

Mathematics of Finance

MATHEMATICS OF FINANCE

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About

This open-access textbook is designed to make the study of financial mathematics both accessible and engaging for everyone. Primarily aimed at first-year college and university students enrolled in business programs, its content is crafted to be of value to learners across all disciplines. Whether learners are encountering the Mathematics of Finance for the first time or seeking a comprehensive refresher, this textbook stands as a versatile resource.

Calculator and Formula Approaches

Recognizing the diverse instructional methods employed in teaching key financial mathematics concepts—such as compound interest and annuities—we have structured this book to accommodate both the financial calculator and the algebraic formula approaches. Topics are organized into distinct sections based on these methodologies, allowing learners to concentrate on the approach that best aligns with their learning goals, without the distraction of unrelated content.

Content and Format

Each chapter introduces fundamental concepts and illustrates their application through practical scenarios, complete with detailed solutions. To further aid in comprehension, the textbook includes ‘Try An Example’ questions. These questions, generated dynamically via the MyOpenMath platform, not only test understanding but also provide instant feedback, enhancing the learning experience with the opportunity to practice and receive immediate support.

Incorporating interactive elements such as videos, dynamic problems, and graphs, the textbook is optimized for web viewing through Pressbook. This enables full interaction with its multimedia content, from watching instructional videos to engaging with dynamic graphs and problem sets. While a downloadable PDF version is available, it does not include the interactive features found in the web format.

Web-based Financial App

In addition to step-by-step guidance on utilizing the Texas Instrument BA II Plus financial calculator, a dedicated web application has been developed for this book. This tool provides a cost-free, user-friendly alternative to purchasing a financial calculator, further democratizing access to financial education.

Sponsor

This project has received support and funding from the Government of Ontario. The views expressed in this publication are the views of the author and do not necessarily reflect those of the Government of Ontario or the Ontario Online Learning Consortium.

Dr. Amir Tavangar – Author and App Developer

Dr. Kate Zhang – Project Lead and Editor



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Special thanks are extended to the project lead and editor Dr. Kate Zhang, whose meticulous attention to detail and commitment to educational excellence have greatly enhanced the quality of this work. The insights, expertise, and tireless effort of Dr. Zhang have been pivotal in achieving the high standard this textbook embodies.

We are also thankful for the invaluable guidance and feedback from our peers and professionals, particularly Professor Colleen Ball, for her insightful feedback on the textbook's content and format, and Amanda Baker Robinson, for her valuable input on accessibility. Their expertise has been essential in shaping this ebook.

Furthermore, our gratitude extends to the learners, educators, and enthusiasts in financial mathematics. Your active engagement and constructive feedback are not just motivational but critical in steering the ongoing improvement and development of our materials.

This ebook is the result of a collective effort, and it is with sincere gratitude that we thank everyone involved in this project. Your support and contributions have been instrumental in bringing the Mathematics of Finance to our readers.

ACCESSIBILITY STATEMENT

The web version of this textbook is fully compliant with the Accessibility for Ontarians with Disabilities Act (AODA) requirements and adheres to the Web Content Accessibility Guidelines (WCAG) 2.0, Level AA standards. Furthermore, it aligns with the comprehensive checklist provided in Appendix A: Checklist for Accessibility of the Accessibility Toolkit – 2nd Edition, ensuring it meets the highest standards of accessibility.

Designed with interactivity at its core, the textbook incorporates videos, dynamic problems, and graphs, making it ideally suited for online learning through Pressbook. Key accessibility features have been meticulously integrated into the web version to accommodate diverse learning needs:

- The content is accessible to users of screen-reader technology, enhancing navigability and usability.
- Keyboard navigation is supported throughout, allowing users to easily move through content without a mouse.
- Formatting for links, headings, and tables is optimized for screen-reader compatibility.
- Mathematical equations are presented in AsciiMath and rendered via MathJax to ensure they are accessible. The JAWS screen reader is recommended for the best experience in accessing these equations.
- All images are accompanied by comprehensive descriptions, provided through text within the main content, alt-text, or detailed image descriptions. These extended alt-texts ensure that all visual information is conveyed clearly to users who rely on screen readers.
- Color is not used as the sole means of conveying information, ensuring content is accessible to users with color vision deficiencies.
- Video content includes captions to support users with hearing impairments.
- The option to adjust font size is available, catering to users with visual impairments.
- While a PDF version of the textbook is offered for download, it's important to note that this format lacks the interactive elements present in the web version, which are central to the enhanced learning experience provided by the online resource.

This holistic approach to accessibility ensures that all learners, regardless of their physical abilities, can effectively engage with and benefit from the rich educational content provided in Mathematics of Finance.

To provide feedback on this text please contact **amir.tavangar@humber.ca**.

PART I

SIMPLE INTEREST

Chapter Outline:

This chapter will provide readers with a comprehensive understanding of simple interest, including

- 1.1 Interest, Principal, Time: This section introduces the basic concept of simple interest and explains how to calculate the interest amount, interest rate, and time.
- 1.2 Future and Present Values: This section delves into the concept of future value and present value in simple interest and provides formulas for the calculation of those values.
- 1.3 Compute Equivalent and Dated Values for Specified Focal Dates: This section explores the concept of equivalent or dated values and discusses how to calculate equivalent values for specific dates in the future or past.
- 1.4 Terminology and Glossary of the Symbols

1.1 PRINCIPAL, INTEREST, TIME

A. Introduction to Simple Interest

Many business interactions involve borrowing and lending money. The fee that is charged to compensate the lender is called **interest**. In other words, interest is the rent paid for an amount of money borrowed. When the amount is returned, interest is added to the original amount borrowed or invested.

Interest may be calculated in two different ways:

- **Simple Interest:** the interest is calculated only on the initial amount borrowed or invested. With this method, the growth of money is linear, meaning it increases at a constant rate. Simple interest is typically applied to short-term financial products, such as short-term personal loans and treasury bills.
- **Compound Interest:** the interest earned in earlier periods, such as previous months, is added to the principal amount. This accumulated total then earns interest in subsequent periods. As a result, money grows exponentially, or at an increasingly rapid rate, under compound interest. This method is commonly applied to long-term financial products like mortgages and car loans.

In simple interest, the interest charged is based on a percentage of the original amount borrowed or invested. The total interest amount depends on three key factors:

- How much: the principal, which is the initial sum of money borrowed or invested.
- How long: the duration for which the money is borrowed or invested.
- What rate: the interest rate applied to the principal

For example, consider the principal amount of \$100 invested at a simple interest rate of 10% per annum (i.e., year) for two years.

- At the end of the first year, the interest is calculated as 10% of \$100, resulting in \$10.
- At the end of the second year, the interest remains 10% of the original principal, again amounting to \$10.

Therefore, over the two years, the initial investment of \$100 grows to \$120, as illustrated in Figure 1.1.1.

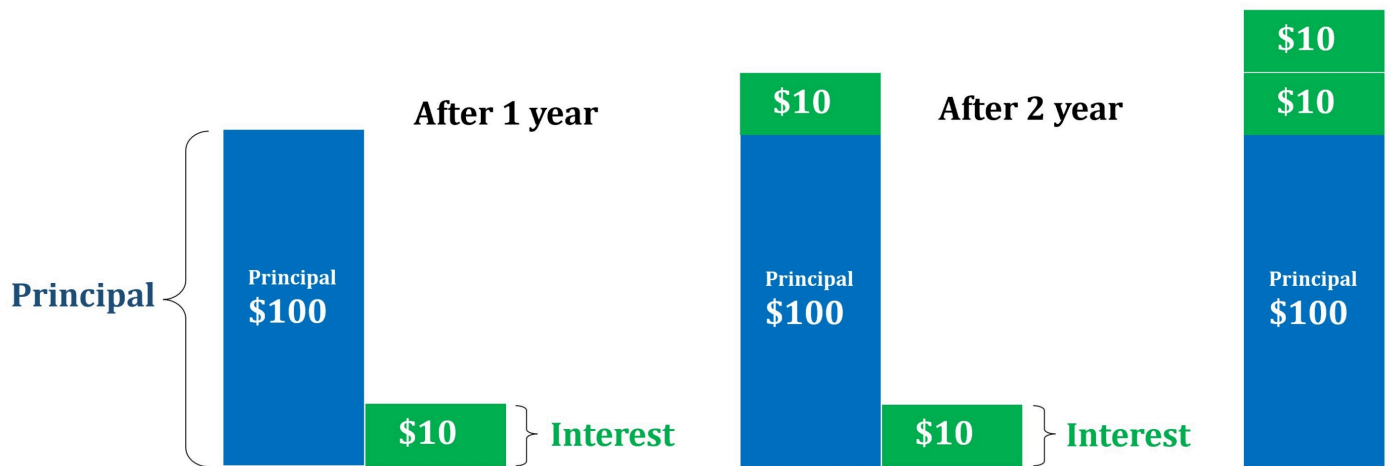


Figure 1.1.1 The amount of simple interest on a \$100 investment at the interest of 10% p.a.

B. Computing the Simple Interest

The amount of simple interest is given by

$$I = P \cdot r \cdot t \quad \text{Formula 1.1}$$

Here P is the principal amount (present value), r the interest rate expressed in percent, and t is the time period for which the principal amount is borrowed or invested.

When applying the simple interest formula, ensure the time period (t) matches the time period of the interest rate (r). For instance, if the interest rate is given per year (per annum or p.a.), you must convert the time to years before using it in the formula. Similarly, if the interest rate is monthly or daily, convert the time to months or days, respectively.

To convert the number of days to the equivalent number of years, divide by 365 since a year typically has 365 days. For converting months into years, divide the number of months by 12, as a year contains 12 months, as shown in Figure 1.1.2. Additionally, if you are given two specific dates, calculate the 'Days Between Dates' (DBD) to find out the number of days for the time period.

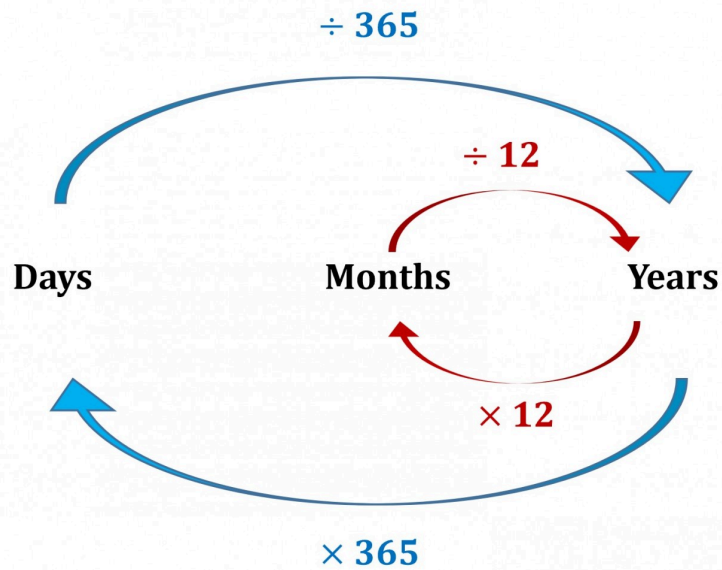


Figure 1.1.2 How to convert the number of days and months to the number of years

How to Compute the Number of Days Between Two Dates (DBD)

Method 1: Using Calendar Days in each month

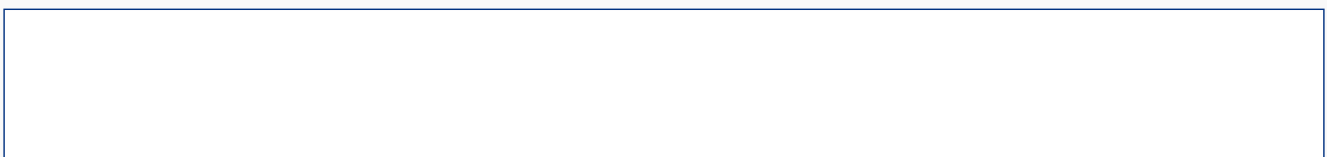
When counting the number of days in a time period, include either the first day or the last day of the investment or loan, but not both days.

Method 2: Using a DBD or date calculator such the Windows calculator or this DBD calculator

Method 3: Using a financial calculator

Financial calculators have a built-in worksheet for calculating the number of days between days. This video shows how to use a financial calculator (TI-BAII Plus) to find DBD.

Watch Video





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Example 1.1.1: Compute Interest

Fill in the table and compute the amount of interest for the given principals.

| | P (\$) | r (%) | t (years) | I (\$) |
|--|--------|-------|-----------|--------|
| \$8,120 at 6.5% p.a. for 3 months. | | | | |
| \$4,000 at 4% p.a. for 146 days. | | | | |
| \$5,000 at $9\frac{3}{4}$ % p.a. between January 20 and April 7, 2020 | | | | |

Show/Hide Solution

a) \$8,120 at 6.5% p.a. for 3 months.

- $P = \$8120$
- $r = 6.5 \%$
- $t = 3 \text{ months} \rightarrow t = \frac{3}{12} = 0.25 \text{ years}$

Applying Formula 1.1 gives

$$I = P \cdot r \cdot t$$

$$I = 8120(6.5\%)(0.25) = \$131.95$$

b) \$4,000 at 4% p.a. for 146 days.

- $P = \$4000$
- $r = 4\%$
- $t = 146 \text{ days} \rightarrow t = \frac{146}{365} = 0.4 \text{ years}$

Applying Formula 1.1 yields

$$I = P \cdot r \cdot t$$

$$I = 4000(4\%)(0.4) = \$64$$

c) \$5,000 at $9\frac{3}{4}\%$ p.a. between January 20 and April 7, 2020.

- $P = \$5000$
- $r = 9\frac{3}{4}\% = 9.75\%$
- $DBD = 78 \text{ days}$
- $t = 78 \text{ days} \rightarrow t = \frac{78}{365} \text{ years}$

Applying Formula 1.1 yields

$$I = P \cdot r \cdot t$$

$$I = 5000(9.75\%)\left(\frac{78}{365}\right) = \$104.18$$

Try an Example



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Computing the Principal, Time, and Interest Rate

Formula 1.1 can be rearranged and solved for the other variables.

- When looking for the principal P given the other variables use $P = \frac{I}{r \cdot t}$.
- When looking for the time t given the other variables use $t = \frac{I}{P \cdot r}$.
- When looking for the interest rate r given the other variables use $r = \frac{I}{P \cdot t}$.

Example 1.1.2: Compute Principal

Maria has to pay \$440 as interest payment on a loan she has taken for a period of 9 months. If the interest rate on the loan is 6.58% p.a., what is the principal amount of the loan?

Show/Hide Solution

Given information:

- Amount of interest: $I = \$440$
- Nominal Interest rate per year: $r = 6.58\%$

- Time period of the loan: $t = 9$ months $= \frac{9}{12}$ years

$$P = \frac{I}{r \cdot t}$$

$$P = \frac{440}{(6.58\%) \left(\frac{9}{12}\right)}$$

$$= 8915.906788\dots$$

$$\approx \$8915.91 \text{ (Rounded to the nearest cents)}$$

Note: Remember to match the time period of the loan and the time period of interest rate.

