Math Skills for Trades

MATH SKILLS FOR TRADES

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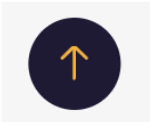

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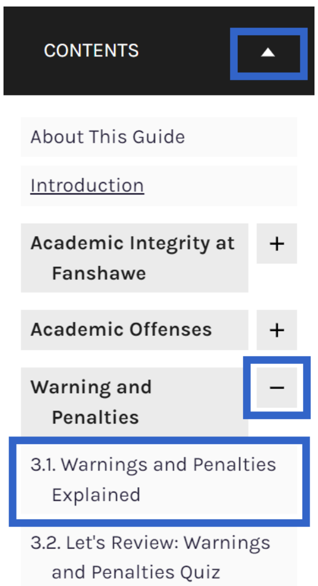


1 MATHEMATICAL OPERATIONS

Chapter Outline

Although mathematics is a massive subject, a common idea behind its study is taking something old and using transformations to make something new. Many of the skills learned by tradespeople are all about transformation. Here are some examples:

1. fastening wood planks into a deck,
2. welding metal sheets into a panel, or
3. assembling fluid lines for an engine.

Since so many trades involve transformation, the mathematics of transformations can help us be better tradespeople. In mathematics, transformations are often called operations, and they are where we begin our study.

1.0 LEARNING OBJECTIVES

Learning Objectives

By the end of this chapter, students will be able to:

- Count forward and backward using whole numbers and write numbers using standard numeral form (1.1)
- Add whole numbers accurately and interpret addition as combining quantities and finding totals (1.2)
- Multiply whole numbers and explain multiplication as repeated addition and as groups of equal size (1.3)
- Evaluate exponent expressions and describe exponentiation as repeated multiplication (1.4)
- Identify and apply key properties of operations (e.g., commutative, associative, distributive) to simplify computations (1.5)
- Use inverse operations (addition/subtraction; multiplication/division) to check work and solve basic equations (1.6)
- Apply the correct order of operations to evaluate expressions involving parentheses, exponents, multiplication/division, and addition/subtraction (1.7)

1.1 COUNTING AND ADDITION

Counting

Counting is likely to be a person's earliest exposure to mathematics. We use counting to express the quantity of something. For example, if on your kitchen table there is a fruit bowl with a banana, an apple, and a pear in it, you can use counting to say: "That bowl has three fruits in it".

Counting is a simple operation. Beginning at zero, you jump to the next highest number for each item that you are counting. To keep track of whether you have counted an item, you could cross it out or physically move it to another area, depending on the situation. We can visually count up to nine in the figure below.

Figure 1.1.1

$$0 \xrightarrow{+1} 1 \xrightarrow{+1} 2 \xrightarrow{+1} 3 \xrightarrow{+1} 4 \xrightarrow{+1} 5 \xrightarrow{+1} 6 \xrightarrow{+1} 7 \xrightarrow{+1} 8 \xrightarrow{+1} 9$$

Starting at zero, each arrow going to the right represents counting to the next number by adding one. Since there are nine arrows, the total that is counted is nine.

Image Description

Nine right-pointing arrows with the numbers one through eight between each arrow. Zero is to the left of the first arrow, and 9 is to the right of the last arrow. Above each arrow is the plus symbol with a one next to it.

Addition

Counting is useful when we do not know how many items we have. However, in situations where we have two groups of items that we have already counted, we do not need to start from scratch to count them all. Instead, we can begin our count starting at the number of items in one of the groups, then count up from there based on the number of items in the other group. Doing this is applying the operation called *addition*, represented by the symbol $+$.

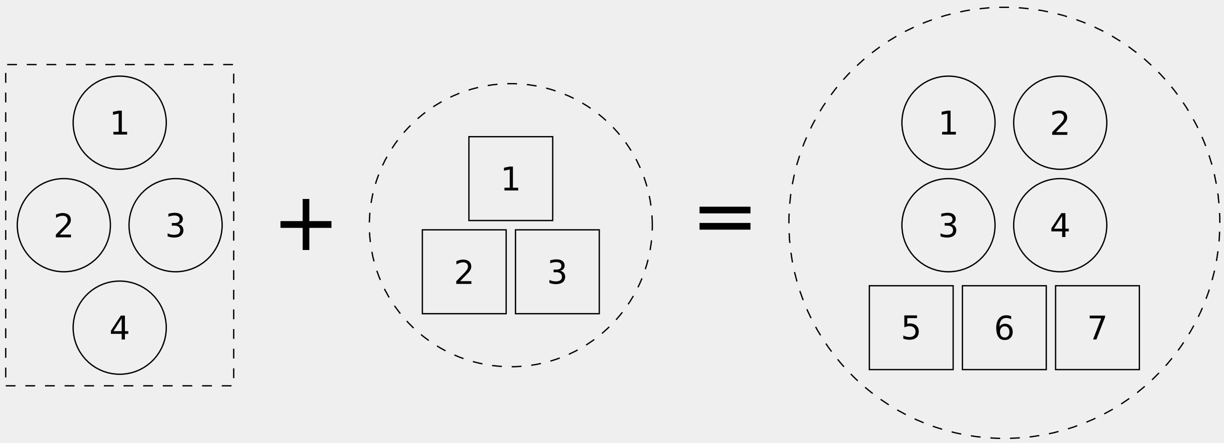
For example, the expression $(4+3)$is shorthand for saying one of two things:

- "The total combined number of items in a group of four and a group of three," or
- $(((4 \ + \ 1) \ + \ 1) \ + \ 1)$.

When we add two numbers together, we call their result the "sum" of those two numbers.

The calculation of $(4 \ + \ 3) \ = \ 7$ is visualized in Figure 1.1.2 below, where the three squares in the second group are added to the group of four circles. The squares are relabeled from $1, \ 2, \ 3$ to $5, \ 6, \ 7$ to represent counting up from four.

Figure 1.1.2



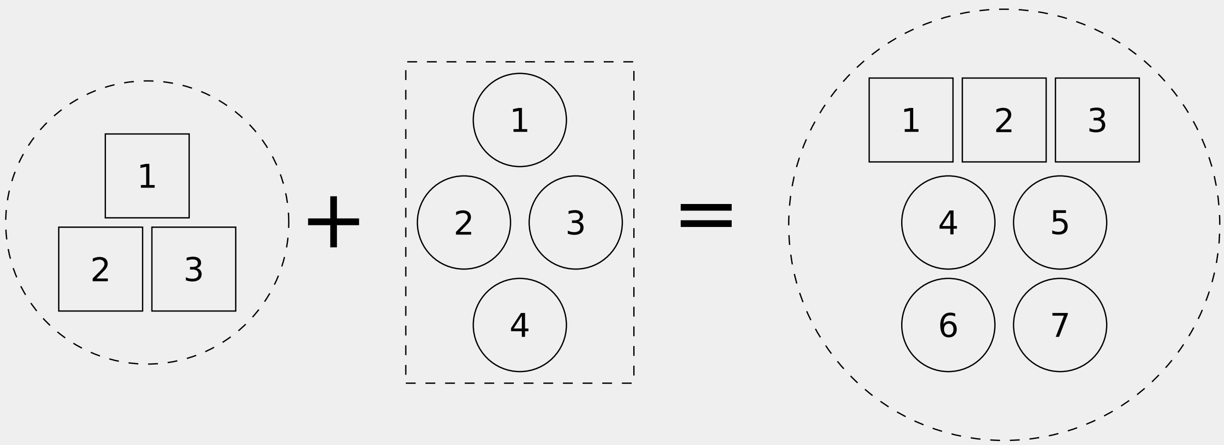
Combined, a group of four and a group of three created a group of seven total.

Image Description

A dashed-line rectangle has four solid-line circles within, labelled one through four. To the right is a plus symbol. To the right is a dashed-line circle with three solid-line squares within, labelled one through three. To the right is an equal symbol. To the right is a large dashed-line circle with four solid line circles within labelled one through four, and three solid line squares within the same circle labelled five through seven.

A reasonable question to ask ourselves is how $(4 + 3)$ compares to $(3 + 4)$. Put mathematically, does $(4 + 3) = (3 + 4)$? The answer is yes, as visualized in Figure 1.1.3 below.

Figure 1.1.3



Combined, a group of three and a group of four created a group of seven total.

Image Description

A dashed-line circle has three solid-line squares within, labelled one through three. To the right is a plus symbol. To the right is a dashed-line rectangle with four solid-line circles within, labelled one through four. To the right is an equal symbol. To the right is a large dashed-line circle with three solid line squares within labelled one through three, and four solid line circles within the same circle labelled four through seven.

Exercises 1.1

1. Looking at Figure 1.1.2 and 1.1.3, what requires more counting, $(4+3)$ or $(3+4)$?

2. Does $(5+2)=(2+5)$? Does $(8+1)=(1+8)$? Based on your answers, think of a reason that could explain these answers without using addition. Does your reason change if the numbers in these expressions change?

3. What are two ways that you could compute $(1+2+3)$? Did either way change your result? What is the reason for this, and does the reason change if the numbers in the expression change?

4. What is $0+1$? What is $0+2$? What is the sum of the number zero with any other number?

1.2 MULTIPLICATION

Addition is useful when we are combining groups of different sizes. That said, there are many situations that arise where we would like to count a number of groups of items that are all the same size. An example of this can be found at the grocery store: if I buy three packs of a dozen eggs, I want to know how many total eggs that is. Doing this is applying the operation called *multiplication*, and it is represented by the symbol $\times$.

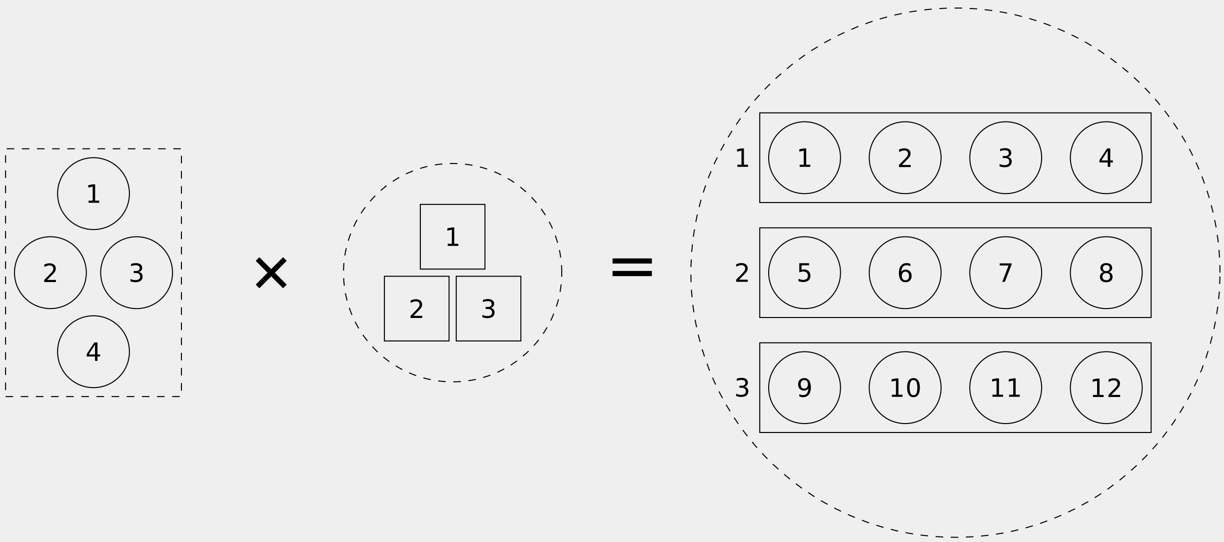
For example, the expression $(4\ \times\ 3)$ is shorthand for saying one of two things:

- "The total combined number of items in three groups of four items," or
- "$((4\ +\ 4)\ +\ 4)$".

When we multiply two numbers together, we call their result the "product" of those two numbers. Notice that multiplication is really just an instruction to add something multiple times.

The calculation of $(4\ \times\ 3)$ is visualized in Figure 1.2.1, where a group of four circles are copied three times into square groups labelled $1,\ 2,\ 3$. The circles in the second group are relabeled $5,\ 6,\ 7,\ 8$ to represent counting up from four by four, and the circles in the third group are relabeled $9, 10,\ 11,\ 12$ to represent counting up from eight by four.

Figure 1.2.1



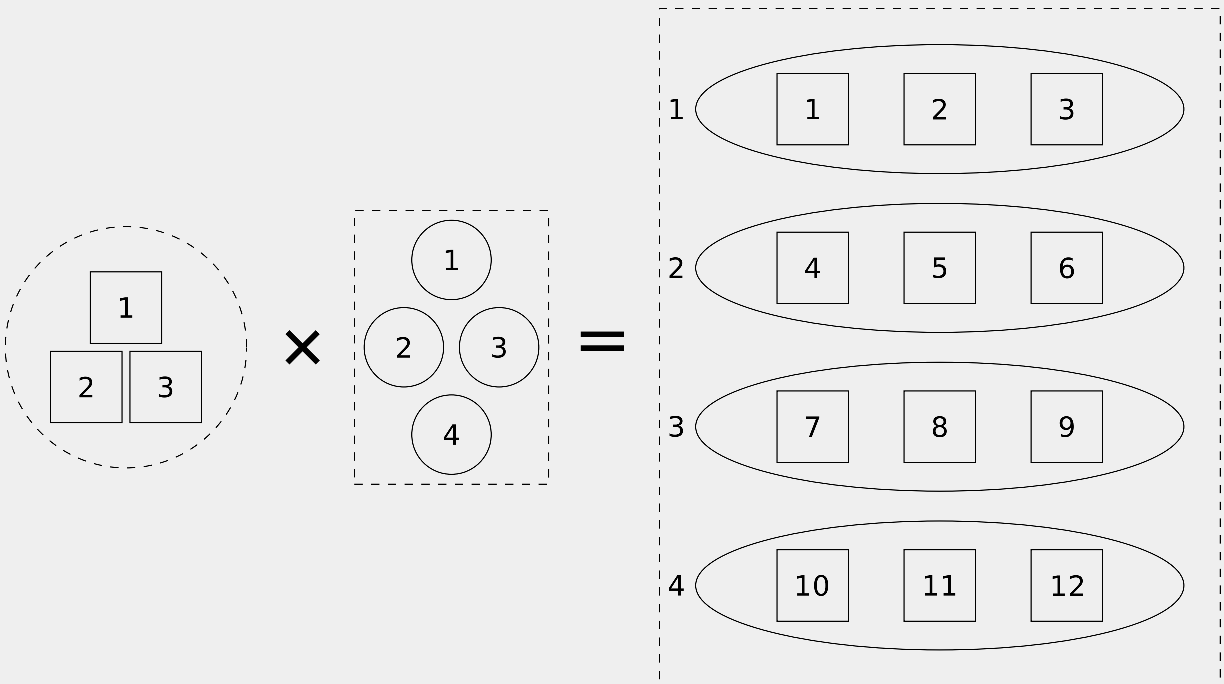
Three copies of a group of four combine to a total of twelve.

Image Description

A dashed-line rectangle contains four solid-line circles labelled 1 through 4. To the right is a multiplication symbol. To the right is a dashed-line circle with three solid-line squares within, labelled one through three. To the right is an equal symbol. To the right is a large dashed circle with three rectangles within, labelled 1 through 3. Within each rectangle is a set of four solid line circles. The first set of circles is labelled one through four, the second set labelled five through eight, and the third set labelled nine through twelve.

Similar to the question we asked ourselves in the last section about addition, how does (4×3) compare to (3×4)? Put mathematically, does $(4 \times 3) = (3 \times 4)$? The answer is yes, as visualized in Figure 1.2.2 below.

Figure 1.2.2



Four copies of a group of three combine to a total of twelve.

Image Description

A dashed-line circle has three solid-line squares within, labelled 1 through 3. To the right is a multiplication symbol. To the right is a dashed-line rectangle with four solid-line circles within, labelled one through four. To the right is an equal symbol. To the right is a large dashed-line rectangle with four ovals labelled 1 through 4. Within each rectangle is a set of three solid line squares. The first set of circles is labelled one through three, the second set labelled four through six, the third set labelled seven through nine, and the last set labelled ten through twelve.

Exercises 1.2

1. Looking at Figures 1.2.1 and 1.2.2, what requires more addition, (4×3) or (3×4)?

2. Does $(5 \times 2) = (2 \times 5)$? Does $(8 \times 1) = (1 \times 8)$? Based on your answers, think of a reason that could explain these answers without using multiplication. Does your reason change if the numbers in these expressions change?

3. What are two ways that you could compute $(2 \times 3 \times 4)$? Did either way change your result? What is the reason for this, and does the reason change if the numbers in the expression change?

4. What is (1×2)? What is (1×3)? What is the product of the number one with any other number?

5. Does $((2 \times 3) + (2 \times 4)) = (2 \times (3 + 4))$? Based on your answer, think of a reason that could explain this without computing the expressions. Does your reason change if the numbers in these expressions change?

6. What is (0×1)? What is (0×2)? What is the product of the number zero with any other number?

1.3 EXPONENTIATION

Multiplication is useful when we are combining many copies of a single thing together. But, there are some situations where, after we make a number of copies of a thing, we need to make the same number of copies again. For example, imagine that you own a farm where every year a pair of sheep gives birth to two more sheep. Every year we add a new copy of all of the previous sheep, i.e., multiply the total by 2. If you begin with only two sheep, you can ask how many you will have by the third year. Doing this is applying the operation called *exponentiation*. Sometimes this operation is represented by the symbol $\wedge$, but more often we write (X^Y) to say X raised to the exponent Y.

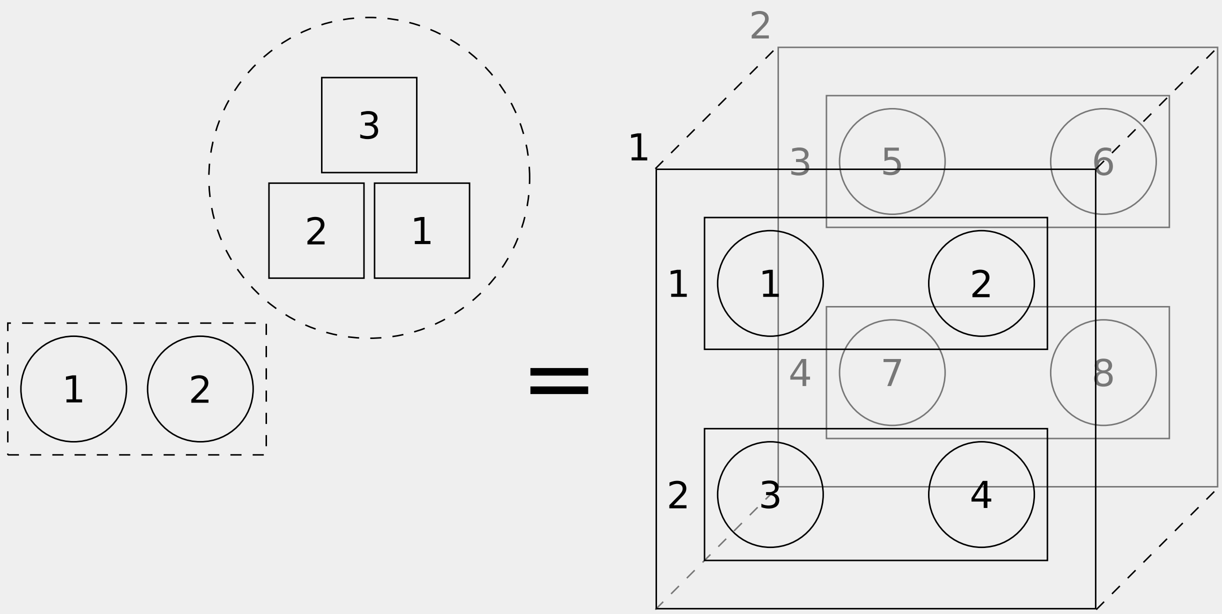
For example, the expression (2^3) is shorthand for saying one of two things:

- "the total combined number of items from repeatedly making two copies of an item, three times," or
- "$((2 \ \times \ 2) \ \times \ 2)$".

When we take the exponent of one number to another number, we call their result the "power" of those two numbers. Notice that exponentiation is really just an instruction to multiply something many times.

The calculation of $(2^3) = 8$ is visualized in Figure 1.3.1. Like with multiplication, a square is made from the first iteration of $(2 \ \times \ 2)$. For the second iteration, we construct a cube by putting our copy of the square at a different depth. For this reason, raising a number to the power of 3 is commonly called *cubing* that number.

Figure 1.3.1



A group of two copied twice, then copied twice again, totals eight items.

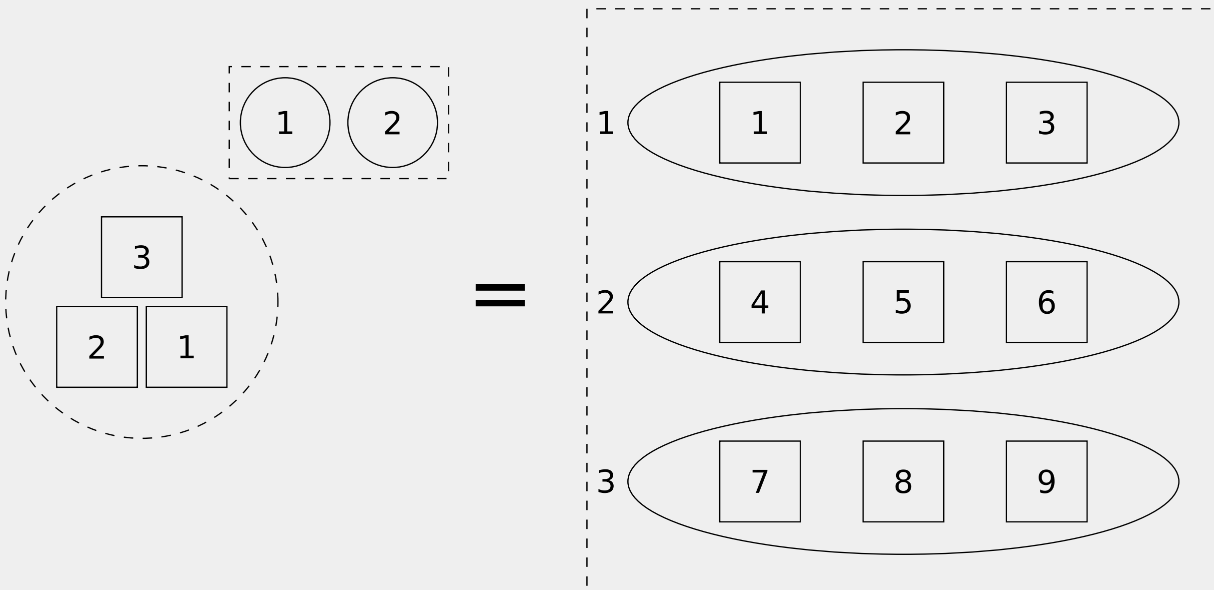
Image Description

Diagram showing how 2 to the power of 3 combines to form 8. On the left, two numbered circles, 1 and 2, are grouped in a dashed rectangle, and three numbered squares, 1, 2, and 3, are grouped in a dashed oval above and to the right of the dashed rectangle. An equals sign points to the right, where the same elements are arranged inside a transparent cube-like structure. The cube shows two horizontal layers labelled 1 and 2, with numbered circles placed across the lower and middle areas and numbered squares or rectangular regions layered behind them. Additional grey numbered circles and outlines indicate overlapping positions and depth within the structure.

Like we have done in the previous sections, how does (3^2) compare to (2^3)? As visualized in Figure 1.3.2, the results are different from one another. Not only does $(3^2) = 9$ instead of 8, but the image we get is a $2D$ square instead of a $3D$ cube. For this reason, raising a number to the power of 2 is called *squaring* a number.

Although we have tried visualizing squaring and cubing numbers, notice that it is hard to imagine what any higher power would look like on paper, since we would need at least a fourth dimension!

Figure 1.3.2



A group of three copied three times totals a group of nine items.

Image Description

Diagram showing how 3 to the power of 2 equals 9. On the left, three numbered squares, 1, 2, and 3, are grouped in a dashed oval, and two numbered circles, 1 and 2, are grouped in a dashed rectangle slightly above and to the right of the dashed circle. An equals sign points to the right, where the result is shown as a dashed rectangle containing three horizontal oval groups labelled 1, 2, and 3. Each oval contains three squares: the first row contains 1, 2, and 3; the second row contains 4, 5, and 6; and the third row contains 7, 8, and 9.

Exercises 1.3

1. As we saw above, $(3^2) \neq (2^3)$. Does$(2^4) = (4^2)$? Does $(2^5) = (5^2)$

2. What are two ways that you could compute (2^{3^2})? Did either way change your result? What is the reason for this, and does the reason change if the numbers in the expression change?

3. What is (2^1)? What is (3^1)? What is any number raised to the power of 1?

4. What is (1^2)? What is (1^3)? What is one raised to the power of another number?

5. What is (0^2)? What is (0^3)? What is zero raised to the power of another number?

6. Does $((3 \times 4)^2) = (3^2) \times (4^2)$?

7. Does $((2^3) \times (2^4)) = (2^7)$?

8. Does $((2+3)^2) = ((2^2) + (3^2))$?

1.4 PROPERTIES OF THE OPERATIONS

The nicest mathematical operations have properties that we can use to help us simplify our calculations and formulas. The exercises in the last few sections have demonstrated some of these properties in action. Right now, we list some of them for reference. To get a better idea of where they come from, try the exercises from that section.

Note that we use variables X, Y, Z to describe these properties. A variable is just a placeholder for a number. For example, the expression $(X + 1)$ means "a number plus one". We could substitute any number for X, such as $X = 1$, which would give the expression $(1 + 1)$. Often we write properties using variables, such as $(X \times 2) = (X + X)$. A property holds if you can substitute any number and the equation is still correct. The property $(X \times 2) = (X + X)$ holds since any number multiplied by 2 is just that number added to itself.

Addition:

Imagine that X, Y, Z are all numbers. Then the following properties hold:

1. $((X + Y) + Z) = (X + (Y + Z))$: we can compute a sum in any order,
2. $(X + Y) = (Y + X)$: we can switch the position of numbers in a sum, and
3. $(0 + X) = X$ and $(X + 0) = X$: zero does nothing in a sum.

Multiplication:

Imagine that X, Y, Z are all numbers. Then the following properties hold:

1. $((X \times Y) \times Z) = (X \times (Y \times Z))$: we can compute a product in any order,
2. $(X \times Y) = (Y \times X)$: we can switch the position of numbers in a product,
3. $(1 \times X) = X$ and $(X \times 1) = X$: one does nothing in a product,
4. $(0 \times X) = 0$ and $(X \times 0) = 0$: zero makes all products zero, and
5. $(X \times (Y + Z)) = ((X \times Y) + (Y \times Z))$: sums 'distribute' over products.

Exponentiation:

Imagine that X, Y, Z are all numbers. Then the following properties hold:

1. $((X^Y) \times (X^Z)) = X^{(Y+Z)}$: products of powers over the same base become powers of sums,
2. $((X^Y)^Z) = (X^{(Y \times Z)})$: powers of powers become powers of products,
3. $((X^Z) \times (Y^Z)) = ((X \times Y)^Z)$: products of the same power are powers of a product,
4. $(X^1) = X$: raising a number to the power of one does nothing,
5. $(X^0) = 1$, $X \neq 0$: raising a number to the power of zero (other than zero) equals one,
6. $(1^X) = 1$: raising one to any power equals one, and
7. $(0^X) = 0$, $X \neq 0$: raising zero to any power (other than zero) equals zero.

Exercises 1.4

1. Rewrite the expression $(2^3) \times (2 + 4)$ into a simpler form, using the multiplication property 5, the exponentiation properties 1 and 4, and the fact that $22 = 4$.

2. (0^0)? Is there any good answer?

1.5 INVERSE OPERATIONS

So far, we have focused on combining things together through different operations. Beginning with counting as the most basic operation of combination, we created new operations by repeatedly applying the simpler operations so that we can tackle common combination problems. What about the opposite focus of separating things, or more generally, reversing the operations we have created?

An inverse operation is similar to undoing something. This section looks at how we can undo all of counting, addition, multiplication, and exponentiation.

Decrement

The opposite of counting is *decrement*. Instead of asking “what is the next number”, decrement says “what is the previous number”. Decrement is how you determine how many apples you have left in your fruit basket if you decide to eat one of them. Another way to think of this is ‘counting down’, which is visualized in Figure 1.5.1 below. Notice that if you ‘count up’ from a number, then ‘count down’, and vice versa, nothing changes to the original number.

Figure 1.5.1

$$9 \xrightarrow{-1} 8 \xrightarrow{-1} 7 \xrightarrow{-1} 6 \xrightarrow{-1} 5 \xrightarrow{-1} 4 \xrightarrow{-1} 3 \xrightarrow{-1} 2 \xrightarrow{-1} 1 \xrightarrow{-1} 0$$

Starting at nine, each arrow going to the right represents counting down to the next number by subtracting one. Since there are nine arrows, we count down to zero.

Image Description

Nine right-pointing arrows with the numbers eight through one between each arrow. Nine is to the left of the first arrow, and zero is to the right of the last arrow. Above each arrow is the minus symbol with a one next to it.

Subtraction

The opposite of addition is *subtraction*. Subtraction takes a group and one of its parts and asks, "How many will be left in the group if I remove a part of the group from it?" For example, if I have seven dollars in my pocket but I pay three dollars of it to buy a soda, how much money do I have left? Doing this is an application of subtraction, which we represent with the symbol $-$.

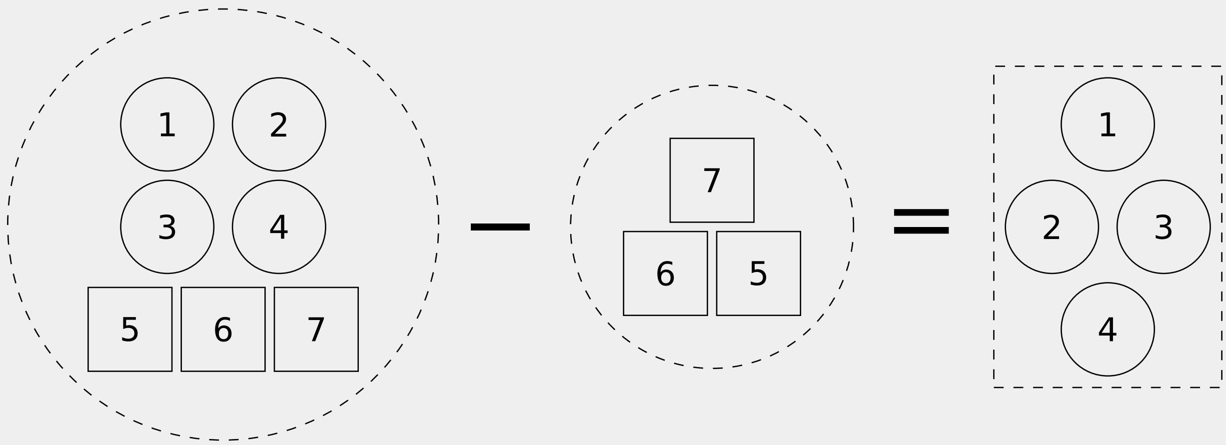
For example, the expression $(7-3)$ is shorthand for saying one of two things:

- "the total number of items left after removing three items from a group of seven" or
- $(((7-1)-1)-1)$

When we subtract two numbers, we call their result the "difference" of those two numbers. The calculation of $(7-3)=4$ is visualized in Figure 1.5.2, where three squares are removed from a group of seven shapes made up of three squares and four circles. Since only four circles remain, the result of the computation is four. Notice that this resembles Figure 1.1.2, where we visualized $(4+3)=7$, and that the same three numbers appear in both of these equations as well. Moreover, we can see that.

$$((7-3)+3)=(4+3)=7$$

Figure 1.5.2



Separating a group of three from a group of seven results in a group of four.

