price. The markup amount represents the dollar amount difference between the cost and the selling price.
- **E is the expenses.** The expenses associated with the product.
- **P is the profit.** The profit earned when the product sells.

Because the markup amount represents the expenses and profit combined, you can substitute M into the selling price formula.

$$S = C + M$$

where

- **S is the selling price.** The regular selling price of the product.
- **C is the cost.** The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.
- **M is the markup.** The markup is the single number that represents the total of the expenses and profit.

EXAMPLE

Recall from the MP3 player example above, that the expenses are \$7.84, the profit is \$6.00 and the cost is \$26.15. Calculate the markup and the selling price.

Solution:**Step 1:** The given information is

$$\begin{array}{l} C = \$26.15 \\ E = \$7.84 \\ P = \$6.00 \end{array}$$

Step 2: Calculate the markup.

$$\begin{aligned} M &= E + P \\ &= 7.84 + 6 \\ &= \$13.84 \end{aligned}$$

Step 3: Calculate the selling price.

$$\begin{array}{l} S = C + M \\ = 26.15 + 13.84 \\ = \$39.99 \end{array}$$

The MP3 player is marked up by \$13.84 and the selling price is \$39.99.

EXAMPLE

A cellular retail store purchases an iPhone with a list price of \$779 less a trade discount of 35% and volume discount of 8%. The store sells the phone at the list price.

1. What is the markup amount?
2. If the store knows that its expenses are 20% of the cost, what is the store's profit?

Solution:**Step 1:** The given information is

$$L = \$779$$

$$d_1 = 35\%$$

$$d_2 = 8\%$$

$$S = \$779$$

$$E = 20\% \text{ of cost} \\ = 0.2 \times C$$

Step 2: Calculate the cost per unit. The cost is the net price.

$$\begin{aligned} C &= L \times (1 - d_1) \times (1 - d_2) \\ &= 779 \times (1 - 0.35) \times (1 - 0.08) \\ &= 779 \times 0.65 \times 0.92 \\ &= \$465.84 \end{aligned}$$

Step 3: Calculate the markup.

$$\begin{aligned} S &= C + M \\ 779 &= 465.84 + M \\ M &= \$313.16 \end{aligned}$$

Step 4: Calculate the expenses.

$$\begin{aligned} E &= 0.2 \times C \\ &= 0.2 \times 465.84 \\ &= \$93.17 \end{aligned}$$

Step 5: Calculate the profit.

$$\begin{aligned} M &= E + P \\ 313.16 &= 93.17 + P \\ 313.16 - 93.17 &= P \\ \$219.99 &= P \end{aligned}$$

The markup amount on the iPhone is \$313.16. When the store sells the iPhone for \$779, the profit is \$219.99.

EXAMPLE

An office supply store purchases a desk chair to sell in its store. The store marks up the chair by \$250. The store's expenses are 25% of the cost and the profit is 13% of the cost. Calculate the cost of the chair and the selling price of the chair.

Solution:

Step 1: The given information is

$$\begin{array}{l} M = \$250 \\ E = 25\% \text{ of cost} \\ P = 13\% \text{ of cost} \end{array}$$

Step 2: Calculate the cost of the chair.

$$\begin{array}{l} M = E + P \\ 250 = 0.25C + 0.13C \\ 250 = 0.38C \\ C = \frac{250}{0.38} \\ C = \$657.89 \end{array}$$

Step 3: Calculate the selling price.

$$\begin{array}{l} S = C + M \\ S = 657.89 + 250 \\ S = \$907.89 \end{array}$$

The cost of the chair is \$657.89. The selling price of the chair is \$907.89.

TRY IT

An office supply store sells a desk lamp in its store. The store marked up the price of the lamp by

\$35. The store's operating expenses are 17% of the selling price and the profit is 10% of the selling price. What is the selling price of the chair?

Click to see Solution

$$\begin{aligned}
 M &= E + P \\
 35 &= 0.17 \times S + 0.1 \times S \\
 35 &= (0.17 + 0.1) \times S \\
 35 &= 0.27 \times S \\
 \frac{35}{0.27} &= S \\
 \$129.63 &= S
 \end{aligned}$$

Rates of Markup

Although it is important to understand markup in terms of the actual dollar amount, it is more common in business practice to calculate the markup as a percentage, of either the cost or the selling price. There are three benefits to converting the markup dollar amount into a percentage.

- Easy comparison of different products having vastly different price levels and costs, to help you see how each product contributes toward the financial success of the company. For example, if a chocolate bar has a \$0.50 markup included in a selling price of \$1 but a car has a \$1,000 markup included in a selling price of \$20,000, it is difficult to compare the profitability of these items. But if these numbers were expressed as a percentage of the selling price, the chocolate bar has a 50% markup and the car has a 5% markup. In this case it is clear that more of every dollar sold for chocolate bars goes toward list profitability.
- Simplified translation of costs into a regular selling price—a task that must be done for each product, making it helpful to have an easy formula, especially when a company carries hundreds, thousands, or even tens of thousands of products. For example, if all products are to be marked up by 50% of cost, an item with a \$100 cost can be quickly converted into a selling price of \$150.
- An increased understanding of the relationship between costs, selling prices, and the list profitability for any given product. For example, if an item selling for \$25 includes a markup on selling price of 40% (which is \$10), then you can determine that the cost is 60% of the selling price (\$15) and that \$10 of

every \$25 item sold goes toward list profits.

The **rate of markup on cost** expresses the markup as a percent of the cost of the item. Many companies use this technique internally because most accounting is based on cost information. The rate of markup on cost allows a reseller to convert easily from a product's cost to its regular unit selling price.

The rate of markup on cost is

$$ROM_C = \frac{M}{C} \times 100\%$$

where

- **ROM_C is the rate of markup on cost.** This is the percentage by which the cost of the product needs to be increased to arrive at the selling price for the product.
- **M is the markup amount.** The total dollars of the expenses and the profits. This total is the difference between the cost and the selling price.
- **C is the cost.** The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.

The rate of markup on selling price expresses the markup as a percent of the selling price of the item. Many other companies use this method because it allows for quick understanding of the portion of the selling price that remains after the cost of the product has been recovered. This percentage represents the list profits before the deduction of expenses and therefore is also referred to as the **list profit margin**.

The rate of markup on selling price is

$$ROM_S = \frac{M}{S} \times 100\%$$

where

- **ROM_S is the rate of markup on selling price.** This is the percentage of the selling price that remains available as list profits after the cost of the product is recovered.
- **M is the markup amount.** The total dollars of the expenses and the profits. This total is the difference between the cost and the selling price.
- **S is the selling price.** The regular selling price of the product.

EXAMPLE

Recall from the MP3 player example above that the cost is \$26.15, the selling price is \$39.99 and the markup is \$13.84. Calculate the rate of markup on cost and the rate of markup on selling price.

Solution:

Step 1: The given information is

$$\begin{array}{l} C = \$26.15 \\ S = \$39.99 \\ M = \$13.84 \end{array}$$

Step 2: Calculate the rate of markup on cost.

$$\begin{aligned} ROM_C &= \frac{M}{C} \times 100\% \\ &= \frac{13.84}{26.15} \times 100\% \\ &= 52.93\% \end{aligned}$$

Step 3: Calculate the rate of markup on selling price.

$$\begin{array}{l} ROM_S = \frac{M}{S} \times 100\% \\ = \frac{13.84}{39.99} \times 100\% \\ = 34.61\% \end{array}$$

The rate of markup on cost is 52.93%. This means that you must add 52.93% of the cost of MP3 player to the cost to arrive at the selling price.

The rate of markup on selling price is 34.61%. This means that you must add 34.61% of the selling price of MP3 player to the cost to arrive at the selling price. In other words, 34.61% of the selling price represents list profits after the business recovers the \$26.15 cost of the MP3 player.

NOTE

Merchandising involves many variables. Several formulas have been established so far, and a few more are yet to be introduced. Though you may feel bogged down by all of these formulas, just remember that you have encountered most of these merchandising concepts since you were very young and that you interact with retailers and pricing every day. This chapter merely formalizes calculations you already perform on a daily basis, whether at work or at home. You know that when a business sells a product, it has to recoup the cost of the product, pay its bills, and make some money.

Do not get stuck in the formulas. Think about the concept presented in the question. Change the scenario of the question and put it in the context of something more familiar. Ultimately, if you really have difficulties then look at the variables provided and cross-reference them to the merchandising formulas. Your goal is to find formulas in which only one variable is unknown. These formulas are solvable. Then ask yourself, “How does knowing that new variable help solve any other formula?”

EXAMPLE

A large national retailer wants to price a calculator for \$39.99. The retailer can acquire the calculator for \$17.23.

1. What is the rate of markup on cost?
2. What is the rate of markup on selling price?

Solution:

Step 1: The given information is

$$\begin{array}{l} C = \$17.23 \\ S = \$39.99 \end{array}$$

Step 2: Calculate the markup.

$$\begin{aligned}
 S &= C + M \\
 39.99 &= 17.23 + M \\
 39.99 - 17.23 &= M \\
 \$22.76 &= M
 \end{aligned}$$

Step 3: Calculate the rate of markup on cost.

$$\begin{aligned}
 ROM_C &= \frac{M}{C} \times 100\% \\
 &= \frac{22.76}{17.23} \times 100\% \\
 &= 132.10\%
 \end{aligned}$$

Step 4: Calculate the rate of markup on selling price.

$$\begin{aligned}
 ROM_S &= \frac{M}{S} \times 100\% \\
 &= \frac{22.76}{39.99} \times 100\% \\
 &= 56.91\%
 \end{aligned}$$

The rate of markup on cost is 132.10%. The rate of markup on selling price is 56.91%.

EXAMPLE

A retailer purchases a product for \$500. The retailer has a rate of markup on selling price of 35%. Calculate the selling price of the product and the amount of markup.

Solution:

Step 1: The given information is

$$\begin{aligned}
 C &= \$500 \\
 M &= 35\% \text{ of selling price} \\
 &= 0.35 \times S
 \end{aligned}$$

Step 2: Calculate the selling price.

$$\begin{aligned}
 S &= C + M \\
 S &= 500 + 0.35 \times S \\
 S - 0.35 \times S &= 500 \\
 (1 - 0.35) \times S &= 500 \\
 0.65 \times S &= 500 \\
 S &= \frac{500}{0.65} \\
 S &= \$769.23
 \end{aligned}$$

Step 3: Calculate the amount of markup.

$$\begin{aligned}
 M &= 0.35 \times S \\
 &= 0.35 \times 769.23 \\
 &= \$369.23
 \end{aligned}$$

The selling price of the product is \$769.23. The amount of markup is \$369.23.

EXAMPLE

An electronics store sells a laptop for \$2,500. The store has a rate of markup on cost of 27%. What is the cost to the store to purchase the laptop?

Solution:

Step 1: The given information is

$$\begin{aligned}
 S &= \$2,500 \\
 M &= 27\% \text{ of cost} \\
 &= 0.27 \times C
 \end{aligned}$$

Step 2: Calculate the cost.

$$\begin{aligned}
 S &= C + M \\
 2,500 &= C + 0.27 \times C \\
 &= (1 + 0.27) \times C \\
 &= 1.27 \times C \\
 C &= \frac{2,500}{1.27} \\
 &= \$1,968.50
 \end{aligned}$$

The cost of the laptop is \$1,968.50.

TRY IT

A furniture manufacturer sells a table for \$470. They have a 37% rate of markup on the cost of the table. What is the cost to the manufacturer to produce the table?

Click to see Solution

$$\begin{array}{l} S = C + M \\ 470 = C + 0.37 \times C \\ 470 = (1 + 0.37) \times C \\ 470 = 1.37 \times C \\ C = \frac{470}{1.37} \\ C = 343.07 \end{array}$$

TRY IT

A retailer purchases a product for \$350.29 less a trade discount of 15%. They have a 20% rate of markup on the selling price. At what price does the retailer sell the product? What is the markup on the product?

Click to see Solution

$$\begin{aligned} C &= 350.29 \times (1 - 0.15) \\ &= 350.29 \times 0.85 \\ &= \$297.75 \end{aligned}$$

$$\begin{aligned}
 S &= C + M \\
 S &= 297.75 + 0.2 \times S \\
 S - 0.2 \times S &= 297.75 \\
 (1 - 0.2) \times S &= 297.75 \\
 0.8 \times S &= 297.75 \\
 S &= \frac{297.75}{0.8} \\
 S &= \$372.19
 \end{aligned}$$

$$\begin{array}{l}
 S = C + M \\
 372.19 = 297.75 + M \\
 372.19 - 297.75 = M \\
 \$74.44 = M
 \end{array}$$

Break-Even Price

In running a business, you must never forget the “bottom line.” In other words, if you fully understand how your products are priced, you will know when you are making or losing money. Remember, if you keep losing money you will not stay in business for long! With your understanding of markup, you now know what it takes to break even in your business. **Break-even** means that you are earning no profit, but you are not losing money either because your profit is zero.

If the regular unit selling price must cover three elements—cost, expenses, and profit—then the break-even price (*BE*) is the selling price when profit is zero.

$$BE = C + E$$

This is not a new formula. It summarizes that at break-even there is no profit or loss, so the profit is eliminated from the formula.

EXAMPLE

Recall from the MP3 player example above, that the expenses are \$7.84 and the cost is \$26.15. Calculate the break-even price.

Solution:

Step 1: The given information is

$$\begin{array}{l} C = \$26.15 \\ E = \$7.84 \end{array}$$

Step 2: Calculate the break-even price.

$$\begin{aligned} BE &= C + E \\ &= 26.15 + 7.84 \\ &= \$33.99 \end{aligned}$$

The break-even price is \$33.99. This means that if the MP3 player is sold for anything more than \$33.99, it is profitable. If it is sold for less, then the business does not cover its costs and expenses and takes a loss on the sale.

EXAMPLE

A jewelry store purchases a necklace for \$300 less discounts of 12% and 5%. The store sells the necklace for \$425 so that the profit on each necklace is 27% of the cost. What is minimum price the store has to sell the necklace at in order to break-even?

Solution:

Step 1: The given information is

$$\begin{aligned}
 L &= \$300 \\
 d_1 &= 12\% \\
 d_2 &= 5\% \\
 S &= \$425 \\
 P &= 27\% \text{ of cost} \\
 &= 0.27 \times C
 \end{aligned}$$

Step 2: Calculate the cost. The cost is the net price.

$$\begin{aligned}
 C &= L \times (1 - d_1) \times (1 - d_2) \\
 &= 300 \times (1 - 0.12) \times (1 - 0.05) \\
 &= 300 \times 0.88 \times 0.95 \\
 &= \$250.80
 \end{aligned}$$

Step 3: Calculate the profit.

$$\begin{aligned}
 P &= 0.27 \times C \\
 &= 0.27 \times 250.80 \\
 &= \$67.72
 \end{aligned}$$

Step 4: Calculate the break-even price.

$$\begin{aligned}
 \text{BE} &= S - P \\
 &= 425 - 67.72 \\
 &= \$357.28
 \end{aligned}$$

The break-even price for the necklace is \$357.28.

EXAMPLE

John is trying to run an eBay business. His strategy has been to shop at local garage sales and find items of interest at a great price. He then resells these items on eBay. On John's last garage sale shopping spree, he only found one item—a Nintendo Wii that was sold to him for \$100. John's vehicle expenses (for gas, oil, wear/tear, and time) amounted to \$40. eBay charges a \$2.00

insertion fee, a flat fee of \$2.19, and a commission of 3.5% based on the selling price less \$25. What is John's minimum list price for his Nintendo Wii to ensure that he at least covers his expenses?

Solution:

Step 1: The given information is

$$\begin{aligned} C &= \$100 \\ E &= 40 + 2 + 2.19 + 0.035 \times (BE - 25) \\ &= 44.19 + 0.035 \times BE - 0.88 \\ &= 43.31 + 0.035 \times BE \end{aligned}$$

Step 2: Calculate the break-even price.

$$\begin{aligned} BE &= C + E \\ BE &= 100 + 43.31 + 0.035 \times BE \\ BE - 0.035 \times BE &= 143.31 \\ (1 - 0.035) \times BE &= 143.31 \\ 0.965 \times BE &= 143.31 \\ BE &= \frac{143.31}{0.965} \\ BE &= \$148.51 \end{aligned}$$

At a price of \$148.51 John would cover all of his costs and expenses but realize no profit or loss. Therefore, \$148.51 is his minimum price.

TRY IT

A retailer sells an item for \$10. The expenses are 30% of the cost and the profit is 17% of costs. What is the break-even price?

Click to see Solution

$$\begin{aligned} S &= C + E + P \\ 10 &= C + 0.3 \times C + 0.17 \times C \\ &= (1 + 0.3 + 0.17) \times C \\ &= 1.47 \times C \\ C &= \frac{10}{1.47} \\ &\approx \$6.80 \end{aligned}$$

$$\begin{aligned} E &= 0.3 \times C \\ &= 0.3 \times 6.80 \\ &= \$2.04 \end{aligned}$$

$$\begin{aligned} BE &= C + E \\ &= 6.80 + 2.04 \\ &= \$8.84 \end{aligned}$$

Exercises

1. The cost of an item is \$188.42. The expenses are \$48.53. The profit is \$85.00. Calculate the following:
 - a. The selling price.
 - b. The markup amount.
 - c. The break-even price.
 - d. The rate of markup on cost.
 - e. The rate of markup on selling price.

Click to see Answer

a. \$321.95; b. \$133.53; c. \$236.95; d. 70.8683%; e. 41.4754%

2. The selling price of an item is \$999.99. The expenses are 30% of cost and the profit is 23% of cost. Calculate the following:
 - a. The cost.
 - b. The amount of markup.
 - c. The break-even price.
 - d. The rate of markup on cost.
 - e. The rate of markup on selling price.

Click to see Answer

a. \$653.59; b. \$346.41; c. \$849.67; d. 53.0011%; e. 34.6413%

3. The profit on an item is 10% of the selling price. The amount of markup is \$183.28. The rate of markup on cost is 155%. Calculate the following:

- a. The selling price.
- b. The cost.
- c. The expenses.
- d. The break-even price.
- e. The rate of markup on selling price.

Click to see Answer

a. \$301.53; b. \$118.25; c. \$153.13; d. \$271.38; e. 60.7833%

4. The selling price of an item is \$274.99. The expenses are 20% of the selling price. The rate of markup on selling price is 35%. Calculate the following:

- a. The cost.
- b. The profit.
- c. The amount of markup.
- d. The break-even price.
- e. The rate of markup on cost.

Click to see Answer

a. \$178.74; b. \$41.25; c. \$96.25; d. \$223.74; e. 53.8492%

5. The expenses on an item are 45% of cost. The markup amount is \$540. The break-even price is \$1,080. Calculate the following:

- a. The selling price.
- b. The cost.
- c. The profit.
- d. The rate of markup on cost.
- e. The rate of markup on selling price.

Click to see Answer

a. \$1, 284.83; b. \$744.83; c. \$204.83; d. 72.4998%; e. 42.0289%

6. The cost of an item is \$200 less a discount of 40%. The profit is 15% of the selling price. The rate of markup on cost is 68%. Calculate the following:
- The selling price.
 - The expenses.
 - The markup amount.
 - The break-even price.
 - The rate of markup on selling price.

Click to see Answer

a. \$201.60; b. \$51.36; c. \$81.60; d. \$171.36; e. 40.4762%

7. An item is marked up by \$275. The expenses are \$100. The rate of markup on selling price is 19%. Calculate the following:
- The selling price.
 - The cost.
 - The profit.
 - The break-even price.
 - The rate of markup on cost.

Click to see Answer

a. \$1,447.37; b. \$1,172.37; c. \$175; d. \$1,272.37; e. 23.4568%

8. The break-even price of an item is \$253. The expenses on an item are 15% of cost. The profit is 12% of the (regular) selling price. Calculate the following:
- The selling price.
 - The cost.
 - The amount of markup.
 - The rate of markup on cost.
 - The rate of markup on selling price.

Click to see Answer

a. \$287.50; b. \$220; c. \$67.50; d. 30.6818%; e. 23.4783\%

9. If a pair of sunglasses sells at a regular unit selling price of \$249.99 and the markup is always 55% of the regular unit selling price, what is the cost of the sunglasses?

Click to see Answer

\$112.50

10. A transit company wants to establish an easy way to calculate its transit fares. It has determined that the cost of a transit ride is \$1.00, with expenses of 50% of cost. It requires \$0.75 profit per ride. What is the rate of markup on cost?

Click to see Answer

125%

11. Daisy is trying to figure out how much negotiating room she has in purchasing a new car. The car has a list price of \$34,995.99. She has learned from an industry insider that most car dealerships have a 20% markup on selling price. What does she estimate the dealership paid for the car?

Click to see Answer

\$27,996.79

12. The markup amount on an eMachines desktop computer is \$131.64. If the machine regularly retails for \$497.25 and expenses average 15% of the selling price, what profit will be earned?

Click to see Answer

\$57.05

13. Manitoba Telecom Services (MTS) purchases an iPhone for \$749.99 less discounts of 25% and 15%. MTS's expenses are known to average 30% of the regular unit selling price.

- What is the regular unit selling price if a profit of \$35 per iPhone is required?
- What are the expenses?
- What is the rate of markup on cost?
- What is the break-even selling price?

Click to see Answer

a. \$733.03; b. \$219.91; c. 53.3151%; d. \$698.03

14. A snowboard has a cost of \$79.10, expenses of \$22.85, and profit of \$18.00.

- What is the regular unit selling price?

- b. What is the markup amount?
- c. What is the rate of markup on cost?
- d. What is the rate of markup on selling price?
- e. What is the break-even selling price? What is the rate of markup on cost at this break-even price?

Click to see Answer

a. \$119.95; b. \$40.85; c. 51.6435%; d. 34.0559%; e. \$101.95, 28.8887%

15. A waterpark wants to understand its pricing better. If the regular price of admission is \$49.95, expenses are 20% of cost, and the profit is 30% of the regular unit selling price, what is the markup amount?

Click to see Answer

\$20.82

16. Sally works for a skateboard shop. The company just purchased a skateboard for \$89.00 less discounts of 22%, 15%, and 5%. The company has standard expenses of 37% of cost and desires a profit of 25% of the regular unit selling price. What regular unit selling price should Sally set for the skateboard?

Click to see Answer

\$102.40

17. If an item has a 75% markup on cost, what is its rate of markup on selling price?

Click to see Answer

42.8571%

18. A product received discounts of 33%, 25%, and 5%. A markup on cost of 50% was then applied to arrive at the regular unit selling price of \$349.50. What was the original list price for the product?

Click to see Answer

\$488.09

19. Mountain Equipment Co-op (MEC) wants to price a new backpack. The backpack can be purchased for a list price of \$59.95 less a trade discount of 25% and a quantity discount of 10%. MEC estimates expenses to be 18% of cost and it must maintain a markup on selling price of 35%.

- a. What is the cost of backpack?
- b. What is the markup amount?
- c. What is the regular unit selling price for the backpack?
- d. What profit will Mountain Equipment Co-op realize?
- e. What happens to the profits if it sells the backpack at the list price instead?

Click to see Answer

a. \$40.47; b. \$21.79; c. \$62.26; d. \$14.51; e. \$12.20, \$2.31 reduction

20. Costco can purchase a bag of Starbucks coffee for \$20.00 less discounts of 20%, 15%, and 7%. It then adds a 40% markup on cost. Expenses are known to be 25% of the regular unit selling price.
- a. What is the cost of the coffee?
 - b. What is the regular unit selling price?
 - c. How much profit will Costco make on a bag of Starbucks coffee?
 - d. What rate of markup on selling price does this represent?
 - e. Repeat questions (a) through (d) if the list price changes to \$24.00.

Click to see Answer

a. \$12.65; b. \$17.71; c. \$0.63; d. 28.5714%; e. \$15.18, \$21.25, \$0.76, 28.5647%

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1.4 MARKDOWN

LEARNING OBJECTIVES

- Calculate the amount of markdown on the selling price of an item.
- Calculate the rate of markdown on the selling price of an item.

Flashy signs in a retail store announce, “40% off, today only!” Excitedly you purchase three tax-free products with regular price tags reading \$100, \$250, and \$150. The cashier processing the transaction informs you that your total is \$325. You are about to hand over your credit card when something about the total makes you pause. The regular total of all your items is \$500. If they are 40% off, you should receive a \$200 deduction and pay only \$300. The cashier apologizes for the mistake and corrects your total.

A **markdown** is a reduction from the regular selling price of a product resulting in a lower price. This lower price is called the **sale price** or **reduced selling price** to distinguish it from the selling price. Many people perceive markdowns as a sign of bad business management decisions. However, in most situations this is not true. Companies must always attempt to forecast the future. In order to stock products, a reseller must estimate the number of units that might sell in the near future for every product that it carries. This is both an art and a science. While businesses use statistical techniques that predict future sales with a relative degree of accuracy, consumers are fickle and regularly change shopping habits. Markdowns most commonly occur under four circumstances.

- **Clearing Out Excess or Unwanted Inventory.** In these situations, the business thought it could sell 100 units. However, consumers purchased only 20 units. In the case of seasonal inventory, such as Christmas items on Boxing Day, the retailer wishes to avoid packing up and storing the inventory until the next season.
- **Clearing Out Damaged or Discontinued Items.** Selling a damaged product at a discount is better than not selling it at all. When products are discontinued, this leaves shelf space underused, so it is better

to clear the item out altogether to make room for profitable items that can keep the shelves fully stocked.

- **Increasing Sales Volumes.** Sales attract customers because almost everyone loves a deal. Though special marketing events such as a 48 hour sale reduce the profitability per unit, by increasing the volume sold these sales can lead to a greater profit overall.
- **Promoting Add-On Purchases.** Having items on sale attracts customers to the store. Many times customers will not only purchase the item on sale but also, as long as they are on the premises, grab a few other items, which are regularly priced and very profitable.

Markdowns are no different from offering a discount. Markdowns are common, so you will find it handy to adapt the discount formulas to the application of markdowns, replacing the symbols with ones that are meaningful in merchandising.

The **sale price or reduced selling price** (S_{red}) of a product is

$$S_{red} = S \times (1 - d)$$

where

- **S_{red} is the sale or reduced selling price.** The sale price is the price of the product after reduction by the markdown percent. Conceptually, the sale price is the same as the net price.
- **S is the regular selling price.** The regular selling price of the product before any discounts.
- **d is the rate of markdown.** A markdown rate is the same as a sale discount rate. Therefore, you use the same discount rate symbol (d) to represent the percentage (in decimal format) by which you reduce the selling price. Note that you are interested in calculating the sale price and not the amount saved. Thus, you take the markdown rate away from 1 to find out the rate owing.

NOTES

1. The formula for the reduced selling price is just an application of the $N = L \times (1 - d)$ where S_{red} is the net price and S is the list price.
2. In markdown situations, the selling price and the reduced selling price are different variables. The reduced selling price is always less than the selling price. In the event that a regular selling price has more than one markdown percent applied to it, you can extend the formula to apply multiple discounts .

Markdown

The **amount of markdown** (D) is the amount by which the regular selling price (S) is reduced to determine the reduced selling price (S_{red}).

$$D = S - S_{red}$$

where

- **D is the markdown amount.** This is the amount the regular selling price is reduced by to arrive at the reduced selling price.
- **S is the regular selling price.** The regular selling price before you apply any markdown percentages.
- **S_{red} is the reduced selling price.** The price after you have deducted all markdown percentages from the regular selling price.

Alternatively, the amount of markdown can be calculated by using the percentage of the of the selling price (S) that is deducted to arrive at the reduced selling price (S_{red}).

$$D = d \times S$$

where

- **D is the markdown amount.** This is the amount the regular selling price is reduced by to arrive at the reduced selling price.
- **S is the regular selling price.** The regular selling price before you apply any markdown percentages.
- **d is the rate of markdown.** The percentage of the selling price to be deducted (in decimal format).

EXAMPLE

A retailer sells an MP3 player for a regular selling price of \$39.99. Assume the retailer has excess inventory and places the MP3 player on sale for 10% off. What is the sale price and markdown amount?

Solution:**Step 1:** The given information is

$$S = \$39.99$$

$$d = 10\%$$

Step 2: Calculate the reduced selling price.

$$\begin{aligned} S_{\text{red}} &= S \times (1-d) \\ &= 39.99 \times (1-0.1) \\ &= 39.99 \times 0.9 \\ &= \$35.99 \end{aligned}$$

Step 3: Calculate the markdown amount.

$$\begin{aligned} D &= S - S_{\text{red}} \\ &= 39.99 - 35.99 \\ &= \$4.00 \end{aligned}$$

The reduced selling price is \$35.99. The amount of markdown is \$4.00.

EXAMPLE

A retailer sells a product for \$189.99. During a sale the product is marked down by 45%. What is the sale price for the product and what dollar amount is saved?

Solution:**Step 1:** The given information is

$$S = \$189.99$$

$$d = 45\%$$

Step 2: Calculate the reduced selling price.

$$\begin{aligned} S_{\text{red}} &= S \times (1-d) \\ &= 189.99 \times (1-0.45) \\ &= 189.99 \times 0.55 \\ &= \$104.49 \end{aligned}$$

Step 3: Calculate the markdown amount.

$$\begin{aligned}
 D &= S - S_{red} \\
 &= 189.99 - 104.49 \\
 &= \$85.50
 \end{aligned}$$

The reduced selling price is \$104.49. The amount of markdown is \$85.50.

TRY IT

A table lamp regularly sells for \$150. The lamp is marked down by 25%. What is the amount of markdown? What is the sale price?

Click to see Solution

$$\begin{aligned}
 D &= d \times S \\
 &= 0.25 \times 150 \\
 &= \$37.50 \\
 S_{red} &= S - D \\
 &= 150 - 37.50 \\
 &= \$112.50
 \end{aligned}$$

Rate of Markdown

Businesses often express markdowns as percentages, facilitating easy comprehension and comparison. The rate of markdown (d) is the percentage of the selling price that is deducted to arrive at the reduced selling price. The rate of markdown expresses the markdown amount as a percent of the regular selling price.

$$d = \frac{D}{S} \times 100\%$$

where

- **d is the rate of markdown.** You always deduct a markdown amount from the regular selling price of the product. Therefore, you always express the rate of markdown as a percentage of the selling price. Use the same symbol for a discount rate because markdown rates are synonymous with sale discounts.
- **D is the markdown amount.** This is the amount the regular selling price is reduced by to arrive at the reduced selling price.
- **S is the regular selling price.** The regular selling price before you apply any markdown percentages.

EXAMPLE

A retailer regularly sells USB drives for \$79.99 each. During a sale, the USB drives are sold for \$58.79. Calculate the rate of markdown.

Solution:

Step 1: The given information is

$$\begin{aligned} S &= \$79.99 \\ S_{red} &= \$58.79 \end{aligned}$$

Step 2: Calculate the amount of markdown.

$$\begin{aligned} D &= S - S_{red} \\ &= 79.99 - 58.79 \\ &= \$21.20 \end{aligned}$$

Step 3: Calculate the rate markdown.

$$\begin{aligned} d &= \frac{D}{S} \times 100\% \\ &= \frac{21.20}{79.99} \times 100\% \\ &= 26.50\% \end{aligned}$$

The rate of markdown is 26.50%.

NOTE

Avoid getting bogged down in formulas. Recall that the three formulas for markdowns are not new formulas, just adaptations of three previously introduced concepts. As a consumer, you are very experienced with endless examples of sales, bargains, discounts, blowouts, clearances, and the like. Every day you read ads in the newspaper and watch television commercials advertising percent savings. This section simply crystallizes your existing knowledge. If you are puzzled by questions involving markdowns, make use of your shopping experiences at the mall!

EXAMPLE

A reseller acquires an Apple iPad for \$650. Expenses are planned at 20% of the cost, and profits are set at 15% of the cost. During a special promotion, the iPad is advertised at \$100 off. What is the sale price and rate of markdown?

Solution:

Step 1: The given information is

$$C = \$650$$

$$E = 0.2 \times C$$

$$P = 0.15 \times C$$

$$D = \$100$$

Step 2: Calculate the expenses.

$$\begin{array}{l} E = 0.2 \times C \\ E = 0.2 \times 650 \\ E = \$130 \end{array}$$

Step 3: Calculate the profit.

$$\begin{array}{l} P = 0.15 \times C \\ P = 0.15 \times 650 \\ P = \$97.50 \end{array}$$

Step 4: Calculate the regular selling price.

$$\begin{array}{l} S = C + E + P \\ \$877.50 \end{array}$$

Step 5: Calculate the reduced selling price.

$$\begin{array}{l} S_{\text{red}} = S - D \\ \$777.50 \end{array}$$

Step 6: Calculate the rate markdown.

$$\begin{array}{l} d = \frac{D}{S} \times 100\% \\ \times 100\% \end{array} = \frac{100}{877.50} \times 100\% = 11.40\%$$

The reduced selling price is \$777.50. The rate of markdown is 11.40%.

EXAMPLE

A clothing retailer purchases shirts for \$17.85. The retailer sells the shirts at a markup of 30% of the selling price. At the end of the season, the retailer wants to clear the inventory, so sells the shirts for \$19.99. Calculate the rate of markdown.

Solution:

Step 1: The given information is

$$\begin{array}{l} C = \$17.85 \\ M = 0.3 \times S \\ S_{\text{red}} = \$19.99 \end{array}$$

Step 2: Calculate the regular selling price.

$$\begin{array}{l} S = C + M \\ S = 17.85 + 0.3 \times S \\ 0.7 \times S = 17.85 \\ S = \frac{17.85}{0.7} \\ S = \$25.50 \end{array}$$

Step 3: Calculate the amount of markdown.

$$\begin{array}{l} D = \$5.51 \\ S = 25.50 - 19.99 = \$5.51 \end{array}$$

Step 4: Calculate the rate markdown.

$$d = \frac{D}{S} \times 100\% = \frac{5.51}{25.50} \times 100\% = 21.61\%$$

The rate of markdown is 21.61%.

TRY IT

A furniture dealer purchases chairs for \$500 less a discount of 15%. The dealer sells the chair with a markup of 30% of the cost. During a sale, the dealer marks down the price by \$100.

1. Calculate the rate of markdown.
2. Calculate the reduced selling price.

Click to see Solution

$$C = 500 \times (1 - 0.15) = 500 \times 0.85 = \$425$$

$$\begin{aligned} S &= C + M \\ &= 425 + 0.3 \times 425 \\ &= \$552.50 \end{aligned}$$

1. Rate of markdown.

$$\begin{aligned} d &= \frac{D}{S} \times 100\% \\ &= \frac{100}{552.50} \times 100\% \\ &= 18.10\% \end{aligned}$$

2. Reduced selling price.

$$\begin{aligned} S_{red} &= S - D \\ &= 552.50 - 100 \\ &= \$452.50 \end{aligned}$$

Exercises

1. The regular selling price of an item is \$439.85. The item is put on sale with a rate of markdown of 35%. Calculate the following:
 - a. The amount of markdown.
 - b. The reduced selling price.

Click to see Answer

a. \$153.95; b. \$285.90

2. An item is marked down by \$100 to a reduced selling price of \$199.95. Calculate the following:
 - a. The regular selling price.
 - b. The rate of markdown.

Click to see Answer

a. \$299.95 b. 33.3389%

3. The regular selling price of an item is \$1,050 and the reduced selling price is \$775. Calculate the following:
 - a. The amount of markdown.
 - b. The rate of markdown.

Click to see Answer

a. \$275; b. 26.1905%

4. The regular selling price of an item is \$28,775. During a sale, the item is markdown by \$3,250.

Calculate the following:

- a. The rate of markdown.
- b. The reduced selling price.

Click to see Answer

a. 11.2945%; b. \$25,525

5. The reduced selling price of an item is \$13,199.95. The rate of markdown is 33%. Calculate the following:

- a. The regular selling price.
- b. The amount of markdown.

Click to see Answer

a. \$19,701.42; b. \$6,501.47

6. An item is marked down by \$38.33. The rate of markdown is 12%. Calculate the following:

- a. The regular selling price.
- b. The reduced selling price.

Click to see Answer

a. \$319.42; b. \$281.09

7. A pair of Nike athletic shoes is listed at a regular selling price of \$89.99. If the shoes go on sale for 40% off, what is the sale price?

Click to see Answer

\$53.99

8. During its special Bay Days, The Bay advertises a Timex watch for \$39.99 with a regular price of \$84.99. Calculate the rate of markdown and markdown amount.

Click to see Answer

\$45, 52.9474%

9. For spring break you are thinking about heading to Tulum, Mexico. In planning ahead, you notice that

a one-week stay at the Gran Bahia Principe Tulum, regularly priced at \$2,349 for air and six nights all inclusive, offers an early-bird booking discount of \$350. What rate of markdown is being offered for booking early?

Click to see Answer

14.9%

10. A Heritage Infusio deep frying pan is advertised at 70% off with a sale price of \$39.99. What is the frying pan's regular selling price, and what markdown amount does this represent?

Click to see Answer

\$133.30, \$93.31

11. Quicky Mart regularly sells its Red Bull sports drink for \$2.99 per can. Quicky Mart noticed that one of its competitors down the street sells Red Bull for \$1.89. What rate of markdown must Quicky Mart advertise if it wants to match its competitor?

Click to see Answer

36.7893%

12. A campus food outlet is advertising a "Buy one, get one 25% off" deal. The 25% off comes off the lower-priced item. If you purchase a chicken dinner for \$8.99 and your friend gets the burger combo for \$6.99, what is the rate of markdown on the total price?

Click to see Answer

10.9512%

13. The Brick advertises that when you purchase a queen-size Tempur-Pedic mattress set for \$2,499.97 it will give you a 51" 3-D plasma television with a 3-D starter kit included. The value of this gift is \$1,199.99. What rate of markdown does this represent?

Click to see Answer

32.4325%

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1.5 WORKING WITH DISCOUNTS, MARKUP, AND MARKDOWN

LEARNING OBJECTIVES

- Solve problems involving discounts, markup, and markdown.

Running a business requires you to integrate discounts, markup, and markdown. From discounts to markups and markdowns, all the numbers must fit together for you to earn profits in the long run. It is critical to understand how pricing decisions affect the various financial aspects of your business and to stay on top of your numbers. Merchandising does not involve difficult concepts, but to do a good job of it you need to keep track of many variables and observe how they relate to one another.

Up to this point, you have examined the various components of merchandising as separate topics. Your study of basic product pricing has included the following:

- Taking a list price and applying discounts to arrive at a product's cost.
- Marking up a product by adding expenses and profits to the cost to arrive at a regular selling price.
- Marking down a product by applying a discount and arriving at a reducing selling price or sale price.
- Working with various percentages in both markup and markdown situations to either simplify calculations or present a clearer pricing picture.

Reduced Profit

When you markdown the selling price of an item to a reduced selling price, what effect does this have on the costs, expenses, and profit? No matter what price you sell an item for, the cost and expenses do not change. The amount that you marked down the selling price reduces your profit because you will still have the same

costs and expenses to cover. The **reduced profit** (P_{red}) is the amount of profit earned when an item is sold at a reduced selling price. Because profit (reduced or not) is what is left over from the selling price after you deduct the costs and expenses:

$$P_{red} = S_{red} - C - E$$

where

- **P_{red} is the reduced profit.** The reduced profit is the amount of profit earned when selling an item at a reduced selling price.
- **S_{red} is the reduced selling price.** The sale price is the price of the product after reduction by the markdown.
- **C is the cost of the item.** The cost is the amount of money that the business must pay to purchase or manufacture the product.
- **E is the expenses.** Expenses are the financial outlays involved in selling the product.

How is the reduced profit related to the markdown amount? Suppose an item has a regular selling price of \$100, costs of \$30, and expenses of \$20. Then the profit earned by selling the product at the regular selling price is \$50. Now, suppose the item is marked down by \$15. The reduced selling price is \$75. The costs and expenses remain the same, \$30 and \$20 respectively. So the reduced profit earned by selling the product at the reduced selling price is \$35. Note that this reduced profit is just the original \$50 profit reduced by the markdown amount of \$15. When you start marking down the selling price, the profit is reduced by an amount equal to the markdown amount.

$$P_{red} = P - D$$

where

- **P_{red} is the reduced profit.** The reduced profit is the amount of profit earned when selling an item at a reduced selling price.
- **P is the profit.** This is the profit when selling an item at the regular selling price.
- **D is the amount of markdown.** This is the amount the regular selling price is reduced by to arrive at the reduced selling price.

From the above formula, you can see that as long as the markdown amount (D) is **smaller** than the profit (P), the reduced profit (P_{red}) will be positive. Consequently, you would still make money when selling the item at the reduced selling price but not as much money if you sold the item at the regular selling price.

What happens when the markdown amount (D) **equals** the profit (P)? Then the reduced selling price would

equal the break-even price and the reduced profit would be 0. In other words, you do not make any money when selling the item at the break-even price but you also do not lose any money.

Is it possible to lose money on the sale of an item? Suppose an item has a regular selling price of \$100, costs of \$30, and expenses of \$20. Then the profit earned by selling the product at the regular selling price is \$50. Now, suppose the item is marked down by \$65. The reduced selling price is \$35. This sale price is less than the costs and expenses combined, so the item would be sold at a loss. In fact, using the formula above, the reduced profit is -\$15, which indicates that every time the item is sold at the reduced selling price of \$35, the business would lose \$15.

When the markdown amount (D) is **larger** than the profit (P), the reduced profit (P_{red}) will be negative. Consequently, you would lose money when selling the item at the reduced selling price.

NOTES

1. The sign (positive, negative, or zero) of the reduced profit is important. A positive reduced profit indicates that you are still making money (a profit) on the sale of the item. A zero reduced profit indicates that you are selling the item at the break-even price. A negative reduced profit indicates that you are losing money on the sale of the item, and the amount of the reduced profit indicates how much you are losing on each sale.
2. You may think it is counter-intuitive to sell an item at a loss, but it is a common practice. For example, after season sales or discounted items are often sold at a loss in order to clear out inventory.

EXAMPLE

A retailer purchases a small kitchen appliance for \$100 with expenses of \$45 for each appliance. The retailer wants a profit of \$80 on each appliance sold. During a sale, the retailer offers a markdown of 35%. At the reduced selling price, what is the retailer's profit or loss?

Solution:**Step 1:** The given information is

$$C = \$100$$

$$E = \$45$$

$$P = \$80$$

$$d = 35\%$$

Step 2: Calculate the regular selling price.

$$\begin{aligned} S &= C + E + P \\ &= 100 + 45 + 80 \\ &= \$225 \end{aligned}$$

Step 3: Calculate the reduced selling price.

$$\begin{aligned} S_{\text{red}} &= S \times (1 - d) \\ &= 225 \times (1 - 0.35) \\ &= 225 \times 0.65 \\ &= \$146.25 \end{aligned}$$

Step 4: Calculate the reduced profit.

$$\begin{aligned} P_{\text{red}} &= S_{\text{red}} - C - E \\ &= 146.25 - 100 - 45 \\ &= \$1.25 \end{aligned}$$

At the reduced selling price, the retailer's profit is \$1.25.

EXAMPLE

A retailer purchases shoes for \$25 per pair and sells them for \$47 per pair, which includes a profit of \$5 per pair. During a sale, the shoes are marked down by 20%. Calculate the profit or loss at the reduced selling price.

Solution:**Step 1:** The given information is

$$C = \$25$$

$$S = \$47$$

$$P = \$5$$

$$d = 20\%$$

Step 2: Calculate the expenses.

$$\begin{aligned} S &= C + E + P \\ 47 &= 25 + E + 5 \\ E &= 47 - 25 - 5 \\ E &= \$17 \end{aligned}$$

Step 3: Calculate the reduced selling price.

$$\begin{aligned} S_{\text{red}} &= S \times (1 - d) \\ &= 47 \times (1 - 0.2) \\ &= 47 \times 0.8 \\ S_{\text{red}} &= \$37.60 \end{aligned}$$

Step 4: Calculate the reduced profit.

$$\begin{aligned} P_{\text{red}} &= S_{\text{red}} - C - E \\ &= 37.60 - 25 - 17 \\ &= -\$4.40 \end{aligned}$$

At the reduced selling price, the retailer loses \$4.40 per pair.

TRY IT

An item costs \$90, the overhead expenses are 30% of cost, and the profit is 15% of cost. During a sale, the item is marked down by 25%. Calculate the profit or loss at the reduced selling price.

Click to see Solution

$$\begin{aligned} S &= C + E + P \\ &= 90 + 0.3 \times 90 + 0.15 \times 90 \\ &= \$130.50 \end{aligned}$$

$$\begin{array}{l} S_{red} = S \times (1-d) \\ S_{red} = 130.50 \times (1-0.25) \\ S_{red} = 130.50 \times 0.75 \\ S_{red} = \$97.88 \end{array}$$

$$\begin{aligned} P_{red} &= S_{red} - C - E \\ &= 97.88 - 90 - 27 \\ &= -\$19.12 \end{aligned}$$

Complete Merchandising Scenarios

Use the following steps to solve complete merchandising scenarios.

1. It is critically important to correctly identify both the known and unknown merchandising variables that you are asked to calculate.
2. For each of the unknown variables, identify the formulas that contain the unknown variables. You must solve one of these formulas to arrive at the answer. Based on the information provided, examine these formulas to determine which formula may be solvable. Write out this formula, identifying which components you know and which components remain unknown.
3. Note the unknown variables among all the formulas you found in the previous step. Are there common unknown variables among these formulas? These common variables are critical variables. Solving for these common unknown is the key to completing the question. Note that these unknown variables may not directly point to the information you were asked to calculate, and they do not resolve the merchandising scenario themselves. However, without these variables you cannot solve the scenario.
4. Apply any of the merchandising formulas to calculate the unknown variables required to solve the formulas. Your goal is to identify all required variables and then solve for the original unknown variables identified in the first step.

NOTE

Before proceeding, take a few moments to review the various concepts and formulas covered earlier in this chapter. A critical and difficult skill is now at hand. As evident in Steps 2 through 4

of the process described above, you must use your problem-solving skills to figure out which formulas to use and in what order. Here are some suggestions to help you on your way.

- Analyze the question systematically.
- If you are unsure of how the pieces of the puzzle fit together, try substituting your known variables into the various formulas. You are looking for
 - Any solvable formulas with only one unknown variable, *or*
 - Any pair of formulas with the same two unknowns, because you can solve this system using your algebraic skills of solving two linear equations with two unknowns.
- Merchandising has multiple steps. Think through the process. When you solve one equation for an unknown variable, determine how knowing that variable affects your ability to solve another formula. There is usually a critical unknown variable. Once you determine the value of this variable, a domino effect allows you to solve any other remaining formulas.

EXAMPLE

A skateboard shop stocks a Tony Hawk Birdhouse Premium Complete Skateboard. Each skateboard costs \$45.46 and the store has overheads expenses of \$25.17 on each skateboard. The shop wants to sell each skateboard for an operating profit of \$10.55. During a sale, it offers a markdown rate of 10%. At the reduced selling price, calculate the store's profit or loss.

Solution:

Step 1: The given information is

$$C = \$45.46$$

$$E = \$25.17$$

$$P = \$10.55$$

$$d = 10\%$$

Step 2: Calculate the regular selling price.

$$\begin{aligned} S &= C + E + P \\ S &= 45.46 + 25.17 + 10.55 \\ S &= \$81.18 \end{aligned}$$

Step 3: Calculate the reduced selling price.

$$\begin{aligned} S_{red} &= S \times (1 - d) \\ S_{red} &= 81.18 \times (1 - 0.1) \\ S_{red} &= 81.18 \times 0.9 \\ S_{red} &= \$73.06 \end{aligned}$$

Step 4: Calculate the reduced profit.

$$\begin{aligned} P_{red} &= S_{red} - C - E \\ &= 73.06 - 45.46 - 25.17 \\ &= \$2.43 \end{aligned}$$

At the reduced selling price, the store's profit is \$2.43.

EXAMPLE

A toy store sells an electronic toy for \$150. The markup on the cost of the toy is 25%. The operating expenses are 15% of cost. During a sale, the toy is marked down by 10%.

1. Calculate the cost of the toy.
2. Calculate the profit if the toy is sold at the regular selling price.
3. Calculate the reduced selling price.
4. Calculate the profit or loss at the reduced selling price.

Solution:

Step 1: The given information is

$$\begin{aligned}
 S &= \$150 \\
 M &= 0.25 \times C \\
 E &= 0.15 \times C \\
 d &= 10\%
 \end{aligned}$$

Step 2: Calculate the cost.

$$\begin{aligned}
 S &= C + M \\
 150 &= C + 0.25 \times C \\
 150 &= (1 + 0.25) \times C \\
 150 &= 1.25 \times C \\
 \frac{150}{1.25} &= \frac{1.25 \times C}{1.25} \\
 \$120 &= C
 \end{aligned}$$

The cost of the toy is \$120.

Step 3: Calculate the expenses.

$$\begin{aligned}
 E &= 0.15 \times C \\
 &= 0.15 \times 120 \\
 &= \$18
 \end{aligned}$$

Step 4: Calculate the profit at the regular selling price.

$$\begin{aligned}
 P &= S - C - E \\
 &= 150 - 120 - 18 \\
 &= \$12
 \end{aligned}$$

The profit at the regular selling price is \$12.

Step 5: Calculate the reduced selling price.

$$\begin{aligned}
 S_{red} &= S \times (1 - d) \\
 &= 150 \times (1 - 0.1) \\
 &= 150 \times 0.9 \\
 &= \$135
 \end{aligned}$$

The reduced selling price is \$135.

Step 6: Calculate the reduced profit.

$$\begin{aligned}
 P_{red} &= S_{red} - C - E \\
 &= 135 - 120 - 18 \\
 &= -\$3
 \end{aligned}$$

At the reduced selling price, the store loses \$3 per toy.

EXAMPLE

A manufacturer makes laptop computers at a cost of \$1800 per machine. The manufacturer's operating profit is 20% on the selling price and the overhead expenses are 35% on the selling price. During a computer trade show, the company offered the computer at a discount of 17.5%.

1. What is the regular selling price of the laptop?
2. What is the profit or loss at the reduced selling price?
3. What rate of markdown does the manufacturer have to offer to sell the laptop at the break-even price?

Solution:

Step 1: The given information is

$$\begin{aligned}C &= \$1800 \\P &= 0.2 \times S \\E &= 0.35 \times S \\d &= 17.5\%\end{aligned}$$

Step 2: Calculate the regular selling price.

$$\begin{aligned}S &= C + E + P \\S &= 1800 + 0.35 \times S + 0.2 \times S \\S &= 1800 + 0.55 \times S \\S - 0.55 \times S &= 1800 \\0.45 \times S &= 1800 \\S &= \frac{1800}{0.45} \\S &= \$4,000\end{aligned}$$

The regular selling price of the laptop is \$4,000.

Step 3: Calculate the expenses.

$$\begin{aligned}E &= 0.35 \times S \\&= 0.35 \times 4,000 \\&= \$1,400\end{aligned}$$

Step 4: Calculate the reduced selling price.

$$\begin{aligned}S_{\text{red}} &= S \times (1 - d) \\&= 4,000 \times (1 - 0.175) \\&= 4,000 \times 0.825 \\&= \$3,300\end{aligned}$$

Step 5: Calculate the reduced profit.

$$\begin{aligned} P_{red} &= S_{red} - C - E \\ &= 3,300 - 1,800 - 1,400 \\ &= \$100 \end{aligned}$$

At the reduced selling price, the manufacturer makes a profit of \$100 per laptop.

Step 6: Calculate the break-even price.

$$\begin{aligned} BE &= C + E \\ &= 1,800 + 1,400 \\ &= \$3,200 \end{aligned}$$

Step 7: Calculate the rate of markdown at the break-even price.

$$\begin{aligned} D &= S - S_{red} \\ &= 4,000 - 3,200 \\ &= \$800 \\ d &= \frac{D}{S} \times 100\% \\ &= \frac{800}{4,000} \times 100\% \\ &= 20\% \end{aligned}$$

At the break-even price, the rate of markdown is 20%.

NOTE

In the previous example, the expenses are given as a percent of the regular selling price, which means the expenses are always based on the regular selling price and not the reduced selling price. Notice in the calculation of the reduced profit that the expenses are the expenses based on the regular selling price. The expenses are NOT recalculated based on the reduced selling price. Remember, the expenses are fixed and based on what you originally plan to sell the item for, and do not change even, if the item is placed on sale.

EXAMPLE

A retailer sells an item for \$110.50 each. The rate of markup on cost is 70%. The operating profit is 18% on cost. During a sale the item's price is set so that the profit at the reduced selling price is \$3.75.

1. What is the cost of the item?
2. What are the expenses?
3. What is the reduced selling price?
4. What is the rate of markdown for the reduced selling price?
5. What is the profit or loss if the item is sold at cost?

Solution:

Step 1: The given information is

$$\begin{aligned} S &= \$110.50 \\ M &= 0.7 \times C \\ P &= 0.18 \times C \\ P_{red} &= 3.75 \end{aligned}$$

Step 2: Calculate the cost.

$$\begin{aligned} S &= C + M \\ 110.50 &= C + 0.7 \times C \\ 110.50 &= (1 + 0.7) \times C \\ 110.50 &= 1.7 \times C \\ C &= \frac{110.50}{1.7} \\ C &= 65 \end{aligned}$$

The cost of the item is \$65.

Step 3: Calculate the profit at the regular selling price.

$$\begin{aligned} P &= 0.18 \times C \\ &= 0.18 \times 65 \\ &= \$11.70 \end{aligned}$$

Step 4: Calculate the markup amount.

$$\begin{aligned}
 M &= 0.7 \times C \\
 &= 0.7 \times 65 \\
 &= \$45.50
 \end{aligned}$$

Step 5: Calculate the expenses.

$$\begin{aligned}
 M + E + P &= 45.50 + 11.70 + 33.80 \\
 &= \$91.00
 \end{aligned}$$

The expenses are \$91.00.

Step 6: Calculate the reduced selling price.

$$\begin{aligned}
 S_{\text{red}} &= C + E + P \\
 &= 65 + 33.80 + 11.70 \\
 &= \$110.50
 \end{aligned}$$

The reduced selling price is \$110.50.

Step 7: Calculate the rate of markdown.

$$\begin{aligned}
 D &= S - S_{\text{red}} \\
 &= 110.50 - 102.55 \\
 &= \$7.95 \\
 d &= \frac{D}{S} \times 100\% \\
 &= \frac{7.95}{110.50} \times 100\% \\
 &= 7.19\%
 \end{aligned}$$

At the reduced selling price, the rate of markdown is 7.19%.

Step 8: Calculate the reduced profit when the reduced selling price is the cost ($S_{\text{red}} = C = 65$).

$$\begin{aligned}
 P_{\text{red}} &= S_{\text{red}} - C - E \\
 &= 65 - 65 - 33.80 \\
 &= -\$33.80
 \end{aligned}$$

If the item is sold at cost, the loss is \$33.80.

EXAMPLE

A retailer of electronic goods purchases TVs from a distributor after trade discounts of 15% and 7% on the list price of \$700 per TV. The retailer then marks up the TVs by 60% on the selling price. The expenses are 25% of the selling price. During a sale, they offer a markdown of 20%.

1. What is the regular selling price?
2. What is the reduced selling price?
3. What is the break-even price?

Solution:

Step 1: The given information is

$$\begin{aligned} L &= \$700 \\ d_1 &= 15\% \\ d_2 &= 7\% \\ M &= 0.6 \times S \\ E &= 0.25 \times S \\ d &= 20\% \end{aligned}$$

Step 2: Calculate the cost.

$$\begin{aligned} C &= L \times (1 - d_1) \times (1 - d_2) \\ C &= 700 \times (1 - 0.15) \times (1 - 0.07) \\ C &= 700 \times 0.85 \times 0.93 \\ C &= \$553.35 \end{aligned}$$

Step 3: Calculate the regular selling price.

$$\begin{aligned} S &= C + M \\ S &= 553.35 + 0.6 \times S \\ S - 0.6 \times S &= 553.35 \\ (1 - 0.6) \times S &= 553.35 \\ 0.4 \times S &= 553.35 \\ S &= \frac{553.35}{0.4} \\ S &= \$1,383.38 \end{aligned}$$

The regular selling price is \$1,383.38

Step 4: Calculate the reduced selling price.

$$\begin{aligned} S_{\text{red}} &= S \times (1-d) \\ &= 1,383.38 \times (1-0.2) \\ &= 1,383.38 \times 0.8 \\ &= \$1,106.70 \end{aligned}$$

The reduced selling price is \$1,106.70.

Step 5: Calculate the expenses.

$$\begin{aligned} E &= 0.25 \times S \\ &= 0.25 \times 1,383.38 \\ &= \$345.85 \end{aligned}$$

Step 6: Calculate the break-even price

$$\begin{aligned} BE &= C + E \\ &= 553.35 + 345.85 \\ &= \$899.20 \end{aligned}$$

The break-even price is \$899.20.

TRY IT

A retailer purchases an item for \$180 less a discount of 20%. The markup on the item is 45% on cost and the operating expenses are 15% on cost.

1. What is the regular selling price of the item?
2. During a sale the item is marked down by 25%. What is the reduced selling price and profit or loss at the sale price?

Click to see Solution

1. Regular selling price.

$$\begin{array}{l}
 C = 180 \times (1 - 0.2) = 180 \times 0.8 = \\
 \$144 \\
 M = 0.45 \times C = 0.45 \times 144 = \\
 \$64.80 \\
 S = C + M = 144 + 64.80 = \$208.80
 \end{array}$$

2. Reduced selling price.

$$\begin{aligned}
 S_{red} &= S \times (1 - d) \\
 &= 208.80 \times (1 - 0.25) \\
 &= \$81.60
 \end{aligned}$$

$$\begin{aligned}
 E &= 0.15 \times C \\
 &= 0.15 \times 144 \\
 &= \$21.60
 \end{aligned}$$

$$\begin{aligned}
 P_{red} &= S_{red} - C - E \\
 &= 81.60 - 144 - 21.60 \\
 &= -\$84
 \end{aligned}$$

TRY IT

A distributor sells water cooling units for \$1000 each. The operating profit is 18% of cost and the expenses are 22% of cost.

1. What is the cost per cooling unit?
2. During a sale, the units are sold at a price so that the profit is \$45 per unit. What is the rate of markdown at the reduced selling price?
3. If the units are marked down by 30%, what is the reduced profit or loss?

Click to see Solution

1. Cost.

$$\begin{aligned} S &= C + E + P \\ 1,000 &= C + 0.22 \times C + 0.18 \times C \\ 1,000 &= 1.4 \times C \\ C &= \$714.29 \end{aligned}$$

2. Rate of markdown.

$$\begin{aligned} E &= 0.22 \times C = 0.22 \times 714.29 = \$157.14 \\ S_{\text{red}} &= C + E + P_{\text{red}} = 714.29 + 157.14 + 45 = \$916.43 \\ D &= S - S_{\text{red}} = 1,000 - 916.43 = \$83.57 \\ d &= \frac{D}{S} \times 100\% = \frac{83.57}{1,000} \times 100\% = 8.357\% \end{aligned}$$

3. Reduced profit or loss.

$$\begin{aligned} S_{\text{red}} &= S \times (1 - d) \\ &= 1,000 \times (1 - 0.3) \\ &= \$700 \end{aligned}$$

$$\begin{aligned} P_{\text{red}} &= S_{\text{red}} - C - E \\ &= 700 - 714.29 - 157.14 \\ &= -\$171.43 \end{aligned}$$

Exercises

1. Ziggy's Skates purchases ice skates for \$30 each pair and sells them at a regular price of \$42 each pair.
 - a. If the profit made is \$5.25 per pair of ice skates, calculate the overhead expense per pair.
 - b. If the discount offered during a Holiday Sale is 20%, calculate the reduced selling price and the profits or loss made on the sale of each pair.

Click to see Answer

- a. \$6.75; b. \$33.60, -\$3.15
2. Belanger Acoustics purchased acoustic guitars for \$80 each and has marked them up by 20% on cost. The overhead expenses were 10% on cost.

- a. Calculate the regular selling price of each guitar and the profit made.
- b. If Belanger decides to offer a markdown of 5% what would be the reduced selling price and profit or loss they would make on the sale of each guitar?

Click to see Answer

a. \$96.00, \$8.00; b. \$91.20, \$3.20

3. Julia makes a 10% profit on the cost of T shirts, which she purchases at \$15.00 each. The overhead expenses are 25% on cost. During a sale, she marked the T shirts down by 10%.
 - a. What is the profit or loss on the sale of this item?
 - b. What is the amount of markup on the sale price of the item?

Click to see Answer

a. -\$0.53; b. \$3.22

4. Guilherme was in the business of purchasing painting from Brazil and selling them in Toronto at his shop. On one shipment listed at \$2,400, he received trade discounts of 10%, 8%, and 6%. The overhead expenses were 15% of his costs and he wanted to make an operating profit of 20% on cost.
 - a. Calculate the regular selling price of the painting.
 - b. Calculate the loss or profit he will make if he decides to markdown the selling prices by 15%.
 - c. Calculate the maximum amount of markdown that he can offer so that he breaks even on the sale.

Click to see Answer

a. \$2521.76; b. 3.52%; c. 25.93%

5. Juanita purchased designer purses for \$242.88 each, less 12% and 8%. The markup is 35% on selling price and the operating profit is 15% on cost. During a sale, the designer purses were marked down to \$260.00.
 - a. What was the regular selling price?
 - b. What was the rate of markdown?
 - c. At the sale price, what was the profit of loss?

Click to see Answer

a. \$302.52; b. 14.06%; c. -\$13.02

6. The regular selling price of cell phones at a store is \$125 each. During a sale, it was sold at a markdown of 45%. Calculate the profit or loss made on the sale of the cell phone if the break-even price is \$75.

Click to see Answer

-\$6.25

7. A retailer purchased shirts for \$50 each, less 10%. The retailer has a markup of 20% on selling price and an operating profit of 10% on cost. During a sale, the shirts were marked down and sold at break-even price.
- What was the regular selling price of each shirt?
 - What was the sale price?
 - What was the rate of markdown offered during the sale?

Click to see Answer

a. \$56.25; b. \$51.75; c. 8%

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1.6 REVIEW EXERCISES

1. An invoice for \$15,000 dated June 16 has terms of $3/15, 2/30, n/45$. What payment in full is required if the payment is received on July 3?

Click to see Answer

\$14,700

2. An invoice for \$37,650 is dated June 24 with terms of $2/10, 1/30, n/60$. If a payment of \$20,000 is made on July 2 and another payment of \$10,000 is made on July 23, what is the balance remaining on the invoice?

Click to see Answer

\$7,140.83

3. What amount must be remitted if invoices dated January 25 for \$1,700, February 1 for \$1,800, and February 15 for \$900, all with terms of $3/25, n/50$ are paid with the same cheque on March 12? Assume it is a non-leap year.

Click to see Answer

\$4,427.84

4. An invoice for \$13,398.25 dated August 27 with terms of $5/10, 2/25, n/45$ was partially paid on September 7. If the invoice balance was reduced to \$2,598.25, what was the amount of the payment?

Click to see Answer

\$10,584

5. An invoice for \$147,477.77 dated July 26 with terms of $5/15, 3.5/25, 2/40, n/60$ was partially paid with two equal payments on August 10 and September 4, reducing the balance owing to \$76,579.81. What amount was paid on each of the two dates?

Click to see Answer

\$34,200

6. The Gap is purchasing an Eminem graphic T-shirt for its stores. If the list price for the T-shirt is \$34.50 and the Gap can receive a 40% trade discount, what net price will the Gap pay?

Click to see Answer

\$20.70

7. Nike has been approached by a retailer who wants to carry their Shox brand. The retailer needs a trade discount of 30% and its wholesaler needs 15%. The retailer also wants a 5% quantity discount. If the list price for the shoes is \$195.00, what is the wholesale (net) price that Nike will charge?

Click to see Answer

\$110.22

8. If the regular price of a product is \$775.00 and it is eligible to receive discounts of 14%, 8%, and 5%, what single discount percentage is equal to the multiple discount percentages? Show calculations that show how the single and multiple discounts produce the same net price.

Click to see Answer

24.84%, \$582.52

9. Subway wants to price a new 12 inch sandwich. The cost of all the ingredients is \$1.23. Expenses are 200% of cost, and Subway wants to earn a profit of 187% of cost. What is the regular selling price of the new sandwich?

Click to see Answer

\$5.99

10. If the cost of a Bose Wave music system is \$27.82 and the list price is \$49.99, determine the markup amount. Express the markup both as the rate of markup on cost and the rate of markup on selling price.

Click to see Answer

\$22.17, 79.69%, 44.35%

11. Rogers Communications is thinking about having a special promotion on its Digital One Rate 250 Plan, which offers 250 minutes of cellular calling. If the current price is \$85.00 per month and Rogers wants to put it on special for \$75.00 per month, what is the rate of markdown?

Click to see Answer

11.76%

12. An iMac has a list price of \$774.99 with available trade discounts of 20% to the retailer and an additional 15% to the wholesaler.
- What price should a retailer pay for the product? What is the dollar value of the discount?
 - What price should a wholesaler pay for the product? How much less than the retailer's price is this?
 - Overall, what single discount percent does the wholesaler receive?

Click to see Answer

a. \$619.99, \$155; b. \$526.99, \$93 less; c. 32%

13. The marketing manager for Tim Hortons is attempting to price a cup of coffee. She knows that the cost of the coffee is \$0.14 per cup, and expenses are 30% of the regular selling price. She would like the coffee to achieve an 88.24% markup on selling price.
- What is the regular selling price for a cup of coffee?
 - What is the profit per cup?
 - What is the rate of markup on cost?

Click to see Answer

a. \$1.19; b. \$0.69; c. 750%

14. Jonathan needs a new business suit for a presentation in his communications course. He heads to Moores and finds a suit on sale for 33% off. The regular price of the suit is \$199.50.
- What is the sale price?
 - What is the dollar amount of the markdown?

Click to see Answer

a. \$133.67; b. \$65.83

15. Birchwood Honda needs to clear out its Honda generators for an end-of-season sale. If the dealership pays \$319.00, has expenses of 15% of cost, and profits of 30% of the regular unit selling price, what rate of markdown can the dealership advertise if it wants to break even during the sale?

Click to see Answer

30%

16. A retailer purchasing the Halo 3 video game is eligible to receive a trade discount, quantity discount, and loyalty discount. If the net price of the game is \$22.94, the list price is \$39.99, the trade discount is 25%, and the quantity discount is 15%, what percentage is the loyalty discount?

Click to see Answer

10.02%

17. Indigo is stocking a new hardcover book. The book has a cover price of \$44.99 and Indigo is eligible to receive discounts of 25%, 8%, 4%, and 1%. Expenses are 20% of cost. The book sells for the cover price.
- What is the profit per book?
 - Some new books are launched at 30% off the cover price. From a strictly financial perspective, would it be wise for Indigo to use this approach for this book? Why or why not?

Click to see Answer

a. \$9.59; b. -\$3.91

18. Mary is shopping for a new George Foreman grill. While shopping at Polo Park Shopping Centre, she came across the following offers at three different retail stores:

- Offer #1: Regular price \$149.99, on sale for 30% off.
- Offer #2: Regular price \$169.99, on sale for 15% off plus an additional 30% off.
- Offer #3: Regular price \$144.95, on sale for 10% off plus an additional 15% off. Mary also gets a 5% loyalty discount at this store.

Which is the best offer for Mary?

Click to see Answer

Offer #2 has lowest price of \$101.14

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1.7 CHAPTER SUMMARY

Key Concepts

- **1.1 Discounts**
 - The relationship between distribution and pricing.
 - Some of the discounts available to businesses and consumers.
 - How to calculate the net price when only one discount is involved.
 - How to calculate the net price when multiple discounts are involved.
 - Converting multiple discounts into single discounts.
 - **1.2 Invoices, Terms of Payments, and Cash Discounts**
 - How invoicing works.
 - Understanding invoice terms and invoice dating.
 - The calculations involved when a full payment amount is made.
 - The calculations involved when a partial payment amount is made.
 - **1.3 Markup**
 - The three components that compose a selling price.
 - Calculating the markup in dollars and the relationship to pricing components.
 - Calculating the markup as a percentage under two different approaches.
 - Determining the price point where all costs are paid but no profits are earned—the break-even point.
 - **1.4 Markdown**
 - How to take a regular selling price and make it into a sale price by applying markdowns.
 - **1.5 Working with Discounts, Markup, and Markdown**
 - Calculating the reduced profit.
 - Combining discounts, markup, and markdown into complete merchandising situations.
-

Glossary of Terms

- **Amount Credited.** The amount deducted from the invoice balance as a result of the payment.
- **Amount Paid.** The actual payment amount made that must be converted into its value toward the invoice total.

- **Cash Discount.** A percentage of the balance owing on an invoice that can be deducted for payment received either in full or in part during the discount period.
- **Cost.** An outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services.
- **Credit Period.** The number of interest-free days from the date of commencement before full payment of the invoice is required.
- **Date of Commencement.** The first day from which the invoice terms extend forward in time and from which all due dates are established.
- **Discount.** A reduction in the price of a product.
- **Discount Period.** The number of days from the date of commencement for which a cash discount is offered.
- **End-of-Month Invoice Dating.** A term of payment where the date of commencement is the last day of the same month as indicated by the invoice date.
- **Expenses.** A business's financial outlays incurred in the selling of a product.
- **List Price.** A price for a product that has been published or advertised in some way.
- **Loyalty Discount.** A discount given from a seller to a purchaser for repeat business.
- **Manufacturer's Suggested Retail Price (MSRP).** A recommended product retail price that a manufacturer sets for a retailer based on market research.
- **Markdown.** A reduction from the regular selling price of a product resulting in a new lower sale price.
- **Markup.** The process of taking a product's cost and increasing it by a certain amount to arrive at a selling price.
- **Markup Amount.** The dollar amount of the expenses and profit combined together into a single number. It represents the difference between the price and cost in dollars.
- **Net Price.** The price of the product after a discount is removed from the list price.
- **Ordinary Invoice Dating.** A term of payment where the date of commencement is the same date as the invoice date.
- **Profit.** The amount of money that remains after a business pays all of its costs and expenses.
- **Quantity Discount.** A discount for purchasing larger quantities of a certain product.
- **Rate of Markup on Cost.** The markup dollars expressed as a rate using cost as the base.
- **Receipt-of-Goods Invoice Dating.** A term of payment where the date of commencement is the day on which the customer physically receives the goods.
- **Rate of Markup on Selling Price.** The markup dollars expressed as a rate using the regular selling price as the base.
- **Reduced Profit.** The reduced profit is the amount of profit earned when selling an item at a reduced selling price.
- **Reduced Selling Price or Sale Price.** A price for a product after a markdown that is lower than its regular selling price.

- **Sale Discount.** A temporary discount lowering the price from a product's regular selling price.
 - **Seasonal Discount.** A discount offered to consumers and businesses for purchasing products out of season.
 - **Single Equivalent Discount.** A single discount rate that is equal to a series of multiple rate discounts.
 - **Trade Discount.** A discount offered to businesses only based on the type of business and its location in the distribution system.
-

Formulas

• Symbols Used

- BE = break-even price
- C = cost
- d = discount rate or rate of markdown
- $d_1, d_2, \dots, d_n$ = multiple discount rates, where the subscript represents each discount up to a count of n discounts.
- D = discount amount or amount of markdown
- d_e = single equivalent discount rate
- E = expenses
- L = list price
- M = amount of markup
- N = net price
- P = profit
- P_{red} = reduced profit
- ROM_C = rate of markup on cost
- ROM_S = rate of markup on selling price
- S = regular selling price
- S_{red} = reduced selling price or sale price

• Formulas Used

- Single Discount: $N = L \times (1 - d)$
- Discount Amount: $D = d \times L$
- Discount Amount: $D = L - N$
- Multiple Discounts: $N = L \times (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$

- Amount Credited: **Formula does not parse**
 - Amount Paid: $\text{Amount Paid} = \text{Amount Credited} \times (1 - d)$
 - Single Equivalent Discount: $d_e = 1 - [(1 - d_1) \times (1 - d_2) \times \cdots \times (1 - d_n)]$
 - Selling Price: $S = C + E + P$
 - Markup Amount: $M = E + P$
 - Selling Price using Markup Amount: $S = C + M$
 - Rate of Markup on Cost: $ROM_C = \frac{M}{C} \times 100\%$
 - Rate of Markup on Selling Price: $ROM_S = \frac{M}{S} \times 100\%$
 - Reduced Profit: $P_{red} = S_{red} - C - E$
 - Reduced Profit: $P_{red} = P - D$
 - Reduced Selling Price: $S_{red} = S \times (1 - d)$
 - Markdown Amount: $D = d \times S$
 - Markdown Amount: $D = S - S_{red}$
 - Rate of Markdown: $d = \frac{D}{S} \times 100\%$
-

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PART II

COST-VOLUME-PROFIT ANALYSIS

When a new retail store opens in your neighbourhood strip mall, you wonder whether it will still be running six months or a year from now. Do the owners know how much merchandise they have to sell to cover their costs? Will any money be left over to help grow the business?

Many new businesses fail. Why? Economic conditions, the fierceness of competition, changing consumer tastes, or even changes in taxes all contribute to the problem, but one of the most common reasons businesses fail is poor tracking of their basic financial numbers. So, if you want your own business to succeed, you need to understand its financial side. You need to be able to answer questions like the following.

- How does the money leave the business through costs and expenses?
- How does the money come into the business through revenue, which is determined by both the price tags on the merchandise and the volume at which that merchandise sells?
- What is the minimum amount of revenue that is needed to cover the costs?
- What is the resulting profitability of the business?

This chapter introduces the global model of the relationship between costs and revenues. It shows the overall cost structure of a business and introduces models for developing total costs and total revenues. It explains how you calculate net income and contribution margins. Armed with all of this knowledge, you can then answer the break-even question: How much does the business need to sell to cover its costs?

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2.1 COST-VOLUME-PROFIT ANALYSIS TERMINOLOGY

LEARNING OBJECTIVES

- Understand the terms used in cost-volume-profit analysis and their relationships to each other.

The end of the month is approaching, and bills are coming due. As you sit at your kitchen table trying to figure out your budget for next month, you wonder whether you will be able to afford that concert you had been planning on attending. Some of your costs remain unchanged from month to month, such as your rent, internet service, gym membership, and insurance. Other costs tend to fluctuate with your usage, such as your utilities, cellphone bill, vehicle fuel, and the amount of money spent on recreational activities. Together, these regular and irregular costs total to next month's costs.

Examining a few recent paycheque stubs, you calculate the average monthly net income you bring home from your hourly cashier position at Sobeys. The exact amount of each paycheque, of course, depends on how many hours you work. Besides your short-term costs, you need to start saving for next year's tuition. Therefore, your budget needs to include regular deposits into your savings account to meet that goal. Once you have put your bills, paycheques, and goals together, you hope that your budget will balance. If there is a shortfall, you will have to miss out on those concert tickets.

Budgeting at work is no different in principle from your home budget. Businesses also need to recognize the different types of costs they incur each month, some of which remain the same and some of which fluctuate. Businesses need to pay for these costs by generating revenues, which correspond to your paycheque. As with your education goals, businesses also require profits to grow. A business needs to understand all of these numbers so it can plan its activities realistically.

A business needs to understand how to relate its total costs to its total revenue in order to determine its total profitability levels. This process is called **break-even analysis**, which is a technique that allows a business to determine the level of sales required to cover its costs and how to set targets to achieve profitability.

Costs

A **cost** is an outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services. In business, examples of costs include rent, employees salaries, taxes, utilities, raw materials, and maintenance, to name a few. Costs can be split into two distinct categories: fixed costs and variable costs.

Fixed costs (FC) are costs that do not change with the level of production or sales (call this “output” for short). In other words, whether the business outputs nothing or outputs 10,000 units, these costs remain the same. Some examples of fixed costs include rent, insurance, property taxes, salaries unrelated to production (such as management), production equipment, office furniture, and much more. **Total fixed costs** are the sum of all fixed costs that a business incurs.

A **variable cost** is a cost that changes with the number of units produced. In other words, if the business produces nothing there are no variable cost. However, if the business produces just one unit (or more) then a variable cost appears. Some examples of variable costs include material costs of products, production labor (hourly or piecework wages), sales commissions, repairs, maintenance, and more. **Total variable costs** (TVC) are the sum of all variable costs that a business incurs at a particular level of output.

In calculating business costs, fixed costs are commonly calculated on a total basis only because the business incurs these costs regardless of any production. However, variable costs are commonly calculated both on a total and per-unit basis to reveal the overall cost along with the cost associated with any particular unit of output. When these variable costs are assigned on an individual basis it is called a **variable cost per unit** (VC). The calculation of the per unit variable cost has a further benefit because it allows managers to explore how the total business costs vary at different levels of output.

The variable cost per unit is

$$VC = \frac{TVC}{x}$$

where

- **VC is the variable cost per unit.** The unit variable cost is the typical or average variable cost associated with an individual unit of output. Being a dollar cost, the unit variable cost is rounded to two decimals.

- ***TVC* is the total variable cost.** The total variable costs is the sum of all the variable costs that were incurred at a particular level of output.
- ***x* is the number of units produced.** This is the number of units produced or sold or the total output that incurred the total variable costs.

Rearranging the variable cost per unit formula, the total variable costs is

$$TVC = VC \times x$$

EXAMPLE

Assume a company produces 10,000 units and wants to know its unit variable cost. It incurs production labour costs of \$3,000, material costs of \$1,875, and other variable costs totaling \$1,625.

Solution:

Step 1: In this case, all of the costs (production labour and material costs) are variable costs. The total variable costs are

$$\begin{aligned} TVC &= 3,000 + 1,875 + 1,625 \\ &= \$6,500 \end{aligned}$$

Step 2: The variable cost for unit is

$$\begin{aligned} VC &= \frac{TVC}{x} \\ &= \frac{6,500}{10,000} \\ &= \$0.65 \end{aligned}$$

The variable cost associated with one unit of production is \$0.65.

NOTE

The definitions of variable and fixed costs along with their typical associated examples

represent a simplified view of how the real world operates. The complexities involved in real-world business costs complicate the fundamentals of managing a business. Here is how this textbook addresses the complexities of atypical operations, changing fixed costs, and decreasing unit variable costs:

- **Atypical Operations.** Although there are “normal” ways that businesses operate, there are also businesses that have atypical operations. What is a fixed cost to one business may be a variable cost to another. For example, rent is usually a fixed cost. However, some rental agreements include a fixed base cost plus a commission on the operational output. This textbook does not venture into any of these atypical costs and instead focuses on common cost categorizations.
- **Changing Fixed Costs.** In real-world applications, fixed costs do not remain flat at all levels of output. As output increases, fixed costs tend to move upwards in steps. For example, at a low level of output only one manager (on salary) may be needed. As output increases, eventually another manager needs to be hired, perhaps one for every 20,000 units produced. In other words, up to 20,000 units the fixed costs would be constant, but at 20,001 units the fixed costs take a step upwards as another manager is added. The model presented in this textbook does not address these upward steps and treats fixed costs as a constant at all levels of output.
- **Decreasing Unit Variable Costs.** Production tends to realize efficiencies as the level of output rises, resulting in the unit variable cost dropping. This is commonly known as achieving **economies of scale**. As a consumer, you often see a similar concept in your retail shopping. If you purchase one can of soup, it may cost \$1. However, if you purchase a bulk tray of 12 cans of soup it may cost only \$9, which works out to \$0.75 per can. This price is lower partly because the retailer incurs lower costs, such as fewer cashiers to sell 12 cans to one person than to sell one can each to 12 different people. Now apply this analogy to production. Producing one can of soup costs \$0.75. However, a larger production run of 12 soup cans may incur a cost of only \$6 instead of \$9 because workers and machines can multitask. This means the unit variable cost would decrease by \$0.25 per can. However, the model in this textbook assumes that unit variable costs always remain constant at any given level of output.

The **total costs** (TC) is the sum of the fixed costs and total variable costs.

$$TC = FC + TVC$$

where

- **TC is the total costs.** The total costs is the sum of all of costs incurred by the business.
- **FC is the fixed costs.** These are the costs that do not change regardless of the level of production.
- **TVC is the total variable costs.** The total variable costs is the sum of all the variable costs that were incurred at a particular level of output.

EXAMPLE

You are considering starting your own home-based internet business. After a lot of research, you have gathered the following financial information.

Dell computer	\$214.48 monthly lease payments
Office furniture	\$186.67 monthly rental
High-speed internet connection	\$166.88 per month
Wages	\$30 per hour
Utilities (fixed)	\$13 per month
Utilities (variable)	\$0.20 per hour
Software plus upgrades	\$20 per month
Licences and permits	\$27 per month
Google click-through rate	\$10 per month

Generating and fulfilling sales of 430 units involves 80 hours of work per month. Based on industry response rates, your research also shows that to achieve your sales you require a traffic volume of 34,890 Google clicks.

On a monthly basis, calculate the total fixed cost, total variable cost, and unit variable cost.

Solution:

Step 1: Sort the costs into fixed and variable costs.

Fixed Costs		Variable Costs	
Dell computer	\$214.48	Wages	\$30 per hour
Office Furniture	\$186.67	Utilities	\$0.20 per hour
Internet	\$166.88		
Utilities	\$13		
Software	\$20		
Licences and permits	\$27		
Google click-through rate	\$10		

Step 2: Add up the fixed costs.

$$\begin{aligned} FC &= 214.48 + 186.67 + 166.88 + 13 + 20 + 27 + 10 \\ &= \$638.03 \end{aligned}$$

Step 3: Calculate the total variable costs based on the 80 hours.

$$\begin{aligned} TVC &= 30 \times 80 + 0.2 \times 80 \\ &= \$2,416 \end{aligned}$$

Step 4: Calculate the variable cost per unit based on the 430 units sold.

$$\begin{aligned} VC &= \frac{2,416}{430} \\ &= \$5.62 \end{aligned}$$

Revenue and Selling Price

The **selling price per unit** (S) is the amount the business charges its customers to purchase one unit of an item produced.

Revenue is how much money or gross income the sale of the product at a certain output level brings into the business. **Total revenue** (TR) is the entire amount of money received by a company for selling its product calculated by multiplying the quantity sold by the selling price (S).

$$TR = S \times x$$

where

- **TR is the total revenue.** This is how much money or gross income the sale of the product at a certain output level brings into the organization.
- **S is the selling price per unit.** The unit selling price of the product.
- **x is the number of units produced.** This is the number of units produced or sold or the total output that incurred the total variable costs.

EXAMPLE

A business sells a product for \$10 each. How much revenue does the business generate if it produces and sell 5,000 units?

Solution:

Step 1: The given information is

$$\begin{aligned} S &= 10 \\ x &= 5,000 \end{aligned}$$

Step 2: Calculate the total revenue.

$$\begin{aligned} TR &= S \times x \\ &= 10 \times 5,000 \\ &= \$50,000 \end{aligned}$$

Net Income and the Break-Even Point

The **net income** (NI) is the difference between the total revenue and the total costs.

$$NI = TR - TC$$

Note that many companies use the terms net earnings or net profit instead of the term net income. Net income is based on a certain level of output. Here, we assume that the number of units produced or purchased exactly matches the number of units that are output or sold by the company. In other words, there is no inventory, and its associated costs, to consider.

There are three scenarios to consider.

- Total revenue is greater than total costs ($TR > TC$). In this case the net income is positive ($NI > 0$) and the business makes a **profit**.
- Total revenue is less than total costs ($TR < TC$). In this case the net income is negative ($NI < 0$) and the business incurs a **loss**.
- Total revenue equals total cost ($TR = TC$). In this case the net income is zero ($NI = 0$). This is the **break-even point**.

If you are starting your own business and head to the bank to initiate a start-up loan, one of the first questions the banker will ask you is your break-even point. You calculate this number through **break-even analysis**, which is the analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost. This **break-even point** is the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero.

EXAMPLE

A business sells an item for \$75 each. The variable costs are \$58 per item and the fixed costs are \$4,250 per month.

1. What is the net income if the business sells 375 units?
2. What is the net income if the business sells 200 units?
3. What is the net income if the business sells 250 units?

Solution:

Step 1: The given information is

$$\begin{aligned} S &= \$75 \\ VC &= \$58 \\ FC &= \$4,250 \end{aligned}$$

The total revenue function is $TR = 75 \times x$. The total cost function is $TC = 4,250 + 58 \times x$.

Step 2: Calculate the net income for 375 units.

$$\begin{aligned}
 NI &= TR - TC \\
 &= 75 \times 375 - (4,250 + 58 \times 375) \\
 &= 28,125 - 26,000 \\
 &= \$2,125
 \end{aligned}$$

The net income at 375 units is \$2,125. Because the net income is positive, at 375 units the business generates a profit.

Step 3: Calculate the net income for 200 units.

$$\begin{aligned}
 NI &= TR - TC \\
 &= 75 \times 200 - (4,250 + 58 \times 200) \\
 &= 15,000 - 15,850 \\
 &= -\$850
 \end{aligned}$$

The net income at 200 units is -\$850. Because the net income is negative, at 200 units the business incurs a loss.

Step 4: Calculate the net income for 250 units.

$$\begin{aligned}
 NI &= TR - TC \\
 &= 75 \times 250 - (4,250 + 58 \times 250) \\
 &= 18,750 - 18,750 \\
 &= \$0
 \end{aligned}$$

The net income at 250 units is \$0. Because the net income is zero, 250 units is the break-even point.

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2.2 REVENUE AND COST FUNCTIONS

LEARNING OBJECTIVES

- Solve cost-volume-profit analysis problems using the revenue and cost function approach.

Most businesses are “for-profit” businesses, meaning that they operate to make money. There are no guarantees in business, and the future is always uncertain. What if you have to pay your employees higher wages? What if the cost of the raw material you use in production decreases? What if you sell more or less items than expected? Successful business managers plan for the future and perform many “what-if” scenarios to answer these types of questions. This section develops a model for calculating total net income based on total revenues and total costs. The model allows managers to analyze various scenarios and determine the impact on profitability.

However, simply looking at the fixed costs, variable costs, potential revenues, contribution margins, and typical net income is not enough. Ultimately, to have any chance of staying in business all costs in a business need to be recovered through sales. Do you know how many units have to be sold to pay your bills? The answer to this question helps assess the feasibility of your business idea.

Revenue and Cost Functions

Recall from the previous section that total costs (TC) are the total costs (fixed and variable) incurred by a business and total revenue (TR) is the amount of money received from the sales of the items produced and sold.

$$TC = FC + VC \times x$$

$$TR = S \times x$$

where

- **TC is the total costs.** The total costs is the sum of all of costs incurred by the business.
- **FC is the fixed costs.** These are the costs that do not change regardless of the level of production.
- **VC is the variable cost per unit.** The unit variable cost is the typical or average variable cost associated with an individual unit of output.
- **TR is the total revenue.** This is how much money or gross income the sale of the product at a certain output level brings into the organization.
- **S is the selling price per unit.** The unit selling price of the product.
- **x is the number of units produced.** This is the number of units produced or sold or the total output that incurred the total variable costs.

Because the number of units, x , varies, we can think of the total costs and total revenue formulas as functions of the number of units, x .

Net Income using Revenue and Cost Functions

The **net income** (NI) is the difference between the total revenue and the total costs.

$$NI = TR - TC$$

Note that many companies use the terms net earnings or net profit instead of the term net income. Net income is based on a certain level of output. Here, we assume that the number of units produced or purchased exactly matches the number of units that are output or sold by the company. In other words, there is no inventory, and its associated costs, to consider.

Substituting in the functions for TR and TC , we can express the net income NI in terms of the number of units x .

$$NI = S \times x - (FC + VC \times x)$$

where

- **NI is the net income.** The amount of money left over after all costs have been paid is the net income. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss.
- **S is the selling price per unit.** The unit selling price of the product.
- **x is the number of units produced.** This is the number of units produced or sold or the total output that incurred the total variable costs.

- **FC is the fixed costs.** These are the costs that do not change regardless of the level of production.
- **VC is the variable cost per unit.** The unit variable cost is the typical or average variable cost associated with an individual unit of output.

There are three scenarios to consider.

- Total revenue is greater than total costs ($TR > TC$). In this case the net income is positive ($NI > 0$) and the business makes a **profit**.
- Total revenue is less than total costs ($TR < TC$). In this case the net income is negative ($NI < 0$) and the business incurs a **loss**.
- Total revenue equals total cost ($TR = TC$). In this case the net income is zero ($NI = 0$). This is the **break-even point**.

EXAMPLE

A company's monthly fixed costs for producing an item are \$12,000. The variable cost per unit is \$10. The company sells the item for \$25 each.

1. What is the net income if the company produces and sells 1,000 units next month? Did they make a profit or loss at this level of output?
2. What is the net income if the company produces and sells 500 units next month? Did they make a profit or loss at this level of output?
3. How many units must be produced and sold for a profit of \$2,400?
4. How many units must be produced and sold for a loss of \$2,850?

Solution:

Step 1: The given information is

$$FC = \$12,000$$

$$VC = \$10$$

$$S = \$25$$

The total revenue function is

$$\begin{array}{l} TR = S \times x \\ TR = 25 \times x \end{array}$$

The total cost function is

$$\begin{aligned} TC &= FC + VC \times x \\ &= 12,000 + 10 \times x \end{aligned}$$

Step 2: Calculate the net income for $x = 1,000$.

$$\begin{aligned} TR &= 25 \times 1,000 \\ &= \$25,000 \end{aligned}$$

$$\begin{aligned} TC &= 12,000 + 10 \times 1,000 \\ &= \$22,000 \end{aligned}$$

$$\begin{aligned} NI &= TR - TC \\ &= 25,000 - 22,000 \\ &= \$3,000 \end{aligned}$$

At 1,000 units, the net income is \$3,000. Because the net income is positive, the company made a profit at 1,000 units.

Step 3: Calculate the net income for $x = 500$.

$$\begin{aligned} TR &= 25 \times 500 \\ &= \$12,500 \end{aligned}$$

$$\begin{aligned} TC &= 12,000 + 10 \times 500 \\ &= \$17,000 \end{aligned}$$

$$\begin{aligned} NI &= TR - TC \\ &= 12,500 - 17,000 \\ &= -\$4,500 \end{aligned}$$

At 500 units, the net income is -\$4,500. Because the net income is negative, the company incurs a loss at 500 units.

Step 4: Calculate the units for $NI = 2,400$. Because this is a profit the net income is **positive**.

$$\begin{aligned}
 NI &= S \times x - (FC + VC \times x) \\
 2,400 &= 25 \times x - (12,000 + 10 \times x) \\
 2,400 &= 25 \times x - 12,000 - 10 \times x \\
 14,400 &= 15 \times x \\
 \frac{14,400}{15} &= x \\
 960 &= x
 \end{aligned}$$

A profit of \$2,400 occurs when 960 units are produced and sold.

Step 5: Calculate the units for $NI = -2,850$. Because this is a loss the net income is **negative**.

$$\begin{aligned}
 NI &= S \times x - (FC + VC \times x) \\
 -2,850 &= 25 \times x - (12,000 + 10 \times x) \\
 -2,850 &= 25 \times x - 12,000 - 10 \times x \\
 9,150 &= 15 \times x \\
 \frac{9,150}{15} &= x \\
 610 &= x
 \end{aligned}$$

A loss of \$2,850 occurs when 610 units are produced and sold.

NOTES

1. In the previous example, a loss occurred at 500 units and a profit occurred at 1,000 units. This means that somewhere between 500 units and 1,000 units is a point where the net income is 0.
2. Remember, net income is positive when you have a profit and net income is negative when you have a loss.

TRY IT

A furniture manufacturer sells tables for \$300 each. The variable cost per table is \$120. The monthly fixed costs are \$5,000.

1. What is the net income if 30 tables are produced next month?
2. How many tables must be produced and sold to incur a loss of \$3,200 next month?

Click to see Solution

1. Net income for 30 tables.

$$\begin{aligned} NI &= 300 \times 30 - (5,000 + 120 \times 30) \\ &= 9,000 - 5,000 - 3,600 \\ &= \$400 \end{aligned}$$

2. Units for loss of \$300.

$$\begin{aligned} NI &= 300 \times x - (5,000 + 120 \times x) \\ -3,200 &= 300 \times x - 5,000 - 120 \times x \\ 1,800 &= 180 \times x \\ x &= 10 \end{aligned}$$

EXAMPLE

You run a small business that produces custom-ordered t-shirts. You have fixed costs of \$2,700 per month. Each t-shirt has a variable cost of \$25. You sell each t-shirt for \$40.

1. What is the net income for a month in which you produce and sell 150 shirts? Would you

incur a profit or loss at this level of production?

2. You decide to increase your selling price to \$45 per shirt. What is the net income for 150 shirts with this higher selling price? Are you making a profit or loss?
3. One of your suppliers has lowered their prices. This decreases the variable cost per shirt to \$18. What is the net income for 150 shirts with this lower variable cost? Are you making a profit or loss? (Assume the original \$40 selling price).

Solution:

Step 1: The given information is

$$FC = \$2,700$$

$$VC = \$25$$

$$S = \$40$$

The total revenue function is

$$\begin{array}{l} TR = S \times x \\ TR = 40 \times x \end{array}$$

The total cost function is

$$\begin{aligned} TC &= FC + VC \times x \\ &= 2,700 + 25 \times x \end{aligned}$$

Step 2: Calculate the net income for $x = 150$.

$$\begin{aligned} TR &= 40 \times 150 \\ &= \$6,000 \end{aligned}$$

$$\begin{aligned} TC &= 2,700 + 25 \times 150 \\ &= \$6,450 \end{aligned}$$

$$\begin{aligned} NI &= TR - TC \\ &= 6,000 - 6,450 \\ &= -\$450 \end{aligned}$$

At 150 shirts, the net income is -\$450. Because the net income is negative, you incur a loss at 150 shirts.

Step 3: Calculate the net income for $x = 150$ with $S = \$45$.

$$\begin{aligned} TR &= 45 \times 150 \\ &= \$6,750 \end{aligned}$$

$$\begin{aligned} TC &= 2,700 + 25 \times 150 \\ &= \$6,450 \end{aligned}$$

$$\begin{aligned} NI &= TR - TC \\ &= 6,750 - 6,450 \\ &= \$350 \end{aligned}$$

With the selling price per shirt at \$45, a profit of \$350 is made when 150 shirts are produced and sold.

Step 4: Calculate the net income for $x = 150$ with $VC = \$18$.

$$\begin{aligned} TR &= 40 \times 150 \\ &= \$6,000 \end{aligned}$$

$$\begin{aligned} TC &= 2,700 + 18 \times 150 \\ &= \$5,400 \end{aligned}$$

$$\begin{aligned} NI &= TR - TC \\ &= 6,000 - 5,400 \\ &= \$600 \end{aligned}$$

With the variable cost per shirt at \$18, a profit of \$600 is made when 150 shirts are produced and sold.

Break-Even Analysis with Revenue and Cost Functions

If you are starting your own business and head to the bank to initiate a start-up loan, one of the first questions the banker will ask you is your break-even point. You calculate this number through **break-even analysis**, which is the analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost ($TR = TC$). This **break-even point** is the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero.

The purpose of break-even analysis is to determine the point at which total cost equals total revenue. To find

the break-even point in terms of the number of units sold (x), solve for x in the equation where $TR = TC$. That is the break-even point is the value of x so that

$$S \times x = FC + VC \times x$$

Because total revenue equals total cost at the break-even point, the net income at the break-even point is zero.

EXAMPLE

A company's monthly fixed costs for producing an item are \$12,000. The variable cost per unit is \$10. The company sells the item for \$25 each.

1. Calculate the number of units the company should produce and sell to break-even.
2. What is the revenue at the break-even point?
3. Suppose the fixed costs increase by \$1,500 with no changes to the variable costs or selling price. What is the new break-even point?
4. Suppose the selling price decreases by \$3 with no changes to the variable costs or fixed costs. What is the new break-even point?
5. Suppose the variable costs increase by 50% with no changes to the fixed costs or selling price. What is the new break-even point?

Solution:

Step 1: The given information is

$$FC = \$12,000$$

$$VC = \$10$$

$$S = \$25$$

The total revenue function is

$$TR = S \times x = 25 \times x$$

The total cost function is

$$\begin{aligned} TC &= FC + VC \times x \\ &= 12,000 + 10 \times x \end{aligned}$$

Step 2: Calculate the break-even point.

$$\begin{aligned} \text{TR} &= \text{TC} \\ 25x &= 12,000 + 10x \\ 25x - 10x &= 12,000 \\ 15x &= 12,000 \\ x &= \frac{12,000}{15} \\ x &= 800 \end{aligned}$$

The break-even point is 800 units.

Step 3: Calculate the revenue for $x = 800$.

$$\begin{aligned} \text{TR} &= 25x \\ &= 25 \times 800 \\ &= \$20,000 \end{aligned}$$

The revenue at the break-even point is \$20,000.

Step 4: Calculate the new break-even point when $FC = 12,000 + 1,500 = \$13,500$.

$$\begin{aligned} \text{TR} &= \text{TC} \\ 25x &= 13,500 + 10x \\ 25x - 10x &= 13,500 \\ 15x &= 13,500 \\ x &= \frac{13,500}{15} \\ x &= 900 \end{aligned}$$

When the fixed costs increase by \$1,500, the break-even point is 900 units.

Step 5: Calculate the new break-even point when $S = 25 - 3 = \$22$.

$$\begin{aligned} \text{TR} &= \text{TC} \\ 22x &= 12,000 + 10x \\ 22x - 10x &= 12,000 \\ 12x &= 12,000 \\ x &= \frac{12,000}{12} \\ x &= 1,000 \end{aligned}$$

When the selling price decreases by \$3, the break-even point is 1,000 units.

Step 6: Calculate the new break-even point when $VC = 10 + 0.5 \times 10 = \$15$.

$$\begin{aligned} \text{TR} &= \text{TC} \\ 25x &= 12,000 + 15x \\ 25x - 15x &= 12,000 \\ 10x &= 12,000 \\ x &= \frac{12,000}{10} \\ x &= 1,200 \end{aligned}$$

When the variable costs increase by 50%, the break-even point is 1,200 units.

NOTE

When you calculate the break-even units, the value of x you get when you solve

$S \times x = FC + VC \times x$ may be a decimal. For example, a break-even point might be 324.39 units. How should you handle the decimal? A partial unit cannot be sold, so the rule is always to round the level of output up to the next integer, regardless of the decimal. Why? The main point of a break-even analysis is to show the point at which you have recovered all of your costs. If you round 324.39 down, you are 0.39 units short of recovering all of your costs. In the long-run, you always operate at a loss, which ultimately puts you out of business. If you round the level of output up to 325, all costs are covered and a tiny dollar amount, as close to zero as possible, is left over as profit. At least at this level of output you can stay in business.

EXAMPLE

You run a small business that produces custom-ordered t-shirts. You have fixed costs of \$2,700 per month. Each t-shirt has a variable cost of \$25. You sell each t-shirt for \$40. Because this is only a part-time business, you can produce a maximum of 500 shirts each month.

1. How many t-shirts do you need to produce and sell each month to break-even?
2. What is the break-even point as a percentage of the maximum capacity?
3. You decide to increase your selling price to \$45 per shirt. What is the new break-even point?
4. Your fixed costs decrease by \$300 with no changes to the selling price or variable costs. What is the new break-even point?
5. Your variable costs increase by \$5 with no changes to the selling price or fixed costs. What is the new break-even point?

Solution:

Step 1: The given information is

$$FC = \$2,700$$

$$VC = \$25$$

$$S = \$40$$

The total revenue function is

$$\begin{array}{l} \text{TR} = S \times x \\ \text{TC} = 40 \times x + 2,700 + 25 \times x \end{array}$$

The total cost function is

$$\begin{aligned} TC &= FC + VC \times x \\ &= 2,700 + 25 \times x \end{aligned}$$

Step 2: Calculate the break-even point.

$$\begin{array}{l} \text{TR} = 40 \times x \\ \text{TC} = 2,700 + 25 \times x \\ \text{Break-even point: } 40 \times x = 2,700 + 25 \times x \\ 40x - 25x = 2,700 \\ 15x = 2,700 \\ x = \frac{2,700}{15} \\ x = 180 \end{array}$$

The break-even point is 180 units.

Step 3: Calculate the break-even point as a percentage of the maximum capacity of 500 units.

$$\begin{array}{l} \text{Break-even as a Percent of Capacity} = \frac{\text{Break-Even Point}}{\text{Maximum Capacity}} \times 100\% \\ = \frac{180}{500} \times 100\% \\ = 36\% \end{array}$$

The break-even point is 36% of the maximum capacity.

Step 4: Calculate the new break-even point when $S = 45$.

$$\begin{array}{l} \text{TR} = 45 \times x \\ \text{TC} = 40 \times x + 2,700 + 25 \times x \\ 45x = 40x + 2,700 + 25x \\ 45x - 40x - 25x = 2,700 \\ -20x = 2,700 \\ x = \frac{2,700}{-20} \\ x = -135 \end{array}$$

When the selling price is \$45, the break-even point is 135 units.

Step 5: Calculate the new break-even point when $FC = 2,700 - 300 = \$2,400$.

$$\begin{array}{l} \text{TR} = 40 \times x \\ \text{TC} = 2,400 + 25 \times x \\ 40x = 2,400 + 25x \\ 40x - 25x = 2,400 \\ 15x = 2,400 \\ x = \frac{2,400}{15} \\ x = 160 \end{array}$$

When the fixed costs decreases by \$300, the break-even point is 160 units.

Step 6: Calculate the new break-even point when $VC = 25 + 5 = \$30$.

$$\begin{array}{l} \text{TR} = 40 \times x \\ \text{TC} = 2,700 + 30 \times x \\ 40x = 2,700 + 30x \\ 40x - 30x = 2,700 \\ 10x = 2,700 \\ x = \frac{2,700}{10} \\ x = 270 \end{array}$$

When the variable costs increase by \$5, the break-even point is 270 units.