Try an Example



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Example 1.1.3: Compute Time

Priya took a short-term loan of \$14,000 at 8.5% p.a. What was the time period of the loan (rounded up to the next day) if she was charged the interest amount of \$1500?

Show/Hide Solution

Given information:

- Amount of interest: $I = \$1500$
- Nominal Interest rate per year: $r = 8.5\%$
- The loan principal: $P = \$14,000$

$$t = \frac{I}{P \cdot r}$$

$$t = \frac{1500}{(14000)(8.5\%)}$$

$$= 1.260504... \text{ years}$$

$$\stackrel{\times 365}{=} 460.08396$$

$$\approx 461 \text{ days (Rounded UP the next day)}$$

Note: For intermediate values, retain at least six decimal places for accuracy.

Note: In time calculations, if the result is a decimal, always **round it up** to the nearest whole number to guarantee that the necessary interest is fully accrued.

Try an Example



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Example 1.1.4: Compute Interest Rate

What was the annual rate of simple interest offered on a savings of \$8500 if the amount of interest earned in 6 months was \$250? Round your answer to two decimal places if needed.

Show/Hide Solution

Given information:

- Amount of interest: $I = \$250$
- The loan principal: $P = \$8500$
- Time period of the savings: $t = 6 \text{ months} = \frac{6}{12} = 0.5 \text{ years}$

$$r = \frac{I}{P \cdot t}$$

$$r = \frac{250}{(8500)(0.5)}$$

$$= 0.058823\dots$$

$$\stackrel{\times 100\%}{=} 5.8823\dots \%$$

$$\approx 5.88 \% \text{ p.a. (Rounded to two decimal places)}$$

Note: Alternatively, you can keep the time in months in the calculation. However, this will give you a monthly rate, and thus you should then multiply by 12 to obtain the annual rate.

Try an Example



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Section 1.1. Exercises

1. On March 9, 2020, Jack borrowed \$62,200 from a bank at 8.62% p.a. If he repaid the amount on July 20, 2020, calculate the amount of interest charged on the loan.

Show/Hide Answer

$$I = \$1953.69$$

2. Kyrie has to pay \$290.36 as an interest payment on a loan he has taken for a period of 6 months. If the interest rate charged on the loan is 7.6% p.a., what is the principal amount of the loan?

Show/Hide Answer

$$P = \$7641.05$$

3. Luke received a loan of \$18,540 from his friend. If he was charged the interest amount of \$994 at the end of 15 months, calculate the annual rate of simple interest on the loan.

Show/Hide Answer

$$r = 4.29\% \text{ p.a.}$$

1.2 FUTURE AND PRESENT VALUES

A. Computing Future Value

When you borrow money, you are to repay, sometime in the future, both the original amount borrowed (the principal or present value) and the amount of interest charged for the loan period. The sum of the original amount and the interest charged is **the future value** S (maturity value or accumulated value).

$$S = P + I \quad \text{Formula 1.2}$$

Substituting $I = Prt$ (Formula 1.1) into Formula 1.2 yields

$$S = P + P \cdot r \cdot t$$

Factoring P gives

$$S = P(1 + r \cdot t) \quad \text{Formula 1.3a}$$

Example 1.2.1: Compute Maturity Value (FV)

You invest \$3000 in a short-term deposit at 4.5% p.a. for 165 days. **a)** What will be the future value of this investment? **b)** What will be the amount of interest earned?

Show/Hide Solution

Given information:

- The investment principal: $P = \$3000$
- Nominal Interest rate per **year**: $r = 4.5\%$
- Time period of the investment: $t = 165 \text{ days} = \frac{165}{365} \text{ years}$

a) Substituting the values into equation (iii), we get

$$S = P(1 + r \cdot t)$$

$$S = 3000 \left(1 + (4.5\%) \left(\frac{165}{365} \right) \right)$$

$$= 3061.027397\dots$$

$$\approx \$3061.03 \text{ (Rounded to two decimal places)}$$

b) The amount of interest can be obtained using either Formula 1.1 or 1.2. Given the future value is known now, we use Formula 1.2 as it involves fewer operations:

$$S = P + I$$

$$\rightarrow I = S - P$$

$$I = 3061.03 - 3000$$

$$= \$61.03$$

Try an Example



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B. Computing Present Value

Sometimes you may know the future value and want to determine the principal amount required to reach that future value at a specific interest rate over a certain period. In that case, Formula 1.3a can be rearranged for P .

$$P = \frac{S}{1 + r \cdot t} \quad \text{Formula 1.3b}$$

which using the negative-exponent notation is equivalent to

$$P = S(1 + r \cdot t)^{-1} \quad \text{Formula 1.3c}$$

To help determine whether a problem is asking you to find the present value (principal) or future value, look for specific keywords or phrases. Below are examples of how questions might be phrased when they are seeking the present value:

- What principal will have a future value of \$8,000?
- What amount of money will accumulate to \$14,000?
- Compute the present value of \$4000.
- Calculate the amount of money that will grow to \$4500.
- Find the amount paid for an investment that will mature to \$20,000.
- How much money was borrowed if the maturity value of the loan is \$12,000?

Example 1.2.2: Compute Present Value

What amount of money will mature to \$4195.25 in 110 days at 4.5% p.a. interest?

Show/Hide Solution

Given information:

- The maturity value: $S = \$4195.25$
- Nominal Interest rate per year: $r = 4.5\%$
- Time period: $t = 110 \text{ days} = \frac{110}{365} \text{ years}$

Substituting the values into Formula 1.3b, we obtain

$$P = \frac{S}{1 + r \cdot t}$$

$$\begin{aligned}
 P &= \frac{4195.25}{1 + (4.5\%) \left(\frac{110}{365}\right)} \\
 &= 4139.116772\dots \\
 &\approx \$4139.12 \text{ (Rounded to the nearest cents)}
 \end{aligned}$$

Try an Example



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Section 1.2. Exercises

1. What is the maturity value of \$37,150 invested in an account earning 2.38% p.a. for 3 years and 11 months?

Show/Hide Answer

$$S = \$40,613.00$$

2. What amount of money will grow to \$76,750 in 2 years and 11 months at the interest rate of 3.95% p.a.?

Show/Hide Answer

$$P = \$68,821.22$$

3. Suppose you invest \$58,650 in a short-term deposit at 6.5% for 42 days. a) Calculate the accumulated value of this amount at the end of the term. b) Calculate the amount of interest earned.

Show/Hide Answer

a) $S = \$59,088.67$; b) $I = \$438.67$

1.3 EQUIVALENT OR DATED VALUES FOR SPECIFIED FOCAL DATES

A. Computing an Equivalent Payment for a Single Payment

Money has time value, implying the potential for it to accrue interest over time. Occasionally, it becomes necessary to modify or reschedule payment due dates for various reasons. Therefore, the payment amount needs to be adjusted. For example, you may need to defer a payment due to financial constraints or opt to make an early payment to reduce interest costs.

When the due date of payment is altered, a new due date, often referred to as the **focal date**, is established, especially in scenarios involving the rescheduling of multiple payments. The value of the money on this new due date is known as the **equivalent value** or **dated value**. If the payment is postponed (shifted to a future date), we calculate its future value. Conversely, if the payment is made earlier (moved to an earlier date), we determine its present value. This concept is illustrated in Figure 1.3.1 below.

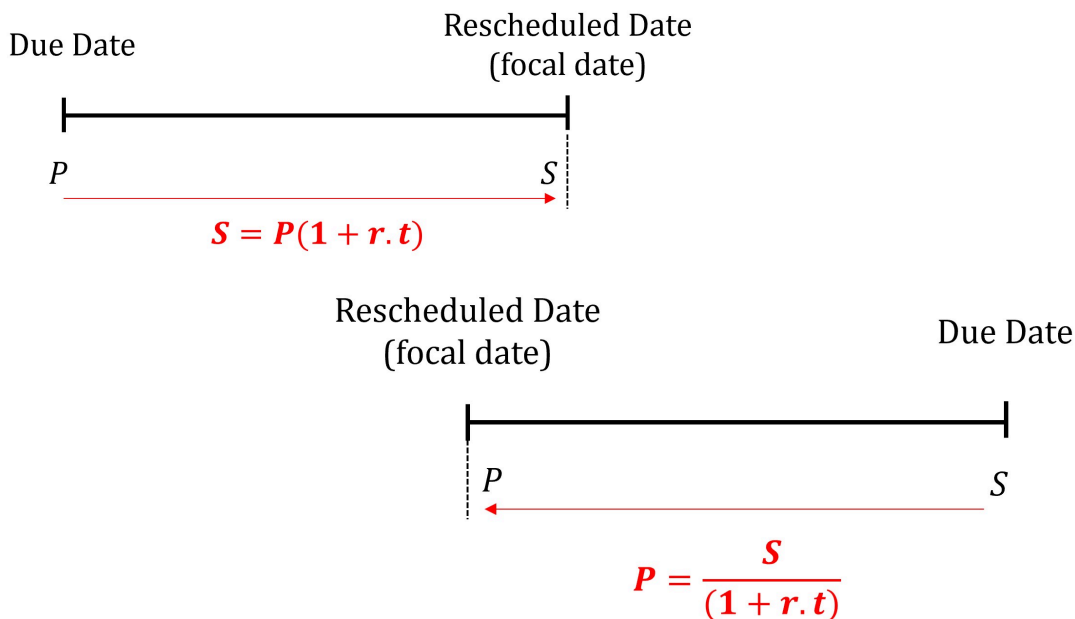


Figure 1.3.1 Rescheduling a single payment: Deferred (left) versus earlier payment (right)

Example 1.3.1: Equivalent Payment of a Single Payment

Aslan is required to repay a matured loan amount of \$1915 in nine months. However, he realizes that he can clear the loan four months earlier. How much will he have to pay to clear the loan four months earlier if the interest charged is 5.2% p.a.?

Show/Hide Solution

Since the payment is rescheduled at an **earlier** time than the due date, the **present value** of the payment should be computed.

- The payment: $S = \$1915$
- Nominal Interest rate per year: $r = 5.2\%$
- Time period payment is paid earlier: $t = 4 \text{ months} = \frac{4}{12} \text{ years}$

Substituting the values into Formula 1.3b, we obtain

$$\begin{aligned}
 P &= \frac{S}{1 + r \cdot t} \\
 P &= \frac{1915}{1 + (5.2\%) \left(\frac{4}{12}\right)} \\
 &= 1882.372215... \\
 &\approx \$1882.37 \text{ (Rounded to the nearest cents)}
 \end{aligned}$$

Note: The new amount is lower than the original payment as the interest of four months is deducted from the original amount of \$1915.

Example 1.3.2: Equivalent Payment of a Single Payment

Nancy realizes that she cannot repay a matured loan amount of \$3400 that is due today and decides to reschedule the payment. If the interest charged is 4.8% p.a., what equivalent payment is required to repay this amount in 10 months?

Show/Hide Solution

Since the payment is deferred to a **later** time than the due date, the **future value** of the payment should be computed.

- The payment: $P = \$3400$
- Nominal Interest rate per year: $r = 4.8\%$
- Time period payment is deferred: $t = 10 \text{ months} = \frac{10}{12} \text{ years}$

Substituting the values into Formula 1.3a, we obtain

$$\begin{aligned}
 S &= P(1 + r \cdot t) \\
 S &= 3400 \left(1 + (4.8\%) \left(\frac{10}{12} \right) \right) \\
 &= \$3536
 \end{aligned}$$

Note: The new amount is higher than the original payment as the interest of 10 months is added to the original amount of \$3400.

B. Computing an Equivalent Payment for a Series of Payments

In the previous section, we calculated the equivalent payment for a single payment. Sometimes we need to calculate an equivalent payment for a series of payments. Since money has time value, making direct comparisons, equating, or adding/deducting different monetary values at various points in time is not feasible. To make valid comparisons, it is essential to find their equivalent values at a common reference date, also known as the focal date. The focal date allows all different cash flows to be adjusted (either discounted or compounded) to a single point in time, making them comparable or combinable.

The time period between 2 months and the focal date is $t_1 = 4 - 2 = 2$ months.

The time period between 9 months and the focal date is $t_2 = 9 - 4 = 5$ months.

Note: The nominal interest rate is 5.4% per annum and thus we need to convert time to years before substituting in the formula.

- $t_1 = 9$ months $= \frac{9}{12}$ years
- $t_2 = 2$ months $= \frac{2}{12}$ years

$$S_1 = 3500 \left(1 + (5.4\%) \left(\frac{2}{12} \right) \right) = \$3531.50$$

$$P_2 = \frac{7200}{1 + (5.4\%) \left(\frac{5}{12} \right)} = \$7041.56$$

3. The sum of the individual dated values is the new single equivalent payment.

$$\begin{aligned} \text{Single equivalent payment} &= S_1 + P_2 \\ &= 3531.50 + 7041.56 \\ &= \$10,573.06 \end{aligned}$$

Therefore, the single equivalent payment of \$10,573.06 made in four months will clear the debt.

Try an Example



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=149>

C. Loan Repayments

The concept of equivalent values is applied in solving loan repayment problems. Loans taken from financial institutions are often repaid through blended payments, which are equal in size and include portions of both principal and interest. For the loan to be fully repaid, the sum of the present values of all these blended payments must be equal to the original amount borrowed (the principal).

How to find the size of the blended payments

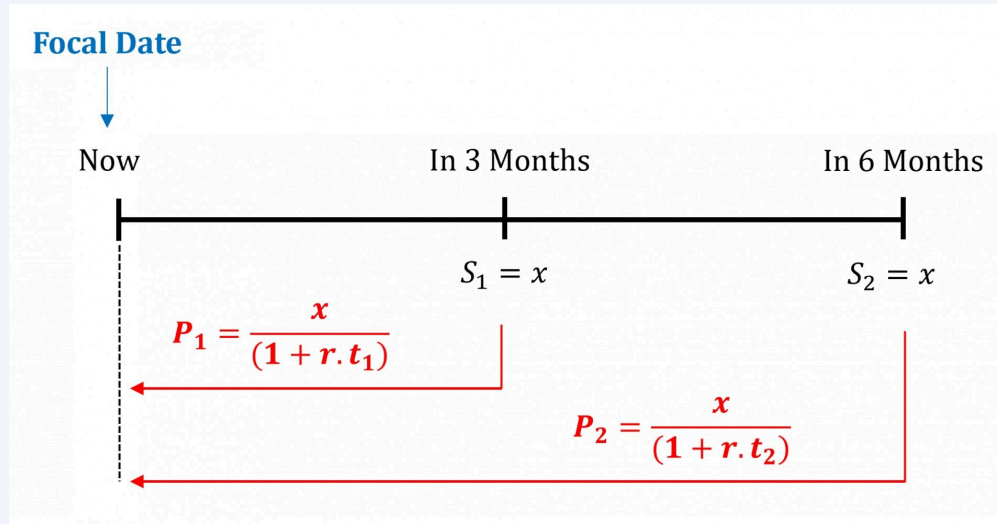
1. Let the size of the payment be represented by x .
2. Find the dated value of each blended payment (i.e., the x payment) on a focal date of “today”. (In this book, the focal date for such problems is always “today”)
3. Equalize the sum of the dated values of the blended payments to the original payment (e.g., the loan amount)
4. Solve for x .