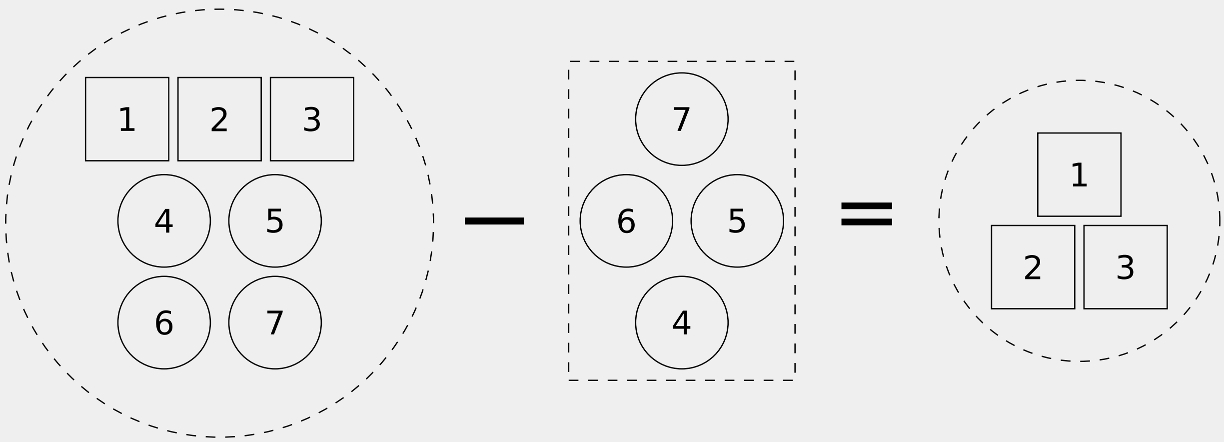
Image Description

A large dashed line circle with four solid line circles labelled one through four and three squares labelled five through seven within. To the right, a minus symbol. To the right, a smaller dashed line circle with three squares within labelled five through seven, to the right of the equals sign. To the right, a large dashed line rectangle with four solid line circles on the inside, labelled one through four.

Likewise, the computation of $(7-4)=3$ is visualized in Figure 1.5.3. It also looks similar to the visualization of $(3+4)=7$ in Figure 1.1.3. The same three numbers appear, and

$$((7-4)+4)=(3+4)=7$$

Figure 1.5.3



Separating a group of four from a group of seven results in a group of three.

Image Description

A large dashed line circle with three solid line squares labelled one through three and four circles labelled four through seven within. To the right, a minus symbol. To the right, a smaller dashed line rectangle with four circles within, labelled four through seven, to the right of the equals sign. To the right, a large dashed line circle with four solid line squares on the inside, labelled one through three.

Division

The opposite of multiplication is *division*. Division takes a group and asks, "If I split this group of items into a number of equal parts, how much will be in each of those parts?" For example, if there are three people who are sharing a pizza with twelve slices, we can ask how many pieces each of us should get so everyone gets the same number of slices. Doing this is an application of division, which we represent with the symbol $\div$.

For example, the expression $(12 \div 3)$ is shorthand for saying one of two things:

- "The total number of items in a part when twelve is broken into three equal parts."

- "the number X such that 12 = $((X+X)+X) = 3 \times X$"

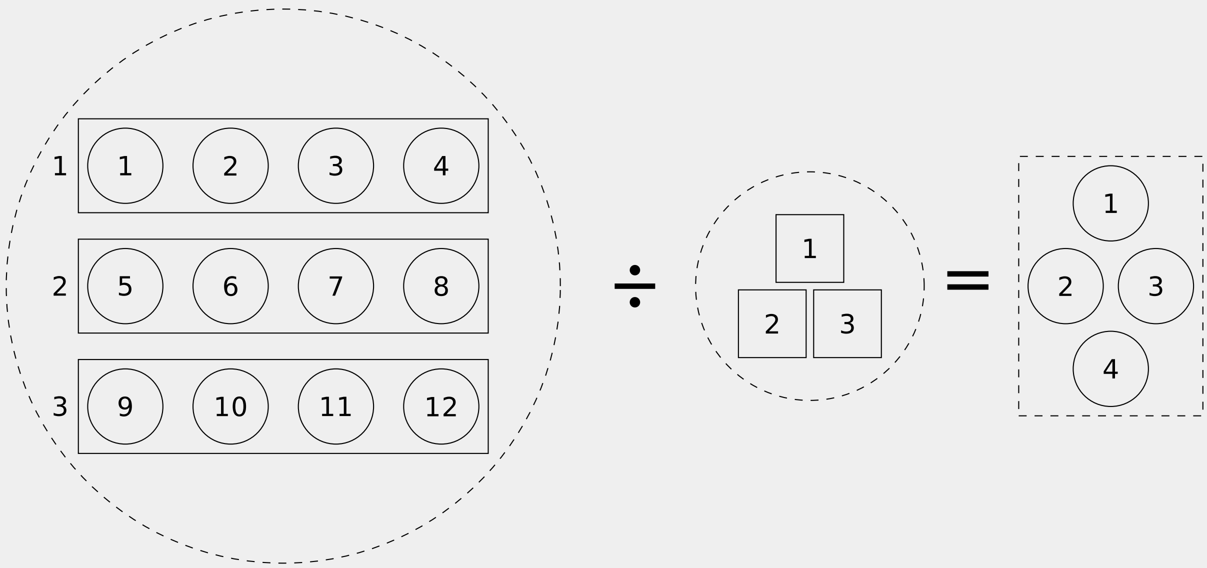
When we divide two numbers, we call their result the "quotient" of those two numbers.

Notice that this second definition is quite a bit different from the other operations because it uses a variable that we have to 'solve' for. Solving for a variable means finding a number that you can substitute for X to make both sides of the equation have the same value.

The computation of $(12 \div 3) = 4$ is visualized in Figure 1.5.4 below. We stretch the three squares that are dividing the group of twelve circles to contain four circles each, which splits all twelve into three equal parts. Notice that this resembles Figure 1.2.1, where $(4 \times 3) = 12$ was visualized, and that both of these equations involve the same three numbers. Moreover, we can see that.

$$((12 \div 3) \times 3) = (4 \times 3) = 12$$

Figure 1.5.4



A group of twelve can be split into three equal parts of size four.

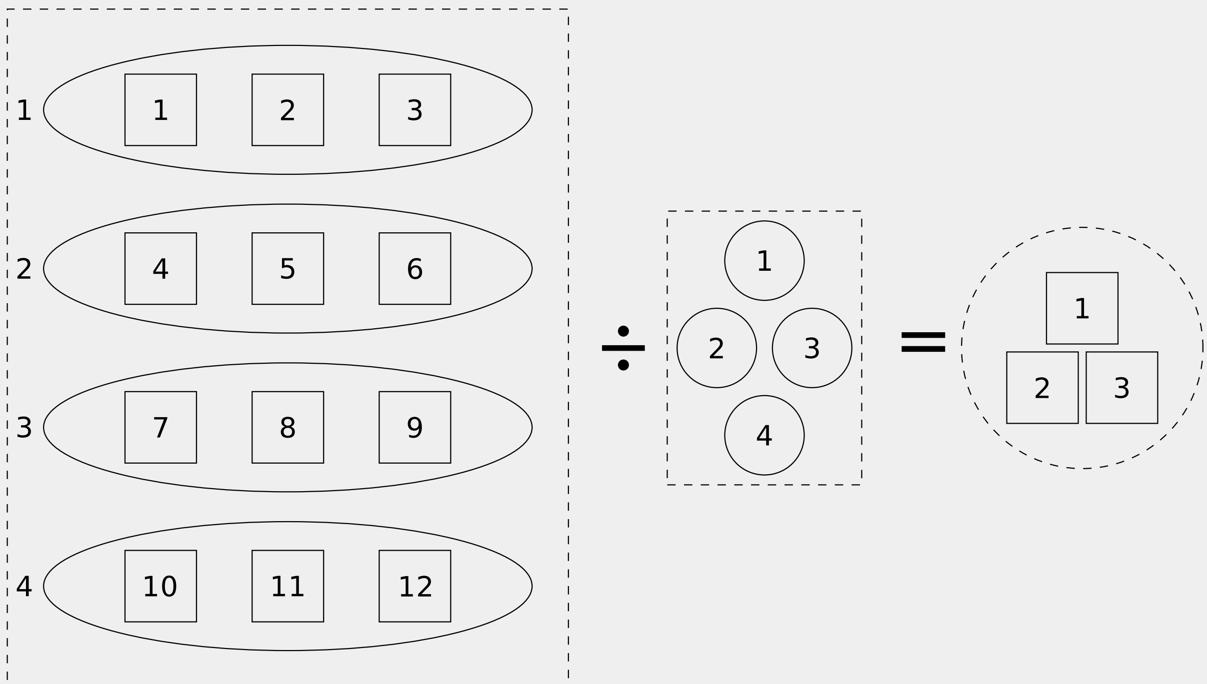
Image Description

A large dashed line circle with three wide rectangles within stacked top to bottom. Each rectangle is labelled 1 through 3, and each rectangle has four solid-line circles within, labelled 1 through 4, 5 through 8, and 9 through 12. To the right is a division symbol. To the right is a small dashed circle with three squares within, labelled 1 through 3. To the right is an equals symbol. To the right is a vertically long rectangle with four circles inside labelled one through four.

Likewise, the computation of $(12 \div 4) = 3$ is visualized in Figure 1.5.5. It looks similar to the computation of $(3 \times 4) = 12$ in Figure 1.2.2. The same three numbers appear in both equations, and

$$((12 \div 4) \times 4) = (3 \times 4) = 12$$

Figure 1.5.5



A group of twelve can be split into four equal parts of size four.

Image Description

Large dashed line rectangle with four labelled ovals. Within each oval are three squares. Oval one has squares one through three, oval two has squares four through six, oval three has squares seven through nine, and oval four has squares 10 through twelve. To the right is the division symbol. To the right is a smaller rectangle with four circles labelled 1 through 4. To the right is an equals sign. To the right, a dashed line circle with three squares within, labelled 1 through 3.

Radicals

The opposite of exponentiation is taking a *radical*. A radical takes a group and asks, "What part of the group would I need to start with and repeatedly multiply by to get back to the total?" For example, if a cold started to spread three days ago at a school, and there are now a total of eight people sick, you could ask what the rate of spread is. Doing this is an application of roots, which we represent with the symbol $\sqrt{}$.

For example, the expression $(\sqrt[3]{8})$. is shorthand for saying one of two things:

- "the number multiplied by itself three times to result in a total of eight," or
- "the number X such that $8 = X \times X \times X = X^3$.

The result of the radicalization of one number by another is called a "root".

Like division, the description of this operation involves a variable X that we are solving for. This makes taking a radical very tricky.

The computation of $(\sqrt[3]{8}) = 2$ is visualized in Figure 1.5.6 below. Since we are working with a radical of three, fit a group of eight into a cube shape. A cube with side lengths of two has enough capacity for all eight, so the solution is two. Notice that this resembles Figure 1.3.1, where $(2^3) = 8$ was visualized, and that both of these equations involve the same three numbers. Moreover, we can see that.

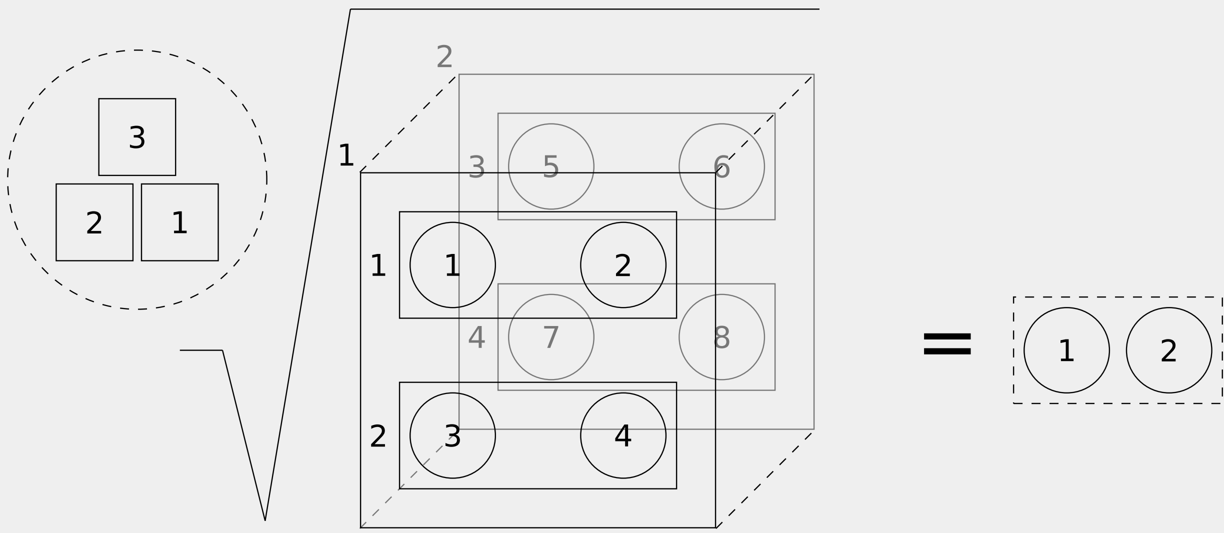
$$((\sqrt[3]{8})^3) = (2^3) = 8$$

When we take a number to a radical of three, we sometimes call it the *cube root* of that number. Likewise, the computation of $(\sqrt[2]{9}) = 3$ can be visualized in Figure 1.5.7. It looks similar to the computation of $(3^2) = 9$ in Figure 1.3.2. The same three numbers appear in both equations, and

$$((\sqrt[2]{9})^2) = (3^2) = 9$$

When we take the radical of a number to two, we call that the *square root* of that number. As a shortcut, instead of writing $\sqrt[2]{}$, we will just write $\sqrt{}$.

Figure 1.5.6

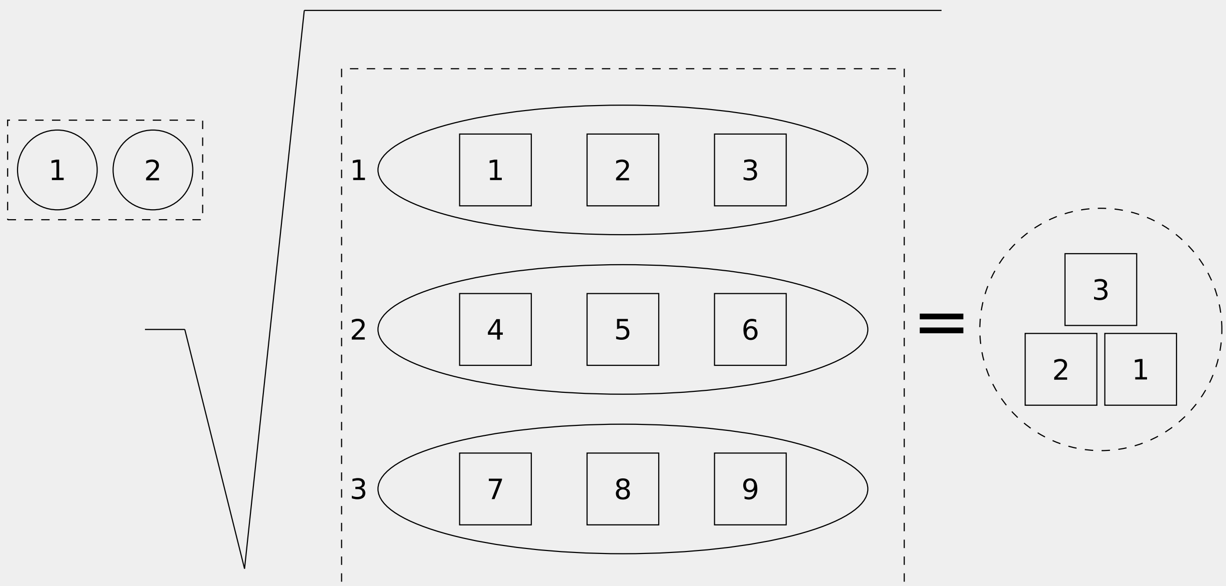


A group of eight can be constructed in three dimensions using a cube of side length two.

Image Description

Diagram showing a grouped set of three numbered squares (1, 2, 3) inside a dashed circle, to the right of a 3D box containing eight numbered circles (1–8), the entirety of this part under a division symbol. An equals sign highlights a selected subset, shown as circles 1 and 2 inside a dashed rectangle.

Figure 1.5.7



A group of nine can be constructed in two dimensions using a square of side length three.

Image Description

Diagram showing a grouped set of two numbered circles inside a dashed rectangle, to the right of a dashed line rectangle containing three numbered ovals (one to three), each with three numbered squares within (one to three, four to six, seven to nine), the entirety of this part under a division symbol. An equals sign highlights a selected subset, shown as squares one through three inside a dashed circle.

Exercises 1.5

1. If the sum of 4 and an unknown number X is 9, then what is X?

2. If the product of 3 and an unknown number Y is 21, then what is Y?

3. If a number Z is squared to get 64, what is Z?

1.6 ORDER OF OPERATIONS

PE(DM)(AS)

1. **P**arentheses: compute expressions inside of parentheses first.
2. **E**xponents: compute all exponents next.
3. (**D**ivision and **M**)ultiplication: compute all division and multiplication operations in the expression, starting from left to right, and
4. (**A**ddition and **S**)ubtraction: compute all additions and subtractions that are left over, starting from left to right.

Figure 1.6.1: An explanation of the acronym **PE(DM)(AS)**

So far in this text, we have been fairly careful to use parentheses around every operation to give an idea of what expression should be evaluated first. Thus, to compute expression $((4 \times (2+4)) \div 4)$, it would go like:

$$\begin{aligned} ((4 \times (2+4)) \div 4) &= ((4 \times 6) \div 4) \\ &= (24 \div 4) \\ &= 6 \end{aligned}$$

If we rewrote the expression without the parentheses, it would be less clear how we should compute it. Also, it could change our answer! For example, if we rewrote the parentheses so that we do the operations from left to right, i.e., $(((4 \times 2) + 3) \div 4)$, we would compute:

$$\begin{aligned} (((4 \times 2) + 4) \div 4) &= ((8 + 4) \div 4) \\ &= (12 \div 4) \\ &= 3 \end{aligned}$$

These two computations have different results, so the order in which we compute an expression matters! But writing parentheses several times for each operation in an expression can be a lot to remember, and can also make the equation hard to read.

Due to this, there is a conventional order in which to compute an expression that does not have brackets around every operation. This order can be remembered by the acronym **PE(DM)(AS)**, which is explained in Figure 1.15.

As an example of how to apply **PE(DM)(AS)**, let's compute the expression $4 \times 2 + 4 \div 4$, which has no parentheses. The computation is shown below, computing one operation at a time on each line, with the arrows on the right showing what was computed on that line.

$$\begin{aligned} & 4 \times 2 + 4 \div 4 && \rightarrow 4 \times 2 \\ = \; & 8 + 4 \div 4 && \rightarrow 4 \div 4 \\ = \; & 8 + 1 && \rightarrow 8 + 1 \\ = \; & 9 && \end{aligned}$$

Notice how we have again used the same set of numbers and operations as before, but achieved a new result based on the order in which we compute the operations in the expression.

Let's try one more example, $4 \times (3-1)^2 + 5 - 1$, which has parentheses and exponents:

$$\begin{aligned} & 4 \times (3-1)^2 + 5 - 1 && \rightarrow 3 - 1 \\ = \; & 4 \times 2^2 + 5 - 1 && \rightarrow 2^2 \\ = \; & 4 \times 4 + 5 - 1 && \rightarrow 4 \times 4 \\ = \; & 16 + 5 - 1 && \rightarrow 16 + 5 \\ = \; & 21 - 1 && \rightarrow 21 - 1 \\ = \; & 20 && \end{aligned}$$

Exercises 1.6

Evaluate the following expressions using the order of operations, i.e., **PE(DM)(AS)**:

1. $((3-1)\times 2)^2 \div 2$

2. $(10+5)\div(3+3-1)$

3. $(10-4\times 2-1)^3$

7. $2\times(10+2)\div 3$

8. $5+2^4\times 3$

9. $(3\times(2+1))-3^2$

10. $1+16\div(8\div(2\times 4))$

11. $(2\times 7)-2^3$

12. $3\times 4-10\div 2+2\times 3$

2 INTEGERS

Chapter Outline

What is $(1-2)$? At first, it might seem like a silly question to ask, since one way of thinking about subtraction is 'separating part of a group from itself'. So, in the expression $(1-2)$, we are asked to separate two items from a group of only one item. Since this group is not big enough to contain a part with two items, we might think that this question does not make any sense. But, in real life, we do see situations where this arises. Think of money, where you might pay only $\$1$ for a $\$2$ item, in which case, you owe $\$1$. Similarly, if you are working on a project that needs two tonnes of cement, but you only have one tonne at the moment, then you are missing one tonne. If we can create a number that represents 'owing' or 'missing' something, then we can try to compute the expression $(1-2)$.

2.0 LEARNING OBJECTIVES

Learning Objectives

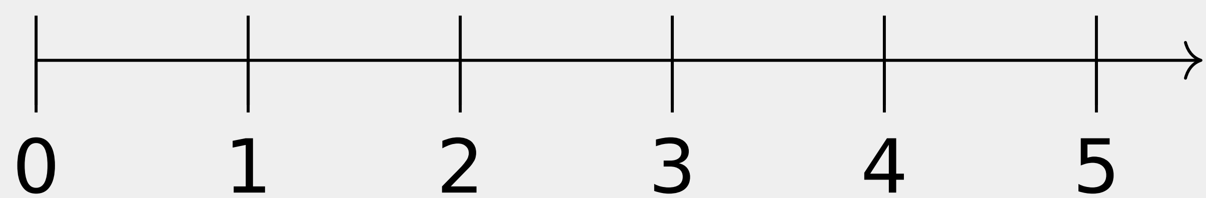
By the end of this chapter, students will be able to:

- Visualize negative numbers using the number line (2.1)
- Compute basic mathematical operations on integers (2.2)

2.1 THE NUMBER LINE

Picture all of the whole numbers as being placed along a line, starting at zero and getting bigger as you go to the right, as shown in Figure 2.1.1. The arrow at the right-hand end of the line indicates that the line continues on without ever ending. We call this the *positive number* line.

Figure 2.1.1



The positive number line.

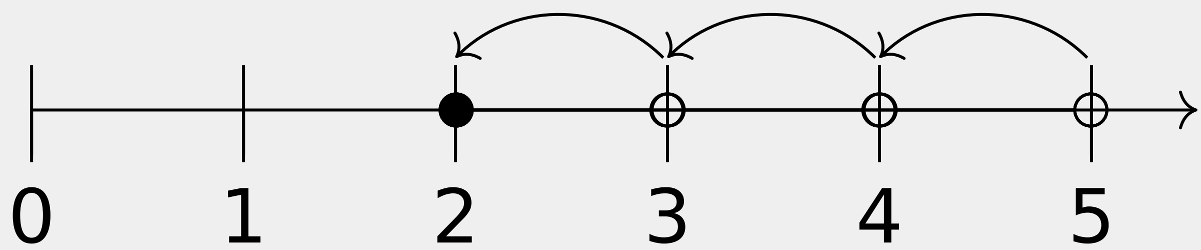
Image Description

A horizontal number line from 0 to 5 with evenly spaced tick marks at each whole number and an arrow extending to the right beyond 5.

When we find the difference of two numbers, say five and three, we can use the number line to visualize what is happening. First, we place a marker at five's place on the line. Next, we move that marker down the number line by shifting it three times to the left, as shown in Figure 2.1.2. To use an analogy, if each number is a house and the number line is a street, subtraction tells us to walk down the street a certain number of houses.

Now, if we use this same strategy but return to the expression $(1 - 2)$, notice that once our walk brings us to house zero, we are at the end of the street on our old number line, as shown in Figure 2.1.3. We would like to keep walking to the next house, if there was one, but on this number line, there is nowhere else to go.

Figure 2.1.2

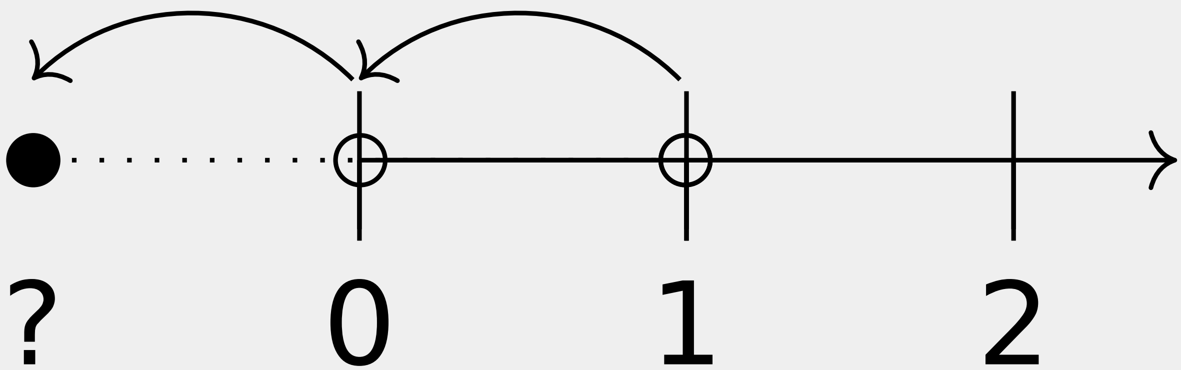


Visualizing $(5-3)$ on the positive number line.

Image Description

A horizontal number line from 0 to 5. The point at 2 is shown with a filled black circle, while 3, 4, and 5 are shown with open circles. Curved arrows above the line show jumps from 5 to 4, from 4 to 3, and from 3 to 2.

Figure 2.1.3



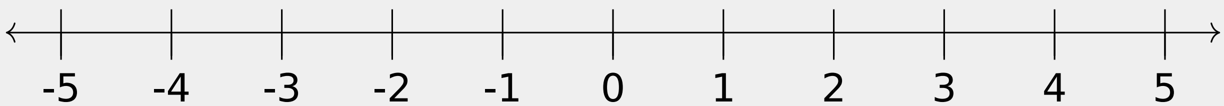
An attempt at $(1 - 2)$ on the positive number line.

Image Description

A horizontal number line showing 0, 1, and 2, with an arrow extending to the right. Open circles mark 0 and 1. Curved arrows above the line show two backward jumps: one from 1 to 0 and another from 0 to an unknown value to the left. A filled black circle appears to the left of 0 with a question mark below it, connected to 0 by a dotted line.

We can fix this if we imagine our number line to be a street that is infinitely long in both the left and right directions. Moreover, we can make the left side of the street a mirror image of the right side by using the 'negative' version of each positive number, which we get by placing a negative sign in front of it. So, since the number one is to the right of zero, the number *negative* one is to the left of zero. We continue like this by increasing the size of the negative numbers as we continue infinitely down to the left. This *number line* is visualized in Figure 2.1.4.

Figure 2.1.4



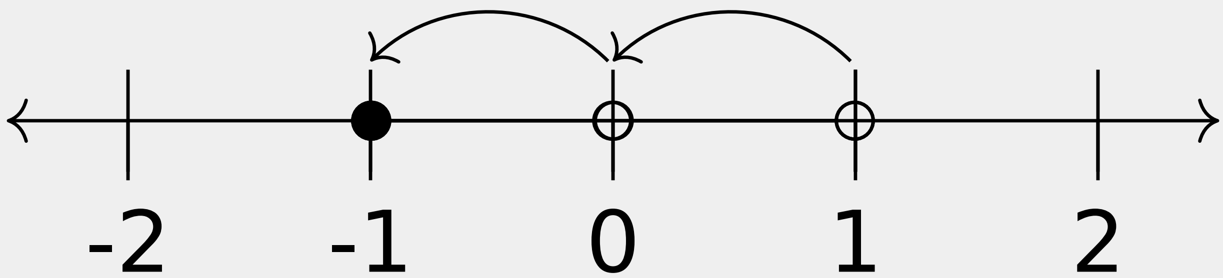
The number line.

Image Description

A horizontal number line from -5 to 5 with evenly spaced tick marks at each integer and arrows extending in both directions.

Now, if we try to use the number line to compute the expression $(1-2)$ again, we have somewhere to walk to after passing zero, as shown in Figure 2.1.5. Thus, $(1-2)=-1$.

Figure 2.1.5



Computing $(1-2)$ on the number line.

Image Description

A horizontal number line from -2 to 2 with arrows extending in both directions. The point at -1 is shown with a filled black circle, while 0 and 1 are shown with open circles. Curved arrows above the line show backward jumps from 1 to 0 and from 0 to -1.

2.2 RULES FOR OPERATIONS WITH NEGATIVE NUMBERS

The set of all the numbers on the number line, i.e., the set of all positive and negative numbers, is called the integers. Using operations on the set of integers is similar to the positive whole numbers, but there are a couple of things to be careful about.

1. *Adding a negative number*: instead, subtract the positive, e.g., $(10 + (-2)) = (10 - 2)$.
2. *Subtracting a negative number*: instead, add the positive, e.g., $(10 - (-2)) = (10 + 2)$.
3. *Multiplying a negative number*: instead, multiply by the positive and make the product
   negative, e.g.,
   $(10 \times (-2)) = -(10 \times 2)$.
4. *Dividing a negative number*: instead, divide by the positive and make the quotient negative.
   e.g.,
   $(10 \div (-2)) = -(10 \div 2)$.
5. Multiplying two negative numbers: instead, multiply by the positives, e.g., $((-10) \times (-2)) = (10 \times 2)$.
6. Dividing two negative numbers: instead, divide by the positives, e.g., $((-10) \div (-2)) = (10 \div 2)$.

Exercises 2.2

Type your exercises here.

1. $6 - 9$
2. $(-9) + 6$
3. $6 + (-9)$
4. $(-5) \times 2$
5. $5 \times (-2)$
6. $(-5) \times (-2)$
7. $12 \div (-3)$
8. $(-2) - (-1)$
9. $(-3) \times (-6)$
10. $(-9) - 1$
11. $(-4) \div 2$
12. $3 \times (-4) + (-2)$

3 FRACTIONS

Chapter Outline

What is $(3 \div 2)$? We are asking to break up a group of three items into two equal parts. That seems impossible, just like subtracting a larger number from a smaller one. But, in real life, we somehow manage to do this all the time. For example, if I have one pizza, I can cut it into any number of equal slices to share with someone else. So, if we try to imagine that whole numbers can be sliced into part numbers like a pizza, we can get new numbers that we call fractions.