TRY IT

A furniture manufacturer sells tables for \$300 each. The variable cost per table is \$120. The monthly fixed costs are \$5,000. The manufacturer can produce a maximum of 38 tables a month.

1. How many tables must be produced and sold to break-even?
2. What is the break-even revenue?
3. What is the break-even point as a percent of capacity?
4. Suppose the fixed costs increase by \$1,300 with no changes to the variables costs or selling price. What is the new break-even point?
5. Suppose the selling price decreases by 10% with no changes to the fixed costs or variable costs. What is the new break-even point?

Click to see Solution

1. Break-even point.

$$\begin{aligned} 300 \times x &= 5,000 + 120 \times x \\ 180 \times x &= 5,000 \\ x &= 27.777 \dots \\ &\rightarrow 28 \end{aligned}$$

2. Break-even revenue.

$$\text{TR} = 300 \times 28 = \$8,400$$

3. Percent capacity.

$$\text{Break-even as a Percent of Capacity} = \frac{28}{38} \times 100\% = 73.68\%$$

4. Break-even point with $FC = \$6,300$.

$$300 \times x = 6,300 + 120 \times x \quad 180 \times x = 6,300 \quad x = 35$$

5. Break-even point with $S = \$330$.

$$\begin{array}{l} 330 \times \text{\&} = \text{\&} 5,000 + 120 \times \text{\&} \\ 210 \times \text{\&} = \text{\&} 5,000 \\ \times \text{\&} = \text{\&} 23.8095 \dots \text{\&} \end{array} \rightarrow \text{\&} 24 \end{array}$$

NOTE

You need to be very careful with the interpretation and application of a break-even number. In particular, the break-even point must have a point of comparison, and it does not provide information about the viability of the business.

Break-Even Points Need to Be Compared.

The break-even number by itself, whether in units or revenue, is meaningless. You need to compare it against some other quantity (or quantities) to determine the feasibility of the number you have produced. The other number needs to be some baseline that allows you to grasp the scope of what you are planning. This baseline could include, but is not limited to, industry sales, number of competitors fighting for market share in your industry, or production capacity of your business.

For example, suppose the break-even point for your internet sales business is 179 units per month. Is that good? Imagine you have eight competitors in this business. If the market can only support a total of 1,000 sales each month, then split evenly between the nine businesses (the eight competitors plus yourself) would mean 111 sales each, which is much lower than the break-even point. To just pay your bills, you would have to sell almost 61% higher than the even split and achieve a 17.9% market share, which is fairly unlikely.

Break-Even Points Are Not Green Lights.

A break-even point alone cannot tell you to do something, but it can tell you *not* to do something. In other words, break-even points can put up red lights, but at no point does it give you the green light. In the above scenario, your break-even point of 179 units put up a whole lot of red lights because it does not seem feasible to obtain. However, what if your industry sold 10,000 units instead of 1,000 units? Your break-even point would now be a 1.79% market share (179 units out of 10,000 units), which certainly seems realistic and attainable. This does not mean “Go for it,” however. It just means that from a strictly financial point of view breaking

even seems possible. Whether you can actually sell that many units depends on a whole range of factors beyond just a break-even number.

Exercises

1. In the current period, Blue Mountain Packers in Salmon Arm, British Columbia, had fixed costs of \$228,000 and a total cost of \$900,000 while maintaining a level of output of 6,720 units. Next period sales are projected to rise by 20%. What total cost should Blue Mountain Packers project?

Click to see Answer

\$1,034,000

2. Fred runs a designer candle-making business out of his basement. He sells the candles for \$15 each, and every candle costs him \$6 to manufacture. If his fixed costs are \$2,300 per month, what is his projected net income or loss next month, for which he forecasts sales of 225 units?

Click to see Answer

-\$275

3. Gayle is thinking of starting her own business. Total fixed costs are \$19,000 per month and unit variable costs are estimated at \$37.50. From some preliminary studies that she completed, she forecasts sales of 1,400 units at \$50 each, 1,850 units at \$48 each, 2,500 units at \$46 each, and 2,750 units at \$44 each. What price would you recommend Gayle set for her products?

Click to see Answer

\$46

4. What level of output would generate a net income of \$15,000 if a product sells for \$24.99, has unit variable costs of \$9.99, and total fixed costs of \$55,005?

Click to see Answer

4,667

5. Franklin has started an ink-jet print cartridge refill business. He has invested \$2,500 in equipment and machinery. The cost of refilling a cartridge including labor, ink, and all other materials is \$4. He charges \$14.95 for his services. How many cartridges does he need to refill to break even?

Click to see Answer

229

6. Hasbro manufactures a line of children's pet toys. If it sells the toy to distributors for \$2.30 each while variable costs are \$0.75 toy, how many toys does it need to sell to recover the fixed cost investment in these toys of \$510,000? What total revenue would this represent?

Click to see Answer

329,033, \$756,775.90

7. If the break-even point is 15,000 units, the selling price is \$95, and the unit variable cost is \$75, what are the company's total fixed costs?

Click to see Answer

\$300,000

8. Louisa runs a secretarial business part time in the evenings. She takes dictation or handwritten minutes and converts them into printed word-processed documents. She charges \$5 per page for her services. Including labor, paper, toner, and all other supplies, her unit variable cost is \$2.50 per page. She invested \$3,000 worth of software and equipment to start her business. How many pages will she need to output to break even?

Click to see Answer

1200

9. Ashley rebuilds old laptops as a home hobby business. Her variable costs are \$125 per laptop and she sells them for \$200. She has determined that her break-even point is 50 units per month. Determine her net income for a month in which she sells 60 units.

Click to see Answer

\$750

10. Shardae is starting a deluxe candy apple business. The cost of producing one candy apple is \$4.50. She has total fixed costs of \$5,000. She is thinking of selling her deluxe apples for \$9.95 each.
- Determine her unit break-even point at her selling price of \$9.95.
 - Shardae thinks her price might be set too high and lowers her price to \$8.95. Determine her new break-even point.
 - An advertising agency approaches Shardae and says people would be willing to pay the \$9.95 if she ran some “upscale” local ads. They would charge her \$1,000. Determine her break-even point.
 - If she wanted to maintain the same break-even units as determined in a., what would the price have to be to pay for the advertising?

Click to see Answer

a. 918; b. 1,124; c. 1,101; d. \$11.04

11. Robert is planning a wedding social for one of his close friends. Costs involve \$865 for the hall rental, \$135 for a liquor license, \$500 for the band, and refreshments and food from the caterer cost \$10 per person. If he needs to raise \$3,000 to help his friend with the costs of his wedding, what price should he charge per ticket if he thinks he can fill the social hall to its capacity of 300 people?

Click to see Answer

\$25

12. École Van Belleghem is trying to raise funds to replace its old playground equipment with a modern, child-safe structure. The Blue Imp playground equipment company has quoted the school a cost of \$49,833 for its 20m by 15m megastructure. To raise the funds, the school wants to sell Show ‘n’ Save books. These books retail for \$15.00 each and cost \$8.50 to purchase. How many books must the school sell to raise funds for the new playground?

Click to see Answer

7,667

13. Suppose the selling price per unit is \$100, the variable costs per unit is \$60 and the fixed costs are

\$250,000

- Calculate the break-even point in both units and revenue.
- Suppose the fixed costs are reduced by 15% and the unit variable costs rise by 5%. What is the new break-even point and break-even revenue?
- Suppose the unit variable costs are reduced by 10% and the fixed costs rise by 5%. What is the new break-even point and break-even revenue?
- Suppose the total fixed costs are reduced by 20% and the unit variable costs rise by 10%. What is the new break-even point and break-even revenue?
- Suppose the unit variable costs are reduced by 15% and the fixed costs rise by 15%. What is the new break-even point and break-even revenue?

Click to see Answer

a. 6,250, \$625,000; b. 5,744, \$574,400; c. 5,707, \$570,700; d. 5,883, \$588,300; e. 5,868, \$586,800

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2.3 CONTRIBUTION MARGIN

LEARNING OBJECTIVES

- Solve cost-volume-profit analysis problems using the contribution margin approach.

Suppose you run a small business that produces custom-ordered t-shirts. You have fixed costs of \$2,700 per month. Each t-shirt has a variable cost of \$25. You sell each t-shirt for \$40. What happens to the \$40 you receive for each t-shirt? Because each t-shirt you sell has a \$25 variable cost, you need to use \$25 of the \$40 selling price to cover the variable costs of each t-shirt. That leaves you with \$15 from each t-shirt sold. Does this mean that you make \$15 in profit for every t-shirt you sell? Not exactly. Do not forget that you also have to cover the \$2,700 fixed costs. Before you can make any profit, the fixed costs must be paid. Until all of the fixed costs are paid, the leftover \$15 from the \$40 selling price is applied to the fixed costs. Once you have sold enough t-shirts to pay off the fixed costs, the \$15 becomes profit. This happens at the break-even point. In other words, you have to use the \$15 to pay the fixed costs until you reach the break-even point, and after the break-even point the \$15 becomes profit. The \$15 is called the **contribution margin**.

Contribution Margin

The **contribution margin** (CM) is the difference between the selling price per unit and the variable cost per unit.

$$CM = S - VC$$

where

- **CM is the contribution margin.** This is the amount of money that remains available to pay for fixed costs once the unit variable cost is removed from the selling price of the product.

- **S is the selling price per unit.** The unit selling price of the product.
- **VC is the variable cost per unit.** The unit variable cost is the typical or average variable cost associated with an individual unit of output.

EXAMPLE

A jewelry store sells watches for \$120 each. The fixed costs are \$6,800 per month and the variable cost per watch is \$70. What is the contribution margin? What does the contribution margin represent?

Solution:

Step 1: The given information is

$$\begin{aligned} S &= \$120 \\ VC &= \$70 \\ FC &= \$6,800 \end{aligned}$$

Step 2: Calculate the contribution margin.

$$\begin{aligned} CM &= S - VC \\ &= 120 - 70 \\ &= \$50 \end{aligned}$$

Step 3: Interpret the contribution margin.

For every watch sold below the break-even point, \$50 is applied to pay off the fixed costs. For every watch sold above the break-even point, a \$50 profit is made.

Net Income using Contribution Margin

In accounting and marketing, the **contribution margin** is the amount that each unit sold adds to the net income of the business. This approach allows you to understand the impact on net income of each unit sold. The contribution margin determines, on a per-unit basis, how much money is left over after unit variable costs are removed from the price of the product. This leftover money is then available to pay for the fixed costs. Ultimately, when all fixed costs have been paid for, the leftover money becomes the profits of the business. If

not enough money is left over to pay for the fixed costs, then the business has a negative net income and loses money.

Recall from the previous section that net income is the different between the total revenue and total costs.

$$NI = S \times x - (FC + VC \times x)$$

where

- **NI is the net income.** The amount of money left over after all costs have been paid is the net income. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss.
- **S is the selling price per unit.** The unit selling price of the product.
- **x is the number of units produced.** This is the number of units produced or sold or the total output that incurred the total variable costs.
- **FC is the fixed costs.** These are the costs that do not change regardless of the level of production.
- **VC is the variable cost per unit.** The unit variable cost is the typical or average variable cost associated with an individual unit of output.

Rearranging this formula, we get

$$NI = (S - VC) \times x - FC$$

Because $CM = S - VC$, we can express the net income in terms of the contribution margin

$$NI = CM \times x - FC$$

where

- **NI is the net income.** The amount of money left over after all costs have been paid is the net income. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss.
- **CM is the contribution margin.** This is the amount of money that remains available to pay for fixed costs once the unit variable cost is removed from the selling price of the product.
- **x is the number of units produced.** This is the number of units produced or sold or the total output that incurred the total variable costs.
- **FC is the fixed costs.** These are the costs that do not change regardless of the level of production.

If you have no units sold, there is no offsetting revenue to pay for the fixed costs. Consequently, your net income is negative and equal to the total fixed costs associated with your business. With each unit sold, the

contribution margin is available to pay off the fixed costs. Once the fixed costs are paid, the contribution margin becomes profit.

EXAMPLE

A manufacturer makes winter boots. The manufacturer sells the boots for \$135 a pair. The fixed costs are \$5,400 per month and the variable cost per pair of boots is \$60.

1. What is the contribution margin?
2. What is the net income when 100 pairs of boots are sold a month?
3. What is the net income when 65 pairs of boots are sold a month?

Solution:

Step 1: The given information is

$$\begin{aligned} S &= \$135 \\ VC &= \$60 \\ FC &= \$5,400 \end{aligned}$$

Step 2: Calculate the contribution margin.

$$\begin{aligned} CM &= S - VC \\ &= 135 - 60 \\ &= \$75 \end{aligned}$$

Step 3: Calculate the net income for $x = 100$ units.

$$\begin{aligned} NI &= CM \times x - FC \\ &= 75 \times 100 - 5,400 \\ &= \$2,100 \end{aligned}$$

At 100 units, the net income is \$2,100. Because the net income is positive, the manufacturer made a profit at 100 units.

Step 4: Calculate the net income for $x = 65$ units.

$$\begin{aligned}
 NI &= CM \times x - FC \\
 &= 75 \times 65 - 5,400 \\
 &= -\$525
 \end{aligned}$$

At 65 units, the net income is -\$525. Because the net income is negative, the manufacturer incurred a loss at 65 units.

In the net income formula show below, x is the number of units produced and sold.

$$NI = CM \times x - FC$$

Rearranging this formula to solve for x , we can express the number of units sold, x , in terms of the net income, contribution margin, and fixed costs.

$$\text{Number of Units Sold} = \frac{FC + NI}{CM}$$

- **NI is the net income.** The amount of money left over after all costs have been paid is the net income. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss.
- **CM is the contribution margin.** This is the amount of money that remains available to pay for fixed costs once the unit variable cost is removed from the selling price of the product.
- **FC is the fixed costs.** These are the costs that do not change regardless of the level of production.

EXAMPLE

A manufacturer makes winter boots. The manufacturer sells the boots for \$135 a pair. The fixed costs are \$5,400 per month and the variable cost per pair of boots is \$60. From the previous example, the contribution margin is \$75.

1. How many pairs of boots must be sold in a month to produce a profit of \$3,600 ?

2. How many pairs of boots must be sold in a month to incur a loss of \$1,275?

Solution:

Step 1: The given information is

$$\begin{aligned} S &= \$135 \\ VC &= \$60 \\ FC &= \$5,400 \\ CM &= \$75 \end{aligned}$$

Step 2: Calculate the number of units for $NI = 3,600$. Because this is a profit, the net income is **positive**.

$$\begin{aligned} \text{Number of Units Sold} &= \frac{FC + NI}{CM} \\ &= \frac{5,400 + 3,600}{75} \\ &= 120 \end{aligned}$$

A profit of \$3,600 occurs when 120 pairs of boots are sold.

Step 3: Calculate the number of units for $NI = -1,275$. Because this is a loss, the net income is **negative**.

$$\begin{aligned} \text{Number of Units Sold} &= \frac{FC + NI}{CM} \\ &= \frac{5,400 - 1,275}{75} \\ &= 55 \end{aligned}$$

A loss of \$1,275 occurs when 55 pairs of boots are sold.

TRY IT

You run a small business making leather backpacks. You sell each backpack for \$150. Your monthly fixed costs are \$7,200 and the variable cost per backpack is \$60.

1. What is the contribution margin?
2. What is your net income when you sell 100 backpacks a month?
3. How many backpacks are sold in a month when you incur a loss of \$1,350?

Click to see Solution

1. Contribution margin.

$$\begin{aligned} CM &= 150 - 60 \\ &= \$90 \end{aligned}$$

2. Net income for 100 backpacks.

$$\begin{aligned} NI &= CM \times x - FC \\ &= 90 \times 100 - 7,200 \\ &= \$1,800 \end{aligned}$$

3. Units sold for loss of \$1,350.

$$\begin{aligned} \text{Number of Units Sold} &= \frac{FC + NI}{CM} \\ &= \frac{7,200 - 1,350}{90} \\ &= 65 \end{aligned}$$

Break-Even Analysis using Contribution Margin

Recall the t-shirt business example at the beginning of this section where the fixed costs are \$2,700, the variable cost is \$25 per unit and the selling price is \$40 per unit. The contribution margin of \$15 is first used to pay the fixed costs. At the break-even point, all of the fixed costs are covered and the \$15 contribution margin becomes profit after the break-even point. How many shirts, x , do you need to sell to cover the fixed costs? That is,

what is the value of x so that $15 \times x = 2,700$? Solving for x yields 180 shirts, which is the break-even point. This means that for every t-shirt you sell **above** the break-even point of 180 shirts, you make \$15 in profit and for every t-shirt you sell **below** the break-even point of 180 shirts, you make no profit and the \$15 contribution margin goes to paying the fixed costs.

The break-even point is the number of units sold so that the net income is zero. From the above formula,

$$\text{Number of Units Sold} = \frac{FC + NI}{CM}$$

To find the break-even point, we set $NI = 0$ in this formula. Then,

$$\text{Break-Even Point} = \frac{FC}{CM}$$

where

- **CM is the contribution margin.** This is the amount of money that remains available to pay for fixed costs once the unit variable cost is removed from the selling price of the product.
- **FC is the fixed costs.** These are the costs that do not change regardless of the level of production.

EXAMPLE

An electronics manufacturer produces an electronic toy. The manufacturer sells the toy for \$145 each. The fixed costs per month are \$15,200. The variable cost to make each toy is \$65.

1. What is the contribution margin?
2. How many toys does the manufacturer need to produce and sell each month to break-even?
3. What is the revenue at the break-even point?
4. What is the net income if the manufacturer sells 75 units above the break-even point?
5. Suppose the fixed costs decrease by \$2,000 with no changes to the variable costs or selling price. What is the new break-even point?
6. Suppose the variable costs increase by \$4 with no changes to the fixed costs or selling price. What is the new break-even point?

Solution:**Step 1:** The given information is

$$\begin{aligned}
 S &= \$145 \\
 VC &= \$65 \\
 FC &= \$15,200
 \end{aligned}$$

Step 2: Calculate the contribution margin.

$$\begin{aligned}
 CM &= S - VC \\
 &= 145 - 65 \\
 &= \$80
 \end{aligned}$$

Step 3: Calculate the break-even point.

$$\begin{aligned}
 \text{Break-Even Point} &= \frac{FC}{CM} \\
 &= \frac{15,200}{80} \\
 &= 190
 \end{aligned}$$

The manufacturer must produce and sell 190 toys a month to break-even.

Step 4: Calculate the revenue for $x = 190$ toys.

$$\begin{aligned}
 TR &= S \times x \\
 &= 145 \times 190 \\
 &= \$27,550
 \end{aligned}$$

Step 5: Calculate the net income for $x = 265$ units (75 units above the break-even point of 190).

$$\begin{aligned}
 NI &= CM \times x - FC \\
 &= 80 \times 265 - 15,200 \\
 &= \$6,000
 \end{aligned}$$

At 75 units above the break-even point of 190 units, the net income is \$6,000. Because the net income is positive, this is a profit.

Step 6: Calculate the break-even point when $FC = 15,200 - 2,000 = 13,200$. Because the selling price and variables costs do not change, the contribution margin is still \$80.

$$\begin{aligned}
 \text{Break-Even Point} &= \frac{FC}{CM} \\
 &= \frac{13,200}{80} \\
 &= 165
 \end{aligned}$$

If the fixed costs decrease by \$2,000, the manufacturer must produce and sell 165 toys a month to break-even.

Step 7: Calculate the break-even point when $VC = 65 + 4 = 69$. Because the variable costs changed, the contribution margin will also change. The contribution margin must be recalculated before finding the new break-even point.

$$\begin{aligned}
 \text{New } CM &= S - VC \\
 &= 145 - 69 \\
 &= \$76
 \end{aligned}$$

$$\begin{aligned}
 \text{Break-Even Point} &= \frac{FC}{CM} \\
 &= \frac{15,200}{76} \\
 &= 200
 \end{aligned}$$

If the variable costs increase by \$4, the manufacturer must produce and sell 200 toys a month to break-even.

NOTE

If either the selling price or variable costs (or possibly both) change, the contribution margin will also change, and so needs to be recalculated in these situations. In the previous example, because the contribution margin is not affected by the fixed costs, we did not have to recalculate the contribution margin in the part of the question where just the fixed costs changed. However, we did have to recalculate the contribution margin when the variable costs changed.

TRY IT

You run a business that produces and sells bicycles. The fixed costs per month are \$35,000 and the variable cost per bike is \$375. You sell each bike for \$500.

1. What is the contribution margin.
2. What is the break-even point?
3. Suppose the fixed costs increase by 20% with no changes to the selling price or variable costs. What is the new break-even point?
4. Suppose the selling price decreases by \$25 with no changes to the fixed costs or variable costs. What is the new break-even point?

Click to see Solution

1. Contribution margin.

$$\begin{aligned} CM &= 500 - 375 \\ &= \$125 \end{aligned}$$

2. Break-even point.

$$\begin{aligned} \text{Break-Even Point} &= \frac{FC}{CM} \\ &= \frac{35,000}{125} \\ &= 280 \end{aligned}$$

3. New break-even point for $FC = \$42,000$.

$$\begin{aligned} \text{Break-Even Point} &= \frac{FC}{CM} \\ &= \frac{42,000}{125} \\ &= 336 \end{aligned}$$

4. New break-even point for $S = \$475$.

$$\begin{aligned}
 CM &= S - VC \\
 &= 475 - 375 \\
 &= \$100
 \end{aligned}$$

$$\begin{aligned}
 \text{Break-Even Point} &= \frac{FC}{CM} \\
 &= \frac{35,000}{100} \\
 &= 350
 \end{aligned}$$

EXAMPLE

A sporting goods manufacturer makes baseball caps. The baseball caps sell for \$28 each. The contribution margin per baseball cap is \$12.35. To break-even each month, the manufacturer needs a total revenue of \$12,600.

1. What is the break-even point?
2. What are the monthly fixed costs?
3. What is the net income if the number of baseball caps sold is 80 units below the break-even point?

Solution:

Step 1: The given information is

$$\begin{aligned}
 S &= \$28 \\
 CM &= \$12.35 \\
 \text{Break-even revenue} &= \$12,600
 \end{aligned}$$

Step 2: Calculate the break-even point.

$$\begin{aligned}
 \text{Break-Even Revenue} &= S \times x \\
 12,600 &= 28 \times x \\
 \frac{12,600}{28} &= x \\
 450 &= x
 \end{aligned}$$

The manufacturer must produce and sell 450 baseball caps a month to break-even.

Step 3: Calculate the fixed costs.

$$\begin{aligned}
 \text{Break-Even Point} &= \frac{FC}{CM} \times 450 \\
 \frac{FC}{12.35} \times 450 &= FC \\
 FC &= \$5,557.50
 \end{aligned}$$

Step 4: Calculate the net income for $x = 370$ units (80 units below the break-even point of 450).

$$\begin{aligned}
 NI &= CM \times x - FC \\
 &= 12.35 \times 370 - 5,557.50 \\
 &= -\$980
 \end{aligned}$$

At 80 units below the break-even point of 450 units, the net income is -\$980. Because the net income is negative, this is a loss.

Exercises

Note: There are other ways to solve most of these questions, but use the contribution margin to answer the following questions.

1. In the current period, Blue Mountain Packers in Salmon Arm, British Columbia, had fixed costs of \$228,000 and a total cost of \$900,000 while maintaining a level of output of 6,720 units. Next period sales are projected to rise by 20%. What total cost should Blue Mountain Packers project?

Click to see Answer

\$1,034,000

2. Fred runs a designer candle-making business out of his basement. He sells the candles for \$15 each, and

every candle costs him \$6 to manufacture. If his fixed costs are \$2,300 per month, what is his projected net income or loss next month, for which he forecasts sales of 225 units?

Click to see Answer

-\$275

3. Gayle is thinking of starting her own business. Total fixed costs are \$19,000 per month and unit variable costs are estimated at \$37.50. From some preliminary studies that she completed, she forecasts sales of 1,400 units at \$50 each, 1,850 units at \$48 each, 2,500 units at \$46 each, and 2,750 units at \$44 each. What price would you recommend Gayle set for her products?

Click to see Answer

\$46

4. What level of output would generate a net income of \$15,000 if a product sells for \$24.99, has unit variable costs of \$9.99, and total fixed costs of \$55,005?

Click to see Answer

4,667

5. What is the unit contribution margin on a product line that has fixed costs of \$1,800,000 with a break-even point of 360,000 units?

Click to see Answer

\$5

6. Franklin has started an ink-jet print cartridge refill business. He has invested \$2,500 in equipment and machinery. The cost of refilling a cartridge including labor, ink, and all other materials is \$4. He charges \$14.95 for his services. How many cartridges does he need to refill to break even?

Click to see Answer

229

7. Hasbro manufactures a line of children's pet toys. If it sells the toy to distributors for \$2.30 each while variable costs are \$0.75 toy, how many toys does it need to sell to recover the fixed cost investment in

these toys of \$510,000? What total revenue would this represent?

Click to see Answer

329,033, \$756,775.90

8. If the break-even point is 15,000 units, the selling price is \$95, and the unit variable cost is \$75, what are the company's total fixed costs?

Click to see Answer

\$300,000

9. Louisa runs a secretarial business part time in the evenings. She takes dictation or handwritten minutes and converts them into printed word-processed documents. She charges \$5 per page for her services. Including labor, paper, toner, and all other supplies, her unit variable cost is \$2.50 per page. She invested \$3,000 worth of software and equipment to start her business. How many pages will she need to output to break even?

Click to see Answer

1200

10. Ashley rebuilds old laptops as a home hobby business. Her variable costs are \$125 per laptop and she sells them for \$200. She has determined that her break-even point is 50 units per month. Determine her net income for a month in which she sells 60 units.

Click to see Answer

\$750

11. Shardae is starting a deluxe candy apple business. The cost of producing one candy apple is \$4.50. She has total fixed costs of \$5,000. She is thinking of selling her deluxe apples for \$9.95 each.
- Determine her unit break-even point at her selling price of \$9.95.
 - Shardae thinks her price might be set too high and lowers her price to \$8.95. Determine her new break-even point.
 - An advertising agency approaches Shardae and says people would be willing to pay the \$9.95 if she ran some "upscale" local ads. They would charge her \$1,000. Determine her break-even point.
 - If she wanted to maintain the same break-even units as determined in a., what would the price have

to be to pay for the advertising?

Click to see Answer

a. 918; b. 1,124; c. 1,101; d. \$11.04

12. Robert is planning a wedding social for one of his close friends. Costs involve \$865 for the hall rental, \$135 for a liquor license, \$500 for the band, and refreshments and food from the caterer cost \$10 per person. If he needs to raise \$3,000 to help his friend with the costs of his wedding, what price should he charge per ticket if he thinks he can fill the social hall to its capacity of 300 people?

Click to see Answer

\$25

13. École Van Belleghem is trying to raise funds to replace its old playground equipment with a modern, child-safe structure. The Blue Imp playground equipment company has quoted the school a cost of \$49,833 for its 20m by 15m megastructure. To raise the funds, the school wants to sell Show 'n' Save books. These books retail for \$15.00 each and cost \$8.50 to purchase. How many books must the school sell to raise funds for the new playground?

Click to see Answer

7,667

14. Calculate the following:
- By what percentage does the unrounded unit break-even point change if the unit contribution margin increases by 1% while all other numbers remain the same?
 - By what percentage does the unrounded unit break-even point change if total fixed costs are reduced by 1% while all other numbers remain the same?
 - What do the solutions to the above questions illustrate?

Click to see Answer

a. -0.99%; b. -1%; c. lowering fixed costs is better.

15. Suppose the selling price per unit is \$100, the variable costs per unit is \$60 and the fixed costs are \$250,000

- What is the contribution margin?

- b. Calculate the break-even point in both units and revenue.
- c. Suppose the fixed costs are reduced by 15% and the unit variable costs rise by 5%. What is the new break-even point and break-even revenue?
- d. Suppose the unit variable costs are reduced by 10% and the fixed costs rise by 5%. What is the new break-even point and break-even revenue?
- e. Suppose the total fixed costs are reduced by 20% and the unit variable costs rise by 10%. What is the new break-even point and break-even revenue?
- f. Suppose the unit variable costs are reduced by 15% and the fixed costs rise by 15%. What is the new break-even point and break-even revenue?

Click to see Answer

a. \$61.73; b. 6,250, \$625,000; c. 5,744, \$574,400; d. 5,707, \$570,700; e. 5,883, \$588,300; f. 5,868, \$586,800

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2.4 CONTRIBUTION RATE

LEARNING OBJECTIVES

- Solve cost-volume-profit analysis problems using the contribution rate approach.

It is difficult to compare different products and their respective dollar amount contribution margins if their selling prices and costs vary widely. For example, how do you compare a unit contribution margin of \$1,390 (selling price of \$2,599.99) on a big screen television to a unit contribution margin of \$0.33 on a chocolate bar (selling price of \$0.67)? On a per-unit basis, which contributes relatively more to fixed costs? To facilitate these comparisons, the products must be placed on equal terms, requiring you to convert all dollar amount contribution margins into percentages, called contribution rates. A **contribution rate** is a contribution margin expressed as a percentage of the selling price.

Contribution Rate

The **contribution rate** (CR) is the contribution margin expressed as a percentage of the selling price. The contribution margin is the percentage of the selling price that is available to first pay for the fixed costs and then to produce profit.

$$CR = \frac{CM}{S} \times 100\%$$

where

- **CR is the contribution rate.** This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business. When all fixed costs are paid for, this percentage is the portion of the selling price that will remain as profit.

- **CM is the contribution margin.** This is the amount of money that remains available to pay for fixed costs once the unit variable cost is removed from the selling price of the product.
- **S is the selling price per unit.** The unit selling price of the product.

EXAMPLE

Suppose the selling price of a product is \$5 and the variable cost is \$3.50. Calculate the contribution rate.

Solution:

Step 1: The given information is

$$\begin{aligned} S &= \$5 \\ VC &= \$3.50 \end{aligned}$$

Step 2: Calculate the contribution margin.

$$\begin{aligned} CM &= S - VC \\ &= 5 - 3.50 \\ &= \$1.50 \end{aligned}$$

Step 3: Calculate the contribution rate.

$$\begin{aligned} CR &= \frac{CM}{S} \times 100\% \\ &= \frac{1.50}{5} \times 100\% \\ &= 30\% \end{aligned}$$

The contribution rate is 30%. This means that 30% of the \$5 selling price is available to pay for the fixed costs. The remaining 70% of the \$5 selling price goes to the variable costs.

NOTES

1. The contribution rate remains the same regardless of the number of units sold and the fixed costs. Changes to the selling price or variable costs (or both) will change the contribution rate.
2. What does the contribution rate really mean? Suppose the contribution rate is 60%. This means that 60% of the selling price goes to pay the fixed costs below the break even point or becomes profit above the break-even point. For example, if the selling price is \$1, a 60% contribution rate means \$0.60 of that \$1 selling price goes to pay the fixed costs. The remaining 40% of the selling price goes to pay the variable costs. For the \$1 selling price, a 60% contribution rate means \$0.40 of that \$1 selling price goes to pay the variable costs.

There may be instances when the unit selling price and unit variable costs are unknown or unavailable. In such circumstances, it would be impossible to calculate the contribution rate using the above formula. Sometime, the only information you have available is aggregate information such as total revenue and total variable costs. In such cases it is still possible to calculate the contribution rate.

$$CR = \frac{TR - TVC}{TR} \times 100\%$$

where

- **CR is the contribution rate.** This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business. When all fixed costs are paid for, this percentage is the portion of the selling price that will remain as profit.
- **TR is the total revenue.** This is the total amount of money that the company has received from the sale of the product.
- **TVC is the total variable cost.** This is the total cost associated with the level of output. When total variable costs are subtracted from the total revenue, the remainder represents the portion of money left over to pay the fixed costs.

EXAMPLE

Suppose a chocolate bar manufacturer has total monthly revenue of \$3,886 and total variable costs of \$1,972. Calculate the contribution rate.

Solution:

Step 1: The given information is

$$\begin{aligned} TR &= \$3,886 \\ TVC &= \$1,972 \end{aligned}$$

Step 2: Calculate the contribution rate.

$$\begin{aligned} CR &= \frac{TR - TVC}{TR} \times 100\% \\ &= \frac{3,886 - 1,972}{3,886} \times 100\% \\ &= 49.25\% \end{aligned}$$

The contribution rate is 49.25%. This means that 49.25% of the selling price of each chocolate bar is available to pay for the fixed costs. The remaining 50.75% of the selling price of each chocolate bar goes to the variable costs.

Net Income using Contribution Rate

Recall that net income is the difference between total revenue (TR) and total costs ($TVC + FC$). So, net income can be expressed as

$$NI = TR - TVC - FC$$

Rearranging this formula, we get

$$TR - TVC = FC + NI$$

From the contribution rate formula, $TR - TVC = CR \times TR$. Substituting into the above formula and solving for TR , we get

$$TR = \frac{FC + NI}{CR}$$

where

- **CR is the contribution rate.** This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business. When all fixed costs are paid for, this percentage is the portion of the selling price that will remain as profit.
- **TR is the total revenue.** This is the total amount of money that the company has received from the sale of the product.
- **TVC is the total variable cost.** This is the total cost associated with the level of output. When total variable costs are subtracted from the total revenue, the remainder represents the portion of money left over to pay the fixed costs.
- **NI is the net income.** The amount of money left over after all costs have been paid is the net income. If the number is positive, then the business is profitable. If the number is negative, then the business suffers a loss.
- **FC is the fixed costs.** These are the costs that do not change regardless of the level of production.

EXAMPLE

The manufacturer of a product has fixed costs of \$200,000 per year. The variable costs are 65% of the selling price.

1. What is the contribution rate?
2. What is the total revenue to earn a profit of \$25,000?
3. What is the total revenue to incur a loss of \$10,000?
4. What is the net income if the total revenue is \$500,000?
5. What is the net income if the total revenue is \$750,000?

Solution:

Step 1: The given information is

$$\begin{array}{l} \text{FC} = \$200,000 \end{array}$$

Step 2: The contribution rate is 35%. Because 65% of the selling price goes to the variable costs, the remaining 35% of the selling price is available for the fixed costs, which is the contribution rate.

Step 3: Calculate the total revenue for $NI = 25,000$. Because this is a profit, the net income is **positive**.

$$\begin{array}{l} \text{TR} = \frac{\text{FC} + \text{NI}}{\text{CR}} \\ \frac{200,000 + 25,000}{0.35} = \$642,857.14 \end{array}$$

The total revenue to earn a profit of \$25,000 is \$642,857.14.

Step 4: Calculate the total revenue for $NI = -10,000$. Because this is a loss, the net income is **negative**.

$$\begin{array}{l} \text{TR} = \frac{\text{FC} + \text{NI}}{\text{CR}} \\ \frac{200,000 - 10,000}{0.35} = \$542,857.14 \end{array}$$

The total revenue to incur a loss of \$10,000 is \$542,857.14.

Step 5: Calculate the net income for $TR = 500,000$.

$$\begin{array}{l} \text{TR} = \frac{\text{FC} + \text{NI}}{\text{CR}} \\ \frac{200,000 + \text{NI}}{0.35} = 500,000 \\ \text{NI} = 500,000 \times 0.35 - 200,000 \\ \text{NI} = 175,000 - 200,000 \\ \text{NI} = -\$25,000 \end{array}$$

A loss of \$25,000 occurs when the total revenue is \$500,000

Step 6: Calculate the net income for $TR = 750,000$.

$$\begin{array}{l} \text{TR} = \frac{\text{FC} + \text{NI}}{\text{CR}} \\ \frac{200,000 + \text{NI}}{0.35} = 750,000 \\ \text{NI} = 750,000 \times 0.35 - 200,000 \\ \text{NI} = 262,500 - 200,000 \\ \text{NI} = \$62,500 \end{array}$$

A profit of \$62,500 occurs when the total revenue is \$750,000

NOTE

When you calculate the contribution rate, you will probably end up with a number that has lots of decimals. When you use the contribution rate in another calculation, such as finding the total revenue, net income, or break-even revenue, keep all of the decimals in the contribution rate. This way you will avoid any round-off error in your next calculation.

TRY IT

Your company has fixed costs \$195,000. Your variable costs are 72% of your sales.

1. What is the contribution rate?
2. What is the total revenue to incur a loss of \$20,000?
3. What is the net income for a total revenue of \$600,000?

Click to see Solution

1. The contribution rate is 28%.
2. Total revenue for loss of \$20,000.

$$\begin{aligned}
 TR &= \frac{FC + NI}{CR} \\
 &= \frac{195,000 - 20,000}{0.28} \\
 &= \$625,000
 \end{aligned}$$

3. Net income for revenue of \$600,000

$$\begin{aligned}
 TR &= \frac{FC + NI}{CR} \\
 600,000 &= \frac{195,000 + NI}{0.28} \\
 168,000 &= 195,000 + NI \\
 -\$27,000 &= NI
 \end{aligned}$$

Break-Even Analysis using Contribution Rate

The break-even revenue is the revenue so that the net income is zero. To find the break-even revenue, we set $NI = 0$ in the above formula. Then,

$$\text{Break-Even Revenue} = \frac{FC}{CR}$$

where

- **CR is the contribution rate.** This represents the percentage of the unit selling price that is available to pay for all of the fixed costs of the business. When all fixed costs are paid for, this percentage is the portion of the selling price that will remain as profit.
- **FC is the fixed costs.** These are the costs that do not change regardless of the level of production.

NOTE

Using the contribution rate, we can only find the revenue at the break-even point, and not the actual number of units sold. This is especially useful when we only have the total revenue and total costs, and not the unit selling price and unit variable costs. If the unit selling price and unit variable costs are known, it is possible to find the number of units to break-even using one of the other methods (revenue/cost functions or contribution margin).

EXAMPLE

Suppose you are starting your own business. You determine that your total fixed costs are \$420,000. After analyzing the financial statements of your competitors, you estimate a contribution rate of 35%. What is your break-even revenue?

Solution:

Step 1: The given information is

$$\begin{array}{l} FC = \$420,000 \\ CR = 0.35 \end{array}$$

Step 2: Calculate the break-even revenue.

$$\begin{array}{l} \text{Break-Even Revenue} = \frac{FC}{CR} \\ \frac{420,000}{0.35} = \$1,200,000 \end{array}$$

The break-even revenue is \$1,200,000.

TRY IT

Your company has fixed costs \$195,000. Your variable costs are 72% of your sales. What is the break-even revenue?

Click to see Solution

$$\begin{array}{l} \text{Break-Even Revenue} = \frac{FC}{CR} \\ \frac{195,000}{0.28} = \$696,428.57 \end{array}$$

EXAMPLE

Based on last year's production, your company recorded the following information.

Total Revenue	\$650,000
Total Fixed Costs	\$125,000
Total Variable Costs	\$409,500

You expect the same level of production next year.

1. What is the contribution rate?
2. What is the break-even revenue?
3. Suppose the fixed costs decrease by \$25,000 with no change in the total revenue and total variable costs. What is the new break-even revenue?
4. Suppose the total variable costs increase by \$10,500 dollar with no change in the total revenue and total fixed costs. What is the new break-even revenue?

Solution:

Step 1: The given information is

$$\begin{array}{l} TR = \$650,000 \\ TVC = 409,500 \\ FC = \$125,000 \end{array}$$

Step 2: Calculate the contribution rate.

$$\begin{aligned} CR &= \frac{TR - TVC}{TR} \times 100\% \\ &= \frac{650,000 - 409,500}{650,000} \times 100\% \\ &= 37\% \end{aligned}$$

Step 3: Calculate the break-even revenue.

$$\begin{array}{l} \text{Break-Even Revenue} = \frac{FC}{CR} \\ \frac{125,000}{0.37} = \$337,837.84 \end{array}$$

The break-even revenue is \$337,837.84.

Step 4: Calculate the break-even revenue with $FC = 125,000 - 25,000 = 100,000$.

Because there is no change in the total revenue and total variable costs, the contribution rate is unchanged.

$$\begin{array}{l} \text{Break-Even Revenue} = \frac{FC}{CR} \\ \frac{100,000}{0.37} = \$270,270.27 \end{array}$$

If the fixed costs decrease by \$25,000, the break-even revenue is \$270,270.27.

Step 5: Calculate the break-even revenue with $TVC = 409,500 + 10,500 = 420,000$.

Because the total variable costs changed but the total revenue did not, the contribution rate will change. The contribution rate must be recalculated to find the new break-even revenue.

$$\begin{array}{l} CR = \frac{TR - TVC}{TR} \times 100\% \\ \frac{650,000 - 420,000}{650,000} \times 100\% = 35.384\% \\ \text{Break-Even Revenue} = \frac{FC}{CR} \\ = \frac{125,000}{0.353854} = \$353,260.87 \end{array}$$

If the total variable costs increase by \$10,500, the break-even revenue is \$353,260.87.

TRY IT

In the annual report to shareholders, Borland Manufacturing reported total gross sales of \$7,200,000, total variable costs of \$4,320,000, and total fixed costs of \$2,500,000.

1. Determine Borland's break-even revenue.
2. Suppose the fixed costs increase by 10% with no changes to the total revenue and total variable costs. What is the new break-even revenue?
3. Suppose the total sales decreases by \$200,000 with no changes to the fixed costs and total variable costs. What is the new break-even revenue?

Click to see Solution

1. Break-even revenue.

$$\begin{aligned}
 CR &= \frac{TR - TVC}{TR} \times 100\% \\
 &= \frac{7,200,000 - 4,320,000}{7,200,000} \times 100\% \\
 &= 40\%
 \end{aligned}$$

$$\begin{aligned}
 \text{Break-Even Revenue} &= \frac{FC}{CR} \\
 &= \frac{2,500,000}{0.4} \\
 &= \$6,250,000
 \end{aligned}$$

2. Break-even revenue for $FC = 2,750,000$.

$$\begin{aligned}
 \text{Break-Even Revenue} &= \frac{FC}{CR} \\
 &= \frac{2,750,000}{0.4} \\
 &= \$6,875,000
 \end{aligned}$$

3. Break-even revenue for $TR = 7,000,000$.

$$\begin{aligned}
 CR &= \frac{TR - TVC}{TR} \times 100\% \\
 &= \frac{7,000,000 - 4,320,000}{7,000,000} \times 100\% \\
 &= 38.285...\%
 \end{aligned}$$

$$\begin{aligned}
 \text{Break-Even Revenue} &= \frac{FC}{CR} \\
 &= \frac{2,500,000}{0.38285...} \\
 &= \$6,529,850.75
 \end{aligned}$$

Exercises

1. Last year, A Child's Place franchise had total sales of \$743,000. If its total fixed costs were \$322,000 and net income was \$81,000, what was its contribution rate?

Click to see Answer

54.24%

2. In the current year, a small Holiday Inn franchise had sales of \$1,800,000, fixed costs of \$550,000, and total variable costs of \$750,000. Next year, sales are forecast to increase by 25% but costs will remain the same. How much will net income change (in dollars)?

Click to see Answer

\$262,000

3. Monsanto Canada reported the following on its income statement for one of its divisions:

- Sales=\$6,000,000
- Total Fixed Costs=\$2,000,000
- Total Variable Costs=\$3,200,000

- a. What is the contribution rate?
- b. What is the break-even revenue?
- c. What is the net income for a revenue of \$5,500,000?
- d. What is the total revenue for a profit of \$200,000?
- e. What is the total revenue for a loss of \$300,000?

Click to see Answer

a. 46.67%; b. \$4,285,714.29; c. \$566,666.67; d. \$4,714,285.71; e. \$3,642,857.14

4. Procter and Gamble is budgeting for next year. For one of its brands, P&G projects it will operate at 80% production capacity next year and forecasts the following:

- Sales=\$80,000,000
- Total Fixed Costs=\$20,000,000
- Total Variable Costs=\$50,000,000

- Net Income=\$10,000,000

Determine the net income if sales are higher than expected and P&G realizes 90% production capacity.

Click to see Answer

\$13,750,000

5. Suppose at a production level of 131,000 units, the total fixed costs are \$3,200,000, the total variable costs are \$5,009,440, and the selling price per unit is \$99.97.
 - a. What is the contribution rate?
 - b. What is the break-even revenue?
 - c. Suppose fixed costs rise by 10% with no changes to total revenue or total variable costs. At 131,000 units, what is the new net income?
 - d. Suppose the selling price decreased by 25% during a sale and the resulting number of units increased by 50% from the current 131,000. What is the new net income?
 - e. Suppose fixed costs are lowered by 5%, total variable costs rise by 3%, the selling price lowered by 5% and the level of output rises by 10%. What is the new net income?

Click to see Answer

a. 61.75%; b. \$5182,310.06; c. \$4,556,650; d. \$4,109,410; e. \$4,969,078

6. You are thinking of starting your own business and want to get some measure of feasibility. You have determined that your total fixed costs would be \$79,300. From annual business reports and competitive studies, you estimate your contribution rate to be 65%. What is your break-even in dollars?

Click to see Answer

\$122,000

7. If a business has total revenue of \$100,000, total variable costs of \$60,000, and total fixed costs of \$20,000, determine its break-even point in dollars.

Click to see Answer

\$50,000

8. If your organization has a contribution rate of 45% and knows the break-even point is \$202,500, what

are your organization's total fixed costs?

Click to see Answer

\$91,125

9. Burton Snowboards reported the following figures last week for its Custom V Rocker Snowboard:

- Sales=\$70,000
- Total Fixed Costs=\$19,000
- Total Variable Costs=\$35,000
- Net Income=\$16,000

If the above numbers represent 70% operational capacity, express the weekly break-even point in dollars as a percentage of maximum capacity.

Click to see Answer

38%

10. Boston Beer Company, the brewer of Samuel Adams, reported the following financial information to its shareholders:

- Total Revenue=\$488,600,000
- Total Variable Costs=\$203,080,000
- Total Fixed Costs=\$182,372,000

What is the break-even revenue?

Click to see Answer

\$312,086,576.10

11. In the beverage industry, PepsiCo and The Coca-Cola Company are the two big players. The following financial information, in millions of dollars, was reported to its shareholders:

	Pepsi Co.	Coca Cola Company
Total Revenue	\$14.296	\$21.807
Total Variable Costs	\$7.683	\$12.663
Total Fixed Costs	\$6.218	\$8.838

Compare the break-even points in total dollars between the two companies based on these reports.

Click to see Answer

Pepsi=\$13,442,088,008.47, Coca-Cola=\$21,077,238,188.98, Pepsi is 36.22% lower.

12. The Puzzle Company had total revenue of \$4,750,000, total fixed costs of \$1,500,000, and total variable costs of \$2,750,000. If the company desires to earn a net income of \$1,000,000, what total sales volume is needed to achieve the goal?

Click to see Answer

\$5,937,500

13. Whirlpool Corporation had annual sales of \$18.907 billion with a net income of \$0.549 billion. Total variable costs amounted to \$16.383 billion.
- What are the fixed costs?
 - Determine Whirlpool Corporation's break-even revenue.
 - Suppose the fixed costs increase by 25%. What is the new break-even revenue?
 - If Whirlpool Corporation managed to increase revenues by 5% the following year while implementing cost-cutting measures that trimmed variable costs by 2%, determine the percent change in the break-even dollars.

Click to see Answer

- a. \$1.975 billion; b. \$14.975 billion; c. \$18.493 billion; d. -30.2%

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2.5 REVIEW EXERCISES

1. Logitech manufactures a surround sound multimedia speaker system for personal computers. The average selling price is \$120 per unit, with unit variable costs totaling \$64. Fixed costs associated with this product total \$980,000.
 - a. Calculate the profit or loss if 25,000 units are sold.
 - b. Calculate the profit or loss if 16,000 units are sold.

Click to see Answer

a. \$420,000; b. -\$84,000

2. Microsoft sells its wireless laser desktop mouse and keyboard for \$70. Unit variable costs are \$45.60 and fixed costs associated with this product total \$277,200.
 - a. What is the break-even point in units?
 - b. What is the net income or loss if 40,000 units are sold?
 - c. Suppose the fixed costs decrease by \$20,000 with no changes to selling price or unit variable costs. What is the new break-even point?
 - d. Suppose the selling price increases by \$5 with no changes to the selling price or unit variable costs. What is the new break-even point?
 - e. Suppose the fixed costs increase by \$10,000 and the unit variable costs increase by \$1.40 with no changes to the selling price. What is the new break-even point?

Click to see Answer

a. 11,361; b. \$698,800; c. 10,541; d. 9429; e. 12,487

3. Frances has a home-based Avon business. She purchases makeup from Avon for an average price of \$12 and sells it to her customers on average for \$20. She assigns \$100 of her monthly computer leasing costs to this product line and spends \$260 per month advertising this product line. In hosting Avon parties, she estimates her labour cost at \$2 per unit sold.
 - a. What is her net income or loss if 100 units are sold?
 - b. What is her net income or loss if 35 units are sold?
 - c. What is the break-even point in both units and dollars?
 - d. What is the new unit break-even point if Frances increases her advertising costs by 50% per month?

Click to see Answer

a. \$240; b. -\$150; c. 60, \$120; d. 82

4. Brynne is developing a business plan. From market research she has learned that her customers are willing to pay \$49.95 for her product. Unit variable costs are estimated at \$23.75, and her monthly total fixed costs are \$12,000. To earn a net income of at least \$5,000, what minimum level of monthly sales (in units) does she need?

Click to see Answer

649

5. Advanced Microcomputer Systems had annual sales of \$31,979,000 with total variable costs of \$20,934,000 and total fixed costs of \$6,211,000.
- What is the company's contribution rate?
 - What annual revenue is required to break even?
 - Suppose the fixed costs increase by \$50,000 with no changes to the selling price or variable costs. What is the new break-even revenue?
 - In the following year, a competitor dealt a severe blow to sales and total revenues dropped to \$15,000,000. Calculate the net income.

Click to see Answer

a. 34.54%; b. \$17,982,939.70; c. \$17,838,172.84; d. -\$1,030,256.39

6. In the North American automotive markets, Toyota and Honda are two of the big players. The following financial information (all numbers in millions of dollars) for each was reported to its shareholders:

	Toyota	Honda
Total Revenue	\$247,734	\$101,916
Total Variable Costs	\$204,135	\$75,532
Total Fixed Costs	\$20,938	\$24,453
Net Income	\$22,661	\$1,931

Compare the break-even revenue between the two companies based on these reports.

Click to see Answer

Toyota=\$118,971.86 million; Honda=\$94,456.94 million; Honda is lower by 20.61%

7. Lagimodiere Industry's unit variable costs are 45% of the unit selling price. Annual fixed costs are \$1.5 million.
- What total revenue results in an income of \$250,000?
 - Calculate the break-even dollars for Lagimodiere Industry.
 - If revenues exceed the break-even dollars by 20%, determine the net income.

Click to see Answer

a. \$3,181,818.18; b. \$2,727,272.73; c. \$300,000

8. Svetlana and Yoric are each thinking of starting the same type of business. Svetlana is young and into technology and mechanization. She has a large fixed cost of \$15,000, but her advanced production machinery minimizes her labour, resulting in a low unit variable cost of \$25. Yoric is a little older and believes the best quality items are handmade. He has a low fixed cost of \$8,000 but incurs more labour costs, resulting in a high unit variable cost of \$60. Regardless of who makes the product, market research shows that consumers would pay \$100 for the product.
- Calculate the unit break-even for both Svetlana and Yoric.
 - Calculate the net income or loss for both if unit sales are 50% higher than the break-even point.
 - Calculate the net income or loss for both if unit sales are 50% lower than the break-even point.

Click to see Answer

a. Svetlana=200, Yoric=200; b. Svetlana=\$9,075, Yoric=\$4,000; c. Svetlana=-\$6,975, Yoric=-\$4,000

9. A surfboard manufacturer lost \$500,000 last year during a recession. Total revenue was \$5,000,000 and total variable costs were 40% of sales. The production facility ran at 50% capacity. The production manager wants to know the following:
- What is the percent capacity required to break even?
 - When the economy recovers this year, if the plant runs at 100% capacity what net income could the company realize?
 - There is a possibility that sales could be so strong this year that the plant may be required to run at 120% capacity by offering a lot of overtime to its production workers. This would result in total variable costs rising by 35%. On a strictly financial basis, should the production manager plan to exceed capacity or should he advise top management to freeze production at 100% capacity? Justify

your answer.

Click to see Answer

a. 58.3%; b. \$2,500,000; c. \$3,100,000

10. The following annual information is known about a company:

Material Inputs to Manufacturing	\$17,400,000	Marketing and Advertising	\$3,500,000
Management Salaries	\$2,750,000	Production Worker Wages	\$9,990,000
Supplies Needed for Manufacturing	\$8,345,000	Utilities Used in Production	\$6,235,000
Utilities Used in Non-Production	\$2,222,000	Property Taxes	\$2,100,000
Number of Units Sold	75,132	Unit Price	\$749

- Calculate the unit variable cost, total fixed costs, unit contribution margin, contribution rate, net income, and breakeven in both units and dollars.
- The marketing research manager shows that if the company lowers its price by 5%, unit sales will rise by 5%. Should the company lower its price? Justify your decision.
- Assume that the price remained the same. Due to economic pressures, fixed costs are forecasted to rise by 6% and variable costs by 4% next year. At the same level of output, by what percentage must the selling price rise to maintain the same level of net income achieved in (a)?

Click to see Answer

a. VC=\$558.62, TFC=\$10,572,000, CM=\$190.38, CR=25.42%, NI=\$3,731,868, BE Units=55,532, BE Revenue=\$41,594,468; b. \$1,492,494.77; c. increase by 4.1108%

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2.6 CHAPTER SUMMARY

Key Concepts

- **2.1 Cost-Volume-Profit Terminology**
 - The different types of costs.
 - How to determine the unit variable cost.
 - Explanation of break-even analysis.
 - **2.2 Revenue and Cost Functions**
 - Calculating the net income using the revenue and cost functions.
 - Calculating the break-even point using the revenue and cost functions.
 - **2.3 Contribution Margin**
 - Calculating the contribution margin.
 - Calculating the net income using the contribution margin.
 - Calculating the break-even point using the contribution margin.
 - **2.4 Contribution Rate**
 - Calculating the contribution rate.
 - Calculating the net income using the contribution rate.
 - Calculating the break-even revenue using the contribution rate.
-

Glossary of Terms

- **Break-Even Analysis.** An analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost.
- **Break-Even Point.** A quantity that represents the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero.
- **Contribution Margin.** The amount each unit sold adds to the net income of the business.
- **Contribution Rate.** A contribution margin expressed as a percentage of the selling price.
- **Cost.** An outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services.
- **Economies of Scale.** The principle that, as production levels rise, per-unit variable costs tend to become lower as efficiencies are achieved.

- **Fixed Cost.** A cost that does not change with the level of production or sales.
 - **Net Income.** The amount of money left over after all costs are deducted from all revenues.
 - **Selling Price.** The amount the business charges its customers to purchase one unit of an item produced.
 - **Total Cost.** The sum of all costs for the company, including both the total fixed costs and total variable costs.
 - **Total Fixed Cost.** The sum of all fixed costs that a business incurs.
 - **Total Revenue.** The entire amount of money received by a company for selling its product, calculated by multiplying the quantity sold by the selling price.
 - **Total Variable Cost.** The sum of all variable costs that a business incurs at a particular level of output.
 - **Unit Variable Cost.** The assignment of total variable costs on a per-unit basis.
 - **Variable Cost.** A cost that changes with the level of production or sales.
-

Formulas

• Symbols Used

- CM = contribution margin
- CR = contribution rate
- NI = net income
- S = selling price per unit
- FC = fixed costs
- TC = total cost
- TR = total revenue
- TVC = total variable cost
- VC = variable cost per unit
- x = number of units

• Formulas Used

- Unit Variable Cost: $VC = \frac{TVC}{x}$
- Total Costs: $TC = FC + VC \times x$
- Total Revenue: $TR = S \times x$
- Net Income: $NI = S \times x - (FC + VC \times x)$

- Contribution Margin: $CM = S - VC$
- Net Income: $NI = CM \times x - FC$
- Number of Units Sold: $\text{Number of Units Sold} = \frac{FC + NI}{CM}$
- Break-Even Point: $\text{Break-Even Point} = \frac{FC}{CM}$
- Contribution Rate: $CR = \frac{CM}{S} \times 100\%$
- Contribution Rate: $CR = \frac{TR - TVC}{TR} \times 100\%$
- Total Revenue: $TR = \frac{FC + NI}{CR}$
- Break-Even Revenue: $\text{Break-Even Revenue} = \frac{FC}{CR}$

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PART III

SIMPLE INTEREST AND ITS APPLICATIONS

If you have a lot of surplus cash, it would hardly make sense to hide it under your mattress or bury it in the backyard of your townhouse. Money has the potential to earn money. As any investment specialist will tell you, you want your money to work for you instead of just sitting idle. So you can invest your money and be rewarded by earning interest.

When you take out a loan, you pay interest to the person, bank, or credit union that loaned you the money. The interest that you pay on the loan is the charge or fee for using someone else's money. **Interest** is the fee paid by a borrower to a lender for using the lender's money. As a borrower, interest paid is an expense. But as a lender or investor, interest earned is income.

Almost all businesses will either invest or borrow money at some time. For example, most new businesses will need to take out a business loan to get the business up and running. A business might issue commercial papers as a way to borrow money for some type of capital project. Similarly, a business might invest their profits in some type of savings account or other investment.

On a personal level, almost all people invest and borrow money at some time in their lives. People take out student loans to pay for their post-secondary education, car loans to purchase a new car, or a line of credit to finance other purchases. To get their money to work for them, people invest money in a variety of investments, such as savings accounts, guaranteed investment certificates, treasury bills, and commercial papers.

Even governments need to borrow money. Have you ever thought about how the government of Canada or your provincial government goes about doing that? Although a government has many methods at its disposal to borrow money, a common method is to use treasury bills—in essence, borrowing the money from investors.

The bottom line is that money does not come free. In this chapter, you will explore the concept of simple interest, learn how to calculate it, and then apply simple interest to various financial tools.

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3.1 SIMPLE INTEREST: PRINCIPAL, RATE, TIME

LEARNING OBJECTIVES

- Understand the concept of simple interest.
- Calculate interest, principal, rate, and time for simple interest transactions.

Interest is the fee paid by a borrower to a lender for using the lender's money. As a borrower, the interest paid is an expense. But as a lender or an investor, interest earned is income. There are two types of interest: simple interest and compound interest.

In a simple interest environment, interest is calculated solely on the initial amount of money invested or borrowed at the beginning of the transaction at a specified simple interest rate for the entire time period. A loan or investment always involves two parties—one giving and one receiving. No matter which party you are in the transaction, the amount of interest remains unchanged. The only difference lies in whether you are earning or paying the interest.

Suppose \$1,000 is placed into an account with 12% simple interest for a period of 12 months. For the entire term of this transaction, the amount of money in the account always equals \$1,000. During this period, interest accrues at a rate of 12%, but the interest is never placed into the account. When the transaction ends after 12 months, the \$120 of interest and the initial \$1,000 are then combined to total \$1,120.

The Simple Interest Formula

The amount of simple interest (I) is calculated as a percent of the amount of money invested or borrow over a specified period of time.

$$I = P \times r \times t$$

where

- **I is the amount of interest.** This is the dollar amount of interest paid or received.
- **P is the present value or principal.** This is the amount borrowed or invested at the beginning of the time period.
- **r is the simple interest rate.** This is the rate of interest that is charged or earned during a specified time period. The simple interest rate is expressed as a percent for a given time period, usually annually or per year, unless otherwise specified.
- **t is the period of time or term of the investment or loan.** The time period is the length of time for which interest is earned or charged on the investment or loan respectively.

NOTE

Recall that algebraic equations require all terms to be expressed with a common unit. This principle remains true for the simple interest formula, particularly with regard to the interest rate and the time period. For example, if you have a 3% annual simple interest rate for nine months, then either

- The time needs to be expressed annually as $\frac{9}{12}$ of a year to match the yearly interest rate, or
- The interest rate needs to be expressed monthly as $\frac{3\%}{12} = 0.25\%$ per month to match the number of months.

It does not matter which conversion you do as long as you express both the interest rate and the time in the same unit. If one of these two variables is your algebraic unknown, the unit of the known variable determines the unit of the unknown variable. For example, assume that you are solving the simple interest formula for the time period. If the interest rate used in the formula is annual, then the time period is expressed in number of years.

EXAMPLE

Julio borrowed \$1,100 from Maria five months ago. When he first borrowed the money, they agreed that he would pay Maria 5% simple interest. If Julio pays her back today, how much interest does he owe her?

Solution:

Step 1: The given information is

$$\begin{array}{l} P = \$1,100 \\ r = 5\% \text{ (per year)} \\ t = 5 \text{ months} \end{array}$$

Step 2: The interest rate is annual, but the time is in months. Convert the time period from months to years.

$$t = \frac{5}{12}$$

Step 3: Solve for the amount of interest, I .

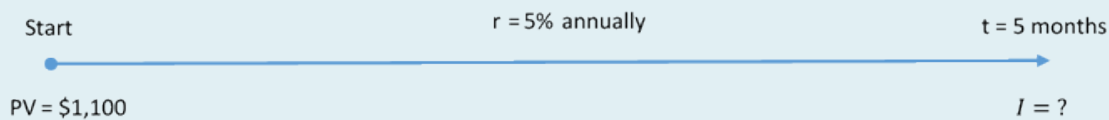


Figure 3.1.1

$$\begin{aligned} I &= P \times r \times t \\ &= 1,100 \times 0.05 \times \frac{5}{12} \\ &= \$22.92 \end{aligned}$$

For Julio to pay back Maria, he must reimburse her for the \$1,100 principal borrowed plus an additional \$22.92 of simple interest as per their agreement.

Solving for P , r , or t

Four variables are involved in the simple interest formula $I = P \times r \times t$, which means that any three can be known quantities and require you to solve for the fourth missing variable.

EXAMPLE

What amount of money invested at 6% annual simple interest for 11 months earns \$2,035 of interest?

Solution:

Step 1: The given information is

$$\begin{array}{l} I = \$2,035 \\ r = 6\% \text{ (per year)} \\ r = 0.06 \\ t = 11 \text{ months} \end{array}$$

Step 2: The interest rate is annual, but the time is in months. Convert the time period from months to years.

$$t = \frac{11}{12}$$

Step 3: Solve for the principal P .

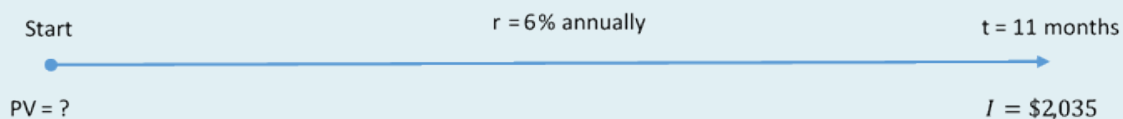


Figure 3.1.2

$$\begin{aligned}
 t &= \frac{I}{P \times r} \\
 &= \frac{1,187.50}{95,000 \times 0.05} \\
 &= 0.25
 \end{aligned}$$

Step 3: The time period found in step 2 is in years, so convert the years to months by multiplying by 12.

$$\begin{aligned}
 \text{Time period in months} &= 0.25 \times 12 \\
 &= 3
 \end{aligned}$$

For \$95,000 to earn \$1,187.50 at 5% simple interest, it must be invested for a three-month period.

TRY IT

If you want to earn \$1,000 of simple interest at a rate of 7% in a span of five months, how much money must you invest?

Click to see Solution

$$\begin{aligned}
 P &= \frac{I}{r \times t} \\
 &= \frac{1000}{0.07 \times \frac{5}{12}} \\
 &= \$34,285.71
 \end{aligned}$$

TRY IT

If you placed \$2,000 into an investment account earning 3% simple interest, how many months does it take for you to have \$2,025 in your account?

Click to see Solution

$$\begin{aligned}
 t &= \frac{I}{P \times r} \\
 &= \frac{25}{2000 \times 0.03} \\
 &= 0.4166...
 \end{aligned}$$

$$\begin{aligned}
 \text{Time in months} &= 0.4166... \times 12 \\
 &= 5
 \end{aligned}$$

TRY IT

A \$3,500 investment earned \$70 of interest over the course of six months. What annual rate of simple interest did the investment earn?

Click to see Solution

$$\begin{array}{l} r = \frac{1}{12} P \times t \\ \frac{70}{3500} \times \frac{6}{12} = 0.04 \end{array} \quad 4\%$$

Time and Dates

In the examples of simple interest so far, the time period was given in months. While this is convenient in many situations, financial institutions and organizations calculate interest based on the exact number of days in the transaction, which changes the interest amount.

To illustrate this, assume you had money saved for the entire months of July and August, where $t = \frac{2}{12}$ or $t = 0.16666... = 0.1\overline{6}$ of a year. However, if you use the exact number of days, the 31 days in July and 31 days in August total 62 days. In a 365-day year that is $t = \frac{62}{365}$ or $t = 0.169863$ of a year. Notice a difference of 0.003196 ($0.169863 - 0.16$) occurs. Therefore, to be precise in performing simple interest calculations, you must calculate the **exact number of days** involved in the transaction. Although you can count the exact number of days by hand, most financial calculators and spreadsheet software like Excel have built-in functions to calculate the number of days between two dates.

USING THE TI BAI PLUS CALCULATOR TO COUNT THE NUMBER OF DAYS BETWEEN DATES.

To count the number of days between two dates:

1. Press 2nd DATE (the 1 button) to enter the date worksheet.
2. At the DT1 screen, enter the first date. Dates are entered in the form mm.ddyy. For example, enter May 19, 2023 as 05.1923. After entering the date press ENTER.
3. Press the down arrow.
4. At the DT2 screen, enter the second date. Dates are entered in the form mm.ddyy. For example, enter August 7, 2023 as 08.0723. After entering the date press ENTER.

5. Press the down arrow.
6. At the DBD screen press CPT to calculate the number of days between the two entered dates. For example, there are 80 days between May 19, 2023 and August 7, 2023.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=26#oembed-1>

[Calculating Dates and Days \(Month-Day-Year\)](#) by Joshua Emmanuel [3:32] (transcript available).

NOTE

When solving for t , decimals may appear in your solution. For example, if calculating t in days, the answer may show up as 45.9978 or 46.0023 days. However, interest is calculated only on the complete number of days. This occurs because the interest amount, I , used in the calculation has been rounded off to two decimals. Because the interest amount is imprecise, the calculation of t is imprecise. When this occurs, round t off to the nearest integer.

EXAMPLE

Mark borrowed \$4,200 from his friend Chris on November 3, 2022. Their agreement required that Mark pay back the loan on April 24, 2023 with 8% simple interest. How much interest did Mark pay Chris?

Solution:

Step 1: The given information is

$$\begin{array}{l} P = \$4,200 \\ r = 8\% \text{ (per year)} \\ t = \frac{294}{365} \end{array}$$

Step 2: Solve for the interest.

$$\begin{aligned} I &= P \times r \times t \\ &= 4200 \times 0.08 \times \frac{294}{365} \\ &= \$270.64 \end{aligned}$$

Mark pays Chris \$270.64 in interest.

EXAMPLE

On September 13, 2011, Aladdin decided to pay back the Genie on his loan of \$15,000 at 9% simple interest. If he paid the Genie the principal plus \$1,283.42 of interest, on what day did he borrow the money from the Genie?

Solution:

Step 1: The given information is

$$\begin{array}{l} P = \$15,000 \\ I = \$1,283.42 \\ r = 9\% \text{ (per year)} \\ \end{array}$$

Step 2: Solve for the time period.

$$\begin{aligned} t &= \frac{I}{P \times r} \\ &= \frac{1,283.42}{15,000 \times 0.09} \\ &= 0.95068... \end{aligned}$$

Step 3: The time period found in step 2 is in years, so convert the years to days by multiplying by 365.

$$\begin{aligned} \text{Time period in days} &= 0.95068... \times 365 \\ &= 346.998... \\ &\rightarrow 347 \end{aligned}$$

Step 4: Use the DATE function to calculate the start date (DT1).

1. Press 2nd DATE.
2. Press the down arrow to skip the DT1 screen and move to the DT2 screen.
3. At the DT2 screen enter 09.13.11 for September 13, 2011. Press ENTER.
4. Press the down arrow.
5. At the DBD screen, enter 347 and press ENTER.
6. Press the up arrow twice to return to the DT1 screen.
7. At the DT1 screen, press CPT. The date is October 1, 2010.

If Aladdin owed the Genie \$1,283.42 of simple interest at 9% on a principal of \$15,000, he must have borrowed the money 347 days earlier, which is October 1, 2010.

TRY IT

Brynn borrowed \$25,000 at 1% per month from a family friend to start her entrepreneurial venture

on December 2, 2011. If she paid back the loan on June 16, 2012, how much simple interest did she pay?

Click to see Solution

$$\begin{aligned} I &= P \times r \times t \\ &= 25,000 \times 0.12 \times \frac{197}{365} \\ &= \$1,619.18 \end{aligned}$$

TRY IT

If \$6,000 principal plus \$132.90 of simple interest was withdrawn on August 14, 2011, from an investment earning 5.5% interest, on what day was the money invested?

Click to see Solution

$$\begin{aligned} t &= \frac{I}{P \times r} \\ &= \frac{132.90}{6000 \times 0.055} \\ &= 0.40272... \end{aligned}$$

$$\begin{aligned} \text{Time in days} &= 0.40272... \times 365 \\ &= 146.9954... \\ &\rightarrow 147 \end{aligned}$$

The money was invested on March 20, 2011.

Exercises

1. Brynn borrowed \$25,000 at 1% per month from a family friend to start her entrepreneurial venture on December 2, 2011. If she paid back the loan on June 16, 2012, how much simple interest did she pay?

Click to see Answer

\$1,619.18

2. How much simple interest is earned on \$50,000 over 320 days if the interest rate is:
 - a. 3%
 - b. 6%
 - c. 9%

Click to see Answer

a. \$1,315.07, b. \$2,630.14, c. \$3,945.21

3. If you placed \$2,000 into an investment account earning 3% simple interest, how many months does it take for you to have \$2,025 in your account?

Click to see Answer

5 months

4. If you want to earn \$1,000 of simple interest at a rate of 7% in a span of five months, how much money must you invest?

Click to see Answer

\$34,285.71

5. If \$6,000 principal plus \$132.90 of simple interest was withdrawn on August 14, 2011, from an investment earning 5.5% interest, on what day was the money invested?

Click to see Answer

March 20, 2011

6. Jessica decided to invest her \$11,000 in two back-to-back three-month term deposits. On the first three-

month term, she earned \$110 of interest. If she placed both the principal and the interest into the second three-month term deposit and earned \$145.82 of interest, how much higher or lower was the interest rate on the second term deposit?

Click to see Answer

1.25% higher

7. Marrina is searching for the best way to invest her \$10,000. One financial institution offers 4.25% on three-month term deposits and 4.5% on six-month term deposits. Marrina is considering either doing two back-to-back three-month term deposits or just taking the six-month deposit. It is almost certain that interest rates will rise by 0.5% before her first threemonth term is up. She will place the simple interest and principal from the first three-month term deposit into the second three-month deposit. Which option should Marrina pursue? How much better is your recommended option?

Click to see Answer

Back-to-back 3 month investments, \$1.26 more

8. Evaluate each of the following \$10,000 investment alternatives and recommend the best alternative for investing any principal regardless of the actual amount. Assume in all cases that the principal and simple interest earned in prior terms are placed into subsequent investments.
- **Alternative 1:** 6% for 1 year
 - **Alternative 2:** 5% for 6 months, then 7% for 6 months
 - **Alternative 3:** 5.25% for 3 months, then 5.75% for 3 months, then 6.25% for 3 months, then 6.75% for 3 months

What percentage more interest is earned in the best alternative versus the worst alternative?

Click to see Answer

Alternative 3, 0.1283% more than Alternative 1

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3.2 SIMPLE INTEREST: MATURITY VALUE

LEARNING OBJECTIVES

- Calculate the maturity value for simple interest transactions.

The **maturity value** of a transaction is the amount of money resulting at the end of a transaction. That is, the maturity value is the sum of the principal and the interest together. It is called a maturity value because, in the financial world, the termination of a financial transaction is known as the “maturing” of the transaction. The amount of principal with interest at some point in the future, but not necessarily the end of the transaction, is known as the future value.

For any financial transaction involving simple interest, the following is true

$$\text{Maturity Value} = \text{Principal} + \text{Interest}$$

Applying algebra, you can summarize this expression by the following equation, where the future value or maturity value is commonly denoted by the symbol S .

$$S = P + I$$

Substituting in $I = P \times r \times t$ for I yields the equation

$$\begin{aligned} S &= P + P \times r \times t \\ &= P \times (1 + r \times t) \end{aligned}$$

The Maturity Value Formula

The maturity value (S) is

$$S = P \times (1 + r \times t)$$

where

- **S is the maturity value.** This is the total principal plus interest.
- **P is the present value or principal.** This is the amount borrowed or invested at the beginning of the time period.
- **r is the simple interest rate.** This is the rate of interest that is charged or earned during a specified time period. The simple interest rate is expressed as a percent for a given time period, usually annually or per year, unless otherwise specified.
- **t is the period of time or term of the investment or loan.** The time period is the length of time for which interest is earned or charged on the investment or loan respectively.

From the maturity value formula $S = P \times (1 + r \times t)$ you can derive the following formula to calculate the principal P :

$$P = \frac{S}{1 + r \times t}$$

Given the maturity value and principal, the amount of simple interest is:

$$I = S - P$$

EXAMPLE

Assume that today you have \$10,000 that you are going to invest at 7% simple interest for 11 months. How much money will you have in total at the end of the 11 months? How much interest do you earn?

Solution:

Step 1: The given information is

$$P = \$10,000$$

$$r = 7\%$$

$$= 0.07$$

$$t = 11 \text{ months}$$

Step 2: Convert the time period from years to months.

$$t = \frac{11}{12}$$

Step 3: Solve for the maturity value, S .

$$\begin{aligned} S &= P \times (1 + r \times t) \\ &= 10,000 \times \left(1 + 0.07 \times \frac{11}{12}\right) \\ &= \$10,641.67 \end{aligned}$$

The total amount after 11 months is \$10,641.67.

Step 4: Solve for the interest amount, I .

$$\begin{aligned} I &= S - P \\ &= 10,641.67 - 10,000 \\ &= \$641.67 \end{aligned}$$

The \$10,000 earns \$641.67 in simple interest over the next 11 months, resulting in \$10,641.67 altogether.

EXAMPLE

You just inherited \$35,000 from your uncle's estate and plan to purchase a house four months from today. If you use your inheritance as your down payment on the house, how much will you be able to put down if your money earns 4.25% simple interest? How much interest will you have earned?

Solution:

Step 1: The given information is

$$\begin{aligned}
 P &= \$35,000 \\
 r &= 4.25\% \\
 &= 0.0425 \\
 t &= 4 \text{ months}
 \end{aligned}$$

Step 2: Convert the time period from years to months.

$$t = \frac{4}{12}$$

Step 3: Solve for the maturity value, S .

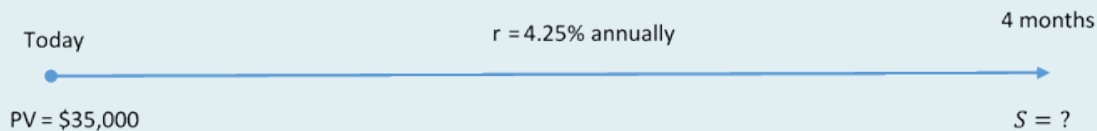


Figure 3.2.1

$$\begin{aligned}
 S &= P \times (1 + r \times t) \\
 &= 35,000 \times \left(1 + 0.0425 \times \frac{4}{12}\right) \\
 &= \$35,495.83
 \end{aligned}$$

Step 4: Solve for the interest amount, I .

$$\begin{aligned}
 I &= S - P \\
 &= 35,495.83 - 35,000 \\
 &= \$495.83
 \end{aligned}$$

Four months from now you will have \$35,495.83 as a down payment toward your house, which includes \$35,000 in principal and \$495.83 in interest.

EXAMPLE

You need \$8,000 for tuition in the fall and the best simple interest rate you could find was 4.5%. Assume you have eight months before you need to pay your tuition. How much money do you need to invest today?

Solution:

Step 1: The given information is

$$\begin{aligned} S &= \$8,000 \\ r &= 4.5\% \\ &= 0.045 \\ t &= 8 \text{ months} \end{aligned}$$

Step 2: Convert the time period from months to years.

$$t = \frac{8}{12}$$

Step 3: Solve for the principal, P .

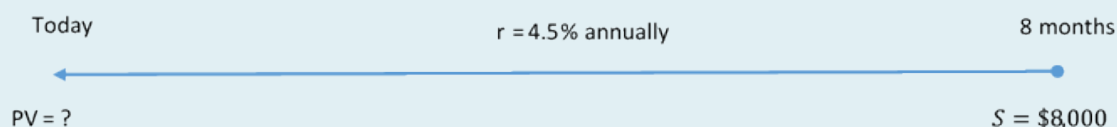


Figure 3.2.2

$$\begin{aligned} P &= \frac{S}{1 + r \times t} \\ &= \frac{8000}{1 + 0.045 \times \frac{8}{12}} \\ &= \$7,766.99 \end{aligned}$$

If you place \$7,766.99 into the investment, it will grow to \$8,000 in eight months.

EXAMPLE

You are sitting in an office at your local financial institution on August 4, 2023. The bank officer says to you, “We will make you a great deal. If we advance that line of credit and you borrow \$20,000 today, when you want to repay that balance on September 1, 2023 you will only have to pay us \$20,168.77, which is not much more!” Before answering, you decide to evaluate the statement. Calculate the simple interest rate that the bank officer used in her calculations.

Solution:

Step 1: Calculate the number of days between August 4, 2023 and September 1, 2023 using the financial calculator.

DT1	8.0423
DT2	9.0123
DBD	28

Step 2: The given information is

$$\begin{array}{l} P = \$20,000 \\ S = \$20,168.77 \\ t = 28 \text{ days} \end{array}$$

Step 3: Convert the time period from days to years.

$$t = \frac{28}{365}$$

Step 4: Solve for the interest, I .

$$\begin{aligned} I &= S - P \\ &= 20,168.77 - 20,000 \\ &= \$168.77 \end{aligned}$$

Step 5: Solve for the rate, r .

$$\begin{aligned}
 r &= \frac{I}{P \times t} \\
 &= \frac{168.77}{20,000 \times \frac{28}{365}} \\
 &= 0.110002 \\
 &\rightarrow 11.0002\%
 \end{aligned}$$

The interest rate on the offered line of credit is 11.0002%. (Note: The rate is probably exactly 11%. The extra 0.0002% is most likely due to the rounded amount of interest used in the calculation.)

NOTE

In the previous example, you could also solve for the interest rate directly from the maturity value formula, without having to calculate the interest separately.

$$\begin{aligned}
 S &= P \times (1 + r \times t) \\
 20,168.77 &= 20,000 \times \left(1 + r \times \frac{28}{365}\right) \\
 1.0084385 &= 1 + r \times \frac{28}{365} \\
 0.0084385 &= r \times \frac{28}{365} \\
 0.110002 &= r
 \end{aligned}$$

But, it is much easier to solve for r in the simple interest formula $I = P \times r \times t$ than it is to solve for r in the maturity value formula $S = P \times (1 + r \times t)$.

TRY IT

An accountant needs to allocate the principal and simple interest on a loan payment into the appropriate ledgers. If the amount received was \$10,267.21 for a loan that spanned April 14, 2023 to July 31, 2023 at 9.1%, how much was the principal and how much was the interest?

Click to see Solution

$$\begin{aligned}
 P &= \frac{S}{1 + r \times t} \\
 &= \frac{10,267.21}{1 + 0.091 \times \frac{108}{365}} \\
 &= \$9,998
 \end{aligned}$$

$$\begin{aligned}
 I &= S - P \\
 &= 10,267.21 - 9,998 \\
 &= \$269.21
 \end{aligned}$$

TRY IT

Suppose Robin borrowed \$3,600 on October 21, 2022 and repaid the loan on February 21, 2023 of the following year. What simple interest rate was charged if Robin repaid \$3,694.63?

Click to see Solution

$$\begin{aligned}
 I &= S - P \\
 &= 3,694.63 - 3,600 \\
 &= \$94.63 \\
 r &= \frac{I}{P \times t} \\
 &= \frac{94.63}{3,600 \times \frac{123}{365}} \\
 &= 0.078 \\
 &\rightarrow 7.8\%
 \end{aligned}$$

Equivalent Payments

- **Late Payments.** If a debt is paid late, then a financial penalty that is fair to both parties involved should

be imposed. That penalty should reflect a current rate of interest and be added to the original payment. Assume you owe \$100 to your friend and that a fair current rate of simple interest is 10%. If you pay this debt one year late, then a 10% late interest penalty of \$10 should be added, making your debt payment \$110. This is no different from your friend receiving the \$100 today and investing it himself at 10% interest so that it accumulates to \$110 in one year.

- **Early Payments.** If a debt is paid early, there should be some financial incentive (otherwise, why bother?). Therefore, an interest benefit, one reflecting a current rate of interest on the early payment, should be deducted from the original payment. Assume you owe your friend \$110 one year from now and that a fair current rate of simple interest is 10%. If you pay this debt today, then a 10% early interest benefit of \$10 should be deducted, making your debt payment today \$100. If your friend then invests this money at 10% simple interest, one year from now he will have the \$110, which is what you were supposed to pay.

Notice in these examples that a simple interest rate of 10% means \$100 today is the same thing as having \$110 one year from now. This illustrates the concept that two payments are equivalent payments if, once a fair rate of interest is factored in, they have the same value on the same day. Thus, in general you are finding two amounts at different points in time that have the same value, as illustrated in the figure below.

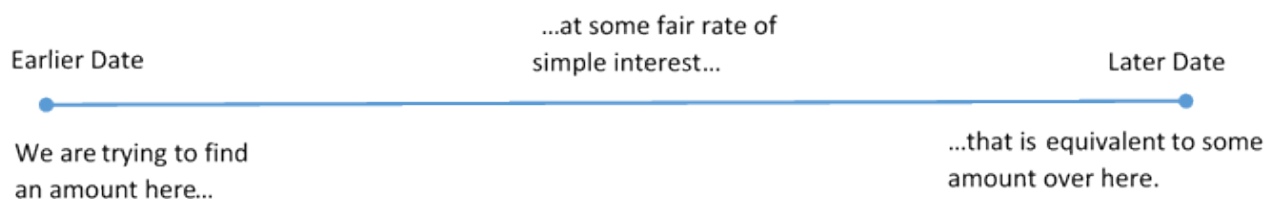


Figure 3.2.3

NOTE

The steps required to calculate an equivalent payment are no different from those for single payments.

- If an early payment is being made, then you know the maturity or future value, so you

solve for the present value or principal, which removes the interest.

- If a late payment is being made, then you know the present value or principal, so you solve for the maturity or future value, which adds the interest penalty.

EXAMPLE

Erin owes Charlotte \$1,500 today. Unfortunately, Erin had some unexpected expenses and is unable to make her debt payment. After discussing the issue, they agree that Erin can make the payment nine months late and that a fair simple interest rate on the late payment is 5%. Use 9 months from now as your focal date and calculate how much Erin needs to pay. What is the amount of her late penalty?

Solution: A late payment is a maturity value amount, S . The late penalty is equal to the interest, I .

Step 1: The given information is

$$\begin{array}{l} P = \$1,500 \\ r = 5\% = 0.05 \\ t = 9 \text{ months} \end{array}$$

Step 2: Convert the time period from months to years.

$$t = \frac{9}{12}$$

Step 3: Solve for the maturity value, S .

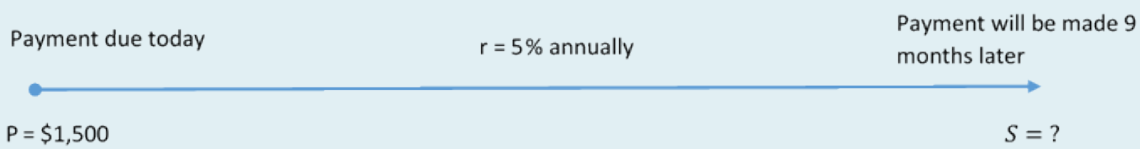


Figure 3.2.4

$$\begin{aligned}
 S &= P \times (1 + r \times t) \\
 &= 1,500 \times \left(1 + 0.05 \times \frac{9}{12}\right) \\
 &= \$1,556.25
 \end{aligned}$$

Step 4: Solve for the interest, I .

$$\begin{aligned}
 I &= S - P \\
 &= 1,556.25 - 1,500 \\
 &= \$56.25
 \end{aligned}$$

Erin's late payment is for \$1,556.25, which includes a \$56.25 interest penalty for making the payment nine months late.

EXAMPLE

Rupert owes Aminata two debt payments: \$600 four months from now and \$475 eleven months from now. Rupert came into some money today and would like to pay off both of the debts immediately. Aminata has agreed that a fair interest rate is 7%. Using today as a focal date, what amount should Rupert pay? What is the total amount of his early payment benefit?

Solution: An early payment is a present value amount, P . Both payments will be moved to today and summed. The early payment benefit will be the total amount of interest removed, I .

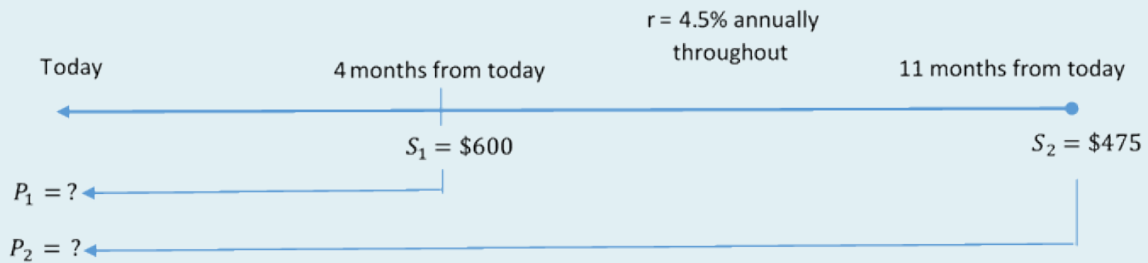
Step 1: The given information is

Payment #1	$S_1 = \$600$ $r = 7\%$ $t = 4 \text{ months}$
Payment #2	$S_2 = \$475$ $r = 7\%$ $t = 11 \text{ months}$

Step 2: Convert the time periods from months to years.

- Payment #1: $t = \frac{4}{12}$
- Payment #2: $t = \frac{11}{12}$

Step 3: Solve for the principal for each payment.



$$\text{Total Amount Paid } (P) = P_1 + P_2$$

Figure 3.2.5

- Payment #1: Solve for P_1

$$\begin{aligned}
 P_1 &= \frac{S_1}{1 + r \times t} \\
 &= \frac{600}{1 + 0.07 \times \frac{4}{12}} \\
 &= \$586.32
 \end{aligned}$$

- Payment #2: Solve for P_2

$$\begin{aligned}
 P_2 &= \frac{S_1}{1 + r \times t} \\
 &= \frac{475}{1 + 0.07 \times \frac{11}{12}} \\
 &= \$446.36
 \end{aligned}$$

The total amount paid today, P , is

$$\begin{aligned}
 P &= P_1 + P_2 \\
 &= 586.32 + 446.36 \\
 &= \$1,032.68
 \end{aligned}$$

Step 4: Calculate the total amount of interest I .

$$\begin{aligned}
 \text{Interest for Payment 1} &= S_1 - P_1 \\
 &= 600 - 586.32 \\
 &= \$13.68 \\
 \text{Interest for Payment 2} &= S_2 - P_2 \\
 &= 475 - 446.36 \\
 &= \$28.64 \\
 I &= 13.68 + 28.64 \\
 &= \$42.32
 \end{aligned}$$

To clear both debts today, Rupert pays \$1,032.68, which reflects a \$42.32 interest benefit reduction for the early payment.

Exercises

- On January 23, 2023, a loan is taken out for \$15,230 at 8.8% simple interest. What is the maturity value of the loan on October 23, 2023?

Click to see Answer

\$16,232.34

- How many weeks will it take \$5,250 to grow to \$5,586 at a simple interest rate of 10.4%? Assume 52 weeks in a year.

Click to see Answer

32 weeks

3. Jayne needs to make three payments to Jade requiring \$2,000 each 5 months, 10 months, and 15 months from today. She proposes instead making a single payment on a focal date eight months from today. If Jade agrees to a simple interest rate of 9.5%, what amount should Jayne pay?

Click to see Answer

\$5,911.32

4. Over a period of nine months, \$40.85 of simple interest is earned at 1.75% per quarter. Calculate both the principal and maturity value for the investment.

Click to see Answer

P=\$778.10, S=\$818.95

5. The Home Depot is clearing out lawn mowers for \$399.75 in an end-of-season sale that ends on September 30, 2022. Alternatively, you could wait until April 1, 2023 and pay \$419.95 on the same lawn mower. Assume your money can earn simple interest of 4.72% on a short-term investment.

- Which option should you choose?
- How much money will you have saved on September 30 by making that choice?

Click to see Answer

a. September, price is lower by \$399.75, b. savings of \$10.49

6. Alia took out three back-to-back short-term simple interest investments. On November 10, 2023, Alia had saved up a total of \$12,986.75. Her interest rate for each investment is listed below:

Date	Interest Rate
March 2, 2023 to June 19, 2023	4.13%
June 19, 2023 to October 7, 2023	4.63%
October 7, 2023 to November 10, 2023	4.43%

How much did Alia originally invest on March 2, 2023? Assume that the principal and interest from a prior investment are both placed into the next investment.

Click to see Answer

\$12,600

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3.3 SAVINGS ACCOUNTS AND SHORT-TERM GUARANTEED INVESTMENT CERTIFICATES

LEARNING OBJECTIVES

- Use simple interest in solving problems involving savings accounts and short-term GICs.

Savings Accounts

A savings account is a deposit account that bears interest and has no stated maturity date. These accounts are found at most financial institutions, such as commercial banks (Royal Bank of Canada, TD Canada Trust, etc.), trusts (Royal Trust, Laurentian Trust, etc.), and credit unions (FirstOntario, Steinbach, Assiniboine, Servus, etc.). Owners of such accounts make deposits to and withdrawals from these accounts at any time, usually accessing the account at an automatic teller machine (ATM), at a bank teller, or through online banking.

A wide variety of types of savings accounts are available. This textbook focuses on the most common features of most savings accounts, including how interest is calculated, when interest is deposited, insurance against loss, and the interest rate amounts available.

- **Calculation of Interest.** There are two common methods for calculating simple interest.
 - Accounts earn simple interest that is calculated based on the daily *closing* balance of the account. The closing balance is the amount of money in the account at the end of the day. Therefore, any balances in the account throughout a single day do not matter. For example, if you start the day with \$500 in the account and deposit \$3,000 at 9:00 a.m., then withdraw the \$3,000 at 4:00 p.m., your closing balance is \$500. That is the principal on which interest is calculated, not the \$3,500 in the account throughout the day.

- Accounts earn simple interest based on a *minimum* monthly balance in the account. For example, if in a single month you had a balance in the account of \$900 except for one day, when the balance was \$500, then only the \$500 is used in calculating the entire month's worth of interest.
- **When Interest Is Deposited.** Interest is accumulated and deposited (paid) to the account once monthly, usually on the first day of the month. Thus, the interest earned on your account for the month of January appears as a deposit on February 1.
- **Insurance against Loss.** Canadian savings accounts at commercial banks are insured by the national Canada Deposit Insurance Corporation (CDIC), which guarantees up to \$100,000 in savings. At credit unions, this insurance is usually provided provincially by institutions such as the Deposit Insurance Corporation of Ontario (DICO), which also guarantees up to \$100,000. This means that if your bank were to fold, you could not lose your money (so long as your deposit was within the maximum limit). Therefore, savings accounts carry almost no risk.
- **Interest Rate Amounts.** Interest rates are higher for investments that are riskier. Savings accounts carry virtually no risk, which means the interest rates on savings accounts tend to be among the lowest you can earn.

Although a wide range of savings accounts are available, these accounts generally follow one of two common structures when it comes to calculating interest. These structures are flat-rate savings accounts and tiered savings accounts. Only flat-rate savings accounts are discussed here.

A **flat-rate savings account** has a single interest rate that applies to the entire balance. The interest rate may fluctuate in sync with short-term interest rates in the financial markets.

EXAMPLE

John invested \$3,000 in a savings account with a simple interest rate of 4.25%. At the end of five years, how much interest did John earn?

Solution:

Step 1: The given information is

$$\begin{array}{l} P = \$3,000 \\ r = 4.25\% \text{ (per year)} \\ = 0.0425 \\ t = 5 \text{ years} \end{array}$$

Step 2: Solve for the amount of interest, I .

$$\begin{aligned} I &= P \times r \times t \\ &= 3,000 \times 0.0425 \times 5 \\ &= \$637.50 \end{aligned}$$

After five years, John earned \$637.50 in interest.

TRY IT

Lea invested \$2,500 in a savings account earning 1.5% simple interest. After three years, Lea invested the maturity value from her savings account into a new savings account with a simple interest rate of 2.1%. Lea left the money in the second savings account for two years. What is the maturity value at the end of the entire five year time period?

Click to see Solution

Maturity value at the end of the first three years.

$$\begin{aligned} S &= P \times (1 + r \times t) \\ &= 2500 \times (1 + 0.015 \times 3) \\ &= \$2,612.50 \end{aligned}$$

Maturity value at the end of the second two years.

$$\begin{aligned} S &= P \times (1 + r \times t) \\ &= 2612.50 \times (1 + 0.021 \times 2) \\ &= \$2,722.23 \end{aligned}$$

At the end of five years, the maturity value is \$2,722.23

To calculate the monthly interest for a flat-rate savings account:

1. Identify the interest rate, opening balance, and the monthly transactions in the savings account.
2. Set up a flat-rate table, as illustrated below. Create a number of rows equaling the number of monthly transactions (deposits or withdrawals) in the account plus one.

Date	Closing Balance in Account	Number of Days	Simple Interest Earned
			$I = P \times r \times t$
Total Interest Earned			

3. For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.
4. Calculate the number of days that the closing balance is maintained for each row.
5. Calculate the simple interest for each row of the table. Ensure that the rate and time are expressed in the same units. Do not round off the resulting interest amounts.
6. Add up the Simple Interest Earned column and round off the answer to two decimals.

NOTE

When you are calculating interest on any type of savings account, pay careful attention to the details on how interest is calculated and any restrictions or conditions on the balance that is eligible to earn the interest.

EXAMPLE

The RBC High Interest Savings Account pays 0.75% simple interest on the daily closing balance in the account and the interest is paid on the first day of the following month. On March 1, 2023, the opening balance in the account was \$2,400. On March 12, 2023, a deposit of \$1,600 was made. On

March 21, 2023, a withdrawal of \$2,000 was made. Calculate the total simple interest earned for the month of March.

Solution:

Step 1: The given information is

$$r = 0.75\%$$

The following transactions dates are known.

- March 1 opening balance = \$2,400
- March 12 deposit = \$1,600
- March 21 withdrawal = \$2,000

Step 2: Set-up the flat-rate table.

1. Determine the date ranges for each balance throughout the month and calculate the closing balances.
2. For each row of the table, calculate the number of days between the two dates. (Note: you can use the built-in count days function on your financial calculator to count the number of days).
3. For each row, calculate the amount of simple interest ($I = P \times r \times t$).
4. Add up the Total Interest Earned column.

Dates	Closing Balance in Account	Number of Days	Simple Interest Earned
March 1 to March 12	\$2400	11	$I = 2400 \times 0.0075 \times \frac{11}{365}$ $= \$0.542465$
March 12 to March 21	$\$2400 + \$1600$ $= \$4000$	9	$I = 4000 \times 0.0075 \times \frac{9}{365}$ $= \$0.739726$
March 21 to April 1	$\$4000 - \$2000$ $= \$2000$	11	$I = 2000 \times 0.0075 \times \frac{11}{365}$ $= \$0.452054$
Total Interest Earned			$I = 0.542465 + 0.739726 + 0.452054$ $= \$1.73$

For the month of March, the savings account earned a total simple interest of \$1.73, which was deposited to the account on April 1.

TRY IT

Michael has a savings account at Canadian Western Bank. The savings account pays 0.95% simple interest on the daily closing balance in the account. The interest is paid on the first day of the following month. On August 1, 2023, the opening balance in Michael's account was \$5,000. On August 10, 2023, Michael withdrew \$1,100 from the account. On August 19, 2023, Michael deposited \$900 into the account. On August 27, 2023, Michael withdrew \$500 from the account. Calculate the total simple interest earned for the month of August.

Click to see Solution

Dates	Closing Balance in Account	Number of Days	Simple Interest Earned
Aug 1 to Aug 10	\$5000	9	$I = 5000 \times 0.0095 \times \frac{9}{365}$ $= \$1.171232877$
Aug 10 to Aug 19	\$3900	9	$I = 3900 \times 0.0095 \times \frac{9}{365}$ $= \$0.913561644$
Aug 19 to Aug 27	\$4800	8	$I = 4800 \times 0.0095 \times \frac{8}{365}$ $= \$0.999452055$
Aug 27 to Sept 1	\$4300	5	$I = 4300 \times 0.0095 \times \frac{5}{365}$ $= \$0.559589041$
Total Interest Earned			$I = 1.171232877 + 0.913561644 + 0.999452055 + 0.559589041$ $= \$3.64$

Short-Term Guaranteed Investment Certificates

A **guaranteed investment certificate** (GIC) is an investment that offers a guaranteed rate of interest over a fixed period of time. GICs are found mostly at commercial banks, trust companies, and credit unions. In this section, you will deal only with short-term GICs, which are defined as those that have a term of less than one year.

Three factors determine the interest rate on a short-term GIC: principal, time, and redemption privileges.

1. **Principal.** Typically, a larger principal is able to realize a higher interest rate than a smaller principal.
2. **Time.** The length of time that the principal is invested affects the interest rate. Short-term GICs range from 30 days to 364 days in length. A longer term usually realizes higher interest rates.
3. **Redemption Privileges.** GICs come in two types: redeemable and nonredeemable. A **redeemable**

GIC can be cashed in at any point before the maturity date, meaning that you can access your money any time you want it. A **nonredeemable** GIC “locks in” your money for the agreed-upon term. Accessing that money before the end of the term usually incurs a stiff financial penalty, either on the interest rate or in the form of a financial fee. Nonredeemable GICs carry a higher interest rate.

NOTE

Short-term GICs involve a lump sum of money (the principal) invested for a fixed term (the time) at a guaranteed interest rate (the rate). Most commonly the only items of concern are the amount of interest earned and the maturity value. But, some instances may require the calculation of an interest rate or the term.

EXAMPLE

Your parents have \$10,000 to invest. They can either deposit the money into a 364-day nonredeemable GIC at Assiniboine Credit Union with a posted rate of 0.75% or they could put their money into back-to-back 182-day nonredeemable GICs with a posted rate of 0.7%. At the end of the first 182 days, they will reinvest both the principal and interest into the second GIC. The interest rate remains unchanged on the second GIC. Which option should they choose?

Solution:

Option 1: 364-day GIC

Step 1: The given information is

$$\begin{array}{l} P = \$10,000 \\ r = 0.75\% \text{ (per year)} \\ r = 0.0075 \\ t = 364 \text{ days} \end{array}$$

Step 2: Convert the time period from days to years.

$$t = \frac{364}{365}$$

Step 3: Solve for the maturity value, S .

$$\begin{aligned} S &= P \times (1 + r \times t) \\ &= 10,000 \times \left(1 + 0.0075 \times \frac{364}{365} \right) \\ &= \$10,074.79 \end{aligned}$$

Option 2: Two consecutive 182-day GICs

Step 1: The given information is

$$\begin{array}{l} P = \$10,000 \\ r = 0.75\% \text{ (per year)} \\ t = 182 \text{ days} \end{array}$$

Step 2: Convert the time period from days to years.

$$t = \frac{182}{365}$$

Step 3: Solve for the maturity value, S , for the first 182-day GIC.

$$\begin{aligned} S &= P \times (1 + r \times t) \\ &= 10,000 \times \left(1 + 0.0075 \times \frac{182}{365} \right) \\ &= \$10,034.90 \end{aligned}$$

Step 4: Solve for the maturity value, S , for the second 182-day GIC.

$$\begin{aligned} S &= P \times (1 + r \times t) \\ &= 10,034.90 \times \left(1 + 0.0075 \times \frac{182}{365} \right) \\ &= \$10,069.93 \end{aligned}$$

The 364-day GIC results in a maturity value of \$10,074.79, while the two back-to-back 182-day GICs result in a maturity value of \$10,069.93. Clearly, the 364-day GIC is the better option as it will earn \$4.86 more in simple interest.

TRY IT

If you place \$25,500 into an 80-day short-term GIC at TD Canada Trust earning 0.55% simple interest, how much will you receive when the investment matures?

Click to see Solution

$$\begin{aligned} S &= 25,500 \times \left(1 + 0.0055 \times \frac{80}{365}\right) \\ &= \$25,530.74 \end{aligned}$$

TRY IT

Interest rates in the GIC markets are always fluctuating because of changes in the short-term financial markets. If you have \$50,000 to invest today, you could place the money into a 180-day GIC at Canada Life earning a fixed rate of 0.4%, or you could take two consecutive 90-day GICs. The current posted fixed rate on 90-day GICs at Canada Life is 0.3%. Trends in the short-term financial markets suggest that within the next 90 days short-term GIC rates will be rising. What does the short-term 90-day rate need to be 90 days from now to arrive at the same maturity value as the 180-day GIC? Assume that the entire maturity value of the first 90-day GIC would be reinvested.