Example 1.3.4: Size of blended payments

The original payment of \$4600 due today is settled by two equal payments due in 3 and 6 months. Assuming the interest rate is 10% p.a., determine the size of the equal payments.

Show/Hide Solution

1. Let the size of the equal payments be represented by x . Then the sum of the present values of the payments on the focal date of today must be equal to the loan amount of \$4600.



The nominal interest rate is 10% per annum and thus we need to convert the time to years before substituting in the formula.

- $t_1 = 3 \text{ months} = \frac{3}{12} = 0.25 \text{ years}$
- $t_2 = 6 \text{ months} = \frac{6}{12} = 0.5 \text{ years}$

2. Next we find the dated value of each payment in terms of x :

$$P_1 = \frac{x}{1 + (10\%)(0.25)} = \left(\frac{1}{1 + (10\%)(0.25)} \right) x = 0.9756098x$$

$$P_2 = \frac{x}{1 + (10\%)(0.5)} = \left(\frac{1}{1 + (10\%)(0.5)} \right) x = 0.9523810x$$

3. The sum of the dated values of the blended payments must be equal to the original payment (e.g., the loan amount).

$$\$4600 = P_1 + P_2$$

$$4600 = 0.9756098x + 0.9523810x$$

4. Finally, we solve the equation for x .

$$4600 = 1.9279908x$$

$$x = \frac{4600}{1.9279908} \approx \$2385.90 \text{ (Rounded to the nearest cents)}$$

Try an Example



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Section 1.3. Exercises

1. Debt payments of \$4,430 and \$5,030 are due in 4 months and 11 months, respectively. Assuming an interest rate of 7% p.a., determine a single equivalent payment that settles the two payments today. Use “today” as the focal date.

Show/Hide Answer

Equivalent payment = \$9,055.69

2. A payment of \$6,750 was due 7 months ago and another payment of \$1,490 is due in 7 months. Assuming an interest rate of 5.9% p.a., determine a single equivalent payment 3 months from now that settles the two payments. Hint: use 3 months from now as the focal date.

Show/Hide Answer

Equivalent Payment = \$8,543.14

3. Thiago borrowed \$6,630 today and will repay the loan in two equal payments, one in 4 months and the other in 10 months. Assuming an interest rate of 7.9% p.a. on the loan, determine the size of the equal payments if a focal date of “today” is used.

Show/Hide Answer

Size of the equal payments = \$3,466.53

1.4 TERMINOLOGY AND GLOSSARY OF THE SYMBOLS

Terminology

- **Dated value or equivalent value:** The value of a sum of money at a specific time relative to its due date, including the amount of interest
- **Focal date:** A reference time chosen to compare the dated value of one or more amounts
- **Future value (accumulated or maturity value):** the value obtained when the amount of interest is added to the original principal
- **Interest:** rent paid on the money borrowed or lent
- **Nominal rate:** the yearly or annual rate of interest charged on the principal of a loan
- **p.a.:** means per annum or per year.
- **Present value (principal or discounted value):** The amount borrowed or invested at the beginning of a (loan or investment) period
- **Term:** refers to either an investment term or loan term, which is the time period an amount is lent or borrowed.

Glossary of Symbols

- I : Interest Amount (\$)
- r : Interest Rate per period (%)
- t : Time period the money is borrowed/lent
- P : Principal (Present Value) (\$)
- S : Future (maturity) value (\$)

PART II

COMPOUND INTEREST

Chapter Outline

This chapter explores compound interest and is divided into two parts. Part one covers the financial calculator approach, while part two discusses algebraic formulas for calculations.

2.1 Instruction to Compound Interest: This section introduces the fundamental concept of compound interest and explains key terms like nominal interest rate, periodic interest rate, and compounding frequency.

Calculator Approach

2.2 Future Value – Calculator Approach: This section focuses on computing the future value of compound interest for loans or investments using a financial calculator.

2.3 Present Value – Calculator Approach: This section discusses computing the present value of compound interest for loans or investments using a financial calculator.

2.4 Interest Rate – Calculator Approach: This section explores how to calculate the nominal and periodic interest rates for compound interest using a financial calculator.

2.5 Number of Compounding Periods and Time – Calculator Approach: This section demonstrates the calculation of the number of compounding periods and the duration of the term for compound interest problems using a financial calculator.

Formula Approach

2.6 Future Value – Formula Approach: This section explains the process of calculating the future value of compound interest for loans or investments using mathematical formulas.

2.7 Present Value – Formula Approach: This section covers the calculation of the present value of compound interest for loans or investments using mathematical formulas.

2.8 Interest Rate – Formula Approach: This section discusses how to calculate the nominal and periodic interest rates of compound interest for loans or investments using mathematical formulas.

2.9 Number of Compounding Periods and Time – Formula Approach: This section details the calculation of the number of compounding periods and the term duration for compound interest problems using mathematical formulas.

2.10 Terminology and Glossary of Symbols

2.1 INTRODUCTION TO COMPOUND INTEREST

A. Simple and Compound Interest Comparison

In the previous chapter, we focused on simple interest, a method where interest is calculated only on the principal amount. This unit shifts to compound interest. Let's review what was discussed before. When money is borrowed or lent, a fee, known as interest, is charged. This fee is akin to rent on the borrowed amount.

There are two main ways to calculate interest:

- **Simple Interest:** the interest is calculated only on the initial amount borrowed or invested. With this method, the growth of money is linear, meaning it increases at a constant rate. Simple interest is typically applied to short-term financial products, such as short-term personal loans and treasury bills.
- **Compound Interest:** the interest earned in earlier periods, such as previous months, is added to the principal amount. This accumulated total then earns interest in subsequent periods. As a result, money grows exponentially, or at an increasingly rapid rate, under compound interest. This method is commonly applied to long-term financial products like mortgages and car loans.

To understand the difference between simple and compound interest, let's use an example. Imagine \$1000 is invested at a 10% annual interest rate for three years. We'll compare two scenarios: one where the investment earns simple interest and another where it earns compound interest. To calculate the interest amount, we will use the formula $I = P \cdot r \cdot t$ provided in Chapter 1.

Simple Interest:

\$1000 invested at 10% p.a.

Interest earned during year 1:

$$I_1 = (\$1000)(10\%)(1 \text{ year}) = \$100$$

Interest earned during year 2:

$$I_2 = (1000)(10\%)(1) = \$100$$

Interest earned during year 3:

$$I_3 = (1000)(10\%)(1) = \$100$$

Total interest earned after 3 years:

$$I_t = \$100 + \$100 + \$100 = \$300$$

Principal
\$1000

Compound Interest:

\$1000 invested at 10% p.a. compounded annually.

Interest earned during year 1:

$$I_1 = (\$1000)(10\%)(1 \text{ year}) = \$100$$

Interest earned during year 2:

$$I_2 = \underbrace{(1100)}_{\text{New } P=1000+100} (10\%)(1) = \$110$$

Interest earned during year 2:

$$I_3 = \underbrace{(1210)}_{\text{New } P=1100+110} (10\%)(1) = \$121$$

Total interest earned after 3 years:

$$I_t = \$100 + \$110 + \$121 = \$331$$

\$121

\$110

\$100

New
Principal
\$1331

Note that in the compound interest scenario, the interest earned during previous periods (for example, the previous year) is added to the original principal. This results in a higher amount on which interest is calculated in subsequent periods (such as the following years). As a result, the same investment accrues more interest and grows quicker over the same time frame with compound interest than it would with simple interest.

B. Determining Compounding Periods

The compounding period, also known as the conversion period, refers to the time interval between successive calculations of compound interest. Essentially, it's the duration between one interest calculation and the next. Compounding frequency (C/Y), on the other hand, indicates how many times interest is compounded within a year. Table 2.1.1 lists various common frequencies for compounding interest.

Table 2.1.1 Compounding Frequencies for Common Compounding Periods

| Compounding Frequency | Compounding Period | Frequency of Compounding Periods per Year (C/Y) |
|-----------------------|--------------------------|---|
| Annually | Every 12 months (1 year) | 1 |
| Semi-annually | Every 6 months | 2 |
| Quarterly | Every 3 months | 4 |
| Monthly | Every month | 12 |
| Daily | Every day | 365 |

The more frequently interest is compounded within a year, the greater the maturity value of the investment will be, given the same principal, rate, and time period. Figure 2.1.1 illustrates this by comparing the graph of the future value of \$1000 as a function of time over 20 years for simple interest and compound interest with the compound interest calculated at varying frequencies.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=305>

Figure 2.1.1 Comparison of the future value of \$1000 as a function of time for simple interest and compounded interest at different compounding frequencies

When working on problems involving compound interest, compounding frequency is typically presented in words,

like “monthly”. Your task is to identify this term and convert it into a numerical value, as outlined in Table 2.1.1, before proceeding with calculations. The relevant information in a problem is typically presented in the format “ I/Y compounded C/Y ”.

Where I/Y represents the nominal interest rate, and C/Y is the compounding frequency expressed in words. For example, in the statement “an investment earning 3.5% compounded semi-annually”, the nominal interest rate is 3.5%, and the compounding frequency, “semi-annually”, corresponds to $C/Y = 2$.

The periodic interest rate, which is the interest rate per compounding period is given by

$$i = \frac{I/Y}{C/Y} \quad \text{Formula 2.1a}$$

The total number of compounding periods over the entire term of an investment or loan is calculated by multiplying the compounding frequency per year by the total number of years in the term. This is represented by

$$N = C/Y \cdot t \quad \text{Formula 2.2a}$$

Where N is the total number of compounding periods, C/Y is the compounding frequency per year, and t is the total number of years in the term.

Example 2.1.1: Components of Compound Interest

For each given scenario, determine the frequency of compounding periods per Year (C/Y), nominal interest rate (I/Y), periodic interest rate (i), term in years (t), and the number of compounding periods in the term (N).

- a) 8.4% compounded semi-annually for 3 years
- b) 7.2% compounded monthly for 3.5 years
- c) 4% compounded quarterly for 4 years and 6 months

Show/Hide Solution

In each scenario, we look for the phrase that includes the keyword *compounded* and information about the nominal interest rate and the compounding frequency.

Then the periodic interest rate can be calculated using Formula 2.1a, $i = \frac{I/Y}{C/Y}$, and the number of compounding periods in the term is calculated using Formula 2.2a, $N = C/Y \cdot t$.

a) 8.4% compounded semi-annually for 3 years

- Frequency of Compounding Periods per Year: Semi-annually so $C/Y = 2$
- Nominal interest rate: $I/Y = 8.4\%$
- Periodic Interest Rate: $i = \frac{8.4\%}{2} = 4.2\%$ per semi-annual
- Term: $t = 3y$
- Number of Compounding Periods in the term: $N = 2 \times 3 = 6$

b) $7\frac{3}{4}\%$ compounded monthly for 3.5 years

- Frequency of Compounding Periods per Year: Monthly so $C/Y = 12$
- Nominal interest rate: $I/Y = 7\frac{3}{4}\% = 7.75\%$
- Periodic Interest Rate: $i = \frac{7.75\%}{12} = 3.875\%$ per month
- Term: $t = 3.5y$
- Number of Compounding Periods in the term: $N = 12 \times 3.5 = 42$

c) 4% compounded quarterly for 4 years and 6 months

- Frequency of Compounding Periods per Year: Quarterly so $C/Y = 4$
- Nominal interest rate: $I/Y = 4\%$
- Periodic Interest Rate: $i = \frac{4\%}{4} = 1\%$ per quarter
- Term: $t = 4 \text{ years} + 6 \text{ months} = 4 \text{ years} + \frac{6}{12} \text{ years} = 4.5 \text{ years}$
- Number of Compounding Periods in the term: $N = 4 \times 4.5 = 18$

Try an Example



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Like simple interest problem solving, this chapter will discuss how to use formulas to compute the future value, present value, and the other related variables. Additionally, this chapter will show you how to use a financial calculator (e.g., Texas Instrument BA II Plus) for the calculations. A financial calculator has all the formulas built into it. It is suggested that you learn both approaches.

Section 2.1 Exercises

1. An investment is earning interest at 7.23% compounded monthly for 8 years and 6 months. **a)** What is the nominal interest rate per year (I/Y)? **b)** What is the number of compounding periods (frequency) per year (C/Y)? **c)** What is the investment period in years (t)? **d)** Calculate the periodic interest rate (i). **e)** Calculate the number of compounding periods in the term (N). *Round your answers to two decimal places if needed.*

Show/Hide Answer

$$I/Y = 7.23\%$$

$$C/Y = 12$$

$$t = 8.50 \text{ years}$$

$$i = 0.6\% \text{ per month}$$

$$N = 102$$

2. A loan is charged an interest of 3.9% compounded quarterly for 5 years and 3 months. **a)** What is the nominal interest rate per year (I/Y)? **b)** What is the number of compounding periods (frequency) per year (C/Y)? **c)** What is the loan period in years (t)? **d)** Calculate the periodic interest rate (i). **e)** Calculate the number of compounding periods in the term (N). *Round your answers to two decimal places if needed.*

Show/Hide Answer

$$I/Y = 3.9\%$$

$$C/Y = 4$$

$$t = 5.25 \text{ years}$$

$$i = 0.98\% \text{ per month}$$

$$N = 21$$

2.2 FUTURE VALUE: CALCULATOR APPROACH

A. Introduction

In the study of compound interest, future value is the total amount that an initial sum, whether invested or borrowed, will accumulate over time at a specific compound interest rate. This includes the original principal and all the interest that accrues, compounded at set intervals. Present value, on the other hand, is the current equivalent of a future sum of money. It is the initial amount that, when interest is applied over a period, will grow to a predetermined future value.

Although solving compound interest problems algebraically improves your understanding of the subjects and the steps involved, if you have to regularly solve such problems, financial calculators offer a faster and more convenient way as they have all the formulas pre-programmed in them.

In this chapter, we will solve problems using the BAII Plus financial calculator. Alternatively, you can use this online calculator as an alternative to the BAII Plus.

TVM Worksheet

To solve compound interest problems using a financial calculator, we use its time-value of money (TVM) worksheet, as shown in Figure 2.2.1.