3.0 LEARNING OBJECTIVES

Learning Objectives

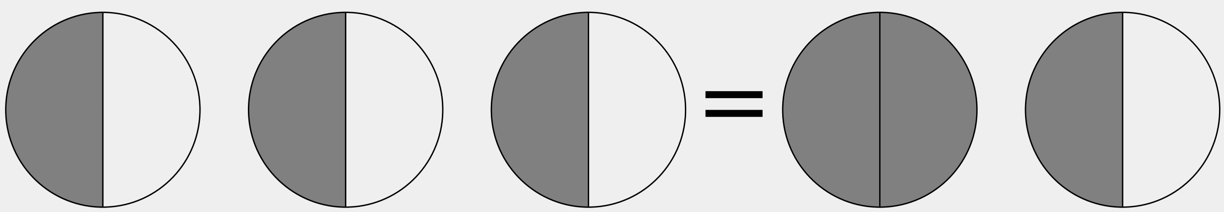
By the end of this chapter, students will be able to:

- Visualize fractions using an evenly sliced circle (3.1)
- Convert between improper and mixed fractions and simplify them (3.2-3)
- Compute basic operations on fractions (3.4-5)
- Rewrite ratios and percentages as fractions (3.6)

3.1 VISUALIZING FRACTIONS

Using the pizza example, let's imagine that every whole number can be represented by a number of circles. So, the number three is given by three circles. Then, division is just splitting each circle into the number of parts that we are dividing by, so, in $(3 \div 2)$, we divide each of the three circles into two parts. Lastly, to find our result, we gather all of the parts to reconstruct as many circles as we can. Thus, the three halves we get from $(3 \div 2)$ can be reassembled into one whole circle and one half circle, as shown in Figure 3.1.1.

Figure 3.1.1



Representing $(3 \div 2)$ using three circles split in half and recombining them.

Image Description

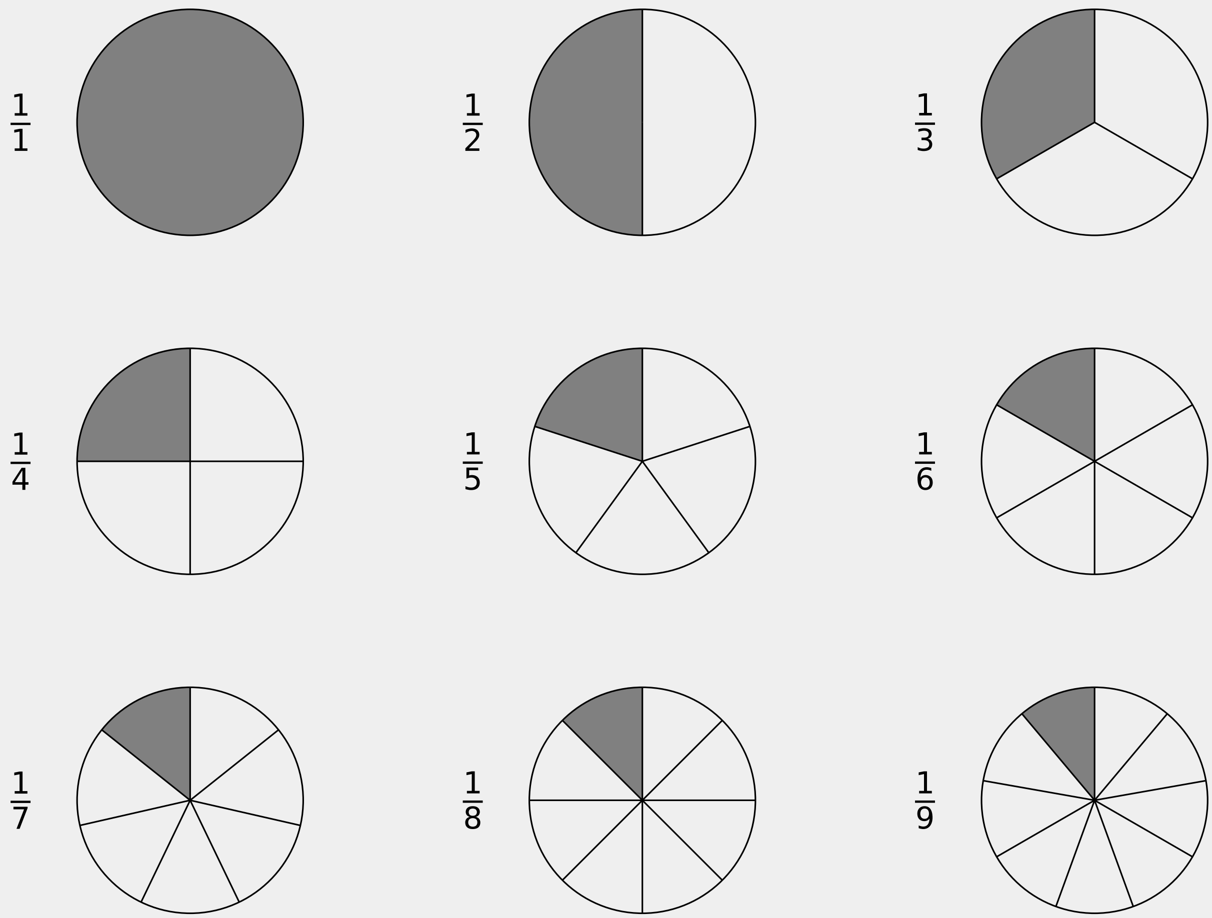
Five circles are divided into equal halves. The first three circles each have the left half shaded grey and the right half unshaded. An equals sign separates these from the final two circles, where the fourth circle is fully shaded grey, and the fifth circle has the left half shaded grey and the right half unshaded.

We can split a whole into any number of parts that we want. To represent the size of that, we write

$$\frac{1}{N}$$

to represent an N^{th} of a whole, where N is a number. We call these numbers *fractions*. The sizes of the first nine fractions are shown in Figure 3.1.2.

Figure 3.1.2



The first nine fractions.

Image Description

The image shows nine fraction models arranged in a three-by-three grid. Each model is a circle divided into equal parts, with one portion shaded grey to represent the fraction shown beside it.

Top row, from left to right:

The first circle is labelled 1/1 and is fully shaded grey. The second circle is labelled 1/2 and is divided into two equal halves, with one half shaded grey. The third circle is labelled 1/3 and is divided into three equal sections, with one section shaded grey.

Middle row, from left to right:

The fourth circle is labelled 1/4 and is divided into four equal sections, with one section shaded grey. The fifth circle is labelled 1/5 and is divided into five equal sections, with one section shaded grey. The sixth circle is labelled 1/6 and is divided into six equal sections, with one section shaded grey.

Bottom row, from left to right:
The seventh circle is labelled 1/7 and is divided into seven equal sections, with one section shaded grey. The eighth circle is labelled 1/8 and is divided into eight equal sections, with one section shaded grey. The ninth circle is labelled 1/9 and is divided into nine equal sections, with one section shaded grey.

In situations where we have multiples of the same-sized part, we can use the top half of the fraction to represent that. For example, the fraction

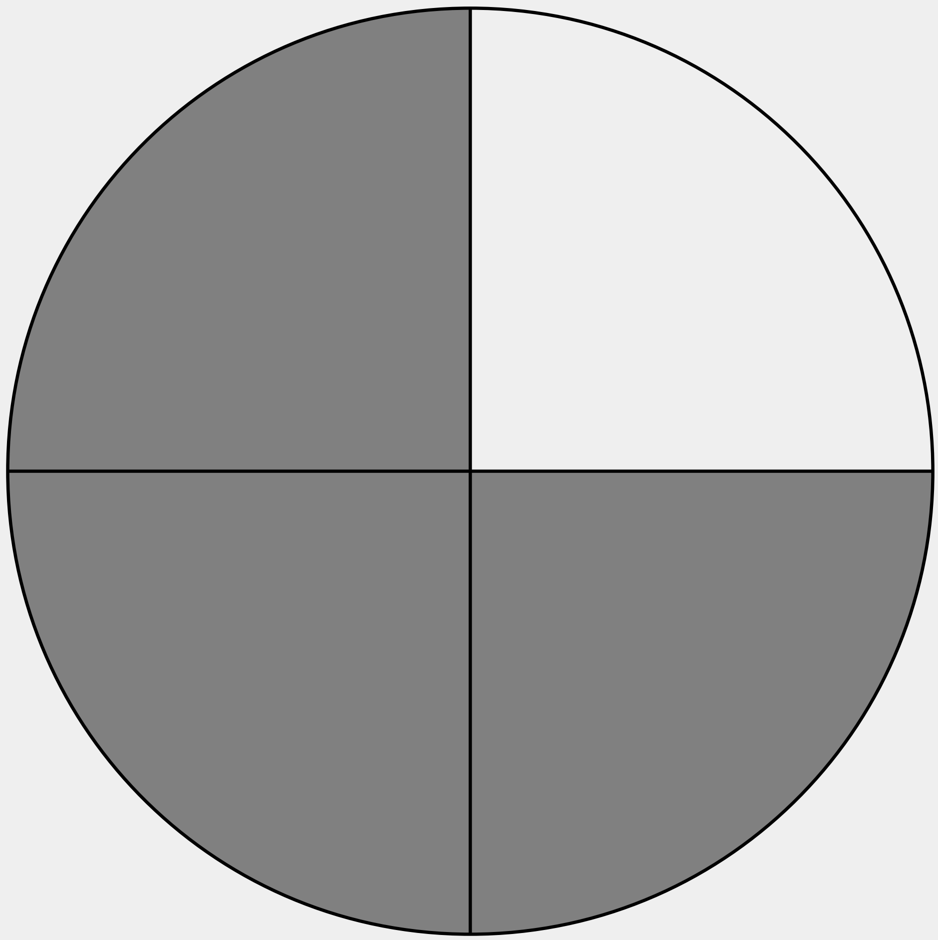
$$\frac{3}{4}$$

represents three $\frac{1}{4}$-sized parts, as shown in Figure 3.1.3.

The top half of a fraction is called the *numerator*, and the bottom half of the fraction is called the *denominator*. The denominator tells you what size the parts are, and the numerator tells you how many parts you have. To write a fraction using words, you write the value of the numerator followed by the numerator with a '-th ' at the end. For example, $\frac{3}{4}$ is written 'three fourths'. The only exception to the '-th ' rule is when the denominator is 2; in that case, you say how many 'halves' there are.

Figure 3.1.3

$\frac{3}{4}$



A visualization of $\frac{3}{4}$

Image Description

The image shows a large circle divided into four equal sections by one vertical line and one horizontal line crossing at the centre. The fraction 3/4 appears to the left of the circle. Three of the four sections are shaded grey: the top-left, bottom-left, and bottom-right sections. The top-right section is unshaded. This represents the fraction three-fourths.

Exercises 3.1

Draw the following fractions as shaded in parts of a circle:

1. $\frac{3}{8}$

2. $\frac{4}{9}$

3. $\frac{2}{5}$

4. $\frac{5}{8}$

5. $\frac{3}{6}$

6. $\frac{3}{12}$

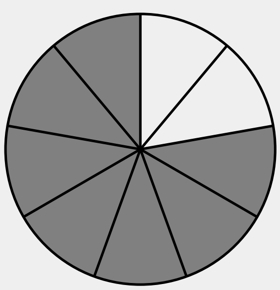
7. $\frac{1}{6}$

8. $\frac{10}{12}$

9. $\frac{6}{7}$

Write the fraction represented by the shaded parts of the circle:

10.



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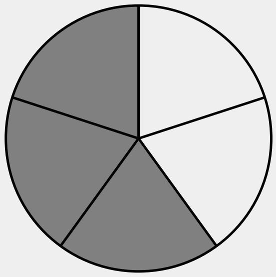
11.



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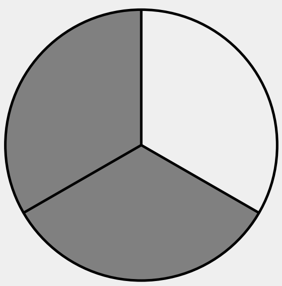
12.



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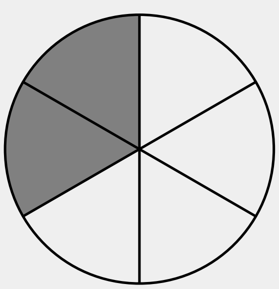
13.



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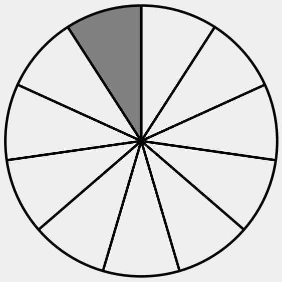
14.



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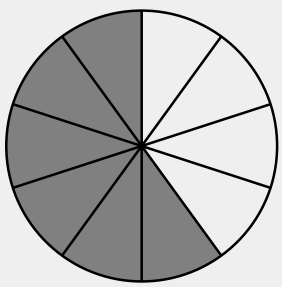
15.



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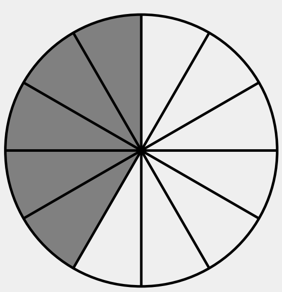
16.



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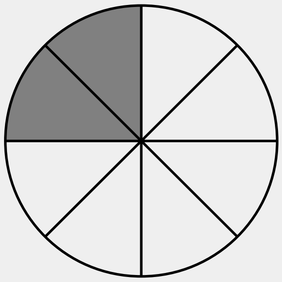
17.



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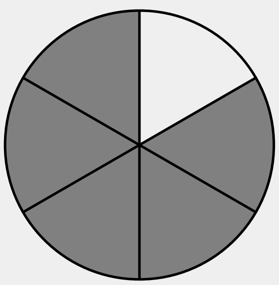
18.



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3.2 IMPROPER AND MIXED FRACTIONS

When the numerator is *smaller* than the denominator, we call it a *proper fraction*. On the other hand, if the numerator is the *same size or larger* than the denominator, we call it an *improper fraction*. For example, the fraction

$$\frac{5}{4}$$

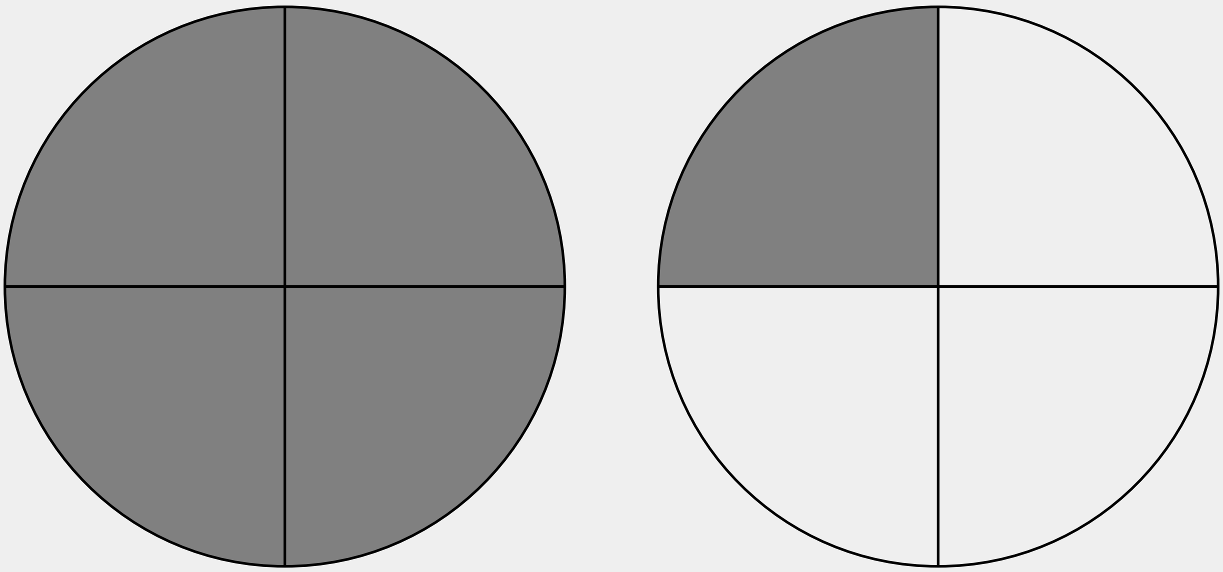
is an improper fraction, and it can be visualized in Figure 3.2.1. Another way to think of an improper fraction is that it is a fraction where there are enough parts to make at least one whole number.

An improper fraction can also be thought of as a whole number plus a proper fraction, i.e.,

$$\frac{5}{4} = 1\frac{1}{4}$$

When a number is written as a combination of a whole number and a fraction, it is called a *mixed number*. To write a mixed fraction in words, you start by (1) writing the whole number in words, (2) writing *and*, then (3) writing the simple fraction in words. For example, $1\frac{1}{4}$ is written 'one and one fourth'.

Figure 3.2.1



A visualization of $\frac{5}{4}$, or $1\frac{1}{4}$, or 'one and one fourth'.

Image Description

The image shows two large circles side by side. Each circle is divided into four equal sections by one vertical line and one horizontal line crossing at the centre.

The circle on the left has all four sections shaded grey, representing one whole or 4/4. The circle on the right has only the top-left section shaded grey, while the remaining three sections are unshaded, representing 1/4. Together, the image shows one whole and one-fourth.

To convert an improper fraction into a mixed number, you check to see what is the greatest number that you can multiply the denominator by without getting larger than the numerator. We call this number the quotient. Then, take the difference of the numerator and the denominator times the quotient. This is called the remainder. The process for this is summarized in the box below.

Converting an Improper Fraction to a Mixed Fraction

1. Calculate the quotient by finding the largest number you can multiply the denominator by without getting larger than the numerator.
2. Calculate the remainder by finding the product of the quotient from step 1 and the denominator, then subtract that product from the numerator.
3. Write the quotient followed by the simple fraction with the remainder as the numerator and the denominator staying the same.

For example, let's convert $\frac{8}{3}$ into a mixed fraction:

1. The numerator is 8 and the denominator is 3. $2 \times 3 = 6$ which is smaller than 8. On the other hand, $3 \times 3 = 9$ which is larger than 8. So, the quotient is 2.
2. The product of the quotient and the denominator is $2 \times 3 = 8$, so taking the difference of this product with the numerator, we get $8 - 6 = 2$. So, the remainder is 2.
3. Using the fact that the quotient is 2 and the remainder is 2, $\frac{8}{3} = 2\frac{2}{3}$.

On the other hand, to convert a mixed number into a fraction, you take the number of wholes you have and find the product of this and the denominator of the fraction. Then, add together this product and the numerator, and set it over the denominator. If you compare this closely to the instructions for converting an improper fraction into a mixed number, they are essentially opposite processes.

Converting a Mixed Fraction to an Improper Fraction

1. Calculate the product of the whole number and the denominator.
2. Calculate the sum of the product from step 1 and the numerator.
3. Write a new fraction with this sum as the numerator, keeping the denominator the same.

For example, let's convert $2\frac{2}{3}$ into an improper fraction:

1. The product of the whole and the denominator is $2 \times 3 = 6$.
2. The sum of this product and the numerator is $6 + 2 = 8$.
3. Using these results, we see that $2\frac{2}{3} = \frac{8}{3}$

Exercises 3.2

Convert the following improper fractions into mixed fractions:

1. $\dfrac{11}{3} =$

2. $\dfrac{9}{2} =$

3. $\dfrac{7}{5} =$

4. $\dfrac{10}{4} =$

5. $\dfrac{9}{6} =$

6. $\dfrac{14}{7} =$

7. $\dfrac{13}{9} =$

8. $\dfrac{16}{3} =$

9. $\dfrac{24}{11} =$

Convert the following mixed fractions into improper fractions:

10. $1\dfrac{4}{5} =$

11. $3\dfrac{3}{4} =$

12. $2\dfrac{2}{5} =$

13. $4\dfrac{2}{3} =$

14. $5\dfrac{1}{2} =$

15. $3\dfrac{4}{7} =$

16. $2\dfrac{4}{9} =$

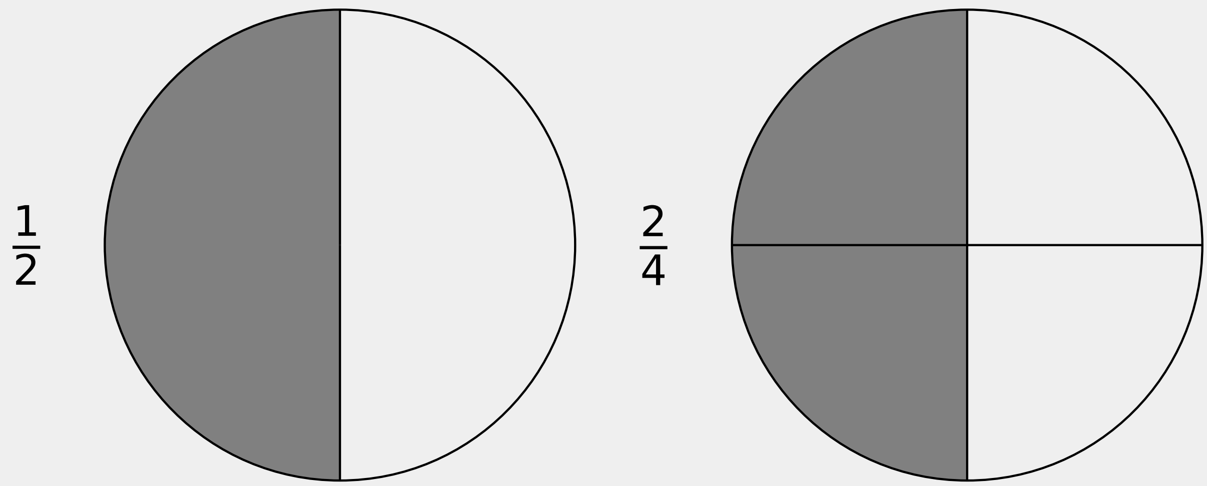
17. $1\dfrac{7}{8} =$

18. $2\dfrac{3}{6} =$

3.3 SIMPLIFYING FRACTIONS

Since we can slice a whole into as many parts as we would like to, we might come across fractions that look different but are actually the same value. For example, is there a difference between one-half vs. two-fourths? In Figure 3.3.1, we see a visualization of $\frac{1}{2}$ and $\frac{2}{4}$. Although one circle is split into pieces that are half as small, there are also double the amount of parts. We can also tell that both circles have the same-sized shaded areas.

Figure 3.3.1



A visualization of the fact that $\dfrac{1}{2} = \dfrac{2}{4}$

Image Description

The image shows two fraction models side by side. On the left, a large circle is divided into two equal halves by a vertical line. The left half is shaded grey, and the right half is unshaded. The fraction 1/2 appears to the left of this circle.

On the right, a second circle is divided into four equal sections by one vertical line and one horizontal line crossing at the centre. The two sections on the left side are shaded grey, while the two sections on the right side are unshaded. The fraction 2/4 appears to the left of this circle.

Together, the image shows that 1/2 and 2/4 represent the same amount.

The reason for this is that in the second fraction, $\dfrac{2}{4}$, the numerator and denominator share a common *factor*. In other words, both 2 and 4 are divisible by the same number. In this case, their common factor is 2. In situations where a numerator and denominator share a factor, you can divide the numerator and denominator by that factor without changing the value of the fraction.

When the numerator and denominator of a fraction have no common factors, we say that the fraction is *simplified*. To simplify a fraction, you must find the *greatest common factor* of the numerator and denominator, and divide both by that factor. After doing this, the fraction is simplified.

Simplifying Fractions

1. Determine the greatest common factor of the numerator and the denominator.
2. Divide the numerator and the denominator by their greatest common factor.
3. Rewrite the fraction using the results of your divisions in step 2.

For example, we follow these steps to simplify $\frac{12}{18}$:

1. $12 = 2 \times 2 \times 3$ and $18 = 2 \times 3 \times 3$, so their greatest common factor is $2 \times 3 = 6$.
2. $12 \div 6 = 2$ and $18 \div 6 = 3$.
3. So, $\frac{12}{18} = \frac{2}{3}$.

The same process applies to improper fractions. For instance, $\frac{15}{10} = \frac{3}{2}$ since 5 is the greatest common factor of 15 and 10, and $15 \div 5 = 3$ and $10 \div 5 = 2$.

Likewise, the mixed fractions can be simplified by focusing on only their fraction part. For instance, $1\frac{5}{10} = 1\frac{1}{2}$ since 5 is the greatest common factor of 5 and 10, and $5 \div 5 = 1$and $10 \div 5 = 2$.

Exercises 3.3

Simplify the following fractions:

1. $\frac{4}{12} =$

2. $\frac{6}{9} =$

3. $\frac{8}{10} =$

4. $\frac{14}{7} =$

5. $\frac{18}{4} =$

6. $\frac{25}{10} =$

7. $2\frac{6}{12} =$

8. $5\frac{8}{12} =$

9. $3\frac{4}{18} =$

3.4 ADDITION AND SUBTRACTION OF FRACTIONS

We started to look at fractions because we wanted to understand what happens when we try to divide a number of items into equal parts, but there is no way to do this only using whole numbers. Now that we have reviewed fractions on their own, we can start to look at how fractions are used in operations.

Adding fractions that have the same denominator is intuitive and simple. For example, let's try to compute:

$$\frac{2}{3}+\frac{5}{3}$$

We remind ourselves that the expression is asking 'what is the sum of two-thirds plus five-thirds?' The answer in this case is just the total number of thirds that we have. Since the numerators tell us how many parts belong to each fraction, we look to the numerators of each and add them together. So,

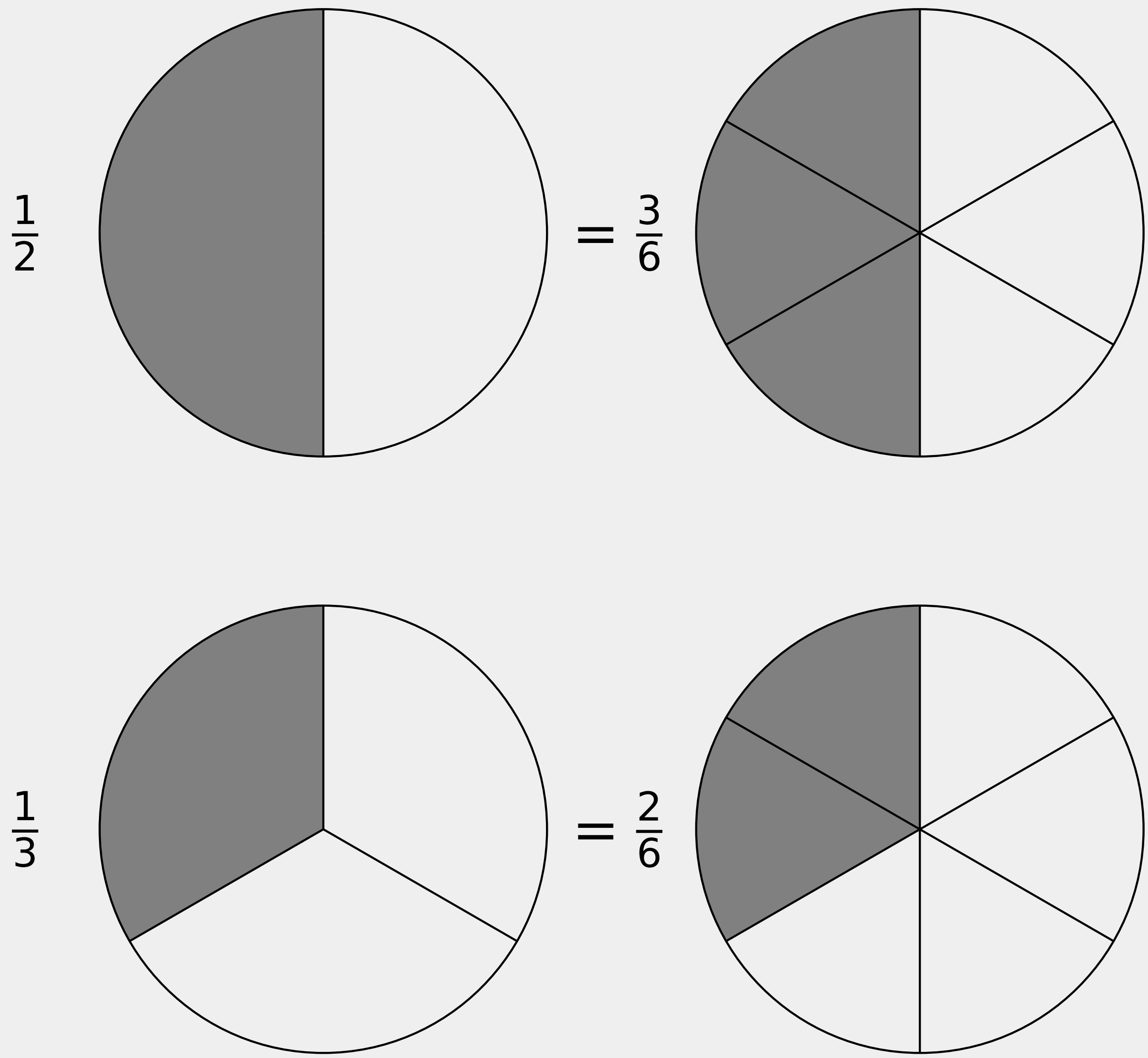
$$\frac{2}{3}+\frac{5}{3}=\frac{(2+5)}{3}=\frac{7}{3}=2\frac{1}{3}$$

But suppose we have two fractions with different denominators, and we want to know their sum. For example, the expression:

$$\frac{1}{2}+\frac{1}{3}$$

This expression is asking 'what is the sum of one half plus one third?' Since one-half and one-third are different sizes, we get stuck here, because there is no easy way to tell how many halves are in one-third, nor how many-thirds are in one-half. However, if we split one half into three parts, and one third into two parts, we end up with parts that are all the same size! This idea is shown in Figure 3.4.1.

Figure 3.4.1



Splitting $\frac{1}{2}$ and $\frac{1}{3}$ into parts of the same size.

Image Description

The image shows two examples of equivalent fractions using circle models.

In the top example, the fraction 1/2 is shown on the left with a circle divided into two equal halves. The left half is shaded grey, and the right half is unshaded. An equals sign connects it to the fraction

3/6 on the right, which is shown with a circle divided into six equal sections. Three of the six sections on the left side are shaded grey, showing that 1/2 is equal to 3/6.

In the bottom example, the fraction 1/3 is shown on the left with a circle divided into three equal sections. One section on the left side is shaded grey, and the other two sections are unshaded. An equals sign connects it to the fraction 2/6 on the right, which is shown with a circle divided into six equal sections. Two of the six sections on the left side are shaded grey, showing that 1/3 is equal to 2/6.

Now it is easy to find the sum since we can convert our two original fractions into equivalent ones with a common denominator.