Click to see Solution

Maturity value of the 180-day GIC.

$$\begin{aligned}
 S &= 50,000 \times \left(1 + 0.004 \times \frac{180}{365} \right) \\
 &= \$50,098.63
 \end{aligned}$$

Maturity value at the end of the first 90-day GIC.

$$\begin{aligned}
 S &= 50,000 \times \left(1 + 0.003 \times \frac{90}{365} \right) \\
 &= \$50,036.99
 \end{aligned}$$

Interest rate for the second 90-day GIC.

$$\begin{aligned}
 50,098.63 &= 50,036.99 \times \left(1 + r \times \frac{90}{365} \right) \\
 1.00123... &= 1 + r \times \frac{90}{365} \\
 0.00123... &= r \times \frac{90}{365} \\
 0.005 &= r
 \end{aligned}$$

The second 90-day GIC needs an interest rate of 0.5%

Exercises

1. A savings account at your local credit union holds a balance of \$5,894 for the entire month of September. The posted simple interest rate is 1.35%. Calculate the amount of interest earned for the month (30 days).

Click to see Answer

\$6.54

2. If you place \$25,500 into an 80-day short-term GIC at TD Canada Trust earning 0.55% simple interest, how much will you receive when the investment matures?

Click to see Answer

\$25,530.74

3. In November of 2022, the opening balance on a savings account was \$12,345. A deposit of \$3,000 was made to the account on November 19, and a withdrawal of \$3,345 was made from the account on

November 8. If simple interest is paid at 0.95% based on the daily closing balance, how much interest for the month of November is deposited to the account on December 1?

Click to see Answer

\$8.57

4. A 320-day short-term GIC earns 0.78% simple interest. If \$4,500 is invested into the GIC, what is the maturity value and how much interest is earned?

Click to see Answer

\$4530.77, \$30.77

5. In February of a leap year, the opening balance on your savings account was \$3,553. You made two deposits of \$2,000 each on February 5 and February 21. You made a withdrawal of \$3,500 on February 10, and another withdrawal of \$750 on February 17. If simple interest is calculated on the daily closing balance at a rate of 1.45%, how much interest do you earn for the month of February?

Click to see Answer

\$3.63

6. An investor places \$30,500 into a short-term 120-day GIC at the Bank of Montreal earning 0.5% simple interest. The maturity value is then rolled into another short-term 181-day GIC earning 0.57% simple interest. Calculate the final maturity value.

Click to see Answer

\$30,636.49

7. An investor with \$75,000 is weighing options between a 200-day GIC or two back-to-back 100-day GICs. The 200-day GIC has a posted simple interest rate of 1.4%. The 100-day GICs have a posted simple interest rate of 1.35%. The maturity value of the first 100-day GIC would be reinvested in the second 100-day GIC (assume the same interest rate upon renewal). Which alternative is best and by how much?

Click to see Answer

200 day GIC is better by \$69.70

8. Sun Life Financial Trust offers a 360-day short-term GIC at 0.65%. It also offers a 120-day short-term GIC at 0.58%. You are considering either the 360-day GIC or three consecutive 120-day GICs. For the 120-day GICs, the entire maturity value would be “rolled over” into the next GIC. Assume that the posted rate increases by 0.1% upon each renewal. If you have \$115,000 to invest, which option should you pursue and how much more interest will it earn?

Click to see Answer

Three back-to-back 120 GICs are better by \$35.75

9. Interest rates in the GIC markets are always fluctuating because of changes in the short-term financial markets. If you have \$50,000 to invest today, you could place the money into a 180-day GIC at Canada Life earning a fixed rate of 0.4%, or you could take two consecutive 90-day GICs. The current posted fixed rate on 90-day GICs at Canada Life is 0.3%. Trends in the short-term financial markets suggest that within the next 90 days short-term GIC rates will be rising. What does the short-term 90-day rate need to be 90 days from now to arrive at the same maturity value as the 180-day GIC? Assume that the entire maturity value of the first 90-day GIC would be reinvested.

Click to see Answer

0.4995%

10. You are considering the following short-term GIC investment options for an amount of \$90,000.
- Option 1: One 360-day GIC at 0.8% simple interest.
 - Option 2: Two 180-day GICs at 0.75% simple interest.
 - Option 3: Three 120-day GICs at 0.72% simple interest.
 - Option 4: Four 90-day GICs at 0.715% simple interest.

In all cases, assume that the posted rates remain unchanged and that the entire maturity value will be reinvested in the next short-term GIC. Calculate the total maturity value for each option at the end of 360 days.

Click to see Answer

\$90,710.14, \$90,666.99, \$90,640.64, \$90,636.36

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3.4 TREASURY BILLS AND COMMERCIAL PAPERS

LEARNING OBJECTIVES

- Use simple interest in solving problems involving treasury bills and commercial papers.

Treasury Bills

Treasury bills, also known as T-bills, are short-term financial instruments that both federal and provincial governments issue with maturities no longer than one year. Approximately 27% of the national debt is borrowed through T-bills.

Here are some of the basics about T-bills:

1. The Government of Canada regularly places T-bills up for auction every second Tuesday. Provincial governments issue them at irregular intervals.
2. The most common terms for federal and provincial T-bills are 30 days, 60 days, 90 days, 182 days, and 364 days.
3. T-bills do not earn interest. Instead, they are sold at a **discount** and redeemed at full value. This follows the principle of “buy low, sell high.” The percentage by which the value of the T-bill grows from sale to redemption is called the **yield** or rate of return. From a mathematical perspective, the yield is calculated in the exact same way as an interest rate is calculated, and therefore the yield is mathematically substituted as the discount rate in all simple interest formulas.
4. The **face value of a T-bill** (also called *par value*) is the maturity value, payable at the end of the term. It includes both the principal and yield together.
5. T-bills do not have to be retained by the initial investor throughout their entire term. At any point

during a T-bill's term, an investor is able to sell it to another investor through secondary financial markets. Prevailing yields on T-bills at the time of sale are used to calculate the price.

Commercial Papers

A **commercial paper** (or paper for short) is the same as a T-bill except that it is issued by a large corporation instead of a government. It is an alternative to short-term bank borrowing for large corporations. Most of these large companies have solid credit ratings, meaning that investors bear very little risk that the face value will not be repaid upon maturity.

Commercial papers carry the same properties as T-bills. The only fundamental differences lie in the term and the yield:

1. The terms are usually less than 270 days but can range from 30 days to 364 days. The most typical terms are 30 days, 60 days, and 90 days.
2. The yield on commercial papers tends to be slightly higher than on T-bills because corporations do carry a higher risk of default than governments.

How Treasury Bills and Commercial Papers Work

Mathematically, T-bills and commercial papers operate in the exact same way. The future value for both of these investment instruments is always known because it is the face value. Commonly, the two calculated variables are either the present value (the price) or the yield (the interest rate).

Calculating the Price

1. The face value, yield, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify the following:
 - The face value (S).
 - The yield (r) on the date of the sale, which is always expressed annually. Remember that mathematically the yield is the same as the discount rate.
 - The number of days remaining between the date of sale and the maturity date. Count the first day, but not the last day.

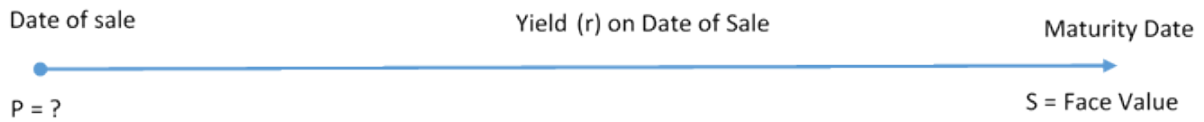


Figure 3.4.1

2. Solve for the present value using $P = \frac{S}{1 + r \times t}$, which is the price of the T-bill or commercial paper. This price is always less than the face value.

Calculating the Yield Rate

1. The face value, price, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify:
 - The face value (S).
 - The price on the date of the sale (P).
 - The number of days remaining between the date of sale and the maturity date. Count the first day, but not the last day.

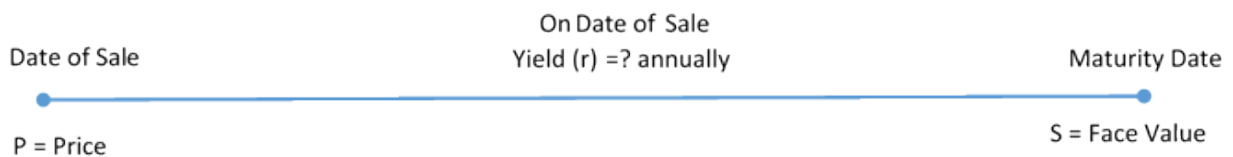


Figure 3.4.2

2. Apply formula $I = S - P$, to calculate the interest earned during the investment.
3. Apply simple interest formula, $I = P \times r \times t$, rearranging for r to solve for the interest rate (or yield or rate of return).

EXAMPLE

A Government of Canada 182-day T-bill has a face value of \$100,000. Market yields on these T-bills are 1.5%. Calculate the price of the T-bill on its issue date.

Solution:

Step 1: The given information is

$$\begin{array}{l} S = \$100,000 \\ r = 1.5\% \text{ (per year)} \\ t = 0.015 \text{ (} \frac{182}{365} \text{)} \end{array}$$

Step 2: Solve for the present value, P .

$$\begin{aligned} P &= \frac{S}{1 + r \times t} \\ &= \frac{100,000}{1 + 0.015 \times \frac{182}{365}} \\ &= \$99,257.61 \end{aligned}$$

An investor will pay \$99,257.61 for the T-bill. If the investor holds onto the T-bill until maturity, the investor realizes a yield of 1.5% and receives \$100,000.

EXAMPLE

Pfizer Inc. issued a 90-day, \$250,000 commercial paper on April 18 when the market rate of return was 3.1%. The paper was sold 49 days later when the market rate of return was 3.63%. Calculate the price of the commercial paper on its date of sale.

Solution:

Note that the historical rate of return of 3.1% is irrelevant to the price of the commercial paper today. The number of days elapsed since the date of issue is also unimportant. The number of days before maturity is the key piece of information.

The original term of the commercial paper was 90 days and 49 days have passed. So, there are 41 (90-49) days remaining until maturity.

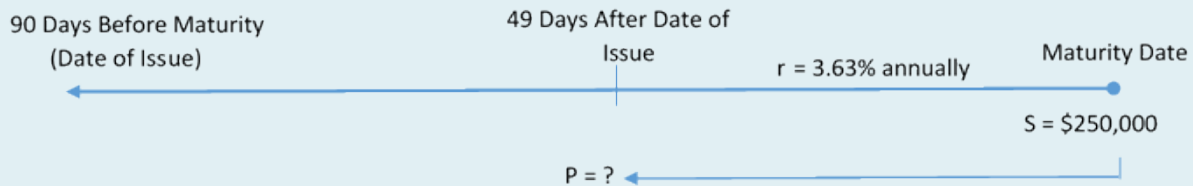


Figure 3.4.3

Step 1: The given information is

$$\begin{array}{l} S = \$250,000 \\ r = 3.63\% \text{ (per year)} \\ t = 0.0363 \times \frac{41}{365} \end{array}$$

Step 2: Solve for the present value, P .

$$\begin{aligned} P &= \frac{S}{1 + r \times t} \\ &= \frac{250,000}{1 + 0.0363 \times \frac{41}{365}} \\ &= \$248,984.76 \end{aligned}$$

An investor pays \$248,984.76 for the commercial paper on the date of sale. If the investor holds onto the commercial paper for 41 more days (until maturity), the investor realizes a yield of 3.63% and receives \$250,000.

TRY IT

A 60-day, \$90,000 face value commercial paper was issued when yields were 2.09%. What was its purchase price?

Click to see Solution

$$\begin{aligned}
 P &= \frac{S}{1 + r \times t} \\
 &= \frac{90,000}{1 + 0.0209 \times \frac{60}{365}} \\
 &= \$89,691.85
 \end{aligned}$$

EXAMPLE

Marlie paid \$489,027.04 on the date of issue for a \$500,000 face value T-bill with a 364-day term. Marlie received \$496,302.21 when he sold it to Josephine 217 days after the date of issue. Josephine held the T-bill until maturity. Determine the following:

1. Marlie's actual rate of return.
2. Josephine's actual rate of return.
3. If Marlie held onto the T-bill for the entire 364 days instead of selling it to Josephine, what would his rate of return have been?

Solution:

Calculate three yields or rates of return (r) involving Marlie and the sale to Josephine, Josephine herself, and Marlie without the sale to Josephine.

Marlie's rate of return for the sale.

Step 1: The given information is

$$P = \$489,027.04$$

$$S = \$496,302.21$$

$$t = \frac{217}{365}$$

Step 2: Calculate the interest, I .

$$\begin{aligned} I &= S - P \\ &= 496,302.21 - 489,027.04 \\ &= \$7,275.17 \end{aligned}$$

Step 3: Solve for the interest rate, r .

$$\begin{aligned} r &= \frac{I}{P \times r \times t} \\ &= \frac{7,275.17}{489,027.04 \times \frac{217}{365}} \\ &= 0.025 \\ &\rightarrow 2.5\% \end{aligned}$$

When Marlie sold the T-bill after holding it for 217 days, he realized a 2.50% rate of return.

Josephine's rate of return.

Step 1: The given information is

$$P = \$496,302.21$$

$$S = \$500,000$$

$$t = \frac{147}{365}$$

Step 2: Calculate the interest, I .

$$\begin{aligned} I &= S - P \\ &= 500,000 - 496,302.21 \\ &= \$3,697.79 \end{aligned}$$

Step 3: Solve for the interest rate, r .

$$\begin{aligned}
 r &= \frac{I}{P \times r \times t} \\
 &= \frac{3,697.79}{496,302.21 \times \frac{147}{365}} \\
 &= 0.0185 \\
 &\rightarrow 1.85\%
 \end{aligned}$$

When Josephine held the T-bill for 148 days, she realized a 1.85% rate of return.

Marlie's rate of return without the sale.

Step 1: The given information is

$$\begin{aligned}
 P &= \$489,027.04 \\
 S &= \$500,000 \\
 t &= \frac{364}{365}
 \end{aligned}$$

Step 2: Calculate the interest, I .

$$\begin{aligned}
 I &= S - P \\
 &= 500,000 - 489,027.04 \\
 &= \$10,972.96
 \end{aligned}$$

Step 3: Solve for the interest rate, r .

$$\begin{aligned}
 r &= \frac{I}{P \times r \times t} \\
 &= \frac{10,972.96}{500,000 \times \frac{364}{365}} \\
 &= 0.0225 \\
 &\rightarrow 2.25\%
 \end{aligned}$$

If Marlie had not sold the note to Josephine and instead held it for the entire 364 days, he would have realized a 2.25% rate of return.

TRY IT

A 90-day Province of Ontario T-bill with a \$35,000 face value matures on December 11, 2022. Farrah works for Hearthplace Industries and notices that the company temporarily has some extra cash available. If she invests the money on October 28, 2022, when the yield is 4.94%, and sells the T-bill on November 25, 2022, when the yield is 4.83%, calculate how much money Farrah earned and the rate of return she realized.

Click to see Solution

Farrah's purchase price on October 28.

$$\begin{aligned} P &= \frac{S}{1 + r \times t} \\ &= \frac{35,000}{1 + 0.0494 \times \frac{44}{365}} \\ &= \$34,792.81 \end{aligned}$$

Farrah's selling price on November 25.

$$\begin{aligned} P &= \frac{S}{1 + r \times t} \\ &= \frac{35,000}{1 + 0.0483 \times \frac{16}{365}} \\ &= \$34,926.05 \end{aligned}$$

Farrah's interest.

$$\begin{aligned} I &= 34,926.05 - 34,792.81 \\ &= \$133.24 \end{aligned}$$

Farrah's rate of return.

$$\begin{aligned}
 r &= \frac{I}{P \times t} \\
 &= \frac{133.24}{34,792.81 \times \frac{28}{365}} \\
 &= 0.0499 \\
 &\rightarrow 4.99\%
 \end{aligned}$$

TRY IT

Philippe purchased a \$100,000 Citicorp Financial 220-day commercial paper for \$96,453.93. He sold it 110 days later to Damien for \$98,414.58, who then held onto the commercial paper until its maturity date.

1. What is Philippe's actual rate of return?
2. What is Damien's actual rate of return?
3. What is the rate of return Philippe would have realized if he had held onto the note instead of selling it to Damien?

Click to see Solution

Philippe's rate of return for the sale.

$$\begin{aligned}
 I &= S - P = 98,414.58 - 96,453.93 = 1,960.65 \\
 r &= \frac{I}{P \times t} = \frac{1,960.65}{96,453.93 \times \frac{110}{365}} = 0.0674 \rightarrow 6.74\%
 \end{aligned}$$

Damien's rate of return.

$$\begin{array}{l} S - P = 100,000 - 98,414.58 = \$1,585.42 \\ \frac{1,585.42}{98,414.58 \times \frac{110}{365}} = 0.0535 \rightarrow 5.35\% \end{array}$$

Philippe's rate of return without the sale.

$$\begin{array}{l} S - P = 100,000 - 96,453.93 = \$3,546.07 \\ \frac{3,546.07}{96,453.93 \times \frac{220}{365}} = 0.061 \rightarrow 6.1\% \end{array}$$

Exercises

1. A 60-day, \$90,000 face value commercial paper was issued when yields were 2.09%. What was its purchase price?

Click to see Answer

\$89,691.85

2. A Government of Canada V39065 issue 90-day T-bill achieved its highest rate of return on May 24, 2000, with a yield of 5.74%. It realized its lowest rate of return on February 26, 2010, with a yield of 0.16%. Calculate the purchase price of a \$100,000 T-bill on each of these dates. In dollars, how much more yield did an investor realize in 2000 than in 2010?

Click to see Answer

\$1356.15

3. A Government of Canada V121780 issue 364-day T-bill achieved its highest rate of return on May 17, 2000, with a yield of 6.31%. It realized its lowest rate of return on May 6, 2009, with a yield of 0.42%. Calculate the purchase price of a \$55,000 T-bill on each of these dates. In dollars, how much more yield did an investor realize in 2000 than in 2009?

Click to see Answer

\$3026.69

4. Pawan is the marketing manager for Cyanamid Canada. His company will execute a marketing program half a year from now that requires a \$500,000 investment. If his finance department had \$485,000 to invest today into a 182-day \$500,000 face value commercial paper yielding 4.55%, would it have enough money to purchase the commercial paper? Show calculations to support your answer.

Click to see Answer

\$488,907.82; cannot purchase for \$485,000; \$3907.82 more is required

5. A 182-day Province of Manitoba T-bill with a face value of \$250,000 was issued 102 days ago, when the yield was 4.3%. What is its purchase price today if the current rate of return is 4.53%?

Click to see Answer

\$247,542.21

6. William purchased a \$110,000 270-day commercial paper on its date of issue, when the yield was 3.89%. He sold it 178 days later when yields had increased to 4.03%. How much money did William earn on his investment?

Click to see Answer

\$1970.63

7. Dollar Thrifty Automotive Group issued a \$1,000,000, 180-day commercial paper. A bank purchased the paper for \$975,560.21 on the issue date. What was the yield for the commercial paper on its issue date?

Click to see Answer

5.08%

8. A 90-day Province of Ontario T-bill with a \$35,000 face value matures on December 11, 2022. Farrah works for Hearthplace Industries and notices that the company temporarily has some extra cash available. If she invests the money on October 28, 2022, when the yield is 4.94%, and sells the T-bill on November 25, 2022, when the yield is 4.83%, calculate how much money Farrah earned and the rate of

return she realized.

Click to see Answer

\$133.24, 4.99%

9. On August 8, 2022, Harriet invested in a \$150,000 Canadian Wheat Board commercial paper on its date of issue with a 220-day term at a yield of 5.98%. On October 15, 2022, she sold the commercial paper to another investor when the current market rate was 5.75%. Calculate the amount of money earned and the rate of return realized.

Click to see Answer

\$1710.69, 6.34%

10. Philippe purchased a \$100,000 Citicorp Financial 220-day commercial paper for \$96,453.93. He sold it 110 days later to Damien for \$98,414.58, who then held onto the commercial paper until its maturity date.
- What is Philippe's actual rate of return?
 - What is Damien's actual rate of return?
 - What is the rate of return Philippe would have realized if he had held onto the note instead of selling it to Damien?

Click to see Answer

a. 6.74%, b. 5.35%, c. 6.10%

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3.5 DEMAND LOANS

LEARNING OBJECTIVES

- Use simple interest in solving problems involving demand loans.

Almost everybody needs a little financial help from time to time. Perhaps you want to purchase a big-ticket item like a big-screen 3D HDTV but do not have the cash in full today to pay for it up front. However, the television is drastically on sale this week and if you wait to purchase it you will lose out and potentially have to pay full price next month. Or maybe as you flip through all of your bills you notice that though you will eventually have enough income to cover them, some of your payments fall due a few days before you will actually receive your income. What will you do for those few days until the income is deposited into your account?

Businesses also find themselves in similar situations. Maybe a supplier is offering a special deal on a product line, but the business does not have the cash to stock up. Also typical of business operations is that they make purchases in advance of sales, so they need to spend the money before they can receive the revenue to pay for their expenses. How can a business get access to short-term financing?

Demand Loans

A **demand loan** is a short-term loan that generally has no specific maturity date, where the borrower can make a payment to settle the loan, either in part or full, at any time without any interest penalty, and where the lender can demand repayment in full at any time. A demand loan allows borrowing when needed and repayment when money permits, subject to the following characteristics.

1. **Credit Limit.** This establishes the maximum amount that can be borrowed.
2. **Variable Interest Rate.** Almost all demand loans use variable simple interest rates based on the prime

rate. Only the best, most secure customers can receive prime, while others usually get “prime plus” some additional amount.

3. **Fixed Interest Payment Date.** Interest is always payable on the same date each and every month. For simplicity, the payment is usually tied to a chequing or savings account, allowing the interest payment to occur automatically.
4. **Interest Calculation Procedure.** Interest is always calculated using a simple interest procedure based on the daily closing balance in the account. This means the first day but not the last day is counted.
5. **Security.** Loans can be secured or unsecured. **Secured loans** are those loans that are guaranteed by an asset such as a building or a vehicle. In the event that the loan defaults, the asset can be seized by the lender to pay the debt. **Unsecured loans** are those loans backed up by the general goodwill and nature of the borrower. Usually a good credit history or working relationship is needed for these types of loans. A secured loan typically enables access to a higher credit limit than an unsecured loan.
6. **Repayment Structure.** The repayment of the loan is either variable or fixed.
 - A variable repayment structure allows the borrower to repay any amount at any time, although a minimum requirement may have to be met such as “at least 2% of the current balance each month.” A **current balance** is the balance in an account plus any accrued interest. **Accrued interest** is any interest amount that has been calculated but not yet placed (charged or earned) into an account.
 - A fixed repayment structure requires a fixed payment amount toward the current balance on the same date each and every month.

Some examples of demand loans include:

1. **Personal Line of Credit (LOC).** A demand loan for individuals, a personal line of credit is generally unsecured and is granted to those individuals who have high credit ratings and an established relationship with a financial institution. Because a line of credit is unsecured, the credit limit is usually a small amount, such as \$10,000. Repayment is variable and usually has a minimum monthly requirement based on the current balance.
2. **Home Equity Line of Credit (HELOC).** This is a special type of line of credit for individuals that is secured by residential homeownership. Typically, an amount not exceeding 80% of the equity in a home is used to establish the credit limit, thus enabling an individual access to a large amount of money. The interest rates tend to follow mortgage interest rates and are lower than personal lines of credit. Repayment is variable, usually involving only the accrued interest every month.
3. **Operating Loans.** An operating loan is the business version of a line of credit. An operating loan may or may not be secured, depending on the nature of the business and the strength of the relationship the business has with the financial institution. Repayment can be either variable or fixed.

4. **Student Loans.** A loan available to students to pursue educational opportunities. Although these are long-term in nature, the calculation of interest on a student loan uses simple interest techniques. These loans are not true demand loans because a student loan cannot be called in at any time. Repayment is fixed monthly.

EXAMPLE

Jesse received a \$8500 demand loan from her bank at 4% simple interest. Three months later, the interest rate on the loan increased to 4.3% simple interest. Jesse repaid the loan in full eight months after taking out the loan. What is the total amount of interest that Jesse paid on the loan?

Solution:

Step 1: Calculate the amount of interest, I , for the first three months when the interest rate is 4%.

$$\begin{aligned} I &= P \times r \times t \\ &= 8500 \times 0.04 \times \frac{3}{12} \\ &= \$85 \end{aligned}$$

Step 2: Calculate the amount of interest, I , for the last five months when the interest rate is 4.3%.

$$\begin{aligned} I &= P \times r \times t \\ &= 8500 \times 0.043 \times \frac{5}{12} \\ &= \$152.29 \end{aligned}$$

Step 3: Find the total interest.

$$\begin{aligned} \text{Total Interest} &= \$85 + \$152.29 \\ &= \$237.29 \end{aligned}$$

The total amount of interest that Jesse paid on the loan was \$237.29.

TRY IT

On March 17, 2023, Mark took out a \$10,000 demand loan at 3.1% simple interest. On May 29, 2023, the interest rate changed to 2.3% simple interest and on July 31, 2023, the interest rate changed to 2.7%. Mark repaid the loan on September 3, 2023. How much interest did Mark pay on the loan?

Click to see Solution

Interest from March 17 to May 29.

$$\begin{aligned} I &= P \times r \times t \\ &= 10,000 \times 0.031 \times \frac{73}{365} \\ &= \$62 \end{aligned}$$

Interest from May 29 to July 31.

$$\begin{aligned} I &= P \times r \times t \\ &= 10,000 \times 0.023 \times \frac{63}{365} \\ &= \$39.70 \end{aligned}$$

Interest from July 31 to September 3.

$$\begin{aligned} I &= P \times r \times t \\ &= 10,000 \times 0.027 \times \frac{34}{365} \\ &= \$25.15 \end{aligned}$$

Total interest.

$$\begin{aligned} \text{Total Interest} &= \$62 + \$39.70 + \$25.15 \\ &= \$126.85 \end{aligned}$$

Partial Payments of Demand Loans

The borrower can make partial payments on a demand loan at any time, without penalty, to reduce the outstanding balance on the loan. When a partial payment is made, the payment is first used to reduce the interest on the loan. If the interest is completely paid off by the payment, then the remainder of the payment is applied to reduce the principal on the loan. This approach is called the **declining balance method**.

NOTE

A partial payment may be more or less than the interest that is due on the loan at the time the payment is made. Each time a partial payment is made, the interest due at the time of the payment is calculated.

- If the partial payment is larger than the amount of interest due, then the interest is paid first and any remaining amount from the payment is used to reduce the principal.
- If the partial payment is smaller than the amount of interest due, then the entire payment is applied to interest due. Any interest that is not paid-off by the payment is carried forward to the next payment. Because there is nothing leftover from the payment, nothing is applied to the principal.

EXAMPLE

Sarah received a demand loan of \$20,000 on April 2, 2023 at 7% simple interest. She made a partial payment of \$100 on May 19, 2023 and another payment of \$3,000 on June 17, 2023. She paid-off the loan on August 5, 2023. Calculate the final payment Sarah made on August 5, 2023 to clear the loan.

Solution:

Step 1: Calculate the amount of interest, I , due at the time of first payment on May 19, 2023.

$$\begin{aligned}
 I &= P \times r \times t \\
 &= 20,000 \times 0.07 \times \frac{47}{365} \\
 &= \$180.27
 \end{aligned}$$

Step 2: Apply the payment on May 19, 2023 to the interest due. Because the payment is smaller than the interest due, there is an interest balance that must be carried forward to the next payment.

$$\begin{aligned}
 \text{Interest Balance} &= 180.27 - 100 \\
 &= \$80.27
 \end{aligned}$$

Step 3: Calculate the amount of interest, I , due at the time of the second payment on June 17, 2023.

$$\begin{aligned}
 I &= P \times r \times t \\
 &= 20,000 \times 0.07 \times \frac{29}{365} \\
 &= \$111.23
 \end{aligned}$$

Step 4: Apply the payment on June 17, 2023 to the interest due, including the unpaid interest from the previous payment. Calculate the leftover amount from the payment after paying the interest.

$$\begin{aligned}
 \text{Leftover Amount from Payment} &= \$3,000 - (80.27 + 111.23) \\
 &= \$2,808.50
 \end{aligned}$$

Step 5: Apply the leftover amount from the June 17, 2023 payment to the loan principal.

$$\begin{aligned}
 \text{Principal Remaining} &= \$20,000 - 2,808.50 \\
 &= \$17,191.50
 \end{aligned}$$

Step 6: Calculate the amount of interest, I , due on August 5, 2023. Note that interest is only charged on the remaining principal from the last payment.

$$\begin{aligned}
 I &= P \times r \times t \\
 &= 17,191.50 \times 0.07 \times \frac{49}{365} \\
 &= \$161.55
 \end{aligned}$$

Step 7: To clear the loan on August 5, 2023, the remaining principal and the interest due must both be paid.

$$\begin{aligned}
 \text{Final Payment} &= \$17,191.50 + 161.55 \\
 &= \$17,353.05
 \end{aligned}$$

To clear the loan on August 5, 2023, Sarah must make a final payment of \$17,353.05.

TRY IT

Tom took out a \$8,000 demand loan at 5%. Three months after he took out the loan Tom repaid \$1,000 towards the loan and eight months after he took out the loan Tom repaid \$3,500 towards the loan. Tom cleared the loan fourteen months after he took out the loan. Calculate the size of Tom's final payment.

Click to see Solution

Balance after the \$1,000 payment in 3 months.

$$\begin{aligned} I &= P \times r \times t \\ &= 8,000 \times 0.05 \times \frac{3}{12} \\ &= \$100 \end{aligned}$$

$$\begin{aligned} \text{Leftover Amount from Payment} &= 1,000 - 100 \\ &= \$900 \end{aligned}$$

$$\begin{aligned} \text{Principal Remaining} &= 8,000 - 900 \\ &= \$7,100 \end{aligned}$$

Balance after the \$3,500 payment in 8 months.

$$\begin{aligned} I &= P \times r \times t \\ &= 7,100 \times 0.05 \times \frac{5}{12} \\ &= \$147.92 \end{aligned}$$

$$\begin{aligned} \text{Leftover Amount from Payment} &= 3,500 - 147.92 \\ &= \$3,352.08 \end{aligned}$$

$$\begin{aligned} \text{Principal Remaining} &= 7,100 - 3,352.08 \\ &= \$3,747.92 \end{aligned}$$

Final payment in 14 months.

$$\begin{array}{l} \text{Interest} = P \times r \times t = 3,747.92 \times 0.05 \times \frac{6}{12} = \$93.70 \\ \text{Final Payment} = 3,747.92 + 93.70 = \$3,841.62 \end{array}$$

Repayment Schedules

A **repayment schedule** is a table that details the financial transactions in an account including the balance, interest amounts, and payments/advances. The table below presents the table structure used for setting up a repayment schedule. Each number in the table corresponds to an entry below that explains how to use each column or row.

	Balance Before Transaction (P)	Annual Interest Rate (r)	Number of Days (t)	Interest Charged (I)	Accrued Interest	Payment or Advance	Principal Amount	Balance After Transaction
Start Date								(1)
(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)

1. The first row appears if the schedule has an opening balance. In these instances, list the start date in the first column and the opening balance in the last column.
2. List the date of any transaction. This could include a payment, advance, interest rate change, or an accrued interest payment.
3. Carry this number forward from (1) (the balance after the transaction from the previous row).
4. Record the interest rate that applies to the date interval in the first column.
5. Calculate the number of days between the date on the previous row and the current row. Use the financial calculator's date function to count the number of days. Express the number annually to match the interest rate (i.e. divide the number of days by 365).
6. Compute the interest charges for the date interval using $I = P \times r \times t$. Use (3), (4), and (5) as your values for the formula.
7. Enter the cumulative total of any unpaid or accrued interest as of the current row's date. This amount is the sum of (6) from the current row plus any number recorded in this column from the row above.
8. Enter the amount of the transaction occurring for this row of the table. Payments should be recorded as positives (credits) as they will decrease the balance, and advances should be recorded as negatives (debits) as they will increase the balance. If this is an interest payment date, one of the following two events will

happen. Regardless of which event happens, cross out the accrued interest amount in (7) as it is considered paid, so the accrued interest balance is reduced to zero once again.

- a. Only the interest payment occurs on this date. Copy the accrued interest from (7) into this column.
 - b. An additional payment or advance is made on this date. Add the accrued interest to the advance or payment, placing the sum into this column.
9. One of two events will happen in this column.
- a. On an interest payment date, subtract the accrued interest from the amount in (8).
 - b. On a date other than an interest payment date, any payment or advance will have its total amount applied to the principal. Therefore, carry the amount in (8) across to this column.
10. Subtract the amount in (9) from the amount in this column on the previous row to create the new balance in the account. Copy this amount to the next row in (3).

Calculating a Repayment Schedule

Although the calculations of a repayment schedule are relatively straightforward, the complexity of the repayment schedule sometimes causes grief. When a repayment schedule is required, follow these steps.

1. Set up the repayment schedule as per the example table.
2. Record the start date and the opening balance for the loan.
3. In chronological order, make new row entries in the schedule by filling in the details provided in the question. You require a new row in the table whenever one of the following three events occurs.
 - a. A payment or advance is made.
 - b. An interest payment date occurs.
 - c. The interest rate changes.
4. Starting with the first row, work left to right across the table, filling in all information. Pay particular attention to the nuances of the “Payment or Advance” and “Principal Amount” columns as discussed previously. Once a row is complete, move to the next row until you fill in the entire table.
5. Calculate any totals requested such as total interest or total principal paid.

NOTES

1. For simplicity in writing the numbers into repayment schedules and performing

calculations, it is this textbook's practice to round all interest calculations to two decimals throughout the table. In real-world applications, you must keep track of all decimals in the account at all times.

2. Because of the size of the repayment schedule and the large amount of information involved in the calculations, the number-one error is what most people call a “silly” error. It means that a wrong date is recorded, a wrong amount is written down, a payment is recorded as an advance or vice versa, or simply the wrong button is pressed on a calculator. Take the time to ensure you read and record the correct numbers and that you pause when performing calculations. For example, advances mean the balance should get bigger, while payments mean the balance should get smaller. Just by thinking for a second about the basic principles of debt you should be able to catch those silly errors.

EXAMPLE

On July 15, 2023, Canadian Footwear took out an operating loan from CIBC for \$8,000 at a rate of 5.25% simple interest. The terms of the loan require a fixed payment of \$1,500 on the 15th of every month until the loan is repaid. On September 29, 2023, the interest rate on the loan increased to 5.75%. Create a repayment schedule for the loan and calculate the total interest paid.

Solution:

Step 1: The given information is

$$\begin{array}{l} \text{Opening balance on July 15} \quad \& = \quad \& \$8,000 \\ \text{Interest rate on July 15} \quad \& = \quad \& 5.25\% \\ \text{Interest rate on September 29} \quad \& = \quad \& 5.75\% \\ \text{Payment on 15th of each month} \quad \& = \quad \& \$1,500 \end{array}$$

Step 2: Set-up the repayment schedule.

Date	Balance Before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment or Advance	Principal Amount	Balance After Transaction
July 15								\$8,000
Aug 15	\$8,000	5.25%	31/365 ¹	\$35.67 ²	~~\$35.67~~	\$1,500	\$1,464.33 ³	\$6,535.67 ⁴
Sept 15	\$6,535.67	5.25%	31/365	\$29.14	~~\$29.14~~	\$1,500	\$1,470.86	\$5,064.81
Sept 29	\$5,064.81	5.75%	14/365	\$10.20	\$10.20	\$0	\$0	\$5,064.81
Oct 15	\$5,064.81	5.75%	16/365	\$12.77	~~\$22.97~~ ⁵	\$1,500	\$1,477.03	\$3,587.88
Nov 15	\$3,587.88	5.75%	31/365	\$17.52	~~\$17.52~~	\$1,500	\$1,482.48	\$2,105.30
Dec 15	\$2,105.30	5.75%	30/365	\$9.95	~~\$9.95~~	\$1,500	\$1,490.05	\$615.25
Jan 15	\$615.25	5.75%	31/365	\$3.00	~~\$3.00~~	\$618.25 ⁶	\$615.25 ⁷	\$0

Selection calculations from the table are provided below.

1. This is the number of days from the date on the previous row to the current row, or July 15 to August 15. Count the first day, but not the last.
2. $I = P \times r \times t = 8000 \times 0.0525 \times \frac{31}{365} = \$35.67$
3. $1,500.00 - 35.67 = \$1,464.33$. The accrued interest of \$35.67 is now paid and can be crossed out.
4. $8,000.00 - 1,464.33 = \$6,535.67$
5. $10.20 + 12.77 = \$22.97$
6. The last payment must clear the balance owing and the accrued interest:
 $615.25 + 3.00 = \$618.25$.
7. $618.25 - 3.00 = \$615.25$

Step 3: Calculate the total interest paid by adding up the Interest Charged column.

$$\begin{aligned} \text{Total Interest} &= 35.67 + 29.14 + 10.20 + 12.77 + 17.52 + 9.95 + 3 \\ &= \$118.25 \end{aligned}$$

EXAMPLE

Lynne has access to a HELOC that requires only the payment of accrued interest on the first of every month. On March 1, 2023, the opening balance on her HELOC was \$15,000. She took advances of \$6,000 and \$10,000 on March 21, 2023 and May 4, 2023, respectively. She made additional payments of \$11,000 and \$15,000 on April 15, 2023 and June 17, 2023 respectively. On March 1, 2023, the interest rate on her HELOC was 5%. On April 26, 2023, the interest rate increased to 5.5%. Construct the repayment schedule from March 1, 2023 to July 1, 2023 and determine the total interest paid from March 1, 2023 to July 1, 2023.

Solution:

Step 1: The given information is

$$\begin{array}{l} \text{Opening balance on March 1} \quad = \quad \$15,000 \\ \text{Interest rate on March 1} \quad = \quad 5\% \\ \text{Interest rate on April 26} \quad = \quad 5.5\% \\ \text{Advance on March 21} \quad = \quad \$6,000 \\ \text{Advance on May 4} \quad = \quad \$10,000 \\ \text{Payment on April 15} \quad = \quad \$11,000 \\ \text{Payment on June 17} \quad = \quad \$15,000 \end{array}$$

Step 2: Set-up the repayment schedule.

Date	Balance Before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment or Advance	Principal Amount	Balance After Transaction
Mar 1								\$15,000
Mar 21	\$15,000 ¹	5%	20/365 ²	\$41.10 ³	\$41.10 ⁴	-\$6,000	-\$6,000 ⁵	\$21,000 ⁶
Apr 1	\$21,000	5%	11/365	\$31.64	~~\$72.74~~ ⁷	\$72.74 ⁸	\$0 ⁹	\$21,000
Apr 15	\$21,000	5%	14/365	\$40.27	\$40.27	\$11,000	\$11,000	\$10,000
Apr 26	\$10,000	5%	11/365	\$15.07	\$55.34	\$0	\$0	\$10,000
May 1	\$10,000	5.5%	5/365	\$7.53	~~\$62.87~~	\$62.87	\$0	\$10,000
May 4	\$10,000	5.5%	3/365	\$4.52	\$4.52	-\$10,000	-\$10,000	\$20,000
June 1	\$20,000	5.5%	28/365	\$84.38	~~\$88.90~~	\$88.90	\$0	\$20,000
June 17	\$20,000	5.5%	16/365	\$48.22	\$48.22	\$15,000	\$15,000	\$5,000
July 1	\$5,000	5.5%	14/365	\$10.55	~~\$58.77~~	\$58.77	\$0	\$5,000

Selection calculations from the table are provided below.

1. This is the balance from the last column in the row above carried forward.
2. The date interval from the previous row to this row is March 1 to March 21: 20 days.
3. $I = P \times r \times t = 15,000 \times 0.05 \times \frac{20}{365} = \$41.10$.
4. The accrued interest from the row above plus (3): $0 + 41.10 = \$41.10$.
5. Not an interest payment date, so the full payment is applied to principal: $-\$6,000$
6. Previous balance in the column above minus the principal portion from (5):
 $15,000 - (-6,000) = \$21,000$.
7. The accrued interest from the row above plus interest from this row:
 $41.10 + 31.64 = \$72.74$.
8. This is an interest payment date. Carry (7) across and cross it out, reducing the accrued interest balance to zero.
9. This is an interest payment date. Take (8) and subtract (7): $72.74 - 72.74 = \$0$

Step 3: Calculate the total interest paid by adding up the Interest Charged column:

$$\begin{array}{l} \text{Total Interest} = 41.10 + 31.64 + 40.27 + 15.07 + 7.53 \\ \quad + 4.52 + 84.38 + 48.22 + 10.55 = \$283.28 \end{array}$$

Exercises

1. Lacy has took out \$40,000 demand loan at 5.25%. Five months after she took out the loan, the interest rate increased to 6% and twelve months after she took out the loan, the interest rate decreased to 5.7%. Lacy repaid the loan fifteen months after she took out the loan. How much interest did Lacy pay on the loan?

Click to see Answer

\$2845

2. Sam received a \$15,000 demand loan at 4.3% on March 20, 2023. On June 12, 2023 the interest rate changed to 4.7% and on August 19, 2023 the interest rate changed to 3.9%. Sam repaid the loan on September 7, 2023. How much interest did Sam pay on the loan? How much did Sam pay on September 7 to clear the loan?

Click to see Answer

\$310.23

3. A local business took out a \$17,500 operating loan on February 15, 2023 at a rate of 3.75%. The business made a \$3,000 payment on March 29, 2023 and a \$4,500 payment on May 4, 2023. The business repaid the loan on June 28, 2023. What is the size of the business's final payment on June 28?

Click to see Answer

\$10,186.66

4. Mohammad received a operating loan \$25,000 operating loan at 4.9%. Mohammad made payments of \$4,000, \$300 and \$2,000 at six months, ten months and fifteen months after he received the loan. Mohammad paid off the loan in eighteen months. What is the size of Mohammad's final payment?

Click to see Answer

\$20,353.07

5. A \$7,500 demand loan was taken out on March 4, 2023 at a fixed interest rate of 7.72% with fixed monthly payments of \$1,200. The first monthly repayment is due April 4, 2023 and the 4th of every month thereafter. Prepare a full repayment schedule for the loan.

Click to see Answer

Date	Balance Before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment or Advance	Principal Amount	Balance After Transaction
Mar 5								\$7,500
Apr 4	\$7,500	7.72%	31/365	\$49.18	~~\$49.18~~	\$1,200	\$1,150.82	\$6,349.18
May 4	\$6,349.18	7.72%	30/365	\$40.29	~~\$40.29~~	\$1,200	\$1,159.71	\$5,189.47
June 4	\$5,189.47	7.72%	31/365	\$34.03	~~\$34.03~~	\$1,200	\$1,165.97	\$4,023.50
July 4	\$4,023.50	7.72%	30/365	\$25.53	~~\$25.53~~	\$1,200	\$1,174.47	\$2,849.03
Aug 4	\$2,849.03	7.72%	31/365	\$18.68	~~\$18.68~~	\$1,200	\$1,181.32	\$1,667.71
Sept 4	\$1,667.71	7.72%	31/365	\$10.93	~~\$10.93~~	\$1,200	\$1,189.07	\$478.64
Oct 4	\$478.64	7.72%	30/365	\$3.04	~~\$3.04~~	\$478.64	\$478.64	\$0

6. Vertical Adventures has an open line of credit with a zero balance at its credit union using a fixed interest rate of 7.35%. On the last day of every month, the accrued interest must be paid. On July 8 and August 14, the company made advances of \$15,000 and \$12,000, respectively. On July 30, it made a payment of \$10,000. Vertical Adventures will restore its zero balance on August 31. Construct a full repayment schedule from July 8 to August 31.

Click to see Answer

Date	Balance Before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment or Advance	Principal Amount	Balance After Transaction
July 8								\$15,000
July 30	\$15,000	7.35%	22/365	\$66.45	~~\$66.45~~	\$10,000	\$9,933.55	\$5,066.45
Aug 14	\$5,066.45	7.35%	15/365	\$15.30	\$15.30	-\$12,000	-\$12,000	\$17,066.45
Aug 31	\$17,066.45	7.35%	17/365	\$58.42	~~\$73.72~~	\$17,140.17	\$17,066.45	\$0

7. Scotiabank approved a \$250,000 line of credit for Buhler Industries at 7.5%. It requires only the repayment of accrued interest on the 27th of each month, which is automatically deducted from the

chequing account of Buhler Industries. Buhler took out an advance on December 2, 2022 for \$200,000. Buhler made payments of \$1,000 on December 27, 2022, \$125,000 on January 12, 2023 and \$5,000 on January 31, 2023. On January 4, 2023, the interest rate changed to 8.5%. Construct a full repayment schedule from December 2, 2022 to January 31, 2023. Calculate the total interest charged to Buhler Industries from December 2, 2022 to January 31, 2023.

Click to see Answer

Date	Balance Before Transaction	Annual Interest Rate	Number of Days	Interest Charged	Accrued Interest	Payment or Advance	Principal Amount	Balance After Transaction
Dec 2								\$200,000
Dec 27	\$200,000	7.5%	25/365	\$1,027.40	\$27.40	\$1,000	\$0	\$200,000
Jan 4	\$200,000	7.5%	8/365	\$328.77	\$356.17	\$0	\$0	\$200,000
Jan 12	\$200,000	8.5%	8/365	\$372.60	~~\$728.88~~	\$125,000	\$124,271.23	\$75,728.77
Jan 31	\$75,728.77	8.5%	19/365	\$335.07	~~\$335.07~~	\$5,000	\$4,664.93	\$71,063.84

Total Interest=\$2063.84

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3.6 REVIEW EXERCISES

1. If \$4,000 is borrowed from April 3 to June 22 at a simple interest rate of 3.8%, how much interest is paid on the loan?

Click to see Answer

\$33.32

2. A savings account pays flat-rate interest of 1.45%. If a balance of \$3,285.40 is maintained for the entire month of August, how much interest does the savings account earn?

Click to see Answer

\$4.05

3. A 182-day \$1,000,000 Government of British Columbia T-bill was issued when the market rate of return was 4.21%. Calculate the purchase price of the T-bill on its issue date.

Click to see Answer

\$979,439.29

4. If you place \$8,000 into a 300-day short-term GIC at Scotiabank earning 0.95% simple interest, how much will you receive when the investment matures?

Click to see Answer

\$8,052.47

5. Proper accounting procedures require accountants to separate principal and interest components on any loan. Allocate the principal and interest portions of a \$24,159.18 payment clearing a 147-day loan at 8.88%.

Click to see Answer

\$23,325, \$834.18

6. As part of your financial plan for retirement, you purchased a 270-day \$25,000 commercial paper on its date of issue, July 14, when market yields were 2.94%. 234 days later, you sold the note when market yields were 2.76%. What rate of return did you realize on your investment?

Click to see Answer

2.96%

7. Sturm put \$48,700 into a 10-month term deposit, but needed to withdraw the funds after five months to deal with a family emergency. The credit union penalized him 2.35% off of his interest rate for the early withdrawal and deposited \$49,602.98 into his account.

- a. What was Sturm's original interest rate?

Click to see Answer

6.8%

- b. How much interest, in dollars, was he penalized for the early withdrawal?

Click to see Answer

\$476.85

8. Three hundred days from now you will be departing on a backpacking trip through Europe. You need \$4,000 in spending money to take with you. Today, you currently have saved \$3,960.

- a. If you place your money in consecutive 100-day short-term GICs earning 1.02%, will you meet your goal? Assume the full maturity values are reinvested into the next GIC.

Click to see Answer

\$6.70 short of meeting the goal

- b. What would the interest rate need to be if you had placed your money into a single 300-day short-term GIC to reach your goal?

Click to see Answer

1.229%

9. A 364-day, \$50,000 face value T-bill is issued when market yields are 2.85%. The T-bill is sold to another investor every 91 days until maturity, with yields of 3.1%, 2.98%, and 3.15% on each of the dates of sale, respectively. Compute the purchase price for each investor, including the date of issue. For each investor, calculate the actual rate of return realized on their investment.

Click to see Answer

91 days after issue: $P = \$48,866.96$; 182 days after issue: $P = \$49,267.92$; 273 days after issue: $P = \$49,610.39$; Investor #1: Date of issue to 91 days: $r = 2.05\%$; Investor #2: 91 days to 182 days: $r = 3.29\%$; Investor #3: 182 days to 273 days: $r = 2.79\%$; Investor #4: 273 days to maturity: $r = 3.15\%$

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3.7 CHAPTER SUMMARY

Key Concepts

- **3.1 Simple Interest: Principal, Rate, Time**
 - Calculating the amount of simple interest either earned or charged in a simple interest environment.
 - Calculating the time period when specific dates or numbers of days are involved.
 - Calculating the simple interest amount when the interest rate is variable throughout the transaction.
 - **3.2 Simple Interest: Maturity Value**
 - Putting the principal and interest together into a single calculation known as maturity value.
 - Altering a financial agreement and establishing equivalent payments.
 - **3.3 Savings Accounts and Short-Term Guaranteed Investment Certificates**
 - How to calculate simple interest for flat-rate and tiered savings accounts.
 - How to calculate simple interest on a short-term GIC.
 - **3.4 Treasury Bills and Commercial Papers**
 - The characteristics of treasury bills.
 - The characteristics of commercial papers.
 - Calculating the price of T-Bills and commercial papers.
 - Calculating the yield of T-Bills and commercial papers.
 - **3.5 Demand Loans**
 - Demand loans, their characteristics, and the common forms they take.
 - Establishing a repayment schedule for demand loans.
 - The characteristics of a student loan and how it is repaid.
-

Glossary of Terms

- **Accrued Interest.** Any interest amount that has been calculated but not yet placed (charged or earned) into an account.
- **Commercial Paper.** A short-term financial instrument with maturity no longer than one year that is issued by large corporations.

- **Compound Interest.** A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more. Interest is periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.
- **Current Balance.** The balance in an account plus any accrued interest.
- **Demand Loan.** A short-term loan that generally has no specific maturity date, may be paid at any time without any interest penalty, and where the lender may demand repayment at any time.
- **Discount Rate.** An interest rate used to remove interest from a future value.
- **Equivalent Payments.** Two payments that have the same value on the same day factoring in a fair interest rate.
- **Face Value of a T-bill.** The maturity value of a T-bill, which is payable at the end of the term. It includes both the principal and interest together.
- **Fixed Interest Rate.** An interest rate that is unchanged for the duration of the transaction.
- **Focal Date.** A single date that is chosen to locate all values in a financial scenario so that equivalent amounts can be determined.
- **Future Value.** The amount of principal with interest at a future point of time for a financial transaction. If this future point is the same as the end date of the financial transaction, it is also called the maturity value.
- **Guaranteed Investment Certificate (GIC).** An investment that offers a guaranteed rate of interest over a fixed period of time.
- **Interest Amount.** The dollar amount of interest that is paid or earned.
- **Interest Rate.** The rate of interest that is charged or earned during a specified time period.
- **Maturity Date.** The date upon which a transaction, such as a promissory note, comes to an end and needs to be repaid.
- **Maturity Value.** The amount of money at the end of a transaction, which includes both the interest and the principal together.
- **Present Value.** The amount of money at the beginning of a time period in a transaction. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some time earlier before the future value was known. In any case, the amount excludes the interest.
- **Prime Rate.** An interest rate set by the Bank of Canada that usually forms the lowest lending rate for the most secure loans.
- **Principal.** The original amount of money that is borrowed or invested in a financial transaction.
- **Proceeds.** The amount of money received from a sale.
- **Repayment Schedule.** A table that details the financial transactions in an account, including the balance, interest amounts, and payments.
- **Savings Account.** A deposit account that bears interest and has no stated maturity date.
- **Secured Loan.** Those loans that are guaranteed by an asset such as a building or a vehicle that can be

seized to pay the debt in case of default.

- **Simple Interest.** A system for calculating interest that primarily applies to short-term financial transactions with a time frame of less than one year.
 - **Time Period.** The length of the financial transaction for which interest is charged or earned. It may also be called the term.
 - **Treasury Bills.** Short-term financial instruments with maturities no longer than one year that are issued by both federal and provincial governments.
 - **Unsecured Loan.** Those loans backed up by the general goodwill and nature of the borrower.
 - **Variable Interest Rate.** An interest rate that is open to fluctuations over the duration of a transaction.
 - **Yield.** The percentage increase between the sale price and redemption price on an investment such as a T-bill or commercial paper.
-

Formulas

• Symbols Used

- S = Maturity value or future value in dollars
- I = Interest amount in dollars
- P = Principal or present value in dollars
- r = Interest rate (in decimal format)
- t = Time or term

• Formulas Used

- Simple Interest: $I = P \times r \times t$
 - Maturity Value: $S = P \times (1 + r \times t)$
 - Present Value: $P = \frac{S}{1 + r \times t}$
-

Calculator

- Date Function
 - 2nd DATE to access.
 - Enter two of the three variables (DT1, DT2, DBD) by pressing ENTER after each input and using the up arrow and the down arrow to scroll through the display. The variables are:
 - DT1 = The starting date of the transaction
 - DT2 = The ending date of the transaction
 - DBD = The days between the dates, counting the first day but not the last, which is the time period of the transaction.
 - ACT / 360 = A setting for determining how the calculator determines the DBD. In Canada, you should maintain this setting on ACT, which is the actual number of days. In other countries, such as the United States, they treat each year as having 360 days (the 360 setting) and each month as having 30 days. If you need to toggle this setting, press 2nd SET.
 - Enter all dates in the format of MM.DDYY, where MM is the numerical month, DD is the day, and YY is the last two digits of the year. DD and YY must always be entered with both digits.
 - Press CPT on the unknown (when it is on the screen display) to compute the answer.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=47#oembed-1>

[Calculating Dates and Days \(Month-Day-Year\)](#) by Joshua Emmanuel [3:32] (transcript available).

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PART IV

COMPOUND INTEREST

Do you dream of owning a home? What about a car or a home theatre system? Big purchases such as these require long-term financial planning. Compound interest means that you will pay substantially more money for your purchases or earn more money on investments than you would with simple interest.

In the previous chapter, you learned about simple interest. In simple interest, all interest is based solely on the original principal amount of the transaction, and the interest is converted to principal at the end of the transaction. But in compound interest, the interest also earns interest. **Compound interest** involves interest being periodically converted to principal throughout the transaction, with the result that the interest itself also accumulates interest. Compound interest is used for most transactions lasting at least one year.

But how significant is the interest when it compounds? Consider the following scenarios:

- A home theatre system with the big-screen TV, Blu-Ray player, stereo surround sound, and more can retail for \$5,000. Did you know that if you put that amount on the retail store's 18% interest credit card and pay it off monthly for three years, you will pay over \$1,500 of compound interest?
- You want to purchase a new car that retails for \$35,000. Unless you have that amount of cash on hand, you will join the ranks of other Canadians who take out car loans to purchase their new car. If you take six years to pay-off this loan at an interest rate of 4.5% compounded monthly, you would have to make monthly payments of \$555.59 and pay over \$5,000 in interest on the loan.
- To purchase a house that is listed for \$600,000, you take out a 25 year mortgage at 6% compounded semi-annually. Over the course of the 25 years, you will pay over \$550,000 in interest on the mortgage, which almost doubles the original cost of the house.

And it is not any different for businesses. Whether they are local companies or multinational conglomerates they must invest and borrow at compound interest rates in their attempt to achieve long-term financial strategies. Some examples of these business activities include the following:

- Borrowing \$480,000 to open a new restaurant.
- Spending \$1,000,000 on a fleet of rigs and semi-trailers for product distribution.
- Constructing new production plants or warehouses costing \$10,000,000 or more.

The money for these types of transactions does not appear out of thin air. It must be borrowed or taken from

savings, and either approach involves compound interest. Throughout the rest of this textbook, you will study compound interest as it relates to three distinct but interconnected concepts:

- Calculating interest on a single amount (called a lump-sum amount or single payment).
 - Calculating interest on a series of regular, equal payments, called annuities.
 - Specialized applications including amortization, mortgages, bonds, sinking funds, net present value, and internal rates of return.
-

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4.1 COMPOUND INTEREST TERMINOLOGY

LEARNING OBJECTIVES

- Identify the differences between compound interest and simple interest.
- Define terms related to compound interest.

Compound interest involves interest being periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest. This means that interest earned in the previous compounding period will earn interest in all subsequent compounding periods.

But how does compound interest compare to simple interest? The critical difference is the placement of interest into the account. Under simple interest, you convert the interest to principal at the end of the transaction's time frame. For example, in a six-month simple interest GIC the balance in its account at any point before the maturity date is the original principal and nothing more. Only upon maturity does the interest appear. In contrast, a five-year compound interest GIC receives an interest deposit into the account at periodic times every year. Once this interest is deposited into the account that interest will begin to earn interest for the remainder of the term of the GIC.

Suppose you invested \$1,000 into two different savings accounts: one account earns 10% simple interest and the other account earns 10% compounded annually. How much is in each account at the end of five years? Which account earns more interest?

From the previous chapter on simple interest, the simple interest savings account will earn \$500 in interest and have a maturity value of \$1,500 at the end of five years. Because this is simple interest, only the original \$1,000 principal earns interest over the five years. The table below shows the balance in the account at the end of each year.

Year	Principal at Start of Year	Accrued Interest	Balance at End of Year
1	\$1,000	\$100	\$1,000
2	\$1,000	\$200	\$1,000
3	\$1,000	\$300	\$1,000
4	\$1,000	\$400	\$1,000
5	\$1,000	\$500	\$1,000

For the compound interest savings account, the 10% interest is compounded annually, which means at the end of every year 10% of the current principal is added to the account. This balance at the end of each year becomes the principal for the next year and is the amount of money that will earn interest in the next year. The table below shows the balance in the account at the end of each year when the interest is calculated and added to the principal.

Year	Principal at Start of Year	Accrued Interest	Balance at End of Year
1	\$1,000	\$100	\$1,100
2	\$1,100	\$110	\$1,210
3	\$1,210	\$121	\$1,331
4	\$1,331	\$133.10	\$1,464.10
5	\$1,461.10	\$146.41	\$1,610.51

The simple interest account earns a total of \$500 in interest. But the compound interest account earns \$610.51 in interest (the sum of the accrued interest column), which is \$110.51 more interest than the simple interest account. This is the power of compounding interest. As an investor, you earn more interest on an investment when the interest is compounded than you would with simple interest. But this also means that as a borrower, you will pay more interest on a loan when the interest is compounded than with simple interest.

Compound Interest Terms

The **compounding period** is the interval of time between two consecutive interest calculations. That is, the compounding period is the period of time between the dates of successive conversions of interest to principal. For example, an interest rate that compounds monthly means that the compounding period is monthly, and so the interest is calculated and added to the principal every month.

The **compounding frequency** (m) is the number of times the interest is compounded every year. That is, the compounding frequency is the number of compounding periods in one year. For example, an interest rate that compounds quarterly means that the interest is calculated and added to the principal four times every year, so $m = 4$.

Compounding Period	Number of Times per Year Interest is Compounded	Compounding Frequency
Annually	Every year/Once a year	1
Semi-annually	Every six months/Twice a year	2
Quarterly	Every three months/Four times a year	4
Monthly	Every month/Twelve times a year	12
Daily	Every day/365 times a year	365

The **term or time period** (t) is the length of time for the investment or loan. That is, the term is the period of time when the interest is calculated. Because the interest rate is given as an annual interest rate and the compounding frequency is the number of compoundings per year, the term must be in years in order to use the term in compound interest calculations. If the term is not in years, the term must be converted to years. If the term is given in months, divide by 12 to convert the term to years. If the term is given in days, divide by 365 to convert the term to years.

Most compound interest calculations require the **total number of compounding periods during the term** (n).

$$\text{Total Number of Compoundings} = \text{Compounding Frequency} \times \text{Time in Years}$$

$$n = m \times t$$

EXAMPLE

For the following interest rates and terms, identify the compounding frequency and the term in years. Then calculate the total number of compounding periods during the term.

1. 3.5% compounded monthly for five years.
2. 7% compounded semi-annually for 18 months.
3. 4.9% compounded quarterly for three years and nine months.

Solution:

1. Compounding frequency = 12
Term = 5 years
 $n = 5 \times 12 = 60$
2. Compounding frequency = 2
Term = $\frac{18}{12} = 1.5$ years
 $n = 1.5 \times 2 = 3$
3. Compounding frequency = 4
Term = $3 + \frac{9}{12} = 3.75$ years
 $n = 3.75 \times 4 = 15$

TRY IT

Complete the following table.

Interest Rate and Time Period	Compounding Frequency (m)	Term (in years) (t)	Number of Compounding Periods for the Term (n)
7.2% compounded quarterly for 27 months			
3.7% compounded semi-annually for 6.5 years			
1.9% compounded daily for 657 days			

Click to see Solution

Interest Rate and Time Period	Compounding Frequency (m)	Term (in years) (t)	Number of Compounding Periods for the Term (n)
7.2% compounded quarterly for 27 months	4	$\frac{27}{12} = 2.25$	$2.25 \times 4 = 9$
3.7% compounded semi-annually for 6.5 years	2	6.5	$6.5 \times 2 = 13$
1.9% compounded daily for 657 days	365	$\frac{657}{365} = 1.8$	$1.8 \times 365 = 657$

The Nominal and Periodic Interest Rates

The **nominal interest rate** (j) is the quoted or stated interest rate annually. It is the rate, expressed as a percent, that precedes the word “compounded”. For example, if the interest rate is 5% compounded quarterly, the nominal interest rate is 5%.

The **periodic interest rate** (i) is interest rate earned or charged for a given compounding period.

$$\text{Periodic Interest Rate} = \frac{\text{Nominal Interest Rate}}{\text{Compounding Frequency}} \quad i = \frac{j}{m}$$

EXAMPLE

Calculate the periodic interest rate for the following nominal interest rates.

1. 9% compounded monthly.
2. 6% compounded quarterly.

Solution:

1. For 9% compounded monthly.

$$\begin{aligned}j &= 9\% \\m &= 12\end{aligned}$$

$$\begin{aligned}i &= \frac{j}{m} \\&= \frac{9\%}{12} \\&= 0.75\% \text{ per month}\end{aligned}$$

This means that 9% compounded monthly is equal to a periodic interest rate of 0.75% per month. Interest is converted to principal 12 times throughout the year at the rate of 0.75% each time.

2. For 6% compounded quarterly.

$$\begin{aligned}j &= 6\% \\m &= 4\end{aligned}$$

$$\begin{aligned}i &= \frac{j}{m} \\&= \frac{6\%}{4} \\&= 1.5\% \text{ per quarter}\end{aligned}$$

This means that 6% compounded quarterly is equal to a periodic interest rate of 1.5% per quarter. Interest is converted to principal 4 times throughout the year at the rate of 1.5% each time.

EXAMPLE

Calculate the nominal interest rate for the following periodic interest rates.

1. 0.7% per month.
2. 0.05% per day.

Solution:

1. For 0.7% per month.

$$i = 0.7\%$$

$$m = 12$$

$$j = i \times m$$

$$= 0.7\% \times 12$$

$$= 8.4\% \text{ compounded monthly}$$

This means that 0.7% per month is equal to 8.4% compounded monthly.

2. For 0.05% per day.

$$i = 0.05\%$$

$$m = 365$$

$$j = i \times m$$

$$= 0.05 \times 365$$

$$= 18.25\% \text{ compounded daily}$$

This means that 0.05% per day is equal to 18.25% compounded daily.

EXAMPLE

Calculate the compounding frequency for the following nominal and periodic interest rates.

1. Nominal rate of 6%, periodic rate of 3%.
2. Nominal rate of 9%, periodic rate of 2.25%.

Solution:

1. For 6% nominal and 3% periodic.

$$\begin{aligned}j &= 6\% \\i &= 3\%\end{aligned}$$

$$\begin{aligned}m &= \frac{j}{i} \\&= \frac{6\%}{3\%} \\&= 2\end{aligned}$$

For the nominal interest rate of 6% to be equal to a periodic interest rate of 3%, the compounding frequency is 2, which means a compounding period of every six months, or semi-annually.

2. For 9% nominal and 2.25% periodic.

$$\begin{aligned}j &= 9\% \\i &= 2.25\% \\m &= \frac{j}{i} \\&= \frac{9\%}{2.25\%} \\&= 4\end{aligned}$$

For the nominal interest rate of 9% to be equal to a periodic interest rate of 2.25%, the compounding frequency is 4, which means a compounded period of every three months, or quarterly.

TRY IT

Calculate the periodic interest rate if the nominal interest rate is 7.75% compounded monthly.

Click to see Solution

$$\begin{aligned}
 i &= \frac{j}{m} \\
 &= \frac{7.75\%}{12} \\
 &= 0.65\% \text{ per month}
 \end{aligned}$$

TRY IT

Calculate the compounding frequency for a nominal interest rate of 9.6% if the periodic interest rate is 0.8%.

Click to see Solution

$$\begin{array}{l}
 m = \frac{j}{i} = \frac{9.6\%}{0.8\%} = 12
 \end{array}$$

TRY IT

Calculate the nominal interest rate if the periodic interest rate is 2.0875% per quarter.

Click to see Solution

$$\begin{aligned}j &= i \times m \\&= 2.0875\% \times 4 \\&= 8.35\% \text{ compounded quarterly}\end{aligned}$$

Exercises

1. Calculate the periodic interest rate if the nominal interest rate is 7.5% compounded monthly.

Click to see Answer

0.625%

2. Calculate the compounding frequency for a nominal interest rate of 9.6% if the periodic interest rate is 0.8%.

Click to see Answer

12

3. Calculate the nominal interest rate if the periodic interest rate is 2.0875% per quarter.

Click to see Answer

8.35%

4. Lori hears her banker state, “We will nominally charge you 10.68% on your loan, which works out to 0.89% of your principal every time we charge you interest.” What is her compounding frequency?

Click to see Answer

12

5. You just received your monthly MasterCard statement and note at the bottom of the form that interest is charged at 11.9355% compounded daily. What interest rate is charged to your credit card each day?

Click to see Answer

0.0327%

6. Calculate the total number of compoundings for an investment at 3.5% compounded monthly for four years and seven months.

Click to see Answer

55

7. Calculate the total number of compoundings for an investment at 4.2% compounded quarterly for 39 months.

Click to see Answer

13

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4.2 CALCULATING THE FUTURE VALUE

LEARNING OBJECTIVES

- Calculate future value for compound interest.

Your company has an employee assistance plan through which employees can borrow funds at 12% compounded semi-annually, much like a loan from a bank, then repay the money at their convenience. An employee who borrowed \$4,000 two years ago has been unable to repay the loan until today. As the human resources manager in charge of the assistance plan, you must tell him the exact amount he needs to pay to return his balance to zero. How do you do this?

In the above scenario, you need to tell the employee the future value of their loan. The **future value** (FV) is the accumulated value or maturity value of a loan or an investment at the end of the term of the loan or investment. The future value includes the principal of the loan or investment plus all of the interest the loan or investment has accumulated over the term. The principal of the loan or investment is called the **present value** (PV). The present value is the amount of money borrowed for a loan or the amount of money invested for an investment at the start of the term.

Calculating the Future Value

Continuing with the scenario described above, suppose \$4000 was borrowed two years ago at 12% compounded semi-annually. At the end of the two years, the borrower will owe two years of compound interest **in addition** to the original principal of \$4,000. The amount of money borrowed for the loan is the present value, so $PV = \$4,000$. The compounding frequency is semi-annually, or twice per year, which makes the periodic interest rate $i = \frac{j}{m} = \frac{12\%}{2} = 6\%$. This means that every six months, the borrower accumulates 6% of current principal in interest charges. The total number of compoundings is

$n = m \times t = 2 \times 2 = 4$, which means that interest will be calculated and added four times over the term of the loan.

At the end of the first six months (one compounding period) of the loan, the current balance or future value of the loan is

$$\begin{array}{l} \text{Balance after first compounding period} \\ \text{Principal} + \text{Interest} \\ FV = PV + i \times PV \\ = 4,000 + 0.06 \times 4,000 \\ = \$4,240 \end{array}$$

In the above calculation, note that the equation $FV = PV + i \times PV$ can be factored and rewritten as $FV = PV(1 + i)$, which can be used to calculate the future value for each compounding period.

The above process will continue for each of the compounding periods: 6% of the current balance will be added to the balance at the end of each compounding period. For the second compounding period, the present value is the future value from the previous compounding period and the future value is the balance on the loan at the end of the second compounding period.

$$\begin{array}{l} FV_{\text{after two compounding periods}} \\ = PV \times (1 + i) \\ = 4,240 \times (1 + 0.06) \\ = \$4,494.40 \end{array}$$

Recall that the $PV = \$4,240$ was the result of the future value calculation of the first compounding period. In fact, $4,240 = 4000 \times (1 + 0.06)$, which can be substituted into the previous calculation.

$$\begin{array}{l} FV_{\text{after two compounding periods}} \\ = PV \times (1 + i) \times (1 + i) \\ = 4,000 \times (1 + 0.06) \times (1 + 0.06) \\ = \$4,494.40 \end{array}$$

The equation $FV = PV \times (1 + i) \times (1 + i)$ can be simplified to the following

$$FV = PV \times (1 + i)^2$$

Continuing the calculations for each of the four compounding periods results in the following.

Compounding Period	Present Value at the Beginning of the Compounding Period	Future Value at the End the Compounding Period
1	\$4,000	$4,000 \times (1 + 0.06) = \$4,240$
2	\$4,240	$\begin{aligned} &4,240 \times (1 + 0.06) = 4,000 \times (1 + 0.06)^2 \\ &= \$4,494.40 \end{aligned}$
3	\$4,494.40	$\begin{aligned} &4,494.40 \times (1 + 0.06) = 4,000 \times (1 + 0.06)^3 \\ &= \$4,764.06 \end{aligned}$
4	\$4,764.06	$\begin{aligned} &4,764.06 \times (1 + 0.06) = 4,000 \times (1 + 0.06)^4 \\ &= \$5,059.91 \end{aligned}$

Do you notice a pattern in these calculations? With the first compounding period, the formula has only one $(1 + i)$, with two compounding periods, the formula has two $(1 + i)$'s in the form of $(1 + i)^2$, with three compounding periods, the formula has three $(1 + i)$'s in the form $(1 + i)^3$, and so on. In general, the exponent of $(1 + i)$ equals the number of the compounding period.

The Future Value Formula

The future value for compound interest is

$$FV = PV \times (1 + i)^n$$

where

- **FV is the future value.** The future value includes the principal plus all of the interest accumulated over the term.
- **PV is the present value or principal.** The present value is the starting amount upon which compound interest is calculated.
- **i is the periodic interest rate.** The periodic interest rate is the interest rate per compounding period:

$$i = \frac{j}{m}$$
where j is the nominal interest rate and m is the compounding frequency.
- **n is the total number of compounding periods over the term.** $n = m \times t$ where m is the compounding frequency and t is the length of the term in years.

The amount of interest (I) accumulated by a loan or investment is

$$I = FV - PV$$

EXAMPLE

If you invested \$5,000 for 10 years at 9% compounded quarterly, how much money would you have? What is the interest earned during the term?

Solution:

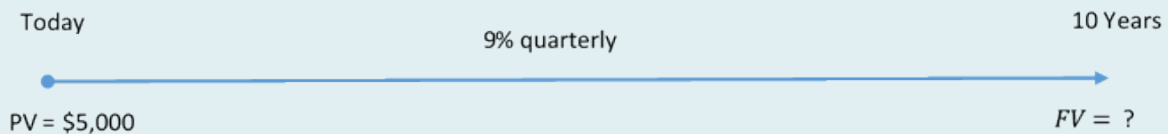


Figure 4.2.1

The timeline for the investment is below.

Step 1: The given information is

$$\begin{aligned} PV &= \$5,000 \\ j &= 9\% \\ m &= 4 \\ t &= 10 \text{ years} \end{aligned}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned} i &= \frac{j}{m} \\ &= \frac{9\%}{4} \\ &= 2.25\% \\ &= 0.0225 \end{aligned}$$

Step 3: Calculate the total number of compoundings.

$$\begin{aligned} n &= m \times t \\ &= 4 \times 10 \\ &= 40 \end{aligned}$$

Step 4: Calculate the future value.

$$\begin{aligned}
 FV &= PV \times (1 + i)^n \\
 &= 5,000 \times (1 + 0.0225)^{40} \\
 &= 5,000 \times (1.0225)^{40} \\
 &= \$12,175.94
 \end{aligned}$$

Step 5: Calculate the interest earned.

$$\begin{aligned}
 I &= FV - PV \\
 &= 12,175.94 - 5,000 \\
 &= \$7,175.94
 \end{aligned}$$

After 10 years, the principal grows to \$12,175.94, which includes your \$5,000 principal and \$7,175.94 of interest.

Using a Financial Calculator

A financial calculator, like the TI BAII Plus, has built-in functions to solve compound interest problems. These functions use the “time value of money” buttons on the calculator. Throughout the remainder of this book, we will focus on using the financial calculator to solve problems involving compound interest, instead of using the formulas as illustrated above.

USING THE TI BAI PLUS CALCULATOR TO FIND THE FUTURE VALUE FOR COMPOUND INTEREST

The time value of money buttons are located in the TVM row (the third row from the top) of the calculator. The five buttons located on the third row of the calculator are five of the seven variables required for time value of money calculations. This row's buttons are different in colour from the rest of the buttons on the keypad. The other two variables are in a secondary menu above the I/Y key and are accessed by pressing 2nd I/Y.

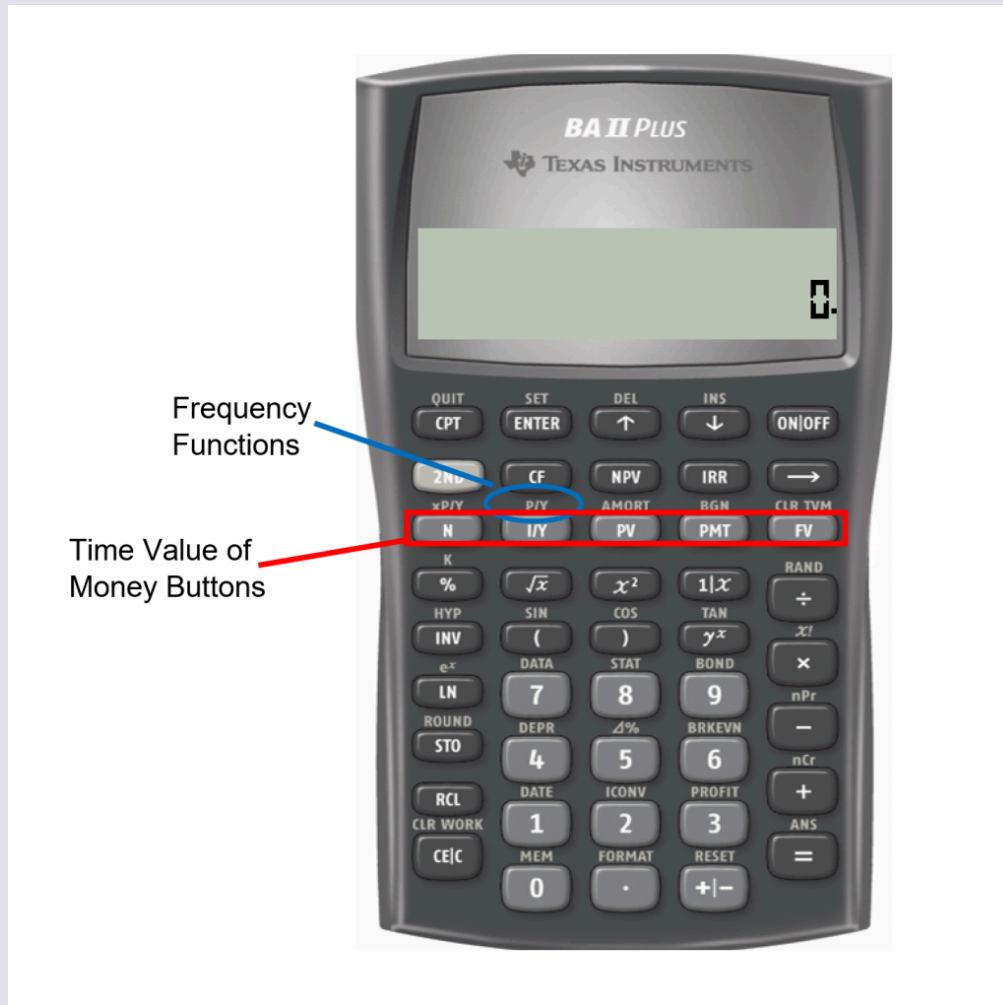


Figure 4.2.2

Altogether, there are seven variables required to complete time value of money calculations. Note that P/Y and C/Y are not main button keys in the TVM row. The P/Y and C/Y variables are located in the secondary function accessed by pressing 2nd I/Y.

Variable	Meaning
N	Total number of compounding periods. This is the same value as n in the future value formula. $N = \text{time in years} \times \text{compounding frequency}$
I/Y	Interest rate per year (i.e. the nominal interest rate). The interest rate is entered in percent form (without the % sign). For example, 5% is entered as 5.
PV	Present value or principal.
PMT	Periodic annuity payment. For compound interest only calculations, PMT=0. (Note: in later chapters you will learn about annuities where PMT will not be 0.)
FV	Future value or maturity value.
P/Y	Payments frequency for annuity payment. For compound interest only calculations, P/Y is set to the same value as C/Y. (Note: in later chapters you will learn about annuities where P/Y will be set to the frequency of the payments.)
C/Y	Compounding frequency. This is the value of m .

To enter values into the calculator:

- For the main button keys in the TVM row (i.e. N, I/Y, PV, PMT, FV), enter the number first and then press the corresponding button.
 - For example, to enter $N=34$, enter 34 on the calculator and then press N.
- For P/Y and C/Y, press 2nd I/Y. At the P/Y screen, enter the value for P/Y and then press ENTER. Press the down arrow to access the C/Y screen. At the C/Y screen, enter the value for C/Y and then press ENTER. Press 2nd QUIT (the CPT button) to exit the menu.
 - For example, to enter $P/Y=4$ and $C/Y=4$, press 2nd I/Y. At the P/Y screen, enter 4 and press ENTER. Press the down arrow. At the C/Y screen, enter 4 and press ENTER. Press 2nd QUIT to exit.

After all of the known quantities are loaded into the calculator, press CPT and then FV to solve for the future value.

NOTES

1. The calculator automatically sets C/Y equal to whatever is entered for P/Y . That is, if you enter 4 for P/Y , the calculator will set $C/Y=4$. This is what we need to complete compound interest only calculations. When we learn about annuities in later chapters, we will have instances where P/Y and C/Y will need to be different.
2. Your calculator has permanent memory. Once you enter data into any of the time value buttons it is permanently stored until
 - You override it by entering another piece of data and pressing the button; or
 - You clear the memory of the time value buttons by pressing 2nd CLR TVM before proceeding with another question; or
 - The reset button on the back of the calculator is pressed.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=61#oembed-1>

[businessfinancialmath/?p=61#oembed-1](https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=61#oembed-1)

[Compound Interest \(Present and Future Values\)](#) by Joshua Emmanuel [6:56] (transcript available).

Cash Flow Convention Signs

An investment and a loan are very different. An investment earns interest and the principal increases over time. A loan is charged interest but is usually paid off through payments, resulting in the principal decreasing over time. Notice that nowhere on the calculator is there a button to enter this critical piece of information. How does the calculator distinguish between the two? You must apply a principle called the **cash flow sign convention** to the PV, PMT, and FV buttons.

1. If you have money leaving your possession and going somewhere else (such as being put into an investment or making a payment against a loan), you must enter the number as a **NEGATIVE** number.
2. If you have money coming into your possession and you are receiving it (such as borrowing money from the bank or having an investment mature and pay out to you), you must enter the number as a **POSITIVE** number.

When doing financial calculations it is important to “be somebody” in the transaction. In any compound interest scenario, two parties are always involved—somebody is investing and somebody is borrowing. From this standpoint, think about whether the money leaves you or comes at you. This will help you place the correct sign in front of the PV, PMT, and FV when using your calculator. Who you are will not change the numbers of the transaction, just the cash flow sign convention.

1. If you borrow money from your friend and then pay it back, the initial loan is received by you and hence entered as a positive PV for you. As you pay back the loan, money leaves you and therefore the FV is negative for you.
2. Taking the other side of the coin and being your friend, the loaning of money is a negative PV for him. When you repay the loan, your friend receives it and therefore results in a positive FV for him.

	Sign of FV	Sign of PV
Investments	When you receive your matured investment at the end of the term this is considered as a cash-inflow for you and the future value should be entered as a positive amount.	When money is invested (paid-out), this amount is considered as a cash-outflow and this amount has to be entered as a negative number for PV.
Loans	When the loan is repaid at the end of the term this is considered as a cash-outflow for you and the future value should be entered as a negative amount.	When money is received (loaned), this amount is considered as a cash-inflow and this amount has to be entered as a positive number for PV.

NOTES

1. Notice that the PV and FV always have opposite signs.
2. When you compute solutions on the BAII Plus calculator, one of the most common error messages displayed is “Error 5.” This error indicates that the cash flow sign convention has been used in a manner that is financially impossible. Some examples of these

financial impossibilities include loans with no repayment or investments that never pay out. In these cases, the PV and FV have been incorrectly set to the same cash flow sign.

EXAMPLE

If you invested \$5,000 for 10 years at 9% compounded quarterly, how much money would you have?

Solution:

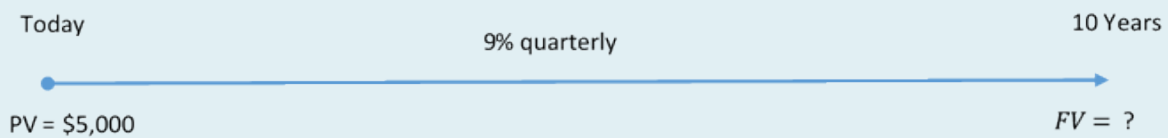


Figure 4.2.3

The timeline for the investment is below.

N	$4 \times 10 = 40$
PV	$-5,000$
FV	?
PMT	0
I/Y	9
P/Y	4
C/Y	4

$$FV = \$12,175.94$$

TRY IT

Find the future value if \$53,000 is invested at 6% compounded monthly for 4 years and 3 months. How much interest did the investment make?

Click to see Solution

N	$12 \times \frac{51}{12} = 51$
PV	$-53,000$
FV	?
PMT	0
I/Y	6
P/Y	12
C/Y	12

$$FV = \$68,351.02$$

$$\begin{aligned}
 I &= FV - PV \\
 &= 68,351.02 - 53,000 \\
 &= \$15,351.02
 \end{aligned}$$

Future Value Calculations with Variable Changes

What happens if a variable such as the nominal interest rate, compounding frequency, or even the principal changes somewhere in the middle of the transaction? When any variable changes, you must break the timeline into separate time fragments at the point of the change. To arrive at the solution, you need to work from left to right one time segment at a time using the future value formula.

1. Read and understand the problem. Identify the present value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the length of the time segment in years.
2. Starting with the present value in the first time segment (starting on the left), solve for the future value.
3. Let the future value calculated in the previous step become the present value for the next step. If the principal changes, adjust the new present value accordingly.
4. Calculate the future value of the next time segment.
5. Repeat the previous steps until you obtain the final future value from the final time segment.

NOTE

Transforming the future value from one time segment into the present value of the next time segment does not require re-entering the computed value. Instead, apply the following technique.

1. Load the calculator with all known compound interest variables for the first time segment.
2. Compute the future value at the end of the segment.
3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the $\pm$ key, and then store the unrounded number back into the present value button by pressing PV. Change the N, I/Y, and C/Y as required for the next segment.
4. Return to step 2 for each time segment until you have completed all time segments.

EXAMPLE

Five years ago Coast Appliances was supposed to upgrade one of its facilities at a quoted cost of \$48,000. The upgrade was not completed, so Coast Appliances delayed the purchase until now. The construction company that provided the quote indicates that prices rose 6% compounded quarterly for the first 1.5 years, 7% compounded semi-annually for the following 2.5 years, and 7.5% compounded monthly for the final year. If Coast Appliances wants to perform the upgrade today, what amount of money does it need?

Solution:

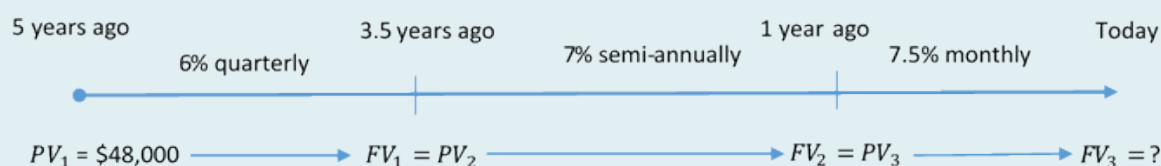


Figure 4.2.4

The timeline below shows the original quote from five years ago until today.

Step 1: Calculate the future value at the end of the first segment.

N	$4 \times 1.5 = 6$
PV	$-48,000$
FV	?
PMT	0
I/Y	6
P/Y	4
C/Y	4

$$FV_1 = \$52,485.27667....$$

Step 2: Calculate the future value at the end of the second segment. The future value from the first segment becomes the present value for the second segment:

$$FV_1 = \$52,485.27667.... = PV_2.$$

N	$2 \times 2.5 = 5$
PV	$-52,485.27667...$
FV	?
PMT	0
I/Y	7
P/Y	2
C/Y	2

$$FV_2 = \$62,336.04435....$$

Step 3: Calculate the future value at the end of the third segment. The future value from the second segment becomes the present value for the third segment:

$$FV_2 = \$62,336.04435.... = PV_3.$$

N	$12 \times 1 = 12$
PV	$-62,336.04435...$
FV	?
PMT	0
I/Y	7.5
P/Y	12
C/Y	12

$$FV_3 = \$67,175.35$$

Coast Appliance requires \$67,175.35 to perform the upgrade today.

EXAMPLE

Two years ago Lorelei placed \$2,000 into an investment earning 6% compounded monthly. Today she makes a deposit to the investment in the amount of \$1,500. What is the maturity value of her investment three years from now?

Solution:

The timeline for the investment is below.

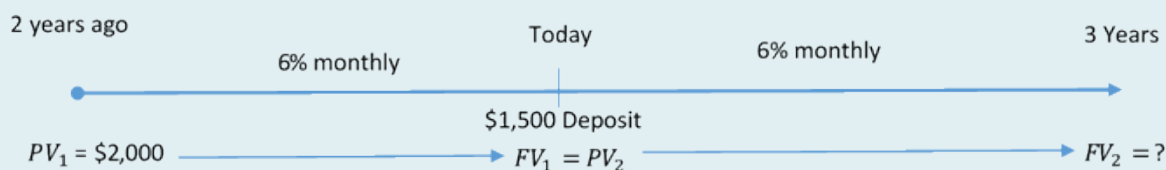


Figure 4.2.5

Step 1: Calculate the future value at the end of the first segment.

N	$12 \times 2 = 24$
PV	$-2,000$
FV	?
PMT	0
I/Y	6
P/Y	12
C/Y	12

$$FV_1 = \$2,254.31955\dots$$

Step 2: Add the \$1,500 deposit to FV_1 to get the present value for the second segment.

$$PV_2 = 2,254.31955\dots + 1,500 = \$3,754.31955\dots$$

Step 3: Calculate the future value at the end of the second segment.

N	$12 \times 3 = 36$
PV	$-3,754.31955\dots$
FV	?
PMT	0
I/Y	6
P/Y	12
C/Y	12

$$FV_2 = \$4,492.72$$

Three years from now Lorelei will have \$4,492.72. This represents \$3,500 of principal and \$992.72 of compound interest.

TRY IT

Find the future value if \$24,500 is invested at 4.1% compounded annually for 4 years; then 5.15% compounded quarterly for 1 year, 9 months; then 5.35% compounded monthly for 1 year, 3 months.

Click to see Solution

N	4	7	15
PV	−24,500	−28,771.930...	−31,467.335...
FV	Formula does not parse	Formula does not parse	Formula does not parse
PMT	0	0	0
I/Y	4.1	5.15	5.35
P/Y	1	4	12
C/Y	1	4	12

$$FV = \$33,638.67$$

TRY IT

Nirdosh borrowed \$9,300 4.25 years ago at 6.35% compounded semi-annually. The interest rate changed to 6.5% compounded quarterly 1.75 years ago. What amount of money today is required to pay off this loan?

Click to see Solution

N	5	7
PV	9,300	10,873.1489...
FV	Formula does not parse	Formula does not parse
PMT	0	0
I/Y	6.35	6.5
P/Y	2	4
C/Y	2	4

$$FV = \$12,171.92$$

Exercises

1. What is the future value of a \$7,500 investment at 8% compounded quarterly for 3 years?

Click to see Answer

\$9,511.81

2. Ruth borrowed \$53,000 at 6% compounded quarterly 4 years and 3 months ago. How much does Ruth have to pay today to clear the loan? How much interest did Ruth pay?

Click to see Answer

FV=\$68,351.02, I=\$15,351.02

3. You invest \$19,000 in a savings account at 5.75% compounded semi-annually. How much is in your account 6.5 years from now?

Click to see Answer

\$27,465.13

4. You invest \$3,750 in an investment that earns 4.75% compounded annually for the first three years and then 5.5% compounded semi-annually for the next two years. How much money do you have at the end of the five years? How much interest did your investment earn?

Click to see Answer

FV=\$4,804.20, I=\$1,054.20

5. James took out a \$11,375 loan today. He will pay interest at 7.5% compounded monthly for the first two years and nine months, and then 8.25% compounded quarterly for the next three years and three months. How much money does James owe six years from now?

Click to see Answer

\$18,218.24

6. You are planning a 16-day African safari to Rwanda to catch a rare glimpse of the 700 remaining mountain gorillas in the world. The estimated cost of this once-in-a-lifetime safari is \$15,000 including the tour, permits, lodging, and airfare. Upon your graduation from college, your parents have promised you a \$10,000 graduation gift. You intend to save this money for five years in a long-term investment earning 8.3% compounded semi-annually. If the cost of the trip will be about the same, will you have enough money five years from now to pay for your trip?

Click to see Answer

\$15,017.33; yes

7. Your investment of \$9,000 that you started six years ago earned 7.3% compounded quarterly for the first $3\frac{1}{4}$ years, followed by 8.2% compounded monthly after that. How much interest has your investment earned so far?

Click to see Answer

\$5,254.44

8. What is the maturity value of your \$7,800 investment after three years if the interest rate was 5%

compounded semiannually for the first two years, 6% compounded quarterly for the last year, and 2½ years after the initial investment you made a deposit of \$1,200? How much interest is earned?

Click to see Answer

FV=\$10,374.33, I=\$1,374.33

9. Cristy borrowed \$4,800 from a family friend 2½ years ago at 7% compounded annually for the first year and 8% compounded semi-annually thereafter. She made a payment 1½ years into the loan for \$2,500. How much should Cristy pay today to clear her loan?

Click to see Answer

\$3,073.30

10. You invested \$5,000 10 years ago and made two further deposits of \$5,000 each after four years and eight years. Your investment earned 9.2% compounded quarterly for the first two years, 8.75% compounded monthly for the next six years, and 9.8% compounded semi-annually for the remaining years. As of today, how much interest has your investment earned?

Click to see Answer

\$11,888.46

11. Suppose you placed \$10,000 into each of the following investments. Rank the maturity values after five years from highest to lowest.
- 8% compounded annually for two years followed by 6% compounded semi-annually.
 - 8% compounded semi-annually for two years followed by 6% compounded annually.
 - 8% compounded monthly for two years followed by 6% compounded quarterly.
 - 8% compounded semi-annually for two years followed by 6% compounded monthly.

Click to see Answer

c is largest with FV=\$14,023.26, d is second largest with FV=\$13,999.47, b is third largest with FV=\$13,933.20, a is smallest with FV=\$13,927.43

12. You made the following three investments:

- \$8,000 into a five-year fixed rate investment earning 5.65% compounded quarterly.

- \$6,500 into a five-year variable rate investment earning 4.83% compounded semi-annually for the first 2½ years and 6.5% compounded monthly for the remainder.
- \$4,000 into a five-year variable rate investment earning 4.75% compounded monthly for the first two years and 5.9% compounded quarterly thereafter, with a \$4,000 deposit made 21 months into the investment.

What is the total maturity value of all three investments after the five years, and how much interest is earned?

Click to see Answer

\$29,270.56, \$6,770.56

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4.3 CALCULATING THE PRESENT VALUE

LEARNING OBJECTIVES

- Calculate present value for compound interest.

The principal of the loan or investment is called the **present value** (PV). The present value is the amount of money borrowed for a loan or the amount of money invested for an investment at the start of the term. The present value is the amount at some earlier point in time than when the future value is known, and so excludes the future interest.

The Present Value Formula

By solving for PV in the future value formula from the previous section, the **present value** for compound interest is

$$PV = FV \times (1 + i)^{-n}$$

where

- **PV is the present value.** The present value is the starting amount upon which compound interest is calculated.
- **FV is the future value.** The future value includes the principal plus all of the interest accumulated over the term.
- **i is the periodic interest rate.** The periodic interest rate is the interest rate per compounding period: $i = \frac{j}{m}$ where j is the nominal interest rate and m is the compounding frequency.
- **n is the total number of compounding periods over the term.** $n = m \times t$ where m is the

compounding frequency and t is the length of the term in years.

EXAMPLE

Castillo's Warehouse will need to purchase a new forklift for its warehouse operations three years from now, when its new warehouse facility becomes operational. If the price of the new forklift is \$38,000 and Castillo's can invest its money at 7.25% compounded monthly, how much money should it put aside today to achieve its goal?

Solution:

The timeline for the investment is shown below.

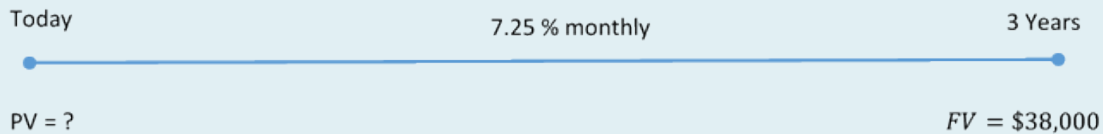


Figure 4.3.1

Step 1: The given information is

$$FV = \$38,000$$

$$j = 7.25\%$$

$$m = 12$$

$$t = 3 \text{ years}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned} i &= \frac{j}{m} \\ &= \frac{7.25\%}{12} \\ &= 0.60416...\% \\ &= 0.0060416... \end{aligned}$$

Step 3: Calculate the total number of compoundings.

$$n = m \times t = 12 \times 3 = 36$$

Step 4: Calculate the present value.

$$\begin{aligned} PV &= FV \times (1 + i)^{-n} \\ &= 38,000 \times (1 + 0.0060416\ldots)^{-36} \\ &= \$30,592.06 \end{aligned}$$

If Castillo's Warehouse places \$30,592.06 into the investment, it will earn enough interest to grow to \$38,000 three years from now to purchase the forklift.

Using a Financial Calculator

As in the previous section, a financial calculator can be used to solve for the present value in compound interest problems. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is PV instead of FV. You must still load the other six variables into the calculator and apply the cash flow sign convention carefully.

USING THE TI BAI PLUS CALCULATOR TO FIND THE PRESENT VALUE FOR COMPOUND INTEREST

The time value of money buttons are located in the TVM row (the third row from the top) of the calculator. The five buttons located on the third row of the calculator are five of the seven variables required for time value of money calculations. This row's buttons are different in colour from the rest of the buttons on the keypad. The other two variables are in a secondary menu above the I/Y key and are accessed by pressing 2nd I/Y.

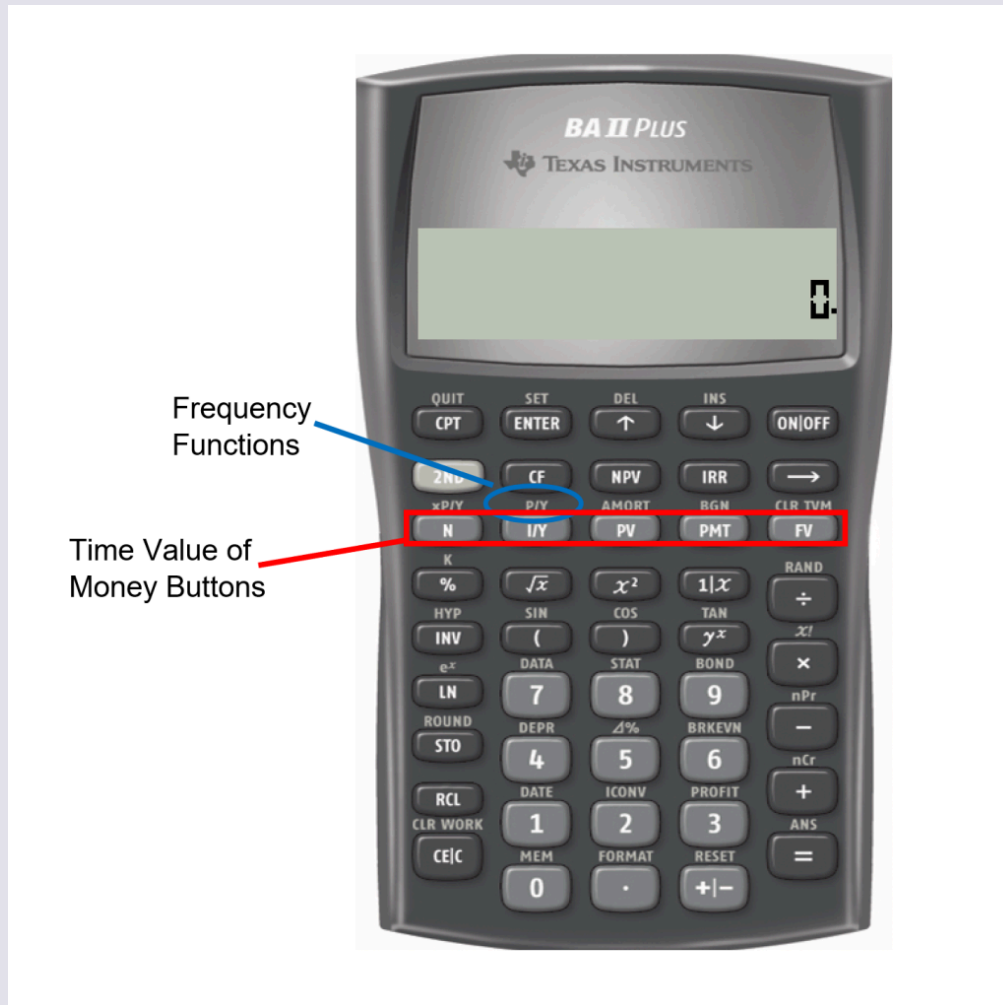


Figure 4.3.2

Altogether, there are seven variables required to complete time value of money calculations. Note that **P/Y** and **C/Y** are not main button keys in the TVM row. The **P/Y** and **C/Y** variables are located in the secondary function accessed by pressing **2nd I/Y**.

Variable	Meaning
N	Total number of compounding periods. This is the same value as n in the future value/present value formulas: $N = \text{time in years} \times \text{compounding frequency}$
I/Y	Interest rate per year (i.e. the nominal interest rate). The interest rate is entered in percent form (without the % sign). For example, 5% is entered as 5.
PV	Present value or principal.
PMT	Periodic annuity payment. For compound interest only calculations, PMT=0. (Note: in later chapters you will learn about annuities where PMT will not be 0.)
FV	Future value or maturity value.
P/Y	Payment frequency for annuity payment. For compound interest only calculations, P/Y is set to the same value as C/Y. (Note: in later chapters you will learn about annuities where P/Y will be set to the frequency of the payments.)
C/Y	Compounding frequency. This is the value of m .

To enter values into the calculator:

- For the main button keys in the TVM row (i.e. N, I/Y, PV, PMT, FV), enter the number first and then press the corresponding button.
 - For example, to enter $N=34$, enter 34 on the calculator and then press N.
- For P/Y and C/Y, press 2nd I/Y. At the P/Y screen, enter the value for P/Y and then press ENTER. Press the down arrow to access the C/Y screen. At the C/Y screen, enter the value for C/Y and then press ENTER. Press 2nd QUIT (the CPT button) to exit the menu.
 - For example, to enter $P/Y=4$ and $C/Y=4$, press 2nd I/Y. At the P/Y screen, enter 4 and press ENTER. Press the down arrow. At the C/Y screen, enter 4 and press ENTER. Press 2nd QUIT to exit.

After all of the known quantities are loaded into the calculator, press CPT and then PV to solve for the present value.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=65#oembed-1>

[Compound Interest \(Present and Future Values\)](#) by Joshua Emmanuel [6:56] (transcript available).

EXAMPLE

Castillo's Warehouse will need to purchase a new forklift for its warehouse operations three years from now, when its new warehouse facility becomes operational. If the price of the new forklift is \$38,000 and Castillo's can invest its money at 7.25% compounded monthly, how much money should it put aside today to achieve its goal?

Solution:

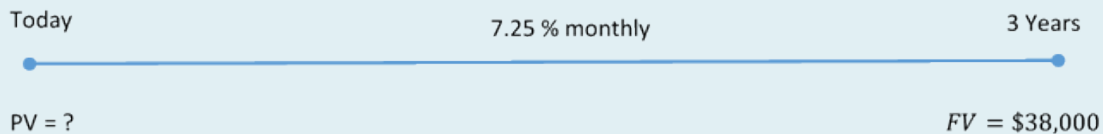


Figure 4.3.3

The timeline for the investment is shown below.