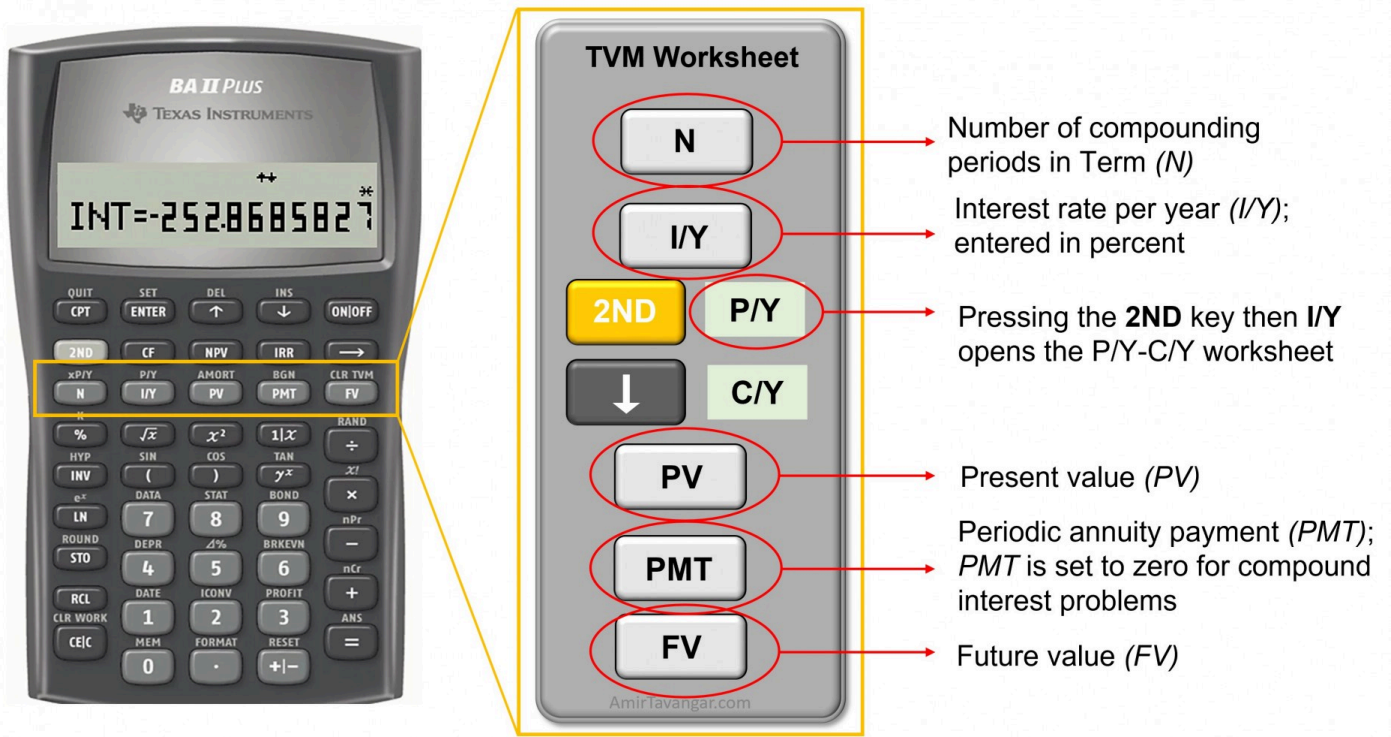


Figure 2.2.1 Keys on the TVM worksheet on a financial calculator

Cash-Flow Sign Convention

When using a financial calculator, you might notice that the signs of FV and PV are different (one negative and another positive). That is because the calculator uses the following cash flow sign convention, which is also summarized in Table 2.2.1.

- **Cash inflow:** when money is received, cash is flowing **in** (to your pocket), thus the amount should be entered as a positive value.
- **Cash outflow:** when money is paid **out** (flowing out of your pocket), the amount is entered as a negative value.

Table 2.2.1 Financial Calculator Cash-Flow Sign Convention

| | PV | FV |
|-------------|---------------|---------------|
| Investments | Outflow (-) | Inflow (+) |
| Loans | Inflow (+) | Outflow (-) |

Note that if try to do calculations with both PV and FV entered with the same sign in the TVM Worksheet, the calculator throws an error, “Error 5”.

B. Computing Interest Amount

The future value of an investment or loan includes both the initial principal and the interest earned or charged over the term. To find the total amount of interest accrued, you subtract the present value (initial amount) from the future value. This interest amount (I), often called compound interest (CI) in the context of compound interest calculations, is given by

$$I = FV - PV \quad \text{Formula 2.3}$$

Where FV is the future value and PV is the present value.

It’s important to note that when working with algebraic formulas, such as Formula 2.3 for calculating the amount of interest, both the present value (PV) and future value (FV) are treated as positive values. This approach differs from the cash flow sign convention used in financial calculators, where the direction of cash flow (inflow or outflow) is indicated by the sign (positive or negative) of the values entered. For algebraic calculations, simply use positive values for both PV and FV, regardless of the cash flow direction.

Example 2.2.1: Compute FV – Time Given in Years

Sarah invested \$25,000 for 5 years at 6.8% compounded monthly. **a)** How much was the accumulated value after 5 years? **b)** How much interest was earned during the investment term?

Watch Video



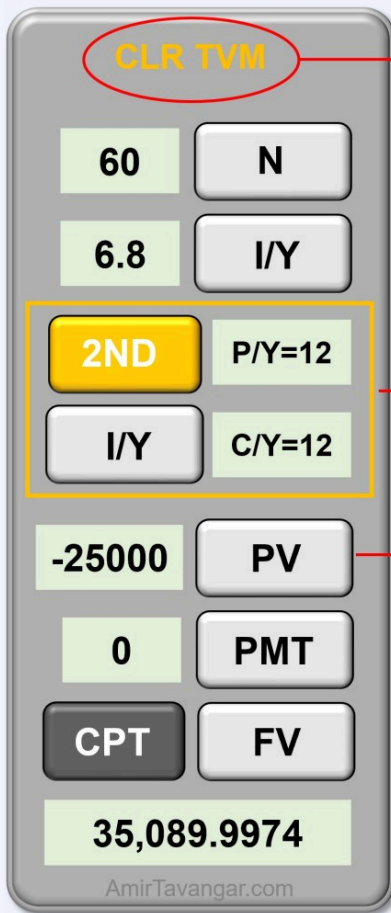
One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=307#oembed-1>

Show/Hide Solution

Given information:

- For investments, present value is a cash outflow: $PV = -\$25,000$
- Nominal interest rate: $I/Y = 6.8\%$
- Interest is compounded monthly, so $C/Y = 12$
- For compound problems, $P/Y = C/Y$, so $P/Y = 12$
- Investment term: $t = 5$ years
- Number of compounding periods in the term: $N = C/Y \cdot t = 12(5) = 60$

a) Note that for investments, the present value (PV) is a cash outflow and is entered in the calculator as a negative value. Also, the nominal interest rate (I/Y) is entered as a percent without the percent symbol.



Always clear the TVM Worksheet before entering new values. To clear the worksheet, access the secondary function of the FV key by pressing the "2ND" function key followed by the "FV" key.

In compound interest problems, P/Y and C/Y are always equal.

In investments, PV is a cash outflow, so it is entered as a negative value.

$$FV \approx \$35,090$$

Therefore, the future value is \$35,090, rounded to the nearest cents.

b) Substituting the PV and FV values into Formula 2.3 gives the amount of interest earned.

$$I = FV - PV$$

$$I = 35,090 - 25,000$$

$$= \$10,090$$

Example 2.2.2: Compute FV – Time Given in Years and Months

Ontario Servers Inc. borrows \$65,400 from a bank at 5.68% compounded quarterly for 4 years and 7 months.

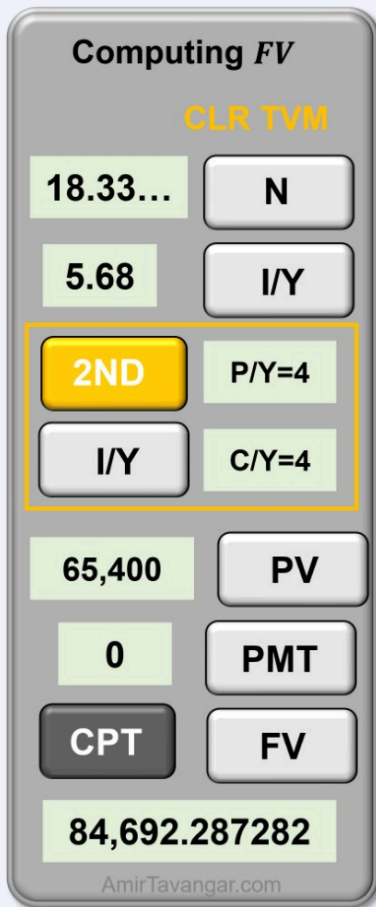
a) How much will the accumulated value of the loan be at the end of the term? **b)** How much interest will be charged on the loan?

Show/Hide Solution

Given information:

- In loans, present value is a cash inflow: $PV = \$65,400$
- Nominal interest rate: $I/Y = 5.68\%$
- Interest is compounded quarterly, so $C/Y = 4$
- For compound problems, $P/Y = C/Y$, so $P/Y = 4$
- Loan term: $t = 4$ years and 7 months $= 4 + \frac{7}{12} = 4.58\bar{3}$ years (The 'overbar' indicates that the digit beneath it is repeated continuously)
- Number of compounding periods in the term: $N = C/Y \cdot t = 4(4.58\bar{3}) = 18.\bar{3}$

a) Compute FV



$$FV \approx \$84,692.29$$

Therefore, the accumulated value of the loan is \$84,692.29, rounded to the nearest cents.

b) Substituting the PV and FV values into Formula 2.3 yields the amount of interest charged.

$$I = FV - PV$$

$$I = 84,692.29 - 65,400$$

$$= \$19,292.29$$

Try an Example



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C. Time Given as Specific Dates

As we saw in the simple interest chapter, sometimes the beginning and end dates of the loan or investment term are given. In that case, unless otherwise stated, the time calculation will be most accurate if the number of days between the given dates is computed and then converted to equivalent years. To convert the number of days to the number of years, divide it by 365 as there are 365 days in a year. There are several ways to compute the days between two dates (DBD), some of which are presented in section 1.2.

Here we use the DATE worksheet of the financial calculator to compute DBD. The worksheet is the secondary function of key 1, so it is opened by pressing 2ND and then key 1. Figure 2.2.2 shows the keys on the DATE worksheet on a financial calculator.

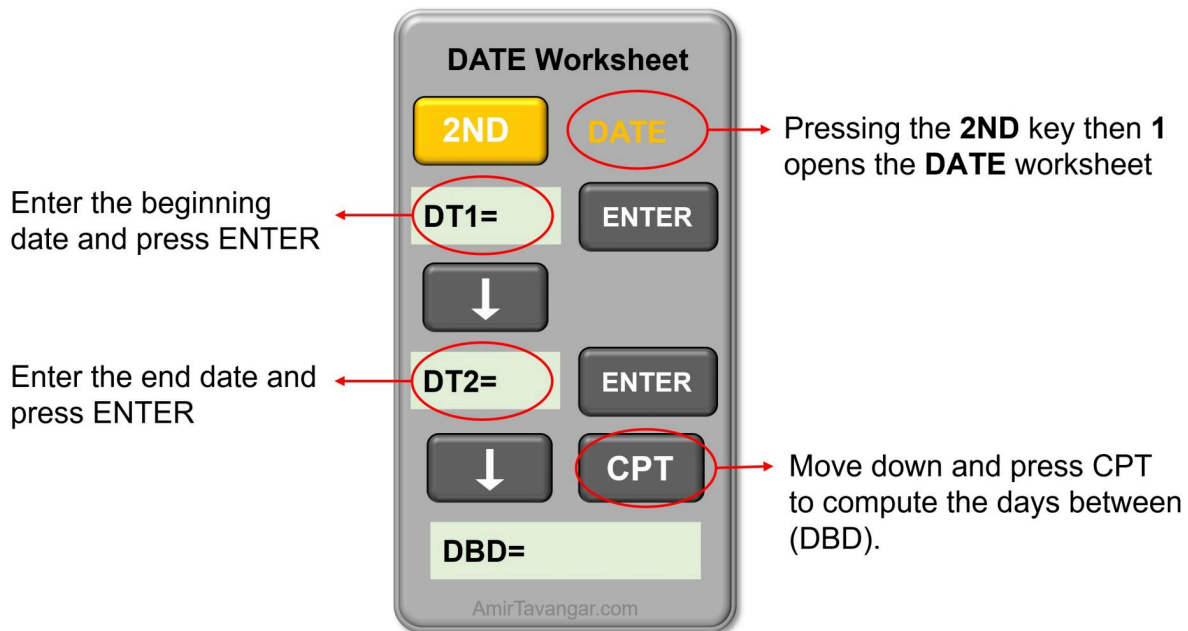


Figure 2.2.2: The keys on the DATE worksheet on a financial calculator

How to Calculate Days Between Dates (DBD) using a Calculator

1. Activate the DATE worksheet by pressing the “2ND” function key followed by “1”.
2. Input the initial date (DT1) using the **MM.DDYY** format. This means you should type the month (MM) followed by a decimal point, then the day (DD) as a two-digit number, and the last two digits of the year (YY). For instance, enter “11.0308” for November 3, 2008. After pressing “ENTER”, the calculator will display the date in the MM-DD-YYYY format.

Remember to use two digits for the day (DD) even if it’s a single-digit day. For example, input “03” for the third day of the month.

3. Navigate to the second date (DT2) using the down arrow key, and enter the second date in the same MM.DDYY format, and then press “ENTER”.
4. To calculate the number of days between the two dates, move down to “DBD” with the down arrow key and press the “CPT” (compute) button to get the result.

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Example 2.2.3: Compute FV Given Specific Dates

A sum of \$36,200.00 was deposited into an account on June 30, 2020. What will be the future value of this sum of money on October 9, 2025, if the interest rate is 2.22% compounded annually?

Show/Hide Solution

The given information:

- Present value (cash outflow): $PV = -\$36,200.00$
- Nominal interest rate: $I/Y = 2.22\%$
- Interest is compounded semi-annually, so $C/Y = 1$
- For compound problems, $P/Y = C/Y$, so $P/Y = 1$
- Beginning date: June 30, 2020
- End date: October 9, 2025

I. We first need to use the DATE worksheet to find the number of days in the investment and find its equivalent years to find time (t). Enter the dates in the MM.DDYY format and compute DBD.