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{(3+2)}{6}=\frac{5}{6}$$

Adding Fractions

1. Identify the least common of the denominators in each fraction.
2. Convert each fraction into an equivalent one by multiplying the top and bottom of each fraction by a factor that sets the denominator of each fraction equal to the least common multiple found in step 1.
3. Add the numerators of the fractions converted in step 2 together and set them over the least common multiple from step 1.

For example, let's compute the sum $\frac{5}{6}+\frac{3}{8}$:

1. The least common multiple of 6 and 8 is 24.
2. Using the least common multiple, we find the factor necessary to convert each fraction and then convert it to the new denominator. $4 \times 6 = 24$ so

$$\frac{5}{6} = \frac{4 \times 5}{4 \times 6} = \frac{20}{24}$$

and $3 \times 8 = 24$ so

$$\frac{3}{8} = \frac{3 \times 3}{3 \times 8} = \frac{9}{24}$$

3. Lastly, we add the numerators of the converted fractions together:

$$\frac{20}{24} + \frac{9}{24} = \frac{20 + 9}{24} = \frac{29}{24} = 1\frac{5}{24}$$

To subtract fractions, it is the exact same process, except in step 3, you take the difference of the numerators instead of their sum.

Subtracting Fractions

1. Identify the least common multiple of the denominators in each fraction.
2. Convert each fraction into an equivalent one by multiplying the top and bottom of each fraction by a factor that sets the denominator of each fraction equal to the least common multiple found in step 1.
3. Subtract the numerators of the fractions converted in step 2 together, and set them over the least common multiple from step 1.

For example, let's compute the sum $\frac{5}{6} - \frac{3}{8}$:

1. The least common multiple of 6 and 8 is 24.
2. Using the least common multiple, we find the factor necessary to convert each fraction and then convert it to the new denominator. $4 \times 6 = 24$ so

$$\frac{5}{6} = \frac{4 \times 5}{4 \times 6} = \frac{20}{24}$$

and $3 \times 8 = 24$ so

$$\frac{3}{8} = \frac{3 \times 3}{3 \times 8} = \frac{9}{24}$$

3. Lastly, we add the numerators of the converted fractions together:

$$\frac{20}{24} - \frac{9}{24} = \frac{20-9}{24} = \frac{11}{24}$$

Exercises 3.4

Compute the following sums and differences, and simplify your result if possible:

1. $\frac{5}{3} + \frac{1}{3}$

2. $\frac{7}{4} + \frac{3}{4}$

3. $\frac{8}{10} - \frac{1}{10}$

4. $\frac{3}{7} + \frac{2}{3}$

5. $\frac{2}{5} + \frac{3}{10}$

6. $\frac{5}{2} - \frac{1}{4}$

7. $1 + \frac{5}{6}$

8. $4\frac{2}{3} + 2\frac{1}{5}$

9. $2\frac{1}{2} - 1\frac{1}{3}$

3.5 MULTIPLICATION AND DIVISION OF FRACTIONS

To round off our study of fractions, let's look at how we should try to apply them within the operations of multiplication and division. Multiplication is an operation that takes the first number of the expression and sums a number of copies of it by the second number in the expression. So, the expression

$$\frac{3}{4} \times \frac{2}{3}$$

is another way of saying 'the amount of two-thirds of a copy of three-fourths'. But what is a two-thirds copy? One way to think of it is to ask yourself what a third copy would look like. Then, once you know that, you just make two of them. But a third of a copy is the same thing as dividing by three. So, the expression can be rewritten like this:

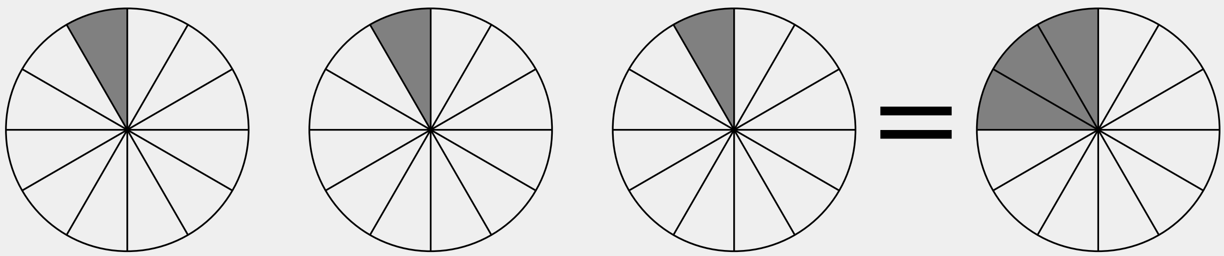
$$\frac{3}{4} \times \frac{2}{3} = \frac{3}{4} \div 3 \times 2$$

This looks more manageable: now all we have to do is find a third of $\frac{3}{4}$ and then double that amount.

To find out what a fraction divided by a whole number is, we can use the same idea that we had with dividing whole numbers by whole numbers, but just start with the fraction of a circle instead of the whole circle.

For $\frac{3}{4} \div 3$, we take three separate fourth circles, split them each into thirds, then recombine them, as shown in Figure 3.5.1. From this graphic, we can see that this expression comes to $\frac{3}{12}$.

Figure 3.5.1



Visualizing $\frac{3}{4} \div 3$ by splitting three fourths into a third of a fourth each.

Image Description

The image shows four circle fraction models arranged in a row. Each circle is divided into twelve equal slices.

The first three circles each have one slice shaded grey and eleven slices unshaded. These three circles are followed by an equals sign.

The fourth circle, after the equals sign, has three adjacent slices shaded grey and nine slices unshaded. The image shows that adding three individual twelfths together is equal to three-twelfths.

Lastly, we need to take the product of $\frac{3}{12}$ and 2, but this is straightforward since it is just addition:

$$\frac{3}{12} \times 2 = \frac{3}{12} + \frac{3}{12} = \frac{3+3}{12} = \frac{6}{12} = \frac{1}{2}$$

The trickiest step here was the division of the fraction. But, in fact, all we did was multiply the denominator by the amount we are dividing by. To check, look at this computation:

$$\frac{3}{4 \times 2} = \frac{3}{12}$$

This wasn't just a coincidence. The size of the parts we started with was a fourth. Then, we split each fourth into three equal parts. To know the size of this new part as a fraction, we need to determine the number of parts needed to make a whole. If we take three parts, we get a fourth, and so four sets of three parts give us a whole. Hence, the size of the part is $\dfrac{1}{4 \times 3} = \dfrac{1}{12}$

Multiplying Fractions

1. Compute the product of the numerators.
2. Compute the product of the denominators.
3. Write the fraction with a numerator equal to the product of step 1 and a denominator equal to the product of step 2.

We can summarize the work that we have done to compute the product of two fractions with the steps summarized in the box below: For example, let's compute $\dfrac{5}{2} \times \dfrac{3}{7}$:

1. The numerators of the fractions are 5 and 3, so their product is $5 \times 3 = 15$.
2. The denominators of the fractions are 2 and 7, so their product is $2 \times 7 = 14$.
3. Our result of step 1 was 15 and our result from step 2 was 14. Thus, $\dfrac{5}{2} \times \dfrac{3}{7} = \dfrac{15}{14}$.

Here it is again with fewer steps:

$$\frac{5}{2} \times \frac{3}{7} = \frac{5 \times 3}{2 \times 7} = \frac{15}{14}$$

Division with fractions is tricky to imagine. For example, consider the expression

$$\frac{3}{4} \div \frac{2}{3}$$

We know that division is about splitting things into equal parts, but how would you split a number evenly into $\dfrac{2}{3}$ parts? One way to think of it is that, given the part that results from this splitting, if I then took a $\frac{2}{3}$ part of it, I would have my original number back. But this means that.

$$\left(\frac{3}{4} \div \frac{2}{3}\right) \times 2 \div 3 = \frac{3}{4}$$

But then, doing the inverse operations gives us.

$$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times 3 \div 2 = \frac{3}{4} \times \frac{3}{2}$$

So, to divide $\frac{3}{4}$ by $\frac{2}{3}$, we multiplied it by $\frac{3}{2}$. Notice that the numerators and denominators of $\frac{2}{3}$ and $\frac{3}{2}$ are reversed. When this happens, we say that two fractions are *reciprocal*. Thus, to compute the division of fractions, just take the reciprocal of the divisor and multiply by that value. It's one more step than multiplication, but other than that, it is very similar.

For example, let's compute $\frac{5}{2} \div \frac{3}{7}$:

1. Rewrite the expression as $\frac{5}{2} \times \frac{7}{3}$
2. The numerators of the fractions are 5 and 7, so their product is $5 \times 7 = 35$.
3. The denominators of the fractions are 2 and 3, so their product is $2 \times 3 = 6$.

Dividing Fractions

1. Rewrite the expression as the product of the first fraction and the reciprocal of the second fraction.
2. Compute this product as normal.

4. Our result of step 2 was 35 and our result from step 3 was 6. Thus, $\frac{5}{2} \div \frac{3}{7} = \frac{35}{6}$.

Here it is again with fewer steps:

$$\frac{5}{2} \div \frac{3}{7} = \frac{5}{2} \times \frac{7}{3} = \frac{5 \times 7}{2 \times 3} = \frac{35}{6}.$$

Exercises 3.5

Type your exercises here.

1. $\frac{2}{3} \times \frac{3}{4} =$
2. $\frac{3}{5} \times \frac{6}{10} =$
3. $\frac{3}{5} \times \frac{1}{2} =$
4. $\frac{3}{10} \times \frac{2}{5} =$
5. $\frac{2}{10} \times \frac{3}{5} =$
6. $\frac{1}{2} \times \frac{6}{10} =$
7. $\frac{1}{4} \div \frac{1}{2} =$
8. $\frac{1}{2} \div \frac{7}{10} =$
9. $\frac{1}{3} \div \frac{3}{4} =$
10. $\frac{1}{2} \div \frac{2}{5} =$
11. $\frac{1}{4} \div \frac{5}{10} =$
12. $\frac{1}{5} \div \frac{1}{2} =$

3.6 RATIOS AND PERCENTAGES

Imagine that you are making some lemonade. A friend shared the following recipe: combine 8 cups of water, 1 cup of sugar, and 1 cup of lemon juice. You are excited to make it, but when you grab your sugar, you realize that you only have $\frac{1}{2}$ a cup left!

In a situation like this, is it still possible to make the lemonade? The answer is yes; all you have to do is look at the ingredient ratios. A ratio is an expression that tells you the quantity of one thing based on the quantity of another. For example, in the lemonade recipe, there are 8 cups of water and 1 cup of sugar, so water and sugar are in an $8:1$ ratio with one another. Therefore, if we have only half a cup of sugar, we should get $8 \times \frac{1}{2} = 4$ cups of water for the recipe. Similarly, the recipe indicates a $1:1$ ratio of sugar and lemon juice, so we only need a $\frac{1}{2}$ cup of lemon juice.

Some ratios are a bit more complex, such as a $3:2$ ratio. This ratio says "for every 3 of an item, there are 2 of another item". For instance, say we have a $3:2$ ratio of water to sugar. Then if there are 3 cups of water, there are 2 cups of sugar. On the other hand, if there is only 1 cup of water, how much sugar is there? One way to look at it is to notice that there is a third of the amount of water in this scenario, so there must also be a third of the amount of sugar! So, if there is 1 cup of water then there is $\frac{2}{3}$ cups of sugar. Similar to fractions, dividing or multiplying both sides of the ratio is still the same ratio, e.g., $3:2 = 1:\frac{2}{3}$.

We can also think of ratios as representing the fraction that an item makes up the total. For example, in a $5:4$ ratio of water to sugar, we know that there will be 5 units of water and 4 units of sugar. This means that there is a total of 9 units between them! Therefore, the water makes up $\frac{5}{9}$ of the total quantity, and the sugar makes up $\frac{4}{9}$ of the total quantity. We can extend this further with ratios using more than two quantities, for example, a $5:4:2$ ratio of water to sugar to lemon juice would give $\frac{5}{11}$ water, $\frac{4}{11}$ sugar, and $\frac{2}{11}$ lemon juice.

Percentages are similar to fractions, because they specify how much a part makes up of the whole. The main difference is that a percentage is always in terms of a fraction with a denominator equal to 100. So, for example, the value 75% is actually $\frac{75}{100} = \frac{3}{4}$. In the lemonade example, we know that the water is $\frac{8}{10}$ of the total recipe. As a percentage, we try to write this fraction using 100 as a denominator, which we can do by multiplying the top and bottom by 10. Therefore, $\frac{8}{10} = \frac{80}{100} = 80\%$. Similarly, the sugar is 10% of the recipe, and the lemon juice is also 10%.

We can convert between all of these different representations using the rules below:

Exercises 3.6a

Write the following ratios as fractions:

1. $2:1$
2. $7:6$
3. $3:2:1$

Converting Fractions, Ratios, and Percentages

- **Ratio to Fraction:** Find the sum of the numbers in the ratio. Set this sum as the denominator, and use the numbers in the ratio as the numerators.
- **Fraction to Ratio:** Write the fractions in terms of a common denominator. The numerators of the fractions are the numbers in the ratio.
- **Percentage to Fraction:** Take the value of the percentage, and use it as the numerator of a fraction with 100 as the denominator.
- **Percentage to Fraction:** Multiply the fraction by 100 and add the percentage symbol. Simplify, and if necessary, use a calculator to compute the value of the fraction.

Exercises 3.6b

Write the following fractions as ratios:

4. $\frac{2}{7}, \frac{5}{7}$

5. $\frac{4}{6}, \frac{1}{3}$

6. $\frac{2}{5}, \frac{1}{10}, \frac{1}{2}$

Write the following percentages as fractions:

7. 50%

8. 40%

9. 65%

Write the following fractions as percentages:

10. $\frac{1}{4}$

11. $\frac{2}{3}$

12. $\frac{3}{2}$

13. Concrete uses a mixture of crushed stone, sand, and cement in a $5:2:1$ ratio. If you need 4000lb. of concrete, how much of each material do you require?

14. A blueprint drawn to scale has a wall that is 15cm long, but which represents a 40m wall.

What is the scale of the blueprint, written in centimetres to metres?

15. You work with a partner who you split the profits with $2:1$, since you tend to do more highly skilled work. As a percentage, how much of the profit do you get? If you and your partner have earned $\$10,000$ in profit so far this year, how much of that is yours? How much of that is your partner's?

4 THE DECIMAL SYSTEM

Chapter Outline

Although we have not yet looked at it closely, we have been using the decimal system to write all of our numbers. That said, it is not the only way that we can write them.

4.0 LEARNING OBJECTIVES

Learning Objectives

By the end of this chapter, students will be able to:

- Explain the purpose of the decimal system and what a decimal number represents (4.1-2)
- Compute basic operations on decimal numbers (4.3-6)
- Apply rounding rules to decimal numbers (4.7)

4.1 THE UNARY SYSTEM

In the first section of this book, all of the operations were built up from the basic operation of counting. Due to this, we can imagine a point in history where counting was one of the only ways people knew how to deal with numbers. During this time, the unary system was used. A unary system uses only one symbol, |, which is called a *tally*. In the unary system, each number can be represented by that number of tallies. For example, the number three would be just three tallies in a row: |||. The number ten would be ||||||||||, and the number fifty would be:

||||||||||||||||||||||||||||||||||||||||||||||||||

As you can see, dealing with large numbers is difficult in the unary system. Imagine trying to write down one million in unary; you literally would need to make one million tallies. Adding numbers in this system would be combining tallies, and multiplication would be making copies of tallies. Overall, a very challenging number system to do math in!

Exercises 4.1

Try to complete the following problems without using any prior knowledge about the decimal system. Reflect on how these tasks would have been different in our usual number system.

1. Write the number 11 and the number 12 in the unary system.

2. Compute $(12 + 11)$ in the unary system.

3. Compute (12×11) using the unary system.

4.2 DIGITS AND PLACE VALUES

Thankfully, at some point in history, people realized that counting to ten is about as much counting as we can do. The reason for this is that most people have ten fingers to count on, so it is simple to count up to ten. Beyond that, it is easy to lose track. The decimal system only requires you to count up to nine, though you might have to do this several times.

The decimal system uses ten familiar symbols, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, called digits, to represent every number. When reading a number in the decimal system, we see these digits arranged in a certain order. The digit furthest to the right tells you how many ones are in the number. The digit to the left of that tells you how many tens there are. The digit to the left of that tells you how many hundreds there are, and so on.

Since each digit tells you how much it represents based on its position in the number, it gives you the place value of the number. The place values of the number *three thousand one hundred ninety-two*, represented in the decimal system, are elaborated on in Figure 4.2.1.

Figure 4.2.1

The place values of 3192

Image Description

Detailed image description: The image shows the number 3,192 represented by place value. Large digits appear across the top: 3, 1, 9, and 2. Under each digit, a diagonal label identifies its place value. The 3 is labelled “three thousands,” the 1 is labelled “one hundred,” the 9 is labelled “nine tens,” and the 2 is labelled “two ones.”

Put another way,

$$3192 = (3 \times 1000) + (1 \times 100) + (9 \times 10) + (2 \times 1).$$

We can write this even more succinctly by using powers of ten, like so:

$$3192 = (3 \times 10^3) + (1 \times 10^2) + (9 \times 10^1) + (2 \times 10^0).$$

When a number is written like this, it is called its *decimal expansion*.

So, in the decimal system, every digit to the left counts ten times more items than the one to its right.

We can also keep track of parts of numbers in the decimal system by counting how many tenths, hundredths, thousandths, etc., there are in a number. We do this by adding additional digits to the right of the ones place, after a decimal mark. For example, the number 3192.574 and its place values are shown in Figure 4.2.2.

We can express this numerically as we have before as well:

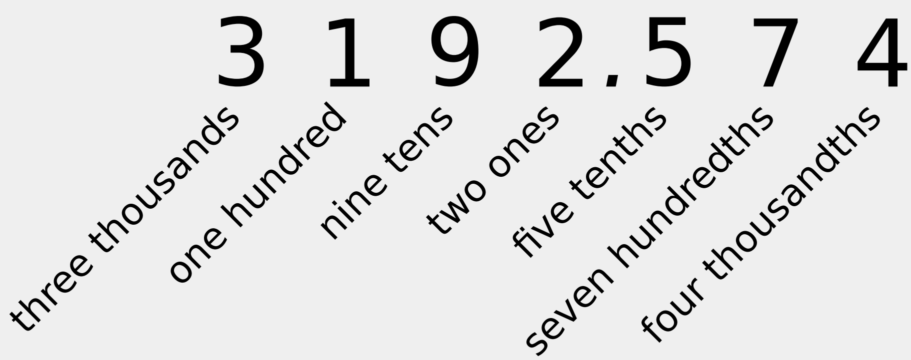
$$3192.574 = (3 \times 10^3) + (1 \times 10^2) + (9 \times 10^1) + (2 \times 10^0) + (5 \times 10^{-1}) + (7 \times 10^{-2}) + (4 \times 10^{-3})$$
.

The difficulty of the decimal system is that some fractions do not play nicely with it. For example:

$$\frac{1}{3} = 0.33333333333333333333333\cdots$$

where that string of 3s after the decimal point never ends. The reason for this is that it is impossible to write one-third in terms of tenths, hundredths, etc., since none of these have a common factor of three. In fact, the decimal system only plays nice with fractions whose denominators are products of 2s and/or 5s (because $10 = 2 \times 5$). Since this complication happens often, we represent repeating strings of digits by using a bar over the top of them. For example:

Figure 4.2.2



The place values of 3192.574.

Image Description

The image shows the decimal number 3,192.574 represented by place value. Large digits appear across the top: 3, 1, 9, 2, decimal point, 5, 7, and 4. Under each digit, a diagonal label identifies its place value. The 3 is labelled "three thousands," the 1 is labelled "one hundred," the 9 is labelled "nine tens," the 2 is labelled "two ones," the 5 is labelled "five tenths," the 7 is labelled "seven hundredths," and the 4 is labelled "four thousandths."

$$\frac{1}{7} = 0.142857142857142857\cdots = 0.\overline{142857}.$$

Exercises 4.2

Identify the place value of the circled digit in the following numbers:

1. $73\textcircled{5}42$
2. $1\textcircled{9}345$
3. $\textcircled{4}5$
4. $\textcircled{6}4299$
5. $1000000\textcircled{2}$
6. $1.\textcircled{7}5$
7. $3\textcircled{3}5.02$
8. $7\textcircled{0}45.5$
9. $3199.4\textcircled{3}$
10. $0.02\textcircled{3}89$
11. $\textcircled{4}450760.\overline{551}$
12. $12.\textcircled{1}\overline{6}$

Write the decimal form of the following decimal expansions:

13. $6 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$
14. $8 \times 10^3 + 5 \times 10^1 + 3 \times 10^0$
15.
16. $1 \times 10^0 + 1 \times 10^{-1} + 6 \times 10^{-3}$

$$2 \times 10^1 + 9 \times 10^0 + 4 \times 10^{-1} + 7 \times 10^{-2}$$

17. Imagine an alien species that counts things using its two antennae. What kind of number system do you think they would develop to do mathematics?

4.3 ADDING DECIMALS

The decimal system helps us add numbers together by breaking up our computations into a sequence of a few simple steps. The idea behind it is this: if I have my numbers organized by place values, then I only need to know the total of the two numbers at that place value to know the sum of the whole number!

For example, let's look at $125 + 342$. This is the same as asking:

$$(1 \times 10^2 + 2 \times 10^1 + 5 \times 10^0) + (3 \times 10^2 + 4 \times 10^1 + 2 \times 10^0)$$

When we look at the equation like this, we realize that we can just group together the digits at the same place between both numbers, which becomes:

$$(1 \times 10^2 + 3 \times 10^2) + (2 \times 10^1 + 4 \times 10^1) + (5 \times 10^0 + 2 \times 10^0)$$

Then, we can just add together the digits of each place value:

$$(1+3) \times 10^2 + (2+4) \times 10^1 + (5+2) \times 10^0 = 4 \times 10^2 + 6 \times 10^1 + 7 \times 10^0 = 467$$

That was a lot quicker than starting at 125 and counting up 342 times past there. In fact, all we had to do was add together each pair of place values in the numbers. We can streamline this method by aligning the numbers in our sums vertically, so that their ones, tens, hundreds, etc., places all line up. Using the same example:

$$\begin{array}{r} 125 \\ +342 \\ 467 \end{array}$$

All that we have done here is add the digits in each column and write our result in the same column.

This works just as well for numbers with parts too:

$$\begin{array}{r} 13.24 \\ +40.63 \\ 55.87 \end{array}$$

The only tricky part is when you have two digits in the same column that add to a number greater than nine. For example, in $15+17$, we might attempt to put a 12 in the ones column of the sum:

$$\begin{array}{r} 15 \\ +17 \\ \hline 2(12) \end{array}$$

But this is incorrect notation, because in the decimal system, we cannot have digits greater than ten in the ones place. Instead, we should think of 12 as $10+2$. By doing this, we focus on the digits in the ones column of our sum and *carry* the digit in the tens column to our next column. We remind ourselves that we carried a digit by writing it on top of the next column, as shown below.

$$\begin{array}{r} 1 \\ 15 \\ +17 \\ \hline 32 \end{array}$$

When we do this, we get 32 instead of $20+12$, because we now have an extra 1 to add in the tens column from our carry.

To make things simple, we always begin at the rightmost column; that way, we always know if there is a carry. We carry whenever the digits in that column sum to a value greater than nine. An example is shown below where each column is carried, including the hundreds column that did not appear in the original expression:

$$\begin{array}{cccccc} 1 & 1 & 1 & & 1 & \\ & 9 & 8 & . & 7 & 6 \\ + & 5 & 5 & . & 5 & 5 \\ \hline 1 & 5 & 4 & . & 3 & 1 \end{array}$$

Adding Decimals

1. Vertically align the numbers so that their ones, tens, tenths, etc., each form a column.
2. Begin at the rightmost column, and take the sum of the digits in that column.
3. Write the ones place value in the same column that the sum came from in step 1, and carry the tens digit to the next column, if necessary.
4. Repeat steps 2 and 3 for the next column to the left.

Exercises 4.3

Compute the following sums:

1. $\begin{array}{r} 23.7 \\ +\,61.5 \\ \hline \end{array}$

2. $\begin{array}{r} 3014.00 \\ +\,21.02 \\ \hline \end{array}$

3. $\begin{array}{r} 56.830 \\ +\,33.003 \\ \hline \end{array}$

4. $\begin{array}{r} 542.20 \\ +\,461.51 \\ \hline \end{array}$

5. $\begin{array}{r} 3.14159 \\ +\,2.71828 \\ \hline \end{array}$

6. $\begin{array}{r} 999.9 \\ +\,111.1 \\ \hline \end{array}$

7. $146.9 + 24.12$

8. $4.534 + 2.455$

9. $1005 + 0.295$

10. $8201.03 + 45.98$

11. $5.041702 + 1.222222$

12. $0.35 + 10.85001$

4.4 SUBTRACTING DECIMALS

Subtraction of two decimal numbers is very similar to addition, but instead of adding the digits at every place value, we take their difference. For example, we can compute $12.34 - 11.22$ as follows:

$$\begin{array}{r} 12.34 \\ -\,11.22 \\ 01.12 \end{array}$$

So, our solution is 1.12 (we do not include the zero in front because it is redundant).

We can run into difficulties when a finding of digits that is negative, as shown here in the computation of $26 - 17$:

$$\begin{array}{r} 26 \\ -\,17 \\ 1(-1) \end{array}$$

Since $6 - 7 = -1$, you can try to write −1 in the ones column, but this is incorrect notation. What we need to do instead is borrow a digit from the column to the left. This is the opposite of carrying in addition. Let's try that computation again:

$$\begin{array}{cccc} & 1 & 1 & 6 \\ & \cancel{2} & \cancel{6} & \\ - & 1 & 7 & \\ & 0 & 9 & \end{array}$$

We can do this multiple times, if necessary. But what if the column that we are borrowing from contains a zero? In that case, we need to borrow from the next column after that, and so on until we find a column with a non-zero digit. For example, if we compute $105 - 26$, we attempt to borrow from the tens column, but find that there is a zero digit. So, then we borrow from the hundreds column into the tens column. Now, we have 10 in the tens column, so we can borrow from it, leaving the hundreds with 0, the tens with 9, and the ones with 15. Then, carrying out the computation in each column, we obtain 79.

$$\begin{array}{cccc} & & 9 & \\ & 0 & \cancel{10} & 15 \\ & \cancel{1} & \cancel{0} & \cancel{5} \\ - & & 2 & 6 \\ & 0 & 7 & 9 \end{array}$$

There is one situation where borrowing will not work, and that is when you are subtracting a larger number from a smaller number. But that's okay. All that we do is switch their order and add a negative sign in front of our answer. For example, $17 - 25 = -(25 - 17)$:

$$\begin{array}{cccc} & 1 & 1 & 5 \\ & \cancel{2} & \cancel{5} & \\ - & 1 & 7 & \\ - & 0 & 8 & \end{array}$$

So, we can use our usual methods as long as we do not forget to add the negative sign.

Subtracting Decimals

1. Vertically align the numbers so that their ones, tens, tenths, etc., each form a column.
2. If the number on top is smaller than the number on the bottom, switch their order, and write a negative sign in front of where your solution will go.
3. Begin at the rightmost column, and take the difference of the digits in that column. If necessary, borrow digits from the columns to the left as many times as required.
4. Repeat steps 2 and 3 for the next column to the left.

Exercises 4.4

Compute the following differences:

1. $\begin{array}{r} 127 \\ +\,15 \\ \hline \end{array}$

2. $\begin{array}{r} 957 \\ +\,732 \\ \hline \end{array}$

3. $\begin{array}{r} 1049 \\ +\,1034 \\ \hline \end{array}$

4. $\begin{array}{r} 365 \\ +\,274 \\ \hline \end{array}$

5. $\begin{array}{r} 871 \\ +\,627 \\ \hline \end{array}$

6. $\begin{array}{r} 4162 \\ +\,3923 \\ \hline \end{array}$

7. $97 - 49$

8. $2678 - 1777$

9. $5163 - 575$

10. $733 - 750$

11. $1225 - 2000$

12. $5498 - 10254$

4.5 MULTIPLYING DECIMALS

To multiply decimals, let's look at computing 25×32. In their expanded forms, this is

$$(2 \times 10^1 + 5 \times 10^0) \times (3 \times 10^1 + 2 \times 10^0)$$

Since 32 groups of 25 is the same as 30 groups of 25 in addition to 2 groups of 25, we can split the expanded form into a sum of two products:

$$(2 \times 10^1 + 5 \times 10^0) \times (3 \times 10^1) + (2 \times 10^1 + 5 \times 10^0) \times (2 \times 10^0)$$

Now, we can bring inside the powers of 10 since they are simple to work with, and notice that this leaves us with the digits of 32 multiplied by 25, or a factor of 25 by 10.

$$(2 \times 10^2 + 5 \times 10^1) \times 3 + (2 \times 10^1 + 5 \times 10^0) \times 2 = 250 \times 3 + 25 \times 2$$

Lastly, re-expanding, we can use the same trick again to rewrite this equation in the form:

$$[(2 \times 3) \times 10^2 + (5 \times 3) \times 10^1] + [(2 \times 2) \times 10^1 + (5 \times 2) \times 10^0] = 750 + 50 = 800$$

As you can see, all we have done is multiply the digits together and shift over a column to the left when multiplying by a digit that came from the tens column.

Let's try this again, building off techniques that we have learned so far.

Since we know that this product splits into a sum of two numbers for each digit of 32, let's begin by multiplying 25 by 2. When doing this, we carry the digits past 9 if they come up, and add them to the product of 2 with the next column.

$$\begin{array}{r} 1 \\ 25 \\ \times 32 \\ 50 \end{array}$$

$$\begin{array}{r} 1 \\ \cancel{1} \\ 25 \\ \times 32 \\ 50 \\ 750 \end{array}$$

After this, we repeat the process again with the digit 3 from 32. But, because it belongs in the tens place, we write a zero in the ones place to indicate this. Then, you proceed as normal, carrying digits whenever necessary.

This leaves you with determining the sum of the two products that you have just found, using the usual rules for addition.

$$\begin{array}{r} \cancel{1} \\ \cancel{1} \\ 25 \\ \times 32 \\ 50 \\ +750 \\ 800 \end{array}$$

If we are multiplying by a decimal with a fraction part, we can use a similar idea to shifting to the left for tens, but instead shift to the right.

For example, we can compute 14×1.3 as follows:

$$\begin{array}{r} \cancel{1} \\ 14 \\ \times 1.3 \\ 4.2 \\ +14.0 \\ 18.2 \end{array}$$

As we can see, the tenths column has shifted the digits over to the right instead of the left. The amount that a digit shifts another in a product is based on how far to the left or right it is of the ones column.

Multiplying Decimals

1. Vertically align the numbers so that their ones, tens, tenths, etc., each form a column.
2. Begin at the rightmost digit of the number on the bottom. Multiply this digit by each digit in the top number, carrying as necessary, and shifting by the appropriate amount. *To verify that you have shifted the correct amount, check that the product from the ones column of the top number is in the same column of the digit from the bottom number that you are multiplying by.*
3. Repeat steps 2 and 3 for the next digit to the left on the bottom number on a new row underneath the last product.
4. Lastly, add all of the products together.