N	$12 \times 3 = 36$
PV	?
FV	38,000
PMT	0
I/Y	7.25
P/Y	12
C/Y	12

$$PV = \$30,592.06$$

TRY IT

A debt of \$37,000 is owed 21 months from today. If prevailing interest rates are 6.55% compounded quarterly, what amount should the creditor be willing to accept today?

Click to see Solution

N	$4 \times 1.75 = 7$
PV	?
FV	$-37,000$
PMT	0
I/Y	6.55
P/Y	4
C/Y	4

$$PV = \$33,023.56$$

Present Value Calculations with Variable Changes

Addressing variable changes in present value calculations follows the same techniques as future value calculations discussed in the previous section. You must break the timeline into separate time segments, each of which involves its own calculations. Solving for the unknown PV at the left of the timeline means you must start at the right of the timeline. You must work from right to left, one time segment at a time using the formula for PV each time. Note that the present value for one time segment becomes the future value for the next time segment to the left.

1. Read and understand the problem. Identify the future value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the length of the time segment in years.
2. Starting with the future value in the last time segment (starting on the right), solve for the present value.
3. Let the present value calculated in the previous step become the future value for the next segment to the left. If the principal changes, adjust the future value accordingly.
4. Calculate the present value of the next time segment.
5. Repeat the previous steps until you obtain the final present value from the leftmost time segment.

NOTE

To use your calculator efficiently in working through multiple time segments, follow a procedure similar to that for future value.

1. Load the calculator with all known compound interest variables for the last time segment (on the right).
2. Compute the present value at the beginning of the segment.
3. With the answer still on your display, adjust the principal if needed, change the cash flow sign by pressing the $\pm$ key, and then store the unrounded number back into the future value button by pressing FV. Change the N, I/Y, and C/Y as required for the next segment.
4. Return to step 2 for each time segment until you have completed all time segments.

EXAMPLE

Sebastien needs to have \$9,200 saved up three years from now. The investment he is considering pays 7% compounded semi-annually, 8% compounded quarterly, and 9% compounded monthly in successive years. To achieve his goal, how much money does he need to place into the investment today?

Solution:

The timeline shows today through to the future value three years from now

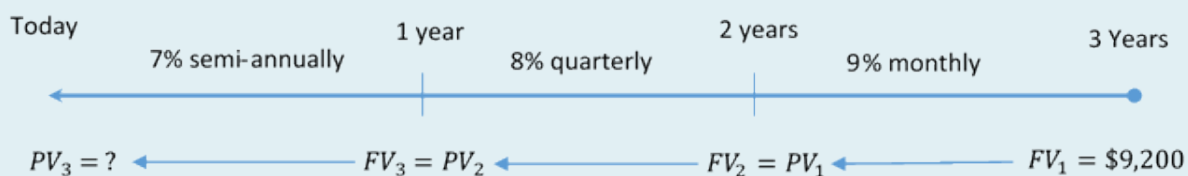


Figure 4.3.4

Step 1: Calculate the present value at the start of the last segment on the right.

N	$12 \times 1 = 12$
PV	?
FV	9,200
PMT	0
I/Y	9
P/Y	12
C/Y	12

$$PV_1 = \$8,410.991\dots$$

Step 2: Calculate the present value at the start of the second segment on the right. The present value from the first step becomes the future value for the second step:

$$PV_1 = \$8,410.991\dots = FV_2.$$

N	$4 \times 1 = 4$
PV	?
FV	8,410.991...
PMT	0
I/Y	8
P/Y	4
C/Y	4

$$PV_2 = \$7,770.4555...$$

Step 3: Calculate the present value at the end of the third segment on the right. The present value from the second step becomes the future value for the third step:

$$PV_2 = \$7,770.455... = FV_3.$$

N	$2 \times 1 = 2$
PV	?
FV	7,770.455..
PMT	0
I/Y	7
P/Y	2
C/Y	2

$$PV_3 = \$7,253.80$$

Sebastien needs to place \$7,253.80 into the investment today to have \$9,200 three years from now.

TRY IT

For the first 4.5 years, a loan was charged interest at 4.5% compounded semi-annually. For the next 4 years, the rate was 3.25% compounded annually. If the maturity value was \$45,839.05 at the end of the 8.5 years, what was the principal of the loan?

Click to see Solution

N	4	9
PV	Formula does not parse	Formula does not parse
FV	−45,839.05	−40,334.378...
PMT	0	0
I/Y	3.25	4.5
P/Y	1	2
C/Y	1	2

$$PV = \$33,014.56$$

Exercises

1. A loan is repaid with \$14,000. If the loan was taken out 14 years ago at 9% compounded semi-annually, how much money was borrowed? How much interest was paid on the loan?

Click to see Answer

$$PV=\$4,081.99, I=\$9,9180.01$$

2. In 9 years and 3 months, you want to have \$97,000 in your savings account. How much money must you invest today if the savings account earns 6% compounded monthly?

Click to see Answer

$$\$55,762.07$$

3. Eight and a half years ago, Tom took out a loan. The interest rate on the loan was 4.5% compounded semi-annually for the first four and half years and 3.25% compounded annually for the last four years. Tom repaid the loan today with a payment of \$45,839.05. How much money did Tom borrow? How much interest did Tom pay?

Click to see Answer

$$PV=\$33,014.56, I=\$12,824.49$$

4. George wants to invest some money today. In 6.5 years, George wants to have \$7,223.83 in his investment. The investment earns 8.05% compounded semi-annually for the first 2 years and 6 months, then 7.95% compounded quarterly for 1 year and 3 months, and then 7.8% compounded monthly for 2 years and 9 months. How much money does George need to invest?

Click to see Answer

$$\$4,340$$

5. Dovetail Industries needs to save \$1,000,000 for new production machinery that it expects will be needed six years from today. If money can earn 8.35% compounded monthly, how much money should Dovetail invest today?

Click to see Answer

$$\$606,976.63$$

6. A debt of \$37,000 is owed 21 months from today. If prevailing interest rates are 6.55% compounded quarterly, what amount should the creditor be willing to accept today?

Click to see Answer

\$33,023.56

7. Rene wants to invest a lump sum of money today to make a \$35,000 down payment on a new home in five years. If he can place his money in an investment that will earn 4.53% compounded quarterly in the first two years followed by 4.76% compounded monthly for the remaining years, how much money does he need to invest today?

Click to see Answer

\$27,736.24

8. In August 2004, Google Inc. made its initial stock offering. The value of the shares grew to \$531.99 by July 2011. What was the original value of a share in August 2004 if the stock has grown at a rate of 26.8104% compounded monthly?

Click to see Answer

\$85

9. If a three-year and seven-month investment earned \$8,879.17 of interest at 3.95% compounded monthly, what amount was originally placed into the investment?

Click to see Answer

\$58,499.97

10. A lottery ticket advertises a \$1 million prize. However, the fine print indicates that the winning amount will be paid out on the following schedule: \$250,000 today, \$250,000 one year from now, and \$100,000 per year thereafter. If money can earn 9% compounded annually, what is the value of the prize today?

Click to see Answer

\$836,206.54

11. Your company is selling some real estate and has received three potential offers:

- **Option A:** \$520,000 today plus \$500,000 one year from now.
- **Option B:** \$200,000 today, \$250,000 six months from now, and \$600,000 15 months from now.
- **Option C:** \$70,000 today, \$200,000 in six months, then four quarterly payments of \$200,000

starting six months later.

Prevailing interest rates are expected to be 6.75% compounded semi-annually in the next year, followed by 6.85% compounded quarterly afterwards. Rank the three offers from best to worst based on their values today.

Click to see Answer

Option B is best with $PV = \$993,846.61$; Option C is second best with $PV = \$993,391.13$; Option A is third best with $PV = \$987,884.82$

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4.4 EQUIVALENT PAYMENTS

LEARNING OBJECTIVES

- Calculate equivalent payments that replace another payment or a set of payments.

Unforeseen events and circumstances can force you to rearrange your financial commitments. When this happens, the new deal has to be fair to all parties concerned.

For example, suppose you and your co-workers rely on your company's annual Christmas bonuses. However, the CEO just announced that because of tough economic conditions no one will get a bonus this year. You already earmarked that money to pay a \$5,000 debt due next week. You want to be financially responsible, but you cannot possibly make the payment. Before picking up the phone to call your creditor, you need to determine what course of action you should pursue. You need to make alternative arrangements that leave your creditor in the same financial position as the original agreement did. You saw in the previous section that if you were going to make an early payment, the payment should be reduced by an agreed-upon discount rate. In this case, though, you are going to make a late payment, so you must grant the creditor interest. Thus, if you propose paying the debt six months late and your creditor agrees to 9% compounded monthly as a fair rate, then you owe \$5,229.26.

This concept applies to all aspects of your personal and professional life. Except for gifts, personal debts to friends or family members should bear some interest. Everyone should be financially fair to each other. A business must be willing to work with its clients in the event they need to alter an agreement. A company that is inflexible tends to find itself writing off bad debt or pursuing unpleasant and sometimes expensive legal action. This section explores the concept of equivalent payment streams, which involves equating two or more alternative financial streams to ensure that neither party is penalized by any choice.

The Fundamental Concept of Equivalency

The **fundamental concept of equivalency** states that two or more payment streams are equal to each other if they have the **same economic value** on the **same focal date**. As illustrated in the figure below, the two alternative financial streams are equivalent if the total of Payment Stream 1 is equal to the total of Payment Stream 2 on the same focal date.

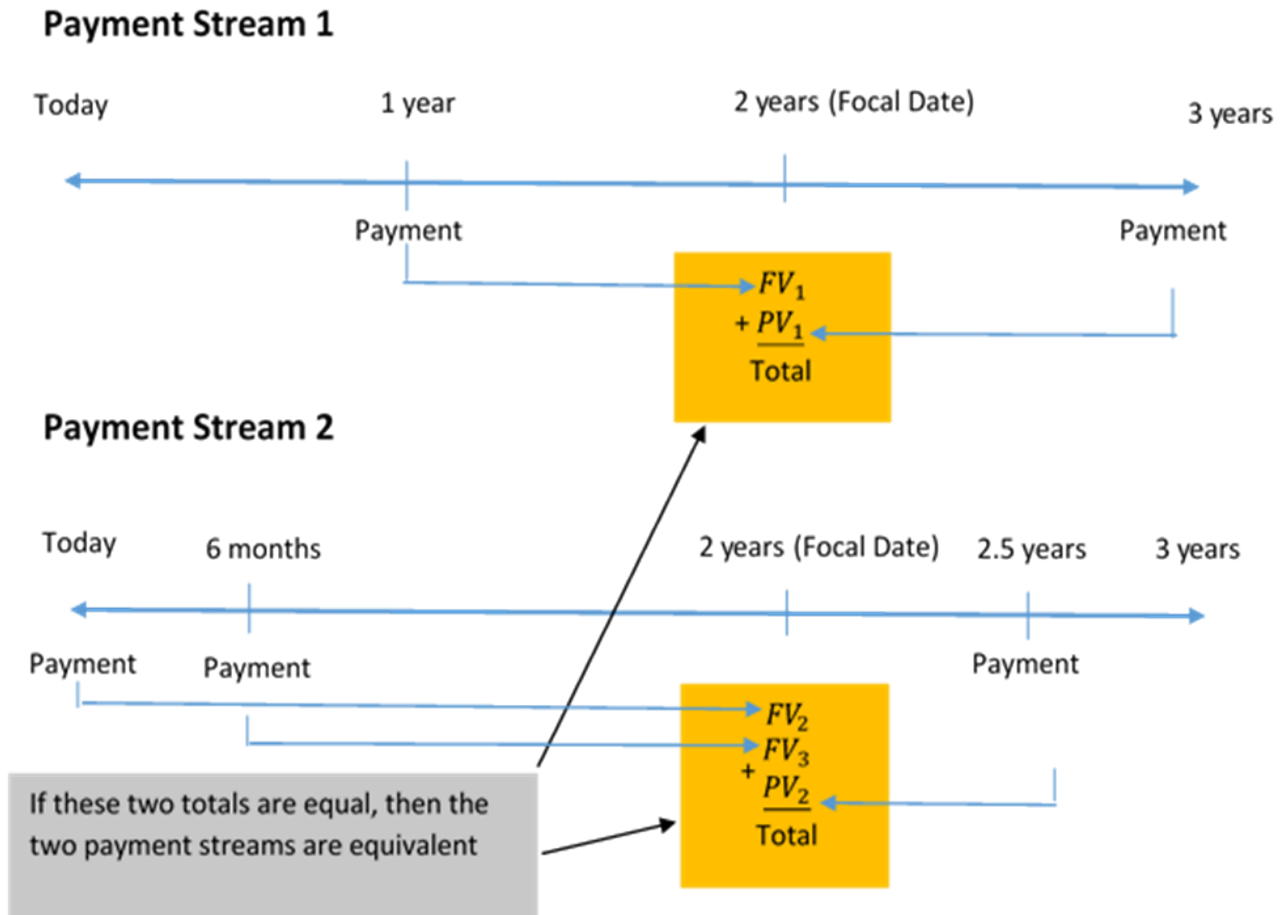


Figure 4.4.1

Replacing a Series of Payment with a Single Equivalent Payment

In this first situation, you are going to replace a series of payments with one single payment that is equivalent to the original series of payments. To find the single payment, follow these steps.

1. Draw a timeline, **in chronological order**, to show the original series of payments along with the times the payments were supposed to be made. Clearly indicate the dates of the payments and the payment amounts.
2. For a single equivalent payment calculation, choose the date of the single payment as the **focal date**.
3. Move all of the payments from the original series of payments to the focal date using a future value or present value calculation.
 - If the payment is **moving forward** in time to get to the focal date, the payment is a present value and you need to find its future value at the focal date.
 - If the payment is **moving backward** in time to get to the focal date, the payment is a future value and you need to find its present value at the focal date.
4. The single equivalent payment is the sum of the payments **at the focal date**.

EXAMPLE

Assume you owe \$1,000 today and \$1,000 one year from now. You find yourself unable to make that payment today, so you indicate to your creditor that you want to make both payments six months from now instead. Prevailing interest rates are at 6% compounded semi-annually. What single payment six months from now (the new payment stream) is equivalent to the two payments (the original payment stream)?

Solution:

Step 1: Draw the timeline.

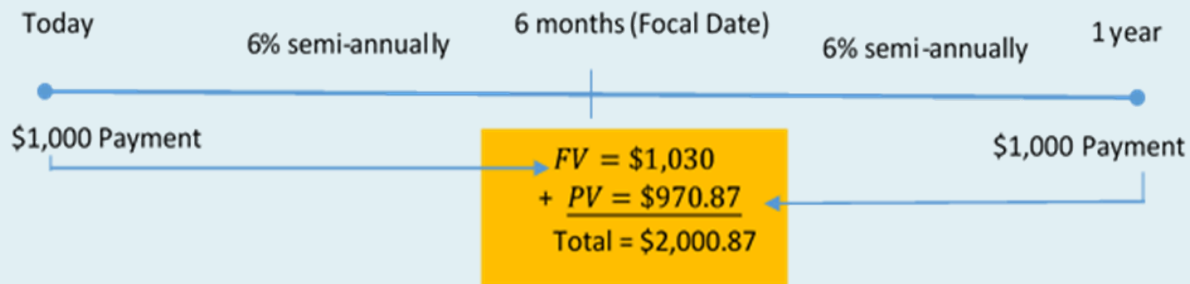


Figure 4.4.2

Step 2: Choose six months from today as the focal date.

Step 3: Calculate the future value of the \$1,000 payment today at the focal date. Because this payment must move forward in time to get to the focal date, the \$1,000 is a present value and its value at the focal date is a future value. This payment must move 6 months (from today to 6 months from today) to get to the focal date, so the time for the future value calculation is 6 months or 0.5 year.

N	$2 \times 0.5 = 1$
PV	$-1,000$
FV	?
PMT	0
I/Y	6
P/Y	2
C/Y	2

$$FV = \$1,030$$

Step 4: Calculate the present value of the \$1,000 payment one year from now at the focal date. Because this payment must move backward in time to get to the focal date, the \$1,000 is a future value and its value at the focal date is a present value. This payment must move 6 months (from 1 year from today to 6 months from today) to get to the focal date, so the time for the present value calculation is 6 months or 0.5 year.

N	$2 \times 0.5 = 1$
PV	?
FV	$-1,000$
PMT	0
I/Y	6
P/Y	2
C/Y	2

$$PV = \$970.87$$

Step 5: The single equivalent payment is the sum of the payments at the focal date.

$$\begin{aligned}\text{Single Payment} &= 1,030 + 970.87 \\ &= \$2,000.87\end{aligned}$$

A single payment of \$2,000.87 six months from now is equivalent to the original payments of \$1,000 today and \$1,000 one year from now.

NOTES

1. A properly drawn timeline, in chronological order, can help you see what you need to do, where the payments need to go (in relation to the focal date), and whether you need to do a future value or present value calculation to move the payments to the focal date.
2. Any date can be chosen as the focal date. But in a single equivalent payment calculation you need to know the value of the single payment at the focal date, so it makes sense to choose the time of the single payment as the focal date.
3. To find the time a payment has to move from its current date to the focal date, use the following technique.
 - If the date of the payment and the focal date are on the **same side** as today,

subtract the times. For example, if the original payment is at 2 years from today and the focal date is at 5 years from today, the time the original payment has to move to get to the focal date is $5-2=3$ years.

- If the date of the payment and the focal date are on **opposite sides** of today, add the times. For example, if the original payment is at 3 years ago and the focal date is 4 years from today, the time the original payment has to move to get to the focal date is $3+4=7$ years.
- Treat today as time 0.

EXAMPLE

Johnson's Garden Centre has recently been unprofitable and concludes that it cannot make two debt payments of \$4,500 due today and another \$6,300 due in three months. After discussions between Johnson's Garden Centre and its creditor, the two parties agree that both payments could be made nine months from today, with interest at 8.5% compounded monthly. What total payment does Johnson's Garden Centre need nine months from now to clear its debt?

Solution:

Step 1: Draw the timeline.

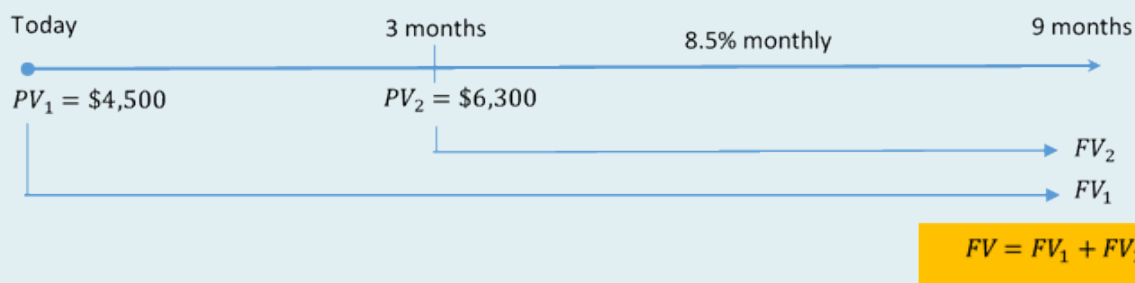


Figure 4.4.3

Step 2: Choose nine months from today as the focal date.

Step 3: Calculate the future value of the \$4,500 payment today at the focal date. Because this payment must move forward in time to get to the focal date, the \$4,500 is a present value and its value at the focal date is a future value. This payment must move 9 months (from today to 9 months from today) to get to the focal date, so the time for the future value calculation is 9 months or 0.75 year.

N	$12 \times 0.75 = 9$
PV	$-4,500$
FV	?
PMT	0
I/Y	8.5
P/Y	12
C/Y	12

$$FV_1 = \$4,795.14$$

Step 4: Calculate the future value of the \$6,300 payment today at the focal date. Because this payment must move forward in time to get to the focal date, the \$6,300 is a present value and its value at the focal date is a future value. This payment must move 6 months (from 3 months from today to 9 months from today) to get to the focal date, so the time for the future value calculation is 6 months or 0.5 year.

N	$12 \times 0.5 = 6$
PV	$-6,300$
FV	?
PMT	0
I/Y	8.5
P/Y	12
C/Y	12

$$FV_2 = \$6,572.54$$

Step 5: The single equivalent payment is the sum of the payments at the focal date.

$$\begin{aligned}\text{Single Payment} &= 4,795.14 + 6,572.54 \\ &= \$11,367.68\end{aligned}$$

A single payment of \$11,367.68 nine months from now is equivalent to the original payments of \$4,500 today and \$6,300 three months from now.

TRY IT

What single payment in 2 years would replace payments of \$5,000 due 1.5 years ago, \$2,500 due in 1 year, and \$1,000 due in 4 years if money earns 3.5% compounded quarterly?

Click to see Solution

Using 2 years as the focal date:

N	14	4	8
PV	−5,000	−2,500	Formula does not parse
FV	Formula does not parse	Formula does not parse	−1,000
PMT	0	0	0
I/Y	3.5	3.5	3.5
P/Y	4	4	4
C/Y	4	4	4

$$\begin{aligned}\text{Single Payment} &= 5,648.59 + 2,588.66 + 932.68 \\ &= \$9,169.93\end{aligned}$$

Replacing a Series of Payments with an Equivalent Series of Payments

In this situation, you are going to replace a series of payments with another series of payments that is equivalent to the original series of payments. To find the new series payment, follow these steps.

1. Draw a timeline, **in chronological order**, to show the original series of payments and the new series of payments, along with the times the payments were supposed to be made. Clearly indicate the dates of the payments and the payment amounts. To keep track of which payment belongs to which payment stream, draw the timeline in different colours or draw multiple timelines, one for each payment stream.
2. Select a **focal date**.
3. Move all of the payments (from both the original payment stream and the new payment stream) to the focal date using a future value or present value calculation.
 - If the payment is **moving forward** in time to get to the focal date, the payment is a present value and you need to find its future value at the focal date.
 - If the payment is **moving backward** in time to get to the focal date, the payment is a future value and you need to find its present value at the focal date.

4. The sum of the payments in the original payment stream at the focal date equals the sum of the payments in the new payment stream at the focal date. This creates an equation that allows you to solve for the unknown in the new payment stream.

EXAMPLE

You have three debts to the same creditor: \$3,000 due today, \$2,500 due in 2.25 years, and \$4,250 due in 3 years 11 months. Unable to fulfill this obligation, you arrange with your creditor to make two alternative payments: \$3,500 in nine months and a second payment due in two years. You agree upon an interest rate of 9.84% compounded monthly. What is the amount of the second payment?

Solution:

Step 1: Draw the timeline. With two payment streams and multiple amounts all on different dates, visualize two timelines, one for the original payment stream and the other for the new payment stream. Alternatively, draw one timeline with the payment streams in different colours. The unknown payment in the new payment stream is represented by x .

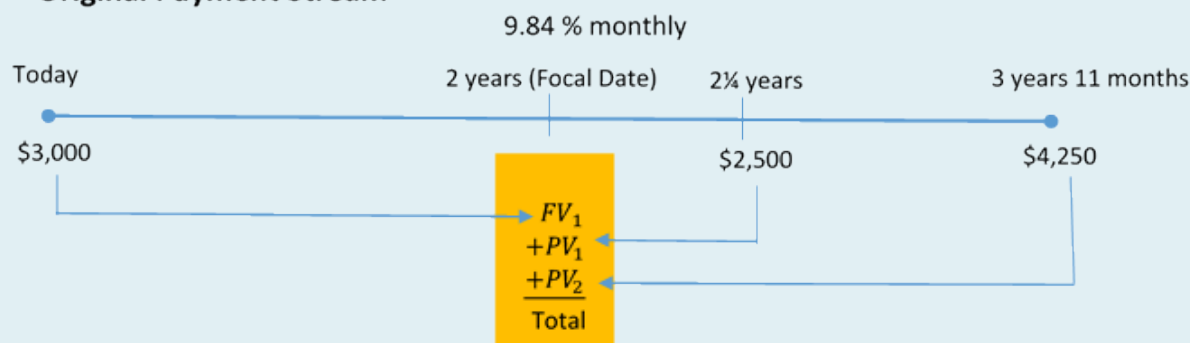
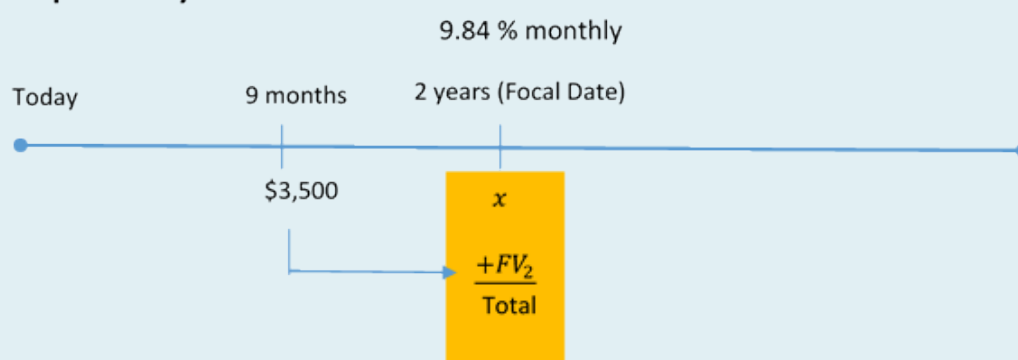
Original Payment Stream**Proposed Payment Stream**

Figure 4.4.4

Step 2: Choose two years from today as the focal date.

Step 3: Calculate the future value of the \$3,000 payment today (in the original payment stream) at the focal date. Because this payment must move forward in time to get to the focal date, the \$3,000 is a present value and its value at the focal date is a future value. This payment must move 2 years (from today to 2 years from today) to get to the focal date, so the time for the future value calculation is 2 years.

N	$12 \times 2 = 24$
PV	$-3,000$
FV	?
PMT	0
I/Y	9.84
P/Y	12
C/Y	12

$$FV_1 = \$3,649.57$$

Step 4: Calculate the present value of the \$2,500 payment 2.25 years from today (in the original payment stream) at the focal date. Because this payment must move backward in time to get to the focal date, the \$2,500 is a future value and its value at the focal date is a present value. This payment must move 0.25 years (from 2.25 years from today to 2 years from today) to get to the focal date, so the time for the present value calculation is 0.25 years.

N	$12 \times 0.25 = 3$
PV	?
FV	$-2,500$
PMT	0
I/Y	9.84
P/Y	12
C/Y	12

$$PV_1 = \$2,439.49$$

Step 5: Calculate the present value of the \$4,250 payment 3 years and 11 months from today (in the original payment stream) at the focal date. Because this payment must move backward in time to get to the focal date, the \$4,250 is a future value and its value at the focal date is a present value. This payment must move 1 year, 11 months (from 3 years, 11 months from today to 2 years

from today) to get to the focal date, so the time for the present value calculation is 1 year, 11 months.

N	$12 \times \frac{23}{12} = 23$
PV	?
FV	-4,250
PMT	0
I/Y	9.84
P/Y	12
C/Y	12

$$PV_2 = \$3,522.21$$

Step 6: Calculate the future value of the \$3,500 payment 9 months from today (in the new payment stream) at the focal date. Because this payment must move forward in time to get to the focal date, the \$3,500 is a present value and its value at the focal date is a future value. This payment must move 1 year, 3 months (from 9 months from today to 2 years from today) to get to the focal date, so the time for the future value calculation is 1 year, 3 months.

N	$12 \times \frac{15}{12} = 15$
PV	-3,500
FV	?
PMT	0
I/Y	9.84
P/Y	12
C/Y	12

$$FV_2 = \$3,956.11$$

Step 7: At the focal date, the sum of the payments in the original payment stream equal the sum of the payments in the new payment stream.

$$\begin{array}{l} 3,649.57 + 2,439.49 + 3,522.21 \text{ \& } = \text{ \& } ; 3,956.11 + x \text{ \& } 9,611.27 \text{ \& } = \text{ \& } ; \\ 3,956.11 + x \text{ \& } 9,611.27 - 3,956.11 \text{ \& } = \text{ \& } ; x \text{ \& } \$5,644.16 \text{ \& } = \text{ \& } ; x \end{array}$$

The amount of the second payment is \$5,644.16.

NOTE

It does not matter what focal date you choose because two values that are equal when moved to one date in common will still be equal when both are moved together to another date. But you should simplify your calculations by selecting a focal date corresponding to the date of an unknown variable. It is always easier to move actual amounts of money than it is to move an unknown variable x . For example, in the previous question, suppose you selected today as the focal date (instead of 2 years). Then the x at 2 years would need to be moved to today (the focal date). Regardless of the chosen focal date, the final answer will be the same.

TRY IT

Loan payments of \$5,000 due two years ago and \$3,000 due today have not been made. Instead, the payments are rescheduled with two payments of \$1,500 in 1 year and 2.5 years, and a final payment in 4 years. What is the size of the final payment if money earns 3.7% compounded semi-annually?

Click to see Solution

Using 4 years as the focal date:

N	12	8	6	3
PV	−5,000	−3,000	−1,500	-1,500
FV	Formula does not parse	Formula does not parse	Formula does not parse	Formula does not parse
PMT	0	0	0	0
I/Y	3.7	3.7	3.7	3.7
P/Y	2	2	2	2
C/Y	2	2	2	2

$$6,230.21 + 3,473.84 = 1,674.39 + 1,584.80 + x$$

$$9,704.05 = 3,259.19 + x$$

$$9,704.05 - 3,259.19 = x$$

$$\$6,444.86 = x$$

EXAMPLE

A \$10,000 loan is to be repaid with three equal payments made in 1 year, 3.5 years and 6 years. If money earns interest at 4.7% compounded quarterly, what is the size of the payments?

Solution:

Step 1: Draw the timeline. The \$10,000 from the original payment stream is in black. The unknown payments in the new payments stream, represented by x , are in red.

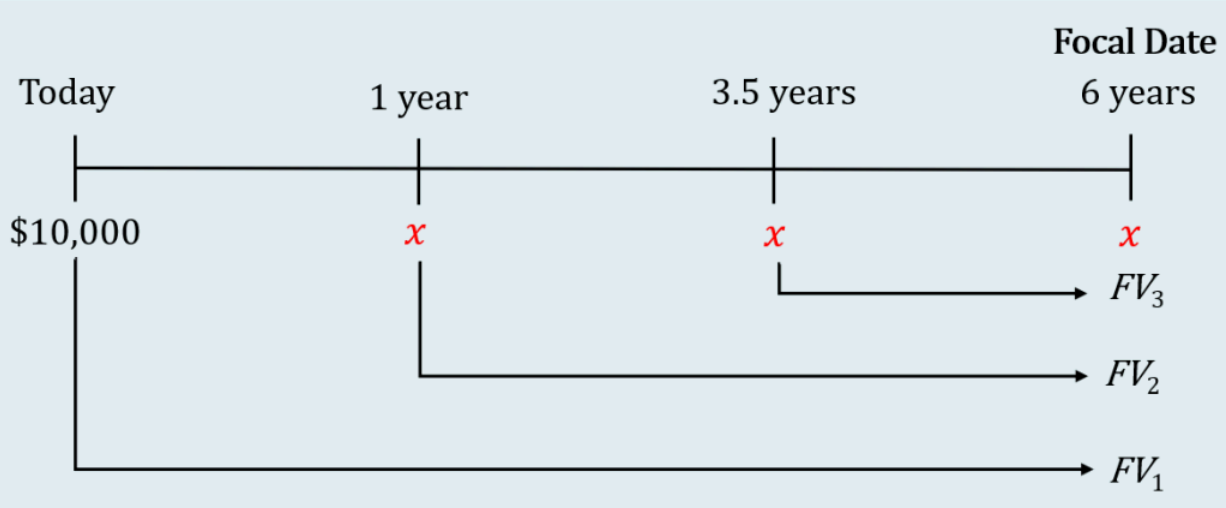


Figure 4.4.5

Step 2: Choose six years from today as the focal date.

Step 3: Calculate the future value of the \$10,000 payment today (in the original payment stream) at the focal date. Because this payment must move forward in time to get to the focal date, the \$10,000 is a present value and its value at the focal date is a future value. This payment must move 6 years (from today to 6 years from today) to get to the focal date, so the time for the future value calculation is 6 years.

N	$4 \times 6 = 24$
PV	$-10,000$
FV	?
PMT	0
I/Y	4.7
P/Y	4
C/Y	4

$$FV_1 = \$13,236.01$$

Step 4: Calculate the future value of the x at 1 year (in the new payment stream) at the focal date. Because this payment must move forward in time to get to the focal date, the x is a present

value and its value at the focal date is a future value. This payment must move 5 years (from 1 year from today to 6 years from today) to get to the focal date, so the time for the future value calculation is 5 years.

Because x cannot be entered in the calculator, you need to find an expression for the future value of x at the focal date. You can think of the unknown payment x as $1 \times x$. The future value of x is the **(future value of 1)** $\times x$. You can find the future value of 1 using the calculator by entering 1 for the present value and calculating out the future value.

N	$4 \times 5 = 20$
PV	-1
FV	Formula does not parse
PMT	0
I/Y	4.7
P/Y	4
C/Y	4

$$FV_2 = 1.2631... \times x$$

Step 5: Calculate the future value of the x at 3.5 year (in the new payment stream) at the focal date. Because this payment must move forward in time to get to the focal date, the x is a present value and its value at the focal date is a future value. This payment must move 2.5 years (from 3.5 years from today to 6 years from today) to get to the focal date, so the time for the future value calculation is 2.5 years.

Because x cannot be entered in the calculator, you need to find an expression for the future value of x at the focal date. You can think of the unknown payment x as $1 \times x$. The future value of x is the **(future value of 1)** $\times x$. You can find the future value of 1 using the calculator by entering 1 for the present value and calculating out the future value.

N	$4 \times 2.5 = 10$
PV	-1
FV	Formula does not parse
PMT	0
I/Y	4.7
P/Y	4
C/Y	4

$$FV_3 = 1.1239... \times x$$

Step 6: At the focal date, the sum of the payments in the original payment stream equal the sum of the payments in the new payment stream.

$$\begin{aligned} 13,236.01 &= x + (1.2631...) \times x + (1.1239...) \times x \\ &= (1 + 1.2631... + 1.1239...) \times x \\ \frac{13,236.01}{3.38708...} &= x \\ \$3,907.78 &= x \end{aligned}$$

The payments are \$3,907.78.

NOTES

1. You can choose any of the dates on the timeline in the previous example as the focal date. A different focal date will require different calculations to move the payments to the focal date, but the final answer will be the same.
2. Keep all of the decimal places throughout the calculation, especially for the future or present value expressions for any unknown payments x , to avoid any round-off error in the final answer.
3. To move an unknown payment x to the focal date, find the value of 1 at the focal date and then multiply this value by x .

- If x moves forward in time to get to the focal date, you need to find the future value of x . To do this, find the future value of 1 and then the value of x at the focal date is **(future value of 1) $\times x$** .
- If x moves backward in time to get to the focal date, you need to find the present value of x . To do this, find the present value of 1 and then the value of x at the focal date is **(present value of 1) $\times x$** .

TRY IT

Payments of \$570 due 1.5 years ago and \$890 due 6 months ago have not been made. Instead, these payments are to be replaced with two equal payments due 9 months and 2 years from today. If money earns 5% compounded monthly, what is the size of the payments?

Click to see Solution

Using 2 years from today as the focal date:

N	42	30	15
PV	−5,70	−890	−1
FV	Formula does not parse	Formula does not parse	Formula does not parse
PMT	0	0	0
I/Y	5	5	5
P/Y	12	12	12
C/Y	12	12	12

$$678.76 + 1,008.24 = x + (1.06435...) \times x$$

$$1,678 = (1 + 1.06435...) \times x$$

$$1,678 = (2.06435...) \times x$$

$$\frac{1,678}{2.06435...} = x$$

$$\$817.21 = x$$

Exercises

1. If money earns 6% compounded monthly, what single payment due in 27 months is equivalent to payments of \$5,000 today and \$5,000 in three years?

Click to see Answer

\$10,501.28

2. If money earns 8.5% compounded quarterly, what single payment due in 2 years is equivalent to payments of \$4,385 due 1 year ago and \$6,000 due in 4 years?

Click to see Answer

\$10,714.59

3. If money earns 6.65% compounded monthly, what single payment due in 16 months is equivalent to payments of \$16,000 due today, \$8,000 due in 10 months, and \$19,000 due in 33 months?

Click to see Answer

\$43,045.16

4. A \$35,000 loan taken out today is to be repaid with a \$15,000 payment due in 6 months and a second unknown payment due in 2 years. If money earns 5.95% compounded semi-annually, find the size of the unknown payment due in 2 years.

Click to see Answer

\$22,975.60

5. A \$51,000 loan borrowed a today is to be repaid with a \$15,000 payment due in 3 months, a \$10,000 payment due in 1 year, and an unknown payment due in 1.5 years. If money earns 6.32% compounded quarterly, what is the size of the unknown payment?

Click to see Answer

\$29,488.31

6. A \$44,000 loan borrowed today is repaid with two equal payments due in 6 months and 15 months. If money earns 5.55% compounded monthly, what is the size of the payments.

Click to see Answer

\$23,087.18

7. If money earns 8.88% compounded monthly, what two equal payments, one due today and one due in 18 months would be equivalent to two equal payments of \$2,500, one due 1 year ago and one due in 6 months?

Click to see Answer

\$2,731.26

8. Payments of \$38,000 due 1.5 years ago, \$17,000 due 6 months ago and \$45,000 due in 1 year are replaced with three equal payments due today, in 18 months and in 2 years. If money earns 9.5% compounded semi-annually, find the size of the payments.

Click to see Answer

\$37,952.43

9. Payments of \$3,750 due 2.5 years ago, \$2,400 due in 3.25 years and \$1,950 in 5 years are replaced with three equal payments due in 1 year, 2 years and 4.5 years. If money earns 10.75% compounded quarterly, find the size of the payments.

Click to see Answer

\$3,322.29

10. A winning lottery ticket offers the following two options:

- Option A: A single payment of \$1,000,000 today.
- Option B: \$250,000 today followed by annual payments of \$300,000 for the next three years.

If money can earn 9% compounded annually, which option should the winner select? How much better is that option in current dollars?

Click to see Answer

Option B by \$9,388.40

11. Darwin is a young entrepreneur trying to keep his business afloat. He has missed two payments to a creditor. The first was for \$3,485 seven months ago and the second was for \$5,320 last month. Darwin has had discussions with his creditor, who is willing to accept \$4,000 one month from now and a second payment in full six months from now. If the agreed upon interest rate is 7.35% compounded monthly, what is the amount of the second payment?

Click to see Answer

\$5,201.24

12. James is a debt collector. One of his clients has asked him to collect an outstanding debt from one of its customers. The customer has failed to pay three amounts: \$1,600 18 months ago, \$2,300 nine months ago, and \$5,100 three months ago. In discussions with the customer, James finds she desires to clear up this situation and proposes a payment of \$1,000 today, \$4,000 nine months from now, and a final payment two years from now. The client normally charges 16.5% compounded quarterly on all outstanding debts. What is the amount of the third payment?

Click to see Answer

\$7,465.59

13. Working in the accounting department, Ariel has noticed that a customer is having trouble paying its bills. The customer has missed a payment of \$2,980 two years ago, \$5,150 14 months ago, and \$4,140 four months ago. The customer proposes making two equal payments in six months and in 18 months. If the agreed-upon interest rate is 6.89% compounded monthly, what are the amounts of each payment?

Click to see Answer

\$7,083.74

14. A \$30,000 loan at 4.9% compounded semi-annually is to be repaid with four equal semi-annual payments. The first payment is one year after the loan. Calculate the amount of each payment.

Click to see Answer

\$8,160.07

15. The Ontario Labour Relations Board is reviewing a human resource complaint. At the time of filing, a construction worker indicated that her employer failed to pay her monthly wages of \$3,400, \$3,200, \$3,600, and \$3,475 starting four months prior. It has taken the Ontario Labour Relations Board nine months since the time of filing to gather the needed information and make a judgment in favour of the complainant. If the standard interest rate used in its judgment is 9% compounded monthly, what amount is awarded to the construction worker?

Click to see Answer

\$14,900.01

16. Larry is a financial adviser helping a client save up \$100,000 in five years' time. The client has the financial means to pursue either of the following two alternatives:
- Option A: Make three equal deposits due today, in 2 years, and 4 years.
 - Option B: Make four equal deposits due today, in 1 year, in 3 years, and $4\frac{1}{2}$ years.

If Larry can invest the funds at 9% compounded semi-annually, which option is in the best interests of the client? In current dollars, how much better for the client is that recommended option?

Click to see Answer

Option A by \$735.14

17. Yi-Fang is the store owner of a franchise. In flipping through her records, she notices the following debts to the same supplier: \$2,389.56 due eight months ago, \$3,478.34 due six months ago, \$1,694.32 due four months ago, \$6,497.98 due two months ago, \$4,611.03 due today, \$5,784.39 due in two months, and \$5,177.44 due in four months. She would like to clear all of these debts with a single payment next month. If the supplier charges 18.1% compounded monthly on overdue balances and provides a credit of 9% compounded monthly on early payments, calculate the amount of the payment.

Click to see Answer

\$30,703.92

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4.5 CALCULATING THE INTEREST RATE

LEARNING OBJECTIVES

- Calculate the interest rate of a loan or investment.

You need to calculate the nominal interest rate under many circumstances including (but not limited to) the following.

- Determining the interest rate on a single payment loan.
- Understanding what interest rate is needed to achieve a future savings goal.
- Calculating the interest rate that generated a specific amount of interest.
- Finding a fixed interest rate that is equivalent to a variable interest rate.

Using a Financial Calculator

Although it is possible to find the interest rate by using the future value or present value formulas, it is much more practical to use a financial calculator. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is I/Y (the nominal interest rate). You must still load the other six variables into the calculator and apply the cash flow sign conventions carefully.

USING THE TI BAI PLUS CALCULATOR TO FIND THE INTEREST RATE FOR COMPOUND INTEREST

Enter values for the known variables (PV, FV, N, PMT, P/Y and C/Y) following the steps below and paying close attention to the cash flow sign convention for PV and FV.

- For the main button keys in the TVM row (i.e. N, I/Y, PV, PMT, FV), enter the number first and then press the corresponding button.
 - For example, to enter $N=34$, enter 34 on the calculator and then press N.
- For P/Y and C/Y, press 2nd I/Y. At the P/Y screen, enter the value for P/Y and then press ENTER. Press the down arrow to access the C/Y screen. At the C/Y screen, enter the value for C/Y and then press ENTER. Press 2nd QUIT (the CPT button) to exit the menu.
 - For example, to enter $P/Y=4$ and $C/Y=4$, press 2nd I/Y. At the P/Y screen, enter 4 and press ENTER. Press the down arrow. At the C/Y screen, enter 4 and press ENTER. Press 2nd QUIT to exit.

After all of the known quantities are loaded into the calculator, press CPT and then I/Y to solve for the interest rate.

NOTE

When entering both PV and FV into the calculator, ensure proper application of cash flow sign convention to PV and FV. One number must be negative and the other must be positive. An ERROR message will appear on the calculator display if PV and FV are entered with the same signs (i.e. both are negative or both are positive).

EXAMPLE

When Sandra borrowed \$7,100 from Sanchez, she agreed to reimburse him \$8,615.19 three years from now including interest compounded quarterly. What nominal quarterly compounded rate of interest is being charged?

Solution:

The timeline for the loan is shown below.

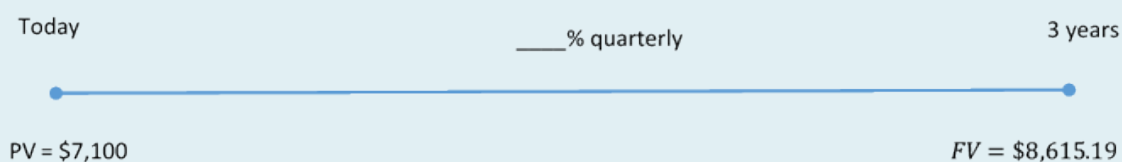


Figure 4.5.1

N	$4 \times 3 = 12$
PV	7,100
FV	−8,615.19
PMT	0
I/Y	?
P/Y	4
C/Y	4

$$I/Y = 6.5\%$$

Sanchez is charging an interest rate of 6.5% compounded quarterly on the loan to Sandra.

N	$12 \times 5 = 60$
PV	15,000
FV	-21,799.42
PMT	0
I/Y	?
P/Y	12
C/Y	12

$$I/Y = 7.5\%$$

Taryn's investment in his RRSP earned 7.5% compounded monthly over the five years.

TRY IT

Your company paid an invoice five months late. If the original invoice was for \$6,450 and the amount paid was \$6,948.48, what monthly compounded interest rate is your supplier charging on late payments?

Click to see Solution

N	$12 \times \frac{5}{12} = 5$
PV	6,450
FV	−6,948.48
PMT	0
I/Y	?
P/Y	12
C/Y	12

$$I/Y = 18\%$$

TRY IT

At what monthly compounded interest rate does it take five years for an investment to double?

Click to see Solution

N	$12 \times 5 = 60$
PV	-1
FV	2
PMT	0
I/Y	$?$
P/Y	12
C/Y	12

$$I/Y = 13.94\%$$

Exercises

1. What is the interest rate compounded monthly if a \$101,000 loan is repaid 10 years later with a payment of \$191,981.42?

Click to see Answer

6.44%

2. You invested \$59,860.48 five and half years ago. Today the investment is worth \$78,500. What interest rate compounded semi-annually did your investment earn?

Click to see Answer

4.99%

3. If a \$5,000 investment grew to \$20,777.73 in five years, what interest rate compounded daily did the investment earn?

Click to see Answer

28.5%

4. In a civil lawsuit, a plaintiff was awarded damages of \$15,000 plus \$4,621.61 in interest for a period of $3\frac{1}{4}$ years. What quarterly compounded rate of interest was used in the settlement?

Click to see Answer

8.35%

5. Muriel just received \$4,620.01 including \$840.01 of interest as payment in full for a sum of money that was loaned 2 years and 11 months ago. What monthly compounded rate of interest was charged on the loan?

Click to see Answer

6.9%

6. At what monthly compounded interest rate does it take five years for an investment to double?

Click to see Answer

13.94%

7. In 2003, a home in Winnipeg was purchased for \$214,000. In 2011, the same home was appraised at \$450,000. What annually compounded rate of growth does this reflect?

Click to see Answer

9.74%

8. On October 1, 1975, the minimum wage in Manitoba was \$2.60 per hour. It rose to \$10 per hour by October 1, 2011. What is the annually compounded growth rate for minimum wage in Manitoba during this period?

Click to see Answer

3.81%

9. Jean-Luc's first month's gross salary in June 1994 was \$800. By June 2012 his monthly gross salary was

\$1,969.23 higher. What monthly compounded rate did his salary increase by over the period?

Click to see Answer

6.92%

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4.6 CALCULATING THE TERM

LEARNING OBJECTIVES

- Calculate the number of compounding periods.
- Calculate the term of a loan or investment.

How long will it take to reach a financial goal? At a casual get-together at your house, a close friend discusses saving for a 14-day vacation to the Blue Bay Grand Esmeralda Resort in the Mayan Riviera of Mexico upon graduation. The estimated cost from Travelocity.ca is \$1,998.94 including fares and taxes. He has already saved \$1,775 into a fund earning 8% compounded quarterly. Assuming the costs remain the same and he makes no further contributions, can you tell him how soon he will be basking in the sun on the beaches of Mexico?

Using a Financial Calculator

Although it is possible to find the number of compounding periods by using the future value or present value formulas, it is much more practical to use a financial calculator. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is N (the number of compoundings). You must still load the other six variables into the calculator and apply the cash flow sign conventions carefully.

USING THE TI BAI PLUS CALCULATOR TO FIND THE NUMBER OF COMPOUNDINGS FOR COMPOUND INTEREST

Enter values for the known variables (PV, FV, I/Y, PMT, P/Y and C/Y) following the steps below and paying close attention to the cash flow sign convention for PV and FV.

- For the main button keys in the TVM row (i.e. N, I/Y, PV, PMT, FV), enter the number first and then press the corresponding button.
 - For example, to enter $N=34$, enter 34 on the calculator and then press N.
- For P/Y and C/Y, press 2nd I/Y. At the P/Y screen, enter the value for P/Y and then press ENTER. Press the down arrow to access the C/Y screen. At the C/Y screen, enter the value for C/Y and then press ENTER. Press 2nd QUIT (the CPT button) to exit the menu.
 - For example, to enter $P/Y=4$ and $C/Y=4$, press 2nd I/Y. At the P/Y screen, enter 4 and press ENTER. Press the down arrow. At the C/Y screen, enter 4 and press ENTER. Press 2nd QUIT to exit.

After all of the known quantities are loaded into the calculator, press CPT and then N to solve for the number of compounding periods.

NOTE

When entering both PV and FV into the calculator, ensure proper application of cash flow sign convention to PV and FV. One number must be negative and the other must be positive. An ERROR message will appear on the calculator display if PV and FV are entered with the same signs (i.e. both are negative or both are positive).

Finding the Term

The value of N is the **number of compounding periods**. For example, $N=9$ means that there are 9 compounding periods. N does not represent the time period or term. Because, $N = C/Y \times t$, where t is the time in years, the number of years can be found by

$$\text{Number of Years} = \frac{N}{C/Y}$$

The whole number portion of the number of years (the part on the left-side of the decimal) represents the number of years. As needed, take the decimal number portion (the part on the right-side of the decimal point) and multiply it by 12 to convert it to months.

NOTES

1. To determine the time period, the value of N must be calculated first.
2. In compound interest, the value of N should **not** be rounded to an integer. Because it is possible to have a fraction of a compounding period, N is not rounded.
3. After calculating the number of months, the months are **rounded up** to the next month.

EXAMPLE

Convert each of the following number of years to years and months.

1. 8.25 years
2. 11.375985 years

Solution:

1. The number of years is 8. The number of months is $0.25 \times 12 = 3$. So 8.25 years is 8 years and 3 months.
2. The number of years is 11. The number of months is $0.375985 \times 12 = 4.51182$, which is rounded up to 5. So 11.375985 years is 11 years and 5 months.

EXAMPLE

Jenning Holdings invested \$43,000 at 6.65% compounded quarterly. A report from the finance department shows the investment is currently valued at \$67,113.46. How long has the money been invested?

Solution:

Step 1: Calculate the value of N .

N	?
PV	−43,000
FV	67,113.46
PMT	0
I/Y	6.65
P/Y	4
C/Y	4

$$N = 26.9999....$$

Step 2: Convert N to years and months.

$$\begin{aligned}
 \text{Number of Years} &= \frac{N}{C/Y} \\
 &= \frac{26.999...}{4} \\
 &= 6.74999... \\
 &\rightarrow 6 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.74999... \times 12 \\
 &= 8.999... \\
 &\rightarrow 9 \text{ months}
 \end{aligned}$$

Jenning Holdings has had the money invested for six years and nine months.

EXAMPLE

Tabitha estimates that she will need at least \$20,000 for her daughter's postsecondary education when she turns 18. If Tabitha is able to save up \$8,500, how far in advance of her daughter's 18th birthday would she need to invest the money at 7.75% compounded semi-annually? Answer in years and months.

Solution:

The timeline for the investment is shown below.

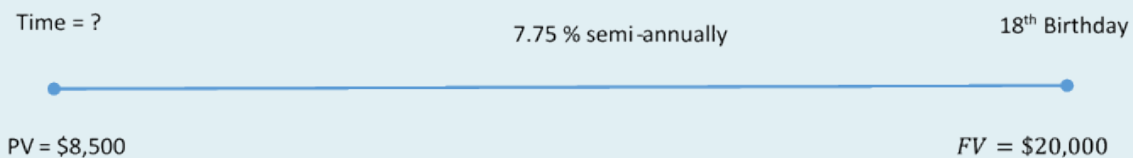


Figure 4.6.1

Step 1: Calculate the value of N.

N	?
PV	−8,500
FV	20,000
PMT	0
I/Y	7.75
P/Y	2
C/Y	2

$$N = 22.50682....$$

Step 2: Convert N to years and months.

$$\begin{aligned}
 \text{Number of Years} &= \frac{N}{C/Y} \\
 &= \frac{22.50682...}{2} \\
 &= 11.2534... \\
 &\rightarrow 11 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.2534... \times 12 \\
 &= 3.0409... \\
 &\rightarrow 4 \text{ months}
 \end{aligned}$$

If Tabitha invests the \$8,500 11 years and 4 months before her daughter's 18th birthday, it will grow to \$20,000.

TRY IT

You just took over another financial adviser's account. A client invested \$15,500 at 6.92% compounded monthly and now has \$24,980. How long (in years and months) has this client had the money invested?

Click to see Solution

N	?
PV	−15,500
FV	24,980
PMT	0
I/Y	6.92
P/Y	12
C/Y	12

$$N = 82.9959....$$

$$\begin{aligned}
 \text{Number of Years} &= \frac{N}{C/Y} \\
 &= \frac{82.9959...}{12} \\
 &= 6.91633... \\
 &\rightarrow 6 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.9163... \times 12 \\
 &= 10.995... \\
 &\rightarrow 11 \text{ months}
 \end{aligned}$$

The money has been invested for 6 years and 11 months.

Exercises

1. How long will it take an investment of \$68,000 to mature to at least \$89,032 if the interest rate is 4.91% compounded monthly?

Click to see Answer

5 years, 6 months

2. What is the term of an investment of \$41,790 that has a maturity value of \$120,000 at 8.36% compounded quarterly?

Click to see Answer

12 years, 9 months

3. How long did it take a \$111,244 investment to earn \$888,756 in interest if the interest rate is 8.8% compounded semi-annually?

Click to see Answer

25 years, 6 months

4. A debt of \$7,500 is owed. Suppose prevailing interest rates are 4.9% compounded quarterly. How far in advance was the debt paid if the creditor accepted a payment of \$6,721.25?

Click to see Answer

2 years, 4 months

5. Wayne was late in making a \$3,500 payment to Dora. If Dora accepted a payment of \$3,801 and charged 5.59% compounded semi-annually, how late was the payment?

Click to see Answer

1 year, 6 months

6. How long will it take \$5,800 to become \$10,000 at 6.25% compounded quarterly?

Click to see Answer

8 years, 10 months

7. A friend of yours just won the 6/7 category on the Lotto Max (matching six out of seven numbers), and her share of the prize was \$275,000. She wants to pay cash for a new home that sells for \$360,000. If she can invest the money at 7.45% compounded semi-annually, how long will she have to wait to purchase the home assuming its sale price remains the same?

Click to see Answer

3 years, 9 months

8. Lakewood Properties anticipates that the City of Edmonton in the future will release some land for a development that costs \$30 million. If Lakewood can invest \$17.5 million today at 9.5% compounded monthly, how long will it take before it will have enough money to purchase the land?

Click to see Answer

5 years, 9 months

9. As marketing manager, you want to pursue a new product development for which you require \$1 million for research. However, budgetary constraints mean you can only receive \$850,000. If you take your budget and invest it at 8.7% compounded monthly, how long will it be before you can pursue the necessary research for the project?

Click to see Answer

1 year, 11 months

10. How long will it take money to triple if it earns 8.2% compounded quarterly?

Click to see Answer

13 years, 7 months

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4.7 EFFECTIVE AND EQUIVALENT INTEREST RATES

LEARNING OBJECTIVES

- Calculate effective interest rates.
- Calculate equivalent interest rates.

How can you compare interest rates posted with different compounding? For example, let's say you are considering the purchase of a new home, so for the past few weeks you have been shopping around for financing. You have spoken with many banks as well as onsite mortgage brokers in the show homes. With semi-annual compounding, the lowest rate you have come across is 6.6%. In visiting another show home, you encounter a mortgage broker offering a mortgage for 6.57% compounded quarterly. You remember from your business math class that the compounding is an important component of an interest rate and wonder which one you should choose—6.6% compounded semi-annually or 6.57% compounded quarterly.

When considering interest rates on loans, you clearly want the best rate. If all of your possible loans are compounded in the same manner, selecting the best interest rate is a matter of picking the lowest number. However, when interest rates are compounded differently the lowest number may in fact not be your best choice. For investments, on the other hand, you want to earn the most interest. However, the highest nominal rate may not be as good as it appears depending on the compounding frequency.

To compare interest rates fairly and select the best, they all have to be expressed with the same compounding frequency. This section explains the concept of an effective interest rate, and you will learn to convert interest rates from one compounding frequency to a different frequency.

Effective Interest Rates

The **effective interest rate** is the interest rate compounded annually that has the same future value of a given present value for a fixed term as an interest rate compounded at some other (non-annual) frequency. In other words, the amount of interest accrued at the effective interest rate once in an entire year exactly equals the amount of interest accrued at the periodic interest rate successively compounded the stated number of times in a year. For example, if you invest \$1,000 at 10.25% effective (compounded annually) for one year you will have the same future value (\$1,102.50) as investing the \$1,000 for one year at 10% compounded semi-annually. Because 10.25% effective and 10% compounded semi-annually result in the same future value, these interest rates are called **equivalent**.

The corresponding effective interest rate (f) for a given nominal interest can be calculated using the following formula.

$$f = (1 + i)^m - 1$$

where

- f is the effective interest.
- i is the periodic interest rate for the given nominal interest rate.
- m is the compounding frequency for the given nominal interest rate.

EXAMPLE

Convert 10% compounded semi-annually to its corresponding effective interest rate.

Solution:

Step 1: Calculate the periodic interest rate.

$$\begin{aligned}
 i &= \frac{j}{m} \\
 &= \frac{10\%}{2} \\
 &= 5\% \\
 &= 0.05
 \end{aligned}$$

Step 2: Calculate the effective interest rate.

$$\begin{aligned}
 f &= (1 + i)^m - 1 \\
 &= (1 + 0.05)^2 - 1 \\
 &= 0.1025 \\
 &= 10.25\%
 \end{aligned}$$

USING THE TI BAI PLUS CALCULATOR TO FIND AN EFFECTIVE INTEREST RATE

The TI BAI Plus calculator has a built-in effective interest rate converter called ICONV. The interest rate converter involves three variables.

Variable	Meaning
NOM	Nominal interest rate, I/Y.
C/Y	Compounding frequency, C/Y.
EFF	Effective interest rate.

To find the effective interest rate:

1. Press 2nd ICONV (the number 2 key).
2. At the NOM screen, enter the nominal interest rate I/Y and press ENTER. The interest rate is entered as a percent (without the percent sign). For example 10% is entered as 10.

3. Press the up arrow.
4. At the C/Y screen, enter the compounding frequency for the nominal interest rate entered in the previous step and press ENTER.
5. Press the up arrow.
6. At the EFF screen, press CPT to calculate the effective interest rate.

The interest converter allows you to solve for any of these three variables, not just the effective interest rate.



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[businessfinancialmath/?p=78#oembed-1](https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=78#oembed-1)

[Nominal and Effective Rate Conversions](#) by Joshua Emmanuel [5:50] (transcript available).

EXAMPLE

Convert 10% compounded semi-annually to its corresponding effective interest rate.

Solution:

NOM	10
I/Y	2
EFF	?

1. Press 2nd ICONV.
2. At the NOM screen, enter 10 and press ENTER.
3. Press the up arrow.
4. At the C/Y screen, enter 2 and press ENTER.
5. Press the up arrow.
6. At the EFF screen, press CPT.

$$EFF = 10.25\%$$

EXAMPLE

You are offered loans at two different interest rates: 6.6% compounded semi-annually and 6.57% compounded quarterly. Which loan should you accept?

Solution:

Step 1: Convert 6.6% compounded semi-annually to its effective interest rate.

NOM	6.6
I/Y	2
EFF	?

$$EFF = 6.71$$

Step 2: Convert 6.75% compounded quarterly to its effective interest rate.

NOM	6.57
I/Y	4
EFF	?

$$EFF = 6.73$$

You should select the loan at 6.6% compounded semi-annually because it has the lower effective interest rate.

NOTES

1. To compare interest rates that compound at different frequencies, you must convert the rates so that they compound at the same frequency. As in the above example, both interest rates were converted to their effective interest rates and then their effective rates are compared.
2. When choosing an interest rate for a loan, you want to pay the least amount of interest. So, select the rate with the lowest effective interest rate.
3. When choosing an interest rate for an investment, you want to receive the most interest. So, select the rate with the highest effective interest rate.

TRY IT

If your investment earns 5.5% compounded monthly, what is the effective rate of interest?

Click to see Solution

NOM	5.5
I/Y	12
EFF	?

$$EFF = 5.64$$

TRY IT

As you search for a car loan, all banks have quoted you monthly compounded rates (which are typical for car loans), with the lowest being 8.4% compounded monthly. At your last stop, the credit union agent says that by taking out a car loan with them, you would be charged 8.65% effective. Should you go with the bank loan or the credit union loan?

Click to see Solution

Convert the bank's rate to its effective rate:

NOM	8.4
I/Y	12
EFF	?

$$EFF = 8.73$$

The credit union's 8.65% effective is the better choice because it is a lower effective rate than the bank's.

Equivalent Interest Rates

Equivalent interest rates are interest rates with different compounding frequencies that result in the same future value for the same given present value and term. For example, if you invest \$1,000 at 10.25% effective (compounded annually) for one year you will have the same future value (\$1,102.50) as investing the \$1,000 for one year at 10% compounded semi-annually. Because 10.25% effective and 10% compounded semi-annually result in the same future value, these interest rates are **equivalent**.

To convert a given nominal interest (j_1) to an equivalent nominal interest (j_2) for a given nominal interest use the following formula.

$$i_2 = \left(1 + i_1\right)^{\frac{m_1}{m_2}} - 1$$

where

- i_1 is the periodic interest rate for the given nominal interest rate j_1 .
- m_1 is the compounding frequency for the given nominal interest rate j_1 .
- m_2 is the compounding frequency for the new nominal interest rate j_2 .
- i_2 is the periodic interest rate for the new nominal interest rate j_2 .

Because this formula only gives the periodic interest rate for the new nominal interest rate, the nominal interest rate must be calculated from the periodic interest rate:

$$j_2 = m_2 \times i_2$$

USING THE TI BAI PLUS CALCULATOR TO FIND AN EQUIVALENT INTEREST RATE

To find an equivalent interest rate:

1. Press 2nd ICONV (the number 2 key).
2. At the NOM screen, enter the nominal interest rate I/Y for the old interest rate and press

ENTER. The interest rate is entered as a percent (without the percent sign). For example 10% is entered as 10.

3. Press the up arrow.
4. At the C/Y screen, enter the compounding frequency for the old interest rate entered in the previous step and press ENTER.
5. Press the up arrow.
6. At the EFF screen, press CPT to calculate the effective interest rate.
7. Press the down arrow.
8. At the C/Y screen, enter the compounding frequency for the new interest rate and press ENTER.
9. Press the down arrow.
10. At the NOM screen, press CPT to calculate the new interest rate.



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[Nominal and Effective Rate Conversions](#) by Joshua Emmanuel [5:50] (transcript available).

NOTE

Two equivalent interest rates have the same effective rate. The calculator uses this fact to find an equivalent rate. The first step to finding an equivalent rate on the calculator requires the

calculation of the effective rate for the old interest rate. Using this effective rate, the calculator finds the new interest rate.

EXAMPLE

What rate compounded quarterly is equivalent to 8% compounded monthly?

Solution:

Step 1: Convert 8% compounded monthly to its effective rate.

NOM	8.8
I/Y	12
EFF	?

$$EFF = 8.2999...\%$$

Step 2: Convert the effective rate from the previous step to the new interest rate compounding quarterly. Make sure that you keep all of the decimal places from the effective interest rate found in the previous step to avoid any round off error in calculating the new interest rate.

NOM	?
I/Y	4
EFF	8.2999....

$$NOM = 8.05\%$$

NOTE

When converting interest rates, the most common source of error lies in confusing the two values of the compounding frequency, or C/Y. When working through the steps, clearly distinguish between the old compounding that you want to convert from and the new compounding that you want to convert to. A little extra time spent on double-checking these values helps avoid mistakes.

EXAMPLE

You are looking at three different investments bearing interest rates of 7.75% compounded semi-annually, 7.7% compounded quarterly, and 7.76% compounded semi-annually. Which investment offers the highest interest rate?

Solution:

Notice that two of the three interest rates are compounded semi-annually while only one is compounded quarterly. Although you could convert all three to effective rates (requiring three calculations), it is easier to convert the quarterly compounded rate to a semi-annually compounded rate. Then all rates are compounded semi-annually and are therefore comparable.

Step 1: Convert 7.7% compounded quarterly to its effective rate.

NOM	7.7
I/Y	4
EFF	?

$$EFF = 7.9252...\%$$

Step 2: Convert the effective rate in the previous step to the new interest rate compounding semi-annually.

NOM	?
I/Y	2
EFF	7.9252...

$$NOM = 7.77\%$$

7.7% compounded quarterly is equivalent to 7.77% compounded semi-annually. In comparison to the semi-annually compounded rates of 7.75% and 7.76%, the 7.7% quarterly rate is the highest interest rate for the investment.

TRY IT

What nominal interest rate compounded semi-annually would result in the same financial position as 9.5% compounded monthly?

Click to see Solution

NOM	9.5	Formula does not parse
C/Y	12	2
EFF	Formula does not parse	9.9247...

TRY IT

You are offered three different investment opportunities at three different interest rates: 6.2% compounded semi-annually, 6.19% compounded quarterly, and 6.07% compounded daily. Which investment should you pick?

Click to see Solution

Convert 6.09% compounded quarterly to a rate compounded semi-annually.

NOM	6.19	Formula does not parse
C/Y	4	2
EFF	Formula does not parse	6.335...

6.19% compounded quarterly is equivalent to 6.24% compounded semi-annually.

Convert 6.07% compounded daily to a rate compounded semi-annually.

NOM	6.07	Formula does not parse
C/Y	365	2
EFF	Formula does not parse	6.257...

6.07% compounded daily is equivalent to 6.16% compounded semi-annually.

Select the investment at 6.19% compounded quarterly.

Note: There are other ways to solve this problem. For example, all of the interest rates could be converted to rates that compound quarterly, or all the interest rates could be converted to their effective rates.

Exercises

1. For each of the following, find their effective rates.

- a. 4.75% compounded quarterly.
- b. 7.2% compounded monthly.
- c. 3.95% compounded semi-annually.

Click to see Answer

a. 4.84%, b. 7.44%, c. 3.99%

2. Convert 10% effective to the equivalent rate compounded semi-annually.

Click to see Answer

9.76%

3. Convert 12% effective to the equivalent rate compounded monthly.

Click to see Answer

11.39%

4. Convert 8% effective to the equivalent rate compounded quarterly.

Click to see Answer

7.77%

5. Convert 6% compounded monthly to the equivalent rate compounded semi-annually.

Click to see Answer

6.08%

6. Convert 4.5% compounded quarterly to the equivalent rate compounded monthly.

Click to see Answer

4.48%

7. Covert 8% compounded semi-annually to the equivalent rate compounded quarterly.

Click to see Answer

7.92%

8. The HBC credit card has a nominal interest rate of 26.44669% compounded monthly. What effective rate is being charged?

Click to see Answer

29.9%

9. What is the effective rate on a credit card that charges interest of 0.049315% per day?

Click to see Answer

19.72%

10. RBC offers two different investment options to its clients. The first option is compounded monthly while the latter option is compounded quarterly. If RBC wants both options to have an effective rate of 3.9%, what nominal rates should it set for each option?

Click to see Answer

3.832% compounded monthly, 3.844 compounded quarterly

11. Louisa is shopping around for a loan. TD Canada Trust has offered her 8.3% compounded monthly, Connexus Credit Union has offered 8.34% compounded quarterly, and ING Direct has offered 8.45% compounded semi-annually. Rank the three offers and show calculations to support your answer.

Click to see Answer

Rank	Company	Rate
1	ING	8.629%
2	TD	8.623%
3	Connexus	8.605%

12. Your three-year monthly compounded investment just matured and you received \$9,712.72, of which \$2,212.72 was interest. What effective rate of interest did you earn?

Click to see Answer

9%

13. The TD Emerald Visa card wants to increase its effective rate by 1%. If its current interest rate is 19.067014% compounded daily, what new daily compounded rate should it advertise?

Click to see Answer

19.89%

14. Five investors reported the following results. Rank the effective rates of return realized by each investor from highest to lowest and show your work.

- **Investor 1:** \$4,000 principal earned \$1,459.10 of interest compounded monthly over four years.
- **Investor 2:** \$9,929.85 maturity value including \$1,429.85 of interest compounded quarterly for two years.
- **Investor 3:** \$14,750 principal maturing at \$19,370.83 after 3½ years of semi-annually compounded interest.
- **Investor 4:** \$3,194.32 of interest earned on a \$6,750 principal after five years of daily compounding.
- **Investor 5:** \$5,321.65 maturity value including \$1,421.65 of interest compounded annually over four years.

Click to see Answer

Investor	Rate
3	8.098%
1	8.085%
2	8.084%
5	8.08%
4	8.057%

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4.8 REVIEW EXERCISES

1. If you invest \$10,000 at 7.74% compounded quarterly for 10 years, what is the maturity value? How much interest did the investment earn?

Click to see Answer

$$FV = \$21,524.50, I = \$11,524.50$$

2. Ford Motor Company is considering an early retirement buyout package for some employees. The package involves paying out today's fair value of the employee's final year of salary. Shelby is due to retire in one year. Her salary is at the company maximum of \$72,000. If prevailing interest rates are 6.75% compounded monthly, what buyout amount should Ford offer to Shelby today?

Click to see Answer

$$PV = \$67,313.13$$

3. Polo Park Bowling Lanes owes the same supplier \$3,000 today and \$2,500 one year from now. The owner proposed to pay both bills with a single payment four months from now. If interest rates are 8.1% compounded monthly, what amount should the supplier be willing to accept?

Click to see Answer

$$\$5,450.83$$

4. What is the effective rate of interest on your credit card if you are being charged 24.5% compounded daily?

Click to see Answer

$$27.7516\%$$

5. Your 2½-year investment of \$5,750 just matured for \$6,364.09. What weekly compounded rate of interest did you earn?

Click to see Answer

4.0604% compounded weekly

6. Vienna just paid \$9,567.31 for an investment earning 5.26% compounded semi-annually that will mature for \$25,000. What is the term of the investment (in years and months)?

Click to see Answer

18 years, 6 months

7. Merryweather's union just negotiated a new four-year contract with her employer. The terms of the contract provide for an immediate wage increase of 3.3%, followed by annual increases of 3.5%, 4.25%, and 2.75%. If she currently earns \$61,500, what will her salary be in the final year of the contract?

Click to see Answer

FV = \$70,432.59

8. Bronco's four-year investment just matured at \$26,178.21. If the investment earned semi-annually compounded interest rates of 4.5% and 4.75% in the first two years, followed by monthly compounded interest rates of 5% and 5.1% in the last two years, how much money did Bronco initially invest? How much interest did the investment earn?

Click to see Answer

PV = \$21,600, I=\$4,578.21

9. Jay's Pharmacy owes the same creditor two debts of \$6,000 due one year ago and \$7,500 due in two years. Jay has proposed making two alternative payments of \$10,000 due in three months and a final payment in $2\frac{1}{2}$ years. If the creditor is agreeable to this proposal and wants an interest rate of 9% compounded quarterly, what is the amount of the final payment? Use $2\frac{1}{2}$ years as the focal date.

Click to see Answer

\$3,817.05

10. Over a 10-year period, the Growth Fund of America had annual returns of 0.02%, 29.8%, 14.84%, 26.86%, 31.78%, 45.7%, 7.49%, -12.2%, -22.02%, and 32.9%. What fixed annually compounded rate of return did the fund realize over the 10 years?

Click to see Answer

13.5131% compounded annually

11. A sum of \$84,100 was invested for $2\frac{1}{2}$ years and matured at \$101,268 using quarterly compounding. What quarterly compounded interest rate was realized? What is this effectively?

Click to see Answer

7.5% compounded quarterly, 7.714% effective

12. A \$10,000 loan at 8.15% compounded quarterly is to be repaid by two equal payments due in 9 months and in 33 months. Determine the size of the payments.

Click to see Answer

\$5,739.43

13. Exactly how long will it take for your money to quadruple at 6.54% compounded monthly? Express answer in years, months, days.

Click to see Answer

21 years, 4 months

14. Six months ago Old Dutch Foods purchased some new machinery for a new product line that they just developed. The supplier agreed to three payments on the machinery of \$40,000 due today, \$85,000 due in six months, and \$75,000 due in 15 months. The new product line has not been as successful as initially planned, so Old Dutch Foods has proposed an alternative agreement involving three equal payments, each due at 3 months, 9 months, and 21 months. If the supplier is agreeable to this and wants an interest rate of 8.55% compounded monthly, determine the payments required in the proposed agreement.

Click to see Answer

\$67,978.25

15. Louisa owns a furniture store and decided to help a friend out by allowing him to purchase \$5,000 of furniture using her credit at 6.25% compounded semi-annually. The furniture loan is to be repaid in

four years. However, after $2\frac{1}{2}$ years Louisa can no longer have the loan outstanding and needs the money. Avco Financial has agreed to purchase the maturity amount of this loan from Louisa using a discount rate of 17.1% compounded monthly. If Louisa proceeds with selling the loan contract to Avco Financial, what sum of money can she expect to receive?

Click to see Answer

$$PV = \$4,957.60$$

16. Jack is considering alternative three-year investment proposals from different banks for his \$20,000 principal. RBC says that if he invests his money with them, he will have \$23,841.78 using quarterly compounding. CIBC indicates that he would have \$23,607.15 using monthly compounding. What nominal and effective rates are being offered by each bank?

Click to see Answer

RBC: 5.925 compounded quarterly, 6.0318% effective; CIBC: 5.54% compounded monthly, 5.6829% effective

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4.9 CHAPTER SUMMARY

Key Concepts

- **4.1 Compound Interest Terminology**
 - How compounding works.
 - How to calculate the periodic interest rate.
 - **4.2 Calculating the Future Value**
 - The basics of taking a single payment and moving it to a future date.
 - Moving single payments to the future when variables change.
 - **4.3 Calculating the Present Value**
 - The basics of taking a single payment and moving it to an earlier date.
 - Moving single payments to the past when variables change.
 - **4.4 Equivalent Payments**
 - The concept of equivalent payments.
 - The fundamental concept of time value of money.
 - The fundamental concept of equivalency.
 - Applying single payment concepts to loans and payments.
 - **4.5 Calculating the Interest Rate**
 - Solving for the nominal interest rate.
 - How to convert a variable interest rate into its equivalent fixed interest rate.
 - **4.6 Calculating the Term**
 - Figuring out the term when n is an integer.
 - Figuring out the term when n is a non-integer.
 - **4.7 Equivalent and Effective Interest Rates**
 - The concept of effective rates.
 - Taking any nominal interest rate and finding its equivalent nominal interest rate.
-

Glossary of Terms

- **Compound Interest.** A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more. Interest is periodically converted to principal

throughout a transaction, with the result that the interest itself also accumulates interest.

- **Compounding Frequency.** The number of compounding periods in a complete year.
 - **Compounding Period.** The amount of time that elapses between the dates of successive conversions of interest to principal.
 - **Discount Rate.** An interest rate used to remove interest from a future value.
 - **Effective Interest Rate.** The true annually compounded interest rate that is equivalent to an interest rate compounded at some other (non-annual) frequency.
 - **Equivalent Payment Streams.** Equating two or more alternative financial streams such that neither party receives financial gain or harm by choosing either stream.
 - **Equivalent Interest Rates.** Interest rates with different compounding that produce the same effective rate and therefore are equal to each other.
 - **Focal Date.** A point in time to which all monies involved in all payment streams will be moved using time value of money calculations.
 - **Fundamental Concept of Equivalency.** Two or more payment streams are equal to each other if they have the same economic value on the same focal date.
 - **Fundamental Concept of Time Value of Money.** All monies must be brought to the same focal date before any mathematical operations, decisions, or equivalencies can be determined.
 - **Nominal Interest Rate.** A nominal number for the annual interest rate, which is commonly followed by words that state the compounding frequency.
 - **Periodic Interest Rate.** The percentage of interest earned or charged at the end of each compounding period.
 - **Present Value Principle for Loans.** The present value of all payments on a loan is equal to the principal that was borrowed.
-

Formulas

• Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- m or C/Y = Number of compounds per year or compounding frequency
- i = Periodic interest rate
- j or I/Y = Nominal interest rate per year
- n or N = Total number of compound periods for the term
- f or EFF = Effective interest rate

- Formulas Used

- Periodic Interest Rate: $i = \frac{j}{m}$
- Total Number of Compounds: $n = m \times \text{time in years}$
- Future Value: $FV = PV \times (1 + i)^n$
- Present Value: $PV = FV \times (1 + i)^{-n}$
- Effective Interest Rate: $f = (1 + i)^m - 1$
- Equivalent Interest Rate: $i_2 = (1 + i_1)^{\frac{m_1}{m_2}} - 1$

Calculator

- Time Value of Money Functions

- The time value of money buttons are the five buttons located on the third row of your calculator.

Calculator Symbol	Characteristic	Data Entry Requirements
N	The number of compounding periods.	An integer or decimal number; no negatives.
I/Y	The nominal interest rate per year.	Percent format without the % sign (i.e., 7% is just 7).
PV	Present value or principal.	An integer or decimal number.
PMT	Used for annuity payment amounts (covered in Chapter 11) and is not applicable to lump-sum amounts; it needs to be set to zero for lump-sum calculations.	An integer or decimal number.
FV	Future value or maturity value.	An integer or decimal number.

To enter any information into any one of these buttons or variables, called loading the calculator, key in the information first and then press the appropriate button.

- The frequency function is logically placed above the I/Y button and is labeled P/Y. This function addresses compound interest frequencies, such as the compounding frequency. Access the

function by pressing 2nd P/Y to find the following entry fields, through which you can scroll using your arrow buttons.

Calculator Symbol	Characteristic	Data Entry Requirements
P/Y	Annuity payments per year (payment frequency is introduced in Chapter 11); when working with lump-sum payments and not annuities, the calculator requires this variable to be set to match the C/Y.	A positive, nonzero number only.
C/Y	Compounds per year (compounding frequency).	A positive, nonzero number only.

To enter any information into one of these fields, scroll to the field on your screen, key in the data, and press ENTER. When you enter a value into the P/Y field, the calculator will automatically copy the value into the C/Y field for you. If in fact the C/Y is different, you can change the number manually. To exit the P/Y window, press 2nd QUIT.

- To key in a question, you must load the calculator with six of the seven variables. To solve for the missing variable, press CPT followed by the variable.
- The cash flow sign convention is used for the PV, PMT, and FV buttons. If money leaves you, you must enter it as a negative. If money comes at you, you must enter it as a positive.



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[Compound Interest \(Present and Future Values\)](#) by Joshua Emmanuel [6:56] (transcript available).

• Interest Conversions

- To find the effective interest rate.
 1. Press 2nd ICONV (the number 2 key).
 2. At the NOM screen, enter the nominal interest rate I/Y and press ENTER. The interest rate is entered as a percent (without the percent sign). For example 10% is entered as 10.
 3. Press the up arrow.

4. At the C/Y screen, enter the compounding frequency for the nominal interest rate entered in the previous step and press ENTER.
 5. Press the up arrow.
 6. At the EFF screen, press CPT to calculate the effective interest rate.
- To find the equivalent rate.
 1. Find the effective interest rate for the original interest rate following the steps above.
 2. Press the down arrow.
 3. At the C/Y Screen, enter the new compounding frequency and press ENTER.
 4. Press the down arrow.
 5. At the NOM screen, press CPT to calculate the new interest rate.



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[Nominal and Effective Rate Conversions](#) by Joshua Emmanuel [5:50] (transcript available).

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PART V

ANNUITIES

Payments show up everywhere. For example, you make payments on bank loans or you make payments into your RRSP. An annuity is anything where you make regularly scheduled payments. Here are some real-world examples of annuities.

- Last week you completed the purchase of your \$150,000 starter home and finished all of the paperwork on your mortgage with the bank. You snagged a great mortgage rate of 5% compounded semi-annually, locking it in for a five-year term with monthly payments of \$872.41 (for the next 25 years) starting one month after you move in.
- You excitedly cruise to the Honda dealership to pick up your new \$22,475 Honda Civic Sport, on which you have a four-year lease. Pulling out your iPhone (for which you pay \$60 monthly), you ensure that you have enough funds in your chequing account to make the \$2,000 down payment today along with the first of your \$242.16 monthly lease payments at 2.9% compounded monthly.
- Driving away in your Civic, you stop in at Sleep Country Canada and purchase a queen size mattress for your new bedroom. You recall a TV ad promoting an \$895 Sealy mattress plus 12% taxes with no money down and 12 easy, interest-free monthly payments of \$83.53. It will be nice to have a comfortable bed to sleep in!

This chapter explores the concept of making regular payments toward savings goals or debt extinguishment. First, it introduces key payment concepts and reviews different types of payment plans. You will then calculate the future and present value of a stream of payments. You extend the application of the concepts further to solve for other variables such as the payment amount, the term, and the nominal interest rate.

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5.1 FUNDAMENTALS OF ANNUITIES

LEARNING OBJECTIVES

- Understand terminology associated with annuities.
- Classify annuities based on the frequency of the payments and the compounding frequency.

An **annuity** is a continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation. An **annuity payment** is the dollar amount of the equal periodic payment in an annuity environment. The payments are continuous, equal, periodic, and occur over a fixed time frame. If any one of these four characteristics is not satisfied, then the financial transaction fails to meet the definition of a singular annuity and requires other techniques and formulas to solve.

Annuity Terms

The **payment interval** is the interval of time between two consecutive payments. For example, if payments are made every month, then the payment interval is monthly and the time period between two successive payments is one month.

The **payment frequency** (P/Y) is the number of times payments are made every year. That is, the payment frequency is the number of payment periods in one year.

Payment Interval	Number of Times per Year Payments are Made	Payment Frequency
Annually	Once a year	1
Semi-annually	Twice a year/Every six months	2
Quarterly	Four times a year/Every three months	4
Monthly	Twelve times a year/Every month	12
Semi-monthly	24 times a year	24
Bi-weekly	26 times a year	26
Weekly	52 times a year	52
Daily	Every day/365 times a year	365

The **term or time period** (t) of an annuity is the length of time from the beginning of the first payment interval to the end of the last payment interval.

Annuity calculations require the **total number of payments during the term** (n). As with compound interest, to calculate the total number of payments, the time must be in years. If the term is not in years, the term must be converted to years. If the term is given in months, divide by 12 to convert the term to years. If the term is given in days, divide by 365 to convert the term to years.

$$\text{Total Number of Payments} = \text{Payment Frequency} \times \text{Time in Years}$$

$$n = P/Y \times t$$

EXAMPLE

For each of the following, calculate the number of payments.

1. \$200 deposited every month for five years.
2. \$750 deposited every quarter for three years and nine months.
3. \$300 loan payments semi-annually for 54 months.

Solution:

1. $n = P/Y \times t = 12 \times 5 = 60$
2. $n = P/Y \times t = 4 \times \frac{45}{12} = 15$
3. $n = P/Y \times t = 2 \times \frac{54}{12} = 9$

NOTE

Annuity calculations do **not** require the total number of compoundings and the calculation of n for an annuity does not involve the compounding frequency.

It now becomes critical to ensure the proper application of the cash flow sign convention on the calculator—failure to do so will result in an incorrect answer. For example, if you borrow money and then make annuity payments on it, you enter the present value (PV) as a positive (you received the money) while you enter the annuity payments as negatives (you paid the money to the bank). This results in future balances getting smaller and you owing less money. If you inadvertently enter the annuity payment as a positive number, this would mean you are borrowing more money from the bank so your future balance would increase and you would owe more money. These two answers are very different!

Types of Annuities

Annuities are classified in two ways—by the timing of the payment (at the end or beginning of the payment interval) and by whether or not the payment frequency and the compounding frequency are equal.

In an **ordinary annuity**, the payments are made at the end of the payment interval. The first payment occurs one interval after the beginning of the annuity and the last payment occurs on the same date as the end of the annuity. For example, if you took out a car loan today that requires you to make monthly payments to repay the loan, your first payment would occur one month from today at the end of the first payment interval and all of your subsequent payments would occur at the end of each month. Common applications of ordinary annuities include bank loans, mortgages, bonds, and Canada Pension Plan payments.

In an **annuity due**, the payments are made at the beginning of the payment interval. The first payment occurs

on the same date as the beginning of the annuity and the last payment occurs one payment interval before the end of the annuity. For example, if you rent an apartment, your rent payment forms an annuity due because the your rent is paid at the beginning of every month. Common applications of annuities due include any kind of lease, any kind of rental, membership dues, and insurance payments.

In a **simple annuity**, the frequency of the payments and the compounding frequency for the interest rate are equal. For example, an annuity with quarterly payments and an interest rate that compounds quarterly is a simple annuity.

In a **general annuity**, the frequency of the payments and the compounding frequency for the interest rate are not equal. For example, an annuity with quarterly payments and an interest rate that compounds monthly is a general annuity.

Altogether, there are four types of annuities—ordinary simple annuity, ordinary general annuity, simple annuity due, general annuity due. The table below summarizes the four types of annuities and their characteristics.

Annuity Type	Timing of Payments in a Payment Interval	Payment Frequency and Compounding Frequency	Start of Annuity and First Payment Same Date?	End of Annuity and Last Payment Same Date?
Ordinary Simple Annuity	End	Equal	No, first payment one interval later	Yes
Ordinary General Annuity	End	Unequal	No, first payment one interval later	Yes
Simple Annuity Due	Beginning	Equal	Yes	No, last payment one interval earlier
General Annuity Due	Beginning	Unequal	Yes	No, last payment one interval earlier

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5.2 FUTURE VALUE OF ANNUITIES

LEARNING OBJECTIVES

- Calculate the future value of an ordinary annuity.
- Calculate the future value of an annuity due.

The **future value of any annuity** equals the sum of the future values for all of the annuity payments when they are moved to the **end** of the last payment interval. For example, assume you will make \$1,000 contributions at the end of every year for the next three years to an investment earning 10% compounded annually. To find out how much money you have in your investment after the three years, you would apply the fundamental concept of the time value of money to move each payment amount to the future date (the focal date) and sum the values at the focal date to arrive at the future value.

Although you could use this technique to solve all future value of an annuity situations, the computations become increasingly cumbersome as the number of payments increases. In the above example, what if the person made quarterly contributions of \$250 for three years? That is 12 payments over three years, resulting in 11 separate future value calculations. Or if they made monthly payments, the 36 payments over three years would result in 35 separate future value calculations! Clearly, solving this would be tedious and time consuming—not to mention prone to error.

Future Value of Ordinary Annuities

The future value for an ordinary annuity is

$$FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

where

- **FV is the future value of the annuity.** The future value is the sum of the future value of all of the annuity payments at the end of the last payment interval.
- **PMT is the annuity payment.** The amount of money that is invested or paid after each payment interval.
- **i_2 is the periodic interest rate for the interest rate whose compounding frequency matches the payment frequency.** If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.
- **n is the total number of payments made during the annuity.** $n = P/Y \times t$ where P/Y is the payment frequency and t is the time in years.

For investment annuities, the amount of interest (I) earned by the annuity is

$$I = FV - n \times PMT$$

NOTES

1. In a simple annuity, the payment frequency and the compounding frequency are equal ($P/Y = C/Y$). In this situation the periodic interest rate $i_2 = \frac{j}{m}$ where j is the nominal interest rate given in the question and m is the compounding frequency (and the payment frequency).
2. In a general annuity, the payment frequency and the compounding frequency are not equal ($P/Y \neq C/Y$). In this situation, the given interest rate must first be converted to the equivalent interest rate where the new compounding frequency equals the payment frequency. Using the equivalent interest rate, calculate the periodic interest rate i_2 .

EXAMPLE

A financial adviser is reviewing one of her client's accounts. The client has been investing \$1,000 at the end of every quarter for the past 11 years in a fund that has averaged 7.3% compounded quarterly. How much money does the client have in his account today? How much interest did the client earn?

Solution:

The timeline shows the client's account.

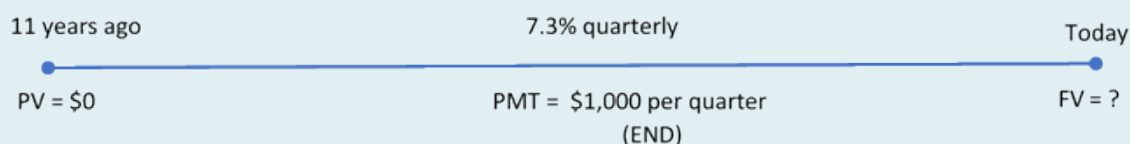


Figure 5.2.1

Step 1: The payments are at the end of the payment intervals, and both the compounding frequency and the payment frequency are the same (both quarterly). This is an ordinary simple annuity. Because this is a simple annuity, an interest rate conversion is not required.

The given information is

$$\begin{array}{l} \text{PMT} = \$1,000 \\ P/Y = 4 \\ I/Y = 7.3\% \\ C/Y = 4 \\ t = 11 \text{ years} \end{array}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned} i_2 &= \frac{I/Y}{C/Y} \\ &= \frac{7.3\%}{4} \\ &= 1.825\% \end{aligned}$$

Step 3: Calculate the number of payments.

$$\begin{aligned}
 n &= P/Y \times t \\
 &= 4 \times 11 \\
 &= 44
 \end{aligned}$$

Step 4: Calculate the future value.

$$\begin{aligned}
 FV &= PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right] \\
 &= 1,000 \times \left[\frac{(1 + 0.01825)^{44} - 1}{0.01825} \right] \\
 &= \$66,637.03
 \end{aligned}$$

Step 5: Calculate the interest earned.

$$\begin{aligned}
 I &= FV - n \times PMT \\
 &= 66,637.03 - 44 \times 1,000 \\
 &= \$22,637.03
 \end{aligned}$$

After 11 years, the client has \$66,637.03 in the account and has earned \$22,637.03 in interest.

Using a Financial Calculator

The above example was a fairly straightforward annuity question. But even this simple example, which did not require an interest conversion, is cumbersome, and time-consuming, to solve using the formula. A financial calculator can quickly solve annuity problems, with the added bonus of not requiring an interest conversion in situations where the payment frequency and compounding frequency are not equal.

USING THE TI BAI PLUS CALCULATOR TO FIND THE FUTURE VALUE FOR ORDINARY ANNUITIES

The time value of money buttons are located in the TVM row (the third row from the top) of the calculator. The five buttons located on the third row of the calculator are five of the seven variables

required for time value of money calculations. This row's buttons are different in colour from the rest of the buttons on the keypad. The other two variables are in a secondary menu above the I/Y key and are accessed by pressing 2nd I/Y.

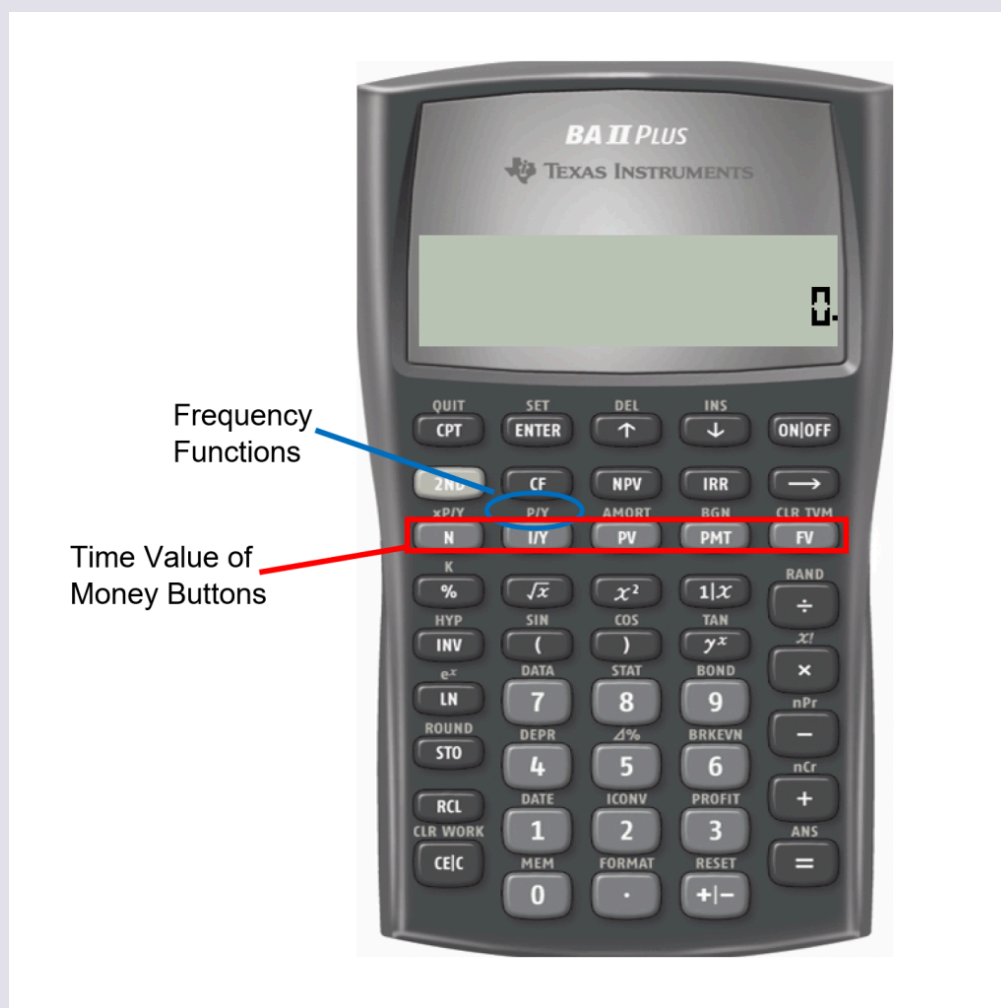


Figure 5.2.2

Altogether, there are seven variables required to complete time value of money calculations. Note that P/Y and C/Y are not main button keys in the TVM row. The P/Y and C/Y variables are located in the secondary function accessed by pressing 2nd I/Y.

Variable	Meaning
N	Total number of payments. This is the same value as n in the future value formula. $N = P/Y \times \text{time in years}$
I/Y	Interest rate per year (i.e. the nominal interest rate). The interest rate is entered in percent form (without the % sign). For example, 5% is entered as 5.
PV	Present value or principal.
PMT	Periodic annuity payment.
FV	Future value or maturity value.
P/Y	Payment frequency.
C/Y	Compounding frequency.

To enter values into the calculator:

- For the main button keys in the TVM row (i.e. N, I/Y, PV, PMT, FV), enter the number first and then press the corresponding button.
- For P/Y and C/Y, press 2nd I/Y. At the P/Y screen, enter the value for P/Y and then press ENTER. Press the down arrow to access the C/Y screen. At the C/Y screen, enter the value for C/Y and then press ENTER. Press 2nd QUIT (the CPT button) to exit the menu.

After all of the known quantities are loaded into the calculator, press CPT and then FV to solve for the future value.

NOTES

1. Be sure to enter PV, FV, and PMT with the correct cash flow sign convention. When you invest, PMT has the same sign as the PV (both negative). When you borrow, PMT has the opposite sign of PV (PMT is negative and PV is positive).
2. Because you can enter P/Y and C/Y, there is no need to do an interest conversion for a general annuity. You simply have to tell the calculator the P/Y and C/Y given in the question, and the calculator will handle any interest conversion internally.

3. If there is no money in the account or investment to start, then $PV=0$.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=125#oembed-1>

[Ordinary Annuity Calculations \(PV, PMT, FV\)](#) by Joshua Emmanuel [4:31] (transcript available).

EXAMPLE

A financial adviser is reviewing one of her client's accounts. The client has been investing \$1,000 at the end of every quarter for the past 11 years in a fund that has averaged 7.3% compounded quarterly. How much money does the client have in his account today?

Solution:

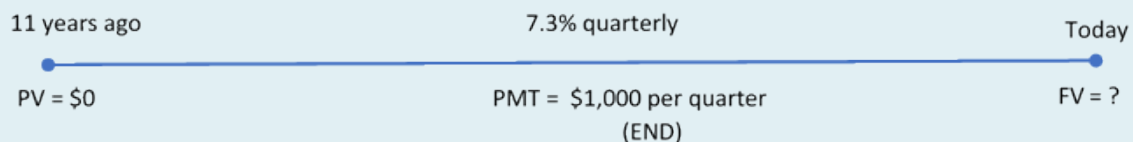


Figure 5.2.3

The timeline shows the client's account.

PMT Setting	END
N	$4 \times 11 = 44$
PV	0
FV	?
PMT	-1,000
I/Y	7.3
P/Y	4
C/Y	4

$$FV = \$66,637.03$$

After 11 years, the client has \$66,637.03 in the account.

TRY IT

You are a financial adviser. Your client is thinking of investing \$600 at the end of every six months for the next six years with the invested funds earning 6.4% compounded semi-annually. Your client wants to know how much money she will have after six years. What do you tell your client?

Click to see Solution

PMT Setting	END
N	$2 \times 12 = 24$
PV	0
FV	?
PMT	−600
I/Y	6.4
P/Y	2
C/Y	2

$$FV = \$8,612.62$$

EXAMPLE

A savings annuity already contains \$10,000. If an additional \$250 is invested at the end of every month at 9% compounded semi-annually for a term of 20 years, what will be the maturity value of the investment?

Solution:

The timeline is shown below.

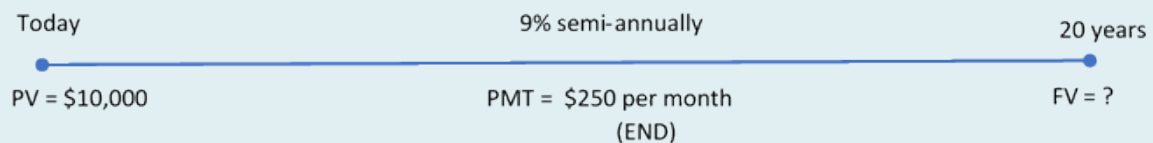


Figure 5.2.4

PMT Setting	END
N	$12 \times 20 = 240$
PV	$-10,000$
FV	?
PMT	-250
I/Y	9
P/Y	12
C/Y	2

$$FV = \$221,693.59$$

The savings annuity will have a balance of \$221,693.59 after the 20 years.

TRY IT

McDonald's major distribution partner, The Martin-Brower Company, needs at least \$1 million to build a new warehouse in Medicine Hat two years from today. To date, it has invested \$500,000. If it continues to invest \$50,000 at the end of every quarter into a fund earning 6% quarterly, will it have enough money to build the warehouse two years from now?

Click to see Solution

PMT Setting	END
N	$4 \times 2 = 8$
PV	$-500,000$
FV	?
PMT	$-50,000$
I/Y	6
P/Y	4
C/Y	4

$$FV = \$984,888.25$$

The fund is \$15,111.75 short of the \$1 million target.

If any of the variables, including I/Y, C/Y, PMT, or P/Y change between the starting point and ending point of the annuity, or if any additional single payment deposit or withdrawal is made, a new time segment is created and must be treated separately. In such cases, there will be multiple time segments that require you to work from left to right through the timeline in order to find the future value at the end of the annuity. The future value at the end of one time segment becomes the present value in the next time segment.

EXAMPLE

Genevieve has decided to start saving up for a vacation in two years, when she graduates from university. She already has \$1,000 saved today. For the first year, she plans on making end-of-month contributions of \$300 and then switching to end-of-quarter contributions of \$1,000 in the second year. If the account can earn 5% compounded semi-annually in the first year and 6% compounded quarterly in the second year, how much money will she have saved when she graduates?

Solution:

The timeline for her vacation saving appears below.

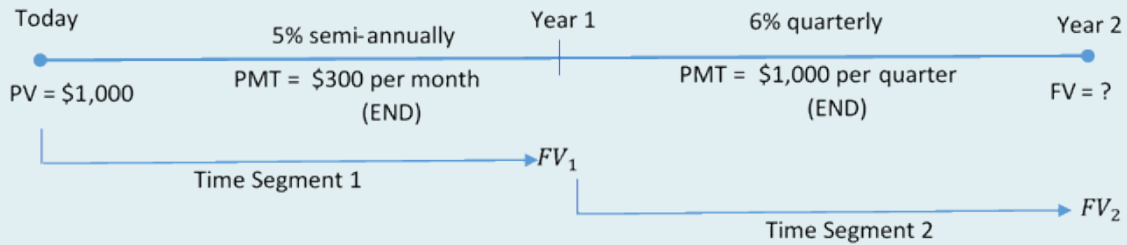


Figure 5.2.5

Step 1: Find the future value at the end of the first segment.

PMT Setting	END
N	$12 \times 1 = 12$
PV	$-1,000$
FV	?
PMT	-300
I/Y	5
P/Y	12
C/Y	2

$$FV_1 = \$4,733.411\dots$$

Step 2: Calculate the future value at the end of the second segment. The future value from the first segment becomes the present value for the second segment:

$$FV_1 = \$4,733.411\dots = PV_2.$$

PMT Setting	END
N	$4 \times 1 = 4$
PV	$-4,733.411\dots$
FV	?
PMT	$-1,000$
I/Y	6
P/Y	4
C/Y	4

$$FV_2 = \$9,114.77$$

When Genevieve graduates she will have saved \$9,114.77 toward her vacation.

TRY IT

Carlyle plans to make month-end contributions of \$400 to his RRSP from age 20 to age 40. From age 40 to age 65, he plans to make no further contributions to his RRSP. The RRSP can earn 9% compounded annually from age 20 to age 60, and then 5% compounded annually from age 60 to age 65. Under this plan, what is the maturity value of his RRSP when he turns 65?