- DT1: 06.3020
- DT2: 10.0925

DATE Worksheet

2ND
DATE

DT1=06.3020

ENTER

↓

DT2=10.0925

ENTER

↓

DBD=1927

CPT

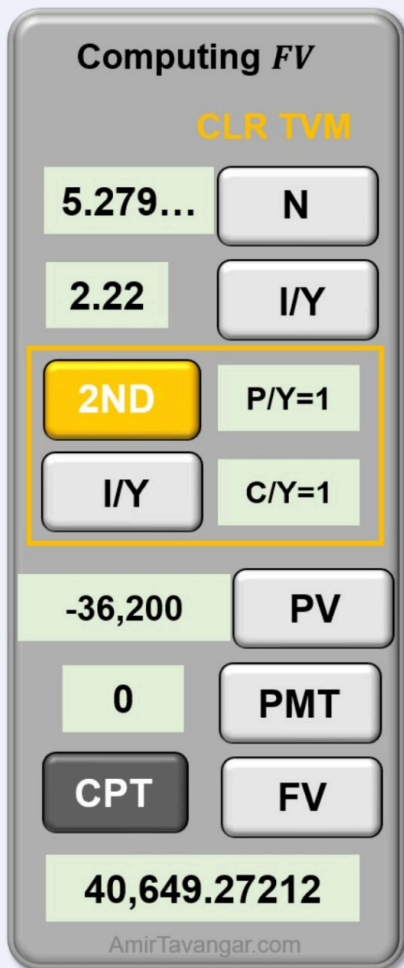
AmirTavangar.com

DBD is 1927 days and we can find time in years by dividing DBD by 365.

$$t = \frac{1927}{365} = 5.279452... \text{ years}$$

$$N = C/Y \cdot t = 1 \left(\frac{1927}{365} \right) = 5.279452... \text{ keeping at least six decimal places for intermediate values}$$

II. We then enter the given values in the TVM worksheet and compute FV.



$$FV \approx \$40,649.27$$

Therefore, the future value is \$40,649.27.

Try an Example



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D. Change in the Interest Rate

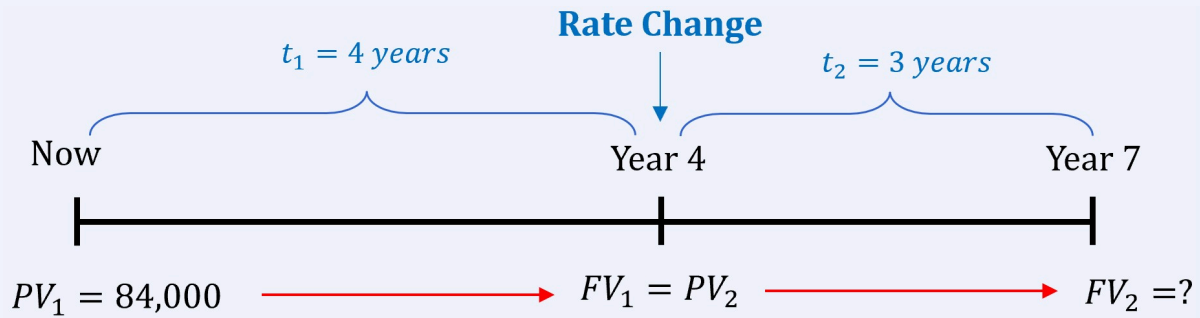
Sometimes during the term of an investment (or a loan), there is a change in one of the variables such as the interest rate or principal amount. In those cases, we should split the term into separate time segments at the point of change. The following example illustrates how the future value is computed when there is a change in the interest rate.

Example 2.2.4: Compute FV with Change in the Interest Rate

Pan's Consulting Inc. invested \$84,000 in a mutual fund at 6.5% compounded semi-annually. After 4 years the interest rate was changed to 4.44% compounded quarterly.

- a) How much was the value of the fund 3 years after the rate change?
- b) How much was the total compound interest earned during 7 years?

Show/Hide Solution



Since there is a change in the interest rate sometime in the middle of the term, we will break the term into two periods at the time of rate change, which is year 4. Then we will calculate the future value of the given principal of \$84,000 at year 4. Then, since the money remains in the account, the future value of the first period (t_1) will become the present value of the second period (t_2). Finally, we use the new interest rate to calculate the future value of the second period.

Given information:

First period: 4 years

- Present value (cash outflow): $PV_1 = -\$84,000$
- Nominal interest rate: $I/Y_1 = 6.5\%$
- Interest is compounded semi-annually so $C/Y_1 = 2$
- For compound problems, $P/Y = C/Y$, so $P/Y_1 = 2$
- Time period: $t_1 = 4$ years
- Number of compounding periods in the term: $N_1 = C/Y_1 \cdot t_1 = 2(4) = 8$

Second period (after the rate change): 3 years

- Present value (cash outflow): $PV_2 = FV_1 = -108,492.513$
- Nominal interest rate: $I/Y_2 = 4.44\%$
- Interest is compounded semi-annually so $C/Y_2 = 4$
- For compound problems, $P/Y = C/Y$, so $P/Y_2 = 4$
- Time period: $t_2 = 3$ years
- Number of compounding periods in the term: $N_2 = C/Y_2 \cdot t_2 = 4(3) = 12$

a) Compute FV

Computing FV_1

CLR TVM

8 N

6.5 I/Y

2ND P/Y=2

I/Y C/Y=2

-84000 PV

0 PMT

CPT FV

108,492.513

AmirTavangar.com

Computing FV_2

CLR TVM

12 N

4.44 I/Y

2ND P/Y=4

I/Y C/Y=4

-108,492.513 PV

0 PMT

CPT FV

123,859.4346

AmirTavangar.com

$FV_2 \approx \$123,859.43$

Therefore, the future value at the end of the second period is \$123,859.43, rounded to the nearest cents.

b) The total interest earned during the 7-year term is the difference between the original principal (PV_1) and the final future value (FV_2):

$$I = FV_2 - PV_1$$

$$I = 123,859.43 - 84,000$$

$$= \$39,859.43$$

Try an Example



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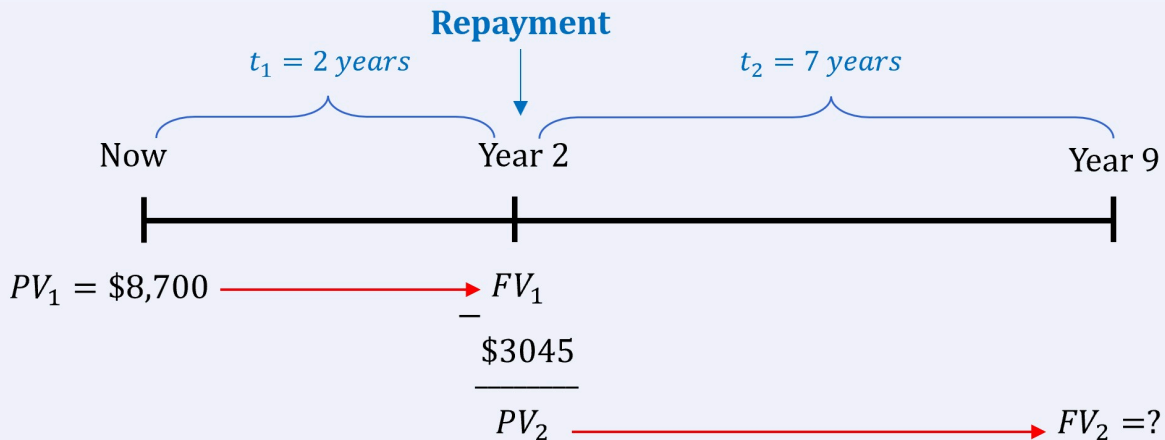
E. Change in the Principal

Just like adjustments made for changes in interest rates, if additional contributions or repayments are made during the period, we need to divide the period into smaller segments that correspond with these transactions. At each transaction point, we calculate the future value up to that moment and then adjust the principal by adding the contribution (if it's an investment) or subtracting the repayment (if it's a loan). After this adjustment, we compute the future value based on the updated principal for the next segment. The example below illustrates the process for calculating the future value when there are changes to the principal amount during the investment or loan term.

Example 2.2.5: Compute FV with Change in principal

Latasha initially borrowed \$8,700 from RBC Bank at 3.59% compounded monthly. After 2 years she repaid \$3,045. If she pays off the debt 9 years after the \$8,700 was initially borrowed, how much should her final payment be to clear the debt completely?

Show/Hide Solution



Since there is a repayment sometime during the term, we break the term into two periods at the time of repayment, which is in year 2. We calculate the future value of the loan principal of \$8,700 in year 2. Then, we find the new principal by deducting the repayment from the future value. This new principal will become the present value of the second period (t_2). Finally, we calculate the future value of the second period.

Given information:

Term 1: 2 years

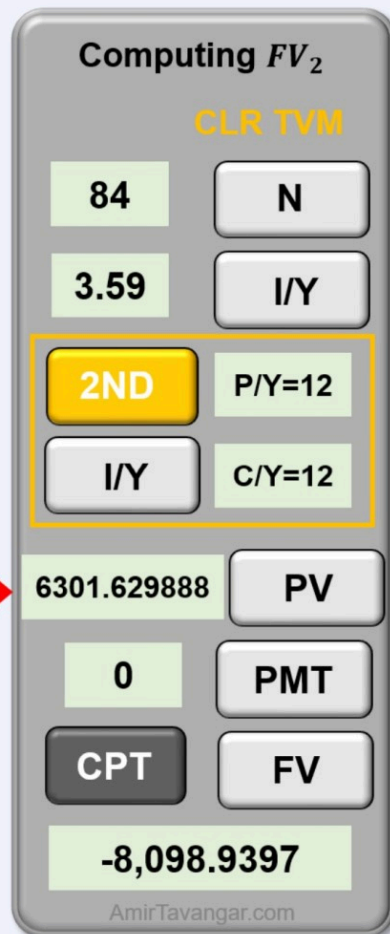
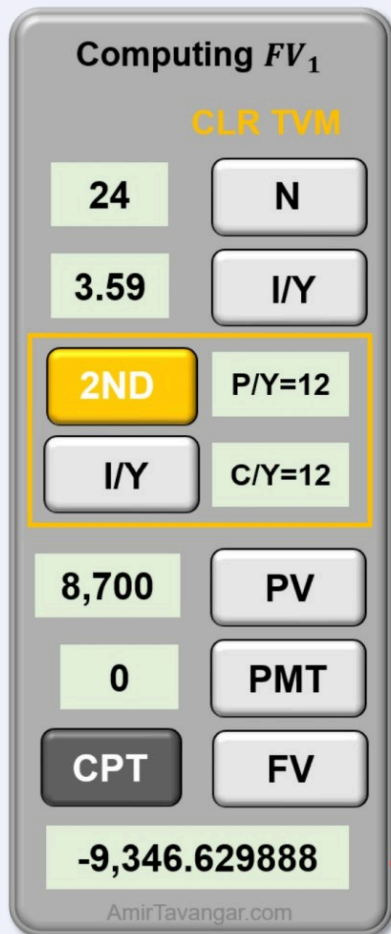
- Present value (cash inflow): $PV_1 = \$8,700$
- Nominal interest rate: $I/Y = 3.59\%$
- Interest is compounded monthly so $C/Y = 12$
- For compound problems, $P/Y = C/Y$, so $P/Y = 12$
- Time period: $t_1 = 2 \text{ years}$
- Number of compounding periods in the term 1: $N_1 = C/Y \cdot t_1 = 12(2) = 24$

Term 2: 7 years

- Present value (cash inflow): $PV_2 = 9346.629888 - \$3,045 = \6301.629888 (keep at least 6 decimal places for intermediate values)
- Nominal interest rate: $I/Y = 3.59\%$
- Interest is compounded monthly so $C/Y = 12$
- For compound problems, $P/Y = C/Y$, so $P/Y = 12$

- Time period: $t_2 = 7$ years
- Number of compounding periods in the term 1: $N_2 = C/Y \cdot t_2 = 12(7) = 84$

Compute FV



$FV_2 \approx \$8,098.94$

Try an Example



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Section 2.2 Exercises

1. Indiana Trust Co borrows \$15,700 from a bank at 6.4% compounded annually for 7 years and 9 months.
a) How much will the accumulated value of the loan be at the end of the term? **b)** How much interest will be charged on the loan?

Show/Hide Answer

a) $FV = \$25,392.02$

b) $I = \$9,692.02$

2. Nevaeh deposited \$12,600 in a savings account at 6.8% compounded annually for 6 years and 10 months.
a) Calculate the accumulated value of this amount at the end of the term. **b)** Calculate the amount of compound interest earned.

Show/Hide Answer

a) $FV = \$19,751.83$

b) $I = \$7,151.83$

3. Hillary invested \$29,400 in a mutual fund at 2.7% compounded semi-annually. After 4 years, the interest rate was changed to 6.1% compounded quarterly. **a)** How much was the value of the fund 8 years after the rate change? **b)** How much was the total compound interest earned during the whole 12-year term?

Show/Hide Answer

a) $FV = \$53,121.86$

b) $I = \$23,721.86$

4. Erika initially borrowed \$46,700 from a bank at 4.3% compounded monthly. After 2 years she repaid \$8,651. If she pays off the debt 9 years after the loan was initially borrowed, how much should the final payment be to clear the debt completely?

Show/Hide Answer

Final payment = \$57,037.81

5. A sum of \$13,480 was deposited into an account on June 14, 2013. What will be the future value of this sum of money on August 21, 2016, if the interest rate is 6.77% compounded semi-annually?

Show/Hide Answer

$FV = \$16,668.68$

2.3 PRESENT VALUE: CALCULATOR APPROACH

To find out the initial amount of money required to achieve a certain future value, we calculate the present value (PV). This amount represents the initial sum of money that, after accruing interest over time, will reach the desired future amount. The present value can also be considered the equivalent value of a given sum of money at an earlier point in time. When using a financial calculator to find the PV, follow the steps outlined in Section 2.2, but in this case, PV is the variable we compute.

The method of calculating the present value of a given future value is called discounting. Thus, the present value is sometimes referred to as the discounted value, and the amount of interest calculated for such cases is called the compound discount. The compound discount is obtained the same way the compound interest is computed using Formula 2.3.

Example 2.3.1: Compute PV

Calculate the present value of \$4320 that is due in 5 years and 9 months given the interest rate is 3.45% compounded semi-annually.

Watch Video



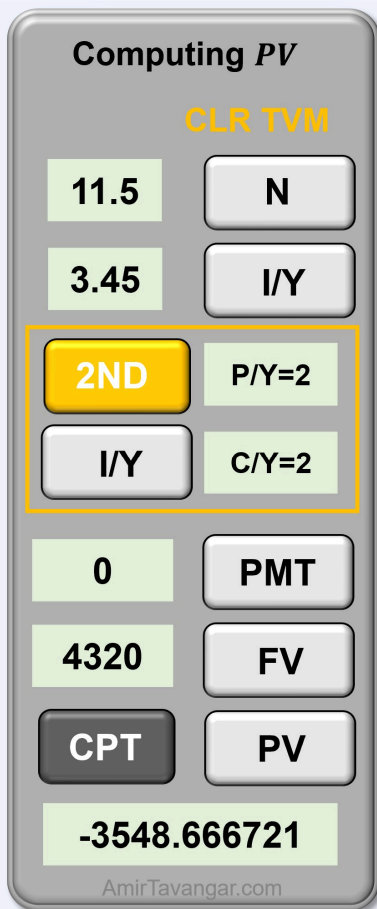
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Show/Hide Solution

Given information:

- Future value (inflow): $FV = \$4320$
- Nominal interest rate: $I/Y = 3.45\%$
- Interest is compounded monthly so $P/Y = C/Y = 2$
- Compound Term: $t = 5 \text{ years} + 8 \text{ months} = 5 + \frac{9}{12} = 5.75 \text{ years}$
- Number of compounding periods in the term: $N = C/Y \cdot t = 2(5.75) = 11.5$

$PV = ?$



$PV \approx \$3548.67$

Therefore, the present value is \$3548.67.