Exercises 4.5

Compute the following products:

1. $\begin{array}{r} 25 \\ \times 7 \\ \hline \end{array}$

2. $\begin{array}{r} 38 \\ \times 9 \\ \hline \end{array}$

3. $\begin{array}{r} 145 \\ \times 6 \\ \hline \end{array}$

4. $\begin{array}{r} 52 \\ \times 34 \\ \hline \end{array}$

5. $\begin{array}{r} 61 \\ \times 78 \\ \hline \end{array}$

6. $\begin{array}{r} 792 \\ \times 41 \\ \hline \end{array}$

7. $\begin{array}{r} 11.0 \\ \times 3.6 \\ \hline \end{array}$

8. $\begin{array}{r} 24.7 \\ \times 8.5 \\ \hline \end{array}$

9. $\begin{array}{r} 12.40 \\ \times 9.21 \\ \hline \end{array}$

4.6 DIVIDING DECIMALS

Division is a difficult operation to compute, even with small numbers. Using fractions, we can make it appear at least a bit simpler. For example, let's look at $63 \div 3 = \frac{63}{3}$. We can use the expanded form to get:

$$\begin{aligned}\frac{63}{3} &= \frac{6 \times 10^1 + 3 \times 10^0}{3} \\ &= \frac{6 \times 10^1}{3} + \frac{3 \times 10^0}{3} \\ &= \frac{6}{3} \times 10^1 + \frac{3}{3} \times 10^0 \\ &= 2 \times 10^1 + 1 \times 10^0 \\ &= 21\end{aligned}$$

This wasn't so bad because all of the digits divided evenly. But what if instead it was $42 \div 3$:

$$\begin{aligned}\frac{42}{3} &= \frac{4 \times 10^1}{3} + \frac{2 \times 10^0}{3} \\ &= \frac{4}{3} \times 10^1 + \frac{2}{3} \times 10^0 \\ &= 1\frac{1}{3} \times 10^1 + \frac{2}{3} \times 10^0\end{aligned}$$

Since the digits do not divide evenly, we are left with a fraction where we want a whole digit. This fraction is called the remainder. If the remainder is not in ones place, we can convert it into 10 times the number of copies of the next lower place value. So, in the equation above, we convert $\frac{1}{3} \times 10^1$ into $\frac{10}{3} \times 10^0$:

$$\begin{aligned}1\frac{1}{3} \times 10^1 + \frac{2}{3} \times 10^0 &= 1 \times 10^1 + \left(\frac{10}{3} + \frac{2}{3}\right) \times 10^0 \\ &= 1 \times 10^1 + \frac{12}{3} \times 10^0 \\ &= 1 \times 10^1 + 4 \times 10^0\end{aligned}$$

$$= 14$$

This idea works even when the number we are dividing by has more than one digit. For example, in $252 \div 12$:

$$\begin{aligned}
\frac{252}{12} &= \frac{2}{12} \times 10^2 + \frac{5}{12} \times 10^1 + \frac{2}{12} \times 10^0 \\
&= \frac{25}{12} \times 10^1 + \frac{2}{12} \times 10^0 \\
&= 2\frac{1}{12} \times 10^1 + \frac{2}{12} \times 10^0 \\
&= 2 \times 10^1 + \frac{12}{12} \times 10^0 \\
&= 2 \times 10^1 + 1 \times 10^0 \\
&= 21
\end{aligned}$$

Sometimes the numbers do not divide evenly, such as $160 \div 15$:

$$\begin{aligned}
\frac{160}{15} &= \frac{1}{15} \times 10^2 + \frac{6}{15} \times 10^1 + \frac{0}{15} \times 10^0 \\
&= \frac{16}{15} \times 10^1 + \frac{0}{15} \times 10^0 \\
&= 1\frac{1}{15} \times 10^1 + \frac{0}{15} \times 10^0 \\
&= 1 \times 10^1 + \frac{10}{15} \times 10^0 \\
&= 1 \times 10^1 + \frac{2}{3} \times 10^0 \\
&= 10\frac{2}{3}
\end{aligned}$$

In which case, we get a fraction in the ones column, in which case we can write our answer as a mixed number.

Lastly, there are times when we have to divide by decimal numbers, like $7.6 \div 1.9$, which looks a lot harder to compute. But, we can actually start off by getting rid of the decimals entirely by multiplying the top and the bottom by a large enough power of ten:

$$\frac{7.6}{1.9} = \tfrac{7.6 \times 10}{1.9 \times 10} = \tfrac{76}{19}$$

Like with addition, subtraction, and multiplication, we can organize our computation so that it is easy to keep track of. However, it looks quite a bit different from the other operations, because we use a tableau. The basic notation looks like that found in Figure 4.6.1. When we use this technique, it is often called long division. So, we would write $12 \div 4 = 3$ as:

Figure 4.6.1

$$\begin{array}{r} 3 \\ 4\overline{)12} \end{array}$$

$$\begin{array}{r} Quotient \\ Denominator\overline{)Numerator} \end{array}$$

Basic setup of long division.

To compute a quotient, we would divide each digit in the numerator by the denominator, keeping track of the quotients on top and the remainders underneath. For example, let's look at $160 \div 15$ again. First, we take the quotient of 9 and 17, which is 0, which we record above the top line in the column we were working in. Then we multiply the denominator by our quotient, which is 0, and write this result underneath 9. Then we take the difference of 9 and 0 to obtain the remainder of the quotient, which is 9.

$$\begin{array}{l} \;\;\;\;\;\;\;0 \\ 17\overline{)9748} \\ \;\;\;\underline{-0} \\ \;\;\;\;\;\;\;9 \end{array}$$

We then drop down the next digit beside the remainder from our last step. Now, we repeat the above process, but this time finding the quotient and remainder of $97 \div 17$. We find that the quotient is 5, which we write in the next column to the right above the top line. Then, we take the product of 5 and 17 to obtain 85, and write this underneath 97. Lastly, we take the difference of the two to get the remainder, we is 12.

$$\begin{array}{l} \;\;\;\;\;\;\;05 \\ 17\overline{)9748} \\ \;\;\;\underline{-0}\downarrow \\ \;\;\;\;\;\;\;97 \\ \;\;\;\underline{-85} \\ \;\;\;\;\;\;\;12 \end{array}$$

Next, we compute $124 \div 17$, which has a quotient of 7 and a remainder of 5.

$$\begin{array}{l} \;\;\;\;\;\;\;057 \\ 17\overline{)9748} \\ \;\;\;\underline{-0}\downarrow\downarrow \\ \;\;\;\;\;\;\;97\downarrow \\ \;\;\;\underline{-85}\downarrow \\ \;\;\;\;\;\;\;124 \\ \;\;\;\underline{-119} \\ \;\;\;\;\;\;\;005 \end{array}$$

Lastly, we compute the quotient and remainder of $58 \div 17$, which is 3 and 7. Since we are in the ones column, we have no more digits to bring down from the numerator. This means that 7 is the remainder of $9748 \div 17$ as well, and we record it next to the quotient after the letter R to signify it as the remainder.

$$\begin{array}{r}
0573R7 \\
17\overline{)9748} \\
\underline{-0}\downarrow\downarrow\downarrow \\
97\downarrow\downarrow \\
\underline{-85}\downarrow\downarrow \\
124\downarrow \\
\underline{-119}\downarrow \\
0058 \\
\underline{-51} \\
7
\end{array}$$

Therefore, $9748 \div 17 = 573\frac{7}{17}$

Dividing Decimals

1. Multiply the denominator and numerator by the same power of 10 to make them both whole numbers, if necessary.
2. Write the denominator and numerator in their correct places within the tableau, as shown in Figure 2.15. Set the first digit of the numerator equal to the *dividend*.
3. Divide the dividend by the denominator, and record this quotient above the tableau.
4. Multiply the denominator by the quotient from step 2, and write the result underneath the dividend.
5. Find the difference of the dividend and the product from step 3 to obtain the remainder.
6. If there are still digits remaining in the numerator, drop the next one down to the right of the remainder from step 4. This is now the dividend. Repeat steps 2 to 4 on this dividend.
7. If there are no digits remaining in the numerator, then the remainder from step 4 is the final remainder and is recorded next to the quotient, using the letter R to separate them.

Exercises 4.6

Compute the following quotients:

1. $5\overline{)65}$
2. $3\overline{)39}$
3. $15\overline{)465}$
4. $4\overline{)145}$
5. $12\overline{)256}$
6. $33\overline{)3471}$
7. $124 \div 7$
8. $276 \div 13$
9. $43.09 \div 1.2$

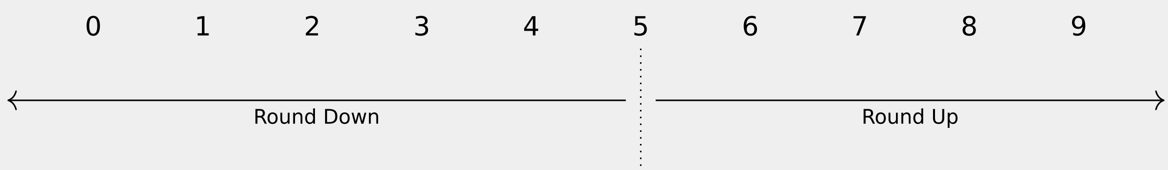
4.7 ROUNDING DECIMALS

There are times when you are working on calculations where the numbers past the decimal point keep getting longer and longer. This is challenging to work with, especially by hand, since we need to keep track of so many digits. On top of this, depending on the situation, the digits past the decimal point may represent a very small value that we are fine to neglect. Rounding is how we choose to neglect digits past a certain point in a number while changing its value as little as possible.

For example, imagine that we have just computed a value as 4.567823511094, but we only care about being accurate up to the nearest *hundredth*. We see that the number up to the hundredths column is 4.56, so we might think that this is what we should round to. But, 4.57 is actually closer to 4.567823511094. This is because the part we are cutting away is 0.007823511094, which is closer to 0.01 than 0.00.

So, the idea of rounding is very simple: look at the column you would like to round to, and then check the next digit to the right. If it is $0, 1, 2, 3$ or 4, then you can round down, which means just cutting off everything after the decimal point. On the other hand, if the digit is $5, 6, 7, 8$ or 9, then we round up by increasing the digit in the column by 1.

Figure 4.7.1



Digits below 5 round down, digits equal to or greater than 5 round up.

Image Description

The image shows a horizontal rounding guide with the digits 0 through 9 evenly spaced along the top. A vertical dotted line appears between 4 and 5 to mark the rounding boundary. Below the digits 0, 1, 2, 3, and 4, the line is labelled “Round Down.” Below the digits 5, 6, 7, 8, and 9, the line is labelled “Round Up.” Arrows at both ends of the horizontal line indicate the scale continues in both directions.

Exercises 4.7

Round the following numbers to the nearest hundredth:

1. 679.6783
2. 4521.11111
3. 0.099

Round the following numbers to the nearest thousandth:

4. 54.89241
5. 97.2315
6. 1000.0009

Compute the following expressions and round to the nearest whole:

7. 1.1×9
8. 2.25×2
9. 14.7×3

Round the numbers in the expression below to the nearest whole, then compute the expression. Are your answers different from questions 7 – 9?

10. 1.1×9
11. 2.25×2
12. 14.7×3

5 MEASUREMENT

Chapter Outline

Measurement is how we quantify and compare different qualities of a thing. For example, we can measure the length, weight, and temperature of an object. Throughout history, there have been many different systems of measurement. This section looks at the two systems of measurement used in the trades, the imperial and metric systems of measurement.

5.0 LEARNING OBJECTIVES

Learning Objectives

By the end of this chapter, students will be able to:

- Use tables to convert between different units in the Imperial and Metric systems (4.1-3)
- Use subtraction to measure lengths from a ruler or tape measure (4.4)
- Use addition to compute the perimeter of an area (4.5)

5.1 THE IMPERIAL SYSTEM

The imperial system of measurement is an older system that is used predominantly in the United States. Historically, its units were based on rudimentary sources of measurement, like the size of a human foot, though in the modern era they have become standardized. Units in this system are often related to one another by simple ratios, like halves or thirds.

The chart below recalls various units used to measure different qualities. Each unit measures a different quantity. Conversion between different units in the same category is given in the next set of tables below.

Imperial Units of Measurement

Measurement	Unit	Symbol
Linear	inch foot yard mile	in. or ” ft. or ’ yd. or yds. mi.
Mass	ounces pounds ton	oz. lb. or lbs. T
Capacity	cup pint quart gallon	c. pt. qt. gal.
Volume	cubic inch cubic yard	$in.^3$ $yd.^3$
Temperature	degrees Fahrenheit	$^\circ F$
Time	seconds minutes	s m

Common Imperial Conversions

Linear	
1ft.	12in.
1yd.	3ft.
1mi.	1760yd.

Mass	
1lb.	16oz.
1T	2000lbs.

Capacity	
1pt.	2c.
1qt	2pt.
1gal.	4qt.

Exercises 5.1

Type your exercises here.

1. Fill in the table below with the correct missing quantity:

Measurement	Conversion
2ft.	_______in.
4lbs.	_______oz.
3gal.	_______qt.
2mi.	_______ft.
2T	_______oz.
1yd.	_______in.
8qt.	_______c.

2. You decide to make a rope swing by tying an $8'$ rope around a thick tree branch. The amount of rope used to tie the knot is about $25''$. How long is your rope swing?

3. You are volunteering at a local fair, serving lemonade to guests. The lemonade comes in a ten-gallon cooler, and each guest is given one cup of lemonade when they come in. If 100 guests

come to the fair, will you have enough lemonade for everyone? What if the jug was only half full?

4. You are taking some waste to the landfill in your truck. When you arrive with the recycling, your truck weighs 2T and 73lbs. When you leave the landfill, your truck weighs 1T and 1923lbs. In pounds, how much waste did you bring to the landfill?

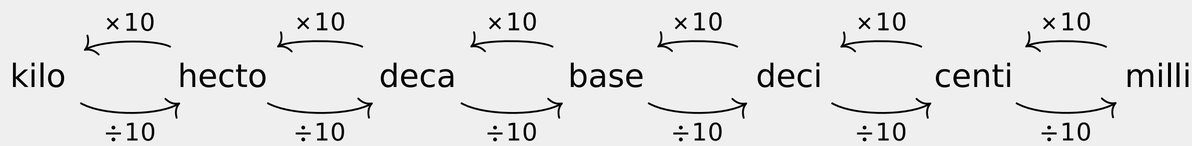
5.2 THE METRIC SYSTEM

In contrast to the imperial system, the metric system is a modern system whose units have always been standardized. The units of this system are based on the decimal system, because they all scale by ten. For example, a centimetre is 10 millimetres, and a litre is 1000 millilitres. There is one basic unit for each type of measurement. The different units of a type of measurement are notated by adding a prefix to the basic unit of that type of measurement. For example, a deca**gram** (da**g**) is ten **grams** (**g**), whereas a milli**gram** (m**g**) is a thousandth of a **gram** (**g**). The basic metric units and the metric prefixes are summarized in the tables below.

Measurement	Unit	Symbol
Linear	metre	m
Mass	gram	g
Capacity	litre	L
Temperature	degrees Celsius	$^\circ\mathrm{C}$

Prefix	Symbol	Scale
milli	m	10^{-3}
centi	c	10^{-2}
deci	d	10^{-1}
deca	da	10^{1}
hecto	h	10^{2}
kilo	k	10^{3}

Figure 5.2.1



You can use this figure above to help remember the relationships between the different prefixes, and as a tool to calculate conversions between the units.

Image Description

The image shows a metric prefix conversion chart arranged horizontally from left to right: kilo, hecto, deca, base, deci, centi, and milli. Between each neighbouring prefix are two curved arrows. The upper arrows point to the left and are labelled "×10," showing that moving one step to the left multiplies by 10. The lower arrows point to the right and are labelled "÷10," showing that moving one step to the right divides by 10.

Exercises 5.2

1. Fill in the table below with the correct missing quantity:

Quantity	Conversion
2m	________ cm
4kg	________ g
3L	________ mL
2km	________ m
2g	________ mg
50mm	________ cm
2000mL	________ L

2. A cookie recipe calls for 500g of flour per batch. But, because you are baking them for a fundraiser, you figure that you should make ten batches. How much flour do you need to prepare this many batches?

3. You are filling a 1000L inflatable pool using your garden hose. You estimate that your garden hose has a flow rate of 35L per minute. How long will it take to fill your pool up?

4. Cars driving at a safe distance on the 401 require about 70 meters of distance between each other. Imagine a 14km stretch of the highway. If everyone is driving at a safe distance, how many cars can fit on this stretch using three lanes?

5.3 CONVERTING BETWEEN SYSTEMS

Although there are many differences between the imperial and metric systems, they both do the same thing: measure things. More importantly, tradespeople in Canada use both of these systems in their work. Due to this, it is important to be able to convert between the two systems. Below is a table which shows the conversion between units of similar sizes. This table and the tables in sections 5.1 and 5.2 can be used as a starting point to convert between all important measurements in the metric and imperial systems.

Metric and Imperial Conversion

Metric	Imperial
1m	3.281ft.
1kg	2.205lbs.
1L	4.227c.

Imperial	Metric
1ft.	0.305m
1lb.	0.454kg
1c.	0.237L

Exercises 5.3

1. You are building a roof using diagrams written in the metric system. It calls for 5m beams. You go to a hardware store that is selling beams at a maximum length of 15ft. Should you buy these beams for your project?

2. The payload (maximum weight you can carry) of your truck is $3310\text{lbs}.$ You are transporting 30 bags of concrete that each weigh 42kg. Can you transport all this material with your truck?

3. You need $1\frac{1}{2}$" nails for a project, but you are not sure what size nails you have. You measure one at about 3.8cm with a metric tape. Is this the right-sized nail?

4. You are making some Kool-Aid for lunch. The packets call for 4 cups of water for each pack. You know that a cup is approximately $\frac{1}{4}\text{L}$, so you use the litre measure that you have to prepare the Kool-Aid. It tastes watered down compared to usual. Why is that?

5.4 ACCURACY IN MEASUREMENT

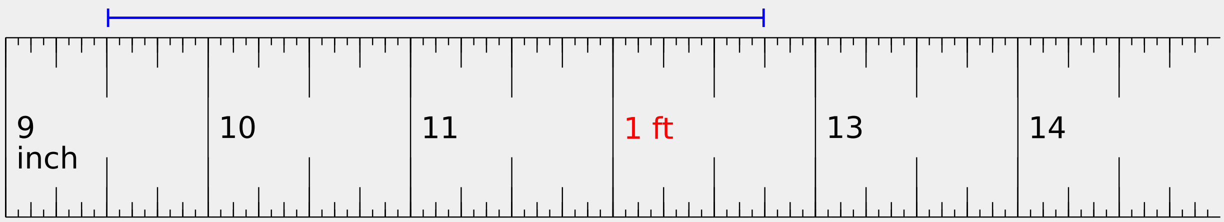
Working in the trades requires you to be highly skilled in measurement. Due to this, it is important to have a strong grasp on the measuring devices that you have at your disposal. The most common measuring tool that you will use is a ruler or tape measure. Rulers will have measurements in the imperial or metric system. An imperial ruler has inches as its basic unit. Since every twelve inches is a foot, those are usually marked as well, sometimes in another colour. On an imperial ruler, each inch is broken up into $\frac{1}{2}''$, $\frac{1}{4}''$, $\frac{1}{8}''$, and sometimes even $\frac{1}{16}''$ increments. We see this pictured in Figure 5.4.1, where there is also a blue line that is being measured. The line begins at $9\frac{1}{2}''$ and ends at

$1'\frac{3}{4}''$.

The total length of the blue line will then be

$$\begin{aligned} & 1'\tfrac{3}{4}'' - 9\tfrac{1}{2}'' \\ = \; & 12\tfrac{3}{4}'' - 9\tfrac{1}{2}'' \\ = \; & 12\tfrac{3}{4}'' - 9\tfrac{2}{4}'' \\ = \; & 3\tfrac{1}{4}'' \end{aligned}$$

Figure 5.4.1



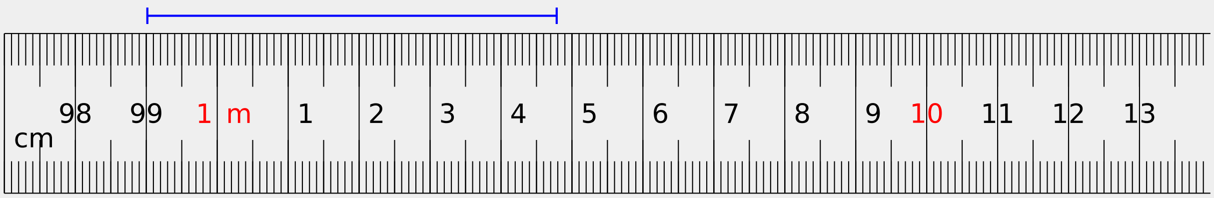
An example of an imperial ruler with $\frac{1}{2}''$, $\frac{1}{4}''$, $\frac{1}{8}''$, and $\frac{1}{16}''$ increments. Feet are in red. A blue line is measured to be a length of $3\frac{1}{4}''$.

Image Description

The image shows a section of an inch ruler from 9 inches to 14 inches. Each inch is divided into smaller tick marks. The numbers 9, 10, 11, 13, and 14 are shown, with the 12-inch position marked by the red label "1 ft" instead of the number 12. A blue bracket above the ruler extends from the 9.5-inch mark to the 12.75-inch mark, showing a measurement of 3.25 inches.

A metric ruler has centimetres as its basic unit. Every 10cm is often marked in a colour, and metres are specially marked. After 1m, the count of centimetres begins again. There are usually millimetre increments marked on a metric ruler as well, with 5mm specially marked. This is shown in Figure 5.4.2, where a blue line is being measured. The line begins at 99cm and ends at 104.8cm. The total length of the blue line will then be $104.8\text{cm} - 99\text{cm} = 5.8\text{cm}$.

Figure 5.4.2



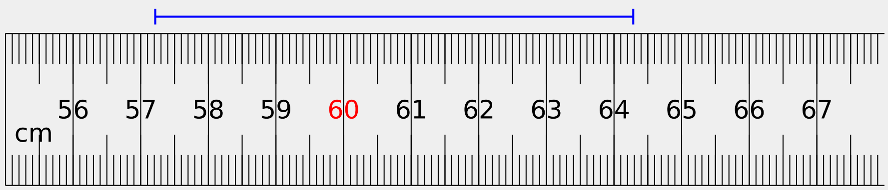
An example of a metric ruler. Every ten centimetres is marked in red. A blue line is measured to be a length of 5.8 cm.

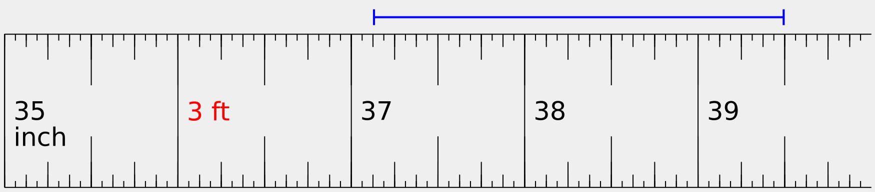
Image Description

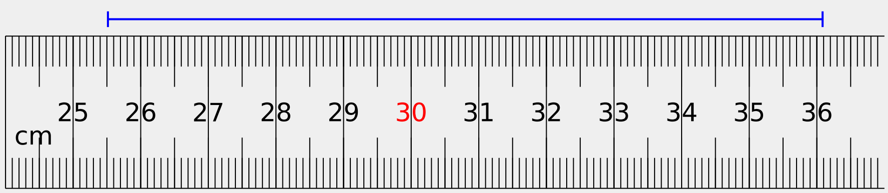
The image shows a section of a centimetre ruler. The scale begins near 98 and 99 centimetres, then reaches the red label "$1m$" at the 100-centimetre mark. After 1 metre, the ruler continues with centimetre labels 1 through 13. Each centimetre is divided into smaller millimetre tick marks. A blue bracket above the ruler extends from the 99-centimetre mark to the 4.8-centimetre mark after 1 metre, showing a measurement of 5.8 centimetres.

Exercises 5.4

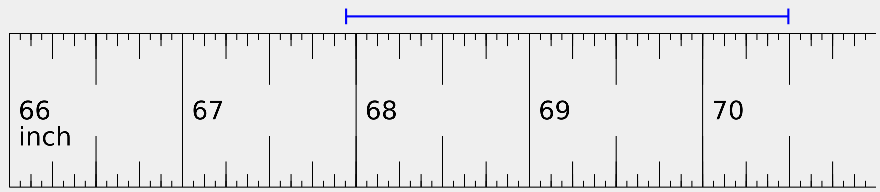
Determine the lengths of the lines below:

1. 

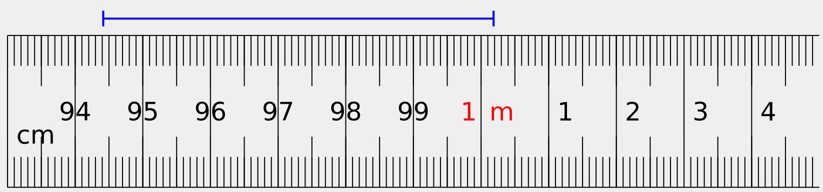
2. 

3. 

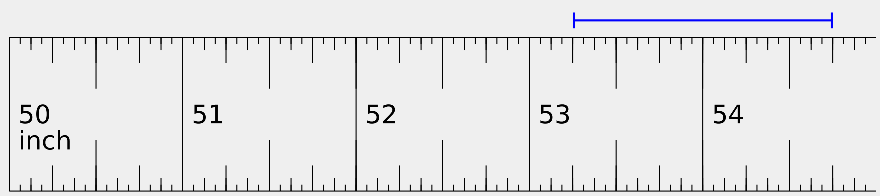
4.


66
inch
67
68
69
70

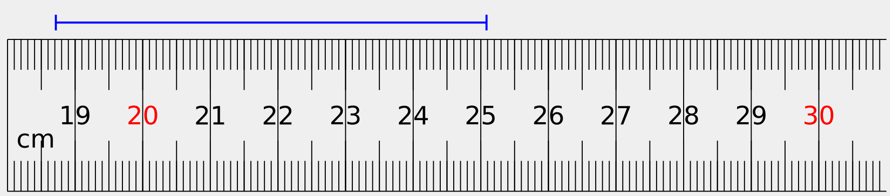
5.


94
cm
95
96
97
98
99
1 m
1
2
3
4

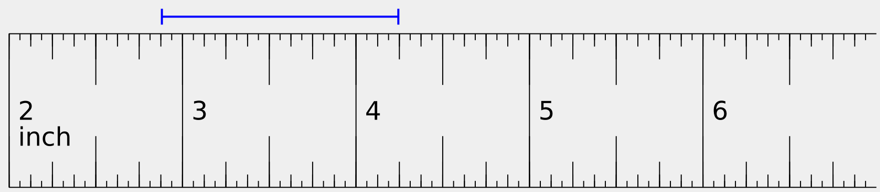
6.


50
inch
51
52
53
54

7.


19
cm
20
21
22
23
24
25
26
27
28
29
30

8.



Draw a straight line from the baseline that is equal to the lengths listed below.

9. $4\frac{1}{4}''$

10. 12.4cm

11. $5\frac{3}{8}''$

12. 8.3cm

13. $6\frac{1}{2}''$

14. 76mm

15. $3\frac{1}{16}''$

16. 4.5cm

17. $1\frac{3}{4}''$

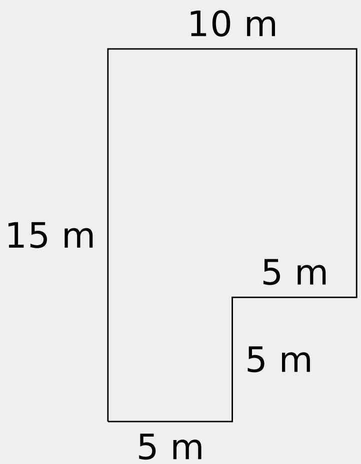
18. 3.9cm

5.5 PERIMETER

There are many situations where we need to measure multiple lengths in order to determine the total length around an area. For example, suppose we are building a chain-link fence around a yard. We need to know how many chain links to purchase, among other things, based on how long the fence will be in total. To do this, we need to calculate the perimeter of the yard.

The perimeter of a shape is the total length of the edges that form that shape. For example, if the yard we are considering has a shape as shown in Figure 5.5.1, the perimeter would be $50\mathrm{m}$. To see this, all we have to do is take the sum of the lengths of the sides. So, we would compute $15\mathrm{m} + 5\mathrm{m} + 5\mathrm{m} + 5\mathrm{m} + 10\mathrm{m} + 10\mathrm{m} = 50\mathrm{m}$. Notice that even though one side was unlabeled, we could determine that its length should be $10\mathrm{m}$ based on the lengths of the other sides.

Figure 5.5.1



An example of a yard with a perimeter of 50m.

Image Description

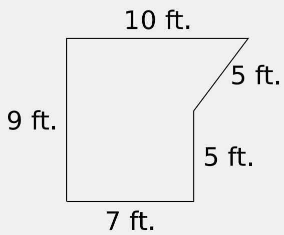
The image shows an irregular L-shaped polygon with labelled side lengths in metres. The top horizontal side is labelled 10 m. The left vertical side is labelled 15 m. The bottom horizontal side is labelled 5 m. On the right side, a horizontal inward section is labelled 5 m, and the short vertical section below it is labelled 5 m. The shape can be interpreted as a 10 m by 15 m rectangle with a 5 m by 5 m square removed from the lower-right corner.

Regardless of the shape, so long as you know how long each part of it is, you can determine its perimeter.

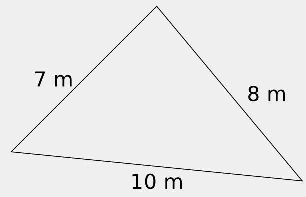
Exercises 5.5

Type your exercises here.

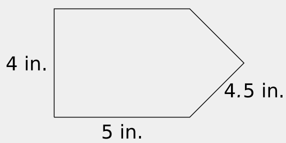
1.



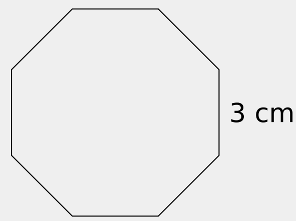
2.



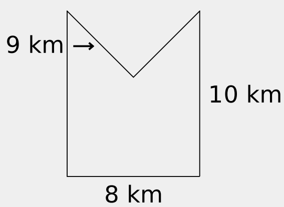
3.



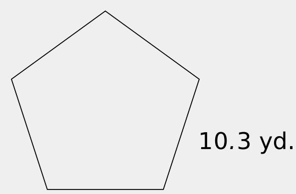
4.



5.



6.



6 TRIGONOMETRY

Chapter Outline

Trigonometry is the study of angles and their relationship to triangles. Trigonometry is important in the trades due to the fact that angles and triangles are the building blocks of nearly all the geometric shapes that we work with. This section will look at the properties of triangles and use some of these properties to study circles and grades.