Click to see Solution

PMT Setting	END	END	END
N	240	20	5
PV	0	−255,540.68...	1,432,254.94...
FV	Formula does not parse	Formula does not parse	Formula does not parse
PMT	−400	0	0
I/Y	9	9	5
P/Y	12	1	1
C/Y	1	1	1

$$FV = \$1,827,832.95$$

Future Value of Annuities Due

The future value for an annuity due is

$$FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$$

where

- **FV is the future value of the annuity.** The future value is the sum of the future value of all of the annuity payments at the end of the last payment interval.
- **i_2 is the periodic interest rate for the interest rate whose compounding frequency matches the payment frequency.** If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.
- **n is the total number of payments made during the annuity.** $n = P/Y \times t$ where P/Y is the payment frequency and t is the time in years.

For investment annuities, the amount of interest (I) earned by the annuity is

$$I = FV - n \times PMT$$

NOTES

1. The steps required to solve the future value of an annuity due are identical to those you use for an ordinary annuity except you use the formula for the future value of an annuity due.
2. If the payment setting is NOT specified in the question, it is assumed that the payments come at the end of the interval.

USING THE TI BAI PLUS CALCULATOR TO FIND THE FUTURE VALUE FOR ANNUITIES DUE

To adapt your calculator to an annuity due, you must toggle the payment setting from **END** to **BGN**. The calculator default is END, which is the ordinary annuity. The payment setting is found on the second shelf above the PMT key (because it is related to the PMT!).

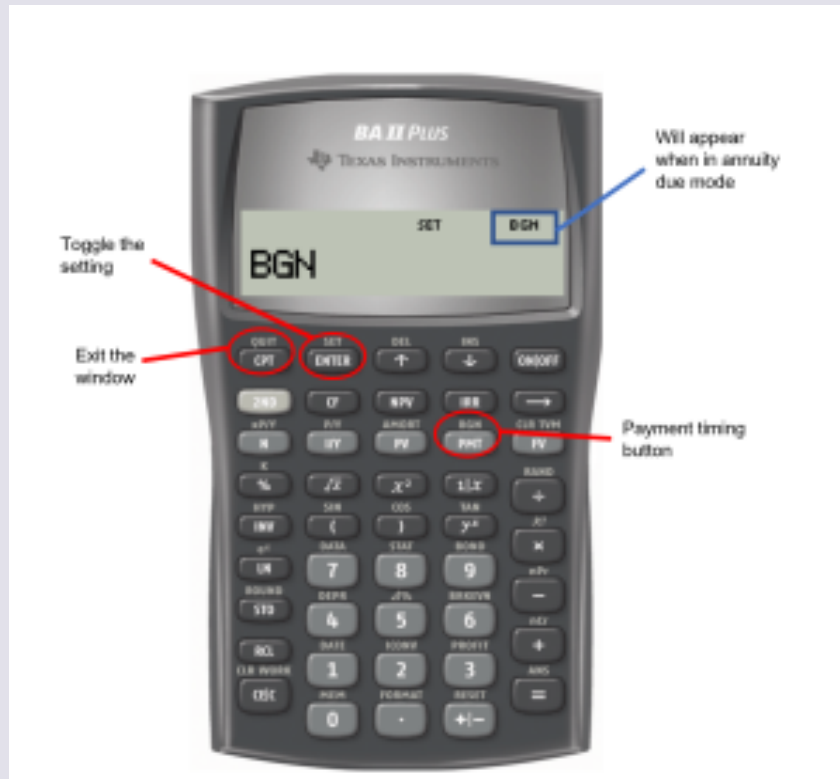


Figure 5.2.6

To change the payment setting, complete the following sequence.

1. Press 2nd BGN (the PMT button).
2. Press 2nd SET (the ENTER button).
3. Press 2nd QUIT (the CPT button).

When the calculator is in annuity due mode, a tiny BGN appears in the upper right-hand corner of your calculator. To return the calculator to ordinary mode, repeat the above keystrokes. When the calculator is in ordinary annuity mode there is nothing in the upper right-hand corner.

To calculate the future value of annuity due, make sure the calculator is in BGN mode. Then enter the values of N, I/Y, PV, PMT, P/Y and C/Y into the TVM money keys on the calculator and compute FV.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=125#oembed-2>

[businessfinancialmath/?p=125#oembed-2](https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=125#oembed-2)

[Annuity Due Calculations Using the BAII Plus](#) by Joshua Emmanuel [2:51] (transcript available).

EXAMPLE

The Set for Life instant scratch n' win ticket offers players a chance to win \$1,000 per week for the next 25 years starting immediately upon validation. If a winner was to invest all of his money into an account earning 5% compounded annually, how much money would he have at the end of his 25-year term? Assume each year has exactly 52 weeks.

Solution:

The timeline for the lottery savings is below.

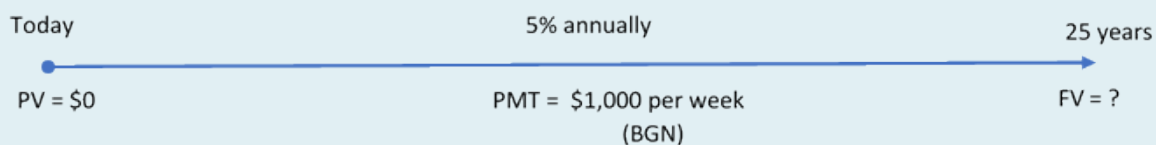


Figure 5.2.7

The payments start immediately, and the compounding period and payment intervals are different. Therefore, this is a general annuity due.

PMT Setting	BGN
N	$52 \times 25 = 1,300$
PV	0
FV	?
PMT	-1,000
I/Y	5
P/Y	52
C/Y	1

$$FV = \$2,544,543.22$$

If the winner was to invest all of his lottery prize money, he would have \$2,544,543.22 after 25 years.

TRY IT

The Saskatchewan Roughriders started a rainy-day savings fund three-and-a-half years ago to help pay for stadium improvements. At the beginning of every quarter the team deposited \$20,000 into the fund, which has been earning 4.85% compounded semi-annually. How much money is in the fund today? How much interest did the fund earn?

Click to see Solution

PMT Setting	BGN
N	$4 \times 3.5 = 14$
PV	0
FV	?
PMT	-20,000
I/Y	4.85
P/Y	4
C/Y	2

$$FV = \$306,680.93$$

$$\begin{aligned}
 I &= FV - n \times PMT \\
 &= 306,680.93 - 14 \times 20,000 \\
 &= \$26,680.93
 \end{aligned}$$

EXAMPLE

When Roberto's son was born, Roberto started making payments of \$1,000 at the beginning of every six months to a trust fund earning 5.75% compounded monthly. After five years, he changed his contributions and started depositing \$500 at the beginning of every quarter. How much money will be in his son's trust fund when his son turns 18?

Solution:

The timeline for the trust fund is shown below.

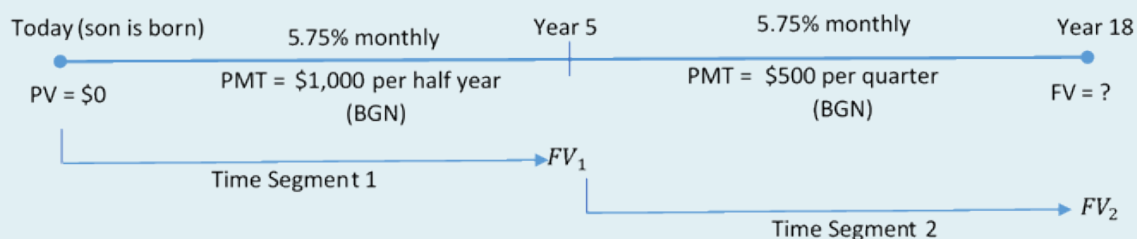


Figure 5.2.8

Step 1: Find the future value at the end of the first segment.

PMT Setting	BGN
N	$2 \times 5 = 10$
PV	0
FV	?
PMT	-1,000
I/Y	5.75
P/Y	2
C/Y	12

$$FV_1 = \$11,748.474\dots$$

Step 2: Calculate the future value at the end of the second segment. The future value from the first segment becomes the present value for the second segment:

$$FV_1 = \$11,748.474\dots = PV_2.$$

PMT Setting	BGN
N	$4 \times 13 = 52$
PV	$-11,748.474\dots$
FV	?
PMT	-500
I/Y	5.75
P/Y	4
C/Y	12

$$FV_2 = \$63,672.39$$

When Roberto's son turns 18, the trust fund will have a balance of \$63,672.39.

Exercises

1. You invest \$2,000 at the beginning of every quarter into a savings account earning 7% compounded quarterly. How much money is in your account in 10 years? How much interest did the account earn?

Click to see Answer

\$116,471.46, \$36,471.46

2. For 20 years, Joan deposits \$375 every quarter into her RRSP. If the RRSP earns 9% compounded monthly, how much money does Joan have in her RRSP at the end of the 20 years?

Click to see Answer

\$250,457.58

3. Your savings account currently has \$15,000. For the next 30 years, you plan to deposit \$3,000 at the end of every year. If the account earns 5.6% compounded quarterly, how much money will you have in the

account at the end of the 30 years?

Click to see Answer

\$305,305.23

4. Jesse's RRSP has a balance of \$38,000. Over the next 8 years, Jesse will invest \$1,500 at the beginning of every month. The RRSP earns 8% compounded semi-annually. How much money will be in Jesse's RRSP in 8 years?

Click to see Answer

\$272,152.25

5. You open a savings account to save money for a down payment on a house. You deposit \$1,000 every quarter for 10 years. For the first four years, the account earns 6% compounded annually. For the last six years, the account earns 7% compounded semi-annually. How much money can you save for your down payment? How much interest did you earn?

Click to see Answer

\$56,486.35, \$16,486.35

6. Phil invests \$100 at the beginning of every month into an RRSP for 10 years. For the first 6.5 years, the RRSP earns 12% compounded quarterly. For the last 3.5 years, the RRSP earns 11% compounded semi-annually. How much money is in Phil's RRSP at the end of the 10 years? How much interest did Phil's RRSP earn?

Click to see Answer

\$22,278.17, \$10,278.17

7. Nikola is currently 47 years old and planning to retire at age 60. She has already saved \$220,000 in her RRSP. If she continues to contribute \$200 at the beginning of every month, how much money will be in her RRSP at retirement if it can earn 8.1% compounded monthly? No deposit is made the day she turns 60.

Click to see Answer

\$683,712.33

8. The human resource department helps employees save by taking preauthorized RRSP deductions from employee paycheques and putting them into an investment. For the first five years, Margaret has had \$50 deducted at the beginning of every biweekly pay period. Then for the next five years, she increased the deduction to \$75. The company has been able to average 8.85% compounded monthly for the first seven years, and then 7.35% compounded monthly for the last three years. What amount has Margaret accumulated in her RRSP after 10 years? Assume there are 26 biweekly periods in a year.

Click to see Answer

\$24,035.26

9. Joshua is opening up a Builder GIC that allows him to make regular contributions to his GIC throughout the term. He will initially deposit \$10,000, then at the end of every month for the next five years he will make \$100 contributions to his GIC. The annually compounded interest rates on the GIC in each year are 0.75%, 1.5%, 2.5%, 4.5%, and 7.25%. What is the maturity value of his GIC?

Click to see Answer

\$18,452.55

10. How much more money would an individual who makes \$300 contributions to her RRSP at the beginning of the month have compared to another individual who makes \$300 contributions to his RRSP at the end of the month? Assume a term of 30 years and that both RRSPs earn 9% annually.

Click to see Answer

\$3680.30

11. Assume a 10-year ordinary annuity earning 10% compounded annually.
- If \$5,000 is deposited annually, what is the maturity value?
 - What is the maturity value if the deposits are doubled to \$10,000? Compared to (a), what is the relationship between the size of the deposit and the maturity value, all other conditions being held equal?
 - What is the maturity value if the \$5,000 deposits are made semi-annually? Compared to (b), what is the relationship between the frequency of payments and the maturity value, all other conditions being held equal?

Click to see Answer

a. \$79,687.12, b. \$159,377.24, c. \$163,263.68

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5.3 PRESENT VALUE OF ANNUITIES

LEARNING OBJECTIVES

- Calculate the present value of an ordinary annuity.
- Calculate the present value of an annuity due.

The **present value of any annuity** is equal to the sum of the present values of all the annuity payments when they are moved to the beginning of the first payment interval. For example, assume you will receive \$1,000 annual payments at the end of every payment interval for the next three years from an investment earning 10% compounded annually. How much money needs to be in the annuity at the start to make this happen? To find out how much money you need in the investment to sustain these payments, you would apply the fundamental concept of the time value of money to move each payment amount to the start of the first payment interval (the focal date) and sum the values to arrive at the present value.

Although you could use this technique to solve all present value of an annuity situations, the computations become increasingly cumbersome as the number of payments increases. In the above example, what if the person instead received quarterly payments of \$250 for three years? That is 12 payments over three years, resulting in 12 separate present value calculations. Or if they received monthly payments, the 36 payments over three years would result in 36 separate present value calculations! Clearly, solving this would be tedious and time consuming—not to mention prone to error.

Present Value of Ordinary Annuities

The present value for an ordinary annuity is

$$PV = PMT \times \left[1 - \frac{(1 + i_2)^{-n}}{i_2} \right]$$

where

- **PV is the present value of the annuity.** The present value is the sum of the present value of all of the annuity payments at the start of the first payment interval.
- **PMT is the annuity payment.** The amount of money that is invested or paid after each payment interval.
- **i_2 is the periodic interest rate for the interest rate whose compounding frequency matches the payment frequency.** If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.
- **n is the total number of payments made during the annuity.** $n = P/Y \times t$ where P/Y is the payment frequency and t is the time in years.

The amount of interest (I) earned by the annuity is

$$I = n \times PMT - PV$$

NOTES

1. In a simple annuity, the payment frequency and the compounding frequency are equal ($P/Y = C/Y$). In this situation the periodic interest rate $i_2 = \frac{j}{m}$ where j is the nominal interest rate given in the question and m is the compounding frequency (and the payment frequency).
2. In a general annuity, the payment frequency and the compounding frequency are not equal ($P/Y \neq C/Y$). In this situation, the given interest rate must first be converted to the equivalent interest rate where the new compounding frequency equals the payment frequency. Using the equivalent interest rate, calculate the periodic interest rate i_2 .

EXAMPLE

Rodriguez is planning on having an annual gross income of \$50,000 at the end of every year when he retires at age 65. He is planning for the account to be emptied by age 78, which is the average life expectancy for a Canadian man. If the account earns 5.1% compounded annually, how much money needs to be in the account when he retires? How much interest will he receive from the fund?

Solution:

The timeline for Rodriguez's account appears below.

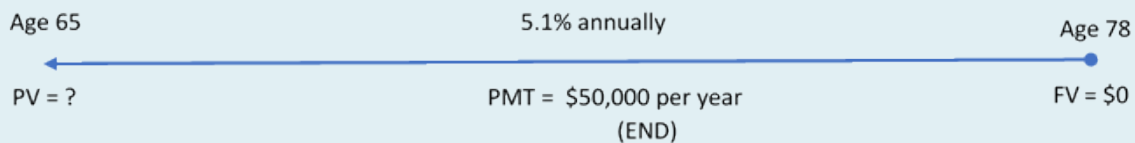


Figure 5.3.1

Step 1: The payments are at the end of the payment intervals, and both the compounding frequency and the payment frequency are the same (both annually). This is an ordinary simple annuity. Because this is a simple annuity, an interest rate conversion is not required.

The given information is

$$\begin{array}{l} \text{PMT} = \$50,000 \\ P/Y = 1 \\ I/Y = 5.1\% \\ C/Y = 1 \\ t = 13 \text{ years} \end{array}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned} i_2 &= \frac{I/Y}{C/Y} \\ &= \frac{5.1\%}{1} \\ &= 5.1\% \end{aligned}$$

Step 3: Calculate the number of payments.

$$\begin{aligned}
 n &= P/Y \times t \\
 &= 1 \times 13 \\
 &= 13
 \end{aligned}$$

Step 4: Calculate the present value.

$$\begin{aligned}
 PV &= PMT \times \left[1 - \frac{(1 + i_2)^{-n}}{i_2} \right] \\
 &= 50,000 \times \left[\frac{1 - (1 + 0.051)^{-13}}{0.051} \right] \\
 &= \$466,863.69
 \end{aligned}$$

Step 5: Calculate the interest earned.

$$\begin{aligned}
 I &= n \times PMT - PV \\
 &= 13 \times 50,000 - 466,863.69 \\
 &= \$183,136.31
 \end{aligned}$$

Rodriguez will need to have \$466,863.69 in his account when he turns 65 if he wants to receive 13 years of \$50,000 payments. He will receive \$183,136.31 in interest from the fund.

Present Value of Annuities Due

The present value for an annuity due is

$$PV = PMT \times (1 + i_2) \times \left[1 - \frac{(1 + i_2)^{-n}}{i_2} \right]$$

where

- **PV is the present value of the annuity.** The present value is the sum of the present value of all of the annuity payments at the start of the first payment interval.
- **i_2 is the periodic interest rate for the interest rate whose compounding frequency matches the payment frequency.** If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.

- **n is the total number of payments made during the annuity.** $n = P/Y \times t$ where P/Y is the payment frequency and t is the time in years.

The amount of interest (I) earned by the annuity is

$$I = n \times PMT - PV$$

NOTES

1. The steps required to solve the present value of an annuity due are identical to those you use for an ordinary annuity except you use the formula for the present value of an annuity due.
2. If the payment setting is NOT specified in the question, it is assumed that the payments come at the end of the interval.

EXAMPLE

Tom has a savings account that earns 6% compounded monthly. How much money does Tom need in his account today in order to receive beginning-of-month payments of \$250 for the next two years? How much interest does Tom receive?

Solution:

Step 1: The payments are at the beginning of the payment intervals, and both the compounding frequency and the payment frequency are the same (both monthly). This is a simple annuity due. Because this is a simple annuity, an interest rate conversion is not required.

The given information is

$$\begin{array}{l} PMT = \$250 \\ P/Y = 12 \\ I/Y = 6\% \\ C/Y = 12 \\ t = 2 \text{ years} \end{array}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned} i_2 &= \frac{I/Y}{C/Y} \\ &= \frac{6\%}{12} \\ &= 0.5\% \end{aligned}$$

Step 3: Calculate the number of payments.

$$\begin{aligned} n &= P/Y \times t \\ &= 12 \times 2 \\ &= 24 \end{aligned}$$

Step 4: Calculate the present value.

$$\begin{aligned} PV &= PMT \times (1 + i_2) \times \left[1 - \frac{(1 + i_2)^{-n}}{i_2} \right] \\ &= 250 \times \left[\frac{1 - (1 + 0.005)^{-24}}{0.005} \right] \\ &= \$5,668.92 \end{aligned}$$

Step 5: Calculate the interest earned.

$$\begin{aligned} I &= n \times PMT - PV \\ &= 24 \times 250 - 5,668.92 \\ &= \$331.08 \end{aligned}$$

Tom needs \$5,668.92 in his account to receive the 24 payments of \$250. He will receive \$331.08 in interest.

Using a Financial Calculator

The above examples were fairly straightforward annuity questions. But even these simple examples, which did not require interest conversions, are cumbersome, and time-consuming, to solve using the formulas. A financial calculator can quickly solve annuity problems, with the added bonus of not requiring an interest conversion in situations where the payment frequency and compounding frequency are not equal.

USING THE TI BAI PLUS CALCULATOR TO FIND THE PRESENT VALUE FOR ANNUITIES

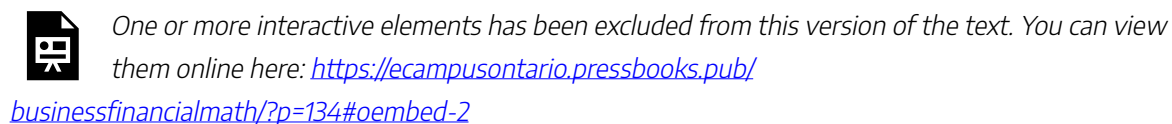
- Set the calculator to the correct payment setting (END or BGN).
- Enter values for the known variables (FV, PMT, N, I/Y, P/Y and C/Y), paying close attention to the cash flow sign convention for PMT and FV.
- After all of the known quantities are loaded into the calculator, press CPT and then PV to solve for the present value.

NOTES

1. Ensure that the calculator is set to the required payment setting. The present value will be different for END and BGN.
2. Because you can enter P/Y and C/Y, there is no need to do an interest conversion for a general annuity. You simply have to tell the calculator the P/Y and C/Y given in the question, and the calculator will handle any interest conversion internally.
3. If there is no money in the account or investment at the end of the term then $FV=0$.
4. The values you enter for PV, FV, and PMT must adhere to the cash flow sign convention.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=134#oembed-1>



[Annuity Due Calculations Using BAII Plus-Part 2](#) by Joshua Emmanuel [2:05] (transcript available).

Rodriguez is planning on having an annual gross income of \$50,000 at the end of every year when he retires at age 65. He is planning for the account to be emptied by age 78, which is the average life expectancy for a Canadian man. If the account earns 5.1% compounded annually, how much money needs to be in the account when he retires?

Age 65

5.1% annually

Age 78

PV = ?

PMT = \$50,000 per year
(END)

FV = \$0

The timeline for the Rodriguez's account appears below.

PMT Setting	END
N	$1 \times 13 = 13$
PV	?
FV	0
PMT	50,000
I/Y	5.1
P/Y	1
C/Y	1

$$PV = \$466,863.69$$

Rodriguez will need to have \$466,863.69 in his account when he turns 65 if he wants to receive 13 years of \$50,000 payments.

TRY IT

When Sinbad retires, he expects his RRSP to pay him \$2,000 at the end of every month for 25 years. If his retirement annuity earns 3.8% compounded quarterly, how much money does he need to have in his RRSP when he retires? How much interest will he receive from the RRSP?

Click to see Solution

PMT Setting	END
N	300
PV	?
FV	0
PMT	2,000
I/Y	3.8
P/Y	12
C/Y	4

$$PV = \$387,444.19$$

$$\begin{aligned}
 I &= n \times PMT - PV \\
 &= 300 \times 2,000 - 387,444.19 \\
 &= \$212,555.81
 \end{aligned}$$

When you purchase an expensive or large-ticket item such as a car or house, you often need to take out a loan to make the purchase. After all, most people do not have large sums of cash available to pay for these items outright. With these expensive items, you often make an up-front partial payment on the purchase price called a **down payment**, and then take out a loan for the outstanding purchase price.

For example, suppose you want to purchase a \$100,000 luxury sports car. You pay \$10,000 as a down payment and take out a loan for the balance. The \$10,000 down payment is cash you are paying up-front on the purchase price. You would then need to take out a loan for the remaining \$90,000 of the purchase price. The \$90,000 is a loan, so you would be charged interest for borrowing the money and then you would need to make regularly scheduled payments to pay back the loan. Because the \$10,000 down payment is paid up-front, there is no interest charged on this \$10,000. By making the \$10,000 down payment, you are reducing the amount of money you need to borrow, and consequently reducing the amount of interest you need to pay.

In general

$$\text{Purchase Price} = \text{Down Payment} + \text{Loan Amount}$$

EXAMPLE

Charlie purchased a new car. To pay for the car, she made a \$5,000 down payment on the purchase price and took out a loan for the balance at an interest rate of 2.9% compounded monthly. Charlie made \$550 monthly payments for six years to settle the loan.

1. What is the purchase price of the car?
2. How much interest did Charlie pay to purchase the car?

Solution:

Step 1: Calculate the purchase price of the car.

PMT Setting	END
N	$12 \times 6 = 72$
PV	?
FV	0
PMT	-550
I/Y	2.9
P/Y	12
C/Y	12

$$PV = \$36,306.08$$

$$\begin{aligned}
 \text{Purchase Price} &= \text{Down Payment} + \text{Loan Amount} \\
 &= 5,000 + 36,306.08 \\
 &= \$41,306.08
 \end{aligned}$$

Step 2: Calculate the interest. Because the \$5,000 is paid upfront, there is no interest charged on the down payment. Interest is only charged on the loan amount.

$$\begin{aligned} I &= n \times PMT - PV \\ &= 72 \times 550 - 36,306.08 \\ &= \$3,293.92 \end{aligned}$$

The purchase price of the car was \$41,306.08. Charlie paid \$3,293.92 in interest to purchase the car.

TRY IT

You want to purchase a piece of vacant land on the river so that you can build your dream home when you retire. The bank agrees to lend you the money but you have to make a \$25,000 down payment and make payments of \$800 every month for ten years. The bank charges you interest at 3.5% compounded semi-annually.

1. What is the purchase price of the land?
2. How much interest will you pay?

Click to see Solution

PMT Setting	END
N	120
PV	?
FV	0
PMT	−800
I/Y	3.5
P/Y	12
C/Y	2

$$PV = \$80,998.20$$

$$\begin{aligned}
 \text{Purchase Price} &= \text{Down Payment} + \text{Loan Amount} \\
 &= 25,000 + 80,998.20 \\
 &= \$105,998.20
 \end{aligned}$$

$$\begin{aligned}
 I &= n \times PMT - PV \\
 &= 120 \times 800 - 80,998.20 \\
 &= \$15,001.80
 \end{aligned}$$

EXAMPLE

Rodriguez is planning on having an annual gross income of \$50,000 at the beginning of every year when he retires at age 65. He wants to leave a \$100,000 inheritance for his children, assuming he dies at age 78. If the account earns 5.1% compounded semi-annually, how much money needs to be in the account when he retires?

Solution:

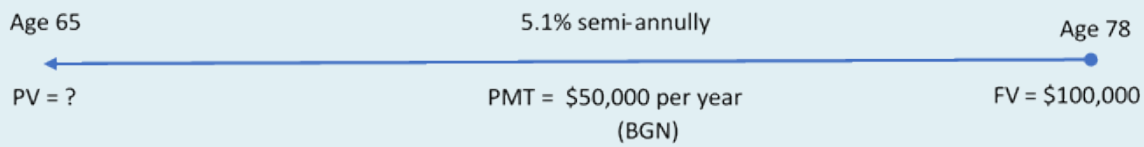


Figure 5.3.3

The timeline for the Rodriguez's account appears below.

PMT Setting	BGN
N	$1 \times 13 = 13$
PV	?
FV	100,000
PMT	50,000
I/Y	5.1
P/Y	1
C/Y	2

$$PV = \$541,027.07$$

Rodriguez will need to have \$541,027.07 in his account when he turns 65 if he wants to receive 13 years of \$50,000 payments and leave a \$100,000 inheritance for his children.

TRY IT

Sandy's parents would like to have an annuity pay her \$500 at the beginning of every month from September 1, 2012 to April 1, 2017 to help with her university tuition and living expenses. On May 1, 2017, they would like to give her a graduation gift of \$5,000. If the annuity can earn 6.15%

compounded quarterly, how much money must be in the account on September 1, 2012? (Use years and months in the calculations.)

Click to see Solution

PMT Setting	BGN
N	56
PV	?
FV	5,000
PMT	500
I/Y	6.15
P/Y	12
C/Y	4

$$PV = \$28,188.42$$

EXAMPLE

Rodriguez is planning on having an annual gross income of \$50,000 at the beginning of every year when he retires at age 65 until age 71. Realizing he needs to make some type of adjustment to his annual gross income to account for the rising cost of living, starting at age 71 he wants his annual payments to increase to \$60,000 until age 78. He wants to leave a \$100,000 inheritance for his children, assuming he dies at age 78. If the account earns 5.1% compounded semi-annually, how much needs to be in the account when he retires?

Solution:

The timeline for the Rodriguez's account appears below.

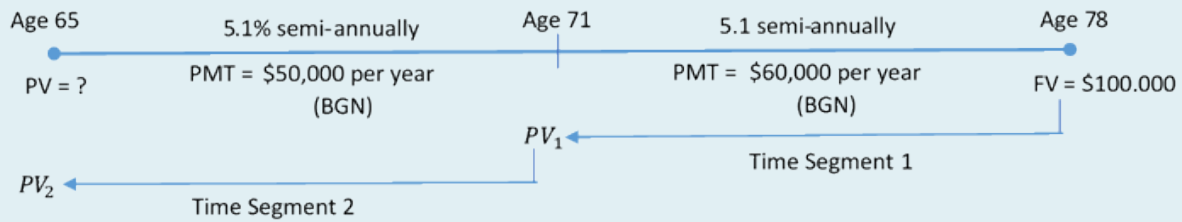


Figure 5.3.4

Step 1: Find the present value at the start of time segment 1.

PMT Setting	BGN
N	$1 \times 7 = 7$
PV	?
FV	100,000
PMT	60,000
I/Y	5.1
P/Y	1
C/Y	2

$$PV_1 = \$433,232.035\dots$$

Step 2: Calculate the present value at the start of time segment 2. The present value from time segment 1 becomes the future value for time segment 2: $PV_1 = \$433,232.035\dots = FV_2$.

PMT Setting	BGN
N	$1 \times 6 = 6$
PV	?
FV	433,232.035...
PMT	50,000
I/Y	5.1
P/Y	1
C/Y	2

$$PV_2 = \$585,742.42$$

Rodriguez needs to have \$585,742.42 invested in his retirement fund at age 65.

TRY IT

You want to purchase an annuity that will provide you with payments over the next twelve years. The annuity will pay you \$1,000 at the end of every quarter for the first seven years and \$3,000 at the end of every six months for the next five years. At the end of the twelve years the annuity is exhausted. If the annuity earns interest at 4.7% compounded monthly, what is the purchase price of the annuity? How much interest does the annuity pay?

Click to see Solution

PMT Setting	END	END
N	10	28
PV	Formula does not parse	Formula does not parse
FV	0	26,429.159...
PMT	3,000	1,000
I/Y	4.7	4.7
P/Y	2	4
C/Y	12	12

$$PV = \$42,756.53$$

$$I = 10 \times 3,000 + 28 \times 1,000 - 42,756.53 \\ = \$15,240.47$$

Exercises

1. You bought a new luxury sports car for \$5,000 down and payments of \$2,000 at the beginning of every quarter for 10 years. If money earns 7% compounded quarterly, what was the purchase price of the car? How much interest did you pay on the loan?

Click to see Answer

\$63,189.28, \$21,810.74

2. You have an investment that pays you \$3,000 every year for 30 years. How much money must you invest today at 5.6% compounded quarterly?

Click to see Answer

\$42,567.80

3. You took out a loan that requires you to pay \$1,500 every month for 8 years. How much money did you borrow if the interest on the loan is 8% compounded semi-annually? How much interest did you pay?

Click to see Answer

\$106,605.18, \$37,394.82

4. Your RIF pays you \$1,000 at the end of every quarter for ten years. For the first six years the RIF earns 7% compounded semi-annually and for the last four years the RIF earns 6% compounded annually. How much money is in your RIF today? How much interest does the RIF pay?

Click to see Answer

\$28,879.80, \$11,129.20

5. You want to purchase an annuity that will pay you \$1,000 every quarter for the first 10 years and then \$750 every month for the next 5 years. If the annuity earns 11% compounded quarterly, what is the purchase price of the annuity? How much interest does the annuity pay?

Click to see Answer

\$35,758.83, \$49,241.17

6. You invest some money into an investment at 10% effective that pays you \$4,000 a year for seven years. At the end of the seven years you take the balance in the investment and place it in a savings account at 8% effective and you will withdraw \$5,000 a year from the account for three years at which point the account is empty. How much money do you need to invest?

Click to see Answer

\$26,085.97

7. When Sinbad retires, he expects his RRSP to pay him \$2,000 at the end of every month for 25 years. If his retirement annuity earns 3.8% compounded quarterly, how much money does he need to have in his RRSP when he retires?

Click to see Answer

\$387,444.19

8. The Workers' Compensation Board has determined that an injury in the workplace was your company's responsibility. As a result, your company has been ordered to pay the employee \$3,000 at the end of every month for the next four years. Your human resource manager wants to set up an annuity to fund this obligation. If the proposed annuity can earn 5.7% compounded monthly for the first two-and-a-half years and then 6% compounded quarterly for the remaining one-and-a half years, how much money should your company set aside today to meet its responsibilities?

Click to see Answer

\$128,398.17

9. Lynne acquired a Sea Ray Sundancer boat and put \$4,000 down. For the past two years, her end-of-month payments have been \$1,049.01 including 9.32% compounded monthly. If she still owes \$22,888.78 today, what was the purchase price of the boat?

Click to see Answer

\$45,898.34

10. Gerald has been granted power of attorney and is now responsible for setting up his aging parents in a seniors' home. The rent will be \$2,490 at the beginning of every month for the first year, then increase by 5% the following year and 4% in the third year. Gerald wants to take money from his parents' estate and set up an annuity to pay their monthly rent. If he can get an annuity that earns 3.75% semi-annually, how much money from his parents' estate needs to be invested today to meet the rental payments over the next three years?

Click to see Answer

\$88,880.80

11. Compare the amount of money that needs to be invested today to provide the required payments from the investment fund annuities earning 9% compounded semi-annually.
- Payments of \$1,000 quarterly for 40 years.
 - Payments of \$1,600 quarterly for 25 years.
 - Payments of \$4,000 quarterly for 10 years.

Note that in all three of these annuities the same total payout occurs. Explain your results and comment on your findings.

Click to see Answer

a. \$43,610.57; b. \$63,942.03; c. \$105,221.32

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5.4 CALCULATING THE PAYMENT

LEARNING OBJECTIVES

- Calculate the payment for an annuity.

Whether you are acquiring merchandise, property or saving up toward some future goal such as retirement, you will deal with annuity payments. When you graduate college and land that promising entry-level position with your employer, a lot of demands are going to get placed on your limited income. If you do not already own a place of your own, perhaps you will get one. This means the purchase of a starter home for which you will make monthly mortgage payments. To fill that home, acquiring some furniture and electronics might take you to The Brick, Sleep Country Canada, Best Buy, or The Home Depot. Then you may be staggered by all the home maintenance items you need. If you make a lot of purchases all at once, you will probably take advantage of various payment plans. These place even more demands on your monthly income. Do not forget that you will need some wheels too. You can either lease or purchase a car. Great, another payment to make! Finally, you remember what your math instructor taught you about the importance of getting started early on your RRSP, so you should begin making those monthly contributions soon, too.

Similarly, businesses also make annuity payments for a wide variety of purposes, such as saving up for future corporate goals or acquiring products and property, businesses have regular bills, too. Marketers develop payment plans for their consumers. Financial agents make investments involving periodic payments. Companies issue marketable bonds that require regular interest payments to investors. Human resource personnel look after employee benefits, including RRSP contributions and pension plan payments. Production departments need expensive machinery, so they must keep payment plans within operating budgets. No matter your choice of profession, as a business manager you will encounter annuity payment calculations.

You need to calculate an annuity payment in many situation:

- Figuring out loan or mortgage payments.
- Determining membership or product payment plans.
- Calculating lease payments.
- Determining the periodic payment necessary to achieve a savings goal.
- Determining the maximum payment that an investment annuity can sustain over period of time.

Using a Financial Calculator

Recall that the annuity payment, PMT, is one of the variables in the future value and present value formulas for annuities. Calculating the payment requires you to determine which formula to use (depending on the type of annuity and whether you have the future value or present value), and then rearranging the formula to solve for PMT.

As we have seen, a financial calculator can quickly find the payment amount. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is PMT (the payment). You must still load the other six variables into the calculator, set the calculator to the correct payment setting, and apply the cash flow sign conventions carefully.

USING THE TI BAI PLUS CALCULATOR TO FIND AN ANNUITY PAYMENT

- Set the calculator to the correct payment setting (END or BGN).
- Enter values for the known variables (PV, FV, N, I/Y, P/Y and C/Y), paying close attention to the cash flow sign convention for PV and FV.
- After all of the known quantities are loaded into the calculator, press CPT and then PMT to solve for the payment.

NOTES

1. Ensure that the calculator is set to the required payment setting. The payment will be different for END and BGN.
2. The values you enter for PV and FV must adhere to the cash flow sign convention.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=140#oembed-1>

[Ordinary Annuity Calculations \(PV, PMT, FV\)](#) by Joshua Emmanuel [4:31] (transcript available).



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=140#oembed-2>

[BAIL Plus Down Payment Questions|PV and PMT](#) by Joshua Emmanuel [3:54] (transcript available).

EXAMPLE

Morgan wants to consolidate a lot of smaller debts into a single three-year loan for \$25,000. If the loan is charged interest at 7.8% compounded monthly, what is her payment amount at the end of every month?

Solution:

The timeline for the loan appears below.

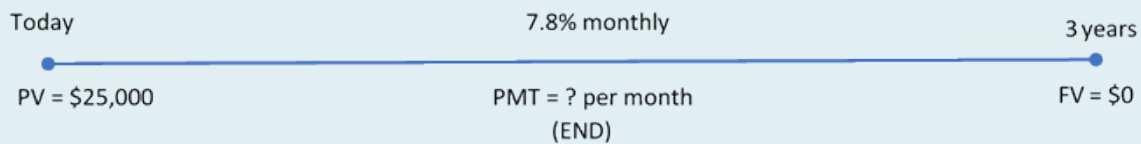


Figure 5.4.1

PMT Setting	END
N	$12 \times 3 = 36$
PV	25,000
FV	0
PMT	?
I/Y	7.8
P/Y	12
C/Y	12

$$PMT = \$781.10$$

To pay off her consolidated loan, Morgan's month-end payments for the next three years are \$781.10.

EXAMPLE

You plan to retire in 20 years. When you do, you want to have saved \$200,000 in an RRSP. To accomplish this goal, you will make quarterly payments into an RRSP that earns 3.1% compounded semi-annually.

1. Calculate the size of the quarterly payments.
2. How much interest does the RRSP earn?

Solution:

Step 1: Calculate the quarterly payments.

PMT Setting	END
N	$4 \times 20 = 80$
PV	0
FV	200,000
PMT	?
I/Y	3.1
P/Y	4
C/Y	2

$$PMT = \$1,816.29$$

Step 2: Calculate the interest.

$$\begin{aligned}
 I &= FV - n \times PMT \\
 &= 200,000 - 80 \times 1,816.29 \\
 &= \$54,696.80
 \end{aligned}$$

You need to make quarterly payments of \$1,816.29 to accomplish your goal. You will earn \$54,696.80 in interest over the 20 years.

EXAMPLE

Franco has placed \$10,000 into an investment fund with the goal of receiving equal amounts at the beginning of every month for the next year while he backpacks across Europe. If the investment fund can earn 5.25% compounded quarterly, how much money can Franco expect to receive each month?

Solution:

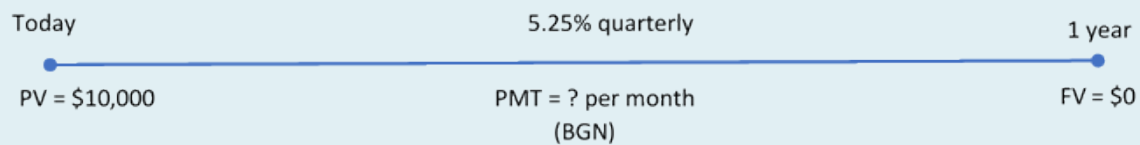


Figure 5.4.2

The timeline for the vacation money appears below.

PMT Setting	BGN
N	$12 \times 1 = 12$
PV	-10,000
FV	0
PMT	?
I/Y	5.25
P/Y	12
C/Y	4

$$PMT = \$853.40$$

While backpacking across Europe, Franco's annuity will pay him \$853.40 at the beginning of every month.

TRY IT

To save approximately \$30,000 for a down payment on a home four years from today, what amount needs to be invested at the end of every month at 4.5% compounded semi-annually?

Click to see Solution

PMT Setting	END
N	48
PV	0
FV	30,000
PMT	?
I/Y	4.5
P/Y	12
C/Y	2

$$PMT = \$572.08$$

EXAMPLE

Kingsley's financial adviser has determined that when he reaches age 65, he will need \$1.7 million in

his RRSP to fund his retirement. Kingsley is currently 22 years old and has saved up \$10,000 already. His adviser thinks that his RRSP will average 9% compounded annually throughout the years. To meet his RRSP goal, how much does Kingsley need to invest every month starting today?

Solution:

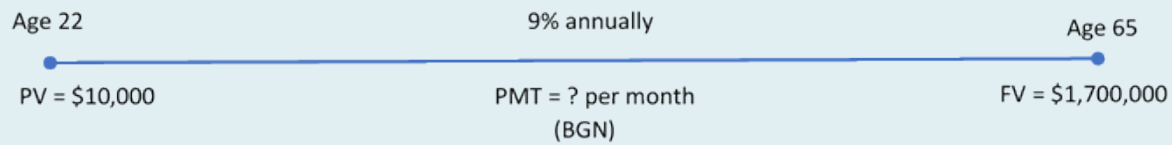


Figure 5.4.3

The timeline for Kingsley's RRSP contributions appears below.

PMT Setting	BGN
N	$12 \times 43 = 516$
PV	-10,000
FV	1,700,000
PMT	?
I/Y	9
P/Y	12
C/Y	1

$$PMT = \$233.24$$

To meet his retirement goals, Kingsley needs to invest \$233.24 at the beginning of every month for the next 43 years.

EXAMPLE

The production department just informed the finance department that in five years' time the robotic systems on the production line will need to be replaced. The estimated cost of the replacement is \$10 million. To prepare for this purchase, the finance department immediately deposits \$1,000,000 into a savings annuity earning 6.15% compounded semi-annually, and it plans to make semi-annual contributions starting in six months. How large do those contributions need to be?

Solution:

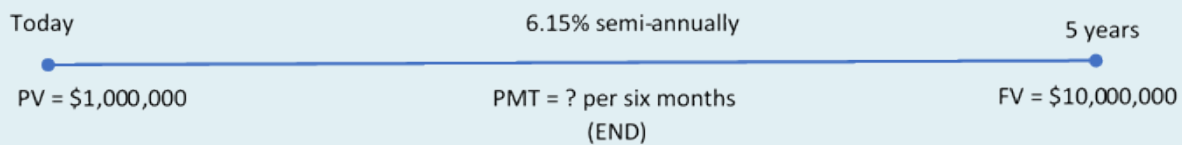


Figure 5.4.4

The timeline for the machinery fund appears below.

PMT Setting	END
N	$2 \times 5 = 10$
PV	-1,000,000
FV	10,000,000
PMT	?
I/Y	6.15
P/Y	2
C/Y	2

$$PMT = \$751,616.87$$

To have adequate funding for the production line machinery replacement five years from now, the finance department needs to deposit \$751,616.87 into the fund every six months.

TRY IT

Sinclair does not believe in debt and will only pay cash for all purchases. He has already saved up \$140,000 toward the purchase of a new home with an estimated cost of \$300,000. Suppose his investments earn 7.5% compounded monthly. How much does he need to contribute at the beginning of each quarter if he wants to purchase his home in five years?

Click to see Solution

PMT Setting	BGN
N	20
PV	−140,000
FV	300,000
PMT	?
I/Y	7.5
P/Y	4
C/Y	12

$$PMT = \$3,943.82$$

EXAMPLE

Over the next 15 years, you will invest \$900 every month into an RRSP that earns 4.3% compounded semi-annually. At the end of the 15 years, you plan to retire and will transfer the money in the RRSP to a RIF earning 2.9% compounded quarterly. Over the next 20 years, you will receive quarterly payments from the RIF. Calculate the size of the quarterly RIF payments.

Solution:

Step 1: Calculate the future value at the end of the RRSP.

PMT Setting	END
N	$12 \times 15 = 180$
PV	0
FV	?
PMT	-900
I/Y	4.3
P/Y	12
C/Y	2

$$FV = \$226,289.88\dots$$

Step 2: Calculate the payment for the RIF. The future value from the RRSP becomes the present value for the RIF: $PV = \$226,289.88\dots$

PMT Setting	END
N	$4 \times 20 = 80$
PV	$-226,289.88\dots$
FV	0
PMT	?
I/Y	2.9
P/Y	4
C/Y	4

$$PMT = \$3,737.74$$

You will receive quarterly payments of \$3,737.74 from the RIF.

EXAMPLE

When you retire, you want to receive beginning-of-month payments of \$750 for 30 years from a RIF earning 4.1% compounded quarterly. To plan for this, you want to make beginning-of-quarter payments for the next 25 years leading up to your retirement into your RRSP that earns 2.7% compounded semi-annually. What payments do you need to make to your RRSP?

Solution:

Step 1: Calculate the present value at the start of the RIF.

PMT Setting	BGN
N	$12 \times 30 = 360$
PV	?
FV	0
PMT	750
I/Y	4.1
P/Y	12
C/Y	4

$$PV = \$156,005.04...$$

Step 2: Calculate the payment for the RRSP. The present value from the RIF becomes the future value for the RRSP: $FV = \$156,005.04...$

PMT Setting	BGN
N	$4 \times 25 = 100$
PV	0
FV	156,005.04...
PMT	?
I/Y	2.7
P/Y	4
C/Y	2

$$PMT = \$1,091.40$$

You will need to make quarterly payments of \$1,091.40 into the RRSP.

TRY IT

The Kowalskis' only child is eight years old. They want to start saving into an RESP so that their son will be able to receive \$5,000 at the end of every quarter for four years once he turns 18 and starts attending postsecondary school. When the annuity is paying out, it is forecast to earn 4% compounded monthly. While they make contributions at the end of every month to the RESP, it will earn 8% compounded semi-annually. What is the monthly contribution payment by the Kowalskis?

Click to see Solution

PMT Setting	END	END
N	16	120
PV	Formula does not parse	0
FV	0	73,569.219...
PMT	5,000	Formula does not parse
I/Y	4	8
P/Y	4	12
C/Y	12	2

$$PMT = \$405.06$$

Exercises

1. In 5 years, Joan wants to have \$25,000 in her savings account. What beginning-of-month payments does

she need to make if the account earns 8.25% compounded monthly? How much interest does the account earn over the 5 years?

Click to see Answer

\$335.72, \$4,856.80

2. You have \$500,000 in an investment fund that earns 5.9% compounded semi-annually. Over the next 15 years, you will receive quarterly payments from the fund. Calculate the size of the payments. How much interest does the fund pay?

Click to see Answer

\$12,580.44, \$254,826.40

3. You retire today with \$1,000,000 saved up in your RRSP. You immediately transfer the funds to an RIF earning 4.75% compounded semi-annually. You will receive beginning-of-month payments from the RIF for 25 years. What is the size of the monthly payments you will receive?

Click to see Answer

\$5,652.40

4. How much does Alex need to deposit every year into his savings account if he wants to have \$1,500,000 in 35 years? The savings account earns interest at 9% compounded annually.

Click to see Answer

\$6,953.76

5. You purchase a \$58,000 car. You pay \$4,500 as a down payment and take out a loan for the balance at 3.65% quarterly. You repay the loan with monthly payments for 6.5 years. What is the size of your monthly loan payment? How much interest did you pay to buy the car?

Click to see Answer

\$771.25, \$6,657.50

6. Your business needs to save up for a major capital expense planned in 4 years that will cost \$5,000,000. Currently the business has \$450,000 saved in an investment fund earning 4.35% compounded monthly.

How much does the business need to deposit into the fund at the end of every year to have enough money saved for the project?

Click to see Answer

\$1,044,548.68

7. At age 60, Tiger has managed to save \$850,000 and decides to retire. He wants to receive equal payments at the beginning of each month for the next 25 years. The annuity can earn 5.4% compounded quarterly.
- If he plans on depleting the annuity, how much are his monthly payments?
 - If he wants to have \$50,000 left over at the end of the annuity, how much are his monthly payments?

Click to see Answer

a. \$5,133.93; b. \$5,054.93

8. To purchase his new \$50,000 car, Scooby-Doo pays \$10,000 down and obtains a six-year loan for the balance at 8.8% compounded semi-annually.
- Determine the monthly payments required on the loan.
 - How much interest does he pay on the loan?

Click to see Answer

a. \$713.95; b. \$11,404.40

9. Gold's Gym wants to offer its clients a monthly payment option on its annual membership dues of \$490. If the gym charges 7.75% compounded quarterly on its membership fee, what beginning-of-month payments should it advertise?

Click to see Answer

\$42.29

10. Carling Industries needs to acquire some real estate to expand its operations. In negotiations with the Province of Nova Scotia, it will be allowed to purchase the \$15 million parcel of land today and start making payments at the end of every six months for the next 10 years. If interest will be charged at 7.6% compounded semi-annually, what will be the required payments? (Round to the nearest dollar.)

Click to see Answer

\$1,084,268

11. You want to have saved \$100,000 in your savings account in 12 years. For the first 5 years, when the account earns 2.3% compounded monthly you make monthly payments of \$235. For the last 7 years the account earns 3.62% compounded quarterly. What monthly payments do you need to make during the last 7 years to accomplish your goal? How much interest does your savings account earn?

Click to see Answer

\$846.83, \$14,766.28

12. In ten years, your daughter will go to university. During the four years she is in university, you want the RESP you set-up for your daughter to pay her \$3,000 at the start of every six months. What beginning-of-month deposits do you need to make into the RESP over the next ten years if the RESP earns 3.89% compounded semi-annually?

Click to see Answer

\$153.16

13. For the next 20 years, you deposit \$1500 every quarter into your RRSP that earns 2.67% compounded semi-annually. At the end of the 20 years, you retire and convert the money saved in your RRSP to a RIF that earns 3.52% compounded quarterly. You want to receive monthly payments from the RIF for 25 years. What are the size of the monthly RIF payments?

Click to see Answer

\$790.63

14. Santana wants his retirement money to pay him \$3,000 at the beginning of every month for 20 years. He expects the annuity to earn 6.15% compounded monthly during this time. If his RRSP can earn 10.25% compounded annually and he contributes for the next 30 years, how much money does he need to invest into his RRSP at the end of every month? He has already saved \$15,000 to date.

Click to see Answer

\$62.65

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5.5 CALCULATING THE TERM OF AN ANNUITY

LEARNING OBJECTIVES

- Calculate the number of payments in an annuity.
- Calculate the term of an annuity.

When saving up for future goals, many people and businesses simply determine how much they can afford to invest each time period and then try to be patient until they meet their savings goal. What they do not know is how long it will take them. For example, if you can put \$75 per month into your vacation fund, how long will it take to save up the \$1,000 needed for a spring break vacation in Puerto Vallarta? How long it takes to fulfill the goal of your annuity depends on your annuity payment, interest rate, and the amount of money involved.

You must calculate the number of annuity payments, and the corresponding term, in a variety of scenarios such as:

- Savings planning.
- Debt extinguishment.
- Sustaining withdrawals from an investment.

Using a Financial Calculator

Although it is possible to find the number of payments by using the appropriate future value or present value formula, it is much more practical to use a financial calculator. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is N (the number of payments

in the annuity). You must still load the other six variables into the calculator and apply the cash flow sign conventions carefully.

USING THE TI BAI PLUS CALCULATOR TO FIND THE NUMBER OF PAYMENTS FOR AN ANNUITY

- Set the calculator to the correct payment setting (END or BGN).
- Enter values for the known variables (PV, FV, PMT, I/Y, P/Y and C/Y), paying close attention to the cash flow sign convention for PV, PMT, and FV.
- After all of the known quantities are loaded into the calculator, press CPT and then N to solve for the number of payments.

NOTE

Because N is the number of payments made during the annuity, N must be a whole number. However, it is unlikely that the calculation of N will result in a whole number. After N is calculated, the value of N must be rounded UP to the next whole number. For example, if $N=10.1397$ after working out the value of N on the calculator, then this value is rounded up to 11.

Finding the Term

The value of N is the **number of payment periods for the annuity**, after the value obtained from the calculator is rounded up to the next whole number. For example, $N=8.3174$ means that there are 9 payments in the annuity. N does not represent the time period or term. Because, $N = P/Y \times t$, where t is the time in years, the number of years can be found by

$$\text{Number of Years} = \frac{\text{rounded up value of } N}{P/Y}$$

The whole number portion of the number of years (the part on the left-side of the decimal) represents the number of years. As needed, take the decimal number portion (the part on the right-side of the decimal point) and multiply it by 12 to convert it to months.

NOTES

1. To determine the term of the annuity, the value of N must be calculated first.
2. In annuity calculations, the value of N is rounded UP to the next whole number **before** converting to years and months.
3. Because the value of N is rounded up to the next whole number, the last payment in the annuity will typically be smaller than the other payments. In a later section, you will learn how to calculate the size of the last payment.

EXAMPLE

Samia has \$500,000 accumulated in her retirement savings when she decides to retire at age 60. If she wants to receive beginning-of-month payments of \$3,000 and her retirement annuity can earn 5.2% compounded monthly, how old is Samia when the fund is depleted?

Solution:

The timeline for the retirement annuity appears below.

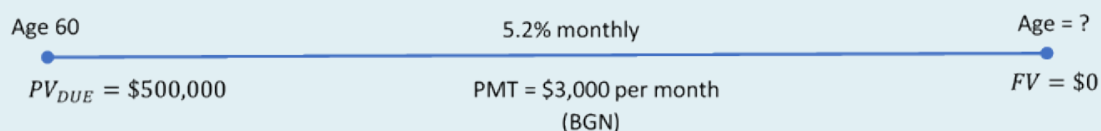


Figure 5.5.1

Step 1: Calculate the value of N .

PMT Setting	BGN
N	?
PV	−500,000
FV	0
PMT	3,000
I/Y	5.2
P/Y	12
C/Y	12

$$N = 293.6601\dots$$

Rounding up, Samia receives 294 monthly payments.

Step 2: Convert (rounded UP) N to years and months.

$$\begin{aligned} \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\ &= \frac{294}{12} \\ &= 24.5 \rightarrow 24 \text{ years} \\ \text{Number of Months} &= 0.5 \times 12 = 6 \end{aligned}$$

Samia can receive payments for 24 years and 6 months. If Samia is currently 60 years old and the annuity endures for 24 years and 6 months, then she will be 84.5 years old when the annuity is depleted.

EXAMPLE

Brendan is purchasing a brand new Mazda MX-5 GT. Including all options, accessories, and fees, the total amount he needs to finance is \$47,604.41 at the dealer's special interest financing of 2.4% compounded monthly. If he makes payments of \$1,000 at the end of every month, how long will it take to pay off his car loan?

Solution:

The timeline for the car payments appears below.

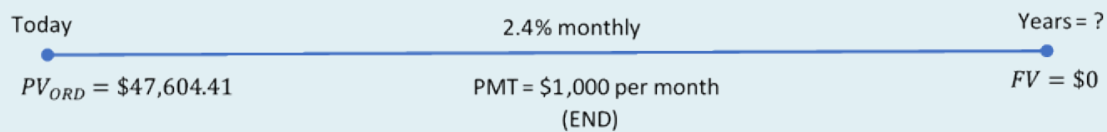


Figure 5.5.2

Step 1: Calculate the value of N .

PMT Setting	END
N	?
PV	47,604.41
FV	0
PMT	−1,000
I/Y	2.4
P/Y	12
C/Y	12

$$N = 50.0755...$$

Rounding up, Brendan makes 51 monthly payments.

Step 2: Convert (rounded UP) N to years and months.

$$\begin{aligned} \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\ &= \frac{51}{12} \\ &= 4.25 \\ &\rightarrow 4 \text{ years} \\ \text{Number of Months} &= 0.25 \times 12 \\ &= 3 \end{aligned}$$

To own his vehicle, Brendan will make payments for 4 years and 3 months.

EXAMPLE

Trevor wants to save up \$1,000,000. He will contribute \$2,500 every six months to an investment earning 5.2% compounded quarterly. How long will it take for Trevor to reach his goal?

Solution:

Step 1: Calculate the value of N .

PMT Setting	END
N	?
PV	0
FV	1,000,000
PMT	−2,500
I/Y	5.2
P/Y	2
C/Y	4

$$N = 94.436...$$

Rounding up, Trevor will need to make 95 payments.

Step 2: Convert (rounded UP) N to years and months.

$$\begin{aligned}\text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\ &= \frac{95}{2} \\ &= 47.5 \\ &\rightarrow 47 \text{ years}\end{aligned}$$

$$\begin{aligned}\text{Number of Months} &= 0.5 \times 12 \\ &= 6\end{aligned}$$

Trevor will need to make the semi-annual payments for 47 years and 6 months to reach his goal.

TRY IT

Amarjit wants to save up for a down payment on his first home. A typical starter home in his area sells for \$250,000 and the bank requires a 10% down payment. If he starts making \$300 month-end contributions to an investment earning 4.75% compounded monthly, how long will it take for Amarjit to have the necessary down payment?

Click to see Solution

PMT Setting	END
N	?
PV	0
FV	25,000
PMT	−300
I/Y	4.75
P/Y	12
C/Y	12

$$N = 72.1612... \rightarrow 73 \text{ payments}$$

$$\begin{aligned} \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\ &= \frac{73}{12} \\ &= 6.0833... \\ &\rightarrow 6 \text{ years} \end{aligned}$$

$$\begin{aligned} \text{Number of Months} &= 0.833... \times 12 \\ &= 1 \end{aligned}$$

It will take 6 years and 1 month.

TRY IT

Hi-Tec Electronics is selling a 52" LG HDTV during a special “no sales tax” event for \$1,995 with end

of month payments of \$100 including interest at 15% compounded semi-annually. How long will it take a consumer to pay off her new television?

Click to see Solution

PMT Setting	END
N	?
PV	1,995
FV	0
PMT	−100
I/Y	15
P/Y	12
C/Y	2

$$N = 22.9783... \rightarrow 23 \text{ payments}$$

$$\begin{aligned}
 \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\
 &= \frac{23}{12} \\
 &= 1.9166... \\
 &\rightarrow 1 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{Number of Months} &= 0.9166... \times 12 \\
 &= 11
 \end{aligned}$$

It will take 1 year and 11 months.

Exercises

1. You make \$3,000 quarterly payments into a savings account earning 6.35% compounded quarterly. How

long will it take you to accumulate \$100,000?

Click to see Answer

6 years, 9 months

2. You make \$2,500 contributions at the beginning of every six months into your RRSP. If the RRSP earns interest at 7% effective, how long will it take your RRSP to reach an accumulated value of \$175,000?

Click to see Answer

18 years

3. You took a \$50,000 loan at 7.2% compounded quarterly. The loan agreement requires you to make monthly payments of \$500 until the loan is paid off. How long will it take you to repay the loan?

Click to see Answer

12 years, 9 months

4. You have \$1,000,000 in your RIF. You want to receive \$40,000 beginning-of-quarter payments for as long as possible from the RIF. If the RIF earns 9% compounded monthly, how long will it take to exhaust the RIF?

Click to see Answer

9 years, 3 months

5. An investment of \$100,000 today will make advance quarterly payments of \$4,000. If the annuity can earn 7.3% compounded semi-annually, how long will it take for the annuity to be depleted?

Click to see Answer

8 years, 3 months

6. The neighbourhood grocery store owned by Raoul needs \$22,500 to upgrade its fixtures and coolers. If Raoul contributes \$3,000 at the start of every quarter into a fund earning 5.4% compounded quarterly, how long will it take him to save up the needed funds for his store's upgrades?

Click to see Answer

2 years

7. Andre has stopped smoking. If he takes the \$80 he saves each month and invests it into a fund earning 6% compounded monthly, how long will it take for him to save \$10,000?

Click to see Answer

8 years, 2 months

8. How much longer will a \$500,000 investment fund earning 4.9% compounded annually last if beginning-of-month payments are \$3,500 instead of \$4,000?

Click to see Answer

3 years, 2 months

9. Consider a \$150,000 loan with month-end payments of \$1,000. How much longer does it take to pay off the loan if the interest rate is 6% compounded monthly instead of 5% compounded monthly?

Click to see Answer

3 years, 6 months

10. You make \$250 month-end contributions to your RRSP, which earns 9% compounded annually.
- How much less time will it take to reach \$100,000 if you increase your payments by 10%?
 - Which alternative requires less principal and by how much? (Assume all payments are equal.)

Click to see Answer

a. 9 months; b. regular payments require \$2250 less principal

11. Most financial institutions tout the benefits of “topping up” your mortgage payments—that is, increasing from the required amount to any higher amount. Assume a 25-year mortgage for \$200,000 at a fixed rate of 5% compounded semi-annually.
- How many fewer payments does it take to pay off your mortgage if you increased your monthly payments by 10%?
 - How much money is saved by “topping up” the payments? Assume that all payments are equal amounts in your calculations.

Click to see Answer

a. 48 fewer payments; b. \$26,521.44

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5.6 CALCULATING THE INTEREST RATE FOR AN ANNUITY

LEARNING OBJECTIVES

- Calculate the interest rate for an annuity.

You must calculate the interest rate on an annuity in a variety of situations:

- To determine the interest rate being charged on any debt.
- To determine the interest rate that an investment is earning.
- To calculate the required interest rate for savings to reach a goal within a certain time period.
- To calculate the required interest rate needed for a series of payments to be sustained over a certain time period.

Using a Financial Calculator

In the future value and present value formulas for annuities, there is no way to algebraic isolate the periodic interest rate. Consequently, calculating the interest rate for an annuity cannot be found by using the formulas. However, a financial calculator can quickly find the required interest rate. You use the financial calculator in the same way as described previously, but the only difference is that the unknown quantity is I/Y (the nominal interest rate). You must still load the other six variables into the calculator, set the calculator to the correct payment setting, and apply the cash flow sign conventions carefully.

USING THE TI BAI PLUS CALCULATOR TO FIND THE INTEREST RATE FOR ANNUITIES

- Set the calculator to the correct payment setting (END or BGN).
- Enter values for the known variables (PV, FV, N, PMT, P/Y and C/Y), paying close attention to the cash flow sign convention for PV, PMT, and FV.
- After all of the known quantities are loaded into the calculator, press CPT and then I/Y to solve for the interest rate.

NOTES

1. Ensure that the calculator is set to the required payment setting. The interest rate will be different for END and BGN.
2. Because the calculator is using a trial-and-error technique to calculate the interest rate, it may take your calculator a few seconds to compute the answer. If your screen goes blank and your calculator hesitates, this is normal.
3. The values you enter for PV, FV, and PMT must adhere to the cash flow sign convention.

EXAMPLE

Smartchoice, a rent-to-own store, offers a Dell 10" Mini Inspiron Netbook for a cash n' carry price of

\$399. Alternatively, under its rent-to-own plan you could make \$59.88 monthly payments in advance and own the laptop after one year. What effective interest rate is being charged on the rent-to-own plan?

Solution:

The timeline for the annuity appears below.

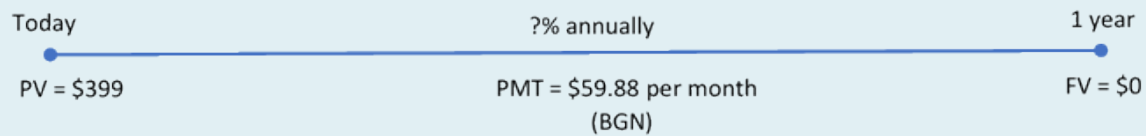


Figure 5.6.1

PMT Setting	BGN
N	$12 \times 1 = 12$
PV	399
FV	0
PMT	−59.88
I/Y	?
P/Y	12
C/Y	1

$$I/Y = 337.98\%$$

The effective interest rate being charged under the rent-to-own payment plan is 337.9759%.

EXAMPLE

Cubonic Industries deposits \$30,000 at the end of every quarter to save up \$550,000 for a capital project in four years. To achieve its goal, what nominal interest rate compounded quarterly does Cubonic Industries require on its investment?

Solution:

PMT Setting	END
N	$4 \times 4 = 16$
PV	0
FV	550,000
PMT	−30,000
I/Y	?
P/Y	4
C/Y	4

$$I/Y = 7.15\%$$

For Cubonic Industries to achieve its savings goal, the savings annuity must earn 7.15% compounded quarterly.

TRY IT

Francisco just changed occupations. Unfortunately, he is not able to transfer his company pension with him to his new company. The administrators of the pension plan offer him the choice of a lump-sum payout of \$103,075 today or beginning-of-month payments of \$535 for the next 25 years. What semi-annually compounded rate of return are the pension administrators using in their calculations?

Click to see Solution

PMT Setting	BGN
N	$12 \times 25 = 300$
PV	103,075
FV	0
PMT	−535
I/Y	?
P/Y	12
C/Y	2

$$I/Y = 3.90\%$$

EXAMPLE

Amadeus has already saved \$5,000 in his RRSP today. Suppose he continues to make \$250 contributions at the beginning of each month for the next 14 years. For him to achieve his goal of having \$100,000, what effective rate of return must his investment earn?

Solution:

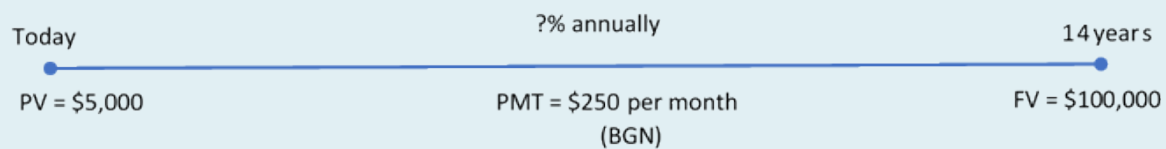


Figure 5.6.2

The timeline for RRSP contributions appears below.

PMT Setting	BGN
N	$12 \times 14 = 168$
PV	-5,000
FV	100,000
PMT	-250
I/Y	?
P/Y	12
C/Y	1

$$I/Y = 9.17\%$$

The nominal rate of interest that Amadeus must earn on his investment is 9.17% effective.

EXAMPLE

Gemma is looking to purchase a new Nissan Pathfinder for \$54,904.64 including all fees and sales taxes. She can afford to pay no more than \$1,500 at the end of every month, and she wants to have the balance owing reduced to \$30,000 after two years, when she can pay off the vehicle with her trust fund. What is the maximum effective rate of interest she could be charged on the car loan to meet her goals?

Solution:

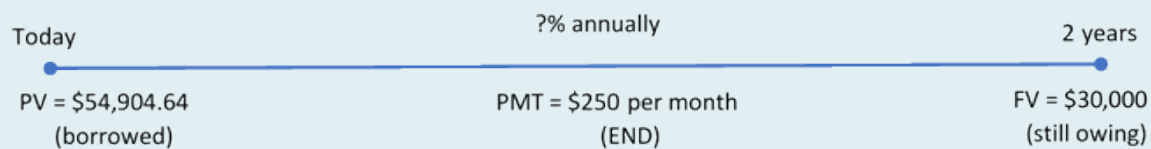


Figure 5.6.3

The timeline for the car loan appears below.

PMT Setting	END
N	$12 \times 2 = 24$
PV	54,904.64
FV	−30,000
PMT	−1,500
I/Y	?
P/Y	12
C/Y	1

$$I/Y = 13.53\%$$

Gemma will be able to purchase the car if she can obtain a car loan that has an effective interest rate lower than 13.53%.

TRY IT

An investment today requires \$1,125.51 to purchase. In return, the investment pays out \$30 after every six months for the next 20 years, along with an additional final lump-sum payout of \$1,000. What semi-annually compounded interest rate is being earned on the investment?

Click to see Solution

PMT Setting	END
N	$2 \times 20 = 40$
PV	$-1,125.51$
FV	1,000
PMT	30
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 5\%$$

TRY IT

When you buy a car, a cash rebate is usually available if you finance the vehicle through your bank instead of the dealership. If you finance the vehicle through the dealership, you are not eligible for the cash rebate. Assume you can purchase a vehicle for \$24,960 and finance it for four years with month-end payments at 0% through the dealership. Alternatively, you could get a loan from a bank and pay cash for your vehicle, which would entitle you to receive a \$3,500 cash rebate. What monthly compounded interest rate would the bank have to charge to arrive at the same monthly payment as the dealership alternative? What decision rule can you create from this calculation?

Click to see Solution

Payments for financing through the dealership at 0% interest:

PMT Setting	END
N	$12 \times 4 = 48$
PV	$-24,960$
FV	0
PMT	?
I/Y	0
P/Y	12
C/Y	12

$$PMT = \$520$$

Interest rate for the bank with monthly payments of \$520 and applying the \$3,500 rebate to the present value:

PMT Setting	END
N	$12 \times 4 = 48$
PV	$-21,460$
FV	0
PMT	-520
I/Y	?
P/Y	12
C/Y	12

$$I/Y = 7.61\%$$

Exercises

1. You purchase an annuity for \$500,000. The annuity will pay you \$8,000 every quarter for 21.5 years. What rate compounded quarterly does the annuity earn?

Click to see Answer

3.12%

2. You invest \$300 at the beginning of every month for 33.5 years. At the end of the 33.5 years, you have saved \$750,000. What rate compounded semi-annually did your investment earn?

Click to see Answer

9.04%

3. You have the opportunity to make an investment of \$100,000 today and receive beginning of the month payments of \$1,200 for 10 years. Calculate the rate compounded monthly on the investment.

Click to see Answer

7.90%

4. Today, your RRSP has \$25,000. Over the next 8 years you plan to deposit \$6,250 every six months into the RRSP with the goal of having \$200,000 in the account. Calculate the rate compounded semi-annually the RRSP must earn for you to achieve your goal.

Click to see Answer

9.76%

5. You have \$300,000 in your RIF. Over the next 13 years and 3 months, you receive beginning-of-quarter payments of \$4,975 from the RIF. At the end of the 13 years and 3 months, the balance in your RIF is \$150,000. What rate compounded quarterly does the RIF earn?

Click to see Answer

3.77%

6. Following his financial adviser's recommendations, Sanchez starts monthly contributions of \$375 today to his RRSP. The plan is to have \$240,000 in his RRSP after 20 years of monthly compounding. What nominal interest rate does the financial adviser think Sanchez's RRSP will be able to realize?

Click to see Answer

8.61%

7. Helen's husband recently passed away. The life insurance company is offering her a lump-sum payout of \$250,000 today, or month-end payments of \$1,585 for 20 years.
- What monthly compounded rate is the life insurance company using in its calculations?
 - Helen thinks she can take the lump sum and invest it herself at 4.75% effectively. How much will her monthly payment increase?

Click to see Answer

a. 4.53%; b. \$16.89

8. Under a wrongful dismissal suit, a court awarded a former employee \$100,835.25 for end-of-month wages of \$4,500 for the past 21 months. What effective interest rate is the court using in the judgment?

Click to see Answer

8%

9. Jake's Electronics wants to match a competitor's advertised payment plan on an identical stereo system. If the system retails for \$1,011.35 including all sales taxes and Jake wants to advertise six equal end-of-month payments of \$174, what effective rate of interest does he need to charge?

Click to see Answer

11.56%

10. The marketing manager for Gold's Gym offers members a two-year membership for \$650 in advance or beginning-of-month payments of \$29. What monthly compounded interest is the marketing manager using in his pricing?

Click to see Answer

7.25%

11. A retail store wants to offer its clients different two-year ordinary payment plans on their product purchases. The marketing manager understands the importance of odd-number pricing in these plans, where \$499 is better than stating \$500. On a typical \$5,000 purchase, the marketing manager wants to offer payments of \$229 monthly, \$699 quarterly, or \$1,399 semi-annually. The Competition Act of Canada requires full disclosure of the annual interest rate being charged on any plan. What interest rate must be published for each plan?

Click to see Answer

monthly plan: 9.95%, quarterly plan: 10.62%, semi-annual plan: 9.54%

12. On a \$3,500 purchase, a company is thinking of offering a year-long month-end payment plan that requires payments of \$299, \$319, \$334, or \$349.
- If the goal of the plan is to offer a competitive interest rate comparable to a bank credit card that averages 18% effectively, which payment plan should be chosen?
 - If the goal of the plan is to offer a competitive interest rate comparable to a department store credit card that averages 28% effectively, which payment plan should be chosen?

Click to see Answer

\$299 payments: 4.71%, \$319 payments: 18.24%, \$334 payments: 29.07%, \$349 payments: 40.51%; a. choose \$319 plan with 18.24% effective; b. choose \$334 plan with 29.07% effective

13. Margarite's goal is to save up \$100,000 after 10 years of monthly contributions into an investment annuity starting today. Depending on the level of risk she chooses, her adviser tells her that under low-risk conditions she would need to contribute \$645.19, under medium-risk conditions her contribution needs to be \$523.32, and if she puts her money into high-risk investments she would need \$401.14 per month. Based on the adviser's calculations, what are the effective interest rates on the low-, medium-, and high-risk investments?

Click to see Answer

low: 5%, medium: 9%, high: 14%

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5.7 DEFERRED ANNUITIES

LEARNING OBJECTIVES

- Solve problems involving a deferred annuity.

Suppose at the age of 21, you took \$50,000 and invested it into an RRSP. You leave the money in the RRSP and do not make any further contributions to the RRSP. When you retire in 40 years, you will take the accumulated money in the RRSP to provide you with monthly payments for 20 years during your retirement. During the 40 years leading up to your retirement, the \$50,000 you invested at age 21 is left to earn interest, with no payments or withdrawals occurring during that time. The payments begin after this 40 year period of interest. This is an example of a **deferred annuity**, where the payments begin after an extended period of time.

What is a Deferred Annuity?

A **deferred annuity** is a financial transaction where annuity payments are delayed until a certain period of time has elapsed. Usually the annuity has two stages, as depicted in this figure.

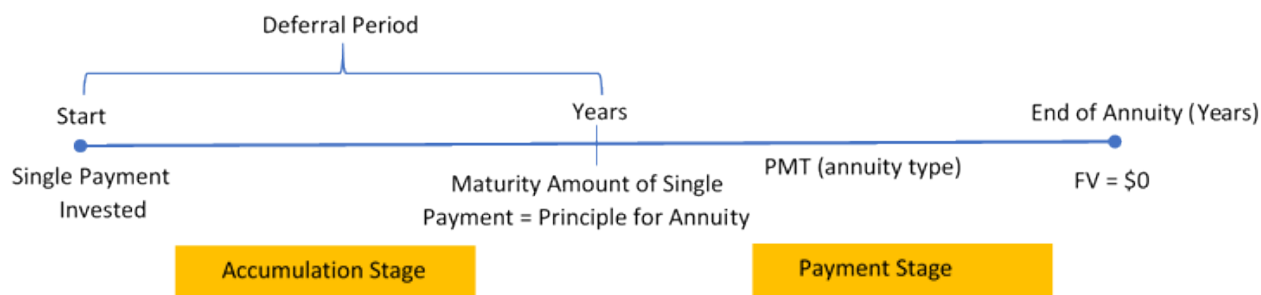


Figure 5.7.1

- **Accumulation Stage.** A single payment is allowed to earn interest for a specified duration. There are no annuity payments during this period of time, which is commonly referred to as the **period of deferral**.
- **Payment Stage.** The annuity takes the form of any of the four annuity types and starts at the beginning of this stage as per the financial contract. Note that the maturity value of the accumulation stage is the same as the principal for the payments stage.

Working with Deferred Annuities

For deferred annuities, the most common unknown variables are either the present value, the length of the period of deferral, the annuity payment amount, or the number of annuity payments that are sustainable for a fixed income payment. Follow this sequence of steps for each of these variables.

	Solving for the Present Value	Solving for the Period of Deferral	Solving for the Annuity Payment	Solving for the Number of Annuity Payments
Step 1	Draw a timeline. Identify the known quantities. Identify the locations of the present value of the annuity and the first payment and place the corresponding times on the timeline.			
Step 2	Starting at the end of your timeline, calculate the present value of the annuity.		Starting at the beginning of the timeline, calculate the future value at the end of the period of deferral.	
Step 3	Calculate the present value at the start of the period of deferral.	Calculate the number of compounding periods and convert to years and months.	Calculate the annuity payment.	Calculate the number of annuity payments and convert to years and months.

NOTES

1. It is an error to treat the period of deferral and the term of the annuity as simultaneous time periods. For example, if a deferred annuity has a three-year period of deferral and a 10-year annuity term, this is sometimes interpreted, mistakenly, as an annuity ending 10 years from today. These time segments are **separate** and **consecutive** on the timeline! The correct interpretation is that the annuity term ends 13 years from today, since the 10-year term does not start until the three-year deferral terminates.
2. A common mistake is to incorrectly determine when the period of deferral ends and the annuity starts. This error usually results from forgetting that the payments on ordinary

annuities start **one payment interval after the annuity starts**, whereas annuity due payments start immediately. Thus, if the first quarterly payment on an ordinary annuity is to be paid 6 years from today, then the period of deferral is 5 years and 9 months. If the deferral is for an annuity due, then the period of deferral is 6 years.

EXAMPLE

You borrow \$10,000 for your business to purchase new computer software. The loan agreement requires you make quarterly payments for 5 years at 3% compounded quarterly. The first loan payment is due in 2 years. Calculate the size of the quarterly loan payments.

Solution:

Step 1: Draw the timeline for the deferred annuity. Because the payment setting is not specified, the payments occur at the end of the payment interval.

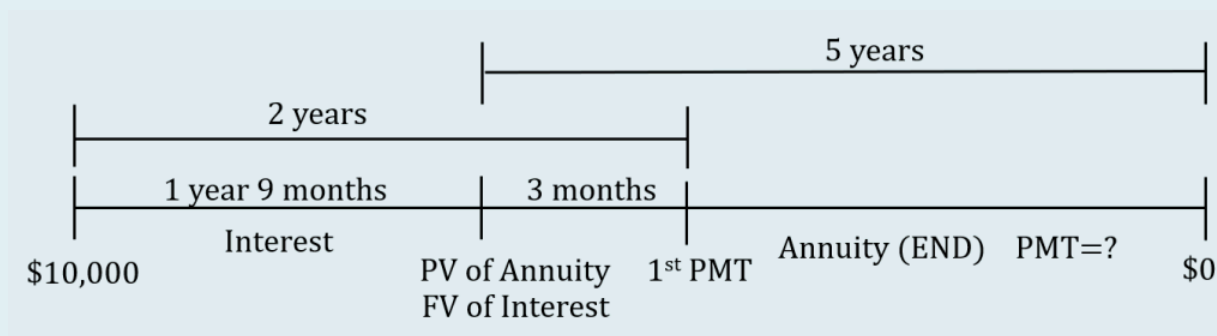


Figure 5.7.2

Some notes about the timeline.

1. The \$10,000 is the present value at the start of the interest period.
2. There is a 2 year time gap between the start of the interest period and first payment.
Because the payments are at the end of the payment interval, the first payment does NOT

occur at the same time as the present value of the annuity, which occurs at the beginning of the first payment interval.

3. Because the payments are at the end of the payment interval, the present value of the annuity occurs one payment interval before the first payment. The payments are quarterly (or 3 months), so the present value of the annuity occurs 3 months before the first payment.
4. The \$10,000 at the start of the interest period must be moved (with a future value calculation) to the location of the present value of the annuity. Because the present value of the annuity occurs 3 months before the first payment and it is 2 years from the start of the interest period to the first payment, the \$10,000 must move 1 year and 9 months (2 years minus 3 months) to get the location of the present value of the annuity.
5. The payments for the annuity last for 5 years.

Step 2: Calculate the future value of the \$10,000. As noted above, the \$10,000 must move to the location of the present value of the annuity, which is 1 year and 9 months from the start of the timeline.

N	$4 \times 1.75 = 7$
PV	10,000
FV	?
PMT	0
I/Y	3
P/Y	4
C/Y	4

$$FV = \$10,536.961...$$

Step 3: Calculate the payment for the annuity. The future value calculated in the previous step becomes the present value for the annuity: $PV = \$10,536.961...$

PMT Setting	END
N	$4 \times 5 = 20$
PV	10,536.961...
FV	0
PMT	?
I/Y	3
P/Y	4
C/Y	4

$$PMT = \$569.32$$

Your quarterly loan payments are \$569.32.

As you can see with the previous example, sometimes adjustments need to be made to the timing of the period of deferral, depending on the wording of the question and the type of annuity. Because the previous question was an ordinary annuity (payments at the end of the interval) and the timing for the period of deferral was to the first payment, we had to adjust the timing of the interest period so that the money was moved to the location that corresponded to the present value of the annuity.

We are going to do the previous example again, but this time we will treat the annuity as an annuity due (payments at the beginning of the interval). We will do this despite the wording in the question which implies that the annuity has payments at the end of the interval.

EXAMPLE

You borrow \$10,000 for your business to purchase new computer software. The loan agreement

requires you make quarterly payments for 5 years at 3% compounded quarterly. The first loan payment is due in 2 years. Calculate the size of the quarterly loan payments.

Solution:

Step 1: Draw the timeline for the deferred annuity. Treat the annuity as an annuity due with the payments at the beginning of the interval.

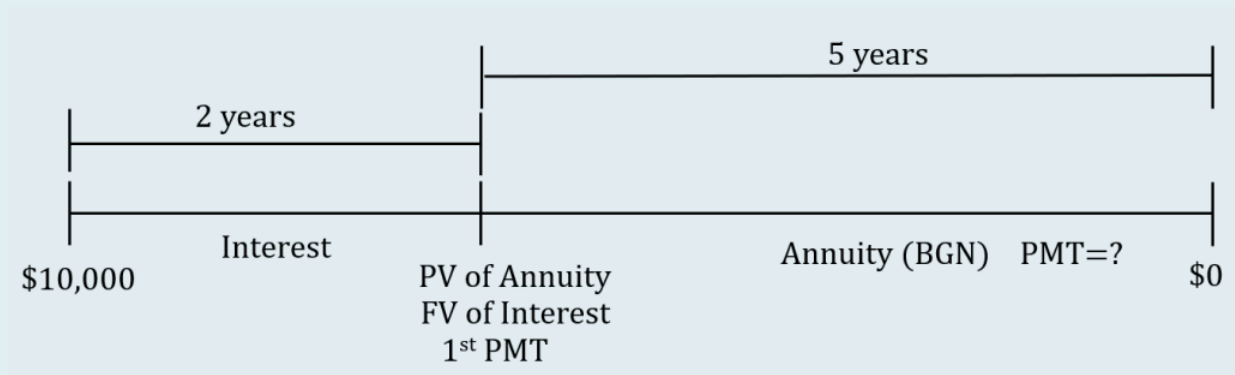


Figure 5.7.3

Some notes about the timeline.

1. The \$10,000 is the present value at the start of the interest period.
2. Because the payments are at the beginning of the payment interval, the present value of the annuity and the first payment occur at the same time, at the start of the annuity. Note that this eliminates the gap between the present value of the annuity and the first payment seen in the previous example.
3. The \$10,000 at the start of the interest period must be moved (with a future value calculation) to the location of the present value of the annuity. Because the present value of the annuity occurs at the time of the first payment, the \$10,000 must move 2 years to get the location of the present value of the annuity.
4. The payments for the annuity last for 5 years.

Step 2: Calculate the future value of the \$10,000. As noted above, the \$10,000 must move to the location of the present value of the annuity, which is 2 years from the start of the timeline.

N	$4 \times 2 = 8$
PV	10,000
FV	?
PMT	0
I/Y	3
P/Y	4
C/Y	4

$$FV = \$10,615.988...$$

Step 3: Calculate the payment for the annuity. The future value calculated in the previous step becomes the present value for the annuity: $PV = \$10,615.988...$

PMT Setting	BGN
N	$4 \times 5 = 20$
PV	10,615.988...
FV	0
PMT	?
I/Y	3
P/Y	4
C/Y	4

$$PMT = \$569.32$$

Your quarterly loan payments are \$569.32.

Note that treating the annuity as an annuity due (payments at the beginning of the interval) had no impact on the final answer. But, by treating the annuity as an annuity due, we eliminated the time adjustments required in the first example to get the future value of the interest aligned with the timing of the present value of

the annuity. You can always treat a deferred annuity as an annuity due, regardless of the information in the question.

NOTE

You can treat any deferred annuity as an ordinary deferred annuity or a deferred annuity due. Regardless of the approach, you will get the same **final** answer, provided you make the correct adjustments based on your approach. However, it is much easier to treat any deferred annuity as a deferred annuity due (payments at the beginning of the interval). The deferred annuity due approach lines up the future value of the interest, the present value of the annuity and the first payment at the same moment in time, which eliminates any time adjustments to the period of deferral.

TRY IT

You are going to invest \$80,000 today at 5.5% compounded quarterly. You want to receive month-end payments from this investment for 25 years. You will receive your first payment in 10 years. What are the size of your monthly payments?

Click to see Solution

PMT Setting		BGN
N	40	300
PV	-80,000	-138,141.661...
FV	Formula does not parse	0
PMT	0	Formula does not parse
I/Y	5.5	5.5
P/Y	4	12
C/Y	4	4

$$PMT = \$842.40$$

EXAMPLE

You want to invest some money today to help fund your retirement. When you retire, you want to receive \$5,000 at the end of every month for 15 years. You want to receive your first payment in 20 years. If your retirement fund earns interest at 5% compounded semi-annually, how much money do you need to invest?

Solution:

Step 1: Draw the timeline for the deferred annuity. Using the trick identified above, treat the annuity as an annuity due with the payments at the beginning of the interval (despite the wording in the question that identifies the payments at the end of the interval).

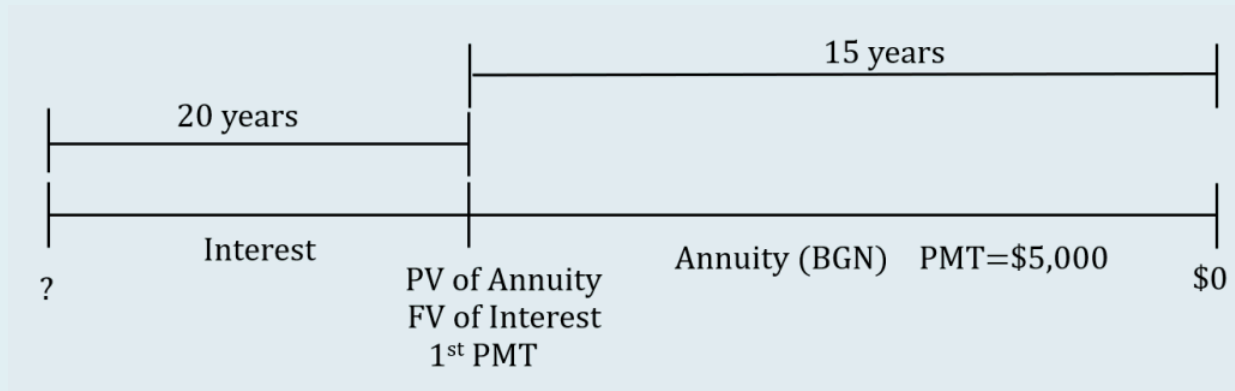


Figure 5.7.4

Some notes about the timeline.

1. The present value at the start of the interest period is unknown.
2. Because the payments are at the beginning of the payment interval, the present value of the annuity and the first payment occur at the same time, at the start of the annuity. There is a 20 year gap from the start of the interest period and the present value of annuity/future value of interest/first payment.
3. The \$5,000 payments for the annuity last for 15 years.
4. The present value of the annuity becomes the future value of the interest period. Then, the future value of the interest period will be moved back 20 years to the start of the interest period.

Step 2: Calculate the present value of the annuity.

PMT Setting	BGN
N	$12 \times 15 = 180$
PV	?
FV	0
PMT	5,000
I/Y	5
P/Y	12
C/Y	2

$$PV = \$637,034.376...$$

Step 3: Calculate the present value for the interest period. The present value calculated in the previous step becomes the future value for the interest period: $FV = \$637,034.376...$

N	$2 \times 20 = 40$
PV	?
FV	637,034.376...
PMT	0
I/Y	5
P/Y	2
C/Y	2

$$PV = \$237,251.11$$

You need to invest \$237,251.11 today to fund your retirement.

TRY IT

You need to borrow some money for your business. You can afford to pay \$1,250 every month for 10 years. You are charged interest at 4.78% compounded semi-annually and will make your first payment in 4 years. How much money can you borrow?

Click to see Solution

PMT Setting	BGN	
N	120	8
PV	Formula does not parse	Formula does not parse
FV	0	−118,784.5266...
PMT	−1,250	0
I/Y	4.78	4.78
P/Y	12	2
C/Y	2	2

$$PV = \$99,160.88$$

EXAMPLE

Bashir has \$50,000 to invest. Eventually, Bashir wants the annuity to pay him \$2,500 at the beginning of every month for 10 years. When can Bashir receive his first payment if the investment earns 8.25% compounded quarterly?

Solution:

Step 1: Draw the timeline for the deferred annuity.

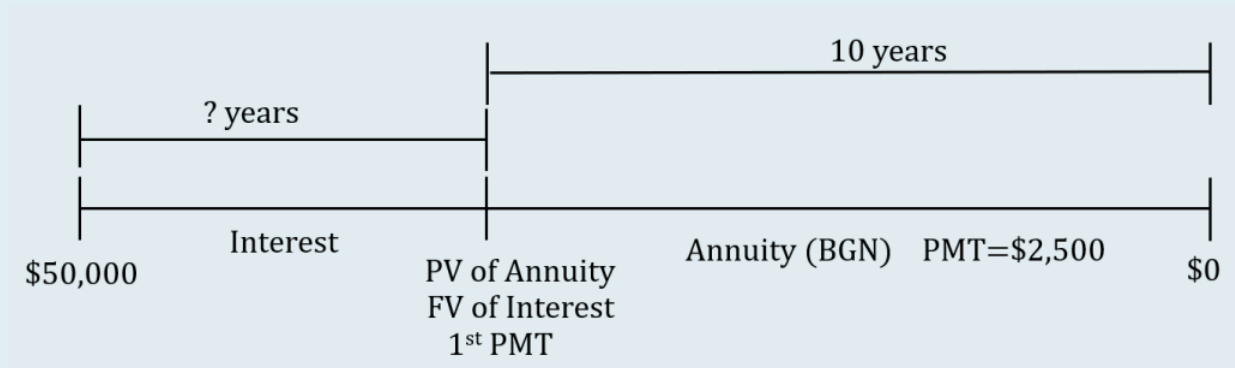


Figure 5.7.5

Some notes about the timeline.

1. The \$50,000 is at the start of the interest period.
2. Because the payments are at the beginning of the payment interval, the present value of the annuity and the first payment occur at the same time, at the start of the annuity.
3. The time from the start of the interest period to the present value of the annuity/future value of the interest period/first payment is unknown.
4. The \$2,500 payments for the annuity last for 10 years.

Step 2: Calculate the present value of the annuity.

PMT Setting	BGN
N	$12 \times 10 = 120$
PV	?
FV	0
PMT	2,500
I/Y	8.25
P/Y	12
C/Y	4

$$PV = \$205,719.154\dots$$

Step 3: Calculate the number of compounding periods for the interest period and convert to years

and months. The present value calculated in the previous step becomes the future value for the interest period: $FV = \$205,719.154...$

N	?
PV	−50,000
FV	205,719.154...
PMT	0
I/Y	8.25
P/Y	4
C/Y	4

$$N = 69.286...$$

$$\begin{array}{l} \text{Number of Years} \times = \frac{N}{C/Y} = \frac{69.286...}{4} = 17.321... \rightarrow 17 \text{ years} \\ \text{Number of Months} \times = 0.321... \times 12 = 3.858... \rightarrow 4 \text{ months} \end{array}$$

Bashir can receive his first payment 17 years and 4 months after he invests the \$50,000.

TRY IT

You want to receive \$5,000 semi-annual payments for 5.5 years from a fund earning 6.82% compounded monthly. If you invest \$20,000 in the fund today, when can you receive your first payment. Give your answer in years and months.

Click to see Solution

PMT Setting	BGN	
N	11	Formula does not parse
PV	Formula does not parse	−20,000
FV	0	46,669.472...
PMT	5,000	0
I/Y	6.82	6.82
P/Y	2	12
C/Y	12	12

$$N = 149.5185....$$

$$\begin{array}{l} \text{Number of Years} \approx \frac{N}{C/Y} \approx \frac{149.5185...}{12} \approx 12.4598... \rightarrow 12 \text{ years} \\ \text{Number of Months} \approx 0.598... \times 12 \approx 5.5185... \rightarrow 6 \text{ months} \end{array}$$

EXAMPLE

Emile received a \$25,000 one-time bonus from his employer today, and he immediately invested it at 3.25% compounded semi-annually. Fourteen years from now, he plans to withdraw \$2,300 every quarter to use as his retirement income. How long can Emile receive these payments?

Solution:

Step 1: Draw the timeline for the deferred annuity. Treat the annuity as an annuity due with the payments at the beginning of the interval (despite the wording in the question that identifies the payments at the end of the interval).

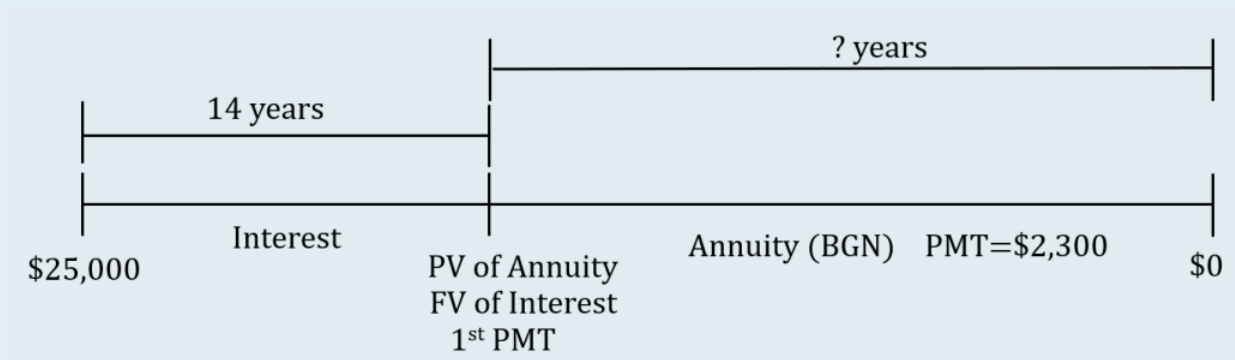


Figure 5.7.6

Some notes about the timeline.

1. The \$25,000 is at the start of the interest period.
2. Because the payments are at the beginning of the payment interval, the present value of the annuity and the first payment occur at the same time, at the start of the annuity. There is a 14 year gap from the start of the interest period and the present value of annuity/future value of interest/first payment.
3. The \$2,300 payments for the annuity last for an unknown number of years.

Step 2: Calculate the future value of the \$25,000.

N	$2 \times 14 = 28$
PV	$-25,000$
FV	?
PMT	0
I/Y	3.25
P/Y	2
C/Y	2

$$FV = \$39,260.484...$$

Step 3: Calculate the number of payment periods for the annuity and convert to years and months. The future value calculated in the previous step becomes the present value for the annuity: $PV = \$39,260.484...$

PMT Setting	BGN
N	?
PV	−39,260.484...
FV	0
PMT	2,300
I/Y	3.25
P/Y	4
C/Y	2

$$N = 18.284... \rightarrow 19 \text{ payments}$$

$$\begin{aligned} \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\ &= \frac{19}{4} \\ &= 4.75 \\ &\rightarrow 4 \text{ years} \\ \text{Number of Months} &= 0.75 \times 12 \\ &= 9 \end{aligned}$$

Emile can receive payments for 4 years and 9 months.

TRY IT

You borrowed \$15,000 today at 4.5% compounded monthly. The loan agreement requires semi-annual payments of \$750 with the first payment due in 3 years. How long will you need to make payments to repay the loan?

Click to see Solution

PMT Setting		BGN
N	36	Formula does not parse
PV	15,000	17,163.717...
FV	Formula does not parse	0
PMT	0	−750
I/Y	4.5	4.5
P/Y	12	2
C/Y	12	12

$$N = 31.602... \rightarrow 32 \text{ payments}$$

$$\begin{aligned} \text{Number of Years} \ \&= \ \& \ \frac{\text{rounded up } N}{P/Y} \ \& \\ \&= \ \& \ \frac{32}{2} \ \&= \ \& \ 16 \text{ years} \end{aligned}$$

Exercises

1. How much money should a company borrow at 7.7% compounded semi-annually in order to repay \$4,200 at the end of every quarter for 12 years. The first payment is due 15.5 years from now.

Click to see Answer

\$41,484

2. You purchase an annuity for \$40,000. The annuity will pay you beginning-of-the-month payments for 10 years and you will receive your first payment in 5 years. If the annuity earns 6.1% compounded quarterly, what is the size of the payments?

Click to see Answer

\$599.92

3. Your company took out a \$100,000 business loan at 8.5% effective. The loan agreement calls for payments of \$20,000 at the beginning of every six months with the first payment made in 7 years. How long will the company need to make payments to repay the loan?

Click to see Answer

5 years, 6 months

4. You invest \$17,500 in an account at 7.5% compounded monthly. You want to withdraw annual payments from the account for 5 years. If you receive the first payment in 14.5 years, what is the size of the payments you will receive?

Click to see Answer

\$11,951.16

5. You want to receive quarterly payments of \$750 from your investment fund for a period of 13 years. The fund earns 2.8% compounded semi-annually. Calculate the amount of money you need to deposit in the fund if the first payment is to be received in 7 years and 9 months.

Click to see Answer

\$26,476.06

6. You invest \$55,000 in a fund earning 6.6% compounded quarterly. You want to withdraw \$900 from the fund at the end of every month with the first withdrawal to be made 4 years from now. How long will it take for the fund to be depleted?

Click to see Answer

8 years, 8 months

7. What is the present value of a deferred annuity with a deferral period of 17 years at 6.7% compounded semi-annually followed by a 10-year annuity due paying \$1,250 every month at 4.78% compounded semi-annually?

Click to see Answer

\$39,070.09

8. If \$38,000 is invested for 15 years at 9.4% compounded quarterly and then pays out \$10,000 at the beginning of each year while earning 2.4% compounded annually, how far from today would the last payment occur?

Click to see Answer

19 years

9. Jeff and Sarah want to invest some money in an RESP earning 4% compounded quarterly for their daughter's education. They want the annuity to pay their daughter \$1,000 monthly for three years and nine months for the duration of her educational studies. What lump-sum amount do they need to invest today if they want their daughter to receive the first payment in 10 years?

Click to see Answer

\$28,123.56

10. Parker invested \$80,000 in a fund earning 3.85% effective. He wants to receive month-end payments from the fund for 25 years. Calculate the size of Parker's payments if he receives his first payment in 10 years.

Click to see Answer

\$600.36

11. Amber plans to retire in 40 years. When she does retire, she would like her RRSP to pay her \$2,500 every month for 20 years. How much money does Amber need to invest today if the RRSP earns 5.9% compounded semi-annually?

Click to see Answer

\$34,735.26

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5.8 PERPETUITIES

LEARNING OBJECTIVES

- Solve problems involving a perpetuity.

Some of the scholarships offered to students at your college were created decades ago, yet these scholarships continue to pay out money to students each and every year. Where does all of the money come from? Did somebody bestow a tremendously large endowment fund all those many years ago to sustain the scholarship over decades? Or does an individual or supporting organization simply pay it out of pocket each year? How is it possible for these scholarships to pay out through all the past years and all of the foreseeable years?

Scholarships, like the one described above, are examples of **perpetuities**, which are annuities that have an infinite term. In other words, the payments last forever. At your individual level, any sum of money you invest as a perpetuity can be used to generate income forever. At the professional level, many companies and not-for-profit organizations such as sports clubs establish scholarship programs and bursaries for their employees, members, or clients. In some contracts, payments such as royalties continue forever. Some divided stocks have their price determined by indefinite future dividend amounts.

What are Perpetuities?

A **perpetuity** is a special type of annuity that has fixed, regular payments that continue indefinitely. If the principal of the investment is never withdrawn, then the interest earned each period can be withdrawn without affecting the future interest earnings of the investment. Therefore, the annuity continues to earn the same amount of interest each and every future interval and can pay out interest forever.

A perpetuity can be an ordinary perpetuity or a perpetuity due. As well, the payment frequency and

compounding frequency create either a simple or general annuity structure. Perpetuities can even occur after a deferral period, and hence deferred perpetuities are possible.

The Perpetuity Formulas

The present value for an ordinary perpetuity is

$$PV = \frac{PMT}{i_2}$$

and the present value for a perpetuity due is

$$PV = (1 + i_2) \times \frac{PMT}{i_2}$$

where

- **PV is the present value of the perpetuity.** The present value is the sum of the present value of all of the annuity payments at the start of the first payment interval.
- **PMT is the annuity payment.** The amount of money that is invested or paid after each payment interval.
- **i_2 is the periodic interest rate for the interest rate whose compounding frequency matches the payment frequency.** If the annuity is a simple annuity, the compounding frequency and the payment frequency are equal and the given interest rate can be used to calculate the periodic interest rate. If the annuity is a general annuity, the compounding frequency and the payment frequency are different and an interest conversion must be done first to find the equivalent interest rate with the compounding frequency equal to the payment frequency.

NOTES

1. There is no future value formula for a perpetuity. Because a perpetuity lasts forever, the future value is an undefined quantity.
2. There is no N value in perpetuity formulas. Because a perpetuity lasts forever, the term of a perpetuity is an undefined quantity.

3. For perpetuity questions, there are only three possible quantities to solve for: PV, PMT or I/Y .
4. In a simple perpetuity, the payment frequency and the compounding frequency are equal ($P/Y = C/Y$). In this situation the periodic interest rate $i_2 = \frac{j}{m}$ where j is the nominal interest rate given in the question and m is the compounding frequency (and the payment frequency).
5. In a general perpetuity, the payment frequency and the compounding frequency are not equal ($P/Y \neq C/Y$). In this situation, the given interest rate must first be converted to the equivalent interest rate where the new compounding frequency equals the payment frequency. Using the equivalent interest rate, calculate the periodic interest rate i_2 .

The steps required to solve for the present value or payment of a perpetuity remain essentially unchanged from solving for the present value or payment of any annuity.

EXAMPLE

Your college wants to provide a \$2,000 scholarship at the end of every quarter. How much money does the college need to invest today at 4% compounded quarterly to sustain this scholarship indefinitely?