Try an Example



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Section 2.3 Exercises

1. Wyatt would like to accumulate \$197,100 for his retirement in 16 years. If his local bank promises him 4.16% compounded quarterly, how much he should invest today?

Show/Hide Answer

$$PV = \$101,651.32$$

2. Find the amount paid for an investment that will mature to \$171,000 in 6 years and 2 months at the interest rate of 3.04% compounded quarterly.

Show/Hide Answer

$$PV = \$141,869.22$$

3. Compute the present value of \$132,300 that is invested for 8 years and 6 months at 4.36% compounded monthly.

Show/Hide Answer

$$PV = \$91,390.70$$

2.4 INTEREST RATE: CALCULATOR APPROACH

A. Computing the Periodic and Nominal Interest Rates

If the nominal interest rate of a loan or investment is not specified, we can determine it by using the present value, the future value, and the number of compounding periods in the term.

In the calculator approach, the nominal interest rate I/Y can be computed directly. Then the periodic interest rate i can be determined by $i = \frac{I/Y}{C/Y}$ (Formula 2.1a).

Example 2.4.1: Compute Interest Rates

Nina invested \$4,900 in an account that grew to \$33,500 over a period of 9 years. Assume that the interest in the account was compounded monthly. **a)** What was the nominal interest rate of the account? **b)** What was the interest rate per month (periodic interest rate)? Give your answer as a percentage rounded to two decimal places.

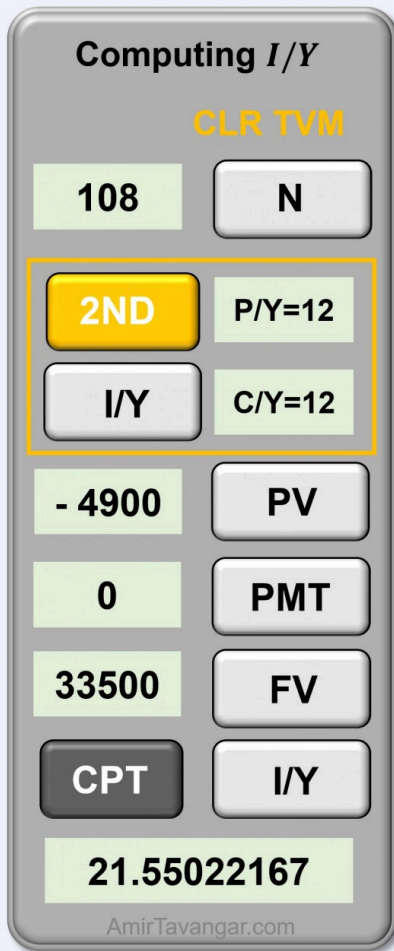
Show/Hide Solution

Given information:

- For investment, present value is a cash outflow: $PV = -\$4900$
- For investment, future value is a cash inflow: $FV = \$33,500$
- Interest was compounded monthly so $C/Y = 12$
- Investment Term: $t = 9$
- Number of compounding periods in the term: $N = C/Y \cdot t = 12(9) = 108$

In the calculator approach, we first compute the nominal interest rate, and then we will find the periodic rate. Note that for investments, PV is considered a cash outflow and entered as a negative value, while FV is a cash inflow and should be entered as a positive value.

a)



$$I/Y \approx 21.55\%$$

Therefore, the nominal interest rate is 21.55% compounded monthly.

b) The periodic (monthly) interest rate i can be determined by Formula 2.1a:

$$\begin{aligned}
 i &= \frac{I/Y}{C/Y} \\
 i &= \frac{21.55022167 \%}{12} \\
 &= 1.795851... \% \\
 &\approx 1.80 \%
 \end{aligned}$$

Try an Example



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B. Calculating Effective Interest Rate

The **effective interest rate** refers to the nominal interest rate that is compounded annually. The effective interest rate is used to make it easier to compare the annual interest rates between loans and investments with different compounding periods.

If the compounding frequency is already one, the given nominal rate is the effective interest rate. Otherwise, we can again use either a formula or a financial calculator to obtain the effective interest rate.

The ICONV (I conversion) worksheet on a financial calculator can be used to compute the effective interest rate. The worksheet is the secondary function of key 2, so it is opened by pressing 2ND and then key 2. Figure 2.4.1 shows the keys on the ICONV worksheet on a financial calculator.

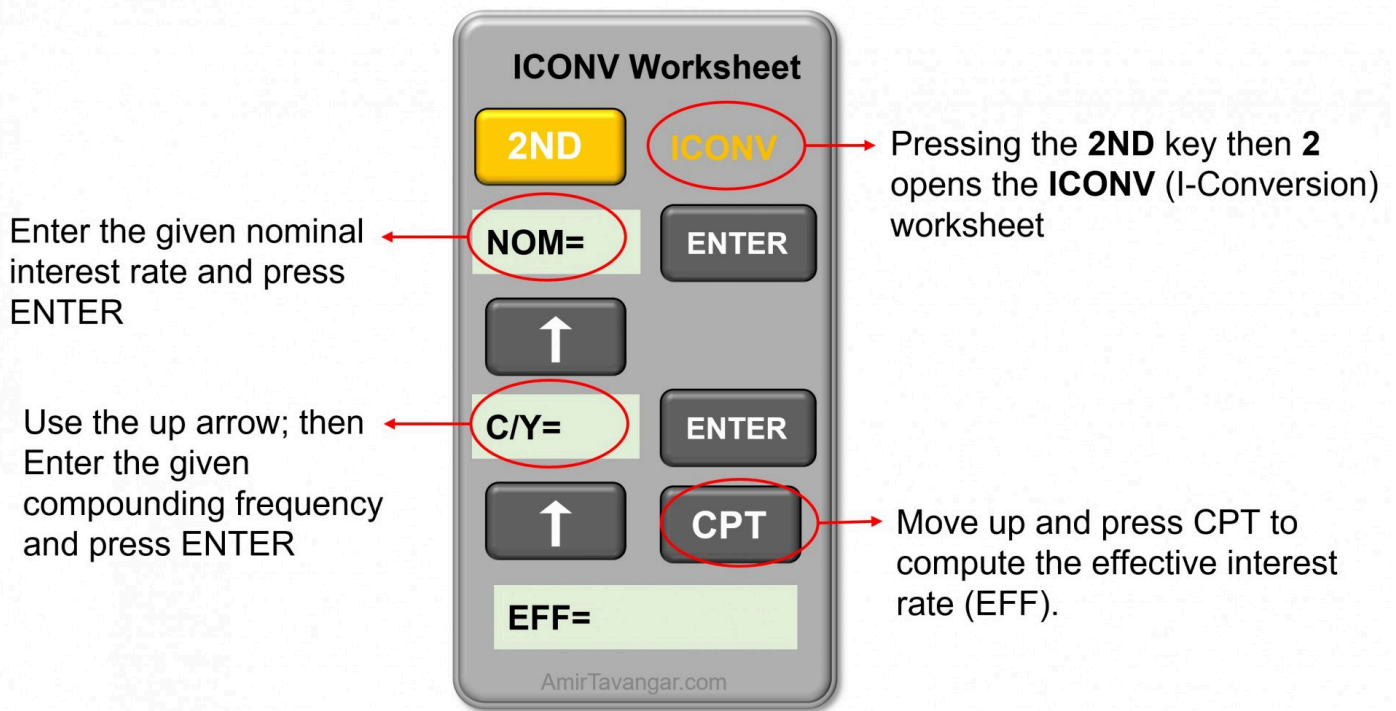


Figure 2.4.1 The keys on the ICONV worksheet on a financial calculator

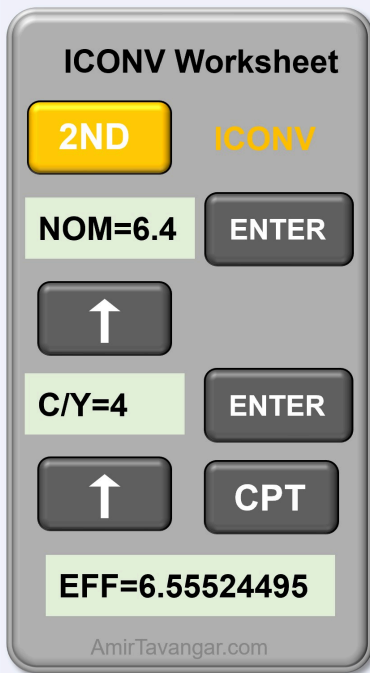
Example 2.4.2: Compute Effective Interest Rate

Find the effective rate of interest of an investment that earns 6.4% compounded quarterly. Express your answer as a percent rounded to two decimal places.

Show/Hide Solution

Given information:

- Interest is compounded quarterly so $C/Y = 4$
- Nominal interest rate: $NOM = 6.4\%$



$EFF \approx 6.56\%$
Compounded annually

Therefore, the effective interest rate is 6.56% compounded annually.

Try an Example



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C. Calculating Equivalent Interest Rates

Equivalent interest rates are nominal interest rates with different compounding periods that result in the same future value given the same principal value and time. We may want to find an equivalent interest rate of a given rate

with different compounding periods for comparison purposes. Similar to the effective interest rate discussed in the previous section, we can use either a formula or a financial calculator to compute the equivalent interest rates.

The ICONV worksheet on a financial calculator can also be used to compute the equivalent interest rates. To find the equivalent interest rate of a given nominal rate and frequency, first, convert the original nominal rate to the effective interest rate (compounded annually), and then use the obtained effective interest rate to compute the equivalent nominal interest rate at the desired compounding frequency.

Example 2.4.3: Equivalent Interest Rates

Convert the interest rate of 7.42% compounded quarterly to an equivalent interest rate compounded monthly. Express your answer as a percent rounded to two decimal places.

Watch Video

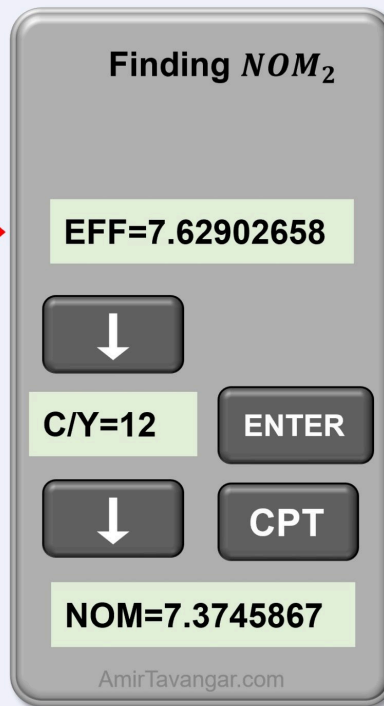
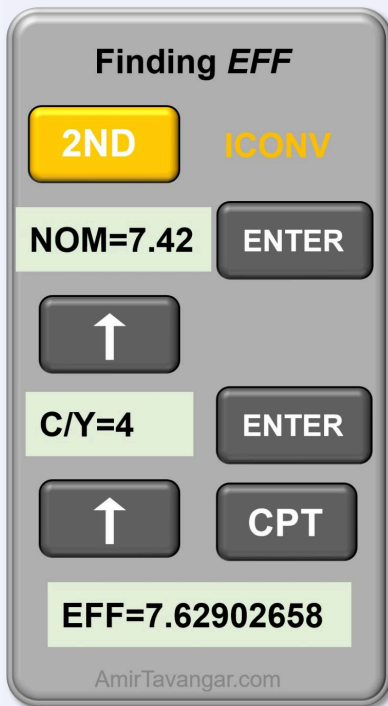


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Show/Hide Solution

Given information:

- Original compounded frequency is quarterly so $C / Y_1 = 4$
- Original nominal interest rate: $NOM_1 = 7.42\%$
- Desired compounded frequency is monthly so $C / Y_2 = 12$



**$NOM_2 \approx 7.37\%$
Compounded monthly**

Thus, the interest rate of 7.37% compounded monthly is equivalent to the effective interest rate of 7.63% (compounded annually) and 7.42% compounded quarterly. That means they all will yield the same future value over the same period given the same principal value.

Try an Example



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Section 2.4 Exercises

1. At what nominal rate of interest compounded quarterly will money grow from \$18,000.00 to \$20,881.85 in 4 years? Round the value of I/Y to two decimal places.

Show/Hide Answer

I/Y = 3.73% compounded quarterly

2. Suppose \$11,500.00 is deposited into an account today, and it is expected to grow to a maturity value of \$13,998.44 in 8 years from now. What is the account nominal interest rate compounded monthly? Round the answer to two decimal places.

Show/Hide Answer

I/Y = 2.46%

3. Find the effective rate of interest of an investment that earns 6.34% compounded quarterly. Round your answer to two decimal places.

Show/Hide Answer

EFF = 6.49% compounded annually

4. Convert the interest rate of 4.59% compounded monthly to an equivalent interest rate compounded semi-annually. Round your answer to two decimal places.

Show/Hide Answer

I/Y = 4.63% compounded semi-annually

2.5 NUMBER OF COMPOUNDING PERIODS AND TIME: CALCULATOR APPROACH

If the duration of a loan or investment is not specified, we can determine the number of compounding periods, N , by using the present value, the future value, and the interest rate. Once we have the number of compounding periods, we can then calculate the total length of time for the term.

We can use the financial calculator to compute N the same way you calculate other variables. Subsequently, in both formula and calculator approaches, the time period of the term t is determined by Formula 2.2b, which is the rearranged form of Formula 2.2a.

$$t = \frac{N}{C/Y} \quad \text{Formula 2.2b}$$

Example 2.5.1: Compute Time

How long will an investment of \$24,000 grow to at least \$96,000 at 5.8% compounded semi-annually? Express your answer in **a)** years, rounded to two decimal places, **b)** years and months, and **c)** years and days.

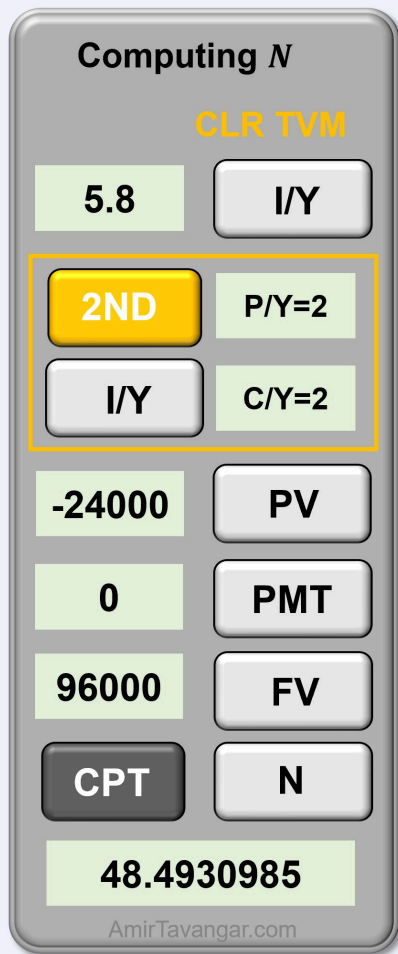
Show/Hide Solution

Given information:

- Present value: $PV = -\$24,000$
- Future value: $FV = \$96,000$
- Interest was compounded semi-annually so $C/Y = 2$
- Nominal interest rate: $I/Y = 5.8\%$

Note that for investments, PV is considered a cash outflow and entered as a negative value, while FV is a cash inflow and should be entered as a positive value.