6.0 LEARNING OBJECTIVES

Learning Objectives

By the end of this chapter, students will be able to:

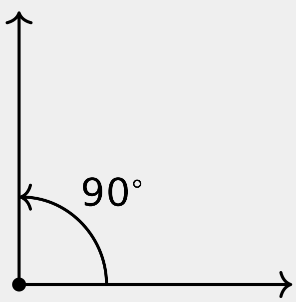
- Measure and combine rotations using degrees (4.1)
- Determine an unknown angle length using the Sum of Angles fact (4.1)
- Use Pythagorean theorem to calculate an unknown side length of a right triangle (4.2)
- Use properties of similar triangles to calculate unknown values (4.3)
- Calculate the circumference of a circle from a measurement of its radius (4.4)
- Use grades to compute the length from the fall of a slope, or vice versa (4.5)

6.1 ANGLES

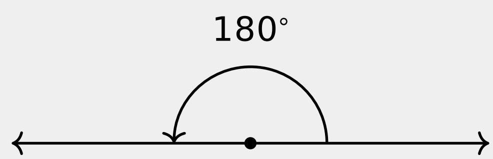
Angles are a measurement of rotation. Like measurements of length or mass, angles are measured in units. The unit of measurement that we will discuss is called a degree, which we represent with the symbol $^\circ$. For example, ninety degrees is written 90°.

When you rotate a quarter turn, that is a 90° rotation. When you rotate to face the opposite direction that you were before, that is a 180° rotation. When you rotate an eighth turn, that is a 45° rotation. These rotations are pictured in Figure 6.1.1. In math, we typically write rotations going counterclockwise.

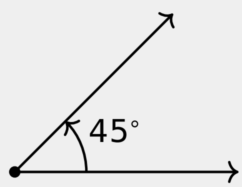
Figure 6.1.1



(a) 90° rotation



(b) 180° rotation



(c) 45° rotation

Image Description

The image shows three angle diagrams arranged horizontally.

On the left, a right angle is shown with one ray pointing upward and one ray pointing to the right from the same vertex. A curved arc between the rays is labelled 90°.

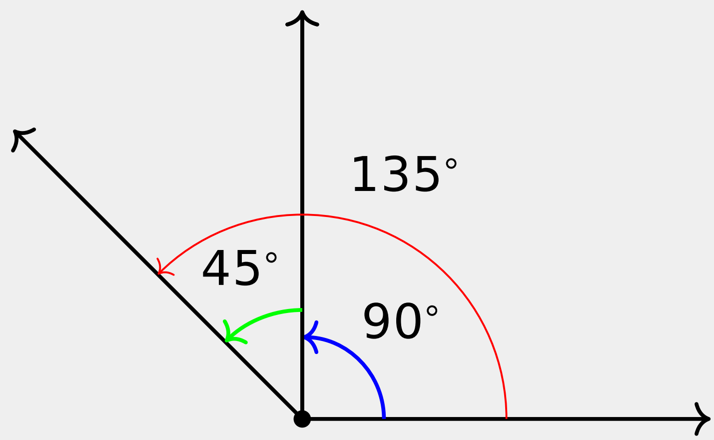
In the centre, a straight angle is shown as a horizontal line with arrows at both ends and a vertex marked in the middle. A semicircular arc above the line is labelled 180°.

On the right, an acute angle is shown with one ray pointing to the right and one ray pointing diagonally upward to the right from the same vertex. A small curved arc between the rays is labelled 45°.

Examples of rotations

You can combine rotations together to get a new rotation. The angle of two rotations combined is the sum of their angles. For example, a rotation of $90°$ followed by a rotation of $45°$ is in total a rotation of $135°$, as pictured in Figure 6.1.2.

Figure 6.1.2



The $135°$ rotation in red is a combination of the $90°$ rotation in blue and the $45°$ rotation in green.

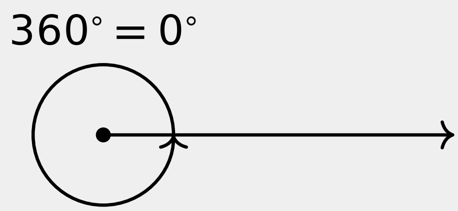
Image Description

The image shows three rays extending from a shared vertex. One ray points horizontally to the right, one ray points vertically upward, and one ray points diagonally upward to the left.

A blue arc marks a 90° angle between the horizontal right ray and the vertical upward ray. A green arc marks a 45° angle between the vertical upward ray and the diagonal ray. A larger red arc marks a 135° angle from the horizontal right ray to the diagonal upper-left ray, showing that the 90° and 45° angles combine to make 135°.

A full rotation is $360°$, as shown in Figure 6.1.3. Notice that the result of a $360°$ turn is the same as a $0°$ rotation. Due to this, we usually write angles as being between $0°$ and $360°$. For example, instead of writing $450°$, we write $90°$ because $450° - 360° = 90°$.

Figure 6.1.3



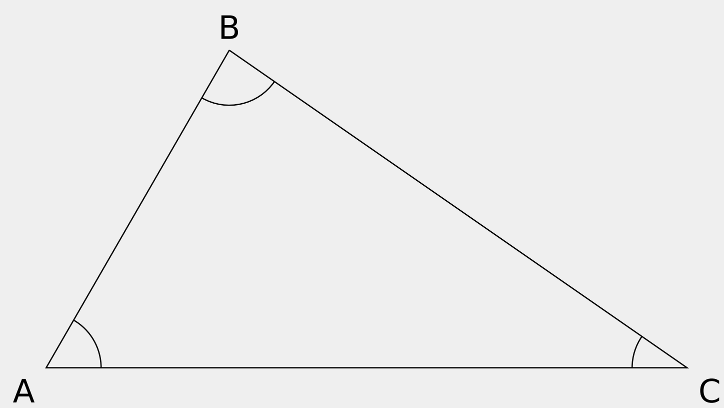
A 360° turn and a 0° turn have the same result.

Image Description

The image shows a full rotation angle diagram. A ray begins at a central point and extends horizontally to the right with an arrow at the end. Around the central point is a complete circular arrow showing one full turn back to the starting position. The label “360° = 0°” appears above the diagram, indicating that a full 360-degree rotation returns to the same direction as 0 degrees.

In addition to measuring rotations, angles can help us understand drawings that are made up of straight lines. For example, a triangle is a shape made up of three straight lines that meet in pairs at three different points. We call these points *vertices* (or *vertex* if there is only one) and can label them with different letters in a diagram. Each vertex creates an angle between its two lines, as shown in Figure 6.1.4.

Figure 6.1.4



A triangle contains three angles at its vertices.

Image Description

The image shows a triangle labelled with vertices A, B, and C. Point A is at the lower left, point B is near the top, and point C is at the lower right. The bottom side, AC, is a long horizontal line. Side AB slopes upward from A to B, and side BC slopes downward from B to C. Curved arcs mark the interior angles at A, B, and C.

When you add up all of the angles in a triangle, you get a total of $180°$, no matter how large or small the triangle is.

Sum of Angles

The sum of all the angles in a triangle is 180°.

Exercises 6.1

From the written description of the rotation, write how many degrees the rotation is.

1. A half-turn.

2. An eighth turn.

3. A three-quarters turn.

Reduce the following angles to a value between zero and 360 degrees:

4. 720°

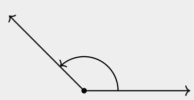
5. 480°

6. -90°

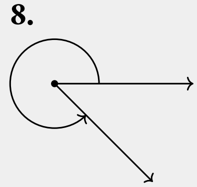
(Hint: a negative angle represents a rotation in the clockwise direction)

Write the approximate size in degrees of the rotations below:

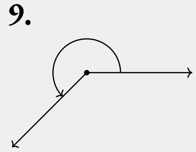
7.



8.



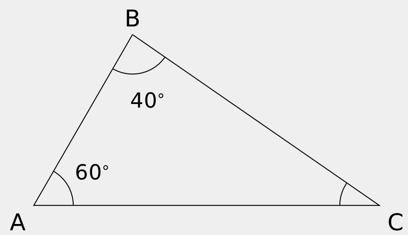
9.

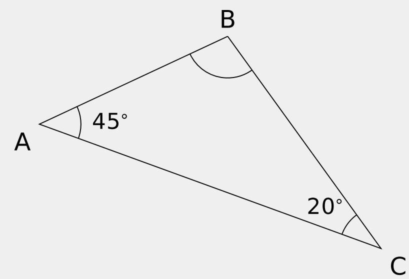


Determine the value of the missing angle of the triangles below:

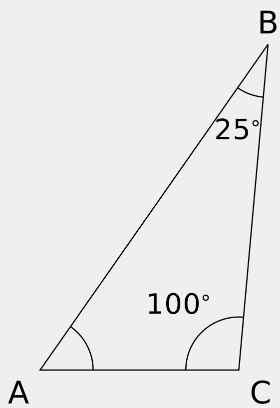
10.

11.

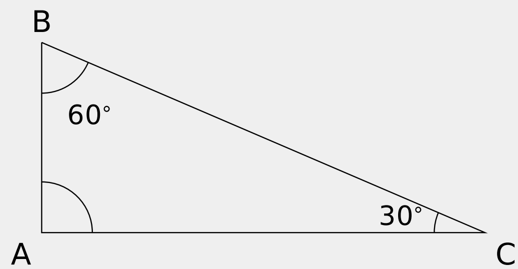




12.



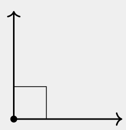
13.



6.2 PYTHAGOREAN THEOREM

A special angle in trigonometry is the $90°$ angle, which is also called a *right angle*. When two lines form a vertex at a right angle, we say that they are *perpendicular*. To mark a right angle in a diagram, we often use a square-shaped arc instead of a rounded one, as shown in Figure 6.2.1.

Figure 6.2.1



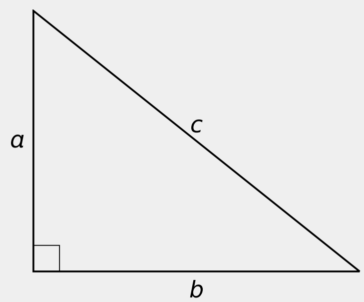
A right angle is formed between perpendicular lines.

Image Description

The image shows a right angle formed by two rays sharing the same vertex. One ray points vertically upward, and the other ray points horizontally to the right. A small square is drawn inside the angle near the vertex to indicate that the angle measures 90 degrees.

When one of the angles in a triangle is a right angle, we call that triangle a *right triangle*. In Figure 6.2.2, the sides are labelled a, b, and c. Clearly, side c is the longest side. In a right triangle, we refer to the longest side as the *hypotenuse*. On the other hand, the shorter sides, a and b, are called the *legs* of the right triangle.

Figure 6.2.2



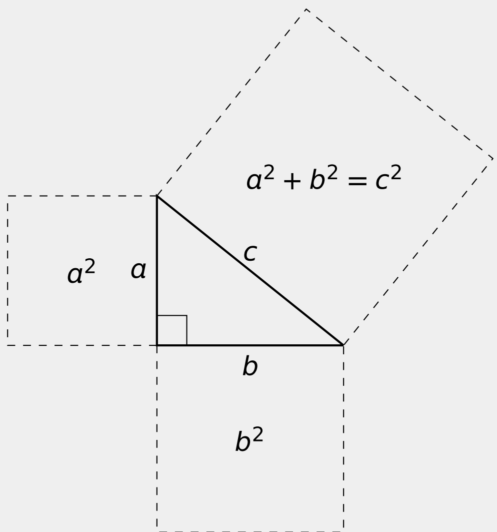
A right triangle with a hypotenuse of c.

Image Description

The image shows a right triangle with the right angle in the bottom-left corner, marked by a small square. The vertical side on the left is labelled a, the horizontal side along the bottom is labelled b, and the diagonal side sloping downward from the top-left to the bottom-right is labelled c. The diagonal side, c, is the hypotenuse.

Right triangles are especially important due to the $a^2 + b^2 = c^2$ property, or the *Pythagorean Theorem*. This says that the square of the side lengths of the legs is equal to the square of the side length of the hypotenuse. In Figure 6.2.3, this is visualized by forming squares of the sides of the right triangle.

Figure 6.2.3



The total size of the squares formed by the legs of a right triangle is equal to the size of the square formed by the hypotenuse.

Image Description

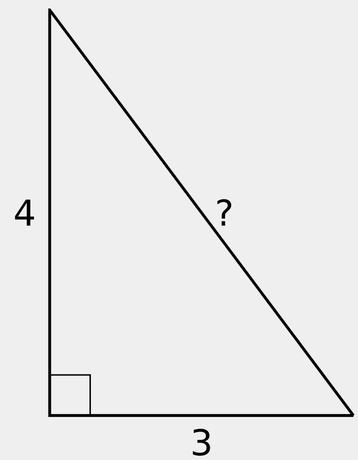
The image shows a right triangle with a vertical side labelled a, a horizontal side labelled b, and a diagonal hypotenuse labelled c. A small square marks the right angle where sides a and b meet.

Dashed squares are drawn outward from each side of the triangle. The square attached to side a is labelled a^2, the square attached to side b is labelled b^2, and the larger tilted square attached to the hypotenuse c contains the equation $a^2 + b^2 = c^2$. The diagram illustrates the Pythagorean theorem, showing that the area of the square on the hypotenuse equals the combined areas of the squares on the two shorter sides.

So, if we know the length of the two legs of a right triangle, then we can also determine the length of the hypotenuse. For example, suppose we know that the triangle has legs of length 3 and 4. Using the formula $a^2 + b^2 = c^2$, we can substitute $a = 3$ and $b = 4$ and simplify to obtain $c = 5$, as shown in in Figure 6.2.4.

Figure 6.2.4

$$
\begin{aligned}
c^2 &= 3^2 + 4^2 \\
c^2 &= 9 + 16 \\
c^2 &= 25 \\
\sqrt{c^2} &= \sqrt{25} \\
c &= 5
\end{aligned}
$$



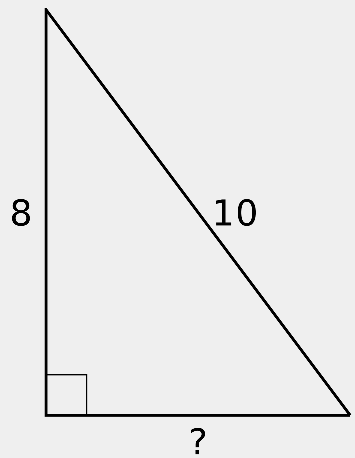
Determining the length of a hypotenuse from the length of its legs.

Image Description

The image shows a right triangle with the right angle in the bottom-left corner, marked by a small square. The vertical side on the left is labelled 4, and the horizontal side along the bottom is labelled 3. The diagonal side slopes downward from the top-left corner to the bottom-right corner and is labelled with a question mark, indicating that the length of the hypotenuse is unknown.

On the other hand, if we were given the length of the hypotenuse and the length of one leg, we could still determine the length of the unknown leg. For example, suppose that we know the length of a hypotenuse is 10, and the length of one of its legs is 8. Using the formula $a^2 + b^2 = c^2$, we can rearrange it to get $c^2 - b^2 = a2$. Then, by substituting $b = 8$ and $c = 10$, we can simplify the formula to get $a = 6$.

Figure 6.2.5



$$
\begin{aligned}
a^2 &= 10^2 - 8^2 \\
a^2 &= 100 - 64 \\
a^2 &= 36 \\
\sqrt{a^2} &= \sqrt{36} \\
a &= 6
\end{aligned}
$$

Determining the length of a leg from the length of its hypotenuse and the other leg.

Image Description

The image shows a right triangle with the right angle in the bottom-left corner, marked by a small square. The vertical side on the left is labelled 8. The diagonal side slopes downward from the top-left corner to the bottom-right corner and is labelled 10, indicating the hypotenuse. The horizontal side along the bottom is labelled with a question mark, indicating that this side length is unknown.

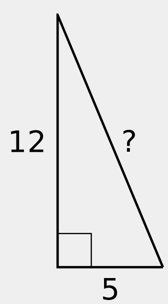
Using $a^2 + b^2 = c^2$

1. If the hypotenuse is unknown, use the formula $a^2 + b^2 = c^2$. If one of the legs is unknown, use the formula $c^2 - b^2 = a^2$.
2. Substitute the known values into the equation, making sure that the legs go into a or b, and that the hypotenuse goes into c.
3. Compute the squares of the known values, then take their sum/difference.
4. Compute the square root of the value from step 3 to obtain the unknown length.

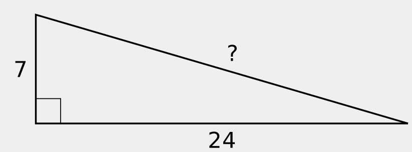
Exercises 6.2

Calculate the unknown lengths in the following right triangles:

1.



2.



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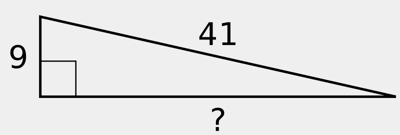

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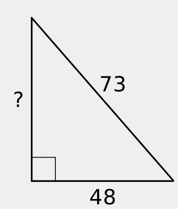


5. I am building a roof for a house. The peak of the roof falls $5\text{ft}.$ to the gutters. The peak is in the center of the house, and the house is $20\text{ft}.$ wide. Draw a diagram of a right triangle that

models this situation. How long is the slanted part of the roof?

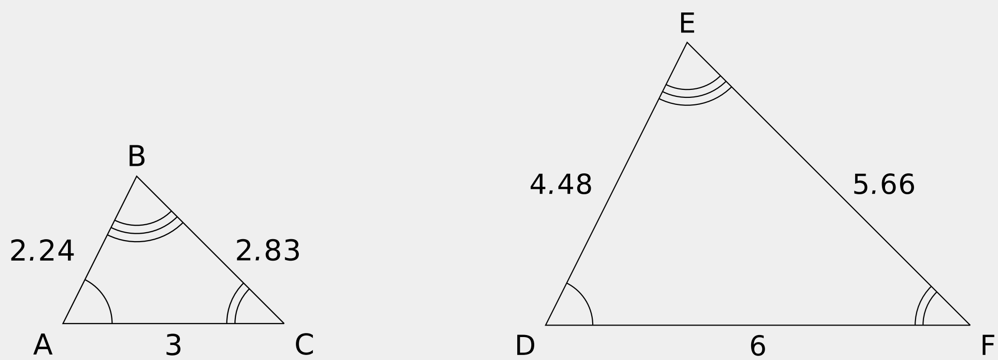
6. Sarnia is about 100k West of London. St. Thomas is about 25km South of London. How far apart are Sarnia and St. Thomas?

6.3 SIMILAR TRIANGLES

In the last section, we gave a special name to right triangles because they all share the same nice property, i.e., $a^2 + b^2 = c^2$. In this section, we will look at *similar triangles*, which are pairs of triangles that have the same sized angles but possibly different sized sides.

For example, in Figure 6.3.1, we see that although the two triangles have different side lengths, they each have the same angles, as marked by the number of lines at each vertex. If you pay even closer attention, you will notice that even though side lengths are different between the two triangles, they have grown proportionally to one another. What this means is that since the length of DF is double the length of AC, every other side has doubled in length. Indeed, the length of DE is double the length of AB, and the length of EF is double the length of BC.

Figure 6.3.1



An example of similar triangles

Image Description

The image shows two similar triangles placed side by side.

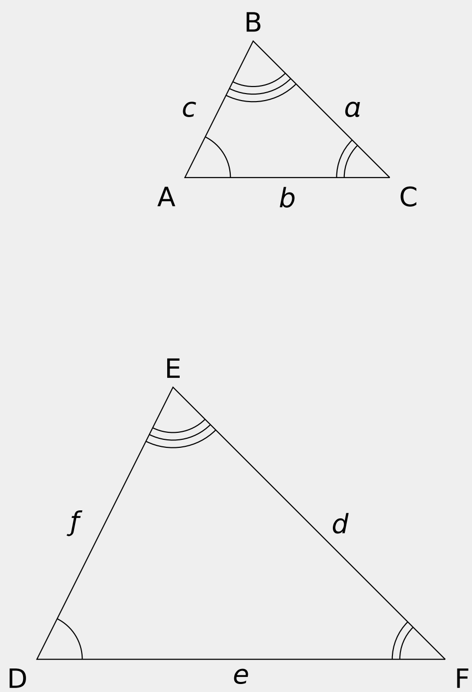
On the left is a smaller triangle labelled A, B, and C. Point A is at the bottom left, B is at the top, and C is at the bottom right. The base AC is labelled 3. The left side AB is labelled 2.24, and the right side BC is labelled 2.83. Angle markings show one arc at angle A, two arcs at angle C, and three arcs at angle B.

On the right is a larger triangle labelled D, E, and F. Point D is at the bottom left, E is at the top, and F is at the bottom right. The base DF is labelled 6. The left side DE is labelled 4.48, and the right side EF is labelled 5.66. Matching angle markings show one arc at angle D, two arcs at angle F, and three arcs at angle E.

The matching angle markings and proportional side lengths show that the two triangles are similar, with the larger triangle scaled up from the smaller triangle.

This was not a fluke: *the side lengths of two similar triangles are always proportional to one another*. To see what this means, let's redraw these triangles without any numbers; instead, we use variables that represent side lengths. This is shown in Figure 6.3.2. Notice that we have used capital letters for the vertices and lowercase letters for the sides. The same letter is used for the side opposite each vertex. What we find is that the side length opposite the same angle has a constant ratio, no matter what pair you look at. That is to say, the ratio of a and d is the same as the ratio of b and e and the ratio of c and f.

Figure 6.3.2



$$\frac{d}{a} = \frac{e}{b} = \frac{f}{c}$$

Similar triangles have their side lengths in proportion.

Image Description

The image shows two similar triangles placed side by side.

On the left is a smaller triangle labelled A, B, and C. Point A is at the bottom left, B is at the top, and C is at the bottom right. The base AC is labelled b, the left side AB is labelled c, and the right side BC is labelled a. Angle markings show one arc at angle A, two arcs at angle C, and three arcs at angle B.

On the right is a larger triangle labelled D, E, and F. Point D is at the bottom left, E is at the top, and F is at the bottom right. The base DF is labelled e, the left side DE is labelled f, and the right side EF is labelled d. Matching angle markings show one arc at angle D, two arcs at angle F, and three arcs at angle E.

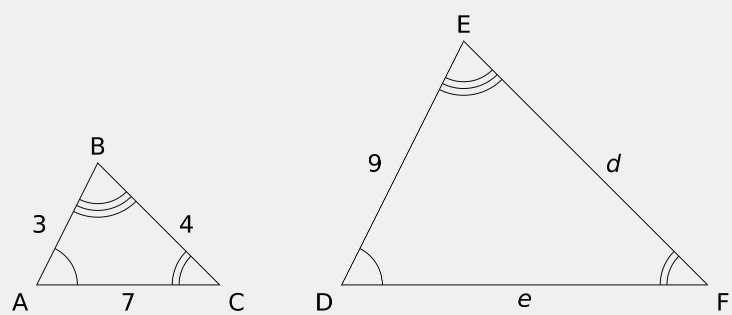
The matching angle markings indicate that the triangles are similar. The corresponding vertices are A ↔ D, B ↔ E, and C ↔ F. The corresponding sides are AB ↔ DE, BC ↔ EF, and AC ↔ DF.

The opposite is true as well: if the ratio between the side lengths of two triangles is the same, then they are similar!

Exercises 6.3

Determine the lengths of the unlabelled sides in the pairs of similar triangles below:

1.



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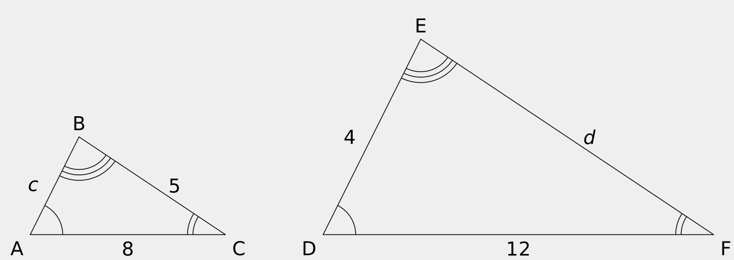


3. Can a triangle with side lengths 1, 2, and 3 be similar to a triangle with side lengths 2, 3, and 4? Why or why not?

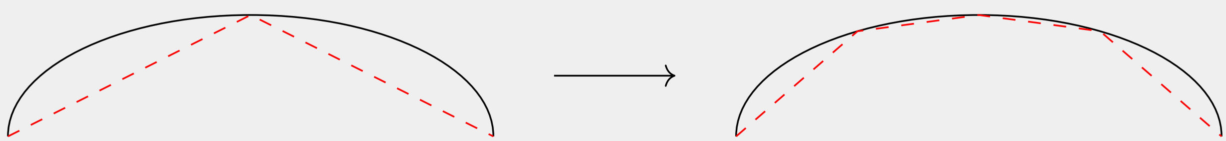
4. A triangle has sides of length $5, 7$, and 13. A second triangle is similar to the first and has one side of length 385. What are the possible lengths of its other sides?

5. A pair of trains leave a railway station at the same time in different directions. Their tracks are fairly straight, so that they will travel in the same direction the entire time. After an hour, they are 25km apart. How far apart will they be after two hours?

6.4 CIRCUMFERENCE OF A CIRCLE

So far, we have only studied how to measure straight lines, but what about curved ones? A ruler will never perfectly align with a curved edge, so we will never get a precise measurement of a curve by hand. But we can always try to approximate them closely with straight lines. As shown in Figure 6.4.1, we can always get a better approximation. With many shapes, we can apply this process to get very good approximations of their length.

Figure 6.4.1



Approximating the length of a curved line.

Image Description

The image shows a curved arch-like shape being changed from one form to another. On the left, a black curved arc connects two bottom endpoints, forming a wide arch. Inside the arch, two red dashed straight lines connect each bottom endpoint to a high point near the top centre, forming a large triangle under the curve.

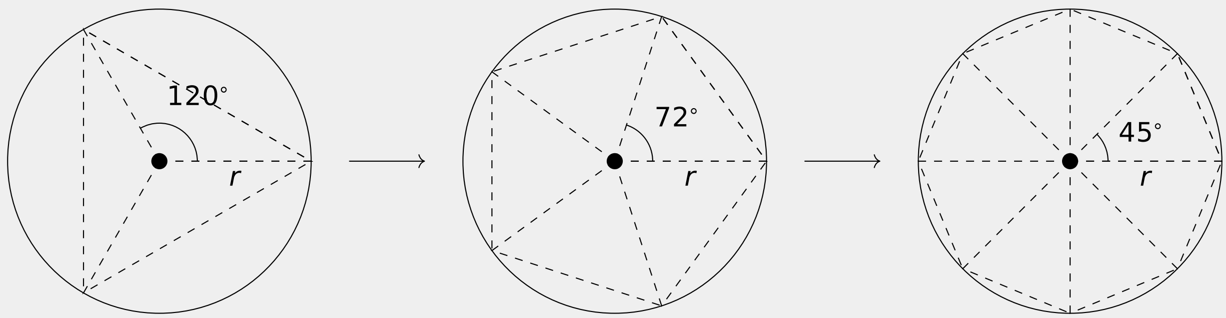
A right-pointing arrow in the middle indicates a transformation or comparison.

On the right, a similar black curved arch is shown. Inside it, the red dashed outline has changed into a multi-sided shape that follows the curve more closely: it rises from the left endpoint to an upper-left point, continues almost horizontally along the top near the curve, then slopes down to the right endpoint. This suggests approximating a curved shape using more straight-line segments.

An important curve to know the length of is the circumference of a circle. The circumference is the special name for a circle's perimeter. A circle is a special shape: every point on the circle's circumference shares the same distance from the circle's center. This distance is called the radius.

We can use ideas from similar triangles to show that a circle has a radius proportional to its radius. Indeed, we can approximate a circumference by dividing the inside of a circle into triangles, as shown in Figure 6.4.2. Closer approximations of the circumference will use more triangles. Using similar triangles, we know that the edges of the triangles approximating the circumference are all proportional to the radius of the circle. Due to this, our approximations themselves are proportional to the radius, and so the circumference itself is proportional to the radius.

Figure 6.4.2



Approximating the circumference of a circle with similar triangles.

Image Description

The image shows three circles arranged horizontally, with arrows between them to show a progression. Each circle contains a regular polygon drawn with dashed lines, along with dashed radius lines from the centre to the polygon's vertices.

In the first circle, a dashed triangle is inscribed inside the circle. A black dot marks the centre, and a dashed radius extends from the centre to the right edge of the circle, labelled r. An angle at the centre is labelled 120°, representing the central angle between two triangle vertices.

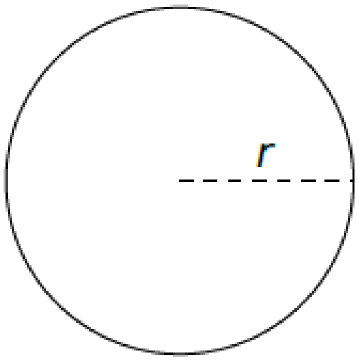
In the second circle, a dashed pentagon is inscribed inside the circle. A black dot marks the centre, and a dashed radius to the right is labelled r. An angle at the centre is labelled 72°, representing the central angle between two pentagon vertices.

In the third circle, a dashed octagon is inscribed inside the circle. A black dot marks the centre, and a dashed radius to the right is labelled r. An angle at the centre is labelled 45°, representing the central angle between two octagon vertices.

The progression shows that as the number of sides in the inscribed polygon increases, the central angle between adjacent vertices becomes smaller and the polygon more closely approximates the circle.

The amount of this proportion turns out to be $2 \times \pi$, (π is a Greek letter that is pronounced like pie). The value of $\pi \approx 3.14$. Therefore, the circumference of a circle is $2 \times \pi \times r \approx 2 \times 3.14 \times r$.

Circumference of a Circle

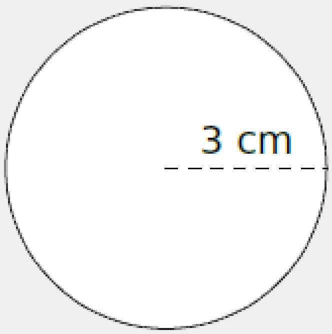


$$C = 2 \times \pi \times r \approx 2 \times 3.14 \times r$$

Exercises 6.4

Type your exercises here.

1.



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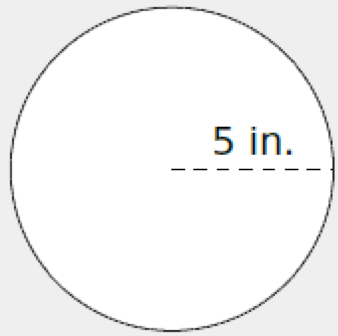

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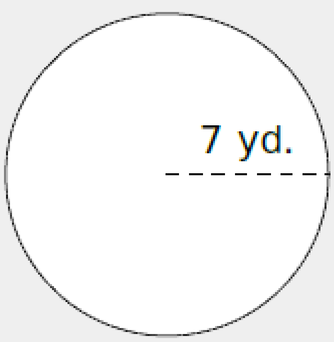

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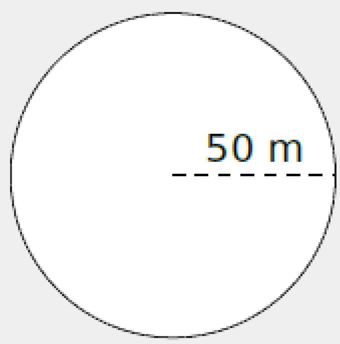


5. The *diameter* of a circle is the distance between two opposite edges going through the center

of the circle. Rewrite the formula for the circumference using diameter instead of radius.