Solution:

Step 1: The payments are at the beginning of the payment intervals, and both the compounding frequency and the payment frequency are the same (both quarterly). This is an ordinary simple perpetuity. Because this is a simple perpetuity, an interest rate conversion is not required.

The given information is

$$\begin{array}{l} \text{PMT} = \$2,000 \\ P/Y = 4 \\ I/Y = 4\% \\ C/Y = 4 \end{array}$$

Step 2: Calculate the periodic interest rate.

$$\begin{aligned} i_2 &= \frac{I/Y}{C/Y} \\ &= \frac{4\%}{4} \\ &= 1\% \end{aligned}$$

Step 3: Calculate the present value.

$$\begin{aligned} PV &= \frac{PMT}{i_2} \\ &= \frac{2,000}{0.01} \\ &= \$200,000 \end{aligned}$$

The college needs to invest \$200,000 today to sustain the scholarship indefinitely.

NOTE

In the previous example, the \$2,000 payments from the perpetuity come from the interest generated by the \$200,000 investment. At 4% compounded quarterly, \$2,000 in interest is accumulated at the end of each quarter. The \$2,000 in interest becomes the payment. None of the \$200,000 principal is ever used to make up the payment. Consequently, the \$2,000 payments will last forever.

Using a Financial Calculator

The TI BAII Plus calculator is set up for **fixed term** annuities only. That is, the calculator can only deal with annuities that have a definite end date. Therefore, the calculator has no specific built-in function or manner in which to handle a perpetuity. However, because a perpetuity is a specialized version of a regular annuity, a few minor adaptations to the annuity inputs will allow you to trick the calculator into calculating answers for perpetuities.

USING THE TI BAI PLUS CALCULATOR ON PERPETUITIES

- Set the calculator to the correct payment setting (END or BGN).
- Set FV to 0.
- Set N to 10,000.
- Enter the values for the other known variables as normal.
- After all of the known quantities are loaded into the calculator, calculate the unknown quantity (PV, PMT or I/Y depending on the question.)

NOTES

1. The future value of a perpetuity is undefined. But you must enter something into the calculator for FV. The convention is to set FV to 0. This will not affect the perpetuity calculation.
2. Setting N to 10,000 will trick the calculator into generating the answer to a perpetuity question. A large value of N places the future value far into the distance future, creating an annuity that lasts a very, very long time. There is nothing special about 10,000, except that it is a very large number. Any very large number for N will work, but it has to be a large enough number to get the calculator to solve the perpetuity question. Using 10,000 is a large enough number to solve the perpetuity question but not so large that it exceeds the computational abilities of the calculator (i.e. sending the calculator into an indefinite loop or producing an error). If you experience an error, try lowering the value of N a little bit. However, try to avoid using a number less than 10,000 if possible to ensure that the calculator finds the answer to the perpetuity question, and not just a fixed term annuity that lasts a long time.
3. Ensure that the calculator is set to the required payment setting. As with any annuity, the answer for a perpetuity will be different for payments at the end or beginning of the interval.

4. Because you can enter P/Y and C/Y, there is no need to do an interest conversion for a general perpetuity. You simply have to tell the calculator the P/Y and C/Y given in the question, and the calculator will handle any interest conversion internally.
5. The values you enter for PV and PMT must adhere to the cash flow sign convention.

EXAMPLE

In memory of his late father, who was a highly successful marketing manager, Brian would like to set up a marketing scholarship program with his university's school of business. He is able to donate \$100,000 from his father's estate to set up the fund. If the perpetuity fund can earn 6.3% compounded semi-annually and the first scholarship is to be provided immediately, what semi-annual scholarship amount can the fund offer?

Solution:

PMT Setting	BGN
N	10,000
PV	−100,000
FV	0
PMT	?
I/Y	6.3
P/Y	2
C/Y	2

$$PMT = \$3,053.81$$

The fund can provide semi-annual scholarships for \$3,053.81.

EXAMPLE

One of the methods for valuing common shares is to determine the present value of its future dividends. If IBM Corporation is forecast to have year-end dividends of \$1.50 per share in perpetuity, how much should an investor be willing to pay for a share if he requires a 12% compounded quarterly rate of return?

Solution:

PMT Setting	END
N	10,000
PV	?
FV	0
PMT	1.50
I/Y	12
P/Y	1
C/Y	4

$$PV = \$11.95$$

An investor should pay \$11.95 per share.

EXAMPLE

A college invests \$500,000 in an endowment fund to provide month-end scholarships of \$800. What interest rate compounded semi-annually does the fund need to earn to provide the scholarships indefinitely?

Solution:

PMT Setting	END
N	10,000
PV	−500,000
FV	0
PMT	800
I/Y	?
P/Y	12
C/Y	2

$$I/Y = 1.44$$

The fund must earn 1.44% compounded semi-annually.

TRY IT

What amount do you need to invest today at 2.9% compounded quarterly to sustain payments of \$450 at the end of every month indefinitely?

Click to see Solution

PMT Setting	END
N	10,000
PV	?
FV	0
PMT	450
I/Y	2.9
P/Y	12
C/Y	4

$$PV = \$186,656.17$$

TRY IT

You invest \$150,000 today into a investment earning 3.72% compounded monthly. You want to receive beginning-of quarter payments in perpetuity. Calculate the size of the payments.

Click to see Solution

PMT Setting	BGN
N	10,000
PV	−150,000
FV	0
PMT	?
I/Y	3.72
P/Y	4
C/Y	12

$$PMT = \$1,386.40$$

TRY IT

An investor purchased shares in a business when the market price per share was \$100. Each share pays quarterly dividends of \$2.50. What effective rate of return does the investor realize on her investment?

Click to see Solution

PMT Setting	END
N	10,000
PV	−100
FV	0
PMT	2.50
I/Y	?
P/Y	4
C/Y	1

$$I/Y = 10.38\%$$

EXAMPLE

You purchase an investment that will pay you \$1,500 every quarter in perpetuity. You will receive your first payment in seven years. If the perpetuity earns 2.3% compounded semi-annually, what is the purchase price of the investment?

Solution:

Step 1: Draw the timeline for the deferred perpetuity. Treat the perpetuity as a perpetuity due with the payments at the beginning of the interval.

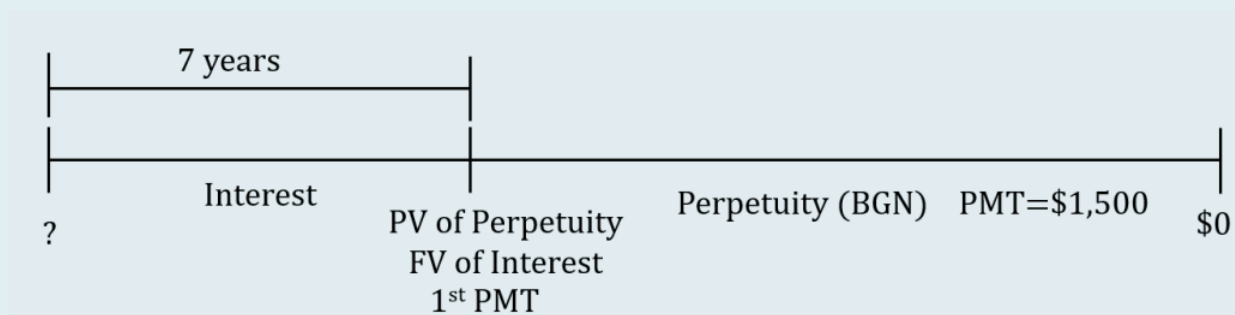


Figure 5.8.1

Some notes about the timeline.

1. The present value at the start of the interest period is unknown.
2. Because the payments are at the beginning of the payment interval, the present value of the perpetuity and the first payment occur at the same time, at the start of the annuity. There is a 7 year gap from the start of the interest period and the present value of annuity/future value of interest/first payment.
3. The \$1,500 payments for the perpetuity last indefinitely.
4. The present value of the perpetuity becomes the future value of the interest period. Then, the future value of the interest period will be moved back 7 years to the start of the interest period.

Step 2: Calculate the present value of the perpetuity.

PMT Setting	BGN
N	10,000
PV	?
FV	0
PMT	1,500
I/Y	2.3
P/Y	4
C/Y	2

$$PV = \$263,117.421\dots$$

Step 3: Calculate the present value for the interest period. The present value calculated in the previous step becomes the future value for the interest period: $FV = \$263,117.421\dots$

N	$2 \times 7 = 14$
PV	?
FV	263,117.421...
PMT	0
I/Y	2.3
P/Y	2
C/Y	2

$$PV = \$224,195.65$$

The purchase price of the investment is \$224,195.65.

TRY IT

A college invests \$500,000 today in an endowment fund to provide semi-annual scholarships in perpetuity. The first scholarship will be given out 10 years from now. If the endowment fund earns 3.9% compounded monthly, what is the scholarship amount?

Click to see Solution

PMT Setting		BGN
N	120	10,000
PV	−500,000	−738,023.537...
FV	Formula does not parse	0
PMT	0	Formula does not parse
I/Y	3.9	3.9
P/Y	12	2
C/Y	12	12

$$PMT = \$14,229.16$$

Exercises

1. How much money must be invested today to sustain beginning-of-six month payments of \$20,000 in perpetuity at 8.4% compounded semi-annually?

Click to see Answer

\$496,190.48

2. What quarterly payments in perpetuity can be received if \$2,400,000 is invested at 4.4% compounded quarterly?

Click to see Answer

\$26,400

3. You want to receive \$10,000 every six months in perpetuity from an investment. How much money do you need to invest if the money earns 5.35% compounded quarterly?

Click to see Answer

\$371,349.38

4. You invest \$500,000 today in an investment at 5.5% compounded annually. What payment can you receive every six months in perpetuity if the first payment is received today?

Click to see Answer

\$13,207.62

5. How much money is required today to fund a perpetuity earning 5.65% annually that needs to pay out \$17,000 at the end of each year?

Click to see Answer

\$300,884.96

6. If \$2,500,000 was invested at 4.8% compounded semi-annually, what payment at the end of every six months could be sustained in perpetuity?

Click to see Answer

\$60,000

7. The dean of the School of Business and Applied Arts at Red River College in Winnipeg wants to establish a scholarship program for the newly created finance major in its business administration stream. He wants to distribute five \$2,000 scholarships annually starting immediately. How much money must he raise from college supporters if the perpetuity can earn 3.6% compounded monthly?

Click to see Answer

\$283,224.19

8. Samson just won the grand prize of \$50 million in the Lotto Max lottery. If he invests the money into a perpetuity fund earning 5.5% compounded annually, what monthly payment can he expect to receive starting today?

Click to see Answer

\$222,589.59

9. An investor states that he would be willing to pay \$25 for common shares to achieve his desired rate of return of 15% annually. What are the forecasted annual dividends starting one year from now?

Click to see Answer

\$3.75

10. The Coca-Cola Company is forecasted to have dividends of \$0.50 per share quarterly for the next five years, followed by dividends of \$0.70 per share quarterly in perpetuity. If an investor desires a 10% compounded annually rate of return, what amount would she be willing to pay per share?

Click to see Answer

\$25.89

11. Indigo's will states that \$150,000 is to be set aside into a fund that will make annual payments to her grandson starting when he turns 18 years old. If Indigo dies when her grandson is six years old and the fund can earn 4.9% compounded quarterly, what annual payment will he receive in perpetuity?

Click to see Answer

\$12,791.51

12. In 2009, the Canadian federal government committed \$1.5 million in annual operating funding for the Canadian Museum for Human Rights in Winnipeg, which opened in 2014. If the government had funded the museum by setting up an ordinary perpetuity in 2009 that could earn prevailing interest rates of 3% compounded annually, how much money would have been required?

Click to see Answer

\$45,757,082.97

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5.9 REVIEW EXERCISES

1. Sangarwe will deposit \$300 every quarter into an investment annuity earning 4.5% compounded quarterly for seven years. What is the difference in the amount of money that she will have after seven years if payments are made at the beginning of the quarter instead of at the end?

Click to see Answer

\$110.36

2. Canseco wants to have enough money so that he could receive payments of \$1,500 every month for the next nine-and-a half years. If the annuity can earn 6.1% compounded semi-annually, how much less money does he need if he takes his payments at the end of the month instead of at the beginning?

Click to see Answer

\$652.42

3. Brianne has a \$21,000 loan being charged 8.4% compounded monthly. What are the month-end payments on her loan if the debt will be extinguished in five years?

Click to see Answer

\$429.84

4. Consider an investment of \$225,000 earning 5% annually. How long could it sustain annual withdrawals of \$20,000 starting immediately?

Click to see Answer

16 years

5. The advertised month-end financing payments on a \$28,757.72 car are \$699 for a four-year term. What semi-annual interest rate is being used in the calculation?

Click to see Answer

7.9%

6. Kubb Bakery estimates it will need \$198,000 at a future point to expand its production plant. At the end of each month, the profits of Kubb Bakery average \$20,000, of which the owner will commit 70% toward the expansion. If the savings annuity can earn 7.3% compounded quarterly, how long will it take to raise the necessary funds?

Click to see Answer

1 year, 2 months

7. An investment fund has \$7,500 in it today and is receiving contributions of \$795 at the beginning of every quarter. If the fund can earn 3.8% compounded semi-annually for the first one-and-a-half years, followed by 4.35% compounded monthly for another one-and three quarter years, what will be the maturity value of the fund?

Click to see Answer

\$19,695.13

8. A \$17,475 Toyota Matrix is advertised with month-end payments of \$264.73 for six years. What monthly compounded rate of return (rounded to one decimal) is being charged on the vehicle financing?

Click to see Answer

2.9%

9. A variable rate loan has a balance remaining of \$17,000 after two years of fixed end-of-month payments of \$655. If the monthly compounded interest rate on the loan was 5.8% for the first 10 months followed by 6.05% for 14 months, what was the initial amount of the loan?

Click to see Answer

\$19,894.24

10. Hank has already saved \$68,000 in his RRSP. Suppose he needs to have \$220,000 saved by the end of 10 years. What are his monthly payments starting today if the RRSP can earn 8.1% compounded annually?

Click to see Answer

\$394.14

11. Many consumers carry a balance each month on their credit cards and make minimal payments toward their debt. If a consumer owes \$5,000 on a credit card being charged 18.3% compounded daily interest, how long will it take him to pay off his debt with month-end payments of \$100?

Click to see Answer

8 years

12. You have a loan for \$20,000 on which you are charged 6% compounded quarterly. What payment amount at the end of every six months would reduce the loan to \$15,000 after two years? What is the interest portion of the total payments made?

Click to see Answer

\$1799.23, \$2,196.92

13. Karen is saving \$1,500 at the end of every six months into an investment that earns 9.4% compounded monthly for the next 20 years. The maturity value will then be rolled into an investment earning 5.85% compounded annually, from which she plans on withdrawing \$23,800 at the beginning of each year. How long will the annuity sustain the withdrawals?

Click to see Answer

9 years

14. Being able to start an RRSP with a lump-sum investment can reduce your end-of-month contributions. For any 35-year term RRSP earning 8.7% compounded annually, calculate the monthly contribution necessary to have a maturity value of \$1,000,000 if the starting lump sums are \$5,000, \$10,000, \$15,000, and \$20,000. What do you observe from your calculations?

Click to see Answer

\$360.92, \$324.06, \$287.19, \$250.31

15. An annuity needs to pay out \$1,000 at the end of every quarter for three years. Using an interest rate of 5% quarterly throughout, what amount of money must be invested today to fund the investment if the

first payment is paid out in four years?

Click to see Answer

\$9,195.75

16. Red Deer College wants to set up a scholarship for students in its business programs such that at the end of every year it could distribute a total of \$50,000. If the perpetuity fund can earn 4.85% compounded semiannually, how much money will need to be raised to fund the scholarship?

Click to see Answer

\$1,019,577.58

17. Procter and Gamble shares are valued at \$61.00 with perpetual year-end dividends of 3.1639%. What dividend payment would a holder of 750 shares receive in perpetuity assuming the share price and dividend rate remain unchanged?

Click to see Answer

\$1447.48

18. The common shares of The Coca-Cola Company are forecast to pay \$1.55 per share at the end of the next four years, and then \$2.05 annually in perpetuity. If the market rate of return on such shares is 2.89%, what price should an investor be willing to pay today?

Click to see Answer

\$69.07

19. A Canadian college plans to implement a new business major in five years. To support the new program, it wants to offer 15 annual \$2,500 scholarships at the beginning of each school year in perpetuity. If the scholarship fund can earn 4.65% compounded annually, what amount of money does the college need to raise today to fund the program?

Click to see Answer

\$672,390.21

20. AVCO Financial is under contract with a national retail chain to purchase its loan contracts on its date

of sale. Under a special promotion, the retail chain allows a customer to defer her payments. If AVCO purchases the contract for \$5,276.83 at its contractual rate of 21% compounded monthly and the consumer is required to make 30 month-end payments of \$425, in how many months will AVCO receive its first payment?

Click to see Answer

37 months

21. Jacques Cousteau just won the \$10 million Powerball lottery. He has been offered the following choices on how to collect his winnings.

- **Option 1:** A one-time lump-sum payment of \$4,289,771.59 today.
- **Option 2:** Year-end payments of \$400,000 for 25 years at 5% effective with the first payment received in five years.
- **Option 3:** Annual payments of \$207,150 in perpetuity starting today earning 5% compounded annually.

Which option is the best financial choice?

Click to see Answer

PV of Option 1: \$4,289,771.59, PV of Option 2: \$4,638,049.33, PV of Option 3: \$4,350,150; Option #2 is best

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5.10 CHAPTER SUMMARY

Key Concepts

- **5.1 Fundamentals of Annuities**
 - Understanding what an annuity is.
 - The four different types of annuities.
 - **5.2 Future Value of Annuities**
 - The future value of ordinary annuities.
 - The future value of annuities due.
 - Variable changes in future value annuity calculations.
 - **5.3 Present Value of Annuities**
 - The present value of ordinary annuities.
 - The present value of annuities due.
 - Variable changes in present value annuity calculations.
 - **5.4 Calculating the Payment**
 - Calculating the annuity payment amount for both ordinary annuities and annuities due.
 - **5.5 Calculating the Term of an Annuity**
 - Calculating the number of annuity payments for both ordinary annuities and annuities due.
 - Calculating the term of an annuity for both ordinary annuities and annuities due.
 - **5.6 Calculating the Interest Rate of an Annuity**
 - Calculating the interest rate for both ordinary annuities and annuities due.
 - **5.7 Deferred Annuities**
 - The stages of deferred annuities.
 - The four common unknown variables and how to solve for them.
 - **5.8 Perpetuities**
 - An explanation of perpetuities.
 - Ordinary perpetuities and perpetuities due.
 - How to solve perpetuity scenarios.
-

Glossary of Terms

- **Annuity.** A continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation.
 - **Annuity Due.** Annuity payments that are each made at the beginning of a payment interval.
 - **Annuity Payment.** The dollar amount of the equal periodic payment in an annuity environment.
 - **Deferred Annuity.** A financial transaction where annuity payments are delayed until a certain period of time has elapsed.
 - **Down Payment.** A portion of the purchase price required up front.
 - **Future Value of an Annuity.** The sum of all the future values for all of the annuity payments when they are moved to the end of the last payment interval.
 - **General Annuity.** An annuity in which the payment frequency and compounding frequency are unequal.
 - **General Annuity Due.** An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.
 - **Ordinary Annuity.** Annuity payments that are each made at the end of a payment interval. This is the most common form of an annuity payment.
 - **Ordinary General Annuity.** An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs one interval after the beginning of the annuity, while the last payment is on the same date as the end of the annuity.
 - **Ordinary Simple Annuity.** An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs one interval after the beginning of the annuity while the last payment is on the same date as the end of the annuity.
 - **Payment Frequency.** The number of annuity payments in a complete year.
 - **Payment Interval.** The amount of time between each continuous and equal annuity payment.
 - **Period of Deferral.** The time segment of a deferred annuity where the single payment earns interest and no contributions are made to the investment.
 - **Perpetuity.** A special type of annuity that has fixed, regular payments continuing indefinitely.
 - **Present Value of an Annuity.** The sum of all the present values for all of the annuity payments when they are moved to the beginning of the first payment interval.
 - **Simple Annuity.** An annuity in which the payment frequency and compounding frequency are equal.
 - **Simple Annuity Due.** An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.
-

Formulas

• Symbols Used

- FV = Future value or maturity value
- PV = Present value of principal
- PMT = Annuity payment amount
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency
- N = Total number of annuity payments

• Formulas Used

- Total Number of Payments: $n = P/Y \times \text{time in years}$
 - Future Value of Ordinary Annuity: $FV = PMT \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$
 - Future Value of Annuity Due: $FV = PMT \times (1 + i_2) \times \left[\frac{(1 + i_2)^n - 1}{i_2} \right]$
 - Present Value of Ordinary Annuity: $PV = PMT \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$
 - Present Value of Annuity Due: $PV = PMT \times (1 + i_2) \times \left[\frac{1 - (1 + i_2)^{-n}}{i_2} \right]$
 - Present Value of Ordinary Perpetuity: $PV = \frac{PMT}{i_2}$
 - Present Value of Perpetuity Due: $PV = (1 + i_2) \times \frac{PMT}{i_2}$
-

Calculator

• Time Value of Money Functions

- The time value of money buttons are the five buttons located on the third row of your calculator.

Calculator Symbol	Characteristic	Data Entry Requirements
N	The number of payment periods	An integer or decimal number; no negatives
I/Y	The nominal interest rate per year	Percent format without the % sign (i.e., 7% is just 7)
PV	Present value or principal	An integer or decimal number
PMT	Annuity payment amounts	An integer or decimal number
FV	Future value or maturity value	An integer or decimal number

To enter any information into any one of these buttons or variables, called loading the calculator, key in the information first and then press the appropriate button.

- The frequency function is logically placed above the I/Y button and is labeled P/Y. This function addresses payment frequencies and compound interest frequencies. Access the function by pressing 2nd P/Y to find the following entry fields, through which you can scroll using your arrow buttons.

Calculator Symbol	Characteristic	Data Entry Requirements
P/Y	Annuity payments per year (payment frequency)	A positive, nonzero number only
C/Y	Compounds per year (compounding frequency)	A positive, nonzero number only

To enter any information into one of these fields, scroll to the field on your screen, key in the data, and press ENTER. When you enter a value into the P/Y field, the calculator will automatically copy the value into the C/Y field for you. If in fact the C/Y is different, you can change the number manually. To exit the P/Y window, press 2nd QUIT.

- To keying in a question, you must load the calculator with six of the seven variables. To solve for the missing variable, press CPT followed by the variable.
- The cash flow sign convention is used for the PV, PMT, and FV buttons. If money leaves you, you must enter it as a negative. If money comes at you, you must enter it as a positive.

• Payment Setting

- The calculator default is for END mode, which is the ordinary annuity.
- The annuity type (payment timing) setting can be found on the second shelf above the PMT key. This function works as a toggle.
- To toggle the setting, complete the following sequence:
 1. Press 2nd BGN (the payment button).
 2. Press 2nd SET (the ENTER button) to switch payment settings.
 3. Press 2nd QUIT (the CPT button) to exit the payment setting menu.
- When the calculator is in annuity due mode, a tiny BGN is displayed in the upper right of your calculator.

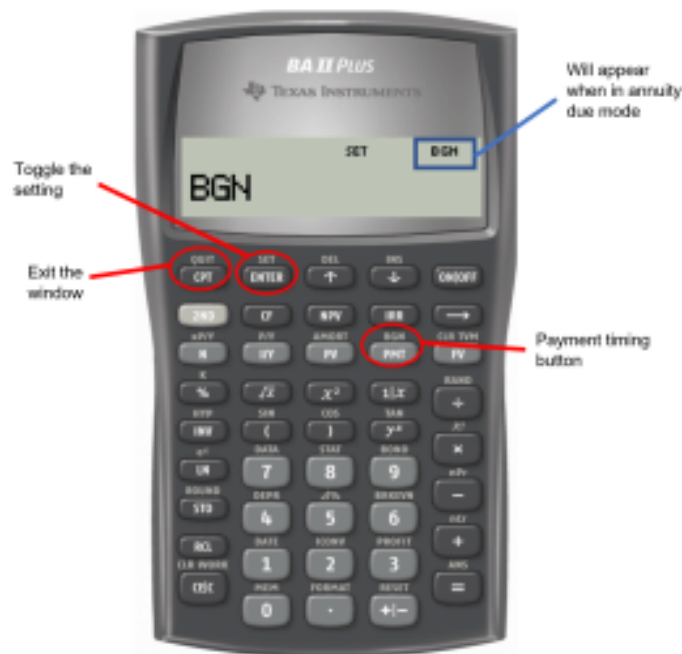


Figure 5.10.1



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[Ordinary Annuity Calculations \(PV, PMT, FV\)](#) by Joshua Emmanuel [4:31] (transcript available).



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[Annuity Due Calculations Using the BAII Plus](#) by Joshua Emmanuel [2:51] (transcript available).



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[Annuity Due Calculations Using BAII Plus-Part 2](#) by Joshua Emmanuel [2:05] (transcript available).

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PART VI

LOANS AND MORTGAGES

It would be nice to pay cash for everything that you acquire. However, goods such as housing, automobiles, furniture, and electronics require more money than most Canadians have lying around. If you tried saving up and paying cash for these large purchases, it would take a long time to accumulate enough funds. Take the example of a new home. If you saved \$300 per month at 5% compounded annually, it would take approximately 30 years to save up \$250,000 in cash, and in the meantime, of course, you would still need to pay for somewhere to live.

So it is not necessarily bad to borrow money at some point. However, to borrow wisely you had better understand where your money is going. Would it shock you to learn that in a typical mortgage arrangement most homeowners pay approximately double for their homes? That is, \$250,000 for the home and \$250,000 in interest!

Businesses borrow money for many of the same reasons as consumers. Loans and mortgages in the business world are commonplace. Certain business activities need to be financed upfront. For example, new products must be researched and developed before a single unit can be sold to consumers. Research and development requires investments that will not be reimbursed until the products turn a profit. All company debts must be accurately recorded onto balance sheets to reflect the balances owing. Income statements must appropriately track interest expenses or earnings. Because businesses can deduct their interest expenses against corporate taxable income and lower their taxes, it is important to see what proportion of their loan payments is going to interest.

When you take out a mortgage for yourself or your business, where does your money go? You need a chart of your loan payments showing how much interest the bank charges and how much is applied against your principal. This chapter takes you through calculating the balance, the principal and interest components of any single payment for a loan. You will now see how to calculate the final payment for a loan. Finally, you will be guided through the largest transactions you personally are likely to make: basic mortgages and renewals.

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6.1 AMORTIZATION SCHEDULES

LEARNING OBJECTIVES

- Construct a full or partial amortization schedule for a loan.

At some point in your life, you will probably take out a loan from a bank, credit union, or some other financial institution. You might need a loan to buy a new car, purchase a home, go back to school, or start a new business. When you borrow money, you have to repay the amount of money you borrowed plus the interest the loan accumulates over the term of the loan. You repay the loan by making regular payments over the term of the loan. Each payment does two things: it pays all of the interest due on the loan at the time the payment is made and the remainder of the payment goes to paying down the loan amount. This process of gradually repaying the loan with periodic payments over the term of the loan is called **amortization**.

What is Amortization?

Amortization is a process by which the principal of a loan is extinguished over the course of an agreed-upon time period through a series of regular payments that go toward both the accruing interest and principal reduction. Two components make up the agreed-upon time component.

1. **Amortization Term.** The amortization term is the length of time for which the interest rate and payment agreement between the borrower and the lender will remain unchanged. Thus, if the agreement is for monthly payments at a 5% fixed rate over five years, it is binding for the entire five years. Or if the agreement is for quarterly payments at a variable rate of prime plus 2% for three years, then interest is calculated on this basis throughout the three years.
2. **Amortization Period.** The amortization period is the length of time it will take for the principal to be reduced to zero. For example, if you agree to pay back your car loan over six years, then after six years you reduce your principal to zero and your amortization period is six years.

In most relatively small purchases, the amortization term and amortization period are identical. For example, a vehicle loan has an agreed-upon interest rate and payments for a fixed term. At the end of the term, the loan is fully repaid. However, larger purchases such as real estate transactions typically involve too much money to be repaid under short time frames. Financial institutions hesitate to agree to amortization terms of much more than five to seven years because of the volatility and fluctuations of interest rates. As a result, a term of five years may be established with an amortization period of 25 years. When the five years elapse, a new term is established as agreed upon between the borrower and lender. The conditions of the new term reflect prevailing interest rates and a payment plan that continues to extinguish the debt within the original amortization period.

Amortization Schedules

An **amortization schedule** shows the payment amount, principal component, interest component, and remaining balance for every payment in the annuity. As the title suggests, it provides a complete understanding of where the money goes, identifying how much of each payment goes to interest and how much goes to principal.

An amortization schedule has five columns:

- **Payment Number.** There is a row for every payment made to repay the loan.
- **Payment.** The periodic payment made to repay the loan. All of the payments are the same (PMT), except for the last payment.
- **Interest Paid.** For each row, the interest paid entry is the amount of interest due on the loan at the time of the corresponding payment.
- **Principal Paid.** For each row, the principal paid entry is the amount of principal repaid at the time of the corresponding payment, after the interest is paid.
- **Balance.** For each row, the balance records how much of the original loan amount remains after the payment is made.

To fill in an amortization schedule, you first need to have all of the details about the loan, including the loan amount (PV), the payment (PMT), the number of payments (N), and the interest rate. If any of these quantities are missing, calculate out the missing value before completing the amortization schedule.

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				Loan Amount ¹
1	PMT ²	INT ³	PRN ⁴	BAL ⁵
2	PMT ²	INT ³	PRN ⁴	BAL ⁵
⋮	⋮	⋮	⋮	⋮
N-1	PMT ²			
N	Final Payment ¹⁰	Final INT ⁹	Final PRN ⁸	0 ⁷
Totals	Total Amount Paid ¹²	Total Interest Paid ¹³	Total Principal Paid ¹¹	

Follow these steps to fill in the amortization schedule.

1. In row 0, the only entry is the loan amount in the balance column.
2. Fill in the **rounded** loan payment down the payment column, except for the last payment.
3. Calculate the interest paid portion of the payment. The interest is the balance from the previous row times the periodic interest rate: **Interest Paid = Balance from Previous Row $\times$ i** . **Note:** this calculation assumes that the payment frequency and the compound frequency for the interest rate are the same (i.e. a simple annuity).
4. Calculate the principal paid portion of the payment. The principal paid is the difference between the payment and the interest paid: **Principal Paid = $PMT - \text{Interest Paid}$** .
5. Calculate the outstanding balance on the loan after the payment is made. The balance is difference between the balance in the previous row and the principal paid:
Balance = Balance from Previous Row $-$ Principal Paid.
6. For each payment, repeat steps 2 through 5, except for the last row.
7. The last balance entry is 0. Because this is the last payment, the loan must be paid off, which means the balance is reduced to 0.
8. The final principal paid entry equals the balance entry from row N-1.
9. Calculate the final interest paid portion in the same way as in step 3.
10. The final payment is the sum of the final interest paid entry and the final principal paid entry:
Final Payment = Final INT + Final PRN.
11. The total principal paid is the sum of the principal paid column, and is just the loan amount:
Total Principal Paid = Loan Amount.
12. The total amount paid is the sum of the payment column:
Total Amount Paid = $(N - 1) \times PMT + \text{Final Payment}$.

13. The total interest paid is the sum of the interest paid column, and equals the difference between the other two column totals:

$$\text{Total Interest Paid} = \text{Total Amount Paid} - \text{Total Principal Paid}.$$

NOTES

1. The manual calculation of the interest paid entry above is based on the assumption that the payment frequency and the compounding frequency are equal. If the payment frequency and the compounding frequency are not equal, an interest conversion is required to convert the interest rate to the equivalent rate with the compounding frequency equal to the payment frequency. However, if you use the TI BAII Plus's built-in amortization worksheet (described below), no interest conversion is required.
2. As you fill in the schedule, round the entries to two decimal places.
3. Make sure that you are using the payment **rounded** to two decimal places in the schedule. In real life, your loan payment is rounded to two decimal places and all of the loan calculations are based on this rounded payment.
4. The final payment must be the exact amount that will result in a 0 balance in the last row. You do not want the last payment to be too large (i.e. an overpayment) because then you are giving the bank more money than necessary. You do not want the last payment to be too small (i.e. an underpayment) because you must have the loan completely paid off with the final payment. Consequently, the last row of the schedule is reverse engineered, working from right to left.

EXAMPLE

A \$3,000 loan at 8% compounded quarterly is repaid with quarterly payments of \$800. Construct the amortization schedule for the loan.

Solution:

Step 1: Calculate the number of payments.

PMT Setting	END
N	?
PV	3,000
FV	0
PMT	−800
I/Y	8
P/Y	4
C/Y	4

$$N = 3.936... \rightarrow 4 \text{ payments}$$

Step 2: Complete the amortization schedule. Because the payment frequency and the compounding frequency are equal, no interest conversion is required. The calculations for each entry are shown in blue. The periodic interest rate is $i = \frac{8\%}{4} = 2\%$.

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$3,000
1	\$800	\$60 Formula does not parse	\$740 Formula does not parse	\$2,260 Formula does not parse
2	\$800	\$45.20 Formula does not parse	\$754.80 Formula does not parse	\$1,505.20 Formula does not parse
3	\$800	\$30.10 Formula does not parse	\$769.90 Formula does not parse	\$735.30 Formula does not parse
4	\$750.37 Formula does not parse	\$15.07 Formula does not parse	\$735.30	\$0
Totals	\$3,150.37 Formula does not parse	\$150.37 Formula does not parse	\$3,000	

Using a Financial Calculator

Although the calculations in an amortization schedule are relatively straightforward, the manual calculations are time-consuming, especially when the schedule has a lot of rows. A financial calculator, such as the TI BAII Plus, has a built-in amortization worksheet that can quickly calculate the entries for each row of the schedule, with the exception of the last row.

USING THE TI BAI PLUS CALCULATOR TO CONSTRUCT AN AMORTIZATION SCHEDULE

The amortization worksheet has five variables (P1, P2, BAL, PRN, INT). You use the up and down arrows to scroll through the amortization worksheet.

- P1 is the starting payment number. The calculator works with a single payment or a series of payments.
- P2 is the ending payment number. This number is the same as P1 when you work with a single payment (i.e. to find the entries for a row of the amortization schedule). When you work with a series of payments later in this chapter, you set it to a number higher than P1.
- BAL is the principal balance remaining after the P2 payment number. The cash flow sign is correct as indicated on the calculator display.
- PRN is the principal portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
- INT is the interest portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.

To use the amortization worksheet to complete an amortization schedule:

1. Solve for any unknown quantities about the loan. You need to know all of the information about the loan first before you can use the amortization worksheet.
2. Enter the values of all seven time value of money variables into the calculator (N, PV, FV, PMT, I/Y, P/Y, C/Y). If you calculated PMT in the first step, you must **re-enter** it **rounded** to two decimals and with the correct cash flow sign. Make sure the payment setting is set to END, and obey the cash flow sign convention. Because this is a loan, PV (the loan amount) is positive and PMT is negative.
3. Go to the amortization worksheet by pressing 2nd AMORT (the PV button).

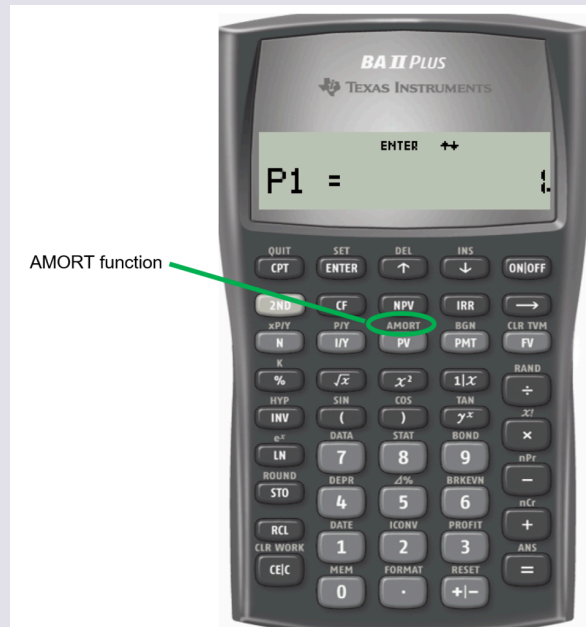


Figure 6.1.1

4. To view the entries for a specific row of the schedule, set P1 and P2 to the row number. For example, to view the entries for row 5, set P1=5 and P2=5.
 - i. At the P1 prompt, enter the row number and press ENTER.
 - ii. Press the down arrow.
 - iii. At the P2 prompt, enter the row number and press ENTER.
 - iv. Press the down arrow.
 - v. The BAL entry is the balance entry for the corresponding row.
 - vi. Press the down arrow.
 - vii. The PRN entry is the principal paid entry for the corresponding row.
 - viii. Press the down arrow.
 - ix. The INT entry is the interest paid entry for the corresponding row.
 - x. Press the down arrow the return to the P1 screen.
5. Repeat the previous step with a different row number to view the entries for a different row.

NOTES

1. Because you can enter P/Y and C/Y, there is no need to do an interest conversion if the payment frequency and compounding frequency are different. You simply have to tell the calculator the P/Y and C/Y given in the question, and the calculator will handle any interest conversion internally.
2. Make sure to re-enter PMT rounded to 2 decimal places before using the amortization worksheet. If you enter PMT with all of the decimal places, you will not get the correct entries for the amortization schedule.
3. The amortization worksheet on the calculator will only work with loans that have payments at the end of the interval (i.e. an ordinary annuity).
4. The amortization worksheet gives you the entries for the schedule in reverse order, from right to left: BAL, PRN, and INT.
5. As you read the entries off of the amortization worksheet on the calculator and put them in the schedule, round the entries to 2 decimal places.
6. For the last row, the amortization worksheet will NOT give you the correct BAL and PRN entries. But, the INT entry for the last row is correct. You will still have to reverse engineer the last row using the steps described above, but you get the INT entry for the last row from the amortization worksheet on the calculator.



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[Amortization Schedule using BAII Plus](#) by Joshua Emmanuel [3:55] (transcript available).

EXAMPLE

You take out a \$50,000 business loan at 5% compounded quarterly. The loan requires semi-annual payments for three years. Construct the amortization schedule for the loan.

Solution:

Step 1: Calculate the payment.

PMT Setting	END
N	$2 \times 3 = 6$
PV	50,000
FV	0
PMT	?
I/Y	5
P/Y	2
C/Y	4

$$PMT = \$9,082.24$$

Step 2: Enter the information into the time value of money buttons on the calculator. Make sure that you re-enter the payment rounded to 2 decimal places.

PMT Setting	END
N	6
PV	50,000
FV	0
PMT	−9,082.24
I/Y	5
P/Y	2
C/Y	4

Step 3: Complete the amortization schedule using the amortization worksheet on the calculator.

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$50,000
1	\$9,082.24	\$1,257.81	\$7,824.43	\$42,175.57
2	\$9,082.24	\$1,060.98	\$8,021.26	\$34,154.31
3	\$9,082.24	\$859.19	\$8,223.05	\$25,931.27
4	\$9,082.24	\$652.33	\$8,429.91	\$17,501.36
5	\$9,082.24	\$440.27	\$8,641.97	\$8,859.39
6	\$9,082.26	\$222.87	\$8,859.39	\$0
Totals	\$54,493.46	\$4,493.46	\$50,000	

- **Row 1:** In the amortization worksheet, set P1=1 and P2=1.
- **Row 2:** In the amortization worksheet, set P1=2 and P2=2.
- **Row 3:** In the amortization worksheet, set P1=3 and P2=3.
- **Row 4:** In the amortization worksheet, set P1=4 and P2=4.
- **Row 5:** In the amortization worksheet, set P1=5 and P2=5.
- **Row 6:**

- The principal paid (PRN) entry is the balance (BAL) entry from row 5.
- Get the interest paid (INT) entry from the calculator by setting P1=6 and P2=6.
- The payment is the sum of the principal paid and interest paid entries:
 $222.87 + 8,859.39 = 9,082.26$.

- **Totals Row:**

- The principal paid total is the loan amount (\$50,000).
- The payments total is the sum of the payments:
 $5 \times 9,082.24 + 9,082.26 = 54,493.46$.
- The interest paid total is the difference in the other two column totals:
 $54,493.46 - 50,000 = 4,493.46$.

TRY IT

A farmer purchased a John Deere combine for \$369,930. The equipment dealership sets up a financing plan to allow for end-of-quarter payments of \$51,000 at 7.8% compounded monthly. Construct a complete amortization schedule including the totals.

Click to see Solution

PMT Setting	END
N	?
PV	369,930
FV	0
PMT	−51,000
I/Y	7.8
P/Y	4
C/Y	12

$N = 7.901... \rightarrow 8 \text{ payments}$

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$369,930
1	\$51,000	\$7,260.63	\$43,739.37	\$326,190.63
2	\$51,000	\$6,402.15	\$44,597.85	\$281,592.78
3	\$51,000	\$5,526.83	\$45,473.17	\$236,119.61
4	\$51,000	\$4,634.33	\$46,365.67	\$189,753.93
5	\$51,000	\$3,724.31	\$47,275.69	\$142,478.24
6	\$51,000	\$2,796.42	\$48,203.58	\$94,274.66
7	\$51,000	\$1,850.33	\$49,149.69	\$45,124.99
8	\$46,010.66	\$885.67	\$45,124.99	\$0
Totals	\$403,010.66	\$33,080.66	\$369,930	

Partial Amortization Schedules

Sometimes, businesses are interested only in creating partial amortization schedules, which are amortization

schedules that show only certain payments or a specified range of payments, and not the entire amortization schedule. This may occur for a variety of reasons. For instance, the complete amortization schedule may be too long (imagine weekly payments on a 25-year loan), or maybe you are solely interested in the principal and interest portions during a specific period of time for accounting and tax purposes.

The TI BAII Plus calculator makes it easy to calculate the entries for a specific row of the schedule by setting P1 and P2 equal to the row number you want to view.

EXAMPLE

You borrowed \$45,000 to purchase a new car. The loan agreement calls for monthly payments of \$900 at 3.5% compounded monthly. Construct a partial amortization schedule showing the details for the first two payments, payment #14, the last two payments, and the totals.

Solution:

Step 1: Calculate the number of payments.

PMT Setting	END
N	?
PV	45,000
FV	0
PMT	−900
I/Y	3.5
P/Y	12
C/Y	12

$$N = 54.122... \rightarrow 55 \text{ payments}$$

Step 2: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	55
PV	45,000
FV	0
PMT	−900
I/Y	3.5
P/Y	12
C/Y	12

Step 3: Complete the partial amortization schedule using the amortization worksheet on the calculator.

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$45,000
1	\$900	\$131.25	\$768.75	\$44,231.25
2	\$900	\$129.01	\$770.99	\$43,460.26
14	\$900	\$101.59	\$798.41	\$34,031.06
54	\$900	\$2.94	\$897.06	\$110.51
55	\$110.83	\$0.32	\$110.51	\$0
Totals	\$48,710.83	\$3,710.83	\$45,000	

- **Row 1:** In the amortization worksheet, set P1=1 and P2=1.
- **Row 2:** In the amortization worksheet, set P1=2 and P2=2.
- **Row 14:** In the amortization worksheet, set P1=14 and P2=14.
- **Row 54:** In the amortization worksheet, set P1=54 and P2=54.
- **Row 55:**
 - The principal paid (PRN) entry is the balance (BAL) entry from row 54.

- Get the interest paid (INT) entry from the calculator by setting P1=55 and P2=55.
- The payment is the sum of the principal paid and interest paid entries:

$$0.32 + 110.51 = 110.83.$$

- **Totals Row:**

- The principal paid total is the loan amount (\$45,000).
- The payments total is the sum of the payments:
$$54 \times 900 + 110.83 = 48,710.83.$$
- The interest paid total is the difference in the other two column totals:
$$48,710.83 - 45,000 = 3,710.83.$$

TRY IT

Ron and Natasha had Oasis Leisure and Spa install an in-ground swimming pool for \$51,000. The financing plan through the company allows for end-of-month payments for five years at 6.9% compounded quarterly. Construct a partial amortization schedule showing the last two rows and the totals.

Click to see Solution

PMT Setting	END
N	60
PV	51,000
FV	0
PMT	?
I/Y	6.9
P/Y	12
C/Y	4

$$PMT = \$1,006.51$$

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
59	\$1,006.51	\$11.41	\$995.10	\$1,000.98
60	\$1,006.70	\$129.01	\$1,000.98	\$0
Totals	\$60,390.79	\$9,390.79	\$51,000	

Exercises

1. A \$27,000 loan at 6.8% compounded quarterly is repaid with quarterly payments over two years. Construct the amortization schedule for the loan.

Click to see Answer

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$27,000
1	\$3,638.26	\$459	\$3,179.26	\$23,820.74
2	\$3,638.26	\$404.95	\$3,233.31	\$20,587.43
3	\$3,638.26	\$349.99	\$3,288.27	\$17,299.16
4	\$3,638.26	\$294.09	\$3,344.17	\$13,954.98
5	\$3,638.26	\$237.23	\$3,401.03	\$10,553.96
6	\$3,638.26	\$179.42	\$3,458.84	\$7,095.12
7	\$3,638.26	\$120.62	\$3,517.64	\$3,577.47
8	\$3,638.29	\$60.82	\$3,577.47	\$0
Totals	\$29,106.11	\$2,106.11	\$27,000	

2. A \$192,000 loan at 4.75% compounded semi-annually is repaid with semi-annual payments of \$26,640. Construct the amortization schedule for the loan.

Click to see Answer

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$192,000
1	\$26,640	\$4,560	\$22,080	\$169,920
2	\$26,640	\$4,035.60	\$22,604.40	\$147,315.60
3	\$26,640	\$3,498.75	\$23,141.25	\$124,174.35
4	\$26,640	\$2,949.14	\$23,690.86	\$100,483.49
5	\$26,640	\$2,386.48	\$24,253.52	\$76,229.97
6	\$26,640	\$1,810.46	\$24,829.54	\$51,400.43
7	\$26,640	\$1,220.76	\$25,419.24	\$25,981.19
8	\$26,598.24	\$617.05	\$25,981.19	\$0
Totals	\$239,718.24	\$47,718.24	\$192,000	

3. A \$450,000 loan at 7.3% compounded semi-annually is repaid with monthly payments for 25 years. Construct a partial amortization schedule showing the details of payment 100, the last three payments and the totals.

Click to see Answer

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
100	\$3,235.66	\$2,262.04	\$973.62	\$376,483.42
298	\$3,235.66	\$57.48	\$3,178.18	\$6,412.91
299	\$3,235.66	\$38.43	\$3,197.23	\$3,215.68
300	\$3,234.95	\$19.27	\$3,215.68	\$0
Totals	\$970.697.29	\$520,697.29	\$450,000	

4. Jennifer purchased a new \$60,000 car for \$5,000 down and monthly payments of \$750 at 11.2% compounded monthly. Construct a partial amortization schedule showing the details of payment 30,

the last two payments and the totals.

Click to see Answer

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
30	\$750	\$440.16	\$309.84	\$46,849.89
124	\$750	\$8.02	\$741.98	\$116.81
125	\$117.90	\$1.09	\$116.81	\$0
Totals	\$93,117.90	\$38,117.90	\$55,000	

5. To purchase a piece of land, Hillary took out a \$118,000 loan at 7.95% compounded semi-annually. The loan agreement required her to make quarterly payments for 15 years. Construct a partial amortization schedule showing the details of the first two payments, payment 28, the last two payments and the totals.

Click to see Answer

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$118,000
1	\$3,368.48	\$2,322.40	\$1,046.08	\$116,953.92
2	\$3,368.48	\$2,301.81	\$1,066.67	\$115,887.24
28	\$3,368.48	\$1,597.93	\$1,770.55	\$79,419.86
59	\$3,368.48	\$128.78	\$3,239.70	\$3,303.30
60	\$3,368.31	\$65.01	\$3,303.30	\$0
Totals	\$202,108.63	\$84,108.63	\$118,000	

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6.2 CALCULATING THE PRINCIPAL AND INTEREST COMPONENTS OF A LOAN PAYMENT

LEARNING OBJECTIVES

- Calculate the principal and interest components of any payment.
- Calculate the loan balance after any payment.
- Calculate the total principal paid and total interest paid for a series of payments.

How much of the principal do you pay off when you make a loan payment? One year ago you purchased your \$250,000 dream home on a 25-year mortgage at a fixed 5% compounded semi-annually interest rate. With monthly contributions of \$1,454.01, or \$17,448.12 in total for the past year, you figure you must have put a serious dent in the balance owing. But you get a rude shock when you inspect your mortgage statement and see that the remaining balance is \$244,806.89, reflecting a principal reduction of only \$5,193.11! The other 70% of your hard-earned money, amounting to \$12,255.01, went solely toward the bank's interest charges.

Many people do not fully understand how their loan payments are portioned out. Over the full course of the 25-year mortgage you will pay \$186,204.46 in interest charges at 5% compounded semi-annually, or approximately 74.5% of the home's price tag. That is a total of \$436,204.46 paid on a \$250,000 home. Term, interest rates, payment amounts, and payment frequency all affect the amount of interest you pay.

These calculations should make it clear that both businesses and consumers need to understand the interest and principal components of loan payments, as well as the remaining balance after each loan payment is made. In the previous section, you already learned how to find the interest, principal, and balance for any loan payment—these are just the entries in the amortization schedule. In this section, we will review how to find these values for any payment, as well as learn how to calculate the total interest paid and total principal paid for a series of payments.

Calculating the Interest Paid, Principal Paid and Balance for a Single Payment

At any point during an amortized loan you can precisely calculate how much any single payment contributes toward principal and interest, and what your outstanding balance is on the loan after a payment is made. This is exactly what each row of the amortization schedule tells you—each row separates the payment into its interest paid and principal paid components, and the balance after the payment is made. Just like in the previous section, you can use the amortization worksheet on your financial calculator to find the interest paid (INT), principal paid (PRN), and balance (BAL) for any payment.

USING THE TI BAI PLUS CALCULATOR TO FIND THE INTEREST PAID, PRINCIPAL PAID, AND BALANCE OF A LOAN PAYMENT

The amortization worksheet has five variables (P1, P2, BAL, PRN, INT). You use the up and down arrows to scroll through the amortization worksheet.

- P1 is the starting payment number. The calculator works with a single payment or a series of payments.
- P2 is the ending payment number. This number is the same as P1 when you work with a single payment (i.e. to find the entries for a row of the amortization schedule). When you work with a series of payments later in this section, you set it to a number higher than P1.
- BAL is the principal balance remaining after the P2 payment number. The cash flow sign is correct as indicated on the calculator display.
- PRN is the principal portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
- INT is the interest portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.

To use the amortization worksheet to find the interest paid, principal paid, and balance of a payment:

1. Solve for any unknown quantities about the loan. You need to know all of the information about the loan first before you can use the amortization worksheet.

2. Enter the values of all seven time value of money variables into the calculator (N, PV, FV, PMT, I/Y, P/Y, C/Y). If you calculated PMT in the first step, you must **re-enter** it **rounded** to two decimals and with the correct cash flow sign. Make sure the payment setting is set to END, and obey the cash flow sign convention. Because this is a loan, PV (the loan amount) is positive and PMT is negative.
3. Go to the amortization worksheet by pressing 2nd AMORT (the PV button).

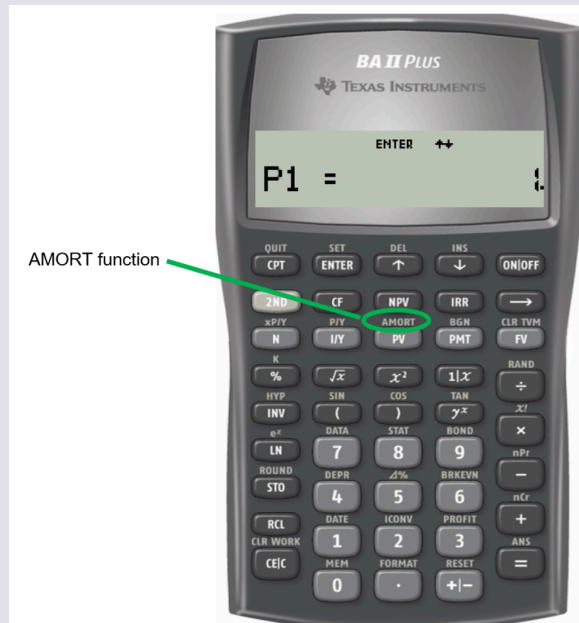


Figure 6.2.1

4. To view the balance, principal paid, and interest paid for a payment, set P1 and P2 to the corresponding payment number. For example, to view the balance, principal paid, and interest paid for payment 5, set P1=5 and P2=5.
 - i. At the P1 prompt, enter the payment number and press ENTER.
 - ii. Press the down arrow.
 - iii. At the P2 prompt, enter the payment number and press ENTER.
 - iv. Press the down arrow.
 - v. The BAL entry is the balance entry for the corresponding row.
 - vi. Press the down arrow.
 - vii. The PRN entry is the principal paid entry for the corresponding row.
 - viii. Press the down arrow.
 - ix. The INT entry is the interest paid entry for the corresponding row.

- x. Press the down arrow the return to the P1 screen.

Remember, if you are interested in a **single** payment, you must set P1 and P2 to the exact same value. For example, if you want to see the details of the 22nd payment then both P1 = 22 and P2 = 22.

NOTE

As described above, you can find the interest paid and principal paid components of any payment by using the amortization worksheet on your calculator. To calculate these values by hand, use the following formulas.

- **Interest Paid for a Payment = Balance from the Previous Row $\times i$**
- **Principal Paid for a Payment = Payment – Interest Paid**

where i is the periodic interest rate for the loan.

But, these formulas require two things: that you know the balance from the previous row and that the payment frequency and the compounding frequency for the loan are equal. Although it is possible to solve both of these problems, such as doing an interest conversion to find the equivalent rate with the compounding frequency equal to the payment frequency, the easiest way to find the interest and principal paid components is to use the amortization worksheet on a financial calculator.

EXAMPLE

Your repaid your student loan of \$20,000 at 3.5% compounded quarterly with month-end payments over five years.

1. What was the outstanding balance on the loan after you made payment 18?
2. How much interest was paid with the 27th payment?
3. How much principal was paid with the 42nd payment?
4. What was the outstanding balance on the loan after three years?

Solution:

Step 1: Calculate the payment.

PMT Setting	END
N	$12 \times 5 = 60$
PV	20,000
FV	0
PMT	?
I/Y	3.5
P/Y	12
C/Y	4

$$PMT = \$363.74$$

Step 2: Calculate the balance for payment 18. To find the balance for payment 18, set P1=18 and P2=18.

PMT Setting	END
N	60
PV	20,000
FV	0
PMT	−363.74
I/Y	3.5
P/Y	12
C/Y	4
P1	18
P2	18

$$BAL = \$14,361.53$$

After 18 payments, the balance on the loan is \$14,361.53.

Step 3: Calculate the interest for payment 27. To find the interest for payment 27, set P1=27 and P2=27.

PMT Setting	END
N	60
PV	20,000
FV	0
PMT	−363.74
I/Y	3.5
P/Y	12
C/Y	4
P1	27
P2	27

$$INT = \$34.20$$

The interest paid for payment 27 is \$34.20.

Step 4: Calculate the principal for payment 42. To find the principal for payment 42, set P1=42 and P2=42.

PMT Setting	END
N	60
PV	20,000
FV	0
PMT	−363.74
I/Y	3.5
P/Y	12
C/Y	4
P1	42
P2	42

$$PRN = \$344.21$$

The principal paid for payment 42 is \$344.21.

Step 5: Calculate the balance after three years. To find the balance after three years, enter the payment number that corresponds to the last payment made in year three. There are 12 payments a year, so the last payment made in year three is 36 (3×12). So, to find the balance after three years, set P1=36 and P2=36.

PMT Setting	END
N	60
PV	20,000
FV	0
PMT	−363.74
I/Y	3.5
P/Y	12
C/Y	4
P1	36
P2	36

$$BAL = \$8,420.49$$

After three years, the balance on the loan is \$8,420.49.

NOTE

The calculator thinks in terms of payment numbers, not years. That is, P1 and P2 must be payment numbers. In the last part of the previous example, a common mistake is to set P1=3 and P2=3 to find the balance at the end of year 3. But P1=P2=3 tells the calculator to find the balance after payment number 3, not after year 3. To find the balance at the end of a particular year, you need the number of the last payment to happen in that year. You can find this payment number by multiplying the payment frequency by the year.

For example, suppose a loan has quarterly payments and you want to find the balance at the end of year 5. The last payment to happen in year five is 20 (4×5). The balance at the end of year 5 is the same as the balance for payment 20. In the amortization worksheet you would set P1=20 and P2=20.

TRY IT

A \$100,000 loan at 4% compounded semi-annually requires monthly payments of \$650.

1. How many payments will it take to pay-off the loan?
2. How much principal is paid with the 97th payment?
3. What is the balance on the loan after the 112th payment?
4. How much interest is paid with the 175th payment?
5. What is the balance on the loan at the end of year 10?

Click to see Solution

1. Calculate the number of payments.

PMT Setting	END
N	?
PV	100,000
FV	0
PMT	−650
I/Y	4
P/Y	12
C/Y	2

$$N = 215.725... \rightarrow 216 \text{ payments}$$

2. Calculate the principal for payment 97.

PMT Setting	END
N	216
PV	100,000
FV	0
PMT	−650
I/Y	4
P/Y	12
C/Y	2
P1	97
P2	97

$$PRN = \$438.48$$

3. Calculate the balance for payment 112.

PMT Setting	END
N	216
PV	100,000
FV	0
PMT	−650
I/Y	4
P/Y	12
C/Y	2
P1	112
P2	112

$$BAL = \$56,789.53$$

4. Calculate the interest for payment 175.

PMT Setting	END
N	216
PV	100,000
FV	0
PMT	−650
I/Y	4
P/Y	12
C/Y	2
P1	175
P2	175

$$INT = \$82.78$$

5. Calculate the balance after 10 years.

PMT Setting	END
N	216
PV	100,000
FV	0
PMT	−650
I/Y	4
P/Y	12
C/Y	2
P1	120
P2	120

$$BAL = \$53,048.37$$

Calculating the Total Interest Paid and Total Principal Paid

for a Series of Payments

Many times in business, you need to know the principal and interest portions for a series of payments. For example, you might need to know the total interest paid or total principal paid on a loan for a particular year. As above, you can use the amortization worksheet on your financial calculator to find the total principal paid or total interest paid for a series of payments.

USING THE TI BAI PLUS CALCULATOR TO FIND THE TOTAL INTEREST PAID OR TOTAL PRINCIPAL PAID FOR A SERIES OF LOAN PAYMENTS

To use the amortization worksheet to find the total interest paid or total principal paid:

1. Solve for any unknown quantities about the loan. You need to know all of the information about the loan first before you can use the amortization worksheet.
2. Enter the values of all seven time value of money variables into the calculator (N, PV, FV, PMT, I/Y, P/Y, C/Y). If you calculated PMT in the first step, you must **re-enter** it rounded to two decimals and with the correct cash flow sign. Make sure the payment setting is set to END, and obey the cash flow sign convention. Because this is a loan, PV (the loan amount) is positive and PMT is negative.
3. Go to the amortization worksheet by pressing 2nd AMORT (the PV button).
4. To view the total principal paid or total interest paid for a series of payments, set P1 to the **first** payment number of the series of payments and set P2 to the **last** payment number of the series of payments. For example, to view the total principal paid or total interest paid for payments four to seven, set P1=4 and P2=7. In the outputs from the amortization worksheet:
 - The BAL entry is the balance after payment number P2. Note that the BAL entry is only tied to the value of P2 and does not depend on the value of P1.
 - The PRN entry is the total principal paid from payment number P1 to payment number P2.
 - The INT entry is the total interest paid from payment number P1 to payment number P2.

P2.

NOTES

1. The PRN entry on the amortization worksheet is the sum of the principal paid entries in the amortization schedule starting at payment number P1 and ending at payment number P2. For example, if P1=4 and P2=7, the PRN entry tells you the sum of the principal paid column in the amortization schedule starting with payment number 4 and ending with payment number 7.
2. The INT entry on the amortization worksheet is the sum of the interest paid entries in the amortization schedule starting at payment number P1 and ending at payment number P2. For example, if P1=4 and P2=7, the INT entry tells you the sum of the interest paid column in the amortization schedule starting with payment number 4 and ending with payment number 7.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=181#oembed-1>

[Mortgage Calculations using BAII Plus](#) by Joshua Emmanuel [4:13] (transcript available).

NOTE

A common mistake occurs in translating years into payment numbers. You often need to find the total interest paid or total principal paid for a particular year. To do this, you need to set P_1 equal to the number of the first payment that occurs in that year and P_2 equal to the number of the last payment that occurs in that year.

For example, suppose you have monthly payments and you want to know the total interest paid in the fourth year. In error, you might calculate that the fourth year begins with payment 36 and ends with payment 48, and so enter $P_1=36$ and $P_2=48$. But the 36th payment is actually the last payment of the third year. The first payment to occur in year four is the 37th. So, if you wanted to find the total interest paid in year 4, $P_1=37$ and $P_2=48$.

When you need to find the first payment number and last payment number for a particular year, there are two methods you can use to calculate the correct payment numbers.

1. Calculate the payment number at the end of the year in question by multiplying P/Y by the year number: $P/Y \times \text{year number}$. To find the payment number at the start of the year, subtract the payment frequency less one ($P/Y - 1$) from the last payment number of the year. For example, suppose the you have monthly payments and want to find the interest paid for year 4. The last payment in year 4 is $12 \times 4 = 48$. The first payment in year 4 is $48 - (12 - 1) = 37$.
2. Calculate the payment number at the end of the year in question by multiplying P/Y by the year number: $P/Y \times \text{year number}$. To find the payment number for the first payment in the year, multiply P/Y by the previous year number and then add one to it: $P/Y \times (\text{year} - 1) + 1$. For example, suppose the you have monthly payments and want to find the interest paid for year 4. The first payment in year 4 is $12 \times 3 + 1 = 37$. The last payment of the fourth year remains at payment 48.

EXAMPLE

You borrowed \$25,000 to renovate your kitchen. The loan agreement calls for quarterly payments for ten years at 3.9% compounded semi-annually.

1. What is the outstanding balance on the loan after four years?
2. How much interest was paid in year two?
3. How much principal was paid in year five?
4. How much interest was paid in year eight?

Solution:

Step 1: Calculate the payment.

PMT Setting	END
N	$4 \times 10 = 40$
PV	25,000
FV	0
PMT	?
I/Y	3.9
P/Y	4
C/Y	2

$$PMT = \$757.10$$

Step 2: Calculate the balance at the end of year four. To find the balance after four years, enter the payment number that corresponds to the last payment made in year four. There are 4 payments a year, so the last payment made in year four is 16 (4×4). So, to find the balance after four years, set P1=16 and P2=16.

PMT Setting	END
N	40
PV	25,000
FV	0
PMT	−757.10
I/Y	3.9
P/Y	4
C/Y	2
P1	16
P2	16

$$BAL = \$16,140.48$$

After four years, the balance on the loan is \$16,140.48.

Step 3: Calculate the interest paid in year two. To find the interest paid in year two, set P1 to first payment number of year two and P2 to last payment of year two. There are 4 payments a year, so the last payment made in year two is 8 (4×2). The first payment made in year two is 5 ($4 \times 1 + 1$). So, to find the interest paid in year two, set P1=5 and P2=8.

PMT Setting	END
N	40
PV	25,000
FV	0
PMT	−757.10
I/Y	3.9
P/Y	4
C/Y	2
P1	5
P2	8

$$INT = \$857.91$$

The total interest paid in year two is \$857.91.

Step 4: Calculate the principal paid in year five. To find the principal paid in year five, set P1 to first payment number of year five and P2 to last payment of year five. There are 4 payments a year, so the last payment made in year five is 20 (4×5). The first payment made in year five is 17 ($4 \times 4 + 1$). So, to find the principal paid in year five, set P1=17 and P2=20.

PMT Setting	END
N	40
PV	25,000
FV	0
PMT	−757.10
I/Y	3.9
P/Y	4
C/Y	2
P1	17
P2	20

$$PRN = \$2,437.15$$

The total principal paid in year five is \$2,437.15.

Step 5: Calculate the interest paid in year eight. To find the interest paid in year eight, set P1 to first payment number of year eight and P2 to last payment of year eight. There are 4 payments a year, so the last payment made in year eight is 32 (4×8). The first payment made in year eight is 29 ($4 \times 7 + 1$). So, to find the interest paid in year eight, set P1=29 and P2=32.

PMT Setting	END
N	40
PV	25,000
FV	0
PMT	−757.10
I/Y	3.9
P/Y	4
C/Y	2
P1	29
P2	32

$$INT = \$291.84$$

The total interest paid in year eight is \$291.84.

TRY IT

Art Industries just financed a \$50,000 purchase at 5.9% effective. It fixes the loan payment at \$450 per month.

1. How many payments will it take to pay-off the loan?
2. How much principal is paid in year three?
3. How much interest is paid in year ten?
4. How much principal is paid in year seven?
5. How much interest is paid in year eleven?

Click to see Solution

1. Calculate the number of payments.

PMT Setting	END
N	?
PV	50,000
FV	0
PMT	−450
I/Y	5.9
P/Y	12
C/Y	1

$$N = 158.969... \rightarrow 159 \text{ payments}$$

2. Calculate the principal paid in year three.

PMT Setting	END
N	159
PV	50,000
FV	0
PMT	−450
I/Y	5.9
P/Y	12
C/Y	1
P1	25
P2	36

$$PRN = \$2,909.70$$

3. Calculate the interest paid in year ten.

PMT Setting	END
N	159
PV	50,000
FV	0
PMT	−450
I/Y	5.9
P/Y	12
C/Y	1
P1	109
P2	120

$$INT = \$1,053.70$$

4. Calculate the principal paid in year seven.

PMT Setting	END
N	159
PV	50,000
FV	0
PMT	−450
I/Y	5.9
P/Y	12
C/Y	1
P1	73
P2	84

$$PRN = \$3,659.58$$

5. Calculate the interest paid in year eleven.

PMT Setting	END
N	159
PV	50,000
FV	0
PMT	−450
I/Y	5.9
P/Y	12
C/Y	1
P1	121
P2	132

$$INT = \$797.27$$

Exercises

1. A \$39,000 loan at 6% compounded semi-annually is repaid with quarterly payments over five years.
 - a. How much interest is paid with the 10th payment?
 - b. How much principal is paid with the 17th payment?
 - c. What is the outstanding balance on the loan after the 7th payment is made?
 - d. How much principal is paid in year 2?
 - e. How much interest is paid in year 3?
 - f. What is the outstanding balance on the loan after year 4?

Click to see Answer

a. \$340.47, b. \$2,138.84, c. \$26,639.82, d. \$7,326.61, e. \$1,303.60, f. \$8,748.41

2. A \$500,000 loan at 6.25% effective is repaid with semi-annual payments of \$35,000.
 - a. How much principal is paid with the 9th payment?

- b. How much interest is paid with the 16th payment?
- c. What is the outstanding balance on the loan after the 3rd payment is made?
- d. How much interest is paid in year 4?
- e. How much principal is paid in year 9?
- f. What is the outstanding balance on the loan after year 7?

Click to see Answer

a. \$24,993.85, b. \$4,098.15, c. \$439,335.29, d. \$22,228.78, e. \$64,686.11, f. \$163,138.14

3. A \$14,000 loan at 6% compounded monthly is repaid by monthly payments over four years.
- a. What is the size of the monthly payment?
 - b. Calculate the principal portion of the 25th payment.
 - c. Calculate the interest portion of the 33rd payment.
 - d. Calculate the total interest paid in the second year.
 - e. Calculate the principal portion of the payments in the third year.
 - f. Calculate the outstanding balance at the end of first year.

Click to see Answer

a. \$328.79, b. \$291.70, c. \$25.22, d. \$556.26, e. \$3,598.25, f. \$10,807.68

4. Quarterly payments are to be made against a \$47,500 loan at 5.95% compounded annually with a six-year amortization.
- a. What is the size of the quarterly payment?
 - b. Calculate the principal portion of the sixth payment.
 - c. Calculate the interest portion of the 17th payment.
 - d. Calculate how much the principal will be reduced in the fourth year.
 - e. Calculate the total interest paid in the third year.

Click to see Answer

a. \$2,359.13, b. \$1,792.76, c. \$257.53, d. \$8,109.20, e. \$1,782.27

5. Cathy and Bill just acquired a new Honda Odyssey Touring Edition minivan for \$60,531.56 under the dealership's purchase financing of 5.65% compounded annually for eight years.
- a. What are their monthly car payments?
 - b. In the first year, what total amount of interest will they pay?
 - c. In the fourth year, by how much will the principal be reduced?

- d. What is the balance on the loan at end of sixth year?

Click to see Answer

a. \$781.07, b. \$3,179.72, c. \$7,303.28, d. \$17,711.66

6. You took out a business loan of \$75,000 at 2.9% compounded quarterly. You repay the loan with monthly payments of \$1300.

- What is the outstanding balance on the loan after the 33rd payment is made?
- How much principal is paid with the 41st payment?
- How much interest is paid with the 29th payment?
- What is the outstanding balance on the loan after year 2?
- How much interest is paid in year 3?
- How much principal is paid in year 4?

Click to see Answer

a. \$36,606.07, b. \$1,232.35, c. \$102.75, d. \$47,381.49, e. \$1,180.56, f. \$14,842.18

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6.3 CALCULATING THE FINAL PAYMENT

LEARNING OBJECTIVES

- Calculate the final payment for a loan.

If you have ever paid off a loan you may have noticed that your last payment was a slightly different amount than your other payments. Whether you are making monthly insurance premium payments, paying municipal property tax instalments, financing your vehicle, paying your mortgage, receiving monies from an investment annuity, or dealing with any other situation where an annuity is extinguished through equal payments, the last payment typically differs from the rest, by as little as one penny or up to a few dollars. This difference can be much larger if you arbitrarily chose an annuity payment as opposed to determining an accurate payment through time value of money calculations.

Why is it important for this final payment to differ from all of the previous payments? From a consumer perspective, you do not want to pay a cent more toward a debt than you have to. In 2011, the average Canadian is more than \$100,000 in debt across various financial tools such as car loans, consumer debt, and mortgages. Imagine if you overpaid every one of those debts by a dollar. Over the course of your lifetime those overpayments would add up to hundreds or even thousands of dollars.

Calculating the Final Payment

When you calculate out the payment for a loan, the actual payment is rounded to two decimal places. It is rare for a calculated loan payment not to require rounding. The rounding up or down of the payment forms the basis for adjusting the final payment.

Consider the situation where the loan payment is rounded up. Suppose the calculated loan payment is \$999.995. Rounding to two decimals, the actual payment made is \$1,000. So with each \$1,000 loan payment,

you are overpaying the debt by $\$1,000 - \$999.995 = \$0.005$. If you make 20 such payments, you end up overpaying the debt by $20 \times \$0.005 = \$0.10$. Therefore, when it comes to the final payment you need to compensate for all of the overpayments made, reducing the final payment by $\$0.10$. Because the principal is slightly smaller at all times as a result of the overpayment, an additional adjustment may be needed because of less interest being calculated.

Now, consider the situation where the loan payment is rounded down. Suppose the calculated loan payment is $\$1,000.0025$. Rounding to two decimals, the actual payment made is $\$1,000$. So, with each $\$1,000$ loan payment, you are underpaying the debt by $\$1,000.0025 - \$1,000 = \$0.0025$. If you make 20 such payments, you end up underpaying by $20 \times \$0.0025 = \$0.05$. Therefore, when it comes to the final payment, you need to increase the final payment by this amount. Because the principal is slightly larger at all times as a result of the underpayment, an additional adjustment may be needed because of more interest being calculated.

We have already learned how to calculate the final payment by completing the last row of the amortization schedule. This method gives the following formula for calculating the final payment.

$$\text{Final Payment} = \text{Balance from Second Last Row} + \text{Interest from Last Row}$$

Alternatively, we can calculate the final payment without constructing the last row of the amortization schedule. This method is based on the assumption that all of the payments, including the final payment are the same. The final payment is found by adjusting the regular payment (PMT) by the amount overpaid or underpaid.

$$\begin{array}{l} \text{Final Payment} = \text{PMT} - \text{Amount Overpaid} \\ \text{or} \\ \text{Final Payment} = \text{PMT} + \text{Amount Underpaid} \end{array}$$

NOTE

1. An overpayment means that you have paid more through the regular payments than is necessary. The total amount overpaid is subtracted from the payment to find the final payment. Consequently, the final payment is smaller than the other payments.
2. An underpayment means that you have not paid enough through the regular payments than is necessary. The total amount underpaid is added to the payment to find the final payment. Consequently, the final payment is larger than the other payments.

USING THE TI BAI PLUS CALCULATOR TO FIND THE FINAL PAYMENT

To use the amortization worksheet to find the final payment:

1. Solve for any unknown quantities about the loan. You need to know all of the information about the loan first before you can use the amortization worksheet.
2. Enter the values of all seven time value of money variables into the calculator (N, PV, FV, PMT, I/Y, P/Y, C/Y). If you calculated PMT in the first step, you must **re-enter** it rounded to two decimals and with the correct cash flow sign. Make sure the payment setting is set to END, and obey the cash flow sign convention. Because this is a loan, PV (the loan amount) is positive and PMT is negative.
3. Go to the amortization worksheet by pressing 2nd AMORT (the PV button).
4. Enter the payment number for the final payment into P1 and P2.
5. Find the BAL entry. Watch the cash flow sign of the BAL entry to properly interpret what to do with it!
6. Calculate the final payment:
 - A **negative** BAL entry indicates an **overpayment**. Then

$$\boxed{\text{Final Payment}} = \text{PMT} - \boxed{\text{Overpayment}}$$
 - A **positive** BAL entry indicates an **underpayment**. Then

$$\boxed{\text{Final Payment}} = \text{PMT} + \boxed{\text{Underpayment}}$$

Example

A \$10,000 loan at 8% compounded quarterly is repaid with month-end payments of \$200. Calculate the final payment.

Solution:

Step 1: Calculate the number of payments.

PMT Setting	END
N	?
PV	10,000
FV	0
PMT	−200
I/Y	8
P/Y	12
C/Y	4

$$N = 60.9273... \rightarrow 61 \text{ payments}$$

Step 2: Calculate the balance for payment 61 (the last payment). To find the balance for payment 61, set P1=61 and P2=61.

PMT Setting	END
N	61
PV	10,000
FV	0
PMT	−200
I/Y	9
P/Y	12
C/Y	4
P1	61
P2	61

$$BAL = -\$14.49$$

Step 3: Calculate the final payment. Because the BAL entry from the previous step is negative, the \$14.49 is an overpayment and must be subtracted from the \$200 payment.

$$\begin{aligned}\text{Final Payment} &= 200 - 14.49 \\ &= \$185.11\end{aligned}$$

The final payment for the loan is \$185.11.

EXAMPLE

A \$50,000 loan at 5% compounded quarterly is repaid with quarterly payments for 6.5 years. Calculate the final payment.

Solution:

Step 1: Calculate the payment.

PMT Setting	END
N	26
PV	50,000
FV	0
PMT	?
I/Y	5
P/Y	4
C/Y	4

$$PMT = \$2,264.36$$

Step 2: Calculate the balance for payment 26 (the last payment). To find the balance for payment 26, set P1=26 and P2=26. Remember to re-enter the payment rounded to two decimal places.

PMT Setting	END
N	26
PV	50,000
FV	0
PMT	−2,264.36
I/Y	5
P/Y	4
C/Y	4
P1	26
P2	26

$$BAL = \$0.13$$

Step 3: Calculate the final payment. Because the BAL entry from the previous step is positive, the \$0.13 is an underpayment and must be added to the \$2,264.36 payment.

$$\begin{aligned}\text{Final Payment} &= 2,264.36 + 0.13 \\ &= \$2,264.49\end{aligned}$$

The final payment for the loan is \$2,264.49.

TRY IT

Semi-annual payments are made against a \$97,500 loan at 7.5% compounded semi-annually with a 10-year amortization. Calculate the final payment.

Click to see Solution

PMT Setting	END
N	20
PV	97,500
FV	0
PMT	?
I/Y	7.5
P/Y	2
C/Y	2

$$PMT = \$7,016.30$$

PMT Setting	END
N	20
PV	97,500
FV	0
PMT	−7,016.30
I/Y	7.5
P/Y	2
C/Y	2
P1	20
P2	20

$$BAL = \$0.13$$

$$\begin{aligned}\text{Final Payment} &= 7,016.30 + 0.13 \\ &= \$7,016.43\end{aligned}$$

TRY IT

A \$65,000 loan at 3.5% compounded quarterly is repaid with \$600 monthly payments. Calculate the final payment.

Click to see Solution

PMT Setting	END
N	?
PV	65,000
FV	0
PMT	−600
I/Y	3.5
P/Y	12
C/Y	4

$$N = 130.309... \rightarrow 131 \text{ payments}$$

PMT Setting	END
N	131
PV	65,000
FV	0
PMT	−600
I/Y	3.5
P/Y	12
C/Y	4
P1	131
P2	131

$$BAL = -\$413.98$$

$$\begin{aligned} \text{Final Payment} &= 600 - 413.98 \\ &= \$186.02 \end{aligned}$$

Exercises

1. A \$15,000 loan at 10% compounded quarterly is repaid with quarterly payments for three years. Calculate the final payment.

Click to see Answer

\$1,462.27

2. A \$85,000 loan at 6.75% compounded monthly is repaid with monthly payments for seven years. Calculate the final payment.

Click to see Answer

\$1,273.03

3. A \$32,000 loan at 8.25% effective is repaid with annual payments for 10 years. Calculate the final payment.

Click to see Answer

\$4,822.82

4. A \$250,000 loan at 5.9% compounded semi-annually is repaid with monthly payments of \$2,400. Calculate the final payment.

Click to see Answer

\$1,265.95

5. A \$28,250 loan at 9% compounded quarterly is repaid by monthly payments over five years. Calculate the final payment.

Click to see Answer

\$585.50

6. You took out a \$90,000 home renovation loan at 2.7% compounded semi-annually. You make quarterly payments of \$3,000 to repay the loan. Calculate your final payment.

Click to see Answer

\$1,8,64.93

7. Stuart and Shelley just purchased a new \$65,871.88 Nissan Titan Crew Cab SL at 8.99% compounded monthly for a seven-year term. What is the amount of the final payment?

Click to see Answer

\$1,059.89

8. Wile E. Coyote owes the ACME Corporation \$75,000 for various purchased goods. Wile agrees to make \$1,000 payments at the end of every month at 10% compounded quarterly until the debt is repaid in full. What is the amount of the final payment?

Click to see Answer

\$512.66

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6.4 MORTGAGES

LEARNING OBJECTIVES

- Calculate mortgage payments for the initial term and renewal term.
- Calculate amortization periods to reflect changes in payment frequency.
- Calculate mortgage balances and amortization periods to reflect prepayments of principal.

A **mortgage** is a special type of loan that is collaterally secured by real estate. In essence, the loan has a lien against the property, which is the right to seize the property for the debt to be satisfied. An individual or business taking out a mortgage is obliged to pay back the amount of the loan with interest based on a predetermined contract. The lender, a bank, or some other financial institution, has a claim on the real estate property in the event that the mortgage goes into default, which means that it is not paid as per the agreement. In these instances, financial institutions will pursue foreclosure of the property, which allows for the tenants to be evicted and the property to be sold. The proceeds of the sale are then used to pay off the mortgage. A mortgage always involves two parties. The individual or business that borrows the money is referred to as the **mortgagor**, and the financial institution that lends the money is referred to as the **mortgagee**.

Types of Mortgages

In the mortgage contract, the two parties can agree to either a fixed interest rate or a variable interest rate. Either way, the mortgage always forms an ordinary annuity because interest is not payable in advance.

- Under a fixed interest rate, the principal is repaid through a number of equal payments that cover both the interest and principal components of the loan. The interest portion is highest at the beginning and gradually declines over the amortization period of the mortgage. In Canada, fixed interest rates are either annually or semi-annually compounded, with the latter being the prevailing choice.
- In a variable interest rate mortgage, the principal is repaid through an agreed-upon number of unequal

payments that fluctuate with changes in borrowing rates. The principal and interest portions of the payment vary as interest rates fluctuate, meaning that the interest portion can rise at any point with any increase in rates. When rates change, a common practice in many financial institutions is to change the variable interest rate as of the first day of the next month. If the rate change does not coincide with a mortgage payment date, then the interest portion is calculated in a way similar to the procedure for a demand loan, where the exact number of days at the different rates must be determined. In Canada, variable interest rates are usually compounded monthly.

The mortgage agreement can be open or closed. An **open mortgage** has very few rules and it allows the mortgagor to pay off the debt in full or make additional prepayments at any given point in any amount without penalty. A **closed mortgage** has many rules that determine how the mortgage is to be paid. It does not allow the mortgagor to pay off the debt in full until the loan matures. As a marketing tool, most closed mortgages have “top-up” options that allow the mortgagor to make additional payments (such as an additional 20% per year) against the mortgage without penalty. Any payments exceeding the maximums or early payment of the mortgage are penalized heavily, with a three-month minimum interest charge that could be increased up to a measure called the interest rate differential, which effectively assesses the bank’s loss and charges the mortgagor this full amount.

Calculating Mortgage Payments and Outstanding Balance

Previously, we learned how to calculate the payment for a loan and how to find the remaining balance on a loan after any payment is made. Because a mortgage is a special type of loan, we can use these same techniques on a mortgage.

EXAMPLE

A new home is purchased for \$408,726 with a \$50,000 down payment and a 25-year mortgage for the outstanding balance. The mortgage is repaid with monthly payments. The interest rate on the mortgage is fixed at 5.29% compounded semi-annually for a five year term.

1. What is the size of the monthly payments?

2. What is the balance on the mortgage at the end of the five year term?

Solution:

Step 1: Calculate the loan amount.

$$\begin{aligned}\text{Loan Amount} &= 408,726 - 50,000 \\ &= \$358,726\end{aligned}$$

Step 2: Calculate the payment.

PMT Setting	END
N	$12 \times 25 = 300$
PV	358,726
FV	0
PMT	?
I/Y	5.29
P/Y	12
C/Y	2

$$PMT = \$2,145.98$$

The monthly mortgage payments are \$2,145.98.

Step 3: Calculate the balance after five years. To find the balance after five years, enter the payment number that corresponds to the last payment made in year five. There are 12 payments a year, so the last payment made in year five is 60 (5×12). So, to find the balance after five years, set P1=60 and P2=60.

PMT Setting	END
N	300
PV	358,726
FV	0
PMT	−2,145.98
I/Y	5.29
P/Y	12
C/Y	2
P1	60
P2	60

$$BAL = \$318,927.89$$

At the end of the five year term, the balance is \$318,927.89.

NOTE

In the previous example, the payment is calculated based on the entire 25 year amortization period, and not the five year term. This is true for any mortgage. The payments are always calculated based on the assumption that the current interest rate will last for the entire amortization period, which determines the length of time over which the loan is repaid. The amortization period forms the basis for calculating the payment. The term has no effect on the payment calculation. The term dictates only the time frame during which the current mortgage arrangement (interest rate, payment frequency, type, and so on) remains in effect.

Renewing the Mortgage

When the term of a mortgage expires, the balance remaining becomes due in full. Typically the balance owing

is still quite substantial, so the mortgage must be renewed. As discussed earlier, this means that the mortgagor assumes another mortgage, not necessarily with the same financial institution, and the amortization term is typically reduced by the length of the first term. The length of the second term of the mortgage then depends on the choice of the mortgagor. Other variables such as payment frequency and the interest rate may or may not change.

For example, assume a mortgage is initially taken out with a 25-year amortization period and a five-year term. After five years, the mortgage becomes due in full. Unable to pay it, the mortgagor renews the mortgage for the remaining 20-year amortization period, and also opts for a three-year term in assuming the new mortgage. When those three years are over, the mortgagor renews the mortgage for the remaining 17-year amortization period and again makes another term decision. This process repeats until the debt is ultimately paid off.

When the mortgage is renewed for another term, the interest rate or payment frequency may be changed. Consequently, the payments on the mortgage will also change when the mortgage is renewed and the new term goes into effect. To find the new payment when the mortgage is renewed, calculate the payment using the remaining balance at the end of the previous term and the remaining time from the amortization period.

The steps to find the new payment upon renewal are:

1. Calculate the original payment for the mortgage before the mortgage is renewed.
2. Calculate the remaining balance at the end of the previous term.
3. Calculate the new payment for the mortgage using the balance from the previous step as the loan amount (the present value), the remaining time on the amortization period, and incorporating any changes that occur in the new term, such as a new interest rate or change in payment frequency.

EXAMPLE

A \$389,000 mortgage is repaid with monthly payments and amortized over 20 years. The interest rate for the first three-year term is 4.9% compounded semi-annually. At the end of the three-year term the mortgage is renewed at 5.85% compounded semi-annually. What is the size of the monthly payment when the mortgage is renewed?

Solution:

Step 1: Calculate the payment for the first term.

PMT Setting	END
N	$12 \times 20 = 240$
PV	389,000
FV	0
PMT	?
I/Y	4.9
P/Y	12
C/Y	2

$$PMT = \$2,535.26$$

The monthly mortgage payments are \$2,535.26.

Step 2: Calculate the balance at the end of the three year term. To find the balance after three years, enter the payment number that corresponds to the last payment made in year three. There are 12 payments a year, so the last payment made in year three is 36 (3×12). So, to find the balance after three years, set P1=36 and P2=36.

PMT Setting	END
N	240
PV	389,000
FV	0
PMT	-2,535.26
I/Y	4.9
P/Y	12
C/Y	2
P1	36
P2	36

$$BAL = \$351,770.37$$

At the end of the three year term, the balance is \$351,770.37.

Step 3: Calculate the payments when the mortgage is renewed. The interest rate changes to 5.85% compounded semi-annually. The time used to find the new payments is based on the **remaining** amortization period. The original amortization period is 20 years and 3 years have past to get to the time of renewal, so the remaining amortization period is $20 - 3 = 17$ years. The new payments are calculated using the new interest rate, 5.85% compounded semi-annually, the 17 years remaining on the amortization period, and the remaining balance, \$351,770.37, from the end of the previous term.

PMT Setting	END
N	$12 \times 17 = 204$
PV	351,770.37
FV	0
PMT	?
I/Y	5.85
P/Y	12
C/Y	2

$$PMT = \$2,711.92$$

The monthly payments when the mortgage is renewed are \$2,711.92.

TRY IT

A \$628,200 house is purchased for \$100,000 down and monthly payments for 30 years. The interest rate for the first four year term is 6.49% compounded semi-annually.

1. Calculate the monthly payments.

2. What is the balance on the mortgage at the end of the four year term?
3. If the mortgage is renewed at 6.19% compounded semi-annually, what is the size of the new monthly payments?

Click to see Solution

1. Calculate the monthly payments.

PMT Setting	END
N	$12 \times 30 = 360$
PV	528,200
FV	0
PMT	?
I/Y	6.49
P/Y	12
C/Y	2

$$PMT = \$3,305.29$$

2. Calculate the balance at the end of the four year term.

PMT Setting	END
N	360
PV	528,200
FV	0
PMT	−3,305.29
I/Y	6.49
P/Y	12
C/Y	2
P1	48
P2	48

$$BAL = \$501,665.54$$

3. Calculate the new payment upon renewal.

PMT Setting	END
N	$12 \times 26 = 312$
PV	501,665.54
FV	0
PMT	?
I/Y	6.19
P/Y	12
C/Y	2

$$PMT = \$3,123.63$$

Changing the Payment Frequency

Although monthly mortgage payments are the most common payment frequency, mortgages can be paid off

with semi-monthly, bi-weekly, or weekly payments. By making more frequency payments, the mortgagor can reduce the amortization period and pay-off the mortgage faster.

To determine how much the amortization period is shortened by changing the payment frequency, calculate the amortization period for the new payment frequency and then calculate the difference in the two amortization periods.

EXAMPLE

A \$280,000 mortgage at 5.3% compounded semi-annually is repaid with monthly payments of \$1678 amortized over 25 years. How much shorter is the amortization period if weekly payments of \$400 are made instead of the monthly payments?

Solution:

Step 1: Calculate the amortization period for the weekly payments.

PMT Setting	END
N	?
PV	280,000
FV	0
PMT	−400
I/Y	5.3
P/Y	52
C/Y	2

$$N = 1,211.952... \rightarrow 1,212 \text{ payments}$$

Convert (rounded UP) N to years and months.

$$\begin{array}{l} \text{Number of Years} \times = \frac{\text{rounded up } N}{P/Y} \\ \times = \frac{1,212}{52} \\ \times = 23.307... \\ \rightarrow 23 \text{ years} \\ \text{Number of Months} \times = 0.307... \times 12 \\ \times = 3.692... \\ \rightarrow 4 \text{ months} \end{array}$$

The amortization period for the weekly payments is 23 years, 4 months.

Step 2: Calculate the difference in the amortization periods.

$$\begin{array}{l} \text{Difference} \times = 25 \text{ years} - (23 \text{ years and } 4 \\ \text{ months}) \\ \times = 1 \text{ year and } 8 \text{ months} \end{array}$$

With the weekly payments, the amortization period is shortened by 1 year and 8 months.

TRY IT

A \$300,000 mortgage at 4.7% compounded semi-annually is repaid with monthly payments of \$1,925 for 20 years. How much shorter is the amortization period if bi-weekly payments of \$1,000 are made instead of the monthly payments?

Click to see Solution

1. Calculate the amortization period for the bi-weekly payments.

PMT Setting	END
N	?
PV	300,000
FV	0
PMT	−1,000
I/Y	4.7
P/Y	26
C/Y	2

$$N = 430.370... \rightarrow 431 \text{ payments}$$

$$\begin{aligned} \text{Number of Years} &= \frac{\text{rounded up } N}{P/Y} \\ &= \frac{431}{26} \\ &= 16.576... \\ &\rightarrow 16 \text{ years} \\ \text{Number of Months} &= 0.5769... \times 12 \\ &= 6.923... \\ &\rightarrow 7 \text{ months} \end{aligned}$$

2. Calculate the difference in the amortization periods.

$$\begin{aligned} \text{Difference} &= 20 \text{ years} - (16 \text{ years and } 7 \text{ months}) \\ &= 3 \text{ year and } 5 \text{ months} \end{aligned}$$

With the bi-weekly payments, the amortization period is shortened by 3 year and 5 months.

Lump-Sum Payments

As noted above, closed mortgages allow the mortgagor to make extra payments, in addition to the regular periodic payments. Any extra payments applied to the mortgage are applied directly to the principal because the interest is paid by the regular periodic payment. Consequently, these extra payments allow the mortgagor to repay the mortgage sooner. What affect do these extra payments have on the amortization period? How much is the amortization period shortened by when extra payments are made?

One of these pre-payment options is to make a lump-sum payment. A lump-sum payment is simply a single amount of money applied to the mortgage. For example, you might make a \$10,000 lump-sum payment to your mortgage at the end of the sixth year. This means that at the end of the sixth year of the amortization

period, you pay \$10,000 to the mortgage principal. What affect does this have on the amount of time it takes to re-pay the mortgage?

To calculate how much the amortization period is shortened by when a lump-sum payment is applied:

1. Find any missing information about the mortgage. For example, you might have to calculate out the payment or the number of payments, depending on what information is missing.
2. Calculate the balance on the mortgage at the time the lump-sum payment is made.
3. Subtract the lump-sum payment from the balance found in step 2. This amount becomes the remaining balance on the mortgage.
4. Calculate the number of regular periodic payments required to pay-off the balance found in step 3.
5. Calculate the difference between the number of payments without the lump-sum payment applied and the number of payments with the lump-sum payment included.
6. Convert the difference found in step 5 to years and months to find how much the amortization period is shortened by with the lump-sum payment included.

NOTE

The lump sum payment is NOT counted as one of the payments in step 4. The number of payments refers to the number of regularly scheduled periodic payments.

EXAMPLE

A \$400,000 mortgage at 3.5% compounded semi-annually is repaid with monthly payments for 25 years. How much is the amortization period shortened by if a \$15,000 lump-sum payment is made at the end of year five?

Solution:

Step 1: Calculate the monthly payments.

PMT Setting	END
N	$12 \times 25 = 300$
PV	400,000
FV	0
PMT	?
I/Y	3.5
P/Y	12
C/Y	2

$$PMT = \$1,997.08$$

The monthly mortgage payments are \$1,997.08.

Step 2: Calculate the balance at the end of five years (the time of the lump-sum payment). To find the balance after five years, enter the payment number that corresponds to the last payment made in year five. There are 12 payments a year, so the last payment made in year five is 60 (5×12). So, to find the balance after five years, set P1=60 and P2=60.

PMT Setting	END
N	300
PV	400,000
FV	0
PMT	-1,997.08
I/Y	3.5
P/Y	12
C/Y	2
P1	60
P2	60

$$BAL = \$345,120.01$$

After five years, the balance is \$345,120.01.

Step 3: Subtract the lump-sum payment from the balance found in the previous step.

$$\begin{aligned}\text{New Balance} &= 345,120.01 - 15,000 \\ &= \$330,120.01\end{aligned}$$

Step 4: Calculate the number of payments needed to re-pay the new balance found in the previous step.

PMT Setting	END
N	?
PV	330,120.01
FV	0
PMT	−1,997.08
I/Y	3.5
P/Y	12
C/Y	2

$$N = 225.263... \rightarrow 226 \text{ payments}$$

Step 5: Calculate the difference between the number of payments without the lump-sum payment and the number of payments with the lump-sum payment. There are 300 payments without the lump-sum payment applied. With the lump-sum payment applied, there are 60 payments before the lump-sum payment and 226 payment after. So with the lump-sum payment, there are a total of $226+60=286$ payments.

$$\begin{aligned}\text{Difference} &= 300 - 286 \\ &= 14 \text{ payments}\end{aligned}$$

With the lump-sum payment applied, it takes 14 fewer payments to pay-off the mortgage.

Step 6: Convert the difference found in the previous step to years and months.

$$\begin{aligned}\text{Number of Years} &= \frac{14}{12} = 1.1666... \\ &\rightarrow 1 \text{ year} \\ &0.1666... \times 12 = 2 \text{ months}\end{aligned}$$

The amortization period is shortened by 1 year and 2 months.

TRY IT

A \$300,000 mortgage at 2.8% compounded semi-annually is repaid with monthly payments for 20 years. How much is the amortization period shortened by if a \$10,000 lump-sum payment is made at the end of year four?

Click to see Solution

1. Calculate the monthly payments.

PMT Setting	END
N	240
PV	300,000
FV	0
PMT	?
I/Y	2.8
P/Y	12
C/Y	2

$$PMT = \$1,631.51$$

2. Calculate the balance at the end of four years.

PMT Setting	END
N	240
PV	300,000
FV	0
PMT	−1,631.51
I/Y	2.8
P/Y	12
C/Y	2
P1	48
P2	48

$$BAL = \$252,555.63$$

3. Subtract the lump-sum payment from the balance.

$$\begin{aligned}\text{New Balance} &= 252,555.63 - 10,000 \\ &= \$242,555.63\end{aligned}$$

4. Calculate the number of payments needed to re-pay the new balance found in the previous step.

PMT Setting	END
N	?
PV	242,555.63
FV	0
PMT	−1,631.51
I/Y	2.8
P/Y	12
C/Y	2

$$N = 182.530... \rightarrow 183 \text{ payments}$$

5. Calculate the difference between the number of payments and convert the difference to years and months.

$$\begin{aligned}\text{Difference} &= 240 - (48 + 183) \\ &= 9 \text{ payments}\end{aligned}$$

$$\begin{aligned}\text{Number of Years} &= \frac{9}{12} = 0.75 \text{ years} \\ \text{Number of Months} &= 0.75 \times 12 = 9 \text{ months}\end{aligned}$$

The amortization period is shortened by 9 months.

Increasing the Amount of the Periodic Payment

Another one of the pre-payment options is to increase the periodic payment amount. For example, you might decide to increase your periodic payment by \$10 at the end of the sixth year. This means that starting with the first payment in year seven, each of the periodic payments will be \$10 higher. Because the interest was already paid by the original periodic payment amount, this extra \$10 gets applied to the mortgage principal. What effect does this have on the amount of time it takes to re-pay the mortgage?

To calculate how much the amortization period is shortened by when the amount of the periodic payment is increased:

1. Find any missing information about the mortgage. For example, you might have to calculate out the payment or the number of payments, depending on what information is missing.
2. Calculate the balance on the mortgage at the time the payment is increased.
3. Calculate the new payment by adding the payment increase to the original payment amount.
4. Calculate the number of new periodic payments required to pay-off the balance found in step 2.
5. Calculate the difference between the number of payments without the payment increase applied and the number of payments with the payment increase included.
6. Convert the difference found in step 5 to years and months to find how much the amortization period is shortened by with the payment increase included.

EXAMPLE

A \$375,000 mortgage at 3.1% compounded semi-annually is repaid with monthly payments for 20 years. How much is the amortization period shortened by if the payments are increased by \$200 at the end of year eight?

Solution:

Step 1: Calculate the monthly payments.

PMT Setting	END
N	$12 \times 20 = 240$
PV	375,000
FV	0
PMT	?
I/Y	3.1
P/Y	12
C/Y	2

$$PMT = \$2,094.82$$

The monthly mortgage payments are \$2,094.82.

Step 2: Calculate the balance at the end of eight years (the time of the payment increase). To find the balance after eight years, enter the payment number that corresponds to the last payment made in year eight. There are 12 payments a year, so the last payment made in year eight is 96 (8×12). So, to find the balance after eight years, set $P1=96$ and $P2=96$.

PMT Setting	END
N	240
PV	375,000
FV	0
PMT	−2,094.82
I/Y	3.1
P/Y	12
C/Y	2
P1	96
P2	96

$$BAL = \$251,916.89$$

After eight years, the balance is \$251,916.89.

Step 3: Add the payment increase to the payment to find the new payment amount.

$$\begin{aligned}\text{New Payment} &= 2,094.82 + 200 \\ &= \$2,294.82\end{aligned}$$

Step 4: Calculate the number of payments needed to re-pay the balance found in step 2 with the new payment found in step 3.

PMT Setting	END
N	?
PV	251,916.89
FV	0
PMT	−2,294.82
I/Y	3.1
P/Y	12
C/Y	2

$$N = 129.108... \rightarrow 130 \text{ payments}$$

Step 5: Calculate the difference between the number of payments without the payment increase and the number of payments with the payment increase. There are 240 payments without the payment increase. With the payment increase, there are 96 payments before the payment increase and 130 payment after. So with the payment increase, there are a total of $96+130=226$ payments.

$$\begin{aligned} \text{Difference} &= 240 - 226 \\ &= 14 \text{ payments} \end{aligned}$$

With the payment increase, it takes 14 fewer payments to pay-off the mortgage.

Step 6: Convert the difference found in the previous step to years and months.

$$\begin{array}{l} \text{Number of Years} \quad = \quad \frac{14}{12} \quad = \quad 1.1666... \\ \quad \rightarrow \quad 1 \text{ year} \\ \text{Number of Months} \quad = \quad 0.1666... \times 12 \quad = \quad 2 \text{ months} \end{array}$$

The amortization period is shortened by 1 year and 2 months.

TRY IT

A \$500,000 mortgage at 4.3% compounded semi-annually is repaid with monthly payments for 25 years. How much is the amortization period shortened by if the payments are increased by \$75 at the end of year three?

Click to see Solution

1. Calculate the monthly payments.

PMT Setting	END
N	300
PV	500,000
FV	0
PMT	?
I/Y	4.3
P/Y	12
C/Y	2

$$PMT = \$2,712.04$$

2. Calculate the balance at the end of three years.

PMT Setting	END
N	300
PV	500,000
FV	0
PMT	−2,712.04
I/Y	4.3
P/Y	12
C/Y	2
P1	36
P2	36

$$BAL = \$464,114.56$$

3. Calculate the new payment.

$$\begin{aligned} \text{New Payment} &= 2,712.04 + 75 \\ &= \$2,787.04 \end{aligned}$$

4. Calculate the number of payments needed to re-pay the balance found in step 2.

PMT Setting	END
N	?
PV	464,114.56
FV	0
PMT	−2,787.04
I/Y	4.3
P/Y	12
C/Y	2

$$N = 252.4769... \rightarrow 253 \text{ payments}$$

5. Calculate the difference between the number of payments and convert the difference to years and months.

$$\begin{aligned} \text{Difference} &= 300 - (36 + 253) \\ &= 11 \text{ payments} \end{aligned}$$

$$\begin{aligned} \text{Number of Years} &= \frac{11}{12} = 0.9166... \\ \text{Number of Months} &= 0.9166... \times 12 = 11 \text{ months} \end{aligned}$$

The amortization period is shortened by 11 months.

Exercises

1. A \$434,693 mortgage is repaid with monthly payments over 30 years. The interest rate for the first three year term is 4.5% compounded semi-annually. If the mortgage is renewed at 5.25% compounded semi-annually, what is the size of the monthly payments upon renewal?

Click to see Answer

\$2,370.32

2. A \$318,222 mortgage is repaid with bi-weekly payments over 25 years. The interest rate for the first two year term is 3% compounded semi-annually. If the mortgage is renewed at 6.8% compounded semi-annually, what is the size of the bi-weekly payments upon renewal?