Finding N



$N = 48.4930985$ Keep at least six decimal places for intermediate values

Finding time (t)

Therefore, the investment term time period is obtained by Formula 2.2b:

$$t = \frac{N}{C/Y}$$

$$t = \frac{48.493098...}{2}$$

$$= 24.246549... \text{ years}$$

a) The time in years rounded to two decimal places is $t = 24.25$ years.

b) To express time in terms of years and months, take the integer portion as the number of years. Then, multiply the decimal part by 12 to convert it into months.

$$\begin{aligned}
 t &= 24.246549\dots \text{ years} \\
 &= 24 \text{ years} + (0.246549\dots \times 12) \text{ months} \\
 &= 24 \text{ years} + (2.958588\dots) \text{ months} \\
 &= 24 \text{ years and } 3 \text{ months}
 \end{aligned}$$

c) To express time in terms of years and days, take the integer portion as the number of years. Then, multiply the decimal part by 365 to convert it into days.

$$\begin{aligned}
 t &= 24.246549\dots \text{ years} \\
 &= 24 \text{ years} + (0.246549\dots \times 365) \text{ days} \\
 &= 24 \text{ years} + (89.990385\dots) \text{ days} \\
 &= 24 \text{ years and } 90 \text{ days}
 \end{aligned}$$

See Section 1.1 to recall how to convert the number of days and months to the number of years and vice versa.

Note: In time calculations for financial problems, the resulting figure often includes a decimal, such as a fraction of a day or month. In such cases, unless instructed otherwise, always round up the value to the nearest whole number, regardless of the decimal value. This rounding-up approach ensures that the required future value is fully accumulated.

Try an Example



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Section 2.5 Exercises

1. Consider an investment of \$13,000.00 that grows to at least \$28,541.83 at 7.97% compounded monthly. **a)** What is the number of compounding periods in the term rounded to two decimal places? **b)** How long does the investment take? Express the time in years and months.

Show/Hide Answer

a) $N = 118.8$

b) $t = 9$ years and 11 months

2. How long will it take for an investment of \$18,200 to grow to \$24,103.80 at an interest rate of 3% compounded quarterly? Express the time in years and days.

Show/Hide Answer

$t = 9$ years and 146 days

2.6 FUTURE VALUE: FORMULA APPROACH

A. Introduction

In the study of compound interest, future value is the total amount that an initial sum, whether invested or borrowed, will accumulate over time at a specific compound interest rate. This includes the original principal and all the interest that accrues, compounded at set intervals. Present value, on the other hand, is the current equivalent of a future sum of money. It's the initial amount that, when interest is applied over a period, will grow to a predetermined future value.

The future value of compound interest FV is determined by

$$FV = PV(1 + i)^N \quad \text{Formula 2.4a}$$

where PV represents the present value i is the periodic interest rate (calculated as $i = \frac{I/Y}{C/Y}$, Formula 2.1a) and

N is the number of compounding periods in the term (computed as $N = C/Y \cdot t$, Formula 2.2a).

B. Computing Interest Amount

The future value of an investment or loan includes both the initial principal and the interest earned or charged over the term. To find the total amount of interest accrued, you subtract the present value (initial amount) from the future value. This interest amount (I), often called compound interest (CI) in the context of compound interest calculations, is given by

$$I = FV - PV \quad \text{Formula 2.3}$$

where FV is the future value and PV is the present value. This formula was initially introduced in Section 2.2.

Example 2.6.1: Compute FV – Time Given in Years

Sarah invested \$25,000 for 5 years at 6.9% compounded monthly. (a) How much was the accumulated value after 5 years? (b) How much interest was earned during the investment term?

Show/Hide Solution

Given information:

- Present value: $PV = \$25,000$
- Nominal interest rate: $I/Y = 6.9\%$
- Interest is compounded monthly so $C/Y = 12$
- Investment Term: $t = 5$ years
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{6.9\%}{12} = 0.575\%$
- Number of compounding periods in the term: $N = C/Y \cdot t = 12(5) = 60$

a)

Substituting the values into Formula 2.4a yields

$$\begin{aligned} FV &= PV(1 + i)^N \\ FV &= 25000(1 + 0.575\%)^{60} \\ &= 25000(1.410595\dots) \\ &= 35264.885920\dots \\ &\approx \$35,264.89 \text{ (Rounded to the nearest cents)} \end{aligned}$$

b) Substituting the PV and FV values into Formula 2.3 gives the amount of interest earned.

$$\begin{aligned} I &= FV - PV \\ I &= 35,264.89 - 25,000 \\ &= \$10,264.89 \end{aligned}$$

Example 2.6.2: Compute FV – Time Given in Years and Months

Ontario Servers Inc. borrows \$65,400 from a bank at 5.68% compounded quarterly for 4 years and 7 months.

a) How much will the accumulated value of the loan be at the end of the term? **b)** How much interest will be charged on the loan?

Show/Hide Solution

Given information:

- Present value: $PV = \$65,400$
- Nominal interest rate: $I/Y = 5.68\%$
- Interest is compounded quarterly so $C/Y = 4$
- Loan term: $t = 4$ years and 7 months $= 4 + \frac{7}{12} = 4.58\bar{3}$ years (The 'overbar' indicates that the digit beneath it is repeated continuously)
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{5.68\%}{4} = 1.42\%$
- Number of compounding periods in the term: $N = C/Y \cdot t = 4(4.58\bar{3}) = 18.\bar{3}$

a)

Substituting the values into Formula 2.4a yields

$$\begin{aligned}
 FV &= PV(1 + i)^N \\
 FV &= 65400(1 + 1.42\%)^{18.\bar{3}} \\
 &= 65400(1.294989\dots) \\
 &= 84,692.287\dots \\
 &\approx \$84,692.29 \text{ (Rounded to the nearest cents)}
 \end{aligned}$$

b) Substituting the PV and FV values into Formula 2.3 yields the amount of interest charged.

$$I = FV - PV$$

$$\begin{aligned}
 I &= 84,692.29 - 65,400 \\
 &= \$19,292.29
 \end{aligned}$$

Try an Example



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C. Time Given as Specific Dates

As we saw in the simple interest chapter, sometimes the beginning and end dates of the loan/investment term are given. In that case, unless otherwise stated, the time calculation will be most accurate if the number of days between the given dates is computed and then converted to equivalent years. To convert the number of days to the number of years, divide it by 365 as there are 365 days in a year. There are several ways to compute the days between two dates (DBD), some of which are presented in section 1.1.

Example 2.6.3: Compute FV Given Specific Dates

A sum of \$36,200.00 was deposited into an account on June 30, 2020. What will be the future value of this sum of money on October 9, 2025, if the interest rate is 2.22% compounded annually?

Show/Hide Solution

The given information:

- Present value: $PV = \$36,200.00$
- Nominal interest rate: $I/Y = 2.22\%$
- Interest is compounded semi-annually so $C/Y = 1$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{2.22\%}{1} = 2.22\%$
- Beginning date: June 30, 2020
- End date: October 9, 2025

I. We first need to find the number of days in the investment and find its equivalent years to find time (t). There are 1927 days between the given dates (DBD). We can find time in years by dividing DBD by 365.

$$t = \frac{1927}{365} = 5.279452... \text{ years}$$

$$N = C/Y \cdot t = 1 \left(\frac{1927}{365} \right) = 5.279452... \text{ keeping at least six decimal places for intermediate values}$$

II. We substitute the values into Formula 2.4a gives

$$\begin{aligned} FV &= PV(1 + i)^N \\ FV &= 36200(1 + 2.22\%)^{5.279452} \\ &= 36200(1.122290...) \\ &= 40,649.272124... \\ &\approx \$40,649.27 \text{ (Rounded to the nearest cents)} \end{aligned}$$

Try an Example



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D. Change in the Interest Rate

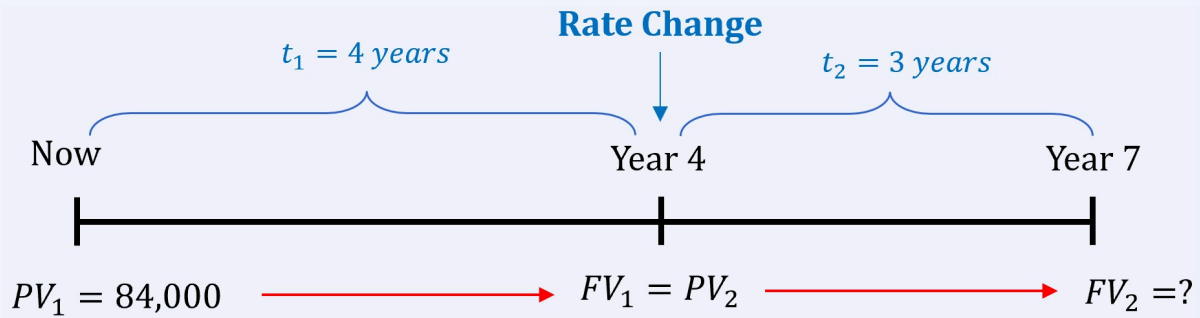
Sometimes during the term of an investment (or a loan), there is a change in one of the variables such as the interest rate or principal amount. In those cases, we should split the term into separate time segments at the point of change. The following example illustrates how the future value is computed when there is a change in the interest rate.

Example 2.6.4: Compute FV with Change in the Interest Rate

Pan's Consulting Inc. invested \$84,000 in a mutual fund at 6.5% compounded semi-annually. After 4 years the interest rate was changed to 4.44% compounded quarterly.

- (a) How much was the value of the fund 3 years after the rate change?
- (b) How much was the total compound interest earned during 7 years?

Show/Hide Solution



Since there is a change in the interest rate sometime in the middle of the term, we will break the term into two periods at the time of rate change, which is year 4. Then we will calculate the future value of the given principal of \$84,000 at year 4. Then, since the money remains in the account, the future value of the first period (t_1) will become the present value of the second period (t_2). Finally, we use the new interest rate to calculate the future value of the second period.

a)

First period: 4 years

- Present value: $PV_1 = \$84,000$
- Nominal interest rate: $I/Y_1 = 6.5\%$
- Interest is compounded semi-annually so $C/Y_1 = 2$
- Time period: $t_1 = 4 \text{ years}$
- Periodic interest rate: $i_1 = \frac{I/Y_1}{C/Y_1} = \frac{6.5\%}{2} = 3.25\%$
- Number of compounding periods in the term: $N_1 = C/Y_1 \cdot t_1 = 2(4) = 8$

Substituting the values into Formula 2.4a yields

$$FV_1 = PV_1(1 + i_1)^{N_1}$$

$$FV_1 = 84,000(1 + 3.25\%)^8$$

$$= 108,492.512\dots \text{ (DO NOT round off the intermediate values)}$$

Second period (after the rate change): 3 years

- Present value: $PV_2 = FV_1 = 108,492.512\dots$
- Nominal interest rate: $I/Y_2 = 4.44\%$
- Interest is compounded semi-annually so $C/Y_2 = 4$
- Time period: $t_2 = 3$ years
- Periodic interest rate: $i_2 = \frac{I/Y_2}{C/Y_2} = \frac{4.44\%}{4} = 1.11\%$
- Number of compounding periods in the term: $N_2 = C/Y_2 \cdot t_2 = 4(3) = 12$

Substituting the values into Formula 2.4a yields

$$\begin{aligned}
 FV_2 &= PV_2(1 + i_2)^{N_2} \\
 FV_2 &= 108,492.512\dots(1 + 1.11\%)^{12} \\
 &= 123,859.434\dots \\
 &\approx \$123,859.43 \text{ (Rounded to the nearest cents)}
 \end{aligned}$$

b) The total interest earned during the 7-year term is the difference between the original principal (PV_1) and the final future value (FV_2):

$$\begin{aligned}
 I &= FV_2 - PV_1 \\
 I &= 123,859.43 - 84,000 \\
 &= \$39,859.43
 \end{aligned}$$

Try an Example





One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=326>

E. Change in the Principal

Just like adjustments made for changes in interest rates, if additional contributions or repayments are made during the period, we need to divide the period into smaller segments that correspond with these transactions. At each transaction point, we calculate the future value up to that moment and then adjust the principal by adding the contribution (if it's an investment) or subtracting the repayment (if it's a loan). After this adjustment, we compute the future value based on the updated principal for the next segment. The example below illustrates the process for calculating the future value when there are changes to the principal amount during the investment or loan term.

Example 2.6.5: Compute FV with Change in principal

Latasha initially borrowed \$8,700 from RBC Bank at 3.59% compounded monthly. After 2 years she repaid \$3,045. If she pays off the debt 9 years after the \$8,700 was initially borrowed, how much should her final payment be to clear the debt completely?

Show/Hide Solution

Term 2 (after the repayment): 7 years

- Present value (cash inflow): $PV_2 = 9346.629888 - 3,045 = 6301.629888\dots$ (*DO NOT round off the intermediate values*)
- Same interest rate
- Time period: $t_2 = 7$ years
- Number of compounding periods in the term 1: $N_2 = C/Y \cdot t_2 = 12(7) = 84$

Substituting the values into Formula 2.4a gives the future value at the end of the loan term:

$$\begin{aligned}
 FV_2 &= PV_2(1 + i)^{N_2} \\
 FV_1 &= 6301.629\dots(1 + 0.2991\bar{6}\%)^{24} \\
 &= 8,098.939\dots \\
 &\approx \$8,098.94 \text{ (Rounded to the nearest cents)}
 \end{aligned}$$

Try an Example



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Section 2.6 Exercises

1. Indiana Trust Co borrows \$15,700 from a bank at 6.4% compounded annually for 7 years and 9 months.
 - a) How much will the accumulated value of the loan be at the end of the term? b) How much interest will be charged on the loan?

Show/Hide Answer

a) $FV = \$25,392.02$

b) $I = \$9,692.02$

2. Nevaeh deposited \$12,600 in a savings account at 6.8% compounded annually for 6 years and 10 months. **a)** Calculate the accumulated value of this amount at the end of the term. **b)** Calculate the amount of compound interest earned.

Show/Hide Answer

a) $FV = \$19,751.83$

b) $I = \$7,151.83$

3. Hillary invested \$29,400 in a mutual fund at 2.7% compounded semi-annually. After 4 years, the interest rate was changed to 6.1% compounded quarterly. **a)** How much was the value of the fund 8 years after the rate change? **b)** How much was the total compound interest earned during the whole 12-year term?