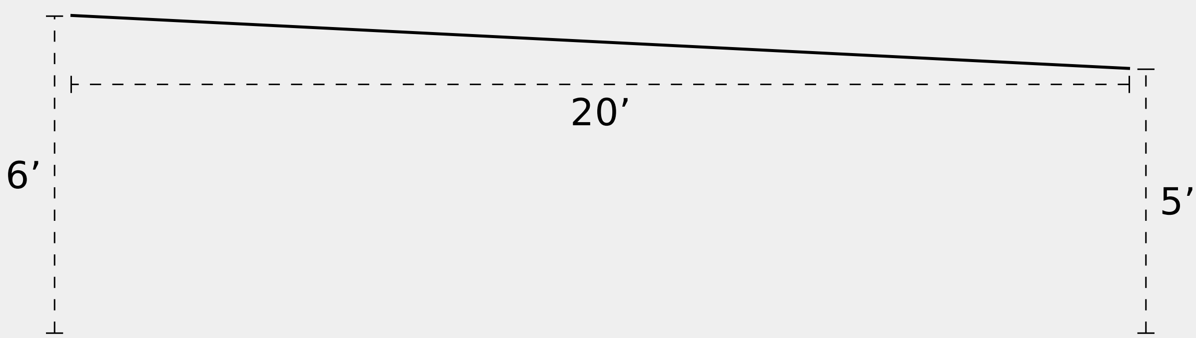
6.5 GRADES

Although we have worked with angles so far to measure rotation, there is an alternative to angles that we call *grade*. Grade is a measurement of *slope*, which is the fall in height over a distance. Due to this, the gradient is a ratio and can be represented with fractions or percentages. For example, a flat slope would have a grade of 0%, since its height does not increase at all over a distance. On the other hand, a slope that is very steep will have a grade larger than 100% since its height falls faster than the distance it travels.

For example, consider a metal rod that starts at an elevation of $6'$, travels a horizontal distance of $20'$, and ends at a height of $5'$. Since it falls $1'$ over a length of $20'$, it has a grade of $\frac{1'}{20'}$ or 5%, as shown in Figure 6.5.1.

Grades can also be negative if the height of the slope is downward. For example, drainage has a negative slope because we want gravity to carry water away from a source to some other location. For example, suppose we have pipes that begin at a height of $10''$, run $10''$, then end at a height of $6''$ above the ground. Then it rises $-4''$ and runs $10''$, so its grade is $-\frac{2}{5}\text{in./ft}$. Notice that since the rise and run were in different units, we have kept the units as usual.

Figure 6.5.1



Computing the grade of a slope.

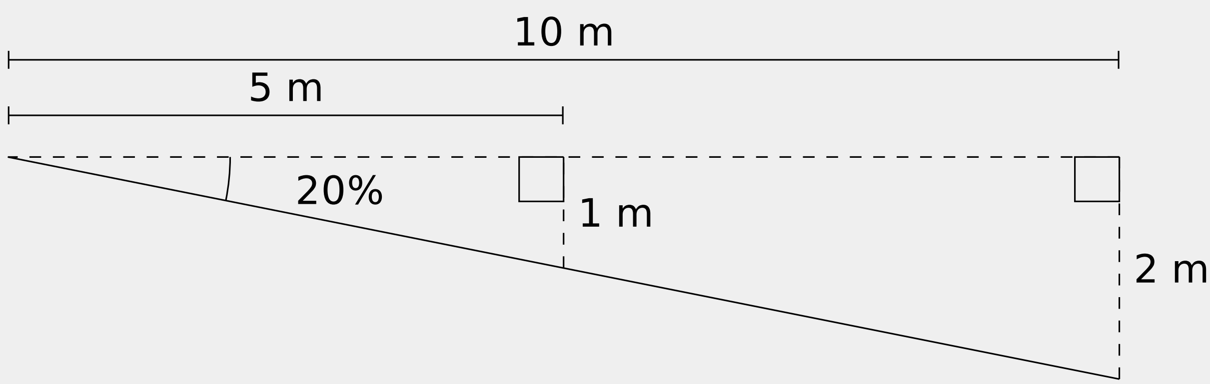
Image Description

The image shows a long, slightly slanted line segment representing the top edge of a shape or surface. The left end is higher than the right end. A dashed vertical measurement line on the left is labelled 6′, and a dashed vertical measurement line on the right is labelled 5′. A dashed horizontal measurement line extends between the two ends and is labelled 20′. The diagram shows a 20-foot horizontal distance with the height decreasing from 6 feet on the left to 5 feet on the right.

do in calculating proportions, since they do not cancel out without further conversion. This grade tells us that every 5ft., our pipe drops 2in.

We can see that grade is a substitute for angles by using the properties of similar triangles and right triangles covered in the last two sections. Suppose that I am measuring two slopes, but one rises 1m over a distance of 5m, and the other rises a distance of 2m over 10m. Then both have a grade of 20%. Also, the angle of the slope is the same between the two: the height and distance of the second slope are both double those of the first. Then, if we use the Pythagorean theorem, we see that the length of the second slope itself is double that of the first slope. Therefore, they are similar triangles and have the same angle. This idea is shown in Figure 6.5.2.

Figure 6.5.2



Slopes of the same grade have the same angle.

Image Description

The image shows a sloped line descending from left to right beneath a horizontal dashed reference line. The full horizontal distance across the diagram is labelled 10 m, and a shorter horizontal distance from the left starting point to the middle point is labelled 5 m.

At the 5 m point, a dashed vertical measurement line extends from the horizontal reference line down to the sloped line and is labelled 1 m. At the 10 m point, another dashed vertical measurement line extends down to the sloped line and is labelled 2 m. Small square markers appear at the top of both vertical measurement lines to indicate right angles with the horizontal reference line.

Near the left side, an angle between the horizontal dashed line and the descending slope is labelled 20%, showing the grade or slope. The diagram illustrates that the slope drops 1 m over 5 m and 2 m over 10 m.

Due to this, if something needs to be installed at a certain angle, a design can specify the grade instead. Using the grade, you can vary the length or the fall based on what is required for the situation at hand. For instance, if you are working with a grade of 10% and you need to run a length of 10ft., then you would calculate $10\% \times 10\text{ft.}$ to compute the necessary fall for that grade, which is 1ft. On the other hand, if you wanted to know how much length you needed to fall 1.5ft. at a 10% grade, you would calculate $1.5\text{ft.} \div 10\%$, which is 15ft.

Sometimes, it is useful to write a grade using units, especially if it is a very small grade. For instance, imagine a length of 8ft that has a fall of 2in. One way to determine the grade of this would be to convert the feet into inches, then take our ratio, so then the grade would be $\frac{2}{96} = \frac{1}{48}$. However, we could keep the units as given and instead take the grade as $\frac{2''}{8ft.} = \frac{1}{4}''$ per foot.

Finding grade, fall, or length

$$grade = \frac{fall}{length}$$

$$fall = grade \times length$$

$$length = \frac{fall}{grade}$$

Remember to keep track of the units you are using.

Exercises 6.5

1. What is the grade of a 45° slope? What is the grade of a 90° slope?

Calculate either the grade, fall, or length based on the available information:

2. **grade = $\frac{1}{4}$″ per foot**
   fall =
   length = 75 ft.

3. **grade = 1%**
   fall =
   length = 125 ft.

4. **grade = 1.04%**
   fall = $\frac{3}{4}$ ft.
   length = _____ ft.

5. **grade = $\frac{1}{8}$″ per foot**
   fall = 11 in.
   length = _____ ft.

6. **grade = _____″ per foot**
   fall = 15 in.
   length = 75 ft.

7. **grade =**
   fall = $4\frac{1}{2}$ ft.
   length = 375 ft.

7 GEOMETRY

Chapter Outline

Geometry is the mathematical study of space. So far, we have touched on aspects of geometry through measurement and trigonometry, through ideas like distance and angle. This section will continue to build off of these ideas by introducing the concepts of area and volume. We will begin each concept by starting at some simple cases, then building up towards more complex scenarios that are applications of these simple cases.

7.0 LEARNING OBJECTIVES

Learning Objectives

By the end of this chapter, students will be able to:

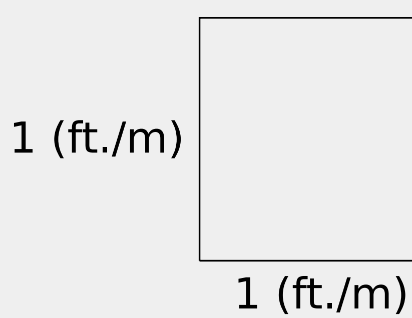
- Compute the areas of simple two-dimensional regions using formulas (5.1)
- Compute the area of a complex two-dimensional region as the sum of several simple two-dimensional areas (5.2)
- Compute the volume of a three-dimensional region using formulas (5.3)

7.1 BASIC AREAS

A common problem in the trades and other aspects of everyday life revolves around comparing the sizes of two-dimensional or flat shapes. For example, if you are painting something, such as a wall, you want to know how much paint is needed to cover the wall. Obviously, walls come in different shapes. But two differently shaped walls might require the same amount of paint. The idea of *area* helps us tackle this problem of comparing the coverage of different shapes.

Like with length, we start with a basic unit that we can use as a building block for measuring other areas. In the imperial system, this basic unit is the *square foot*, written ft^2, and in the metric system, it is the *square meter*, written m^2. Both are represented by a two-dimensional square, with side lengths of $1 \mathrm{ft.}$ or $1 \mathrm{m}$, respectively, as shown in Figure 7.1.1.

Figure 7.1.1

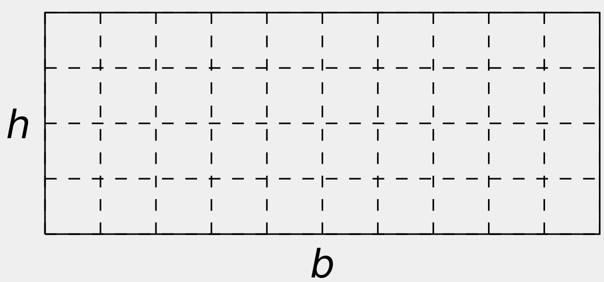


A square unit.

Starting from this basic unit, we measure the areas of other shapes by finding exactly (or approximately) how many of these basic units can fit inside it, perhaps with the need to cut the units up in order to fit properly. For example, the area of a square with side length s is s^2, because you can fit an $s \times s$ grid of square units inside this square.

More generally, the area of a rectangle is the product of its base (b) and height (h) because we can make a $b \times h$ grid of units square to cover it, as seen in Figure 7.1.2.

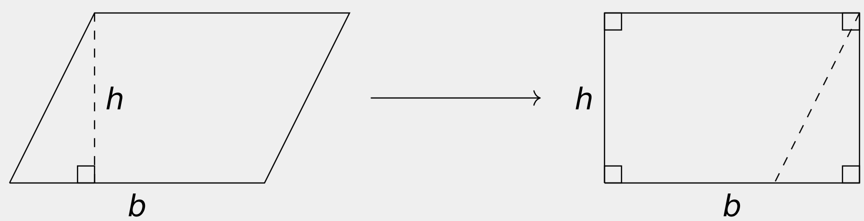
Figure 7.1.2



The area of a rectangle is the product of its base and height.

A parallelogram is a shape that has two pairs of equal sides that are opposite from one another. Due to this, we can cut it at a right angle to its base at one of its vertices, then reattach its pieces to get a rectangle. In doing this, we have not changed the area, so that we can use the formula for the area of the rectangle. So, the area of a parallelogram is given by the product of its base and height.

Figure 7.1.3



The area of a parallelogram is the product of its base and height.

Image Description

The image shows how a parallelogram can be rearranged into a rectangle to show the relationship between their areas.

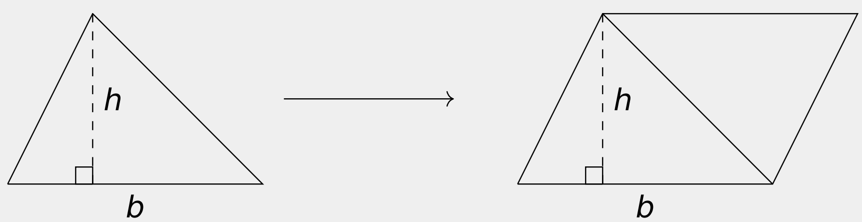
On the left, there is a slanted parallelogram with the bottom side labelled b for base. A dashed vertical line inside the parallelogram is labelled h for height, and a small square at the bottom of the dashed line marks a right angle between the height and the base.

A right-pointing arrow in the centre shows the transformation from the parallelogram into a rectangle.

On the right, the shape has been rearranged into a rectangle with the bottom side labelled b and the left side labelled h. Small squares mark the four right angles of the rectangle. A dashed diagonal line near the right side shows where a triangular section from the original parallelogram was moved to form the rectangle.

Lastly, we will determine the area of a triangle using the formula for the area of a parallelogram. Given a triangle, we can copy it by reflecting it over one of its sides that is not its base. Doing this creates a parallelogram with double the area of the original triangle. This parallelogram has the same base and height as the triangle. Thus, the area of a triangle is half of the product of its base and height.

Figure 7.1.4



The area of a triangle is half of the product of its base and height.

Image Description

The image shows how a triangle can be related to a parallelogram to explain the area formula.

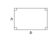
On the left, there is a triangle with a horizontal base labelled b. A dashed vertical line from the top vertex down to the base is labelled h, representing the height. A small square at the bottom of the dashed line marks a right angle between the height and the base.

A right-pointing arrow in the centre shows a transformation or comparison.

On the right, two congruent triangles are arranged together to form a parallelogram. The bottom side of the parallelogram is labelled b, and a dashed vertical height is labelled h. A small square marks the right angle where the height meets the base. A diagonal line inside the parallelogram shows the boundary between the two triangles, illustrating that the original triangle is one-half of a parallelogram with the same base and height.

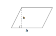
The results of this section are summarized in the box below:

Basic Area Formulas



(a)

$A = b \times h$



(b)

$A = b \times h$

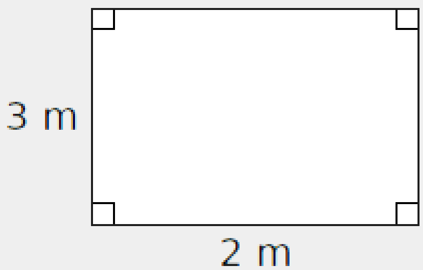
(c)

$A = \frac{b \times h}{2}$

Exercises 7.1

Calculate the areas of the following shapes. Make sure to include units in your answer.

1.



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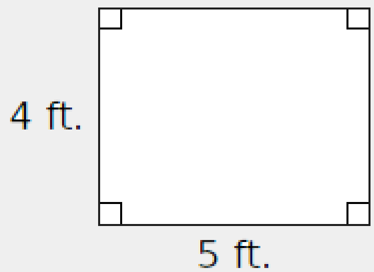

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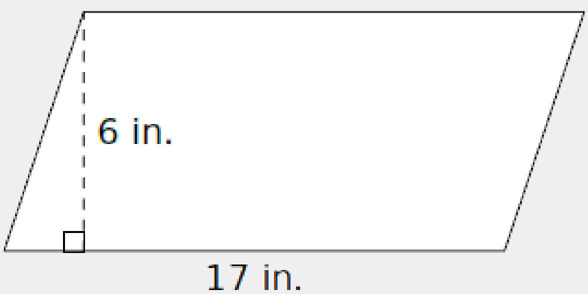

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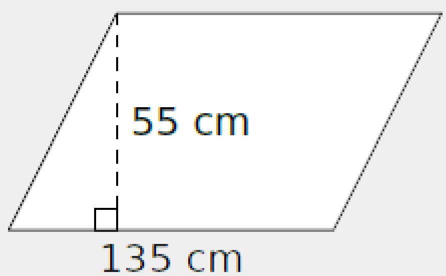
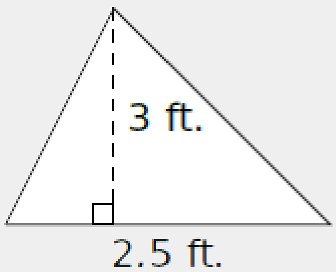
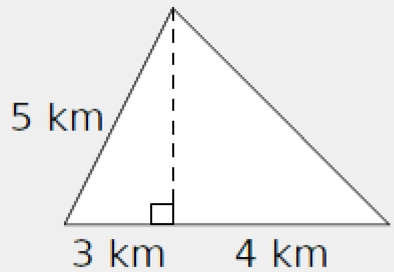


5.

6.





7. How many cm^2 are in a m^2?

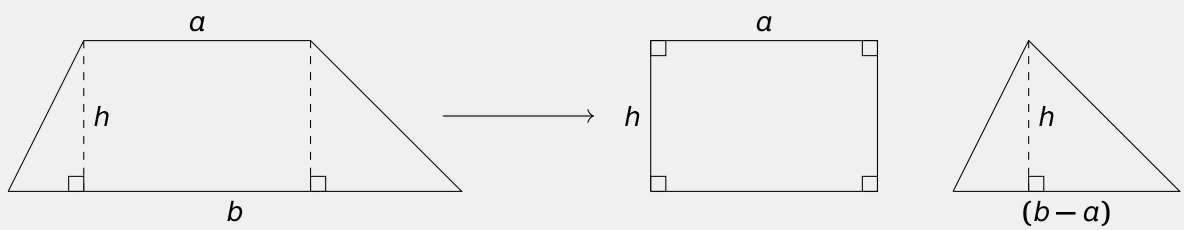
8. How many $in.^2$ are in a $ft.^2$?

7.2 COMPLEX AREAS

The last section discussed how to determine the areas of some basic shapes: rectangles, parallelograms, and triangles. A remarkable fact is that most other shapes can be easily built up from these shapes.

For example, a trapezoid is a four-sided shape with two parallel sides, but these sides can be different lengths. These parallel sides are called the *bases* of the trapezoid, and they have lengths of a and b. The distance between them is the height, h. If the trapezoid is cut perpendicular to the base of length a at its vertices, we obtain a rectangle and two triangles with the same height that can be glued together. The rectangle has base a and height h, while the triangle has base $(b-a)$ and height h. This is shown in Figure 7.2.1.

Figure 7.2.1



The area of a trapezoid, split into a rectangle and a triangle.

Image Description

A grayscale mathematical diagram shows how a trapezoid can be decomposed into a rectangle and a triangle.

On the left, there is a trapezoid with a shorter horizontal top base labelled a and a longer horizontal bottom base labelled b. Two vertical dashed lines drop from the endpoints of the top base down to the bottom base, marking off a central rectangular region inside the trapezoid. A vertical dashed segment inside the shape is labelled h, indicating the height of the trapezoid. Small right-angle markers appear where the dashed height lines meet the bottom base, showing that the height is perpendicular to the base.

A right-pointing arrow leads to the decomposed shapes on the right. First, there is a rectangle labelled a across the top and h along the left side. Right-angle markers appear at all four corners of the rectangle. Beside it is a triangle representing the two slanted side portions of the trapezoid combined. The triangle has a vertical dashed height labelled h, with a right-angle marker at the bottom where the height meets the base. The base of the triangle is labelled (b – a), showing that the triangle's base equals the difference between the longer bottom base and the shorter top base of the original trapezoid.

So, the total area is:

$$h \times a + \frac{h \times (b-a)}{2} = h \times \left(a + \frac{b-a}{2}\right)$$

$$= h \times \left(\frac{2a + (b-a)}{2}\right)$$

$$= h \times \frac{(a+b)}{2}$$

Notice that the quantity $\frac{(a+b)}{2}$ is the average size of the bases, and the area of a trapezoid is
the product of its height and the average size of its bases.

What about the area of a circle? This does not seem like something that we can measure with square units, since it is round. While this is true to some extent, we can try to approximate the area of a circle using equal-sized triangles. For instance, we could first try 5, then get a better approximation with 8, as shown in Figure 7.2.2. At each stage, the triangles get closer and closer to covering the circle. If we were to carry on doing this forever, we would tend to an area of $\pi \times r^2$. We can round π to 3.14 to make calculations simpler.

Figure 7.2.2

Approximating the area of a circle using triangles. This tends to $\pi \times r^2$.

Image Description

A grayscale mathematical diagram shows a progression of regular polygons inscribed inside circles, suggesting that as the number of polygon sides increases, the polygon more closely approximates the circumference of the circle.

On the left, a circle contains an inscribed regular pentagon drawn with dashed lines. The centre of the circle is marked with a solid black dot. Dashed radii extend from the centre to the pentagon's vertices, dividing the pentagon into five triangular sections. One radius extends horizontally to the right edge of the circle and is labelled r. An angle at the centre between two adjacent radii is marked with a curved arc and labelled 72°, representing the central angle of one sector of the pentagon.

A right-pointing arrow leads to a second circle on the right. This circle contains an inscribed regular octagon drawn with dashed lines. The centre is again marked with a solid black dot, with dashed radii extending to the octagon's vertices. The horizontal radius to the right is labelled r. A smaller central angle is marked with a curved arc and labelled 45°, showing the central angle between adjacent radii in the octagon.

Another arrow points to an ellipsis, indicating the pattern continues with polygons that have more sides and smaller central angles, increasingly resembling the circle.

Lastly, let's work through a concrete example of a complex shape, such as the shape in Figure 7.2.3. It looks like a capsule, with rounded ends and a straight center. We can break this into a rectangle with a base of 3m and a height of 2m, as well as a circle with a radius of 1m. Using the formulas for the area of a circle and a rectangle, we compute that:

$$\begin{aligned} A &= (3m) \times (2m) + \pi \times (1m)^2 \\ &= 6m^2 + 3.14m^2 \\ &= 9.14m^2 \end{aligned}$$

Figure 7.2.3

Computing the area of a complex shape.

Image Description

A grayscale mathematical diagram shows a compound shape being decomposed into simpler shapes to calculate its area.

On the left, there is a capsule-like shape made from a central rectangle with a semicircle attached to each end. The central rectangular portion is marked by two vertical dashed lines. The rectangle is labelled 3 m along the bottom and 2 m vertically inside the shape, indicating that the rectangle is 3 metres long and 2 metres high. A small right-angle marker appears at the lower-right corner of the rectangular section. On the left semicircle, a dashed horizontal radius from the curved edge to the dashed vertical line is labelled 1 m, showing that each semicircle has a radius of 1 metre.

A right-pointing arrow indicates that the compound shape can be separated into its component parts. On the right side of the diagram, the shape is shown as a 3 m by 2 m rectangle and a separate circle with a dashed radius labelled 1 m. This represents the two semicircular ends combined into one full circle.

We summarize the formulas for a trapezoid and a circle below:

Complex Area Formulas

(a)

$A = h \times \frac{(a+b)}{2}$

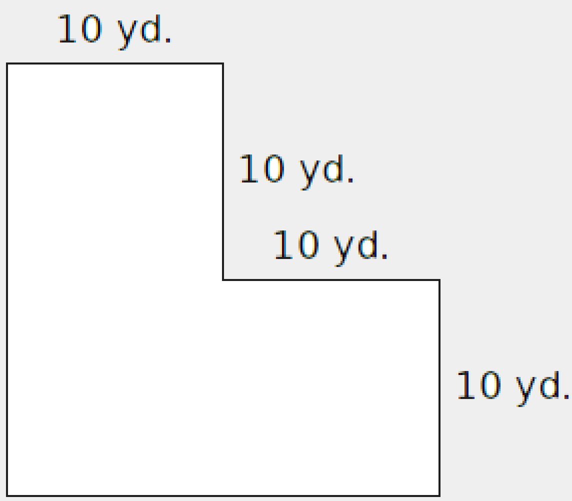
(a)

$A = \pi \times r^2 \approx 3.14 \times r^2$

Exercises 7.2

Calculate the areas of the following complex shapes:

1.



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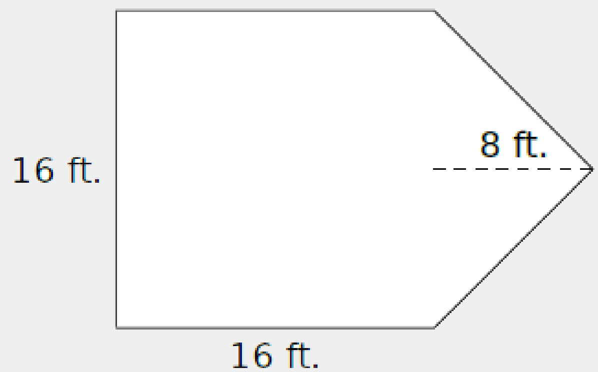

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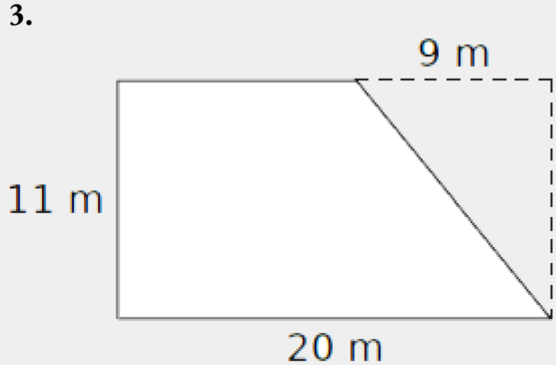

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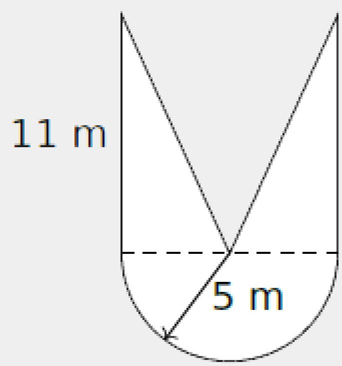

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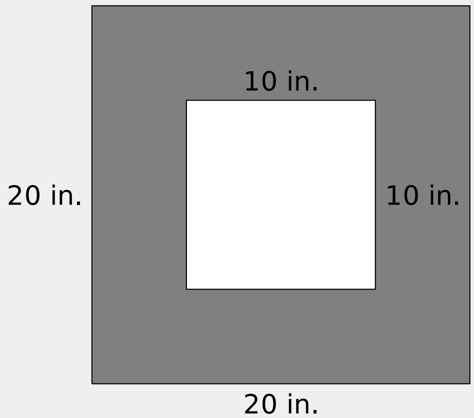

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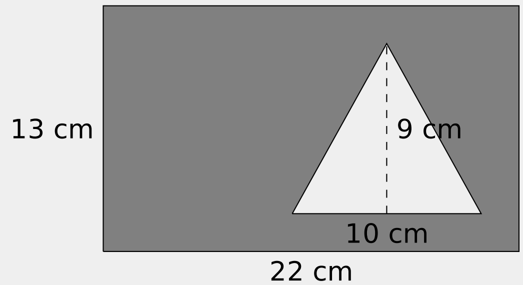
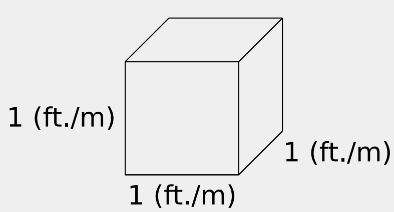


7.3 VOLUMES

Like with two-dimensional objects, we are interested in comparing the size of three-dimensional ones as well. For instance, being able to determine how much fluid a hot water tank can hold. The size of a three-dimensional shape is called *volume*.

We begin by defining our basic unit of volume, which is a *cubed unit*. In the imperial system, this would be a cubed foot, written $ft.^3$, and in the metric system, this is a *cubed metre*, written m^3. As its name implies, this is a cube with sides of length 1ft. or m, as shown in Figure 7.3.1.

Figure 7.3.1

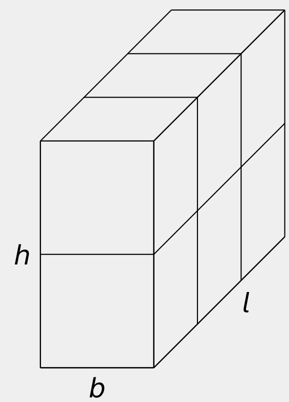


A cubed unit.

A common three-dimensional shape is the *prism*, which you can imagine as a two-dimensional shape that has been stretched like an accordion into the third dimension. For example, a rectangular prism is a rectangle that has been stretched out into the third dimension to form a
brick shape. The rectangle has a base of length b, a height of h, and it has been stretched to a length of l. To calculate its area, we notice that we can create a

three-dimensional $b \times h \times l$ grid of unit cubes. Thus, the area of a rectangular prism is $b \times h \times l$.

Figure 7.3.2



The volume of a rectangular prism is the product of its base, height, and length.

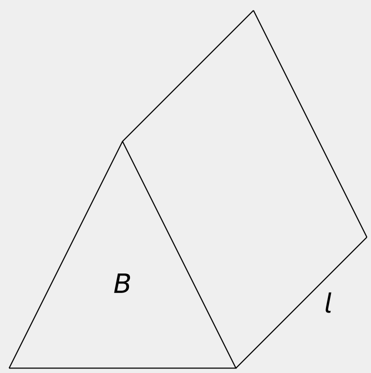
Image Description

The image shows a rectangular prism drawn in perspective. The front face is a rectangle, and the top and right faces are visible, giving the shape a three-dimensional appearance. The front bottom edge is labelled b for base or width, the left vertical edge is labelled h for height, and the depth extending backward along the right side is labelled l for length. Several interior lines divide the prism into smaller rectangular sections, suggesting layers or unit blocks within the solid.

More generally, the volume of any prism is the product of the area of its base and its length. To understand this, picture the base being covered by unit squares for a total area of B. Then, stretch them out along the length of the prism. Then, the total volume of each stretched unit
square is the length, l. Since there are a total of B unit squares, we can just multiply $B \times l$ to get the total volume. This is visualized in Figure 7.3.2.

A pyramid is a three-dimensional shape that has the base of a two-dimensional shape, but instead of stretching evenly into the third dimension, the base is gradually *shrunk* until it becomes a single point, called the *apex* of the pyramid. The volume of a pyramid is given by $\frac{1}{3}(B \times h)$, where B is the area of its base, and h is the height of the apex. A diagram of this is shown in Figure 7.3.3.

Figure 7.3.3

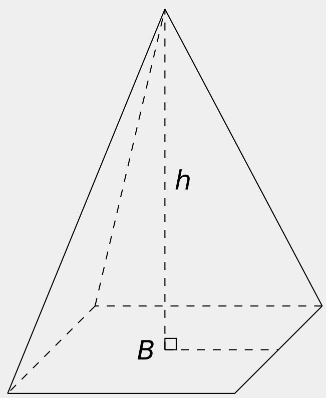


The volume of a prism is the product of the area of its base and its length.

Image Description

The image shows a triangular prism drawn in perspective. The front face is a triangle labelled B, representing the area of the triangular base. From the triangle, the prism extends diagonally upward and to the right, forming a three-dimensional solid. The slanted side edge of the prism is labelled l, representing the length of the prism.