Click to see Answer

\$985.74

3. Five years ago, Asia purchased her \$322,000 home in Edmonton with a 25-year amortization. In her first five-year term, she made monthly payments and was charged 4.89% compounded semi-annually. She will renew the mortgage on the same amortization timeline for a three year term at 5.49% compounded semi-annually with monthly payments.
- Calculate the balance remaining after the first term.
 - What is the new mortgage payment amount for the second term?
 - What is the balance at the end of the second term?

Click to see Answer

a. \$284,498.75; b. \$1,945.52; c. \$258,813.11

4. Luke took out a \$350,000 mortgage at 3.7% compounded semi-annually and repaid the mortgage with monthly payments of \$1,700.
- How long will it take Luke to repay the mortgage?
 - How much would the amortization period be shortened by if Luke made weekly payments of \$425 instead of the monthly payments?

Click to see Answer

a. 27 years, 2 months; b. 3 years, 5 months

5. Sally took out a \$475,000 mortgage at 2.9% compounded semi-annually. She repaid the mortgage with monthly payments for 25 years. How much would the amortization period be shortened by if Sally made bi-weekly payments of \$1,100 instead of the monthly payments?

Click to see Answer

2 years, 4 months

6. Sam received a \$280,000 mortgage at 4.5% compounded semi-annually. She repaid the mortgage with

monthly payments for 20 years. How much would the amortization period be shortened by if Sam made a \$12,000 lump-sum payment at the end of year two?

Click to see Answer

1 year, 2 months

7. A \$420,000 mortgage at 3.47% compounded semi-annually is repaid with monthly payments for 30 years. How much would the amortization period be shortened by if a lump-sum payment of \$20,000 is made at the end of year six?

Click to see Answer

1 year, 11 months

8. Jesse received a \$570,000 mortgage at 2.9% compounded semi-annually. He repaid the mortgage with monthly payments for 25 years. How much would the amortization period be shortened by if Jesse increases his monthly payments by \$175 at the end of year five?

Click to see Answer

1 year, 7 months

9. A \$190,000 mortgage at 3.2% compounded semi-annually is repaid with monthly payments for 15 years. How much would the amortization period be shortened by if the monthly payments are increased by 10% at the end of year three?

Click to see Answer

1 year, 3 months

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6.5 REVIEW EXERCISES

1. Annanya took out a \$42,500 loan at 6.6% compounded monthly with monthly payments over the six-year amortization period. Calculate the total principal and interest portions for the third year.

Click to see Answer

PRN=\$6,810.95, INT=\$1,786.45

2. Two years ago, Sumandeep borrowed \$20,000 at 9.45% compounded monthly. She has been paying end-of-month payments since, and the last payment will be today. Calculate the amount of the final payment.

Click to see Answer

\$917.82

3. Hogwild Industries borrowed \$75,000 to purchase some new equipment. The terms of the ordinary loan require quarterly payments for three years with an interest rate of 7.1% compounded semi-annually. Calculate the total interest and principal portions for the third year.

Click to see Answer

PRN=\$26,763.03, INT=\$1,187.52

4. Dr. Strong of Island Lakes Dental Centre acquired a new Panoramic X-ray machine for his practice. The \$7,400 for the machine, borrowed at 8.8% compounded annually, is to be repaid in four end-of-quarter instalments. Develop a complete amortization schedule and total the interest paid.

Click to see Answer

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
0				\$7,400
1	\$1,949.59	\$157.69	\$1,791.90	\$5,608.10
2	\$1,949.59	\$119.50	\$1,830.09	\$3,778.01
3	\$1,949.59	\$80.51	\$1,869.08	\$1,908.93
4	\$1,949.61	\$40.68	\$1,908.93	\$0
Totals	\$7,798.38	\$398.38	\$7,400	

5. Dr. Miller acquired a new centrifuge machine from Liaoyang Longda Pharmaceutical Machinery Company (LLPMC) for his medical practice. He is to pay off the \$60,341 through 20 month-end payments. LLPMC has set the interest rate on the loan at 9.5% compounded quarterly. Develop a partial amortization schedule for the third to sixth payments.

Click to see Answer

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
3	\$3,272.05	\$429.84	\$2,842.21	\$51,880.64
4	\$3,272.05	\$407.51	\$2,864.54	\$49,016.11
5	\$3,272.05	\$385.01	\$2,887.04	\$46,129.07
6	\$3,272.05	\$362.33	\$2,909.72	\$43,219.35

6. Kerry, who is a pharmacist, just became a new franchisee for Shoppers Drug Mart. As part of her franchising agreement, her operation is to assume a \$1.2 million mortgage to be financed over the next 15 years. She is to make payments after every six months. Head office will charge her a rate of 14.25% compounded annually. Determine the amount of her mortgage payment.

Click to see Answer

\$95,615.95

7. Alibaba took out a 25-year amortization \$273,875 mortgage five years ago at 4.85% compounded semi-annually and has been making monthly payments. He will renew the mortgage for a three-year term today at an interest rate of 6.1% compounded semi-annually on the same amortization schedule. What are his new monthly mortgage payments?

Click to see Answer

\$1,735.84

8. Monthly payments are to be made against an \$850,000 loan at 7.15% compounded annually with a 15-year amortization.
- What is the size of the monthly payment?
 - Calculate the principal portion of the 100th payment.
 - Calculate the interest portion of the 50th payment.
 - Calculate how much the principal will be reduced in the second year.
 - Calculate the total interest paid in the fifth year.

Click to see Answer

a. \$7,604.85; b. \$4,771.37; c. \$4,026.56; d. \$35,827.23; e. \$47,183.46

9. An investment annuity of \$100,000 earning 4.5% compounded quarterly is to make payments at the end of every three months with a 10-year amortization.
- What is the size of the quarterly payment?
 - Calculate the principal portion of the 20th payment.
 - Calculate the interest portion of the 33rd payment.
 - Calculate how much the principal will be reduced in the second year.
 - Calculate the total interest paid in the seventh year.

Click to see Answer

a. \$3,118.35; b. \$2,465.45; c. \$166.96; d. \$8,480.07; e. \$1,866.95

10. Four years ago, Katrina became a landlord and opened her new four-unit apartment housing unit with an initial mortgage at 6.83% compounded semi-annually in the amount of \$971,000 less a \$100,000 down payment. She amortized over 30 years and opted for monthly payments. Upon renewing her mortgage today, she is taking a two-year term at 5.1% compounded semi-annually while continuing with monthly payments and the original amortization timeline.
- Calculate the interest and principal portions in her first term.

- b. What is the balance remaining after the first term?
- c. What is the new mortgage payment amount in the second term?
- d. What is the balance remaining after the second term?

Click to see Answer

a. PRN=\$ 41,301.88, INT=\$229,441.64; b. \$829,698.12; c. \$4,779.82; d. \$797,181.08

11. You have a \$50,000 student loan at 3.1% compounded quarterly. You repay the loan with monthly payments of \$750.
- a. How much interest is paid with the 20th payment?
 - b. How much principal is paid with the 47th payment?
 - c. What is the balance on the loan after five years?
 - d. How much interest is paid in year four?
 - e. How much principal is paid in year six?
 - f. What is the size of your final payment?

Click to see Answer

a. \$97.71; b. \$699.23; c. \$9,750.63; d. \$705.58; e. \$8,822.85; f. \$180.64

12. John received a \$80,000 loan at 4.96% compounded semi-annually. He made quarterly payments of \$3,000 to repay the loan. Construct a partial amortization schedule showing the details of the last two payments and the totals.

Click to see Answer

Payment Number	Payment	Interest Paid (INT)	Principal Paid (PRN)	Balance (BAL)
32	\$3, 000	\$55.70	\$2, 944.30	\$1, 575.34
33	\$1, 594.75	\$19.41	\$1, 575.34	\$0
Totals	\$97, 594.75	\$17, 594.75	\$80, 000	

13. You purchased a \$400,000 house. You paid 10% as a down payment and took out a mortgage for the balance at 4.5% compounded semi-annually. You repaid the mortgage with monthly payments for 25 years. How much is the amortization period shortened by if a lump-sum payment of \$15,000 is made at the end of the fourth year?

Click to see Answer

1 year, 6 months

14. A \$525,000 mortgage at 3.9% compounded semi-annually is repaid with monthly payments for 25 years. How much is the amortization period shortened by if the monthly payments are increased by \$200 at the end of year three?

Click to see Answer

2 years, 3 months

15. A \$600,000 mortgage at 2.75% compounded semi-annually is repaid with monthly payments of \$2,445 for 30 years. How much is the amortization period shortened by if mortgage is repaid with semi-monthly payments of \$1,300 instead of the monthly payments?

Click to see Answer

2 years, 8 months

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6.6 CHAPTER SUMMARY

Key Concepts

- **6.1 Amortization Schedules**
 - The concept of amortization.
 - The development of an amortization schedule.
 - The development of a partial amortization schedule.
 - **6.2 Calculating the Principal and Interest Components of a Loan**
 - How to calculate interest and principal components for a single payment.
 - How to calculate interest and principal components for a series of payments.
 - Variable changes in future value annuity calculations.
 - **6.3 Calculating the Final Payment**
 - Understanding why the final payment is different than all the other payments.
 - How to calculate the exact amount of the final payment.
 - **6.4 Mortgages**
 - The language and concepts involved in mortgages.
 - How to calculate the mortgage payment.
 - The procedure involved in renewing a mortgage.
 - The procedure involved with prepayment options.
-

Glossary of Terms

- **Amortization.** A process by which the principal of a loan is extinguished over the course of an agreed-upon time period through a series of regular payments that go toward both the accruing interest and principal reduction.
- **Amortization Period.** The length of time it will take for the principal of a loan to be reduced to zero.
- **Amortization Schedule.** A table that shows the payment amount, principal component, interest component, and remaining balance for every payment in the annuity.
- **Amortization Term.** The length of time for which the interest rate and payment agreement between the borrower and the lender will remain unchanged.
- **Closed Mortgage.** A type of mortgage that has many strict rules and does not allow the mortgager to

pay off the debt in full until the loan matures. Early payment incurs substantial penalties.

- **Foreclosure.** A process that allows a financial institution to evict tenants from a mortgaged property and put the property up for sale. The proceeds of the sale are then used to pay off the mortgage.
 - **Mortgage.** A special type of loan that is collaterally secured by real estate property.
 - **Mortgagee.** The financial institution that lends the money for a mortgage.
 - **Mortgagor.** The individual or business that borrows the money for a mortgage.
 - **Open Mortgage.** A type of mortgage that has very few rules and allows the mortgagor to pay off the debt in full or with additional prepayments at any given point without penalty.
 - **Partial Amortization Schedule.** Amortization schedules that show only a specified range of payments and not the entire annuity.
-

Formulas

• Symbols Used

- BAL = Principal balance immediately after a loan payment
 - INT = Interest portion of a loan payment or a series of payments
 - PRN = Principal portion of a loan payment or a series of payments
 - PMT = Annuity payment amount
 - I/Y = Nominal interest rate
 - P/Y = Number of payments per year or payment frequency
 - C/Y = Number of compounds per year or compounding frequency
 - N = Total number of annuity payments
-

Calculator

- [Amortization Worksheet](#)
- AMORT is located on the 2nd shelf above the PV button.

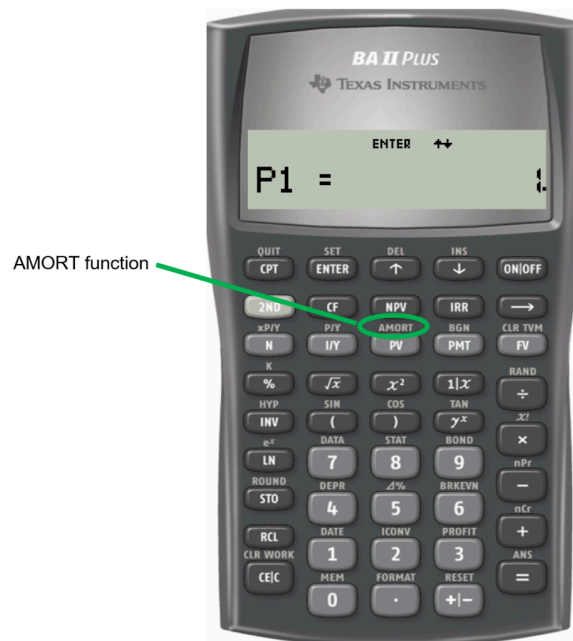


Figure 6.6.1

- There are five variables in the amortization worksheet:
 - P1 is the starting payment number. The calculator can work with a single payment or a series of payments.
 - P2 is the ending payment number. This number is the same as P1 when you are concerned with just a single payment. When working with a series of payments, you can set it to a higher number.
 - BAL is the principal balance remaining after the payment number entered into the P2 variable. The cash flow sign is correct as indicated on the calculator display.
 - PRN is the principal portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
 - INT is the interest portion of the payments from P1 to P2 inclusive. Ignore the cash flow sign.
- To use the amortization worksheet:
 - Enter all seven of the time value of money variables accurately (N, I/Y, PV, PMT, FV, P/Y, and C/Y). If PMT was computed, you must re-enter it with only two decimals while retaining the correct cash flow sign convention.
 - Press 2nd AMORT.
 - Enter a value for P1, press ENTER and then the down arrow.
 - Enter a value for P2, press ENTER and then the down arrow. Note that the higher the numbers entered in P1 or P2, the longer it will take the calculator to compute the outputs. It is possible that the calculator will go blank and take a few moments before displaying the

outputs.

- Press the down arrow to scroll through BAL, PRN, and INT to read the output.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=1454#oembed-1>

[Amortization Schedule using BAII Plus](#) by Joshua Emmanuel [3:55] (transcript available).



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[Mortgage Calculations using BAII Plus](#) by Joshua Emmanuel [4:13] (transcript available).

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PART VII

BONDS AND SINKING FUNDS

One of the most powerful ways for an institution to raise large amounts of capital is by issuing marketable bonds. A company or a government will essentially borrow the required financing from investors such as you and me. Individually we cannot provide the needed funds, but the bond issuer can raise a large sum of money from us collectively. One of the largest bond issuers in Canada is the Government of Canada. This money is primarily used to fund Canada's national debt.

On the consumer side, marketable bonds represent another investment opportunity you can include in your RRSP portfolio. They feature regular interest payments and a chance to grow your savings through trading on the bond market. This chapter begins with the basics of how bonds operate. Then you will learn how to calculate the purchase price of a bond and work with bond interest rates. Bond issuance usually comes with a sinking fund provision. Finally, we will explore the financial and accounting responsibilities associated with amortization of bond premiums and accrual of bond discounts.

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7.1 BOND TERMINOLOGY

LEARNING OBJECTIVES

- Understand terminology used in bonds and bond transactions.

A marketable **bond** is a debt that is secured by a specific corporate asset, that establishes the issuer's responsibility toward a creditor for paying interest at regular intervals, and for repaying the principal at a fixed later date. A **debenture** is the same as a marketable bond, except that the debt is not secured by any specific corporate asset. Mathematically, the calculations are identical for these two financial tools, which this textbook refers to as *bonds* for simplicity.

Bond Terminology

- **Issue Date.** The **bond issue date** is the date that the bond is issued and available for purchase by creditors. Interest accrues from this date.
- **Face Value.** Also called the *par value* or *denomination of the bond*, the **face value** is the principal amount of the debt. It is what the investor lent to the bond-issuing corporation. The amount, usually a multiple of \$100, is found in small denominations up to \$10,000 for individual investors and larger denominations up to \$50,000 or more for corporate investors.
- **Coupon Rate.** Also known as the *bond rate* or *nominal rate*, the **coupon rate** is the nominal interest rate paid on the face value of the bond. The coupon rate is fixed for the life of the bond. Most commonly the interest is calculated semi-annually and payable at the end of every six-month period over the entire life of the bond, starting from the issue date. All coupon rates used in this textbook are assumed to be semi-annually compounded, unless stated otherwise.
- **Yield Rate.** The **yield rate**, or *market rate*, is the prevailing nominal rate of interest in the open bond market. Because bonds are actively traded, this rate fluctuates based on economic and financial conditions. On the issue date, the market rate determines the coupon rate that is tied to the bond.

Market rates are usually compounded semi-annually, as will be assumed in this textbook unless otherwise stated. Therefore, marketable bonds form ordinary simple annuities, because the interest payments and the market rate are both compounded semi-annually, and the payments occur at the end of the interval.

- **Redemption Value.** Also called the *redemption price* or *maturity value*, the **redemption value** is the amount the bond issuer will pay to the bondholder upon maturity of the bond. The redemption price normally equals the face value of the bond, in which case the bond is said to be “redeemable at par” because interest on the bond has already been paid in full periodically throughout the term, leaving only the principal in the account. In some instances a bond issuer may in fact redeem the bond at a premium, which is a price greater than the face value. The redemption price is then stated as a percentage of the face value, such as 103%. For introductory purposes, this textbook sticks to the most common situation, where the redemption price equals the face value.
- **Maturity Date.** Also known as the *redemption date* or *due date*, the **maturity date** is the day upon which the redemption price will be paid to the bondholder (along with the final interest payment), thereby extinguishing the debt.
- **Selling Date.** The date that a bond is actively traded and sold to another investor through the bond market is known as the **selling date**. In the timeline, the selling date can appear anywhere on the timeline between the issue date and maturity date, and it may occur more than once as the bond is sold by one investor after another.
- **Purchase Price.** The **purchase price** is the price the bond holder pays to purchase the bond. Although the redemption value and the periodic interest payments remain fixed throughout the lifetime of the bond, the purchase price fluctuates depending on various market conditions, such as the current yield rate.

EXAMPLE

A \$10,000 bond was issued on March 1, 2019 with a 7% coupon and 16 years to maturity. The bond was purchased on July 13, 2023 when the yield to maturity was 5%.

- Face Value (or Redemption Value) = \$10,000
- Issue Date = March 1, 2019

- Purchase Date (or Selling Date) = July 23, 2023
- Maturity Date (or Redemption Date) = March 1, 2035
- Coupon Rate = 7% compounded semi-annually
- Yield Rate = 5% compounded semi-annually

NOTES

1. Unless otherwise stated, the coupon rate and the yield rate are compounded semi-annually and the coupon payments are made every six months.
2. In this chapter we only deal with bonds that are redeemed for their face value at the time of maturity. So, the redemption value of the bond will equal its face value.

Premium, Discount, and At Par Bonds

The price at which a bond is purchased in the market may not be the face value of the bond. In fact, the price of a bond fluctuates with the market rate over time. There are three scenarios relating the purchase price of the bond to its face value.

- **Bonds purchased at par.** The purchase price of the bond **equals** its face value. This happens when the coupon rate equals the yield rate.
- **Bonds purchased at premium.** The purchase price of the bond is **higher** than its face value. This happens when the coupon rate is greater than the yield rate. In such cases, the bond is providing a higher rate of return (through the coupons) than an investment in the market (earning the lower yield rate). Consequently, the bond will be in demand and will sell at a higher price than its face value. The difference between the purchase price and the face value is called the **premium**.
- **Bonds purchased at discount.** The purchase price of the bond is **lower** than its face value. This happens when the coupon rate is less than the yield rate. In such cases, the bond is providing a lower rate of return (through the coupons) than an investment in the market (earning the higher yield rate). Consequently, the bond will not be in demand and will sell at a lower price than its face value. The difference between the face value and the purchase price is called the **discount**.

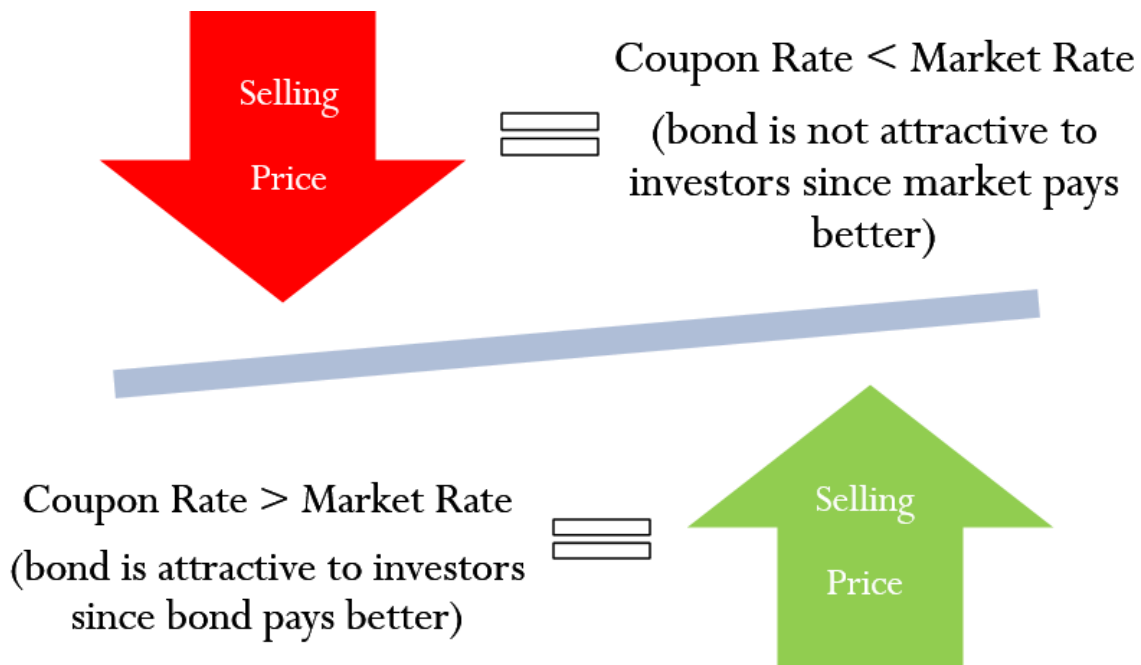


Figure 7.1.1

EXAMPLE

Identify each of the following bonds as premium, discount, or at par.

1. A \$5,000 bond with a 3% coupon is purchased when there are five years to maturity and the yield to maturity is 4.5%.
2. A \$12,000 bond with a 4% coupon is purchased when there are ten years to maturity and the yield to maturity is 4%.
3. A \$7,000 bond with a 6% coupon is purchased when there are twelve years to maturity and the yield to maturity is 3.7%.

Solution:

1. Discount because the coupon rate (3%) is less than the yield rate (4.5%)
2. At par because the coupon rate and yield rate are equal.
3. Premium because the coupon rate (6%) is higher than the yield rate (3.7%).

Calculating the Bond Payment

The **bond payment**, or coupon payment, is the payment the bond holder received semi-annually throughout the investment. This interest is **not** converted to the principal of the bond, and therefore does not compound. Instead, the bond payment is directly paid to the bond holder. The amount of the bond payment depends only on the face value of the bond and the coupon rate.

$$\text{Bond Payment} = FV \times b$$

where

- FV is the face value of the bond when the bond is redeemed at maturity.
- b is periodic coupon rate where $b = \frac{\text{coupon rate}}{\text{number of coupons per year}}$

EXAMPLE

A \$5,000 bond has a 3% coupon. Calculate the bond payment.

Solution:

$$\begin{aligned} \text{Bond Payment} &= FV \times b \\ &= 5,000 \times \frac{0.03}{2} \\ &= \$75 \end{aligned}$$

NOTES

1. Remember, the bond payments and coupon rate are assumed to be semi-annual, unless

stated otherwise.

2. For the bond in the previous example, the bond holder will receive a \$75 payment at the end of every six months over the life time of the bond.

TRY IT

A \$12,000 bond has a 7% coupon. Calculate the bond payment.

Click to see Solution

$$\begin{aligned}\text{Bond Payment} &= FV \times b \\ &= 12,000 \times \frac{0.07}{2} \\ &= \$420\end{aligned}$$

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7.2 CALCULATING THE PURCHASE PRICE OF A BOND ON AN INTEREST PAYMENT DATE

LEARNING OBJECTIVES

- Calculate the purchase price of a bond on an interest payment date.

Exactly what is your marketable bond worth? When you inspect the financial section of your local newspaper, you pay particular attention to the quotes on bonds, into which you invested a portion of your RRSP portfolio. Ten years ago, when prevailing bond rates were 5.5%, you purchased 10 Government of Canada \$5,000 face value bonds with a 5% coupon and 20 years remaining to maturity. At the time you paid \$4,699.02 each for them. Today's prevailing bond rates have dropped to 3.35% but are expected to rise in the near future. Should you hold on to those bonds? Should you cash them in by selling them in the bond market?

Marketable bonds and debentures are nonredeemable, which means the only way to cash these bonds in before the maturity date is to sell them to another investor. Therefore, the key mathematical calculation is what to pay for the bond. The selling date, maturity date, coupon rate, redemption price, and market rate together determine the bond's purchase price.

On the bond's issue date, the market rate determines the coupon rate, so these two rates are identical. As a result, the price of the bond equals its face value. After the bond is issued, interest starts to accrue on it, and the market rate begins to fluctuate based on market conditions. This changes the price of the bond.

Because the bond pays interest semi-annually, two days of the year are defined as the interest payment dates. For now, we are interested in finding out the purchase price of a bond if the bond is purchased on one of these two interest payment dates. Because the bond sells on its interest payment date, the most recent interest payment date is the selling date.

When you purchase a bond, you are buying two promises: to be paid the face value of the bond at maturity and to be paid the periodic bond payments. The purchase price takes both of these promises into account. The purchase price of the bond on an interest payment date is the sum of the present value of all of the remaining payments on the bond on this date and the present value of the face value of the bond on this date, using the yield rate to calculate the present values.

$$\text{Purchase Price} = PV \text{ of Payments} + PV \text{ of Face Value}$$

NOTES

1. The bond payment issued on the date of purchase (which in this case is an interest payment date) goes to the seller of the bond, and not the purchaser. The purchaser only starts to receive payments on the first interest payment date after the purchase of the bond.
2. The price the seller sells the bond for equals the price the purchaser pays for the bond.
3. Similar to T-bills and commercial papers, the time for the purchase price calculation is based on the remaining time to maturity (i.e. the time from the interest payment date to the maturity date) at the time of purchase.

Recall from the previous section that a bond can be bought at a premium, discount, or a par.

- **Bonds purchased at par.** The purchase price of the bond **equals** its face value. This happens when the coupon rate is the same as the yield rate.
- **Bonds purchased at premium.** The purchase price of the bond is **higher** than its face value. The difference between the purchase price and the face value is called the **premium**.

$$\text{Premium} = \text{Purchase Price} - \text{Face Value}$$

- **Bonds purchased at discount.** The purchase price of the bond is **lower** than its face value. The difference between the face value and the purchase price is called the **discount**.

$$\text{Discount} = \text{Face Value} - \text{Purchase Price}$$

You can use the financial calculator to find the purchase price of the bond. After all, the purchase price is just

the sum of two present value calculations. You can calculate each present value separately and then add the results together. Alternatively, the calculator can do both calculations simultaneously and output the purchase price.

USING THE TI BAI PLUS CALCULATOR TO FIND THE PURCHASE PRICE OF A BOND ON AN INTEREST PAYMENT DATE

- Set the calculator to the END payment setting.
- Enter values for the known variables (FV, PMT, N, I/Y, P/Y and C/Y), paying close attention to the cash flow sign convention for PMT and FV.
 - Because you are purchasing the bond, you will become the bond holder and receive the payments and the face value. So PMT (the bond payment) and FV (the face value) are both entered with a positive cash flow sign.
- After all of the known quantities are loaded into the calculator, press CPT and then PV to solve for the purchase price.

NOTE

There are two interest rates associated with a bond: the yield rate and the coupon rate. They are NOT interchangeable.

- The coupon rate is only used to calculate the bond payment. The coupon rate is NEVER used as the value for I/Y.
- The yield rate is the prevailing interest rate in the market. In the purchase price calculation, the yield rate is the interest rate entered for I/Y.

EXAMPLE

A \$50,000 bond with a 10.15% coupon is purchased when there are 10.5 years remaining to maturity. If the yield to maturity at the time of purchase is 4.31%, what is the purchase price? What is the amount of premium or discount?

Solution:

Step 1: Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$\begin{array}{l} FV = \$50,000 \\ P/Y = 2 \\ I/Y = 4.31\% \\ C/Y = 2 \\ t = 10.5 \text{ \textit{years}} \\ \text{Coupon Rate} = 10.15\% \end{array}$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 50,000 \times \frac{0.1015}{2} \\ &= \$2,537.50 \end{aligned}$$

Step 3: Calculate the purchase price.

PMT Setting	END
N	$2 \times 10.5 = 21$
PV	?
FV	50,000
PMT	2,537.50
I/Y	4.31
P/Y	2
C/Y	2

$$PV = \$74,452.86$$

Step 4: Calculate the premium or discount. Because the purchase price is greater than the face value, this is a premium bond.

$$\begin{array}{l} \text{Premium} = \text{Purchase Price} - \text{Face Value} \\ = 74,452.86 - 50,000 = \$24,452.86 \end{array}$$

The purchase price of the bond is \$74,452.86 and the premium is \$24,452.86.

EXAMPLE

A \$25,000 bond with a 8.92% coupon is purchased when there are 22.5 years remaining to maturity. If the yield to maturity at the time of purchase is 9.46%, what is the purchase price? What is the amount of premium or discount?

Solution:

Step 1: Because no other information is given, the frequency of the payments and the

compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$\begin{array}{l} FV = \$25,000 \\ P/Y = 2 \\ I/Y = 9.46\% \\ C/Y = 2 \\ t = 22.5 \text{ years} \\ \text{Coupon Rate} = 8.92\% \end{array}$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 25,000 \times \frac{0.0892}{2} \\ &= \$1,115 \end{aligned}$$

Step 3: Calculate the purchase price.

PMT Setting	END
N	$2 \times 22.5 = 45$
PV	?
FV	25,000
PMT	1,115
I/Y	9.46
P/Y	2
C/Y	2

$$PV = \$23,751.28$$

Step 4: Calculate the premium or discount. Because the purchase price is less than the face value, this is a discount bond.

$$\begin{array}{l} \text{Discount} = \text{Face Value} - \text{Purchase Price} \\ = 25,000 - 23,751.28 \\ = \$1,248.72 \end{array}$$

The purchase price of the bond is \$23,751.28 and the discount is \$1,248.72.

TRY IT

A \$7,000 bond with a coupon rate of 5% is redeemable in ten years. Calculate the purchase price and the premium or discount if the yield rate is 7%.

Click to see Solution

PMT Setting	END
N	$2 \times 10 = 20$
PV	?
FV	7,000
PMT	175
I/Y	7
P/Y	2
C/Y	2

$$PV = \$6,005.13$$

$$\begin{array}{l} \text{Discount} = 7,000 - 6,005.13 \\ \text{Discount} = \$994.87 \end{array}$$

EXAMPLE

An investor purchased a \$10,000 bond with a coupon rate of 6.5% when there was 15 years to maturity and the yield rate was 4%. Four years later, the investor decided to sell the bond when the yield to maturity was 5.2%.

1. At what price did the investor purchase the bond?
2. At what price did the investor sell the bond?
3. What was the investor's gain or loss on the sale of the bond?

Solution:

Step 1: Calculate the investor's purchase price.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 10,000 \times \frac{0.065}{2} \\
 &= \$325
 \end{aligned}$$

PMT Setting	END
N	$2 \times 15 = 30$
PV	?
FV	10,000
PMT	325
I/Y	4
P/Y	2
C/Y	2

$$PV = \$12,799.56$$

The investor paid \$12,799.56 for the bond.

Step 2: Calculate the selling price of the bond. The selling price of the bond for the investor equals the purchase price of the bond for the person the investor is selling the bond to. The purchase price is based on the time to maturity and the yield rate at the time the bond is sold. The investor purchased the bond when there was 15 years to maturity and then sold the bond four years later. So at the time of the sale there was 11 (15-4) years to maturity.

PMT Setting	END
N	$2 \times 11 = 22$
PV	?
FV	10,000
PMT	325
I/Y	5.2
P/Y	2
C/Y	2

$$PV = \$11,078.66$$

The investor sold the bond for \$11,078.66.

Step 3: Calculate the gain or loss on the sale. Because the investor sold the bond for less than they paid for it, the investor realized a loss on the sale.

$$\text{Loss} = 12,799.56 - 11,078.66 = \$1,720.90$$

The investor's lost \$1,720.90 on the sale of the bond.

TRY IT

You purchased a \$3,000 bond paying a 2.5% coupon rate when there was eight years to maturity

and the yield rate was 5.7%. After two years, you decided to sell the bond when the yield to maturity was 4.9%.

1. What price did you pay for the bond?
2. What price did you sell the bond for?
3. What was your gain or loss on the sale of the bond?

Click to see Solution

1. Calculate the purchase price of the bond.

PMT Setting	END
N	$2 \times 8 = 16$
PV	?
FV	3,000
PMT	37.50
I/Y	5.7
P/Y	2
C/Y	2

$$PV = \$2,390.09$$

2. Calculate the selling price of the bond.

PMT Setting	END
N	$2 \times 6 = 12$
PV	?
FV	3,000
PMT	37.50
I/Y	4.9
P/Y	2
C/Y	2

$$PV = \$2,629.60$$

3. Calculate the gain on the sale of the bond.

$$\begin{array}{l} \text{Gain} = 2,629.60 - 2,390.09 \\ \text{Gain} = \$239.51 \end{array}$$

Exercises

1. A \$1,000 bond with a 7.16% coupon is redeemable in 27 years.
 - a. Calculate the purchase price if the yield to maturity is 6.3%.
 - b. Calculate the premium or discount.

Click to see Answer

a. \$1,110.93; b. Premium=\$110.93

2. A \$5,000 bond with a 7.64% coupon is redeemable in 6.5 years.
 - a. Calculate the purchase price if the yield to maturity is 3.96%.
 - b. Calculate the premium or discount.

Click to see Answer

a. \$6,045.43; b. Premium=\$1,045.53

3. With 17 years until maturity, Julio purchased an \$8,000 Government of Saskatchewan bond with a coupon rate of 5.55%. The market yield was 8.65%.
- Calculate the purchase price of the bond.
 - Calculate the premium or discount.

Click to see Answer

a. \$5,777.23; b. Discount=\$2,222.77

4. A \$5,000 with 14 years to maturity has a coupon rate of 4.42% and a yield of 4.07%.
- Calculate the purchase price of the bond.
 - Calculate the premium or discount.

Click to see Answer

a. \$5,185.37; b. Premium=\$185.37

5. A \$25,000 bond has a coupon rate of 5.12%. James purchased the bond when there was seven years to maturity and the yield rate was 4.18%. Five years later, James sold the bond when the yield to maturity was 3.87%.
- Calculate the price James paid for the bond.
 - Calculate the price James sold the bond for.
 - Calculate James' gain or loss on the sale of the bond.

Click to see Answer

a. \$26,413.52; b. \$25,595.90; c. Loss=\$817.62

6. William purchased a \$18,000 bond with a coupon rate of 11.25%, twelve years to maturity and a yield rate of 8.3%. After holding the bond for seven years, William sold the bond when the yield rate was 5.7%.
- At what price did William purchase the bond?
 - At what price did William sell the bond?
 - What was William's gain or loss on the sale of the bond?

Click to see Answer

a. \$21,986.61; b. \$22,293.64; c. Gain=\$307.03

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7.3 CALCULATING THE YIELD RATE

LEARNING OBJECTIVES

- Calculate the yield to maturity of a bond on an interest payment date.
- Calculate the investor's yield when a bond is sold before maturity.

The **yield to maturity** is a bond's overall rate of return when purchased at a market price and held until maturity. It includes both the semi-annual interest that the bondholders earn on their investment along with the gain or loss resulting from the difference between the market price on the selling date and the redemption price. This yield to maturity is exactly equal to the market rate of return on the date of purchase. For example, if you purchase a bond today with a yield rate of 5.5% and you hold onto the bond until maturity, you will realize a yield to maturity of 5.5%.

You can use the financial calculator to find the yield rate for a bond. To calculate the yield rate, you need to know the purchase price of the bond, along with the other information about the bond.

USING THE TI BAI PLUS CALCULATOR TO FIND THE YIELD RATE FOR A BOND

- Set the calculator to the END payment setting.
- Enter values for the known variables (PV, FV, PMT, N, P/Y and C/Y), paying close attention to

the cash flow sign convention for PV, PMT and FV.

- Because you are paying out the purchase price, PV (the purchase price) is entered with a negative cash flow sign.
- Because you are the bond holder, you receive the payments and the face value. So PMT (the bond payment) and FV (the face value) are both entered with a positive cash flow sign.
- After all of the known quantities are loaded into the calculator, press CPT and then I/Y to solve for the yield rate.

NOTES

1. The calculated yield rate is the market rate of return that the bond holder earns if the bond is purchased at the given purchase price and held until maturity. It represents the rate of return the bond holder will receive on their investment, provided they hold the bond until maturity. If the bond holder decides to sell the bond at any point before the maturity date, they will realize a different rate of return on the bond.
2. To calculate a yield rate, the purchase date must be on an interest payment date.

EXAMPLE

Suppose a \$10,000 face value bond is purchased for \$7,688.52 with 20 years until maturity and has a coupon rate of 4% semi-annually. What yield will the bondholder realize if she holds onto it until maturity?

Solution:

Step 1: Because no other information is given, the frequency of the payments and the compounding frequency for the yield rate are assumed to be semi-annual. The given information is

$$\begin{array}{l} PV = \$7,688.52 \quad FV = \$10,000 \quad P/Y = 2 \\ C/Y = 2 \quad t = 20 \text{ years} \quad \text{Coupon Rate} = 4\% \end{array}$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 10,000 \times \frac{0.04}{2} \\ &= \$200 \end{aligned}$$

Step 3: Calculate the yield rate.

PMT Setting	END
N	$2 \times 20 = 40$
PV	-7,688.52
FV	10,000
PMT	200
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 6\%$$

The yield rate is 6% compounded semi-annually.

In order to calculate the yield rate, you need to know the purchase price of the bond. As in the above example, you could be told the purchase price directly. An alternative way to be given information about the purchase price is through a market quotation.

The **market quotation** of a bond is the purchase price expressed as a percentage of the face value. For

example, if a bond has a market quotation of 97.5, it means that the bond's purchase price is 97.5% of its face value. If the face value of the bond is \$5,000, the purchase price is $0.975 \times 5,000 = \$4,875$.

$$\text{Purchase Price} = \text{Face Value} \times \text{Market Quotation}$$

NOTE

Although market quotations are percents, they generally do not include a percent sign when written. It is common practice to see phrases like “The bond is purchased at 102.3.” or “The bond has a market quotation of 102.3.” This does NOT mean that the bond is purchased for \$102.30. In this type of phrasing, it is understood that the “102.3” is a percent and that the purchase price of the bond is 102.3% of the face value of the bond.

EXAMPLE

A \$5,000 bond has a 7% coupon. The bond is purchased at 103.7 when there are 10 years to maturity. What is the yield rate on the bond?

Solution:

Step 1: Because no other information is given, the frequency of the payments and the compounding frequencies are assumed to be semi-annual. The given information is

$$\begin{array}{l} \text{FV} = \$5,000 \\ \text{P/Y} = 2 \\ \text{C/Y} = 2 \\ t = 10 \text{ years} \\ \text{Coupon Rate} = 7\% \\ \text{Market Quotation} = 103.7\% \end{array}$$

Step 2: Calculate the bond payment.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 5,000 \times \frac{0.07}{2} \\
 &= \$175
 \end{aligned}$$

Step 3: Calculate the purchase price (PV).

$$\begin{aligned}
 PV &= FV \times \text{market quotation} \\
 &= 5,000 \times 1.037 \\
 &= \$5,185
 \end{aligned}$$

Step 4: Calculate the yield rate.

PMT Setting	END
N	$2 \times 10 = 20$
PV	-5,185
FV	5,000
PMT	175
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 6.49\%$$

The yield rate is 6.49% compounded semi-annually.

TRY IT

A \$13,000 bond with a coupon rate of 4.5% is redeemable in 15 years. The bond is purchased today at 95.4. Calculate the yield rate on the bond.

Click to see Solution

PMT Setting	END
N	$2 \times 15 = 30$
PV	$-12,402$
FV	13000
PMT	292.50
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 4.94\%$$

The Investor's Yield

The yield-to-maturity calculation requires that the bond is held until its maturity date, at which point the future value redemption price is known and equal to its face value. However, this condition may not hold. The bond holder may sell the bond at any point before the maturity date. If the bond is sold, the future value is based on the prevailing bond rate at that time, and the price is generally not equal to its face value. What then is the bond's yield for the investor? To find the investor's yield, calculate the rate of return (I/Y) with the present value set to the purchase price, the future value set to the selling price, the payment set to the bond payment, and time based on the length of time the investor held the bond.

You can use the financial calculator to find the investor's yield. To calculate the investor's yield, you need to know the purchase price of the bond, the selling price of the bond, and the other information about the bond.

USING THE TI BAI PLUS CALCULATOR TO FIND THE INVESTOR'S YIELD

- Set the calculator to the END payment setting.
- Enter values for the known variables (PV, FV, PMT, N, P/Y and C/Y), paying close attention to the cash flow sign convention for PV, PMT and FV.
 - Because you paid out the purchase price, PV (the purchase price) is entered with a negative cash flow sign.
 - Because you received the bond payments as the bond holder, PMT (the bond payment) is entered with a positive cash flow sign.
 - Because you are selling the bond, you will receive the selling price. So FV (the selling price) is entered with a positive cash flow sign.
- After all of the known quantities are loaded into the calculator, press CPT and then I/Y to solve for the investor's yield.

NOTES

1. To calculate the investor's yield, the purchase date and the selling date must be on an interest payment dates.
2. For an investor's yield calculation, the time is based on the length of time between the purchase date and the selling date. The time to maturity is not involved in an investor's yield calculation.
3. In an investor's yield calculation, the present value is the purchase price of the bond and the future value is the selling price of the bond. The face value is not involved in an

investor's yield calculation (except to find the bond payment).

EXAMPLE

A \$1,000 face value bond with a 7% coupon and 12 years to maturity was purchased for \$1,084.68. It sold seven years later for \$920.87. What rate of return did the bond holder realize on the sale?

Solution:

Step 1: Because no other information is given, the frequency of the payments and the compounding frequencies are assumed to be semi-annual. The given information, related to the sale, is

$$\begin{array}{l} PV = \$1,084.68 \\ FV = \$920.87 \\ P/Y = 2 \\ C/Y = 2 \\ t = 7 \text{ years} \\ \text{Coupon Rate} = 7\% \end{array}$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 1,000 \times \frac{0.07}{2} \\ &= \$35 \end{aligned}$$

Step 3: Calculate the bond holder's rate of return on the sale.

PMT Setting	END
N	$2 \times 7 = 14$
PV	-1,084.68
FV	920.87
PMT	35
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 4.6\%$$

The bond holder's rate of return on the sale is 4.6% compounded semi-annually.

EXAMPLE

Sara purchased a \$25,000 bond paying 2.8% when the yield rate was 3.7% and there was 10 years to maturity. After three years, Sara decides to sell the bond when the yield rate was 4.5%.

1. What price did Sara pay to purchase the bond?
2. At what price did Sara sell the bond?
3. What rate of return did Sara realize on sale of the bond?

Solution:

Step 1: Calculate the Sara's purchase price.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 25,000 \times \frac{0.028}{2} \\
 &= \$350
 \end{aligned}$$

PMT Setting	END
N	$2 \times 10 = 20$
PV	?
FV	25,000
PMT	350
I/Y	3.7
P/Y	2
C/Y	2

$$PV = \$23,133.56$$

Sara paid \$23,133.56 for the bond.

Step 2: Calculate the selling price of the bond. The selling price of the bond for the investor equals the purchase price of the bond for the person the investor is selling the bond to. The purchase price is based on the time to maturity and the yield rate at the time the bond is sold. The investor purchased the bond when there was 10 years to maturity and then sold the bond three years later. So at the time of the sale there was 7 (10-3) years to maturity.

PMT Setting	END
N	$2 \times 7 = 14$
PV	?
FV	25,000
PMT	350
I/Y	4.5
P/Y	2
C/Y	2

$$PV = \$22,472.11$$

Sara sold the bond for \$22,472.11.

Step 3: Calculate Sara's rate of return on the sale.

PMT Setting	END
N	$2 \times 3 = 6$
PV	-23,133.56
FV	22,472.11
PMT	350
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 2.10\%$$

Sara's rate of return on the sale is 2.10% compounded semi-annually.

TRY IT

John purchased a \$12,000 bond with a 4.7% coupon for \$13,750.09 when there was 17 years to maturity and the yield rate was 3.55%. After 11 years, John sold the bond when the yield rate was 5.3%.

1. At what price did John sell the bond?
2. What rate of return did John realize on the sale of the bond?

Click to see Solution

1. Calculate the sale price of the bond.

PMT Setting	END
N	$2 \times 6 = 12$
PV	?
FV	12,000
PMT	282
I/Y	5.3
P/Y	2
C/Y	2

$$PV = \$11,634.05$$

2. Calculate the rate of return on the sale.

PMT Setting	END
N	$2 \times 11 = 22$
PV	-13,750.09
FV	11,634.05
PMT	282
I/Y	?
P/Y	2
C/Y	2

$$I/Y = 2.9\%$$

Exercises

1. A \$1,000 bond with a 6% coupon is purchased for \$1,546.16 when there was 25 year to maturity. Calculate the yield to maturity.

Click to see Answer

2.91%

2. A \$5,000 bond paying 5.75% is purchased for 76.4822 when there was 18.5 years to maturity. Calculate the yield to maturity.

Click to see Answer

8.25%

3. A \$25,000 bond has a 7.25% coupon rate. The bond was purchased for \$23,123.77 and sold six years later for \$21,440.13. Calculate the investor's rate of return on the sale price.

Click to see Answer

6.8363%

4. A \$100,000 bond with a 8.65% coupon rate is purchased for \$106,590. The bond is sold 11.5 years later for \$109,687.75. Calculate the investor's rate of return on the sale price.

Click to see Answer

8.271%

5. A \$500,000 bond has a 9% coupon rate and is redeemable in 15 years. The bond is purchased at 105.72. Calculate the yield rate.

Click to see Answer

8.33%

6. A \$495,000 face value Province of Ontario bond issued with a coupon rate of 7.5% is sold for \$714,557.14. If 20 years remain until maturity, what was the yield to maturity?

Click to see Answer

4.2%

7. A \$100,000 face value bond is issued with a 5% coupon and 22 years until maturity. If it sells for \$76,566.88 9.5 years later, what was the yield to maturity?

Click to see Answer

8%

8. A \$50,000 face value Government of Canada bond is purchased for \$48,336.48 and sold nine years later for \$51,223.23. If the coupon rate is 4.985%, calculate the investor's yield.

Click to see Answer

5.6742%

9. Great-West Life (GWL) issued a \$100,000 face value bond carrying a coupon rate of 6.14% and 20 years to maturity. Eight years later, GWL decides to buy back some of its outstanding bonds when current

market rates are 7.29%. If the bond was held from the issue date until the selling date, what yield did the bondholder realize on her investment?

Click to see Answer

5.2091%

10. A \$50,000 face value bond with a 5.1% coupon is purchase 8.5 years before maturity for \$53,336.24. After 1.5 years, the bond is sold for \$53,109.69.
- Calculate the yield to maturity on the purchase date.
 - Calculate the yield to maturity on the selling date.
 - Calculate the investor's yield.

Click to see Answer

a. 4.16%; b. 4.07%; c. 4.5041%

11. A \$75,000 bond carrying a 4.08% coupon is purchased 19.5 years before maturity when the yield rate was 4.2%. After 3.5 years, the bond is sold when yield rate is 4.05%.
- Calculate the purchase price of the bond along with the amount of the premium or discount.
 - Calculate the selling price of the bond along with the amount of the premium or discount.
 - Calculate the rate of return when the bond was sold.

Click to see Answer

a. Purchase Price=\$73,809.92, Discount=\$1,190.08; b. Purchase Price=\$75,263.06, Premium=\$263.06; c. 4.6701%

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7.4 BOND SCHEDULES

LEARNING OBJECTIVES

- Construct an amortization of the premium bond schedule.
- Construct an accumulation of the discount bond schedule.

Recall that the purchase price of a bond does not necessarily equal its face value. The bond may be purchased for more than its face value, resulting in a premium, or the bond may be purchased for less than its face value, resulting in a discount. When an investor purchases a bond, the investor receives the bond payments as well as the bond's face value at maturity. What are the tax implications associated with owning a bond? For example, the bond payments are taxable income for the investor. But, what are the tax implications of the premium or discount that is realized when the bond is redeemed at maturity?

The **book value of a bond** is the face value of the bond plus or minus any unamortized premium or discount. The initial book value of a bond is its purchase price. A common practice to deal with the tax implications of the premium or discount is to make adjustments to the book value of the bond over the holding period of the bond. The adjustments to the book value involve amortizing small parts of the premium or discount during each payment interval until the book value of the bond equals the bond's face value.

Amortization of the Premium

When a bond is purchased for more than its face value, the result is a premium. For example, suppose an investor pays \$1,050 for a \$1,000 face value bond. If the investor holds onto the bond until maturity, only the redemption price of \$1,000 is returned. The bond premium of \$50 represents a capital loss for the investor because the premium is not recovered at maturity. A **capital loss** is the amount by which the current value of an asset falls short of the original purchase price. What does the investor do with this capital loss? One option is for the investor to apply the capital loss against their income at the time of redemption, which results in a lower

taxable amount at the time of redemption. But, commonly accepted practices allow the investor to amortize the capital loss over the period of time that the bond is held, and not just in the period during which the capital loss actually occurs (at maturity). This allows the investor to spread the capital loss over each payment interval, claiming a little bit of the capital loss against each bond payment in order to lower the taxable amount for each payment interval. This process is called the **amortization of the premium**.

The amortization of the premium schedule records the bond payment, the amount of the premium that is claimed with each payment, the book value, and the remaining premium to be amortized. The amortization of the premium schedule shows how much premium is claimed against each payment and the resulting book value.

An amortization of the premium schedule has six columns:

- **Payment Number.** There is a row for every bond payment during the holding period of the bond.
- **Bond Payment.** The periodic bond payment (PMT). All of the payments are the same, including the last payment.
- **Interest on Book Value at Yield.** For each row, the interest on book value entry is the interest portion of the current bond payment based on the book value from the previous row.
- **Amortized Premium.** For each row, the amortized premium entry is the amount of the premium that is claimed as a capital loss in that payment interval.
- **Book Value.** For each row, the book value is the face value of the bond plus the remaining unamortized premium.
- **Remaining Premium to be Amortized.** For each row, the remaining premium is the amount of premium that is left to claim against the payments.

To fill in an amortization of the premium schedule, you first need to have all of the details about the bond, including the face value (FV), the purchase price (PV), the bond payment (PMT), the number of payments (N), and the yield rate. If any of these quantities are missing, calculate out the missing value before completing the amortization of the premium schedule.

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				Purchase Price ¹	Premium ¹
1	PMT ²	INT on BV ³	AP ⁴	BV ⁵	RP ⁶
2	PMT ²	INT on BV ³	AP ⁴	BV ⁵	RP ⁶
⋮	⋮	⋮	⋮	⋮	⋮
N-1	PMT ²	INT on BV ³	AP ⁴	BV ⁵	RP ⁶
N	PMT ²	INT on BV ³	AP ⁴	Face Value ⁸	0 ⁹
Totals	Total Payments ¹⁰	Total Interest ¹²	Total Amortized Premium ¹¹		D

Follow these steps to fill in the amortization of the premium schedule.

1. In row 0, the only entries are in the book value and remaining premium to be amortized columns. The initial book value is the purchase price of the bond and the initial remaining premium is the premium.
2. Each entry in the bond payment column is the bond payment. All of the payments in this column are the same, including the last payment.
3. Calculate the interest on the book value at yield. The interest is the book value from the previous row times the periodic yield rate:

Interest on Book Value = Book Value from Previous Row $\times i$. **Note:** this calculation uses the periodic yield rate, not the periodic bond rate.

4. Calculate the amortized premium. The amortized premium is the difference between the payment and the interest on the book value:

Amortized Premium = $PMT - \text{Interest on Book Value}$.

5. Calculate the new book value. The book value is the difference between the book value in the previous row and the amortized premium:

Book Value = Book Value from Previous Row $- \text{Amortized Premium}$

6. Calculate the remaining premium to be amortized. The remaining premium is the difference between the remaining premium from the previous row and the amortized premium:

Remaining Premium = Remaining Premium from Previous Row $- \text{Amortized Premium}$

.

7. For each payment, repeat steps 2 through 6, including for the last row.
8. The last book value is the face value of the bond. If the calculations are done correctly, this will automatically happen.
9. The last remaining premium to be amortized is 0. If the calculations are done correctly, this will

automatically happen.

10. The total payments is the sum of the payment column: **Total Payments** = $N \times PMT$.
11. The total amortized premium is the sum of the amortized premium column, and is just the premium:
Total Amortized Premium = **Premium**.
12. The total interest is the sum of the interest on book value column, and equals the difference between the other two column totals:
Total Interest = **Total Payments** – **Total Amortized Premium**.

NOTES

1. The manual calculation of the interest on the book value entry above is based on the assumption that the payment frequency and the compounding frequency are equal. For bonds this is generally not an issue because most bonds have semi-annual payments and a yield rate that compounds semi-annually. If the payment frequency and the compounding frequency are not equal, an interest conversion would be required to convert the interest rate to the equivalent rate with the compounding frequency equal to the payment frequency. However, if you use the TI BAII Plus's built-in amortization worksheet (described below), no interest conversion is required.
2. As you fill in the schedule, round the entries to two decimal places.
3. The amortized premium column represents how much of the premium is claimed as a capital loss against the payment for that payment interval.
4. The amortization of the premium schedule presented here assumes the bond is purchased on an interest payment date. Bonds that are not purchased on an interest payment date have added complications that will not be addressed here.

EXAMPLE

A \$2,000 bond with a coupon rate of 8% is redeemable in two years. The bond was purchased when the yield rate was 5%. Construct the amortization of the premium schedule.

Solution:

Step 1: Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$\begin{array}{l} FV = \$2,000 \\ P/Y = 2 \\ I/Y = 5\% \\ C/Y = 2 \\ t = 2 \text{ years} \\ \text{Coupon Rate} = 8\% \end{array}$$

Step 2: Calculate the bond payment.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 2,000 \times \frac{0.08}{2} \\ &= \$80 \end{aligned}$$

Step 3: Calculate the purchase price.

PMT Setting	END
N	$2 \times 2 = 4$
PV	?
FV	2,000
PMT	80
I/Y	5
P/Y	2
C/Y	2

$$PV = \$2,112.86$$

The purchase price of the bond is \$2,112.86.

Step 4: Calculate the premium.

$$\begin{aligned} \text{Premium} &= \text{Purchase Price} - \text{Face Value} \\ &= 2,112.86 - 2,000 = \$112.86 \end{aligned}$$

The premium is \$112.86.

Step 5: Complete the amortization of premium schedule. Because the payment frequency and the compounding frequency are equal, no interest conversion is required. The calculations for each entry are shown in blue. The periodic yield rate is $i = \frac{5\%}{2} = 2.5\%$.

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$2,112.86	\$112.86
1	\$80	\$52.82 Formula does not parse	\$27.18 Formula does not parse	\$2,085.68 Formula does not parse	\$85.68 Formula does not parse
2	\$80	\$52.14 Formula does not parse	\$27.86 Formula does not parse	\$2,057.82 Formula does not parse	\$57.82 Formula does not parse
3	\$80	\$51.45 Formula does not parse	\$28.55 Formula does not parse	\$2,029.27 Formula does not parse	\$29.27 Formula does not parse
4	\$80	\$50.73 Formula does not parse	\$29.27 Formula does not parse	\$2,000	\$0 Formula does not parse
Totals	\$320 Formula does not parse	\$207.14 Formula does not parse	\$112.86		

NOTES

- 1. In the previous example, the result of the amortization of the premium schedule is that the investor ends up paying taxes only on \$207.14 (the total of the interest on the book

value at yield column) instead of \$320 (the payments total). That is, by spreading the \$112.86 capital loss over the holding period of the bond, the total taxable amount is reduced from \$320 down to \$207.14.

2. Sometimes there is a minor difference in the face value and the remaining premium to be amortized in the last row, where the last entries in these columns are off by a cent or two. This is a normal occurrence that happens because of the rounding of the entries in the schedule.

Although the calculations in an amortization of premium schedule are relatively straightforward, the manual calculations are time-consuming, especially when the schedule has a lot of rows. The amortization worksheet on a financial calculator, such as the TI BAII Plus, can be used to quickly calculate the entries for each row of the schedule.

USING THE TI BAI PLUS CALCULATOR TO CONSTRUCT AN AMORTIZATION OF THE PREMIUM SCHEDULE

To use the amortization worksheet to complete an amortization of the premium schedule:

1. Solve for any unknown quantities about the bond. You need to know all of the information about the bond first before you can use the amortization worksheet.
2. Enter all the value of all seven time value of money variables into the calculator (N, PV, FV, PMT, I/Y, P/Y, C/Y). If you calculated the purchase price (PV) in the first step, you must **re-enter** it rounded to two decimals and with the correct cash flow sign. Make sure the payment setting is set to END, and obey the cash flow sign convention. Because this is a bond, PV (the purchase price) is negative, FV (the face value) is positive, and PMT (the bond payment) is positive.
3. Go to the amortization worksheet by pressing 2nd AMORT (the PV button).
4. To view the entries for a specific row of the schedule, set P1 and P2 to the row number. For

example, to view the entries for row 5, set P1=5 and P2=5:

- i. At the P1 prompt, enter the row number and press ENTER.
 - ii. Press the down arrow.
 - iii. At the P2 prompt, enter the row number and press ENTER.
 - iv. Press the down arrow.
 - v. The BAL entry is the book value entry for the corresponding row.
 - vi. Press the down arrow.
 - vii. The PRN entry is the amortized premium entry for the corresponding row.
 - viii. Press the down arrow.
 - ix. The INT entry is the interest on book value at yield entry for the corresponding row.
 - x. Press the down arrow the return to the P1 screen.
5. Repeat the previous step with a different row number to view the entries for a different row.

NOTES

1. On the amortization worksheet, BAL is the book value entry, PRN is the amortized premium entry, and INT is the interest on book value at yield entry.
2. You cannot get the entries for the last column, remaining premium to be amortized, from the amortization worksheet on the calculator. This entry will still need to be calculated manually.
3. Make sure to re-enter PV rounded to 2 decimal places before using the amortization worksheet. If you enter PV with all of the decimal places, you will not get the correct entries for the amortization of the premium schedule.
4. As you read the entries off of the amortization worksheet on the calculator and put them in the schedule, round the entries to 2 decimal places.

EXAMPLE

A \$5,000 bond has a 4% coupon and is redeemable in two years. It was purchased to yield 3%. Construct the amortization of the premium schedule.

Solution:

Step 1: Calculate the purchase price.

$$\begin{aligned} PMT &= FV \times \frac{\text{coupon rate}}{2} \\ &= 5,000 \times \frac{0.04}{2} \\ &= \$100 \end{aligned}$$

PMT Setting	END
N	$2 \times 2 = 4$
PV	?
FV	5,000
PMT	100
I/Y	3
P/Y	2
C/Y	2

$$PV = \$5,096.36$$

Step 2: Calculate the premium.

$$\begin{aligned} \text{Premium} &= \text{Purchase Price} - \text{Face Value} \\ &= 5,096.36 - 5,000 \\ &= \$96.36 \end{aligned}$$

Step 3: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	4
PV	−5,096.36
FV	5,000
PMT	100
I/Y	3
P/Y	2
C/Y	2

Step 4: Complete the amortization of the premium schedule using the amortization worksheet on the calculator. Remember, the calculator will not tell you the entries for the last column (remaining premium to be amortized), so you will need to complete this column manually.

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$5,096.36	\$96.36
1	\$100	\$76.45	\$23.55	\$5,072.81	\$72.81
2	\$100	\$76.09	\$23.91	\$5,048.90	\$48.90
3	\$100	\$75.73	\$24.27	\$5,024.63	\$24.63
4	\$100	\$75.37	\$24.63	\$5,000	\$0
Totals	\$400	\$303.63	\$96.36		

- **Row 1:** In the amortization worksheet, set P1=1 and P2=1. The entry for the last column (remaining premium to be amortized) is **96.36 — 23.55**.
- **Row 2:** In the amortization worksheet, set P1=2 and P2=2. The entry for the last column (remaining premium to be amortized) is **72.81 — 23.91**.
- **Row 3:** In the amortization worksheet, set P1=3 and P2=3. The entry for the last column (remaining premium to be amortized) is **48.90 — 24.27**.
- **Row 4:** In the amortization worksheet, set P1=4 and P2=4. The entry for the last column

(remaining premium to be amortized) is $24.63 - 24.63$.

• **Totals Row:**

- The amortized premium total is the premium (**\$96.36**).
- The payments total is the sum of the payments: $4 \times 100 = 400$.
- The interest on book value total is the difference in the other two column totals:
 $400 - 96.36 = 303.63$.

TRY IT

A \$10,000 bond has a coupon rate of 6.5%. The bond was purchased when there was three years to maturity and the yield rate was 5%. Construct the amortization of the premium schedule.

Click to see Solution

PMT Setting	END
N	$2 \times 3 = 6$
PV	?
FV	10,000
PMT	325
I/Y	5
P/Y	2
C/Y	2

$$PV = \$10,413.11$$

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$10,413.11	\$413.11
1	\$325	\$260.33	\$64.67	\$10,348.44	\$348.44
2	\$325	\$258.71	\$66.29	\$10,282.15	\$282.15
3	\$325	\$257.05	\$67.95	\$10,214.20	\$214.20
4	\$325	\$255.36	\$69.64	\$10,144.56	\$144.56
5	\$325	\$253.61	\$71.39	\$10,073.17	\$73.17
6	\$325	\$251.83	\$73.17	\$10,000	\$0
Totals	\$1,950	\$303.63	\$413.11		

Accumulation of the Discount

When a bond is purchased for less than its face value, the result is a discount. For example, suppose an investor pays \$950 for a \$1,000 face value bond. If the investor holds onto the bond until maturity, the investor receives the redemption price of \$1,000. The bond discount of \$50 represents a capital gain for the investor. A **capital gain** is the amount by which the current value of an asset exceeds the original purchase price. What does the investor do with this capital gain? One option is for the investor to claim all of the capital gain as taxable income at the time of redemption, which results in a higher taxable amount at the time of redemption. But, commonly accepted practices allow the investor to distribute the capital gain over the period of time that the bond is held, and not just in the period during which the capital gain actually occurs (at maturity). This allows the investor to spread the capital gain over each payment interval, claiming a little bit of the capital gain along with each bond payment. This results in a higher taxable amount for each payment interval, but results in lower taxes overall because the capital gain is not claimed as a single lump-sum taxable expense on the maturity date. This process is called the **accumulation of the discount**.

The accumulation of the discount schedule records the bond payment, the amount of the discount that is claimed with each payment, the book value, and the remaining discount to be accumulated. The accumulation of the discount schedule shows how much discount is claimed with each payment and the resulting book value.

An accumulation of the discount schedule has six columns:

- **Payment Number.** There is a row for every bond payment during the holding period of the bond.
- **Bond Payment.** The periodic bond payment (PMT). All of the payments are the same, including the last payment.
- **Interest on Book Value at Yield.** For each row, the interest on book value entry is the interest portion of the current bond payment based on the book value from the previous row.
- **Accumulated Discount.** For each row, the accumulated discount entry is the amount of the discount that is claimed as a capital gain in that payment interval.
- **Book Value.** For each row, the book value is the face value of the bond minus the remaining unaccumulated discount.
- **Remaining Discount to be Accumulated.** For each row, the remaining discount is the amount of discount that is left to claim with the payments.

To fill in an accumulation of the discount schedule, you first need to have all of the details about the bond, including the face value (FV), the purchase price (PV), the bond payment (PMT), the number of payments (N), and the yield rate. If any of these quantities are missing, calculate out the missing value before completing the accumulation of the discount schedule.

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				Purchase Price ¹	Discount ¹
1	PMT ²	INT on BV ³	AD ⁴	BV ⁵	RD ⁶
2	PMT ²	INT on BV ³	AD ⁴	BV ⁵	RD ⁶
⋮	⋮	⋮	⋮	⋮	⋮
N-1	PMT ²	INT on BV ³	AD ⁴	BV ⁵	RD ⁶
N	PMT ²	INT on BV ³	AD ⁴	Face Value ⁸	0 ⁹
Totals	Total Payments ¹⁰	Total Interest ¹²	Total Accumulated Discount ¹¹		

Follow these steps to fill in the accumulation of the discount schedule.

1. In row 0, the only entries are in the book value and remaining discount to be accumulated columns. The initial book value is the purchase price of the bond and the initial remaining discount is the

discount.

2. Each entry in the bond payment column is the bond payment. All of the payments in this column are the same, including the last payment.
3. Calculate the interest on the book value at yield. The interest is the book value from the previous row times the periodic yield rate:

$$\text{Interest on Book Value} = \text{Book Value from Previous Row} \times i.$$
Note: this calculation uses the periodic yield rate, not the periodic bond rate.
4. Calculate the accumulated discount. The accumulated discount is the difference between the interest on the book value and the payment:

$$\text{Accumulated Discount} = \text{Interest on Book Value} - PMT.$$
5. Calculate the new book value. The book value is sum of the book value in the previous row and the accumulated discount:

$$\text{Book Value} = \text{Book Value from Previous Row} + \text{Accumulated Discount}.$$
6. Calculate the remaining discount to be accumulated. The remaining discount is the difference between the remaining discount from the previous row and the accumulated discount:

$$\text{Remaining Discount} = \text{Remaining Discount from Previous Row} - \text{Accumulated Discount}.$$
7. For each payment, repeat steps 2 through 6, including for the last row.
8. The last book value is the face value of the bond. If the calculations are done correctly, this will automatically happen.
9. The last remaining discount to be accumulated is 0. If the calculations are done correctly, this will automatically happen.
10. The total payments is the sum of the payment column: $\text{Total Payments} = N \times PMT.$
11. The total accumulated discount is the sum of the accumulated discount column, and is just the discount: $\text{Total Accumulated Discount} = \text{Discount}.$
12. The total interest is the sum of the interest on book value column, and equals the sum of the other two column totals: $\text{Total Interest} = \text{Total Payments} + \text{Total Accumulated Discount}.$

NOTES

1. The manual calculation of the interest on the book value entry above is based on the assumption that the payment frequency and the compounding frequency are equal. For bonds this is generally not an issue because most bonds have semi-annual payments and

a yield rate that compounds semi-annually. If the payment frequency and the compounding frequency are not equal, an interest conversion would be required to convert the interest rate to the equivalent rate with the compounding frequency equal to the payment frequency. However, if you use the TI BAII Plus's built-in amortization worksheet (described below), no interest conversion is required.

2. As you fill in the schedule, round the entries to two decimal places.
3. The accumulated discount column represents how much of the discount is claimed as a capital gain with the payment for that payment interval.
4. The accumulation of the discount schedule presented here assumes the bond is purchased on an interest payment date. Bonds that are not purchased on an interest payment date have added complications that will not be addressed here.

EXAMPLE

A \$3,000 bond with a coupon rate of 5% is redeemable in two years. The bond was purchased when the yield rate was 8%. Construct the accumulation of the discount schedule.

Solution:

Step 1: Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$\begin{array}{l} \text{FV} = \$3,000 \\ \text{P/Y} = 2 \\ \text{I/Y} = 8\% \\ \text{C/Y} = 2 \\ \text{t} = 2 \text{ years} \\ \text{Coupon Rate} = 5\% \end{array}$$

Step 2: Calculate the bond payment.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 3,000 \times \frac{0.05}{2} \\
 &= \$75
 \end{aligned}$$

Step 3: Calculate the purchase price.

PMT Setting	END
N	$2 \times 2 = 4$
PV	?
FV	3,000
PMT	75
I/Y	8
P/Y	2
C/Y	2

$$PV = \$2,836.65$$

The purchase price of the bond is \$2,836.65.

Step 4: Calculate the discount.

$$\begin{aligned}
 \text{Discount} &= \text{Face Value} - \text{Purchase Price} \\
 &= 3,000 - 2,836.65 \\
 &= \$163.35
 \end{aligned}$$

The discount is \$163.35.

Step 5: Complete the accumulation of discount schedule. Because the payment frequency and the compounding frequency are equal, no interest conversion is required. The calculations for each

entry are shown in blue. The periodic yield rate is $i = \frac{8\%}{2} = 4\%$.

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Premium to be Amortized
0				\$2,836.65	\$163.35
1	\$75	\$113.47 Formula does not parse	\$38.47 Formula does not parse	\$2,875.12 Formula does not parse	\$124.88 Formula does not parse
2	\$75	\$115 Formula does not parse	\$40 Formula does not parse	\$2,915.12 Formula does not parse	\$84.88 Formula does not parse
3	\$75	\$116.60 Formula does not parse	\$41.60 Formula does not parse	\$2,956.72 Formula does not parse	\$43.28 Formula does not parse
4	\$75	\$118.27 Formula does not parse	\$43.27 Formula does not parse	\$2999.99 Formula does not parse	\$0.01 Formula does not parse
Totals	\$300 Formula does not parse	\$463.34 Formula does not parse	\$163.34		

NOTES

1. In the previous example, there is a minor difference of \$0.01 in the final book value and the final remaining discount to be accumulated in row 4 and the totals in the interest on the book value and accumulated discount totals. This sometimes happens, and is caused by rounding the entries at each step.
2. In the previous example, the result of the accumulation of discount schedule is that the investor ends up paying taxes on \$463.34 (the total of the interest on the book value at yield column) instead of \$300 (the payments total). That is, by spreading the \$163.35 capital gain over the holding period of the bond, the total taxable amount is increased from \$300 to \$463.34. Because the capital gain is spread out over the payments, this results in a lower tax amount overall.

Although the calculations in an accumulation of the discount schedule are relatively straightforward, the manual calculations are time-consuming, especially when the schedule has a lot of rows. The amortization

worksheet on a financial calculator, such as the TI BAI Plus, can be used to quickly calculate the entries for each row of the schedule.

USING THE TI BAI PLUS CALCULATOR TO CONSTRUCT AN ACCUMULATION OF THE DISCOUNT SCHEDULE

To use the amortization worksheet to complete an accumulation of the discount schedule:

1. Solve for any unknown quantities about the bond. You need to know all of the information about the bond first before you can use the amortization worksheet.
2. Enter all the value of all seven time value of money variables into the calculator (N, PV, FV, PMT, I/Y, P/Y, C/Y). If you calculated the purchase price (PV) in the first step, you must **re-enter** it rounded to two decimals and with the correct cash flow sign. Make sure the payment setting is set to END, and obey the cash flow sign convention. Because this is a bond, PV (the purchase price) is negative, FV (the face value) is positive, and PMT (the bond payment) is positive.
3. Go to the amortization worksheet by pressing 2nd AMORT (the PV button).
4. To view the entries for a specific row of the schedule, set P1 and P2 to the row number. For example, to view the entries for row 5, set P1=5 and P2=5:
 - i. At the P1 prompt, enter the row number and press ENTER.
 - ii. Press the down arrow.
 - iii. At the P2 prompt, enter the row number and press ENTER.
 - iv. Press the down arrow.
 - v. The BAL entry is the book value entry for the corresponding row.
 - vi. Press the down arrow.
 - vii. The PRN entry is the accumulated discount entry for the corresponding row.
 - viii. Press the down arrow.
 - ix. The INT entry is the interest on book value at yield entry for the corresponding row.
 - x. Press the down arrow the return to the P1 screen.
5. Repeat the previous step with a different row number to view the entries for a different row.

NOTES

1. On the amortization worksheet, BAL is the book value entry, PRN is the accumulation of discount entry, and INT is the interest on book value at yield entry.
2. You cannot get the entries for the last column, remaining discount to be accumulated, from the amortization worksheet on the calculator. This entry will still need to be calculated manually.
3. Make sure to re-enter PV rounded to 2 decimal places before using the amortization worksheet. If you enter PV with all of the decimal places, you will not get the correct entries for the accumulation of the discount schedule.
4. As you read the entries off of the amortization worksheet on the calculator and put them in the schedule, round the entries to 2 decimal places.

EXAMPLE

A \$7,000 bond has a 3% coupon and is redeemable in two years. It was purchased to yield 5%. Construct the accumulation of the discount schedule.

Solution:

Step 1: Calculate the purchase price.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 7,000 \times \frac{0.03}{2} \\
 &= \$105
 \end{aligned}$$

PMT Setting	END
N	$2 \times 2 = 4$
PV	?
FV	7,000
PMT	105
I/Y	5
P/Y	2
C/Y	2

$$PV = \$6,736.66$$

Step 2: Calculate the discount.

$$\begin{aligned} \text{Premium} &= \text{Face Value} - \text{Purchase Price} \\ &= 7,000 - 6,736.66 \\ &= \$263.34 \end{aligned}$$

Step 3: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	4
PV	-6,736.66
FV	7,000
PMT	105
I/Y	5
P/Y	2
C/Y	2

Step 4: Complete the accumulation of the discount schedule using the amortization worksheet on the calculator. Remember, the calculator will not tell you the entries for the last column (remaining discount to be accumulated), so you will need to complete this column manually.

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$6,736.66	\$263.34
1	\$105	\$168.42	\$63.42	\$6,800.08	\$199.92
2	\$105	\$170	\$65	\$6,865.08	\$134.92
3	\$105	\$171.63	\$66.63	\$6,931.71	\$68.29
4	\$105	\$173.29	\$68.29	\$7,000	\$0
Totals	\$420	\$683.34	\$263.34		

- **Row 1:** In the amortization worksheet, set P1=1 and P2=1. The entry for the last column (remaining discount to be accumulated) is $263.34 - 63.42$.
- **Row 2:** In the amortization worksheet, set P1=2 and P2=2. The entry for the last column (remaining discount to be accumulated) is $199.92 - 65$.
- **Row 3:** In the amortization worksheet, set P1=3 and P2=3. The entry for the last column (remaining discount to be accumulated) is $134.92 - 66.63$.
- **Row 4:** In the amortization worksheet, set P1=4 and P2=4. The entry for the last column (remaining discount to be accumulated) is $68.29 - 68.29$.
- **Totals Row:**
 - The accumulated discount total is the discount (\$263.34).
 - The payments total is the sum of the payments: $4 \times 105 = 420$.
 - The interest on book value total is the sum of other two column totals:
 $420 + 263.34 = 683.34$.

TRY IT

A \$5,000 bond has a coupon rate of 3.6%. The bond was purchased when there was 2.5 years to maturity and the yield rate was 6%. Construct the accumulation of the discount schedule.

Click to see Solution

PMT Setting	END
N	$2 \times 2.5 = 5$
PV	?
FV	5,000
PMT	90
I/Y	6
P/Y	2
C/Y	2

$$PV = \$4,725.22$$

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$4,725.22	\$274.78
1	\$90	\$141.76	\$51.76	\$4,776.98	\$223.02
2	\$90	\$143.31	\$53.31	\$4,830.29	\$169.71
3	\$90	\$144.91	\$54.91	\$4,885.19	\$114.80
4	\$90	\$146.56	\$56.56	\$4,941.75	\$58.25
5	\$90	\$148.25	\$58.25	\$5,000	\$0
Totals	\$450	\$703.33	\$253.33		

Exercises

1. A \$20,000 bond has a 5% coupon rate. The bond was purchased when there are three years to maturity and the yield rate was 6.75%. Construct the appropriate bond schedule for the bond.

Click to see Answer

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$19,063.66	\$936.34
1	\$500	\$643.40	\$143.40	\$19,207.06	\$792.94
2	\$500	\$648.24	\$148.24	\$19,355.30	\$644.70
3	\$500	\$653.24	\$153.24	\$19,508.54	\$491.46
4	\$500	\$658.41	\$158.41	\$19,666.95	\$333.05
5	\$500	\$663.76	\$163.76	\$19,830.71	\$169.29
6	\$500	\$669.29	\$169.29	\$20,000	\$0
Totals	\$3,000	\$3,936.34	\$936.34		

2. A \$50,000 bond with an 8% coupon is redeemable in two years. The bond was purchased when the yield to maturity was 5%. Construct the appropriate bond schedule for the bond.

Click to see Answer

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$52,821.48	\$2,821.48
1	\$2,000	\$1,320.54	\$679.46	\$52,142.02	\$2,142.02
2	\$2,000	\$1,303.55	\$696.45	\$51,445.57	\$1,445.57
3	\$2,000	\$1,286.14	\$713.86	\$50,731.71	\$731.71
4	\$2,000	\$1,268.29	\$731.71	\$50,000	\$0
Totals	\$8,000	\$5,178.52	\$2,821.48		

3. Three years before maturity, a \$55,000 face value bond carrying a 5.5% coupon is acquired when posted market rates are 4.77% compounded semi-annually. Construct the appropriate bond schedule for the bond.

Click to see Answer

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$56,110.02	\$1,110.02
1	\$1,512.50	\$1,338.22	\$174.28	\$55,935.74	\$935.74
2	\$1,512.50	\$1,334.07	\$178.43	\$55,757.31	\$757.31
3	\$1,512.50	\$1,329.81	\$182.69	\$55,574.62	\$574.62
4	\$1,512.50	\$1,325.45	\$187.05	\$55,387.58	\$387.58
5	\$1,512.50	\$1,320.99	\$191.51	\$55,196.07	\$196.07
6	\$1,512.50	\$1,316.43	\$196.07	\$55,000	\$0
Totals	\$9,075	\$57,964.98	\$1,110.02		

4. When the current bond market is yielding 5.89% compounded semi-annually, Jennifer purchases a \$10,000 face value bond carrying a 4.2% coupon with three years until maturity for her RRSP. Construct the appropriate bond schedule for the bond.

Click to see Answer

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$9,541.41	\$458.59
1	\$210	\$280.99	\$70.99	\$9,612.40	\$387.60
2	\$210	\$283.09	\$73.09	\$9,685.49	\$314.52
3	\$210	\$285.24	\$75.24	\$9,760.73	\$239.27
4	\$210	\$287.45	\$77.45	\$9,838.19	\$161.82
5	\$210	\$289.73	\$79.73	\$9,917.92	\$82.08
6	\$210	\$292.08	\$82.08	\$10,000	\$0
Totals	\$1,260	\$1,718.59	\$458.59		

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7.5 QUOTATION OF BONDS

LEARNING OBJECTIVES

- Calculate the purchase price of a bond on a non-interest payment date.
- Calculate the quoted price of a bond.

Calculating the Purchase Price of a Bond on a Non-Interest Payment Date

In a previous section, we learned how to calculate the purchase price of a bond when the bond is purchased on an interest payment date. But, bonds can be bought or sold on any day of the year. How do we calculate the purchase price when a bond is purchased on a non-interest payment date?

For each day that elapses after an interest payment date, interest for the next payment starts to accrue. When the bond is sold in between interest payment dates, the interest, called the **accrued interest**, that accrues on the bond since the interest payment date before the purchase date up to but not including the purchase date is paid to the seller. This accrued interest is included in the purchase price paid to the seller. That is, the buyer has to pay the seller the purchase price of the bond on the previous interest payment date, called the **market price**, plus the accrued interest. The market price plus the accrued interest is called the **flat price**, and is the actual amount the buyer pays the seller on the purchase date.

Why does the buyer have to pay the seller the accrued interest? Suppose a bond makes semi-annual interest payments of \$50 and the buyer acquires the bond two months after the last interest payment date. Because the seller held the bond for two months of the six-month payment interval, it is fair and reasonable for the seller to receive the interest earned during those two months. However, the next interest payment occurs four months after the sale of the bond, at which time the buyer, who now owns the bond, will receive the full \$50 interest payment for the full six month payment interval. Thus, at the time of buying the bond, the buyer has

to pay the seller the bond's market price plus the portion of the next interest payment that legally belongs to the seller. In this example, an interest amount representing two of the six months needs to be paid.

Arriving at the bond's purchase price in between interest payment dates is a little complex because the flat price is increasing according to a compound interest formula, but in practice the accrued interest on the bond is increasing according to a simple interest formula. If this seems peculiar to you, you would be right, but that is just how bonds happen to work!

Follow these steps to find the flat price, the actual purchase price of a bond between interest payment dates.

1. Identify four important dates:
 - Purchase date.
 - Maturity date.
 - Interest payment date immediately before the purchase date.
 - Interest payment date immediately after the purchase date.
2. Calculate the market price on the interest payment date immediately before the purchase date. The market price is the purchase price if the bond is actually purchased on the interest payment date immediately before the purchase date. This calculation is the same purchase price calculation we learned in a previous section.
3. Calculate the flat price on the purchase date by adding the accrued interest to the market price. This is a compound interest only calculation that moves the market price from the payment date immediately before the purchase date to the purchase date. In this calculation, you calculate the future value with the present value set to the market price from the previous step, the payment is 0, the interest rate is the yield rate, and N is

$$N = \frac{\text{Number of days from previous interest payment date to purchase date}}{\text{Number of days in between interest payment dates}}$$

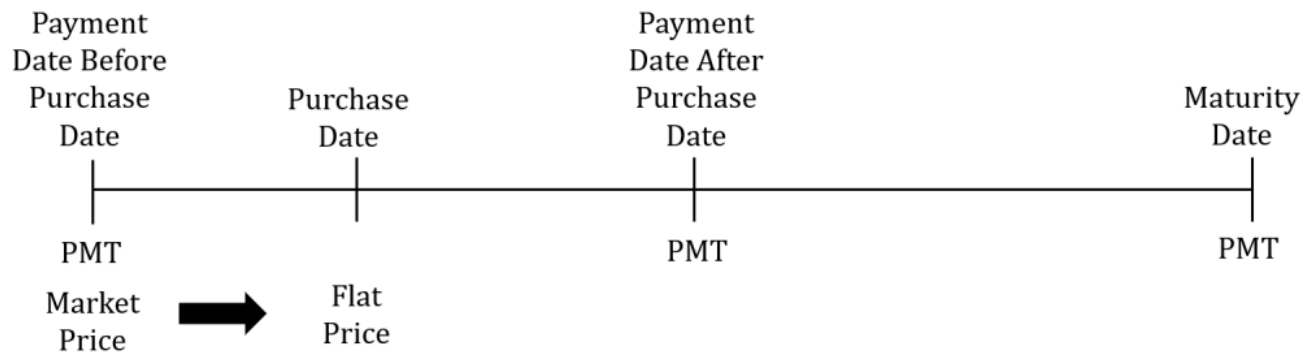


Figure 7.5.1

NOTES

1. The market price is the purchase price on the payment date before the purchase price. That is, if the actual purchase date was this payment date, you would pay the market price to buy the bond.
2. Two things happen on the maturity date: the investor receives the face value of the bond and the last bond payment. Consequently, the maturity date is a payment date. Because the payments happen on the same two dates of every year, the month and day of the maturity date tells you one of the payment dates. The other payment date happens six months later (or six months earlier depending on how you think about it) on the same day. For example, suppose the maturity date is January 1, 2040. This means that January 1 is one of the two payment dates. The other payment date happens six months later, in this case July 1.

EXAMPLE

A \$1,000 bond with a 7% coupon has a maturity date of February 1, 2035. The bond was purchased on May 20, 2023 when the yield to maturity was 5%. Calculate the purchase price.

Solution:

Step 1: Identify the four dates. Because February 1 is the maturity date, February 1 is one of the two payment dates. The other payment date is six months later, in this case August 1.

- Purchase date: May 20, 2023
- Maturity date: February 1, 2035
- Payment date before purchase date: February 1, 2023
- Payment date after purchase date: August 1, 2023



Figure 7.5.2

Step 2: Calculate the market price on February 1, 2023. The time for this calculation is the number of years between February 1, 2023 (the payment date before the purchase date) and February 1, 2035 (the maturity date), which is 12 years.

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$\begin{array}{l} \text{FV} = \$1,000 \\ \text{P/Y} = 2 \\ \text{I/Y} = 5\% \\ \text{C/Y} = 2 \\ \text{t} = 12 \text{ \boxed{years}} \\ \text{\boxed{Coupon Rate}} = 7\% \end{array}$$

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 1,000 \times \frac{0.07}{2} \\
 &= \$35
 \end{aligned}$$

PMT Setting	END
N	$2 \times 12 = 24$
PV	?
FV	1,000
PMT	35
I/Y	5
P/Y	2
C/Y	2

$$PV = \$1,178.85$$

The market price on February 1, 2023 is \$1,178.85.

Step 3: Calculate the flat price on May 20, 2023. This is a compound interest calculation that moves the market price from the previous step forward in time to the purchase date. The value of N for this compound interest calculation is

$$\begin{aligned}
 N &= \frac{\text{Number of days from February 1, 2023 to May 20, 2023}}{\text{Number of days from February 1, 2023 to August 1, 2023}} \\
 &= \frac{108}{181}
 \end{aligned}$$

N	$\frac{108}{181}$
PV	1,178.85
FV	?
PMT	0
I/Y	5
P/Y	2
C/Y	2

$$FV = \$1,196.35$$

The flat price on May 20, 2023 is \$1,196.35. This is the purchase price of the bond on May 20, 2023.

NOTES

1. In the above example, the market price of \$1,178.85 is the purchase price if you purchased the bond on February 1, 2023.
2. To calculate the number of days between the dates in step 3, use the DATE function on the financial calculator.
3. In step 3 above, you will need to convert the fraction $\frac{108}{181}$ to a decimal in order to enter it into your calculator. Remember to keep **all** of the decimals when you enter this number into N to avoid any possible round-off error.
4. The flat price calculation in step 3 is a compound interest only calculation, which means the payment is 0. This is a future value calculation because the market price is moving forward in time to get to the purchase date. The interest rate for this calculation is the yield rate.

USING THE TI BAI PLUS CALCULATOR TO COUNT THE NUMBER OF DAYS BETWEEN DATES.

To count the number of days between two dates:

1. Press 2nd DATE (the 1 button) to enter the date worksheet.
2. At the DT1 screen, enter the first date. Dates are entered in the form mm.ddyy. For example, enter May 19, 2023 as 05.1923. After entering the date press ENTER.
3. Press the down arrow.
4. At the DT2 screen, enter the second date. Dates are entered in the form mm.ddyy. For example, enter August 7, 2023 as 08.0723. After entering the date press ENTER.
5. Press the down arrow.
6. At the DBD screen press CPT to calculate the number of days between the two entered dates. For example, there are 80 days between May 19, 2023 and August 7, 2023.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=395#oembed-1>

[businessfinancialmath/?p=395#oembed-1](https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=395#oembed-1)

[Calculating Dates and Days \(Month-Day-Year\)](#) by Joshua Emmanuel [3:32] (transcript available).

EXAMPLE

A 25-year, \$2,000 bond with a 3% coupon was issued on April 1, 2018. The bond was purchased on January 27, 2023 when the yield to maturity was 8%. Calculate the purchase price.

Solution:

Step 1: Identify the four dates. The bond was issued on April 1, 2018 and reaches maturity 25 years later. So, the maturity date is April 1, 2043. Because April 1 is the maturity date, April 1 is one of the two payment dates. The other payment date is six months later, in this case October 1.

- Purchase date: January 27, 2023
- Maturity date: April 1, 2043
- Payment date before purchase date: October 1, 2022
- Payment date after purchase date: April 1, 2023

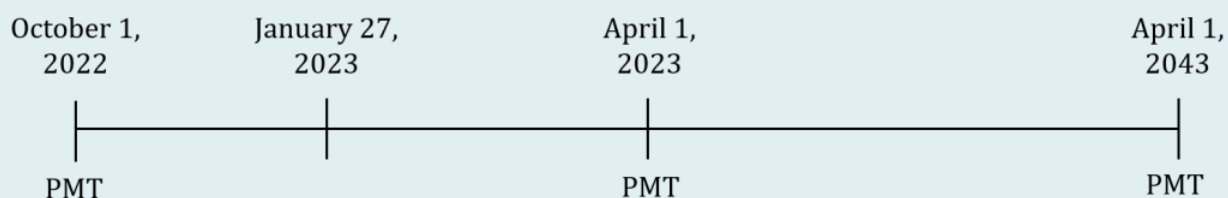


Figure 7.5.3

Step 2: Calculate the market price on October 1, 2022. The time for this calculation is the number of years between October 1, 2022 (the payment date before the purchase date) and April 1, 2043 (the maturity date), which is 20.5 years.

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$\begin{array}{l} FV = \$2,000 \\ P/Y = 2 \\ I/Y = 8\% \\ C/Y = 2 \\ t = 20.5 \text{ years} \\ \text{Coupon Rate} = 3\% \end{array}$$

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 2,000 \times \frac{0.03}{2} \\
 &= \$30
 \end{aligned}$$

PMT Setting	END
N	$2 \times 20.5 = 41$
PV	?
FV	2,000
PMT	30
I/Y	8
P/Y	2
C/Y	2

$$PV = \$1,000.35$$

The market price on October 1, 2022 is \$1,000.35.

Step 3: Calculate the flat price on January 27, 2023. This is a compound interest calculation that moves the market price from the previous step forward in time to the purchase date. The value of N for this compound interest calculation is

$$\begin{aligned}
 N &= \frac{\text{Number of days from October 1, 2022 to January 27, 2023}}{\text{Number of days from October 1, 2022 to April 1, 2023}} \\
 &= \frac{118}{182}
 \end{aligned}$$

N	$\frac{118}{182}$
PV	1,000.35
FV	?
PMT	0
I/Y	8
P/Y	2
C/Y	2

$$FV = \$1,026.11$$

The flat price on January 27, 2023 is \$1,026.11. This is the purchase price of the bond on January 27, 2023.

TRY IT

A \$5,000 bond with a 4.7% coupon has a maturity date of December 1, 2040. The bond was purchased on September 19, 2023 when the yield rate was 5.9%. Calculate the purchase price.

Click to see Solution

1. The four dates are

- Purchase date: September 19, 2023
- Maturity date: December 1, 2040
- Payment date before purchase date: June 1, 2023
- Payment date after purchase date: December 1, 2023

2. Calculate the market price on June 1, 2023.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 5,000 \times \frac{0.047}{2} \\
 &= \$117.50
 \end{aligned}$$

PMT Setting	END
N	$2 \times 17.5 = 25$
PV	?
FV	5,000
PMT	117.50
I/Y	5.9
P/Y	2
C/Y	2

$$PV = \$4,350.65$$

3. Calculate the flat price on September 19, 2023.

N	$\frac{110}{183}$
PV	4,350.65
FV	?
PMT	0
I/Y	5.9
P/Y	2
C/Y	2

$$FV = \$4,427.35$$

Quotation of Bonds

The flat price we calculated in the previous examples is the actual price a buyer would pay to purchase those bonds on the given purchase date. The flat price is the market price (on the payment date before the purchase date) plus any interest that has accrued since the previous payment. However, two different bonds purchased on the same day can have very different amounts of accrued interest, depending on where the purchase date is in relation to each bond's payment date. This can make it very difficult for an investor to directly compare the price of the bonds.

To make a meaningful comparison of bond prices, the flat prices are adjusted for their differing amount of accrued interest. The **quoted price** of bond is the flat price excluding the accrued coupon interest.

$$\text{Quoted Price} = \text{Flat Price} - \text{Accrued Interest}$$

$$\text{Accrued Interest} = PMT \times \frac{\text{Number of days from previous interest payment date to purchase date}}{\text{Number of days in between interest payment dates}}$$

NOTE

The fraction used to find the accrued interest is the same fraction calculated in step 3 of the flat price calculations above.

Quoted prices for bonds are normally given as a percent of their face value. This is the **market quotation** of the bond. Market quotations allow investors to easily compare bond prices.

$$\text{Percent of Face Value} = \frac{\text{Quoted Price}}{\text{Face Value}} \times 100\%$$

EXAMPLE

A \$1,000 bond with a 7% coupon has a maturity date of February 1, 2035. The bond was purchased on May 20, 2023 when the yield to maturity was 5%. Calculate the quoted price as a percentage of face value.

Solution:

The flat price of the bond on May 20, 2023 is \$1,196.35 (found in the first example above).

Step 1: Calculate the accrued interest.

$$\begin{aligned}\text{Accrued Interest} &= PMT \times \frac{\text{Number of days from February 1, 2023 to May 20, 2023}}{\text{Number of days from February 1, 2023 to August 1, 2023}} \\ &= 35 \times \frac{108}{181} \\ &= \$20.88\end{aligned}$$

Step 2: Calculate the quoted price.

$$\begin{aligned}\text{Quoted Price} &= \text{Flat Price} - \text{Accrued Interest} \\ &= 1,196.35 - 20.88 \\ &= \$1,175.47\end{aligned}$$

Step 3: Calculate the quoted price as a percentage of face value.

$$\begin{aligned}\text{Percent of Face Value} &= \frac{\text{Quoted Price}}{\text{Face Value}} \times 100\% \\ &= \frac{1,175.47}{1,000} \times 100\% \\ &= 117.55\%\end{aligned}$$

The quoted price as a percentage of face value is 117.55%.

EXAMPLE

A \$8,000 bond with a 4% coupon has a maturity date of November 1, 2037. The bond was purchased on February 5, 2023 when the yield to maturity was 7.5%. Calculate the quoted price as a percentage of face value.

Solution:

Step 1: Identify the four dates. Because November 1 is the maturity date, November 1 is one of the two payment dates. The other payment date is six months later, in this case May 1.

- Purchase date: February 5, 2023
- Maturity date: November 1, 2037
- Payment date before purchase date: November 1, 2022
- Payment date after purchase date: May 1, 2023

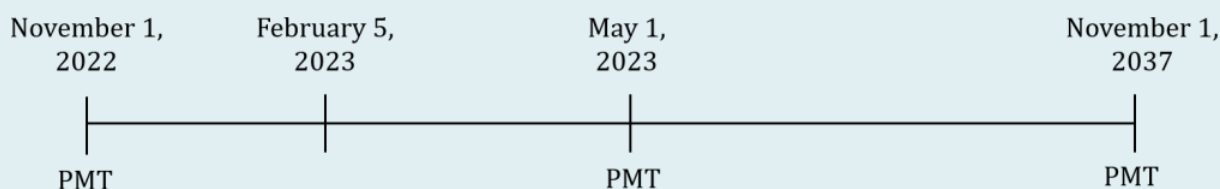


Figure 7.5.4

Step 2: Calculate the market price on November 1, 2022. The time for this calculation is the number of years between November 1, 2022 (the payment date before the purchase date) and November 1, 2037 (the maturity date), which is 15 years.

Because no other information is given, the frequency of the payments and the compounding frequencies (for the coupon rate and the yield rate) are assumed to be semi-annual. The given information is

$$\begin{array}{l} \text{FV} = \$8,000 \\ P/Y = 2 \\ I/Y = 7.5\% \\ C/Y = 2 \\ t = 15 \text{ years} \\ \text{Coupon Rate} = 4\% \end{array}$$

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 8,000 \times \frac{0.04}{2} \\
 &= \$160
 \end{aligned}$$

PMT Setting	END
N	$2 \times 15 = 30$
PV	?
FV	8,000
PMT	160
I/Y	7.5
P/Y	2
C/Y	2

$$PV = \$5,503.91$$

The market price on November 1, 2022 is \$5,503.91.

Step 3: Calculate the flat price on February 5, 2023. This is a compound interest calculation that moves the market price from the previous step forward in time to the purchase date. The value of N for this compound interest calculation is

$$\begin{aligned}
 N &= \frac{\text{Number of days from November 1, 2022 to February 5, 2023}}{\text{Number of days from November 1, 2022 to May 1, 2023}} \\
 &= \frac{96}{181}
 \end{aligned}$$

N	$\frac{96}{181}$
PV	5,503.91
FV	?
PMT	0
I/Y	7.5
P/Y	2
C/Y	2

$$FV = \$5,612.43$$

The flat price on February 5, 2023 is \$5,612.43.

Step 4: Calculate the accrued interest.

$$\begin{aligned}
 \text{Accrued Interest} &= PMT \times \frac{\text{Number of days from November 1, 2022 to February 5, 2023}}{\text{Number of days from November 1, 2022 to May 1, 2023}} \\
 &= 160 \times \frac{96}{181} \\
 &= \$84.86
 \end{aligned}$$

Step 5: Calculate the quoted price.

$$\begin{aligned}
 \text{Quoted Price} &= \text{Flat Price} - \text{Accrued Interest} \\
 &= 5,612.43 - 84.86 \\
 &= \$5,527.57
 \end{aligned}$$

Step 6: Calculate the quoted price as a percentage of face value.

$$\begin{aligned}
 \text{Percent of Face Value} &= \frac{\text{Quoted Price}}{\text{Face Value}} \times 100\% \\
 &= \frac{5,527.57}{8,000} \times 100\% \\
 &= 69.09\%
 \end{aligned}$$

The quoted price as a percentage of face value is 69.09%.

TRY IT

A 15-year, \$7,000 bond with a 2.9% coupon was issued on September 1, 2015. The bond was purchased on April 15, 2023 when the yield to maturity was 4.1%. Calculate the quoted price as a percentage of face value.

Click to see Solution

1. The four dates are

- Purchase date: April 15, 2023
- Maturity date: September 1, 2030
- Payment date before purchase date: March 1, 2023
- Payment date after purchase date: September 1, 2023

2. Calculate the market price on March 1, 2023.

$$\begin{aligned}
 PMT &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 7,000 \times \frac{0.029}{2} \\
 &= \$101.50
 \end{aligned}$$

PMT Setting	END
N	$2 \times 7.5 = 15$
PV	?
FV	7,000
PMT	101.50
I/Y	4.1
P/Y	2
C/Y	2

$$PV = \$6,462.34$$

3. Calculate the flat price on April 15, 2023.

N	$\frac{45}{184}$
PV	6,462.34
FV	?
PMT	0
I/Y	4.1
P/Y	2
C/Y	2

$$FV = \$6,494.50$$

4. Calculate the accrued interest.

$$\begin{aligned}
 \text{Accrued Interest} &= PMT \times \frac{\text{Number of days from March 1, 2023 to April 15, 2023}}{\text{Number of days from March 1, 2023 to September 1, 2023}} \\
 &= 101.50 \times \frac{45}{184} \\
 &= \$24.82
 \end{aligned}$$

5. Calculate the quoted price.

$$\begin{aligned}
 \text{Quoted Price} &= \text{Flat Price} - \text{Accrued Interest} \\
 &= 6,494.50 - 24.82 \\
 &= \$6,469.68
 \end{aligned}$$

6. Calculate the quoted price as a percentage of face value.

$$\begin{aligned}
 \text{Percent of Face Value} &= \frac{\text{Quoted Price}}{\text{Face Value}} \times 100\% \\
 &= \frac{6,469.68}{7,000} \times 100\% \\
 &= 92.42\%
 \end{aligned}$$

Exercises

1. A \$50,000 bond with a 6.33% coupon rate is redeemable on January 1, 2038. The bond was purchased on May 2, 2023 when the yield rate was 4.19%.
 - a. Calculate the flat price on the purchase date.
 - b. Calculate the quoted price on the purchase date.
 - c. Calculate the quoted price as a percentage of face value on the purchase date.

Click to see Answer

a. \$62,689.98; b. \$61,632.07; c. 123.26%

2. A 30-year, \$2,500 bond with a 10.4% coupon rate was issued on November 1, 2006. The bond was purchased on September 13, 2023 when the yield to maturity was 12.39%.
 - a. Calculate the flat price on the purchase date.
 - b. Calculate the quoted price on the purchase date.
 - c. Calculate the quoted price as a percentage of face value on the purchase date.

Click to see Answer

a. \$2,275.89; b. \$2,180.51; c. 87.22%

3. A \$15,000 bond with a 9.75% coupon was purchased on January 2, 2023 when the yield rate was 10.1%.

The bond matures on April 1, 2040.

- a. Calculate the flat price on the purchase date.
- b. Calculate the quoted price on the purchase date.
- c. Calculate the quoted price as a percentage of face value on the purchase date.

Click to see Answer

a. \$14,944.40; b. \$14,570.74; c. 97.14%

4. A 20-year, \$10,000 bond was issued on September 1, 2019 with a 2% coupon. The bond was purchased on November 17, 2023 when the yield rate was 5.25%.

- a. Calculate the flat price on the purchase date.
- b. Calculate the quoted price on the purchase date.
- c. Calculate the quoted price as a percentage of face value on the purchase date.

Click to see Answer

a. \$6,582.90; b. \$6,540.59; c. 130.81%

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7.6 SINKING FUNDS

LEARNING OBJECTIVES

- Construct a sinking fund schedule.

A **sinking fund** is a special account into which an investor, either an individual or a business, makes annuity payments so that sufficient funds are on hand by a specified date to meet a future savings goal or debt obligation. In its simplest terms, a sinking fund is a financial savings place. As the definition indicates, a sinking fund has is used for one of two main purposes:

- **Capital Savings.** When your goal is to acquire some form of a capital asset by the end of the fund, you have a capital savings sinking fund. What is a capital asset? It is any tangible property that is not easily converted into cash. For example, saving up to buy a home, car, warehouse, or even new production machinery are all capital savings.
- **Debt Retirement.** When your goal is to pay off some form of debt by the end of the fund, you have a debt retirement sinking fund. Perhaps as a consumer, if you were able to get a 0% interest plan with no payments for one year, you might want to make monthly payments into your own savings account so that you would have the needed funds to pay off your purchase when it comes due. Businesses usually set up sinking funds for the retirement of stocks, bonds, and debentures.

Sinking Funds and Debt Retirement

Whether the sinking fund is for capital savings or debt retirement, the mathematical calculations and procedures are identical. However, this section will focus on using sinking funds for debt retirement. Now, why discuss sinking funds in the chapter about bonds? Many bonds carry a sinking fund provision. Once the bond has been issued, the company must start regular contributions to a sinking fund because large sums of

money have been borrowed over a long time frame, and investors need assurance that the bond issuer will be able to repay its debt upon bond maturity.

To provide further assurance to bondholders, the sinking fund is typically managed by a neutral third party rather than the bond-issuing company. This third-party company ensures the integrity of the fund, working toward the debt retirement in a systematic manner according to the provisions of the sinking fund. Investors much prefer bonds or debentures that are backed by sinking funds and third-party management because they are less likely to default.

Sinking funds are an alternative way to pay off a loan or debt. A sinking fund is used to accumulate the principal only owed on a debt so that the principal of the debt can be repaid in its entirety on the maturity date. For example, sinking funds are used to accumulate the face value of bonds so that money is available to pay the face value at maturity. Sinking funds are NOT used to pay the interest due on the debt. For example, sinking funds are not used to pay the periodic bond payments.

In the case of bonds or debentures, sinking funds are most commonly set up as ordinary simple annuities that match the timing of the bond interest payments. Thus, when a bond issuer makes a bond interest payment to its bondholders, it also makes an annuity payment to its sinking fund. In other applications, any type of annuity is possible, whether ordinary or due, general or simple.

When a sinking fund is used to retire a debt, there are two interest rates associated with the debt.

- **Interest rate for the loan or debt.** Because sinking funds are used to pay off loans, bonds, or other debts, there is an interest rate for calculating the amount of interest charged on the loan. For example, if the sinking fund is used to accumulate the face value of a bond, the interest rate on the debt is the coupon rate of the bond. This interest rate is only for calculating the interest due on the loan or debt, and does not have anything to do with the money deposited into the sinking fund.
- **Interest rate for the sinking fund.** Because a sinking fund is an interest-earning account, there is an interest rate for calculating the interest earned by the money in the sinking fund.

When a sinking fund is established to retire a debt, there are two different periodic costs or expenses made in relation to the debt.

- **Periodic interest payments on the debt.** The interest rate for the loan or debt is used to calculate the periodic interest payment. For example, if a sinking fund is used to accumulate the face value of a bond, the bond issuer still has to make the periodic bond payments to the bond holders. These periodic interest payments are not directly related to the sinking fund, but are a cost associated with the debt.
- **Periodic payments made to the sinking fund.** When a periodic interest payment is made on the debt, a payment is made into the sinking fund with the goal of accumulating the loan amount. The

interest rate for the sinking fund is used to calculate the periodic annuity payment for the sinking fund. For example, if a sinking fund is used to accumulate the face value of a bond, the bond issuer makes a deposit into the sinking fund at the same time they make the bond payment. Because these periodic sinking fund payments are made as a result of a debt, they are also costs associated with the debt.

The periodic cost of a debt retired with a sinking fund is the total amount paid each payment interval as a result of the debt.

Periodic Cost of the Debt = Periodic Interest Payment + Periodic Sinking Fund Payment

The payments made into a sinking fund form an annuity, and are calculated the same as any other annuity payment with the future value of the sinking fund set to the loan amount and using the interest rate associated with the sinking fund. Because the goal of the sinking fund is to accumulate **at least** the required amount, sinking fund payments are always rounded **UP** to the next cent. Consequently, all of the payments made to a sinking fund, including the last payment, are the same.

EXAMPLE

A bank issued a \$10,000,000 face value bond carrying a 5.1% coupon and 30 years until maturity. The bank set up a sinking earning 4.5% to accumulate the face value of the bond.

1. Calculate the sinking fund payment.
2. Calculate the periodic expense of the debt.

Solution:

Step 1: Because no other information is given, the frequency of the payments (for both the bond and the sinking fund) and the compounding frequencies (for the coupon rate and the sinking fund rate) are assumed to be semi-annual. The given information for the sinking fund is

$$\begin{array}{l} FV = \$10,000,000 \\ I/Y = 4.5\% \\ P/Y = 2 \\ C/Y = 2 \\ t = 30 \text{ years} \end{array}$$

Step 2: Calculate the sinking fund payment.

PMT Setting	END
N	$2 \times 30 = 60$
PV	0
FV	10,000,000
PMT	?
I/Y	4.5
P/Y	2
C/Y	2

$$PMT = \$80,353.2748... \rightarrow \$80,353.28$$

The sinking fund payment is \$80,353.28. Remember, sinking fund payments always get rounded UP to the next cent.

Step 3: Calculate the bond payment.

$$\begin{aligned}
 \text{Bond Payment} &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 10,000,000 \times \frac{0.051}{2} \\
 &= \$255,000
 \end{aligned}$$

The bond payments are \$255,000.

Step 4: Calculate the periodic cost of the debt.

$$\begin{aligned}
 \text{Periodic Cost of the Debt} &= \text{Periodic Interest Payment} + \text{Periodic Sinking Fund Payment} \\
 &= 255,000 + 80,353.28 \\
 &= \$335,353.28
 \end{aligned}$$

The periodic cost of the debt is \$335,353.28. This means that every six months the bank must pay out a total of \$335,353.28 because of the debt. Of this amount, \$255,000 goes to paying the bond payments and \$80,353.28 goes to the sinking fund to accumulate the \$10,000,000 face value of the bonds.

NOTES

1. The goal of a sinking fund is to accumulate the loan amount so that the loan amount can be paid off in one lump-sum payment at the end of the term. So, the loan amount becomes the future value of the sinking fund.
2. Sinking fund payments always get rounded UP to the next cent. This ensures that the final amount in the sinking fund will be at or over the loan amount. Rounding the payment up guarantees that the balance in the sinking fund at the end of the term will always be at or over the loan amount.

TRY IT

A company issued bonds worth \$200,000 to raise money to build an expansion to its factory. The bonds have a coupon rate of 3.9% compounded semi-annually and ten years to maturity. The company established a sinking fund earning 2.7% compounded semi-annually to accumulate the face value of the bonds.

1. Calculate the sinking fund payment.
2. Calculate the periodic expense of the debt.

Click to see Solution

1. Calculate the sinking fund payment.

PMT Setting	END
N	$2 \times 10 = 20$
PV	0
FV	200,000
PMT	?
I/Y	2.7
P/Y	2
C/Y	2

$$PMT = \$8,777.63$$

2. Calculate the periodic expense of the debt.

$$\begin{aligned}
 \text{Bond Payment} &= FV \times \frac{\text{coupon rate}}{2} \\
 &= 200,000 \times \frac{0.039}{2} \\
 &= \$3,900
 \end{aligned}$$

$$\begin{aligned}
 \text{Periodic Cost of the Debt} &= \text{Periodic Interest Payment} + \text{Periodic Sinking Fund Payment} \\
 &= 3,900 + 8,777.63 \\
 &= \$12,677.63
 \end{aligned}$$

Sinking Fund Schedules

When a company takes out a loan or issues bonds, these are debts to the company. Through a sinking fund the company saves up money to extinguish that debt. The **book value of the debt** is the difference between the principal amount owing on the debt (i.e. the loan amount or face value of the bond) and the accumulated balance in the sinking fund at any point in time. For example, if the company issued \$10 million in bonds and has accumulated \$1 million in its sinking fund, the book value of the debt is \$9 million.

$$\text{Book Value} = \text{Principal} - \text{Fund Balance}$$

A sinking fund schedule is a table that records the sinking fund contribution, the interest earned by the fund, the increase in the fund, the accumulated balance for every payment, and the current book value of the debt.

A sinking fund schedule is very similar to an amortization schedule except that the balance increases instead of decreases and the interest is earned instead of being paid.

A sinking fund schedule has six columns:

- **Payment Number.** There is a row for every payment into the sinking fund.
- **Payment.** The sinking fund payment (PMT). Because the sinking fund payment is rounded up, all of the payments are the same, including the last payment.
- **Interest.** The interest earned by the fund at the end of each payment interval.
- **Increase.** The total amount added to the fund with each payment interval.
- **Balance.** The current amount accumulated in the fund for each payment interval.
- **Book Value.** The book value of the debt.

To fill in a sinking fund schedule, you first need to have all of the details about the fund, including the loan amount (FV), the sinking fund payment (PMT), the number of payments (N), and the sinking fund's interest rate. If any of these quantities are missing, calculate out the missing value before completing the sinking fund schedule.

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				0	Loan Amount ¹
1	PMT ²	INT ³	INC ⁴	BAL ⁵	BV ⁶
2	PMT ²	INT ³	INC ⁴	BAL ⁵	BV ⁶
⋮	⋮	⋮	⋮	⋮	⋮
N-1	PMT ²	INT ³	INC ⁴	BAL ⁵	BV ⁶
N	PMT ²	INT ³	INC ⁴	BAL ⁵	BV ⁶
Totals	Total Payments ⁸	Total Interest ¹⁰	Total Increase ⁹		

Follow these steps to fill in a sinking fund schedule.

1. In row 0, the only entries are in the balance and book value columns. The initial balance is 0 and the initial book value is the loan amount (the future value of the sinking fund).
2. Each entry in the payment column is the sinking fund payment. If you have to calculate out the payment, remember to round the payment up to the next cent. All of the payments in this column are the same, including the last payment.

3. Calculate the interest. The interest is the balance from the previous row times the periodic interest rate:

$$\text{Interest} = \text{Balance from Previous Row} \times i.$$
Note: this calculation uses the periodic sinking fund rate, not the periodic interest rate associated with the loan.
4. Calculate the increase. The increase is the sum of the payment and the interest:

$$\text{Increase} = PMT + \text{Interest}.$$
5. Calculate the new balance. The balance is the sum of the balance in the previous row and the increase:

$$\text{Balance} = \text{Balance from Previous Row} + \text{Increase}$$
6. Calculate the new book value. The book value is the difference between the book value from the previous row and the increase:

$$\text{Book Value} = \text{Book Value from Previous Row} - \text{Increase}.$$
7. For each payment, repeat steps 2 through 6, including for the last row.
8. The total payments is the sum of the payment column: $\text{Total Payments} = N \times PMT.$
9. The total increase is the sum of the increase column, and is the last balance entry:

$$\text{Total Increase} = \text{Balance in Last Row}.$$
10. The total interest is the sum of the interest column, and equals the difference between the other two column totals: $\text{Total Interest} = \text{Total Payments} - \text{Total Increase}.$

NOTES

1. The manual calculation of the interest entry above is based on the assumption that the payment frequency and the compounding frequency are equal. If the payment frequency and the compounding frequency are not equal, an interest conversion would be required to convert the interest rate to the equivalent rate with the compounding frequency equal to the payment frequency. However, if you use the TI BAII Plus's built-in amortization worksheet (described below), no interest conversion is required.
2. As you fill in the schedule, round the entries to two decimal places.
3. The sinking fund schedule presented here assumes the payments are made at the end of the payment interval. That is, the sinking fund schedule presented above is for an ordinary annuity. If the sinking fund is an annuity due (payments at the beginning), the calculations are the same except for the interest column, where the interest is based on both the balance from the previous row and the payment.

EXAMPLE

A company has to repay a \$20,000 loan in two years. The company establishes a sinking fund earning 4% compounded semi-annually and makes end-of-six-month payments into the fund to accumulate the loan amount. Construct the sinking fund schedule.

Solution:

Step 1: The given information is

$$\begin{array}{l} FV = \$20,000 \\ I/Y = 4\% \\ P/Y = 2 \\ C/Y = 2 \\ t = 2 \text{ years} \end{array}$$

Step 2: Calculate the sinking fund payment.

PMT Setting	END
N	$2 \times 2 = 4$
PV	0
FV	20,000
PMT	?
I/Y	4
P/Y	2
C/Y	2

$$PMT = \$4,852.48$$

The sinking fund payment is \$4,852.48.

Step 3: Complete the sinking fund schedule. Because the payment frequency and the compounding frequency are equal, no interest conversion is required. The calculations for each entry are shown in blue. The periodic interest rate is $i = \frac{4\%}{2} = 2\%$.

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				\$0	\$20,000
1	\$4,852.48	\$0 Formula does not parse	\$4,852.48 Formula does not parse	\$4,852.48 Formula does not parse	\$15,147.52 Formula does not parse
2	\$4,852.48	\$97.05 Formula does not parse	\$4,949.53 Formula does not parse	\$9,802.01 Formula does not parse	\$10,197.99 Formula does not parse
3	\$4,852.48	\$196.04 Formula does not parse	\$5,048.52 Formula does not parse	\$14,850.5 Formula does not parse	\$5,149.47 Formula does not parse
4	\$4,852.48	\$297.01 Formula does not parse	\$5,149.49 Formula does not parse	\$20,000.02 Formula does not parse	−\$0.02 Formula does not parse
Totals	\$19,409.92 Formula does not parse	\$590.10 Formula does not parse	\$20,000.02		

NOTE

In the previous example, the final balance is slightly more than required \$20,000 because the sinking fund payment was rounded up to the next cent. By rounding the payment up, we ensure that the sinking fund has at least \$20,000 at the end of the term.

Although the calculations in a sinking fund schedule are relatively straightforward, the manual calculations are time-consuming, especially when the schedule has a lot of rows. The amortization worksheet on a financial calculator, such as the TI BAII Plus, can be used to quickly calculate the entries for each row of the schedule.

USING THE TI BAI PLUS CALCULATOR TO CONSTRUCT A SINKING FUND SCHEDULE

To use the amortization worksheet to complete a sinking fund schedule:

1. Solve for any unknown quantities about the sinking fund. You need to know all of the information about the sinking fund first before you can use the amortization worksheet.
2. Enter all the value of all seven time value of money variables into the calculator (N, PV, FV, PMT, I/Y, P/Y, C/Y). If you calculated the payment in the first step, you must **re-enter** it rounded up to the next cent and with the correct cash flow sign. Make sure the payment setting is set to END, and obey the cash flow sign convention. Because PMT (the sinking fund payment) is paid out to the fund, PMT is negative. At the end of the term, FV is received, so FV is positive.
3. Go to the amortization worksheet by pressing 2nd AMORT (the PV button).
4. To view the entries for a specific row of the schedule, set P1 and P2 to the row number. For example, to view the entries for row 5, set P1=5 and P2=5.
 - i. At the P1 prompt, enter the row number and press ENTER.
 - ii. Press the down arrow.
 - iii. At the P2 prompt, enter the row number and press ENTER.
 - iv. Press the down arrow.
 - v. The BAL entry is the balance entry for the corresponding row.
 - vi. Press the down arrow.
 - vii. The PRN entry is the increase entry for the corresponding row.
 - viii. Press the down arrow.
 - ix. The INT entry is the interest entry for the corresponding row.
 - x. Press the down arrow the return to the P1 screen.
5. Repeat the previous step with a different row number to view the entries for a different row.

NOTES

1. On the amortization worksheet, BAL is the balance entry, PRN is the increase entry, and INT is the interest entry.
2. You cannot get the entries for the last column, the book value, from the amortization worksheet on the calculator. This entry will still need to be calculated manually. You can find the book value for any row by subtracting the balance for the row from the loan amount: **Book Value = Principal – Fund Balance**
3. Make sure to re-enter PMT rounded up to the next cent before using the amortization worksheet. Otherwise, you will not get the correct entries for the sinking fund schedule.
4. As you read the entries off of the amortization worksheet on the calculator and put them in the schedule, round the entries to 2 decimal places.

EXAMPLE

A company set up a sinking fund to accumulated the \$30,000 they need to repay a loan. The sinking fund earns 3.5% compounded semi-annually. The company made semi-annual deposits into the sinking fund for 2.5 years. Construct the sinking fund schedule.

Solution:

Step 1: Calculate the sinking fund deposit.

PMT Setting	END
N	$2 \times 2.5 = 5$
PV	0
FV	30,000
PMT	?
I/Y	3.5
P/Y	2
C/Y	2

$$PMT = \$5,793.65$$

Step 2: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	5
PV	0
FV	30,000
PMT	−5,793.65
I/Y	3
P/Y	2
C/Y	2

Step 4: Complete the sinking fund schedule using the amortization worksheet on the calculator.

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				\$0	\$30,000
1	\$5,793.65	\$0	\$5,793.65	\$5,793.65	\$24,206.35
2	\$5,793.65	\$101.39	\$5,895.04	\$11,688.69	\$18,311.31
3	\$5,793.65	\$204.55	\$5,998.20	\$17,686.89	\$12,313.11
4	\$5,793.65	\$309.52	\$6,103.17	\$23,790.06	\$6,209.94
5	\$5,793.65	\$416.33	\$6,209.98	\$30,000.04	—\$0.04
Totals	\$28,968.25	\$1,031.79	\$30,000.04		

- **Row 1:** In the amortization worksheet, set P1=1 and P2=1. The entry for the last column (the book value) is $30,000 - 5,793.65$.
- **Row 2:** In the amortization worksheet, set P1=2 and P2=2. The entry for the last column (the book value) is $30,000 - 11,688.69$.
- **Row 3:** In the amortization worksheet, set P1=3 and P2=3. The entry for the last column (the book value) is $30,000 - 17,686.89$.
- **Row 4:** In the amortization worksheet, set P1=4 and P2=4. The entry for the last column (the book value) is $30,000 - 23,790.06$.
- **Row 5:** In the amortization worksheet, set P1=5 and P2=5. The entry for the last column (the book value) is $30,000 - 30,000.04$.
- **Totals Row:**
 - The increase total is the last balance entry (\$30,000.04).
 - The payments total is the sum of the payments: $5 \times 5,793.65 = 28,968.25$.
 - The interest total is the difference in the other two column totals:
 $30,000.04 - 28,968.25 = 1,031.79$.

TRY IT

A \$10,000 loan was repaid using a sinking fund that was earning 4.5% compounded semi-annually. Deposits were made every six months into the fund for three years to accumulate the loan amount. Construct the sinking fund schedule.

Click to see Solution

PMT Setting	END
N	$2 \times 3 = 6$
PV	0
FV	10,000
PMT	?
I/Y	4.5
P/Y	2
C/Y	2

$$PMT = \$1,575.35$$

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				\$0	\$10,000
1	\$1,575.35	\$0	\$1,575.35	\$1,575.35	\$8,424.65
2	\$1,575.35	\$35.45	\$1,610.80	\$3,186.15	\$6,813.85
3	\$1,575.35	\$71.69	\$1,647.04	\$4,833.18	\$5,166.82
4	\$1,575.35	\$108.75	\$1,684.10	\$6,517.28	\$3,482.72
5	\$1,575.35	\$146.64	\$1,721.99	\$8,239.27	\$1,760.73
6	\$1,575.35	\$185.38	\$1,760.73	\$10,000	\$0
Totals	\$9,452.10	\$547.90	\$10,000		

Other Sinking Fund Calculations

Similar to working with loans, the amortization worksheet on the financial calculator can be applied to sinking funds to find a partial sinking fund schedule, to find the total interest or the total increase for a series of payments, to find the balance in the fund after any payment, or to find the book value after any payment.

EXAMPLE

A company took out a \$25,000 loan and establishes a sinking fund earning 2.7% compounded semi-annually to accumulate the loan amount. The company makes semi-annual payments into the fund for ten years. Construct a partial sinking fund schedule showing the details of payment #7, the last two payments and the totals.

Solution:

Step 1: Calculate the sinking fund deposit.

PMT Setting	END
N	$2 \times 10 = 20$
PV	0
FV	25,000
PMT	?
I/Y	2.7
P/Y	2
C/Y	2

$$PMT = \$1,097.21$$

Step 2: Enter the information into the time value of money buttons on the calculator.

PMT Setting	END
N	20
PV	0
FV	25,000
PMT	-1,097.21
I/Y	2.7
P/Y	2
C/Y	2

Step 4: Complete the partial sinking fund schedule using the amortization worksheet on the calculator.

Payment Number	Payment	Interest	Increase	Balance	Book Value
7	\$1,097.21	\$91.93	\$1,189.14	\$7,988.62	\$17,011.38
19	\$1,097.21	\$299.54	\$1,396.75	\$23,584.57	\$1,415.43
20	\$1,097.21	\$318.39	\$1,415.60	\$25,000.17	−\$0.17
Totals	\$21,944.20	\$3,055.97	\$25,000.17		

- **Row 7:** In the amortization worksheet, set P1=7 and P2=7.
- **Row 19:** In the amortization worksheet, set P1=19 and P2=19.
- **Row 20:** In the amortization worksheet, set P1=20 and P2=20.
- **Book Value for each Row:** $\text{Book Value} = 25,000 - \text{Balance}$
- **Totals Row:**
 - The increase total is the last balance entry (\$25,000.17).
 - The payments total is the sum of the payments: $20 \times 1,097.21 = 21,944.20$.
 - The interest total is the difference in the other two column totals:
 $25,000.17 - 21,944.20 = 3,055.97$.

USING THE TI BAI PLUS CALCULATOR FOR OTHER SINKING FUND CALCULATIONS

To use the amortization worksheet to find the total interest, the total increase or the balance for a sinking fund:

1. Solve for any unknown quantities about the sinking fund. You need to know all of the information about the sinking fund first before you can use the amortization worksheet.
2. Enter all the value of all seven time value of money variables into the calculator (N, PV, FV,

PMT, I/Y, P/Y, C/Y). If you calculated the payment in the first step, you must **re-enter** it rounded up to the next cent and with the correct cash flow sign. Make sure the payment setting is set to END, and obey the cash flow sign convention. Because PMT (the sinking fund payment) is paid out to the fund, PMT is negative. At the end of the term, FV is received, so FV is positive.

3. Go to the amortization worksheet by pressing 2nd AMORT (the PV button).
4. To find the balance in the fund after a payment, set P1 and P2 equal to the payment number. The BAL entry is the balance after payment number P2. Note that the BAL entry is only tied to the value of P2 and does not depend on the value of P1.
5. To view the total increase or the total interest for a series of payments, set P1 to the **first** payment number of the series of payments and set P2 to the **last** payment number of the series of payments. For example, to view the total increase or the total interest for payments four to seven, set P1=4 and P2=7. In the outputs from the amortization worksheet:
 - The PRN entry is the total increase from payment number P1 to payment number P2.
 - The INT entry is the total interest from payment number P1 to payment number P2.

NOTES

1. The PRN entry on the amortization worksheet is the sum of the increase entries in the sinking fund schedule starting at payment number P1 and ending at payment number P2. For example, if P1=4 and P2=7, the PRN entry tells you the sum of the increase column in the sinking fund schedule starting with payment number 4 and ending with payment number 7.
2. The INT entry on the amortization worksheet is the sum of the interest entries in the sinking fund schedule starting at payment number P1 and ending at payment number P2. For example, if P1=4 and P2=7, the INT entry tells you the sum of the interest column in the sinking fund schedule starting with payment number 4 and ending with payment number 7.
3. The calculator thinks in terms of payment numbers, not years. That is, P1 and P2 must be payment numbers.

EXAMPLE

A company issued \$500,000 worth of bonds with 15 years to maturity. The company set up a sinking fund earning 4.7% compounded semi-annually to accumulate the face value of the bonds and made semi-annual payments into the sinking fund.

1. What is the sinking fund payment?
2. What was the balance in the fund after payment 18?
3. By how much did the sinking fund increase with payment 9?
4. How much interest was paid in year 10?
5. What was the book value in the fund at the end of year 8?

Solution:

Step 1: Calculate the payment.

PMT Setting	END
N	$12 \times 15 = 30$
PV	0
FV	500,000
PMT	?
I/Y	4.7
P/Y	2
C/Y	2

$$PMT = \$11,663.61$$

Step 2: Calculate the balance for payment 18. To find the balance for payment 18, set P1=18 and P2=18.

PMT Setting	END
N	30
PV	0
FV	500,000
PMT	−11,663.61
I/Y	4.7
P/Y	2
C/Y	2
P1	18
P2	18

$$BAL = \$257,632.82$$

After 18 payments, the balance in the fund is \$257,632.82.

Step 3: Calculate the increase for payment 9. To find the interest for payment 9, set P1=9 and P2=9.

PMT Setting	END
N	30
PV	0
FV	500,000
PMT	−11,663.61
I/Y	4.7
P/Y	2
C/Y	2
P1	9
P2	9

$$PRN = \$14,045.45$$

The increase for payment 9 is \$14,045.45.

Step 4: Calculate the interest paid for year ten. To find the interest paid in year two, set P1 to first payment number of year ten and P2 to last payment of year ten. There are 2 payments a year, so the last payment made in year two is 20 (2×10). The first payment made in year ten is 19 ($2 \times 9 + 1$). So, to find the interest paid in year ten, set P1=19 and P2=20.

PMT Setting	END
N	30
PV	0
FV	500,000
PMT	-11,663.61
I/Y	4.7
P/Y	2
C/Y	2
P1	10
P2	20

$$INT = \$12,525.11$$

The interest paid in year ten is \$12,525.11.

Step 5: Calculate the book value after eight years. To find the book value, calculate the balance after eight years and then subtract the balance from \$500,000. To find the balance after eight years, enter the payment number that corresponds to the last payment made in year eight. There are 2 payments a year, so the last payment made in year eight is 16 (8×2). So, to find the balance after eight years, set P1=16 and P2=16.

PMT Setting	END
N	30
PV	0
FV	500,000
PMT	−11,663.61
I/Y	4.7
P/Y	2
C/Y	2
P1	16
P2	16

$$BAL = \$223,407.95$$

$$\begin{aligned}\text{Book Value} &= 500,000 - 223,407.95 \\ &= \$276,592.05\end{aligned}$$

After eight years, the book value in the fund is \$276,592.05.

NOTE

A common mistake occurs in translating years into payment numbers. You often need to find the total interest or the total increase in the sinking fund for a particular year. To do this, you need to set P1 equal to the number of the first payment that occurs in that year and P2 equal to the number of the last payment that occurs in that year.

For example, suppose you have monthly payments and you want to know the total interest in the fourth year. In error, you might calculate that the fourth year begins with payment 36 and ends with payment 48, and so enter P1=36 and P2=48. But the 36th payment is actually the last payment of the third year. The first payment to occur in year four is the 37th. So, if you wanted to find the total interest in year 4, P1=37 and P2=48.

When you need to find the first payment number and last payment number for a particular year, there are two methods you can use to calculate the correct payment numbers.

1. Calculate the payment number at the end of the year in question by multiplying P/Y by the year number: $P/Y \times \text{year number}$. To find the payment number at the start of the year, subtract the payment frequency less one ($P/Y - 1$) from the last payment number of the year. For example, suppose the you have monthly payments and want to find the interest paid for year 4. The last payment in year 4 is $12 \times 4 = 48$. The first payment in year 4 is $48 - (12 - 1) = 37$.
2. Calculate the payment number at the end of the year in question by multiplying P/Y by the year number: $P/Y \times \text{year number}$. To find the payment number for the first payment in the year, multiply P/Y by the previous year number and then add one to it: $P/Y \times (\text{year} - 1) + 1$. For example, suppose the you have monthly payments and want to find the interest paid for year 4. The first payment in year 4 is $12 \times 3 + 1 = 37$. The last payment of the fourth year remains at payment 48.

TRY IT

A company sold \$100,000 worth of bonds and set up a sinking fund earning 4% compounded semi-annually to retire the bonds in nine years. The company made semi-annual deposits into the sinking fund.

1. What is the sinking fund payment?
2. How much interest did the sinking fund earn with the 10th payment?
3. By how much did the sinking fund increase with the 7th payment?
4. What is the balance on the loan after eight years?
5. How much interest did the sinking fund earn in year three?
6. By how much did the sinking fund increase in year six?
7. What is the book value in the fund after year five?

Click to see Solution

1. Calculate the payment.

PMT Setting	END
N	$2 \times 9 = 18$
PV	0
FV	100,000
PMT	?
I/Y	4
P/Y	2
C/Y	2

$$PMT = \$4,670.22$$

2. Calculate the interest for the 10th payment.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	$-4,670.22$
I/Y	4
P/Y	2
C/Y	2
P1	10
P2	10

$$INT = \$911.13$$

3. Calculate the increase for the 7th payment.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	−4,670.22
I/Y	4
P/Y	2
C/Y	2
P1	7
P2	7

$$PRN = \$5,259.43$$

4. Calculate the balance at the end of year 8.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	−4,670.22
I/Y	4
P/Y	2
C/Y	2
P1	16
P2	16

$$BAL = \$87,049.56$$

5. Calculate the interest for year 3.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	−4,670.22
I/Y	4
P/Y	2
C/Y	2
P1	5
P2	6

$$INT = \$871.06$$

6. Calculate the increase for year 6.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	−4,670.22
I/Y	4
P/Y	2
C/Y	2
P1	11
P2	12

$$PRN = \$11,499.80$$

7. Calculate the book value after year 5.

PMT Setting	END
N	18
PV	0
FV	100,000
PMT	−4,670.22
I/Y	4
P/Y	2
C/Y	2
P1	10
P2	10

$$BAL = \$51,137.61$$

$$\begin{aligned} \text{Book Value} &= 100,000 - 51,137.61 \\ &= \$48,862.39 \end{aligned}$$

Exercises

1. A company set up a sinking fund earning 6.2% effective to accumulate \$10,000 to repay a loan. The company makes quarterly payments into the sinking fund for two years. Construct the sinking fund schedule for the loan.

Click to see Answer

Payment Number	Payment	Interest	Increase	Balance	Book Value
0				\$0	\$10,000
1	\$1,185.21	\$0	\$1,185.21	\$1,185.21	\$8,814.79
2	\$1,185.21	\$17.96	\$1,203.17	\$2,388.38	\$7,611.62
3	\$1,185.21	\$36.19	\$1,221.40	\$3,609.78	\$6,390.22
4	\$1,185.21	\$54.70	\$1,239.91	\$4,849.68	\$5,150.32
5	\$1,185.21	\$73.48	\$1,258.69	\$6,108.38	\$3,891.62
6	\$1,185.21	\$92.55	\$1,277.76	\$7,386.14	\$2,613.86
7	\$1,185.21	\$111.92	\$1,297.13	\$8,683.27	\$1,316.73
8	\$1,185.21	\$131.57	\$1,316.78	\$10,000.05	—\$0.05
Totals	\$9,481.68	\$518.37	\$10,000.05		

2. A company issued \$5,000,000 worth of bonds and set up a sinking fund earning 7% compounded semi-annually to retire the bonds in six years. The company made semi-annual payments into the fund. Construct a sinking fund schedule showing the details of third year, the last two payments and the totals.

Click to see Answer

Payment Number	Payment	Interest	Increase	Balance	Book Value
5	\$342,419.75	\$50,514.79	\$392,934.54	\$1,836,214.23	\$3,163,785.77
6	\$342,419.75	\$64,267.50	\$406,687.25	\$2,242,901.47	\$3,757,098.53
11	\$342,419.75	\$140,597.13	\$483,016.88	\$4,500,077.59	\$49,922.41
12	\$342,419.75	\$157,502.72	\$499,922.47	\$5,000,000.05	—\$0.05
Totals	\$4,109,037	\$890,963.05	\$5,000,000.05		

3. A company borrowed \$800,000 to expand their factory. They establish a sinking fund earning 5%

compounded quarterly to accumulate the loan amount in 8 years. The company made monthly payments into the sinking fund. Construct a sinking fund schedule showing the details payment #37, the last two payments and the totals.

Click to see Answer

Payment Number	Payment	Interest	Increase	Balance	Book Value
37	\$6,800.52	\$1,093.21	\$7,893.73	\$271,355.37	\$528,644.63
95	\$6,800.52	\$3,236.08	\$10,036.60	\$789,922.04	\$10,077.96
96	\$6,800.52	\$3,277.72	\$10,078.24	\$800,000.29	—\$0.29
Totals	\$652,849.92	\$890,963.05	\$5,000,000.05		

4. A \$500,000 bond with a 6.5% coupon is redeemable in ten years. The bond comes with a sinking fund that requires deposits timed to match the bond payments. The sinking fund earns 4% compounded semi-annually.
- What is the sinking fund payment?
 - What is the periodic expense of the debt?
 - What is the balance in the fund after two years?
 - How much interest did the fund earn with the 17th payment?
 - By how much did the fund increase with the 8th payment?
 - How much interest did the fund earn in year nine?
 - By how much did the fund increase in year six?
 - What is the book value at the end of year seven?

Click to see Answer

a. \$20,578.40; b. \$36,828.40; c. \$84,816.10; d. \$7,671.33; e. \$23,638.11; f. \$15,907.66; g. \$50,671.61; h. \$171,281.91

5. A \$125,000 face value bond is issued with 15 years to maturity and a coupon rate of 5%. A sinking fund is established to accumulate the face value of the bond. The sinking fund earns 7.4% compounded semi-annual and the payments are time to match the bond payments.
- What is the sinking fund payment?
 - What is the periodic expense of the debt?

- c. What is the balance in the fund half way through the term?
- d. How much interest did the fund earn with the 20th payment?
- e. By how much did the fund increase with the 10th payment?
- f. How much interest did the fund earn in year seven?
- g. By how much did the fund increase in year three?
- h. What is the book value at the end of year twelve?

Click to see Answer

a. \$2,342.79; b. \$5,467.79; c. \$45,878.92; d. \$2,329.50; e. \$3,248.94; f. \$2,694.64; g. \$5,518.73; h. \$36,885.21

6. A company borrowed \$750,000 at 3.8% compounded quarterly which required them to make quarterly interest payments. The company set up a sinking fund earning 4.9% compounded quarterly to retire the debt in eight years. The company makes quarterly payments into the sinking fund at the same time they make the quarterly interest payments on the loan.
- a. What is the sinking fund payment?
 - b. What is the periodic expense of the debt?
 - c. What is the balance in the fund at the end of year five?
 - d. How much interest did the fund earn with the 13th payment?
 - e. By how much did the fund increase with the 25th payment?
 - f. How much interest did the fund earn in year four?
 - g. By how much did the fund increase in year six?
 - h. What is the book value at the end of year two?

Click to see Answer

a. \$19,284.57; b. \$33,534.57; c. \$434,053.97; d. \$3,033.85; e. \$25,829.55; f. \$13,789.23; g. \$100,229.96; h. \$588,944.27

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7.7 REVIEW EXERCISES

1. A \$100,000 face value bond has a 5.03% coupon. The bond is purchased when there was 13.5 years to maturity and the yield to maturity was 4.29%.
 - a. What is the purchase price of the bond?
 - b. What is the premium or discount?

Click to see Answer

a. \$107,523.91; b. Premium=\$7,523.91

2. Greg purchased a \$5,000 bond with a 4.3% coupon when there was 12 years to maturity and the yield to maturity was 5.7%. Four years later, Greg sold the bond when the yield rate was 3.9%.
 - a. What did Greg pay to purchase the bond?
 - b. What is the premium or discount?
 - c. At what price did Greg sell the bond?
 - d. What was Greg's gain or loss on the sale of the bond?

Click to see Answer

a. \$4,397.56; b. Discount=\$602.44; c. \$5,136.32; d. Gain=\$738.76

3. A 25-year, \$50,000 bond with a 5.75% coupon was issued on June 1, 2008. The bond was purchased on November 21, 2019 when the yield to maturity was 3.85%.
 - a. Calculate the flat price on the purchase date.
 - b. Calculate the quoted price as a percentage of face value.

Click to see Answer

a. \$61,302.61; b. 119.89%

4. A \$35,000 bond with a 7% coupon is purchased for 132.92 when there was 12 years to maturity. Calculate the yield rate.

Click to see Answer

3.6%

5. An investor purchases a \$55,000 bond with a 4% coupon for \$33,227.95 when there was 10.5 years to maturity. After 4 years, the investor sold the bond for \$60,231.63.
- Calculate the yield to maturity on the purchase date.
 - Calculate the yield to maturity on the selling date.
 - Calculate the investor's rate of return when the bond was sold.

Click to see Answer

a. 10.24% b. 2.41%; c. 20.66%

6. A \$17,000 bond with a 3.8% coupon has a maturity date of October 1, 2035. The bond is purchased on May 30, 2023 when the yield to maturity was 4.19%.
- Calculate the flat price on the purchase date.
 - Calculate the quoted price as a percentage of face value.

Click to see Answer

a. \$16,469.68; b. 96.27%

7. A \$6,500 bond with a 6.7% coupon is purchased three years before maturity when the yield rate was 8%. Construct the appropriate bond schedule for the bond.

Click to see Answer

Payment Number	Bond Payment	Interest on Book Value at Yield	Accumulated Discount	Book Value	Remaining Discount to be Accumulated
0				\$6,278.52	\$221.48
1	\$217.75	\$251.14	\$33.39	\$6,311.91	\$188.09
2	\$217.75	\$252.48	\$34.73	\$6,346.64	\$153.36
3	\$217.75	\$253.87	\$36.12	\$6,382.75	\$117.25
4	\$217.75	\$255.31	\$37.56	\$6,420.31	\$79.69
5	\$217.75	\$256.81	\$39.06	\$6,459.38	\$40.62
6	\$217.75	\$258.38	\$40.63	\$6,500	\$0
Totals	\$1,306.50	\$1,527.98	\$221.48		

8. A \$4,000 bond with a 5.2% coupon is purchase two years before maturity when the yield rate was 3.7%. Construct the appropriate bond schedule for the bond.

Click to see Answer

Payment Number	Bond Payment	Interest on Book Value at Yield	Amortized Premium	Book Value	Remaining Premium to be Amortized
0				\$4,114.65	\$114.65
1	\$104	\$76.12	\$27.88	\$4,086.77	\$86.77
2	\$104	\$75.61	\$28.39	\$4,058.38	\$58.38
3	\$104	\$75.08	\$28.91	\$4,029.46	\$29.46
4	\$104	\$74.54	\$29.46	\$4,000	\$0
Totals	\$416	\$301.35	\$114.65		

9. A \$62,000 face value bond carrying an 8.88% coupon is purchased on July 15, 2023. The bond matures on November 1, 2037. At the time of purchase, the market rate on the bond was 4.44%.
- Calculate the flat price on the purchase date.
 - Calculate the quoted price.

Click to see Answer

a. \$92,021.62; b. \$90,899.55

10. A company with a \$150,000 debt establishes a sinking fund earning 3.89% compounded quarterly to retire the debt in full in ten years. The company makes quarterly payments into the sinking fund. Construct a partial sinking fund schedule showing the details of the payments in year three, the last two payments and the totals.

Click to see Answer

Payment Number	Payment	Interest	Increase	Balance	Book Value
9	\$3,085.78	\$248.41	\$3,334.19	\$28,877.23	\$121,122.77
10	\$3,085.78	\$280.83	\$3,366.61	\$32,243.84	\$117,756.16
11	\$3,085.78	\$313.57	\$3,399.35	\$35,643.19	\$114,356.81
12	\$3,085.78	\$346.63	\$3,432.41	\$39,075.60	\$110,924.40
39	\$3,085.78	\$1,371.64	\$4,457.42	\$145,499.62	\$4,500.38
40	\$3,085.78	\$1,414.98	\$4,500.76	\$150,000.38	—\$0.38
Totals	\$123,431.20	\$26,569.18	\$150,000.38		

11. A \$300,000 face value bond with a 4% coupon is issued with 15 years to maturity. A sinking fund earning 6.35% compounded semi-annually is set-up to accumulate the face value of the bonds.
- Calculate sinking fund payment.
 - What is the periodic expense of the debt?
 - How much interest does the fund earn with the 17th payment?
 - What is the book value of the fund after the 9th payment?
 - What is the balance in the fund after three years?
 - By how much does the fund increase with the 23rd payment?
 - On which payment does fund reach the half way point to the \$300,000?
 - By how much does the fund increase in the 10th year?
 - How much interest does the fund earn in the 12th year?
 - What is the book value of the fund after the 6th year?

Click to see Answer

a. \$6,129.04; b. \$12,129.04; c. \$3,977.07; d. \$237,288.64; e. \$39,819.74; f. \$12,190.77; g. 19; h. \$21,857.64; i. \$12,510.51; j. \$212,146.63

12. A company borrowed \$400,000 and set up a sinking fund earning 3.7% compounded semi-annually to retire the debt in seven years. The company made monthly deposits into the fund and rounded the payment up to the next dollar.
- Calculate sinking fund payment.
 - What is the balance in the fund at the half way point?

- c. How much interest does the fund earn in year five?
- d. Calculate the amount by which the sinking fund increased in year two.
- e. What is the book value in the fund at the end of year three?
- f. What is the final balance in the fund?
- g. What is the total increase in the fund?
- h. What is the total amount of interest accumulated by the fund?

Click to see Answer

a. \$4,184; b. \$187,214.25; c. \$8,918.61; d. \$52,968.39; e. \$241,023.63; f. \$400,060.56; g. \$400,060.56; i. \$48,604.56;

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7.8 CHAPTER SUMMARY

Key Concepts

- **7.1 Bond Terminology**
 - Bond definition, characteristics, and key terminology.
 - **7.2 Calculating the Purchase Price of a Bond on an Interest Payment Date**
 - Calculating a marketable bond price when the selling date is the interest payment date.
 - Calculating premiums and discounts.
 - **7.3 Calculating the Yield Rate**
 - Calculating the yield on the bond to maturity.
 - Calculating the yield realized by an investor if a bond is purchased and sold before maturity.
 - **7.4 Bond Schedules**
 - How to amortize a bond premium or accrue a bond discount.
 - **7.5 Quotation of Bonds**
 - Calculating a marketable bond price when the selling date is a noninterest payment date.
 - Calculating the quoted price of a bond.
 - **7.6 Sinking Funds**
 - Sinking funds and their purposes.
 - How to construct complete ordinary sinking fund schedules.
 - How to construct partial ordinary sinking fund schedules.
 - Other sinking fund calculations.
-

Glossary of Terms

- **Accrued Interest.** The amount of interest that the bond has earned but has not yet paid out since the previous interest payment date.
- **Bond.** A debt that is secured by a specific corporate asset and that establishes the issuer's responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date.
- **Book Value of the Debt.** The difference between the principal amount owing on the bond and the accumulated balance in the sinking fund at any point in time.
- **Capital Gain.** The amount by which the current value of an asset exceeds the original purchase price.

- **Capital Loss.** The amount by which the current value of an asset falls short of the original purchase price.
 - **Coupon Rate.** Also known as the bond rate or nominal rate, this is the nominal interest rate paid on the face value of the bond.
 - **Debenture.** A debt that is not secured by a specific corporate asset and that establishes the issuer's responsibility toward a creditor for paying interest at regular intervals and repaying the principal at a fixed later date.
 - **Discount.** The amount by which a bond's selling price falls short of its face value.
 - **Face Value.** Also called the par value or denomination of the bond, this is the principal amount of the debt that the investor lent to the bond-issuing corporation.
 - **Issue Date.** The date that a bond is issued and available for purchase by creditors.
 - **Market Quotation.** The purchase price expressed as a percentage of face value.
 - **Market Rate.** This is the prevailing nominal rate of interest in the open bond market.
 - **Maturity Date.** Also known as the redemption date or due date, this is the day upon which the redemption price will be paid to the bondholder (along with the final interest payment), thereby extinguishing the debt.
 - **Periodic Cost of the Debt.** The periodic total of the interest payments and the sinking fund payments.
 - **Premium.** The amount by which a bond's selling price exceeds its face value.
 - **Purchase Price.** Also known as the cash price or flat price, this is the amount of money an investor must directly pay out to acquire the bond. It represents the total of the market price and the accrued interest.
 - **Quoted Price.** This is the actual value of the bond excluding any accrued interest.
 - **Redemption Price.** Also called the redemption value or maturity value, this is the amount the bond issuer will pay to the bondholder upon maturity of the bond.
 - **Selling Date.** This is the date that a bond is actively traded and sold to another investor through the bond markets.
 - **Sinking Fund.** A special account into which an investor, whether an individual or a business, makes annuity payments such that sufficient funds will be on hand by a specified date to meet a future savings goal or debt obligation.
 - **Sinking Fund Schedule.** A table that shows the sinking fund contribution, interest earned, and the accumulated balance for every payment in the annuity.
 - **Yield to Maturity.** A bond's overall rate of return when purchased at a market price and held until maturity. It includes both the semi-annual interest that the bondholders earn on their investment along with the gain or loss resulting from the difference between the market price and the redemption price.
-

Formulas

• Symbols Used

- BAL = Principal balance
- BV = Book value
- FV = Face value of bond
- PMT = Annuity payment amount
- N = Number of annuity payments
- I/Y = Nominal interest rate
- P/Y = Number of payments per year or payment frequency
- C/Y = Number of compounds per year or compounding frequency

• Formulas Used

- Bond Payment: $\text{Face Value} \times \text{periodic bond rate}$
- Purchase Price on an Interest Payment Date:

$$\text{Present Value of Face Value} + \text{Present Value of Bond Payments}$$
- Premium: $\text{Purchase Price} - \text{Face Value}$
- Discount: $\text{Face Value} - \text{Purchase Price}$
- Quoted Price: $\text{Flat Price} - \text{Accrued Interest}$
- Quoted Price as a Percentage of Face Value: $\frac{\text{Quoted Price}}{\text{Face Value}} \times 100\%$
- Periodic Cost of Debt: $\text{Sinking Fund Payment} + \text{Periodic Interest Payment}$
- Book Value: $\text{Loan Amount} - \text{Balance}$

Calculator

• Date Function

- 2nd DATE to access.
- Enter two of the three variables (DT1, DT2, DBD) by pressing ENTER after each input and using the up arrow and down arrow to scroll through the display. The variables are:
 - DT1 = The starting date of the transaction
 - DT2 = The ending date of the transaction

- DBD = The days between the dates, counting the first day but not the last, which is the time period of the transaction.
- $ACT / 360 = A$ setting for determining how the calculator determines the DBD. In Canada, you should maintain this setting on ACT, which is the actual number of days. In other countries, such as the United States, they treat each year as having 360 days (the 360 setting) and each month as having 30 days. If you need to toggle this setting, press 2nd SET.
- Enter all dates in the format of MM.DDYY, where MM is the numerical month, DD is the day, and YY is the last two digits of the year. DD and YY must always be entered with both digits.
- Press CPT on the unknown (when it is on the screen display) to compute the answer.

• Amortization Worksheet for Bond Schedules

- To use the amortization worksheet for a bond schedule:
 - Enter all seven of the time value of money variables accurately (N, I/Y, PV, PMT, FV, P/Y, and C/Y). Because money is received by the bond holder, PMT and FV are positive. Because the purchase price is paid out by the bond holder, PV is negative.
 - Press 2nd AMORT.
 - Enter a value for P1, press ENTER and then the down arrow.
 - Enter a value for P2, press ENTER and then the down arrow. Note that the higher the numbers entered in P1 or P2, the longer it will take the calculator to compute the outputs. It is possible that the calculator will go blank and take a few moments before displaying the outputs.
 - Press the down arrow to scroll through BAL, PRN, and INT to read the output. The BAL number is the book value entry of the bond schedule. The PRN number is the amortized premium or accumulated discount entry of the bond schedule. The INT number is the interest on the book value at yield entry of the bond schedule.

• Amortization Worksheet for Sinking Fund Schedules

- To use the amortization worksheet for a sinking fund schedule:
 - Enter all seven of the time value of money variables accurately (N, I/Y, PV, PMT, FV, P/Y, and C/Y). If PMT was computed, you must re-enter it rounded up to the next cent while retaining the correct cash flow sign convention. Because money is invested into the account, PMT is negative. Because the money is withdrawn from the fund, FV is positive.
 - Press 2nd AMORT.
 - Enter a value for P1, press ENTER and then the down arrow.
 - Enter a value for P2, press ENTER and then the down arrow. Note that the higher the

numbers entered in P1 or P2, the longer it will take the calculator to compute the outputs. It is possible that the calculator will go blank and take a few moments before displaying the outputs.

- Press the down arrow to scroll through BAL, PRN, and INT to read the output. The BAL number is the balance entry of the sinking fund schedule. The PRN number is the increase entry of the sinking fund schedule. The INT number is the interest entry of the sinking fund schedule.

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PART VIII

BUSINESS INVESTMENT DECISIONS

You must make financial decisions throughout your professional career and everyday personal life. Some of these decisions are easy. To make the best choice you need a little intuition and simple calculation. Other decisions are very challenging, confronting you with a great array of competing options, each of which is bundled with numbers projected from now until ... whenever. If you tried just to wing it in these latter scenarios, you could make a catastrophic mistake.

For example, suppose you are the production manager for a company that soon needs to replace a critical machine that is nearing the end of its useful life. At your invitation, salespeople from three competing companies have paid you a visit this week. Each of them showed you an impressive replacement machine. The three machines appear to be equal in design and performance, but each carries a different price tag. Each machine also differs widely in operating costs such as power consumption, consumables, and labour costs. The maintenance requirements also follow different timetables and have different costs. In all cases, you can either purchase the machines or lease them through the supplier's leasing plan.

These machines are not cheap. Because the one you select represents such a significant investment on your company's part, good financing for it is an integral part of the decision. Will the money to pay for it come from a bank loan, a bond issuance, or by issuing some common shares? Perhaps the finance department can withdraw some money from your organization's current investments. Each of these funding sources is associated with a different interest rate.

Your company relies on you to choose the best machine at the lowest possible cost. From a strictly financial perspective, and assuming that all machines are equally productive, which of the three machines is your best choice?

This business investment decision should not be guesswork. In previous chapters you have already learned many of the fundamental financial skills required to make effective monetary decisions. In this chapter you will amalgamate and apply your existing skills in dealing with compound interest, ordinary annuities, annuities due, leases, loans, and much more. Using a concept known as cash flows, there are two different mechanisms you will learn about to help make these types of business investment decisions: net present value and internal rate of return. Each of these techniques allow you to analyze a decision-making scenario, determine if a particular project is profitable, and select the best option from several choices.

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8.1 NET PRESENT VALUE

LEARNING OBJECTIVES

- Calculate the net present value of an investment.
- Evaluate and compare investment decisions using the net present value.

If you decide to pursue a course of action, what is the net benefit or cost of making that decision? How can you tell systematically which choice is financially superior?

Suppose that your marketing department thinks it has stumbled onto the next hot product to hit the market. The estimated research and development costs are \$10 million today with an expected \$5 million revitalization cost in five years. The product hits shelves in two years. Estimated year-end profits are \$4 million in the first three years and \$3 million for the subsequent two years. At that time, the rapid pace of technology will end the life cycle of the new product. This last year will earn an estimated \$1 million in profits. The funds needed for this project will be obtained at 12% compounded annually. Does this project make financial sense? Your co-worker argues that the decision is clear: You would spend \$15 million to earn \$19 million, so the project should go forward. What do you think?

Your co-worker's analysis summed the nominal numbers. This clearly violates the fundamental concept of time value of money—for example, you know that the projected profit of \$1 million in the last year of the product is not worth \$1 million today. To get an accurate picture of the potential benefit of this product, we need a technique that takes into account the time value of money.

What Is Involved in Making Monetary Business Decisions?

Investment decisions must consider both the type of decision being made along with the source of funding, which influences the interest rate used in financial calculations.

What type of decision is begin made?

Although there is no such thing as a “standard” or “normal” decision, many business decisions seem to fall into the same three categories.

- **Making One Choice from Multiple Options.** When you face a variety of options that will solve your problem, which option is financially the best?
- **Deciding Whether to Pursue One Course of Action.** When you face only one particular course of action, do you do it or not?
- **Making Multiple Choices under Constraints.** If you have limited resources and many different opportunities from which you can choose more than one, how do you select which opportunities to pursue and which to let pass by?

What monetary source is being used and what does it cost?

The structure of many decisions follows the old adage that you have to spend money to make money. Many courses of action require an investment up front (costs) to receive the reward (profits) in the future. This means you need to tap a source of money before you can go forward with any course of action. Individuals and corporations can access many of the same monetary sources, but businesses have additional options that are not available to individuals.

- **Personal Funding Sources.** You have limited options for obtaining money. Typically, you can either use debt financing such as a loan from a bank or a line of credit, or you can withdraw money from an investment fund or savings account.
 - If you borrow, the financial cost of the project is the interest rate that is being charged on the borrowed funds. For example, if you fund the project using a loan at 8.5% compounded annually, the loan’s interest rate becomes the nominal rate you should use in the required time value of money calculations.
 - If you withdraw money from an investment, the cost of this project is the interest rate you forego on those funds. For example, if you take the money out of a GIC earning 4% compounded annually, you should use the GIC’s interest rate in the decision calculations.
- **Business Funding Sources.** Businesses have substantially more choices than individuals when obtaining financing. In addition to debt financing and current investments, businesses also can issue bonds or use equity financing, such as offering company shares. Bartering with business partners is also possible. The same principle applies, though: whatever interest rate the funding source involves carries forward into any decision-making calculations.

Rate of return and cost of capital

The **rate of return** is the required rate of return that a business expects to receive from investing in the project. That is, the rate of return is what the business would earn by investing the same amount into a similar investment with the same risk. The rate of return is also known as the **cost of capital**, which is the cost of investing. Unless otherwise specified, the cost of capital is an interest rate that compounds annually.

As mentioned above, if the business does not have the funds on hand to invest in the project, some alternative funding source is required to carry out the project. In these cases the source of funding you use determines the cost of capital or interest rate in these calculations. If you use multiple sources to gather the needed funds, your interest rate is a weighted average of all the debt and equity financing rates. If you use a single source then the weighted average is simply the one source's rate. This weighted interest rate is the cost of capital.

Cash Flows

A **cash flow** is a movement of money into or out of a particular project. Therefore, any time you incur a cost or receive a benefit on a particular project, you have a cash flow.

- When money is received, such as revenue or profit, you have a cash in-flow. Cash in-flows are recorded as positive.
- When money is paid out, such as an expense or a cost, you have a cash out-flow. Cash out-flows are recorded as negative.

As you work with these continuous cash flows, where to put an amount on the timeline depends on whether it is a cost or a benefit.

- **Costs.** Unless otherwise noted, capital costs are spent at the beginning of any time segment. Thus, if there are capital costs in year two, you can assume you incur them at the beginning of year two, which is the same as the end of year one.
- **Benefits.** Revenues, net profits (product sales minus product costs), or other benefits are realized at the end of any time segment. Thus, if there are net profits in year two, you can assume you realize them at the end of year two.

Net Present Value

The **net present value** (*NPV*) method calculates the difference between all benefits and costs for any given project in **today's dollars**. This decision-making technique calculates the value of all future cash in-flows and

cash out-flows brought back to today (which serves as the focal date), using the rate of return or cost of capital as the interest rate. The word *net* means that the technique considers the invested money, all costs or expenses, along with all rewards such as profits in its calculations.

$$NPV = \text{Sum of the } PV \text{ of Future Cash Flow} - \text{Initial Investment}$$

To calculate the sum of the present value of the future cash flows:

1. Using today, or year 0, as the focal date, find the present value of all **future** cash flows, inflows and outflows, where the interest rate is the cost of capital.
2. Add up the present values of the cash inflows and subtract the present values of the cash outflows.

The initial investment is money spent today, and so is already at the focal date. Because most projects require advance funding, this amount represents a cash outflow today, against which you must net the present value of all future cash flows.

An alternative way to think of the net present value is that NPV is the sum of the present value of the cash inflows minus the sum of the present value of the cash outflows. Because the initial investment is a cash outflow, the initial investment is included in the second sum. As above, today is used as the focal date and the present value of each cash flow is found at the focal date.

$$NPV = \text{Sum of the } PV \text{ of Cash Inflows} - \text{Sum of the } PV \text{ of Cash Outflows}$$

The net present value nets out all costs and benefits. Here is how to interpret the NPV :

- $NPV = 0$. The project repays the amounts of its investments plus the required cost of capital. In other words, the project breaks even. The business should accept the project.
- $NPV > 0$. The project provides returns that exceed its investments and the cost of capital. The project makes financial sense because it will add value to the company. The business should accept the project.
- $NPV < 0$. The project does not provide enough return to pay for its investments and the cost of capital. The project does not make financial sense because it will lose money for the business. The business should reject the project.

Of course, businesses often have many nonfinancial reasons for pursuing or not pursuing certain courses of action. These reasons are beyond the scope of this textbook.

Observe how NPV relates to each of the three common types of business decision categories.

- **Making One Choice from Multiple Options.** You should choose the project with the highest NPV.
- **Choosing Whether to Pursue One Course of Action.** You should pursue the project if the NPV is greater than or equal to \$0.
- **Making Multiple Choices under Constraints.** You should choose the projects that maximize the total *NPV*.

EXAMPLE

A local business is considering purchasing a new machine. The machine costs \$80,000. The machine will reduce costs by \$25,000 in year 1, \$17,000 in year 2 and \$10,000 in year 3. At the end of year 3, the business can sell the machine for \$20,000. The business's cost of capital is 8%.

1. Calculate the net present value for the machine.
2. Should the business purchase the machine?

Solution:

Step 1: Record the cash flows in a table. Calculate the net cash flow for each year.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-80,000	-80,000
1	25,000	0	25,000
2	17,000	0	17,000
3	10,000 20,000	0	30,000

Step 2: Calculate the *PV* for each future net cash flow.

	Year 1	Year 2	Year 3
N	1	2	3
PV	23,148.15	14,574.76	23,814.97
FV	-25,000	-17,000	-30,000
PMT	0	0	0
I/Y	8	8	8
P/Y	1	1	1
C/Y	1	1	1

Step 3: Calculate the NPV .

$$\begin{aligned}
 NPV &= \text{Sum of the } PV \text{ of future cash flows} - \text{Initial Investment} \\
 &= 23,148.15 + 14,574.76 + 23,814.97 - 80,000 \\
 &= -\$18,462.12
 \end{aligned}$$

The NPV for the machine is $-\$18,462.12$.

Because the $NPV < 0$, the business should not purchase the machine.

NOTES

1. Create a table to record the cash flows for each time period. The table should record the cash inflows, the cash outflows, and the net cash flow for each time period. Alternatively, use a timeline to illustrate the cash flows for each time period.
2. Because cash flows can be inflows or outflows, write all numbers with their cash flow sign conventions in the table to avoid any confusion. That is, cash inflows are recorded as positive and cash outflows are recorded as negative. This helps you keep track of the different flows and minimizes the possibility of incorrectly netting out all present values.

Although the net present value can be found by finding the present value of all the future cash flows, as illustrated in the above example, this can be a cumbersome calculation when there are numerous cash flows over an extended period of time, even with the aid of a financial calculator. Most financial calculators include a cash flow worksheet and a built-in function to calculate the net present value.

USING THE TI BAI PLUS CASH FLOW WORKSHEET TO CALCULATE NET PRESENT VALUE

The photo illustrates the buttons used in the cash flow function of your BAI Plus calculator. Access the cash flow function by pressing the CF button on the keypad. Immediately clear the memory using 2nd CLR WORK to delete any previously entered data. Use the up arrow and down arrow to scroll through the various lines. You must strictly obey the cash flow sign convention and press ENTER after keying in any data.

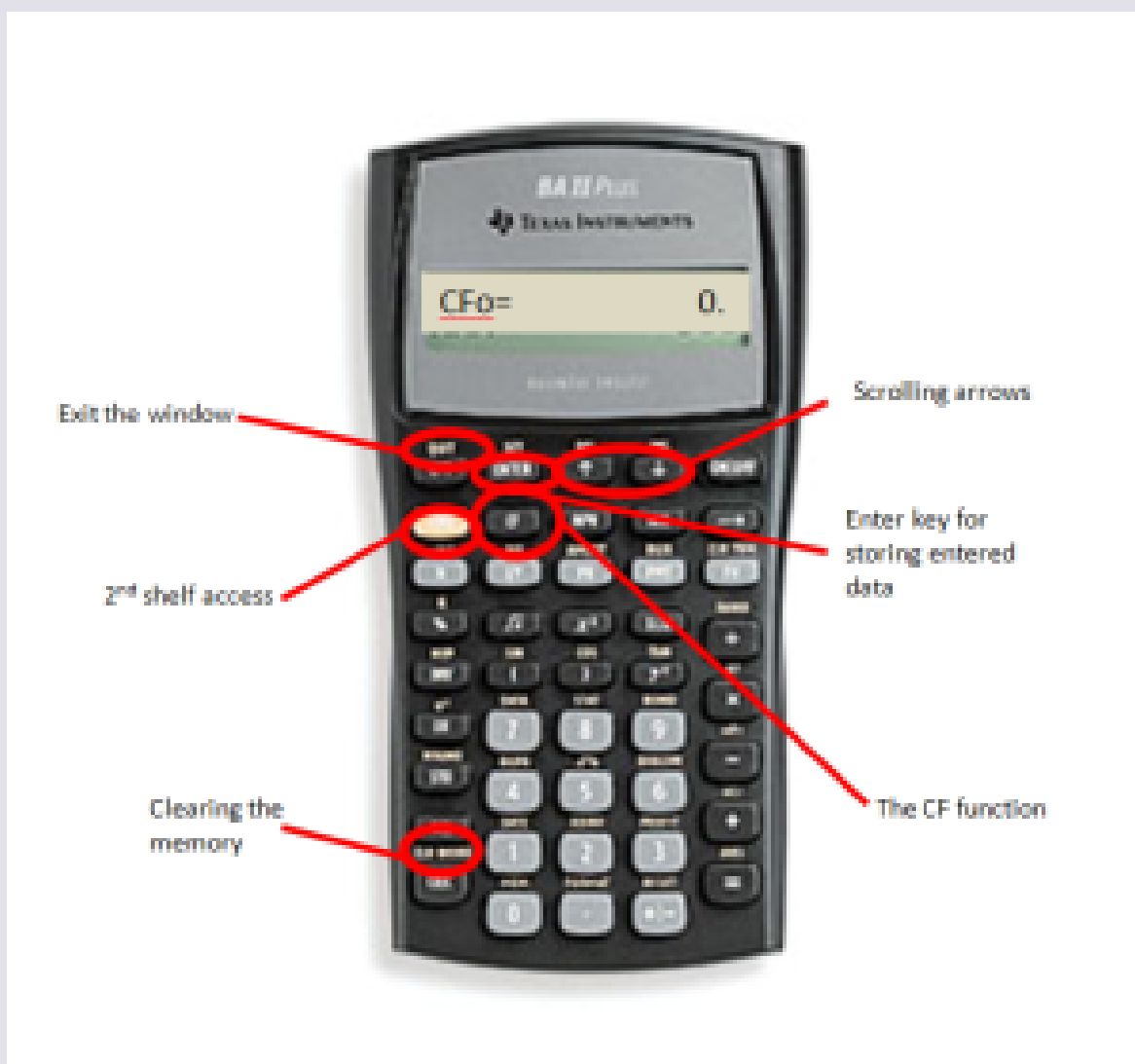


Figure 8.1.1

Here are the various lines appearing in the cash flow function:

- CF_0 = any cash flow today (i.e. the initial investment).
- CXX = a particular cash flow, where XX is the cash flow number starting with 01. You must key in the cash flows in order from the first time segment to the last. You cannot skip a time segment even if it has a value of zero. Each time segment is a placeholder on the timeline.
- FXX = the frequency of a particular cash flow, where XX is the cash flow number. It is how many times **in a row** the corresponding cash flow amount occurs. This allows you to enter recurring amounts instead of keying them in separately. By default, the calculator sets this variable to 1.

Once you have entered all cash flows, you must access other functions to generate the output. For net present value, press NPV on the keypad as illustrated in the photo.

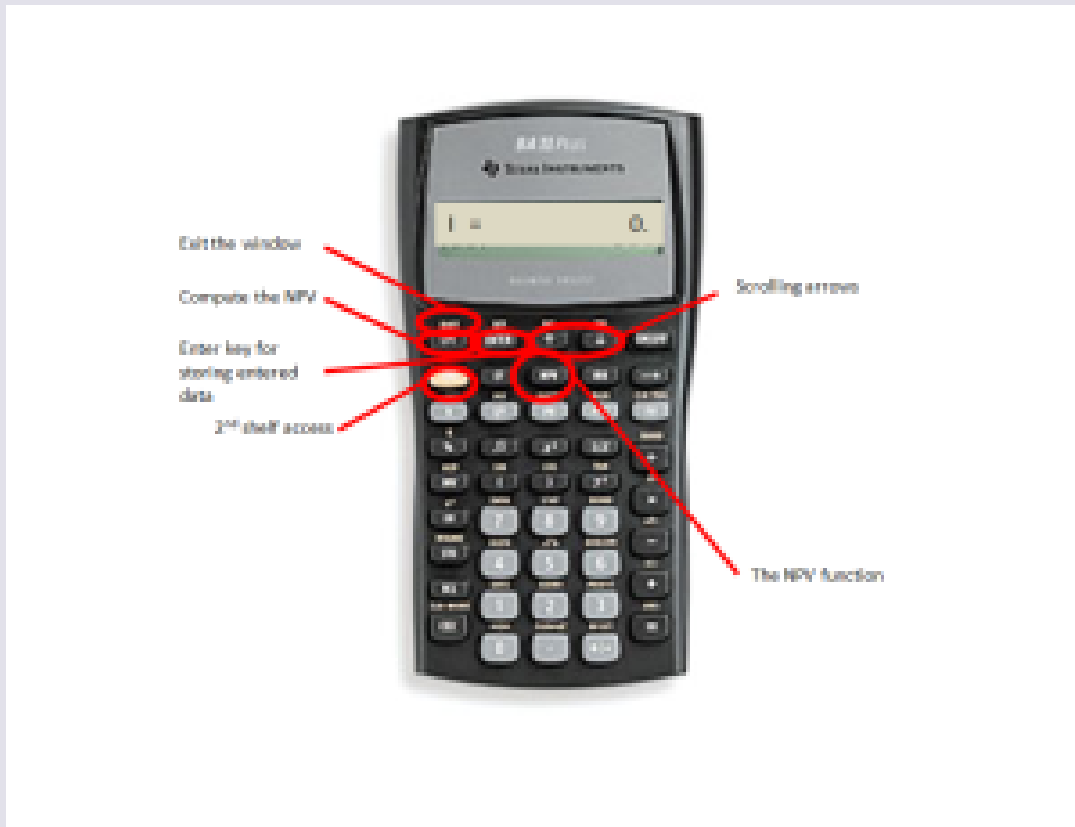


Figure 8.1.2

In the NPV menu:

- I = the matching periodic interest rate for the interval of each time segment. If the timeline is drawn with yearly intervals, then this must be the annual cost of capital. If the timeline is drawn with semi-annual intervals, this needs to be the matching semi-annual periodic cost of capital.
- NPV = net present value. Press CPT to calculate this amount. The output nets all future cash flows against any initial investment.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=488#oembed-1>

[businessfinancialmath/?p=488#oembed-1](https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=488#oembed-1)

[Cash Flows 1: Net Present Value \(NPV\) and IRR Calculations](#) by Joshua Emmanuel [2:52] (transcript available).

NOTES

1. For each period, only one cash flow can be entered in the financial calculator. If more than one cash flow exists for a certain period, enter the NET cash flow for that period. The net cash flow for a period is the sum of the cash inflows minus the sum of the cash outflows.
2. When using the cash flow worksheet on a financial calculator, you must enter a cash flow for EVERY period. If there is no cash flow for a certain period, a 0 is recorded as the net cash flow for that period. Enter 0 for CO on the calculator for that period and 1 for F0.

EXAMPLE

A local business is considering purchasing a new machine. The machine costs \$80,000. The machine will reduce costs by \$25,000 in year 1, \$17,000 in year 2 and \$10,000 in year 3. At the end of year 3, the business can sell the machine for \$20,000. The business's cost of capital is 8%.

Find the net present value.

Solution:

Step 1: Record the cash flows in a table. Calculate the net cash flow for each year.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-80,000	-80,000
1	25,000	0	25,000
2	17,000	0	17,000
3	10,000 20,000	0	30,000

Step 2: Enter the net cash flow data into the cash flow worksheet on the financial calculator.

1. Press CF.
2. Press 2nd CLR WORK. (This step clears any previously recorded data from the CF worksheet.)
3. At the CFo screen, enter -80,000 and press ENTER. (This step enters the -80,000 net cash flow for year 0.)
4. Press the down arrow.
5. At the C01 screen, enter 25,000 and press ENTER. (This step enters the 25,000 net cash flow for year 1.)
6. Press the down arrow.
7. At the F01 screen, enter 1 and press ENTER. (This records the frequency of the 25,000 net cash flow from year 1.)
8. Press the down arrow.
9. At the C02 screen, enter 17,000 and press ENTER. (This step enters the 17,000 net cash flow for year 2.)
10. Press the down arrow.
11. At the F02 screen, enter 1 and press ENTER. (This records the frequency of the 17,000 net cash flow from year 2.)
12. Press the down arrow.
13. At the C03 screen, enter 30,000 and press ENTER. (This step enters the 30,000 net cash flow for year 3.)
14. Press the down arrow.
15. At the F03 screen, enter 1 and press ENTER. (This records the frequency of the 30,000 net cash flow from year 3.)
16. Press 2nd QUIT. (This step exits the CF worksheet.)

Step 3: Calculate the NPV on the financial calculator.

I	8
NPV	-\$18,462.12

1. Press NPV.
2. At the I screen, enter 8 and press ENTER. (This step enters the 8% cost of capital.)
3. Press the down arrow.
4. At the NPV screen, press CPT.

TRY IT

A company wants to develop an app for its business over the course of five years. The development requires an investment of \$900,000 today and another investment of \$200,000 one year from now. Starting in year three, the company forecasts that the app will generate profits of \$150,000 every year for three years. At the end of the fifth year, the company plans to sell the app to a competitor for \$1,250,000. The company's cost of capital is 7%.

1. Calculate the NPV for this project.
2. Is this a sound investment decision?

Click to see Solution

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-900,000	-900,000
1	0	-200,000	-200,000
2	0	0	0
3	150,000	0	150,000
4	150,000	0	150,000
5	150,000 1,250,000	0	1,400,000

I	7
NPV	\$148,143.73

Because the $NPV = \$148,143.73 > 0$, this is a sound investment decision.

EXAMPLE

Lethbridge Community College is considering purchasing new industrial photocopier equipment. The college has narrowed down the choices to two comparable machines from Xerox and Canon.

- The Xerox machine can be acquired for \$81,900 today and is expected to have a life span of four years. The machine is expected to reduce labour and materials costs by \$31,000 each year. A \$10,000 maintenance package is expected to be purchased at the end of year two. After the four years, the machine can be sold for \$16,000.
- The Canon machine retails for \$74,800 and is expected to have a life span of four years. The machine is forecasted to save \$26,000 in labour and materials each year. A \$4,000 maintenance procedure is needed at the end of both years two and three. The salvage value of the machine is estimated at \$11,750.

The cost of capital is 11%. Which machine should be purchased, and how much money will it save over the alternative?

Solution:

Step 1: Calculate the *NPV* for the Xerox machine.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-81,900	-81,900
1	31,000	0	31,000
2	31,000	-10,000	21,000
3	31,000	0	31,000
4	31,000 16,000	0	47,000

I	11
NPV	\$16,699.29

Step 2: Calculate the *NPV* for the Canon machine.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-74,800	-74,800
1	26,000	0	26,000
2	26,000	-4,000	22,000
3	26,000	-4,000	22,000
4	26,000 11,750	0	37,750

I	11
NPV	\$7,432.42

The college should pick the Xerox machine because it has the higher *NPV*. By choosing the Xerox machine, the college will save $\$16,699.29 - \$7,432.42 = \$9,266.87$

TRY IT

The operations team at an engineering firm is considering investing in two different projects, but they only have the funds to invest in one project.

- Project 1 requires an initial investment of \$650,000 and another investment of \$250,000 at the end of year 2. The project will generate annual profits of \$200,000 starting in year 2 until the end of the project in year 5. At the end of the project, the firm can sell the project for \$500,000.
- Project 2 requires an initial investment of \$800,000 and another investment of \$200,000 at the end of year 3. The project will generate annual profits of \$150,000 starting in year 1 until the end of the project in year 5. At the end of the project, the firm can sell the project for \$600,000.

The firm's cost of capital is 9%. Which project should the firm pick?

Click to see Solution

Project 1:

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-650,000	-650,000
1	0	0	0
2	200,000	-250,000	-50,000
3	200,000	0	200,000
4	200,000	0	200,000
5	200,000 500,000	0	700,000

I	9
NPV	\$58,989.71

Project 2:

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-800,000	-800,000
1	150,000	0	150,000
2	150,000	0	150,000
3	150,000	-200,000	-50,000
4	150,000	0	150,000
5	150,000 600,000	0	750,000

I	9
NPV	\$18,969.83

The firm should select Project 1 because it has the higher NPV.

Exercises

1. A new diamond deposit has been found in northern Alberta. Your researchers have determined that it will cost \$2.5 million to purchase the land and prepare it for mining. At the end of both the first and second years, another \$1 million investment will be required to establish the mining operations. Starting at the end of the second year, the deposit is expected to earn net profits of \$3 million, which will be sustained for three years before the deposit is depleted. If the cost of capital is 16%, should your company pursue this venture? Provide calculations to support your decision.

Click to see Answer

NPV=\$1,703,103.15; pursue project

2. A company can pursue only one of two available projects, both of which require a \$3 million investment today. In project A, another investment of \$2 million is required at the end of the first year, and three years of \$2.5 million in year-end profits will start in the second year. In project B, another investment of \$2 million is required at the end of the third year, and four years of \$1.7 million in year-end profits will

start in the first year. If the cost of capital is 12%, which project should be chosen? How much better is your chosen alternative?

Click to see Answer

NPV of Project A=\$575,516.22; NPV of Project B=\$739,933.36; Project B

3. A new marketing project requires initial research and development costs of \$550,000 today with further investments of \$100,000 and \$75,000 in the third and fifth year, respectively. Initially the project will lose \$50,000 in its first year and then earn profits of \$200,000 for the next four years followed by profits of \$140,000 in the last two years. At the end of the project, its capital goods can be sold for an estimated \$50,000. If the cost of capital is 13%, should the marketing project be pursued? Show calculations to support your recommendation.

Click to see Answer

NPV=-\$29,798.58; do not pursue project

4. One of your company's clients has proposed a contract offering an estimated \$150,000 in net profits for the next six years; however, your company would be required to invest \$585,000 today to acquire the needed resources for the project. Determine whether the project should be accepted if the cost of capital is
 - a. 10%
 - b. 13%
 - c. 16%

Click to see Answer

- a. NPV=\$68,289.10; accept project b. NPV=\$14,632.47; accept project c. NPV=-\$32,289.61; do not accept project.
5. To replace an aging production machine, Johnston Distributors is considering the purchase of a new robotic machine for \$950,000. The machine is expected to have a service life of eight years and an estimated residual value of \$75,000. In each of the first four years, the machine is expected to produce production efficiencies and labour savings of \$250,000. In the last four years, the annual savings are expected to rise to \$300,000. Hydro costs for the machine are estimated at \$25,000 starting at the end of the first year and increasing by 5% each year thereafter. Maintenance costs are estimated at the end of the second, fourth, and sixth years in the amounts of \$40,000, \$60,000, and \$80,000, respectively. If the cost of capital is 16%, determine if the new robotic machine should be purchased. Provide calculations to

support your answer.

Click to see Answer

NPV=\$15,496.12; purchase machine

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8.2 INTERNAL RATE OF RETURN

LEARNING OBJECTIVES

- Calculate the internal rate of return of an investment.
- Evaluate and compare investment decisions using the internal rate of return.

The net present value presented in the previous section provides an exact monetary magnitude of a project. However, the common mindset in business focuses on profitability as a percentage and not as a dollar amount. Thus, decisions are based on the **internal rate of return** for a project, or *IRR* for short. The *IRR* is the annual percentage rate of return on the investment so that the present value of all cash inflows equals the present value of all cash outflows. That is, the *IRR* is the rate of return required to make $NPV = 0$.

To interpret the *IRR* and how the *IRR* can be used to make a decision about a project, consider the relationship between the value of *NPV* and the *IRR*.

- $NPV = 0$. By definition, the cost of capital and the *IRR* are the same value.
- $NPV > 0$. Because the *IRR* is the rate of return that makes $NPV = 0$, this means that a higher rate of return must be used in order for the $NPV > 0$. In other words, the *IRR* is greater than the cost of capital.
- $NPV < 0$. Because the *IRR* is the rate of return that makes $NPV = 0$, this means that a lower rate of return must be used in order for the $NPV < 0$. In other words, the *IRR* is less than the cost of capital.

In summary, here is how to interpret the *IRR* and use it to make a decision about the viability of a project.

- *IRR* = cost of capital. The project breaks even. This is the minimum financial condition to pursue the project. The business should accept the project.
- *IRR* > cost of capital. The project is profitable because the project makes enough money to

cover the costs. The business should accept the project.

- **$IRR < \text{cost of capital}$.** The project is not profitable because the project does not make enough money to cover the costs. The business should reject the project.

Solving for the internal rate of return is difficult without the use of a financial calculator or other technology. Similar to calculating the NPV in the previous section, the IRR can be calculated by using the cash flow worksheet on the financial calculator and a built-in function to calculate the internal rate of return.

USING THE TI BAI PLUS CASH FLOW WORKSHEET TO CALCULATE THE INTERNAL RATE OF RETURN

Enter the cash flow data into the cash flow worksheet, as discussed in the previous section. Once you have entered all cash flows, you must access other functions to generate the output. For internal rate of return, press IRR on the keypad, as illustrated in the photo, followed by the CPT button to perform the calculation. The output is a percentage in percent format. To exit the window, press $2nd$ $QUIT$.

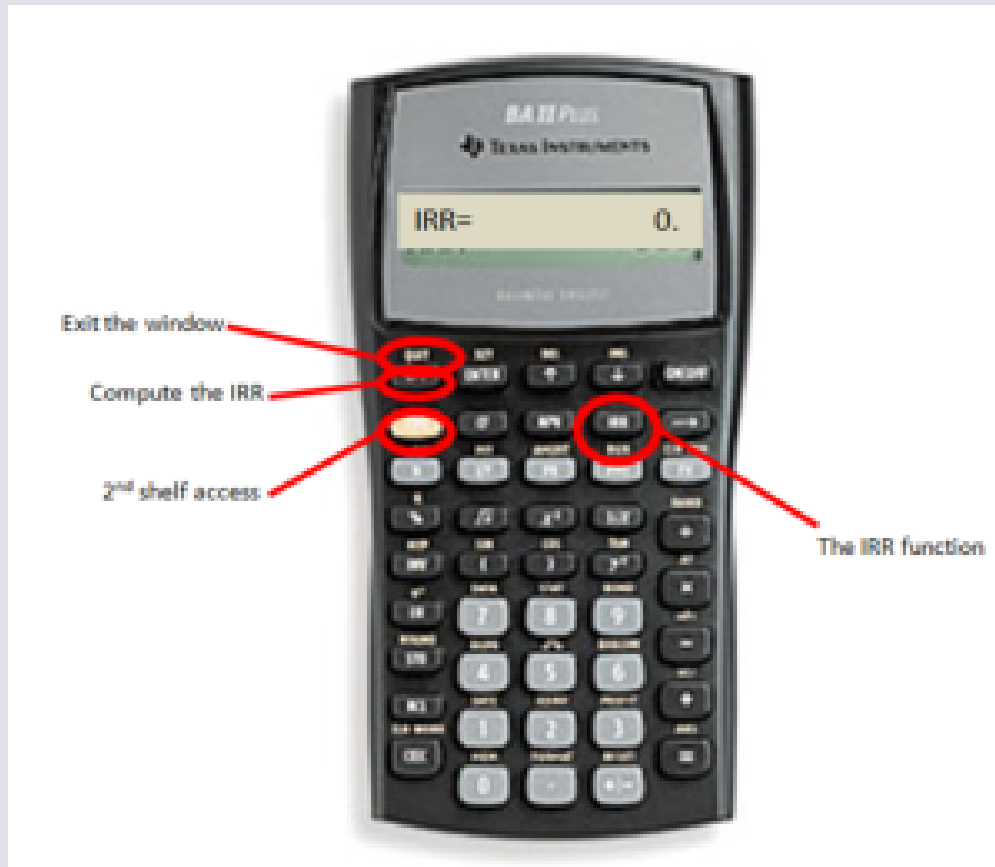


Figure 8.2.1

NOTES

1. Create a table to record the cash flows for each time period. The table should record the cash inflows, the cash outflows, and the net cash flow for each time period. Alternatively, use a timeline to illustrate the cash flows for each time period.
2. Because cash flows can be inflows or outflows, write all numbers with their cash flow sign conventions in the table to avoid confusion. That is, cash inflows are recorded as positive and cash outflows are recorded as negative. This helps you keep track of the different flows and minimizes the possibility of incorrectly netting out all present values.

3. For each period, only one cash flow can be entered in the financial calculator. If more than one cash flow exists for a certain period, enter the NET cash flow for that period. The net cash flow for a period is the sum of the cash inflows minus the sum of the cash outflows.
4. When using the cash flow worksheet on a financial calculator, you must enter a cash flow for EVERY period. If there is no cash flow for a certain period, a 0 is recorded as the net cash flow for that period. Enter 0 for CO on the calculator for that period and 1 for FO.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=506#oembed-1>

[Cash Flows 1: Net Present Value \(NPV\) and IRR Calculations](#) by Joshua Emmanuel [2:52] (transcript available).

EXAMPLE

Tim Hortons has purchased the lease on a three-year onsite concession space in the cafeteria at a local college for \$750,000. The franchise is expected to earn \$400,000, \$500,000, and \$600,000 in profits per year for the first three years, respectively.

1. Calculate the internal rate of return for this investment.
2. If the cost of capital is 20%, did Tim Hortons make a smart financial decision?

Solution:**Step 1:** Record the cash flows in a table.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-750,000	-750,000
1	400,000	400,000	400,000
2	500,000	500,000	500,000
3	600,000	600,000	600,000

Step 2: Calculate the *IRR*.

IRR	40.92%

1. Press IRR.
2. At the IRR screen, press CPT.

Because the $IRR = 40.92\% > 20\%$, this was a smart decision.

TRY IT

A research project requires an investment of \$350,000 today and another investment of \$200,000 one year from today. Starting from the third year, the project will reduce costs by \$250,000 every year for three years.

1. Calculate the internal rate of return for the project.
2. If the cost of capital is 12%, is the project financially viable?

Click to see Solution

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-350,000	-350,000
1	0	-200,000	-200,000
2	0	0	0
3	250,000	0	250,000
4	250,000	0	250,000
5	250,000	0	250,000

IRR	8.95%

Because the $IRR = 8.95\% < 12\%$, the project is not financially viable.

Like the net present value, the internal rate of return can be used to choose one project from several different projects. Relative to their initial investment, projects with higher *IRRs* are more profitable. Consequently, *IRR* can be used to rank projects. When making a choice from several options, the option with the highest *IRR* should be accepted because it will be the most profitable.

EXAMPLE

A company is consider two different investments.

- Investment A requires an initial investment of \$300,000 today and will provide returns of \$100,000 in year two, \$150,000 in year three, and \$400,000 in year five.
- Investment B requires an initial investment of \$500,000 today and will provide returns of \$350,000 in year one, \$200,000 in year four, and \$75,000 in year five.

Calculate the internal rate of return for each investment. Which investment should the company select?

Solution:**Step 1:** Calculate the *IRR* for Investment A.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-300,000	-300,000
1	0	0	0
2	100,000	0	100,000
3	150,000	0	150,000
4	0	0	0
5	400,000	0	400,000

IRR	21.76%

Step 2: Calculate the *IRR* for Investment B.

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-500,000	-500,000
1	350,000	0	350,000
2	0	0	0
3	0	0	0
4	200,000	0	200,000
5	75,000	0	75,000

IRR	10.14%

The company should select Investment A because it has the higher *IRR*.

TRY IT

A company is considering two different projects to deal with an equipment issue. The company can either upgrade the equipment (Project A) or they can retrofit the equipment (Project B).

- Project A requires an initial investment of \$300,000 in new equipment that will result in increased revenue of \$90,000 each year for five years.
- Project B requires an initial investment of \$125,000 to retrofit existing equipment that will result in increased revenue of \$30,000 each year for five years.

Calculate the internal rate of return for each project. Which project should the company select?

Click to see Solution

Project A:

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-300,000	-300,000
1	90,000	0	90,000
2	90,000	0	90,000
3	90,000	0	90,000
4	90,000	0	90,000
5	90,000	0	90,000

IRR	15.24%

Project B:

Year	Cash Inflows	Cash Outflows	Net Cash Flows
0	0	-125,000	-125,000
1	30,000	0	30,000
2	30,000	0	30,000
3	30,000	0	30,000
4	30,000	0	30,000
5	30,000	0	30,000

IRR	6.40%

The company should select Project A because it has the higher *IRR*.

NOTE

As mentioned above, the internal rate of return can be used to rank projects. However, a couple of things need to be considered about using the internal rate of return in this way.

1. **The Cost of Capital is Ignored.** The internal rate of return does not factor in the cost of capital in its computations. Recall that the fundamental concept of time value of money requires all money to be on the same date using an appropriate rate of interest—the cost of capital—before any decision can be made. Therefore, if you have not factored in the cost of capital, then your analysis is incomplete.
2. **The Magnitude of the Decision is Ignored.** It can easily happen that an alternative has a high *IRR* but a low *NPV*. For example, using a cost of capital of 10% consider two alternatives. Alternative A invests \$1 and one year later returns \$1.50. The *IRR* is 50%, while the *NPV* is \$0.36. Alternative B invests \$1,000 and one year later returns \$1,250. The *IRR* is 25%, while the *NPV* is \$136.36. If choosing between these two options based on the *IRR*, you select Alternative A, resulting in a net present value that is \$136 lower than for Alternative B.

Exercises

1. A project requires an investment of \$225,000 today and is projected to have annual profits of \$34,000 for nine years. The capital assets can be sold at the end of the ninth year for \$23,500. Calculate the IRR for this project. If capital could be acquired at a cost of 12%, should the project be pursued?

Click to see Answer

IRR=7.91%; do not pursue project

2. A new diamond deposit has been found in northern Alberta. Your researchers have determined that it will cost \$4.5 million to purchase the land and prepare it for mining. Starting in the second year, the lode is expected to earn net profits of \$3 million, which will be sustained for three years before the deposit is depleted. Calculate the IRR on the diamond mine. If the cost of capital is 21%, should the deposit be acquired?

Click to see Answer

IRR=26.78%; pursue project

3. Schick is considering launching only one of two new products to compete with the Gillette Fusion ProGlide Power razor.
 - Product A requires a \$10 million research and development investment with forecasted profits are expected to be \$3 million in years two to four, \$1.5 million in years five to seven, and \$0.5 million in years eight to ten.
 - Product B requires a \$6.5 million research and development investment with forecasted profits of \$2.75 million in years one to two, \$1 million in years three to five, and \$0.35 million in years six to ten.

Calculate the IRR for each product. Which product should Schick launch?

Click to see Answer

IRR of Product A=10.59%; IRR of Product B=16.89%; launch Product B

4. The Dragons (from CBC's *Dragon's Den*) are considering a venture from an Albertan entrepreneur. His proposal requires the Dragons to invest \$500,000 immediately. Based on market conditions, the product is estimated to have year-end profits of \$100,000, \$200,000, \$300,000, \$200,000, and finally \$100,000.

The Dragons will invest only if the internal rate of return exceeds their cost of capital of 20%. Should they invest?

Click to see Answer

IRR=22.78%; pursue project

5. A Western Canadian producer is considering purchasing a five-year product licence from Sunkist Growers Inc. The product licence will cost \$1,500,000, and the required equipment will cost \$1 million. Equipment upgrades will be \$100,000 at the end of the second and fourth years. Expected profits are \$1 million in each of the first three years and \$750,000 in the last two years. The producer will purchase product licences only if the IRR is greater than 25%. Should the product licence be purchased?

Click to see Answer

IRR=22.91%; do not pursue project

6. Consider the following two projects, of which only one can be selected:

- Project A. Immediate investment of \$225,000, profits in years four to six of \$160,000.
- Project B. Immediate investment of \$112,500, profits in years one to three of \$55,000 followed by profits of \$20,000 in years four to six.

Calculate the IRR for each project. Which project should be selected?

Click to see Answer

IRR of Project A=16.54%; IRR of Project B=32%; select Project B

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8.3 REVIEW EXERCISES

1. Jeremy is thinking of starting up a candle manufacturing business. The initial outlay for equipment, moulds, and other required production equipment is \$15,000. Working part time on this hobby business, Jeremy estimates that he will lose \$2,000 in the first year, break even in the second year, and earn annual profits of \$5,000, \$10,000, and \$15,000 in subsequent years. After the five years, he hopes to sell the business to an investor for \$17,500. If his cost of capital is 8.25% compounded annually, should he pursue this venture? Provide net present value calculations to support your answer.

Click to see Answer

NPV=\$16,241.50; pursue venture

2. A company pursues all projects that exceed its 15% cost of capital. Project “Affinity” is available to the company with an initial investment of \$655,000. Forecasted annual profits are expected to be \$200,000 for the first three years and \$160,000 for the subsequent three years. Based on the IRR, should the company pursue Project “Affinity”?

Click to see Answer

IRR=17.47%; pursue project

3. Coca-Cola is considering two new flavours—cherry and vanilla—for expansion of one of its product lines. Based on historical and competitive information, the cherry flavour isn’t quite as popular as the vanilla flavour, but it will endure in the market longer. Both options require an investment of \$2.25 million in equipment and modifications. The cherry flavour is expected to have annual profits of \$750,000 for the first two years and \$500,000 for five more years. The vanilla flavour is expected to initially have annual profits of \$950,000 for two years, \$650,000 for two years, and \$250,000 for the last three years. The cost of capital is 16%. Calculate the NPV for each flavour. Which flavour should Coca-Cola launch?

Click to see Answer

NVP of Cherry=\$170,590.69; NPV of Vanilla=\$360,483.17; launch Vanilla

4. A saleswoman informs the head librarian that investing in her new automated library product is a steal at

\$103,000. Before purchasing, the head librarian investigates the benefits of the automated product, which is expected to have a useful life of seven years and a 15% cost of capital. Each year, \$24,000 in labour savings and reduced book shrinkage is projected. However, the automated products require regular maintenance of \$1,000 in the first four years and \$2,000 in the last three years. Electricity bills will also rise by \$1,500 each year in the first three years and \$2,100 each year in the last four years. Calculate the NPV for this product. Is the saleswoman correct? Should the automated system be purchased?

Click to see Answer

NPV=-\$15,982.73; do not pursue automation

5. Hershiser has been researching different methods of management leadership that are designed to increase employee motivation and performance. If he pursues a new leadership style, it will require an investment of \$165,000 to take the training. A new executive assistant will be hired at a cost of \$55,000 annually with expected raises of 5% each year. The techniques and tactics from the style are expected to have an impact on employees for five years before needing to be refreshed and updated with more modern techniques. Increased motivation and performance will produce net profits of \$100,000 in the first year, rising by 10% in each subsequent year.
 - a. Calculate the internal rate of return.
 - b. If the cost of capital is 10%, should Hershiser take the management leadership course?

Click to see Answer

IRR=22.36%; take course

6. Mariners Inc. in Halifax has the option to purchase only one of two available offshore fishing rights. Both rights are available for purchase on a four-year contract before they have to be renewed.
 - Fishing right #1 costs \$600,000 to acquire, and \$64,000 is needed in new equipment to harvest the fish in the area. For each of the first two years, \$21,000 will be spent on equipment maintenance and replacement followed by \$33,000 in both of the last two years. Annual profits are expected at \$300,000.
 - Fishing right #2 costs \$400,000 to acquire, and \$82,000 is needed in new equipment, which requires annual costs of \$30,000 for each of the first two years and \$41,000 for both of the last two years. Annual profits are expected to be \$250,000.

Calculate the net present value for both projects if the cost of capital is 14%. What decision do you recommend?

Click to see Answer

NPV of #1=\$133,721.11; NPV of #2=\$145,079.14; choose fishing right #2

7. The following projects are available, but only one can be chosen. The cost of capital in all cases is 16%.
- Project A requires an initial investment of \$180,000 followed by profits of \$30,000 in years one to four, \$40,000 in years five to seven, and \$50,000 in years eight to ten with a residual value of \$100,000 in year ten.
 - Project B requires an initial investment of \$335,000 followed by profits of \$65,000 in years one to three, \$85,000 in years four to six, and \$110,000 in years seven to eight with a residual value of \$60,000 in year ten.
 - Project C requires an initial investment of \$372,000 followed by profits of \$150,000 in years four to nine and a residual value of \$70,000 in the tenth year.
- a. If the cost of capital is 16%, calculate the net present value for each project. Which project should be selected?
 - b. If the cost of capital is 11%, calculate the net present value for each project. What decision should be selected?

Click to see Answer

a. NPV of A=\$15,962.27; NPV of B=\$19,359.83; NPV of C=-\$2,034; choose Project B b. NPV of A=\$71,533.63; NPV of B=\$97,566.83; NPV of C=\$116,652.84; choose Project C

8. The Banff Gondola is looking to increase its operational capacity through modifications to its gondola equipment. If it invests \$2.3 million it can upgrade the motors, strengthen the necessary supports, and add an additional cable car to its line. The expected life of the project is eight years before the equipment will need to be replaced. Annual increased maintenance costs are \$50,000 at the end of every year for four years and then \$100,000 at the end of every year during the last four years. Because of construction and down times, Banff Gondola is forecasting a loss of \$415,000 in the first year. Afterwards, it projects that the equipment will result in increased profits of \$850,000 per year. Calculate the internal rate of return for the project.

Click to see Answer

IRR=16.07%

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8.4 CHAPTER SUMMARY

Key Concepts

- **8.1 Net Present Value**
 - The characteristics of making decisions including decision types, monetary sources, and interest rates.
 - Making decisions through net present value.
 - **8.2 Internal Rate of Return**
 - Using internal rate of return to choose whether to pursue one course of action.
-

Glossary of Terms

- **Cash Flow.** A movement of money into or out of a particular project.
 - **Cost of Capital.** A weighted average of all of the debt and equity financing rates used to provide needed funds for a project.
 - **Internal Rate of Return (IRR).** The annual rate of return on the investment being made such that the net present value of all cash flows in a particular project equals zero.
 - **Net Present Value (NPV).** The difference in today's dollars between all benefits and costs for any given project.
-

Formulas

- **Symbols Used**
 - NPV = net present value
 - IRR = internal rate of return

- Formulas Used

- Net Present Value:

$NPV = (\text{Sum of PV of All Future Cash Flows}) - (\text{Initial Investment})$ or

$NPV = (\text{Sum of PV of Cash Inflows}) - (\text{Sum of PV of Cash Outflows})$

Calculator

- Cash Flow Worksheet

- Access the cash flow function by pressing CF on the keypad.
- Always clear the memory using 2nd CLR WORK so that any previously entered data is deleted.
- Use the up arrow and down arrow to scroll through the various lines.
- Strictly adhere to the cash flow sign convention when using this function and press ENTER after keying in the data.
- To exit the window, press 2nd QUIT.
- The various lines are summarized below:
 - CF₀ = any cash flow today.
 - CXX = a particular cash flow, where XX is one of a series of cash flow numbers starting with 01. You must key in cash flows in order from the first time segment to the last. You cannot skip a time segment, even if it has a value of zero, because each time segment is a placeholder on the timeline.
 - FXX = the frequency of a particular cash flow, where XX is the cash flow number. It is how many times *in a row* the corresponding cash flow amount occurs. This allows you to enter recurring amounts together instead of keying them in separately. By default, the calculator sets this variable to 1.

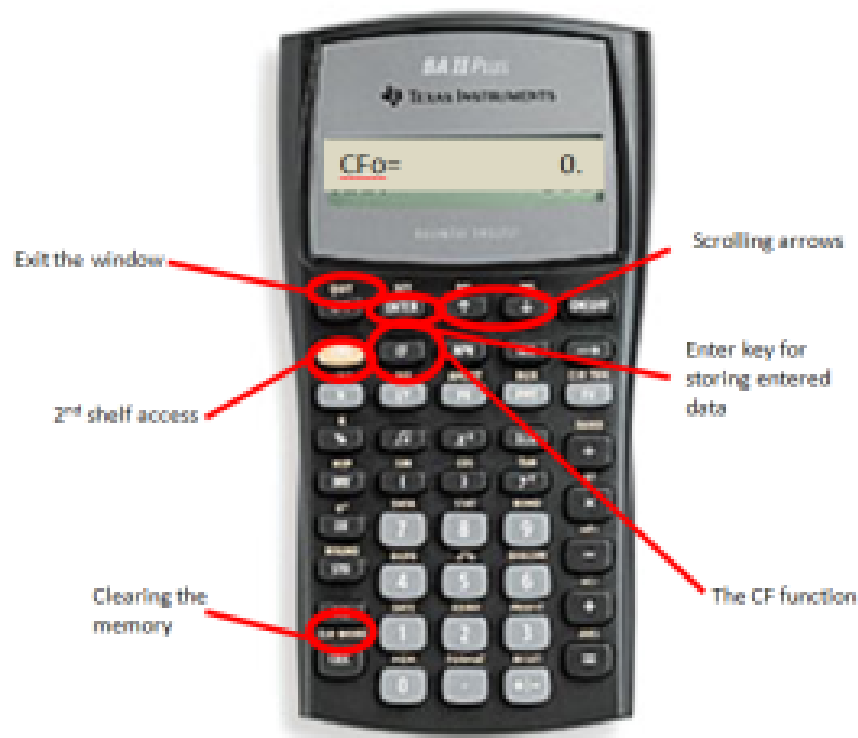


Figure 8.4.1

• Net Present Value

- Use this function after you have entered all cash flows.
- Press NPV on the keypad to access the function.
- Use the up arrow and down arrow to scroll through the window.
- To exit the window, press 2nd QUIT.
- This window has two lines:
 - I = the matching periodic interest rate for the interval of each time segment.
 - NPV = the net present value. Press CPT to calculate this amount.

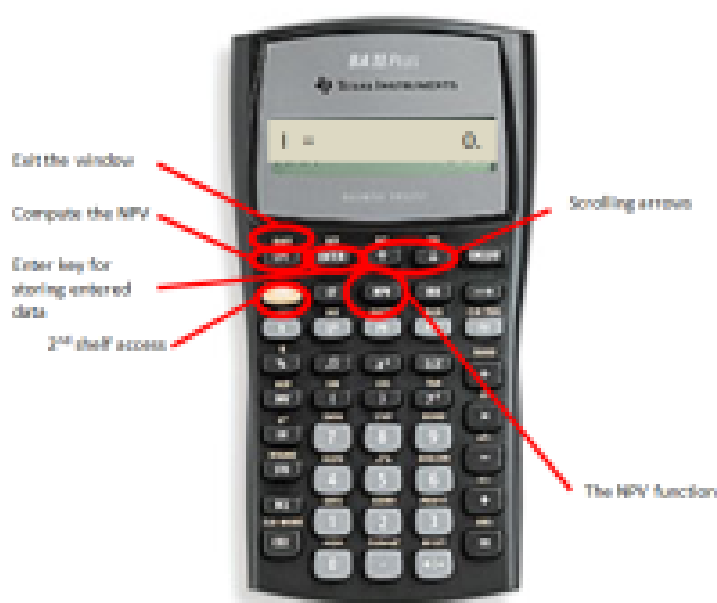


Figure 8.4.2

• Internal Rate of Return

- Use this function after you have entered all cash flows.
- Press IRR on the keypad to access the function.
- Press CPT button to perform the calculation. The output is in percent format.
- To exit the window, press 2nd QUIT.

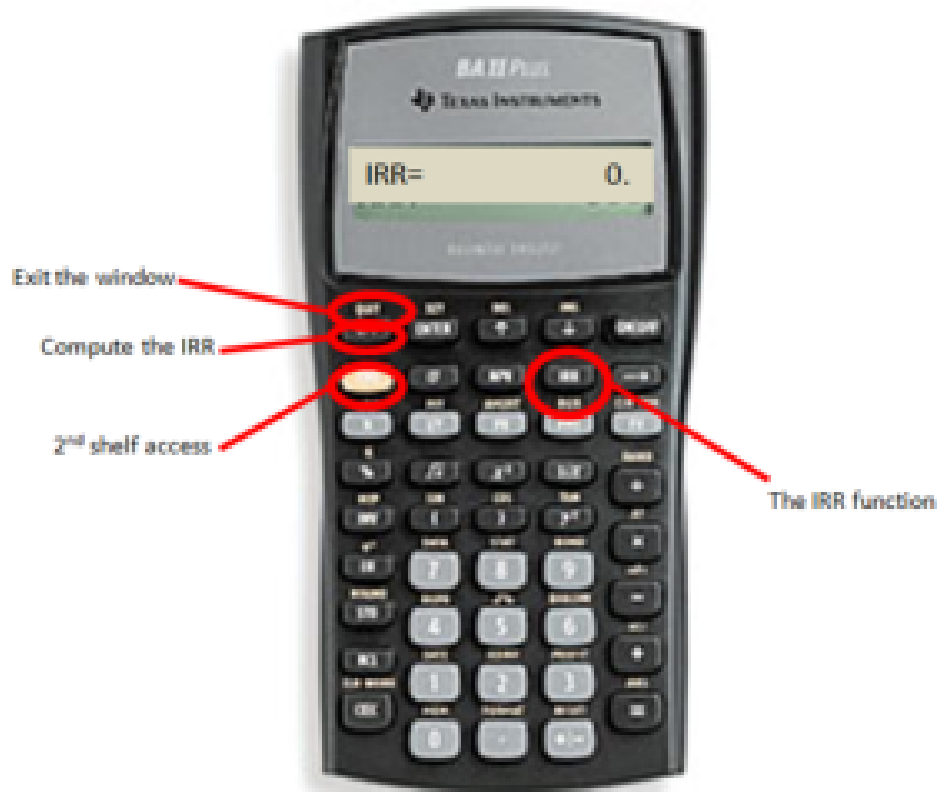


Figure 8.4.3



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/businessfinancialmath/?p=510#oembed-1>

[Cash Flows 1: Net Present Value \(NPV\) and IRR Calculations](#) by Joshua Emmanuel [2:52] (transcript available).

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VERSIONING HISTORY

This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.1. If the edits involve a number of changes, the version number increases to the next full number.

The files posted alongside this book always reflect the most recent version.

Version	Date	Change	Affected Web Page
1.0	June 1, 2023	First Publication	N/A