Figure 7.3.4



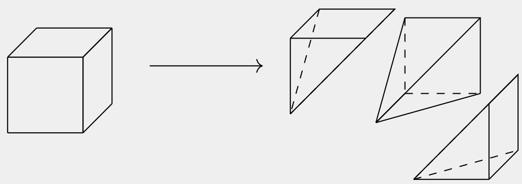
The volume of a pyramid is a third of the product of its base area and its height.

Image Description

The image shows a pyramid drawn in perspective. The base is shown as a slanted quadrilateral and is labelled B, representing the area of the base. The pyramid's triangular sides meet at a single top vertex. A dashed vertical line runs from the top vertex down to the base and is labelled h, representing the height of the pyramid. A small square at the bottom of the dashed line marks a right angle where the height meets the base. Additional dashed lines show hidden edges inside the pyramid.

The formula for the volume of a pyramid tells us that its volume is a third of the volume of the prism with the same base. Why is this? Admittedly, it's actually hard to give a simple explanation for why this is. The most intuitive reason is that the unit square can be split into three pyramids with a square base. Notice that each of these pyramids shares the same vertex of the cube as its apex, and is all the same volume. So, each of these is a third of a cube.

Figure 7.3.5



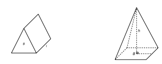
A pyramid is a third of a prism.

Image Description

The image shows a cube on the left with an arrow pointing to the right. On the right side, the cube is shown separated into three congruent pyramid-like pieces. Each piece has triangular faces, slanted edges, and dashed lines to indicate hidden edges. The diagram illustrates that a cube can be decomposed into three equal pyramids.

The main results of this section are summarized below:

Volume Formulas



(a) $V = B \times h$

(b) $V = \frac{1}{3} B \times h$

Image Description

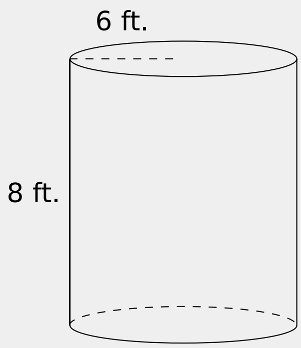
On the left is a triangular prism drawn in perspective. Its front face is a triangle labelled B, representing the area of the triangular base. The prism extends diagonally upward and to the right, and one slanted side edge is labelled l, representing the length of the prism.

On the right is a pyramid drawn in perspective. Its base is shown as a slanted quadrilateral labelled B, representing the area of the base. The triangular faces meet at a single top vertex. A dashed vertical line from the top vertex to the base is labelled h, representing the height. A small square marks the right angle where the height meets the base, and dashed lines show hidden edges inside the pyramid.

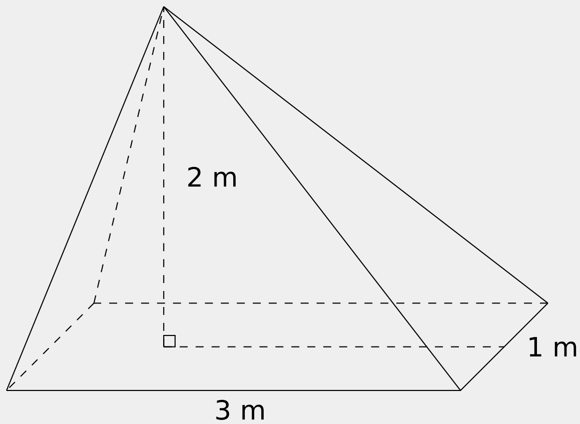
Exercises 7.3

Calculate the volumes of the following solids:

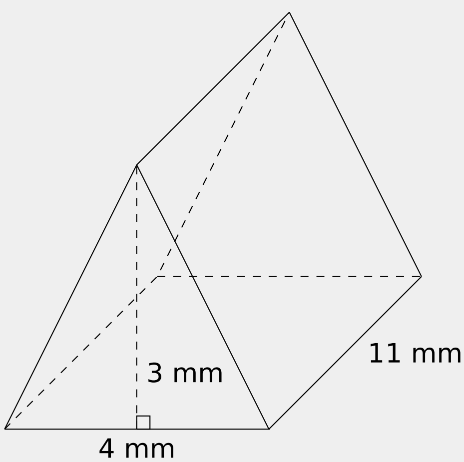
1.



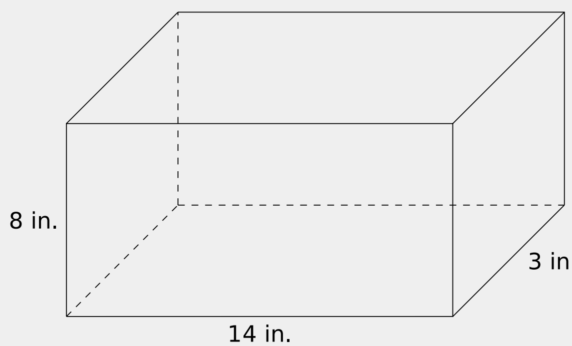
2.



3.



4.



5. How many cm^3 are in a m^3?

6. How many $in.^3$ are in a $ft.^3$?

8 FINANCIAL MATH

Chapter Outline

As long as money is the way that we exchange for materials and how we are compensated for our work, it will be important for tradespeople to have an understanding of their finances, especially when it comes time to take out a loan from a bank so you can fund your work.

This section looks at the mathematics behind lending money and the primary way that the lender is compensated for their loan, interest, which is a rate at which additional money accumulates on your loan, based on how much you have borrowed.

8.0 LEARNING OBJECTIVES

Learning Objectives

By the end of this chapter, students will be able to:

- Identify the difference between simple and compound interest (6.1-2)
- Use the simple and compound interest formulas to compute the cost of a loan (6.1-2)

8.1 SIMPLE INTEREST

Let's imagine that you have borrowed $\$1000$ from a banker to finance some new tools. The banker says that you owe a 5% annual simple interest with monthly installments. What does this mean?

Well, since you owe the banker 5% annually, each month you owe $5\% \div 12 = 0.417\%$. So, a month from now, the banker will say that you owe them $0.417\% \times \$1000 = \$4.17$. Since the interest is simple, you will owe this same amount again the next month. So, the equation for how much you owe the bank after N months is:

$$\$1000 + N \times \$4.17$$

Therefore, after 18 months, you will owe $\$1000 + 18 \times \$4.16 = \$1075.06$.

The equation for the amount that you owe on a loan with simple interest is given by the equation:

Simple Interest Formula

$$\begin{aligned} A &= P + N \times \frac{r}{k} \times P \\ &= P\left(1 + \frac{N \times r}{k}\right) \end{aligned}$$

- A is the amount owed,
- P is the principle amount borrowed,
- r is the annual interest rate,
- k is number of time periods/year,
    - $k = 1$ for years,
    - $k = 12$ for months,
    - $k = 365$ for days.
- N is the number time periods.

Exercises 8.1

Type your exercises here.

1. You take out a loan for $\$25,000$ with a 2% annual simple interest, calculated monthly. How much do you owe after 2 years?

2. You take out a loan for $\$5,000$ with a 7% annual simple interest, calculated daily. How much do you owe after 160 days?

3. You take out a loan for $\$14,000$ with a 1% annual simple interest, calculated yearly. How much do you owe after 50 years?

8.2 COMPOUND INTEREST

Let's imagine again that we have borrowed $\$1000$ from a banker to finance new tools. But, this time, the banker says that you owe a 5% annual interest that is *compounded* monthly. What does the word *compounded* mean?

Compound interest is a situation where not only do you owe interest on the amount that you borrowed, but you also owe interest on the interest that you have already accumulated. In this situation, the monthly interest rate is again $5\% \div 12 = 0.417\%$. In the first month, you have not accumulated any interest, so the interest payment for that month is $0.417\% \times \$1000 = \$4.17$. The next month, however, you owe interest on what has accumulated so far; that is, you owe 0.417% of $\$1004.17$, or $\$4.19$. Notice that, although this is not a massive increase in the monthly payment, the amount has increased. Moreover, it will continue to increase more rapidly as it continues to compound with itself. For example, after five years, you will now owe $\$5.33$, which is over a dollar more than you originally owed.

The equation for compound interest is given by:

Compound Interest Formula

$$A = P\left(1 + \frac{r}{k}\right)^N$$

- A is the amount owed,
- P is the principle amount borrowed,
- r is the annual interest rate,
- k is number of time periods/year,
    - $k = 1$ for years,
    - $k = 12$ for months,
    - $k = 365$ for days.
- N is the number time periods.

Exercises 8.2

Type your exercises here.

1. You take out a loan for $\$25,000$ with a 2% annual compound interest, compounded monthly. How much do you owe after 2 years?

2. You take out a loan for $\$5,000$ with a 7% annual compound interest, compounded daily. How much do you owe after 160 days?

3. You take out a loan for $\$14,000$ with a 1% annual compound interest, compounded yearly. How much do you owe after 50 years?

APPENDIX 1: SOLUTIONS

Solutions

1 MATHEMATICAL OPERATIONS

1.1 Addition

1. $3+4$ requires more counting than $4+3$. In computing $3+4$, you begin at 3 and then count 4 additional items. In computing $4+3$, you begin at 4 and then count 3 additional items. Thus, computing $3+4$ requires counting more numbers than $4+3$.
2. The expressions in the first pair of sums both equal 7. The expressions in the second pair of sums equal both 9. Since addition is the total number of items between two groups of items, the order of the two numbers does not matter, as the combination of the two groups remains the same. This applies to any two numbers, not just the ones found in the expressions above, i.e., $(X+Y)=(Y+X)$for all numbers X and Y. Operations like $+$ where this is true are called *commutative*.
3. The expression does not make it clear whether you should compute $(1+2)$ or $(2+3)$ first. So, you could compute:
   $((1+2)+3)=(3+3)=6$,
   Or, you could compute:
   $(1+(2+3))=(1+5)=6$.
   Both methods result in the same outcome. This is because it does not matter in what order you combine a list of groups together; the final group will always be the same. This applies to all numbers, i.e., $(X+(Y+Z))=((X+Y)+Z)$ for all numbers X,Y,Z. Operations like $+$ where this is true are called *associative*.
4. $0+1=1$ and $0+2=2$ since you start from zero and count up to the second number in the expression for each expression. This logic applies to all expressions of this type, i.e., $(0+X)=X$ for all numbers X. We called 0 the *additive identity*.

1.2 Multiplication

1. (3×4) requires more addition than (4×3). In computing (3×4), you compute $(((3+3)+3)+3)$,

which uses addition three times. In computing (4×3), you compute
$((4+4)+4)$,
which uses addition twice. Thus, computing (3×4) requires adding more numbers than (4×3).

2. The expressions in the first pair of sums both equal 10. The expressions in the second pair of sums equal both 8. Since multiplication is the total number of items from combining a group with itself some number of times, we can represent it in a rectangle where the width is the size of the group, and the height is the number of groups. When you rotate the rectangle by a quarter turn, the height becomes the width, and vice versa, but its overall size remains the same. Since this is the rectangle we get from taking the product in the other order, the products are the same. This applies to any two numbers, not just the ones found in the expressions above, i.e., $(X \times Y) = (Y \times X)$ for all numbers X and Y. Operations like $\times$ where this is true are called *commutative*.
3. The expression does not make it clear whether you should compute (2×3) or (3×4) first. So, you could compute:
$((2 \times 3) \times 4) = (6 \times 4) = 24$,
Or, you could compute:
$(2 \times (3 \times 4)) = (2 \times 12) = 24$.
Both methods result in the same outcome. In computing a product of two numbers, we can construct a rectangle with the height of one number and the width of the second number. If we take a product with one other number, we can construct a rectangular prism with a depth equal to this last number. Looking at it this way, it does not matter what order the operations are completed in because they all create a congruent rectangular prism. This applies to all numbers, i.e., $(X \times (Y \times Z)) = ((X \times Y) \times Z)$ for all numbers X, Y, Z. Operations like $\times$ where this is true are called *associative*.
4. $(1 \times 2) = 2$ and $(1 \times 3) = 3$ since we are copying groups of size equal to one by the number of times found in the second part of the expression, which is the same as counting up to the second number in the expression. This logic applies to all expressions of this type, i.e., $(1 \times X) = X$ for all numbers X. We called 1 the *multiplicative identity*.
5. $((2 \times 3) + (2 \times 4)) = (6 + 8) = 14$ and $(2 \times (3 + 4)) = (2 \times 7) = 14$, so the expressions have the same value. In the first expression, we can imagine that we are combining two rectangles together, both with width 2, but one with height 3 and the other with height 4. Combining these two rectangles gives us a single rectangle of width 2 and height 7. Since this is the rectangle we get

from the second expression, the expressions are equal. This logic applies to all expressions of this type, i.e., $((X \times Y) + (X \times Z)) = (X \times (Y + Z))$ for all numbers X, Y, Z.

6. $(0 \times 1) = 0$ and $(0 \times 2) = 0$ because in each expression we are counting up groups of size zero, which never changes. This logic applies to all expressions of this type, i.e., $(0 \times X) = 0$ for all numbers X. The *zero property of multiplication* states that if X and $Y \neq 0$, but $(X \times Y) = 0$, then $X = 0$.

1.3 Exponentiation

1. It is true that $(2^4) = (4^2)$. It is not true that $(2^5 = 5^2)$.
2. Notice that it is unclear what exponent to resolve first: (2^3) or (3^2). If we choose the first option, we compute $((2^3)^2) = (8^2) = 64$. On the other hand, $(2^{(3^2)}) = (2^9) = 512$. So, we get different results depending on how we compute this expression. This is because exponentiation is not commutative.
3. $(2^1) = 2$ and $(3^1) = 3$. In general, $X^1 = X$ for all numbers X.
4. $(1^2) = 1$ and $(1^3) = 1$. In general, $1^X = 1$ for all numbers X.
5. $(0^2) = 0$ and $(0^3) = 0$. In general, $0^X = 0$, for all numbers X as long as $X \neq 0$.
6. First we see that $((3 \times 4)^2) = (12^2) = 144$. Next, we see that $((3^2) \times (4^2)) = (9 \times 16) = 144$. Therefore, these expressions are equal. In fact $((X \times Y)^Z) = X^Z \times Y^Z$ for all numbers X, Y, Z.
7. First we see that $((2^3) \times (2^4)) = (8 \times 16) = 128$. Next, we see that $(2^7) = 128$. Therefore, these expressions are equal. In fact, $((X^Y) \times (X^Z)) = (X^{(Y+Z)})$ for all numbers X, Y, Z.
8. First we see that $((2 + 3)^2) = (5^2) = 25$. Next, we see that $((2^2) + (3^2)) = (4 + 9) = 13$. Therefore, these expressions are not equal. Generally, $((X + Y)^Z) \neq ((X^Z) + (Y^Z)$ for most numbers X, Y, Z.

1.4 Properties of Operations

1. $$\begin{aligned} ((2^3) \times (2 + 4)) &= ((2^3) \times (2^1 + 2^2)) \\ &= (2^3 \times 2^1) + (2^3 \times 2^2) \\ &= (2^4) + (2^5) \end{aligned}$$
2. The exponentiation property 4 states that a number raised to the power of 0 equals 1. On the other hand, the exponentiation property 7 states that a power of 0 is 0. Mathematicians look at these two contradictory properties of 0, and state that (0^0) is *undefined*.

1.5 Inverse Operations

1. We can rewrite this statement as $(4+X)=9$. The inverse operation of addition is subtraction, so we may subtract 4 from both sides of the equation to remove addition from the left-hand side to obtain: $X=(9-4)=5$. Therefore, X equals 5.
2. We can rewrite this statement as $(3\times Y)=21$. The inverse operation of multiplication is division, so, we may divide 3 from both sides of the equation to remove multiplication from the left-hand side of the equation to obtain: $Y=(21\div 3)=7$. Therefore, Y equals 7.
3. We can rewrite this statement as $(Z^2)=64$. The inverse operation of exponentiation is a root, so we may take the square root of both sides of the equation to remove exponentiation from the left-hand side to obtain: $Z=\sqrt{64}=8$. Therefore, Z equals 8.

1.6 Inverse Operations

1. We can rewrite this statement as $(4+X)=9$. The inverse operation of addition is subtraction, so we may subtract 4 from both sides of the equation to remove addition from the left-hand side to obtain: $X=(9-4)=5$. Therefore, X equals 5.
2. We can rewrite this statement as $(3\times Y)=21$. The inverse operation of multiplication is division, so we may divide 3 from both sides of the equation to remove multiplication from the left-hand side of the equation to obtain: $Y=(21\div 3)=7$. Therefore, Y equals 7.
3. We can rewrite this statement as $(Z^2)=64$. The inverse operation of exponentiation is a root, so we may take the square root of both sides of the equation to remove exponentiation from the left-hand side to obtain: $Z=\sqrt{64}=8$. Therefore, Z equals 8.

1.7 Order of Operations

1. 8
2. 3
3. 1
4. 8
5. 53

6. 0
7. 17
8. 6
9. 13

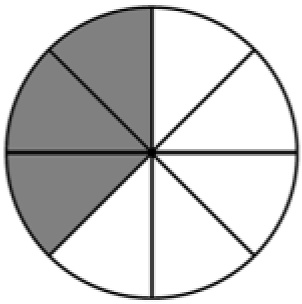
2 INTEGERS

2.2 Rules for Operations with Negative Numbers

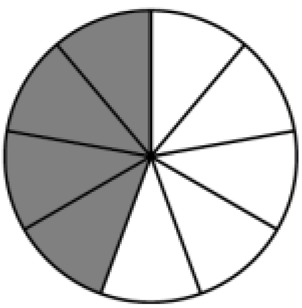
1. (-3)
2. (-3)
3. (-3)
4. (-10)
5. (-10)
6. 10
7. (-4)
8. (-1)
9. 18
10. (-10)
11. (-2)
12. (-14)

3 FRACTIONS

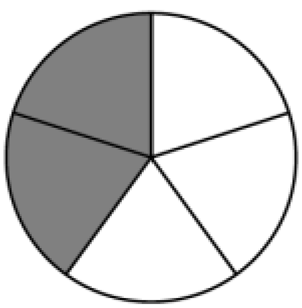
3.1 Visualizing Fractions



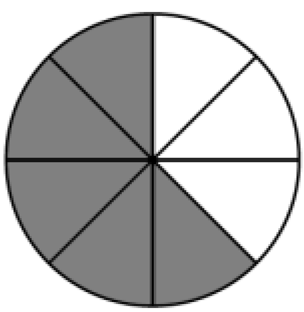
A visual depiction of [latex]\frac{3}{8}[/latex]



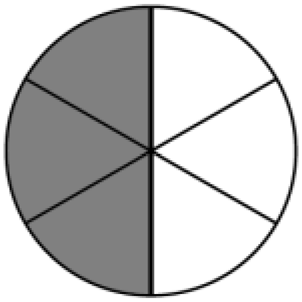
A visual depiction of [latex]\frac{4}{9}[/latex]



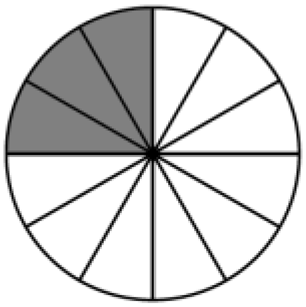
A visual depiction of [latex]\frac{2}{5}[/latex]



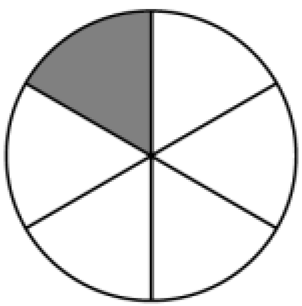
A visual depiction of [latex]\frac{5}{8}[/latex]



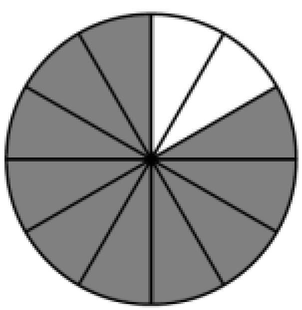
A visual depiction of [latex]\frac{3}{6}[/latex]



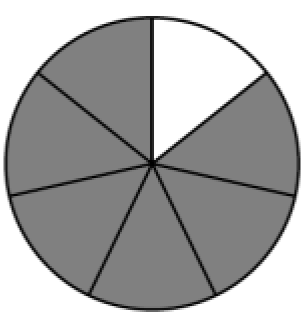
A visual depiction of [latex]\frac{3}{12}[/latex]



A visual depiction of [latex]\frac{1}{6}[/latex]



A visual depiction of [latex]\frac{10}{12}[/latex]



A visual depiction of [latex]\frac{6}{7}[/latex]

10. $\frac{7}{9}$
11. $\frac{3}{5}$
12. $\frac{2}{3}$
13. $\frac{2}{6}$
14. $\frac{1}{11}$
15. $\frac{6}{10}$
16. $\frac{5}{12}$
17. $\frac{2}{8}$
18. $\frac{5}{6}$

3.2 Improper and Mixed Fractions

1. $3\frac{2}{3}$
2. $4\frac{1}{2}$
3. $1\frac{2}{5}$
4. $2\frac{2}{4}$
5. $1\frac{3}{6}$
6. 2
7. $1\frac{4}{9}$
8. $5\frac{1}{3}$
9. $2\frac{2}{11}$
10. $\frac{9}{5}$
11. $\frac{15}{4}$
12. $\frac{12}{5}$
13. $\frac{14}{3}$

14. $\frac{11}{2}$
15. $\frac{25}{7}$
16. $\frac{22}{9}$
17. $\frac{15}{8}$
18. $\frac{15}{6}$

3.3 Simplifying Fractions

1. $\frac{1}{3}$
2. $\frac{2}{3}$
3. $\frac{4}{5}$
4. 2
5. $\frac{9}{2}$
6. $\frac{5}{2}$
7. $2\frac{1}{2}$
8. $5\frac{2}{3}$
9. $3\frac{2}{9}$

3.4 Addition and Subtraction of Fractions

1. 2
2. $\frac{5}{2}$
3. $\frac{7}{10}$
4. $\frac{23}{21}$
5. $\frac{7}{10}$
6. $\frac{9}{4}$
7. $\frac{11}{6}$

8. $6\frac{13}{15}$
9. $1\frac{1}{6}$

3.5 Multiplication and Division of Fractions

1. $\frac{1}{2}$
2. $\frac{9}{25}$
3. $\frac{3}{10}$
4. $\frac{3}{25}$
5. $\frac{3}{25}$
6. $\frac{3}{10}$
7. $\frac{1}{2}$
8. $\frac{5}{7}$
9. $\frac{4}{9}$
10. $\frac{5}{4}$
11. $\frac{1}{2}$
12. $\frac{2}{5}$

3.6 Ratios and Percentages

1. $\frac{2}{3}, \frac{1}{3}$
2. $\frac{7}{13}, \frac{6}{13}$
3. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$
4. $5:2$
5. $2:1$
6. $4:1:5$
7. $\frac{1}{2}$

8. $\frac{2}{5}$
9. $\frac{13}{20}$
10. 25%
11. $66.66\ldots\%$
12. 150%
13. Begin by converting the ratio into fractions. Doing this, we find that we need $\frac{5}{8}$ to be crushed stone, $\frac{1}{4}$ to be sand, and $\frac{1}{8}$ to be cement. Since the total concrete required is $4000lb.$, we need $\frac{5}{8} \times 4000lb. = 2500lb.$ of crushed stone, $\frac{1}{4} \times 4000lb. = 1000lb.$ of mixed stone, and $\frac{1}{8} \times 4000lb. = 500lb.$ of cement.
14. Based on the scale, there are $15cm$ on the blueprint for every $40m$ in reality. Therefore, there is a $15:40$ ratio (in centimetres to metres). Simplifying this, we see that the scale of the blueprint is $3:8$ centimetres to metres.
15. First, convert the ratio $2:1$ into the fractions $\frac{2}{3}$ and $\frac{1}{3}$. Therefore, as a percentage, you get 66.66% of the profits. If we made $\$10,000$ in profit, then I would get $\$6,666.67$ while my partner gets $\$3,333.33$.

4 THE DECIMAL SYSTEM

4.2 Digits and Place Values

1. 500
2. 9000
3. 40
4. 6000
5. 2
6. 0.7
7. 30
8. 0
9. 0.03
10. 0.03
11. 4000000
12. 0.1
13. 624
14. 8013
15. 29.47
16. 1.106
17. Since they have two antennae, they might use a base 2 number system, which would use only the digits 0 and 1. Such a system is called *binary*. It is the same number system that computers use.

4.3 Adding Decimals

1. 85.2
2. 3035.02
3. 89.833
4. 1003.71
5. 5.85987
6. 1111
7. 171.02
8. 6.989

9. 1005.295
10. 8247.01
11. 6.263924
12. 11.20001

4.4 Subtracting Decimals

1. 112
2. 225
3. 15
4. 91
5. 244
6. 239
7. 48
8. 901
9. 4588
10. -17
11. -775
12. -4756

4.5 Multiplying Decimals

1. 175
2. 342
3. 870
4. 1768
5. 4758
6. 32472
7. 39.6
8. 209.95
9. 114.204

4.6 Dividing Decimals

1. 13
2. 13
3. 31

4. $36\frac{1}{4}$
5. $21\frac{1}{3}$
6. $105\frac{2}{11}$
7. $17\frac{5}{7}$
8. $21\frac{3}{13}$
9. $35\frac{109}{120}$

4.7 Rounding Decimals

1. 679.68
2. 4521.11
3. 0.1
4. 54.892
5. 97.232
6. 1000.001
7. 10
8. 5
9. 44
10. 9
11. 4
12. 45
13. Notice that these answers are different, so rounding before vs after calculating an expression changes your result.

5 MEASUREMENT

5.1 The Imperial System

1.

Measurement	Conversion
2ft.	24in.
2lbs.	64oz.
3gal.	12qt.
2mi.	3520ft.
2T	64000oz.
1yd.	36in.
8qt.	32c.

2. Since $25''$ of the rope is lost due to the branch, you have $5'11''$ left for the swing.
3. Yes, you will. There are 16 cups in a gallon, so with 10 gallons of lemonade, you can serve 160 guests. On the other hand, if the jug is half full, then you can only serve 80 guests, which is not enough for all 100.
4. We can convert everything to pounds to simplify the calculation. 2T and 73lbs is 4073lbs. On the other hand, 1T and 1923lbs is 3923lbs. Taking the difference of these values, we obtain 150lbs., which is the amount of waste you brought to the landfill.

5.2 The Metric System

1.

Measurement	Conversion
2m	200cm
4kg	4000g
3L	3000mL
2km	2000m
2g	2000mg
50mm	5cm
2000mL	2L

2. We need 5000g or 5kg of flour.
3. $1000 \div 35 \approx 28.5$, so it will take about 28 and a half minutes.
4. $14km = 14000\text{m}$, and $14000 \div 70 = 200$, so 200 cars can fit safely in one lane. In three lanes, 600 can fit safely.

5.3 Converting Between Systems

1. Converting 5m to feet, we get 16.4ft. Hence, the beams at the hardware store are too short for your project.
2. The total weight of the concrete is $30 \times 42kg = 1260kg$. Therefore, the concrete weighs $2.205 \times 1260 = 2778.3lbs$. So, your truck can carry this material.
3. $1\text{ft.} = 0.305\text{m}$ and $1\text{ft.} = 12\text{in.}$, therefore $1\text{in.} = 0.0254\text{m} = 2.54\text{cm.}$ Therefore, $\frac{1}{2}in = 1.29cm$, so $1\frac{1}{2}in. = 3.83cm$. Therefore, the nail is likely the correct size!
4. From our conversion chart, an imperial cup is a bit less than a $\frac{1}{4}L$, so there is more water in this recipe than what is called for in the recipe.

5.4 Accuracy in Measurement

1. 7.1cm
2. $2\frac{3}{8}in.$
3. 10.6cm

4. $2\frac{9}{16}in.$
5. 5.8cm
6. $1\frac{1}{2}in.$
7. 6.4cm
8. $1\frac{3}{8}in.$

5.5 Perimeter

1. 36ft.
2. 25m
3. 23in.
4. 24cm
5. 46km
6. 51.5yd.

6 TRIGONOMETRY

6.1 Angles

1. 180°
2. 45°
3. 270°
4. 0°
5. 120°
6. 270°
7. 135°
8. 315°
9. 225°
10. 80°
11. 115°
12. 55°
13. 90°

6.2 Pythagorean Theorem

1. 13
2. 25
3. 40
4. 55
5. Using the Pythagorean theorem, we see that the slant is $\sqrt{10^2 + 5^2} \approx 11.18$.
6. Using the Pythagorean theorem, we see that the distance is $\sqrt{100^2 + 25^2}$ km ≈ 103 km.

6.3 Similar Triangles

1. $e = 21$ and $d = 12$.
2. $c \approx 2.66$ and $d = 7.5$.

3. No, it cannot, due to the ratios of the sides of the triangles. In the first triangle, the ratios are $1:2:3$, whereas in the second, the ratios are $2:3:4$, which are not equal. Indeed, the first triangle's ratios can also be written as $2:4:6 \neq 2:3:4$.
4. They will be 50km apart. This is because the triangles formed by the train station and the two trains will all be similar since the trains travel at constant speeds. After an hour, each train will have doubled its distance from the station, which implies that they have doubled their distance from each other as well!

6.4 Circumference of a Circle

1. 18.84cm
2. 31.4in.
3. 43.96yd.
4. 314m
5. Let r be the radius of a circle, d the diameter, and C the circumference. Since $C = 2 \times \pi = \pi \times (2 \times r)$ and $d = 2 \times r$, we see that $C = \pi \times d$.

6.5 Grades

1. A right triangle with a $45°$ angle has equal length and fall, therefore, its grade is 100%. On the other hand, a right triangle with an additional $90°$ angle is a straight line pointing upwards. It has no length and some fall, so the grade formula gives $\frac{1}{0}$ which we say is ∞.
2. $18\frac{3}{4}''$
3. 1.25ft.
4. 72.12ft.
5. 88ft.
6. $\frac{1}{5}''$ per ft.
7. 1.2%

7 GEOMETRY

7.1 Basic Areas

1. $6m^2$
2. $20ft.^2$
3. $102in.^2$
4. $7425cm^2$
5. $3.75ft.^2$
6. $14km^2$
7. $10,000cm^2 = 1m^2$
8. $144in.^2 = 1ft.^2$

7.2 Complex Areas

1. $300yd.^2$
2. $320ft.^2$
3. $170.5m^2$
4. $133.5m^2$
5. $300in.^2$
6. $241cm^2$

7.3 Volumes

1. $904.32ft.^3$
2. $2m^3$
3. $66mm^3$
4. $336in.^3$
5. $1,000,000cm^3 = 1m^3$
6. $1728in.^3 = 1ft.^3$

8 FINANCIAL MATH

8.1 Simple Interest

1. $\$26,000$
2. $\$5,153.42$
3. $\$21,000$

8.2 Compound Interest

1. $\$26,019.40$
2. $\$5,155.79$
3. $\$23,024.85$

VERSION HISTORY

This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.1. If the edits involve a number of changes, the version number increases to the next full number.

The files posted alongside this book always reflect the most recent version.

Version	Date	Change	Affected Web Page
1.0	May 14 2026	First Publication	N/A