Show/Hide Answer

a) $FV = \$53,121.86$

b) $I = \$23,721.86$

4. Erika initially borrowed \$46,700 from a bank at 4.3% compounded monthly. After 2 years she repaid \$8,651. If she pays off the debt 9 years after the loan was initially borrowed, how much should the final payment be to clear the debt completely?

Show/Hide Answer

Final payment = \$57,037.81

5. A sum of \$13,480 was deposited into an account on June 14, 2013. What will be the future value of this sum of money on August 21, 2016, if the interest rate is 6.77% compounded semi-annually?

Show/Hide Answer

$$\text{FV} = \$16,668.68$$

2.7 PRESENT VALUE: FORMULA APPROACH

To find out the initial amount of money required to achieve a certain future value, we calculate the present value (PV). This amount represents the initial sum of money that, after accruing interest over time, will reach the desired future amount. The present value can also be considered the equivalent value of a given sum of money at an earlier point in time. To calculate the present value, Formula 2.4a can be rearranged for PV .

$$PV = \frac{FV}{(1 + i)^N} \quad \text{Formula 2.4b}$$

which using the negative-exponent notation is equivalent to

$$PV = FV(1 + i)^{-N} \quad \text{Formula 2.4c}$$

The method of calculating the present value of a given future value is called discounting. Thus, the present value is sometimes referred to as the discounted value, and the amount of interest calculated for such cases is called the compound discount. The compound discount is obtained the same way the compound interest is computed using Formula 2.3.

Example 2.7.1: Compute PV

Calculate the discounted value of \$4320 that is due in 5 years and 8 months given the interest rate is 3.45% compounded monthly.

Show/Hide Solution

Given information:

- Future value: $FV = \$4320$
- Nominal interest rate: $I/Y = 3.45\%$

- Interest is compounded monthly so $C/Y = 12$
- Compound term: $t = 5 \text{ years} + 8 \text{ months} = 5 + \frac{8}{12} = 5\frac{8}{12}$ years
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{3.45\%}{12} = 0.2875\%$
- Number of compounding periods in the term: $N = C/Y \cdot t = 12 \left(5\frac{8}{12} \right) = 68$

Substituting the values into Formula 2.4c yields

$$\begin{aligned}
 PV &= FV(1 + i)^{-N} \\
 PV &= 4320(1 + 0.2875\%)^{-68} \\
 &= 3553.865541\dots \\
 &\approx \$3553.87 \text{ (Rounded to the nearest cents)}
 \end{aligned}$$

Try an Example



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Section 2.7 Exercises

1. Wyatt would like to accumulate \$197,100 for his retirement in 16 years. If his local bank promises him 4.16% compounded quarterly, how much he should invest today?

Show/Hide Answer

$$PV = \$101,651.32$$

2. Find the amount paid for an investment that will mature to \$171,000 in 6 years and 2 months at the interest rate of 3.04% compounded quarterly.

Show/Hide Answer

$$PV = \$141,869.22$$

3. Compute the present value of \$132,300 that is invested for 8 years and 6 months at 4.36% compounded monthly.

Show/Hide Answer

$$PV = \$91,390.70$$

2.8 INTEREST RATE: FORMULA APPROACH

A. Computing the Periodic and Nominal Interest Rates

If the nominal interest rate of a loan or investment is not specified, we can determine it by using the present value, the future value, and the number of compounding periods in the term.

Since the periodic interest rate i appears in the formula, using the formula approach, we find i first, and then we calculate the nominal interest rate I/Y as follows.

Rearranging Formula 2.4a for i gives

$$FV = PV(1 + i)^N$$

$$\text{Formula does not parse } (1 + i)^N = \frac{FV}{PV}$$

$$1 + i = \left(\frac{FV}{PV}\right)^{\frac{1}{N}}$$

$$i = \left(\frac{FV}{PV}\right)^{\frac{1}{N}} - 1 \quad \text{Formula 2.5}$$

The nominal interest rate can be then found by rearranging Formula 2.1a.

$$I/Y = i \cdot C/Y \quad \text{Formula 2.1b}$$

Example 2.8.1: Compute Interest Rates

Nina invested \$4,900 in an account that grew to \$33,500 over a period of 9 years. Assuming that the interest in the account was compounded monthly, **a)** What was the monthly interest rate? **b)** what was the annual interest rate of the account? Give your answer as a percentage rounded to two decimal places.

Show/Hide Solution

Given information:

- Present value: $PV = \$4900$
- Future value: $FV = \$33,500$
- Interest was compounded monthly so $C/Y = 12$
- Investment Term: $t = 9$
- Number of compounding periods in the term: $N = C/Y \cdot t = 12(9) = 108$

Substituting the values into Formula 2.5 gives the periodic (monthly) interest rate:

a)

$$\begin{aligned}
 i &= \left(\frac{FV}{PV} \right)^{\frac{1}{N}} - 1 \\
 i &= \left(\frac{33500}{4900} \right)^{\frac{1}{108}} - 1 \\
 &= 1.0179585... - 1 \\
 &= 0.0179585... \\
 &= 1.79585... \% \\
 &\approx 1.80 \% \text{ per month}
 \end{aligned}$$

b) The nominal interest rate I/Y is then given by Formula 2.1b

$$\begin{aligned}
 I/Y &= i \cdot C/Y \\
 &= 1.79585... \% (12) \text{ (Note that the unrounded } i \text{ is used here for accuracy)} \\
 &= 21.550221... \% \\
 &\approx 21.55 \%
 \end{aligned}$$

Try an Example



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B. Calculating Effective Interest Rate

The **effective interest rate** refers to the nominal interest rate that is compounded annually. The effective interest rate is used to make it easier to compare the annual interest rates between loans and investments with different compounding periods.

If the compounding frequency is already one, the given nominal rate is the effective interest rate. Otherwise, we can again use either a formula or a financial calculator to obtain the effective interest rate.

By definition, given the same principal and time period, the effective interest rate EFF should result in the same future value as a nominal rate with a given compounding frequency. Therefore, the future values can be made equal, and the equation can be solved for EFF , which yields

$$EFF = (1 + i)^{C/Y} - 1 \quad \text{Formula 2.6}$$

where EFF is the effective interest rate and i and C/Y are the stated periodic interest rate and the compounding frequency, respectively.

Example 2.8.2: Compute Effective Interest Rate

Find the effective rate of interest of an investment that earns 6.4% compounded quarterly. Express your answer as a percent rounded to two decimal places.

Show/Hide Solution

Given information:

- Interest is compounded quarterly so $C/Y = 4$
- Nominal interest rate: $I/Y = 6.4\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{6.4\%}{4} = 1.6\%$

Substituting the values into Formula 2.6, we obtain

$$\begin{aligned}
 EFF &= (1 + i)^{C/Y} - 1 \\
 EFF &= (1 + 1.6\%)^4 - 1 \\
 &= 0.065552\dots \\
 &= 6.5552\dots\% \\
 &\approx 6.56\% \text{ compounded annually}
 \end{aligned}$$

Try an Example



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C. Calculating Equivalent Interest Rates

Equivalent interest rates are nominal interest rates with different compounding periods that result in the same future

value given the same principal value and time. We may want to find an equivalent interest rate of a given rate with different compounding periods for comparison purposes. Similar to the effective interest rate discussed in the previous section, we can use either a formula or a financial calculator to compute the equivalent interest rates.

By definition, given the same principal value and time period, the equivalent interest rates result in the same future value. Therefore, we can equate the future values and rearrange the equation for an equivalent periodic interest rate i_2 :

$$i_2 = (1 + i_1)^{\frac{C/Y_1}{C/Y_2}} - 1 \quad \text{Formula 2.7}$$

where i_1 and C/Y_1 are the original periodic interest rate and the compounding frequency, respectively, and C/Y_2 is the compounding frequency of the equivalent interest rate. Note that if $C/Y_2 = 1$, Formula 2.7 will become Formula 2.6, and thus i_2 will be the effective interest rate.

Once the periodic interest rate i_2 is known, the equivalent nominal interest rate is then determined by Formula 2.1b.

$$I/Y_2 = C/Y_2 \cdot i_2$$

Example 2.8.3: Equivalent Interest Rates

Convert the interest rate of 7.42% compounded quarterly to an equivalent interest rate compounded monthly. Express your answer as a percent rounded to two decimal places.

Show/Hide Solution

Given information:

- Original compounded frequency is quarterly so $C/Y_1 = 4$
- Original nominal interest rate: $I/Y_1 = 7.42\%$
- Original periodic interest rate: $i_1 = \frac{I/Y_1}{C/Y_1} = \frac{7.42\%}{4} = 1.855\%$
- Desired compounded frequency is monthly so $C/Y_2 = 12$

Substituting the values into Formula 2.7 gives

$$i_2 = (1 + i_1)^{\frac{C/Y_1}{C/Y_2}} - 1$$

$$i_2 = (1 + 1.855\%)^{\frac{4}{12}} - 1$$

$$= 0.0061454\dots$$

$$= 0.61454\dots\%$$

So the equivalent nominal interest rate is obtained by

$$I/Y_2 = C/Y_2 \cdot i_2$$

$$I/Y_2 = 12(0.61454\dots\%)$$

$$= 7.374586\dots\%$$

$$\approx 7.37\% \text{ compounded monthly}$$

Therefore, the interest rate of 7.37% compounded monthly is equivalent to 7.42% compounded quarterly, meaning that they will both result in the same future value over the same time period given the same principal value.

Try an Example



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Section 2.8 Exercises

1. At what nominal rate of interest compounded quarterly will money grow from \$18,000.00 to \$20,881.85 in 4 years? Round the value of I/Y to two decimal places.

Show/Hide Answer

I/Y = 3.73% compounded quarterly

2. Suppose \$11,500.00 is deposited into an account today, and it is expected to grow to a maturity value of \$13,998.44 in 8 years from now. What is the account nominal interest rate compounded monthly? Round the answer to two decimal places.

Show/Hide Answer

I/Y = 2.46%

3. Find the effective rate of interest of an investment that earns 6.34% compounded quarterly. Round your answer to two decimal places.

Show/Hide Answer

EFF = 6.49% compounded annually

4. Convert the interest rate of 4.59% compounded monthly to an equivalent interest rate compounded semi-annually. Round your answer to two decimal places.

Show/Hide Answer

I/Y = 4.63% compounded semi-annually

2.9 NUMBER OF COMPOUNDING PERIODS AND TIME: FORMULA APPROACH

If the duration of a loan or investment is not specified, we can determine the number of compounding periods, N , using Formula 2.4a and the given present value, future value, and interest rate. Once we have the number of compounding periods, we can then calculate the total length of time for the term.

Rearranging Formula 2.4a for N yields

$$\begin{aligned}
 FV &= PV(1 + i)^N \\
 (1 + i)^N &= \frac{FV}{PV} \\
 \ln(1 + i)^N &= \ln\left(\frac{FV}{PV}\right) \\
 N \ln(1 + i) &= \ln\left(\frac{FV}{PV}\right) \\
 N &= \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1 + i)}
 \end{aligned}
 \tag{Formula 2.8}$$

Note that N should not be rounded.

Subsequently, the time period of the term t is determined by Formula 2.2b, which is the rearranged form of Formula 2.2a.

$$t = \frac{N}{C/Y}
 \tag{Formula 2.2b}$$

Example 2.9.1: Compute Time

How long will an investment of \$24,000 grow to at least \$96,000 at 5.8% compounded semi-annually? Express your answer in **a)** years, rounded to two decimal places, **b)** years and months, and **c)** years and days.

Show/Hide Solution

Given information:

- Present value: $PV = \$24,000$
- Future value: $FV = \$96,000$
- Interest was compounded semi-annually so $C/Y = 2$
- Nominal interest rate: $I/Y = 5.8\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{5.8\%}{2} = 2.9\%$

Finding N

Substituting the values into Formula 2.8, we calculate the number of compounding periods in the investment term:

$$N = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1 + i)}$$

$$N = \frac{\ln\left(\frac{96000}{24000}\right)}{\ln(1 + 2.9\%)}$$

$$= \frac{1.386294\dots}{0.028587\dots}$$

$$N = 48.4930985 \text{ Keep at least six decimal places for intermediate values}$$

Finding time (t)

Therefore, the investment term time period is obtained by Formula 2.2b:

$$t = \frac{N}{C/Y}$$

$$t = \frac{48.493098\dots}{2}$$

$$= 24.246549\dots \text{ years}$$

a) The time in years rounded to two decimal places is $t = 24.25$ years.

b) To express time in terms of years and months, take the integer portion as the number of years. Then, multiply the decimal part by 12 to convert it into months.

$$t = 24.246549\dots \text{ years}$$

$$= 24 \text{ years} + (0.246549\dots \times 12) \text{ months}$$

$$= 24 \text{ years} + (2.958588\dots) \text{ months}$$

$$= 24 \text{ years and } 3 \text{ months}$$

c) To express time in terms of years and days, take the integer portion as the number of years. Then, multiply the decimal part by 365 to convert it into days.

$$t = 24.246549\dots \text{ years}$$

$$= 24 \text{ years} + (0.246549\dots \times 365) \text{ days}$$

$$= 24 \text{ years} + (89.990385\dots) \text{ days}$$

$$= 24 \text{ years and } 90 \text{ days}$$

See Section 1.1 to recall how to convert the number of days and months to the number of years and vice versa.

Note: In time calculations for financial problems, the resulting figure often includes a decimal, such as a fraction of a day or month. In such cases, unless instructed otherwise, always round up the value to the nearest whole number, regardless of the decimal value. This rounding-up approach ensures that the required future value is fully accumulated.

Try an Example



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Section 2.9 Exercises

1. Consider an investment of \$13,000.00 that grows to at least \$28,541.83 at 7.97% compounded monthly. **a)** What is the number of compounding periods in the term rounded to two decimal places? **b)** How long does the investment take? Express the time in years and months.

Show/Hide Answer

a) $N = 118.8$

b) $t = 9$ years and 11 months

2. How long will it take for an investment of \$18,200 to grow to \$24,103.80 at an interest rate of 3% compounded quarterly? Express the time in years and days.

Show/Hide Answer

$t = 9$ years and 146 days

2.10 TERMINOLOGY AND GLOSSARY OF SYMBOLS

Terminology

- **Dated value or equivalent value:** The value of a sum of money at a specific time relative to its due date, including the amount of interest.
- **Effective interest rate:** refers to the nominal interest rate that is compounded annually.
- **Future value (accumulated or maturity value) (FV):** The value obtained when the amount of interest is added to the original principal.
- **Nominal rate (I/Y):** the yearly or annual rate of interest.
- **Present value (principal or discounted value) (PV):** The amount borrowed or invested at the beginning of a (loan or investment) period.
- **Term (t):** Refers to either an investment term or loan term, which is the time period an amount is lent or borrowed.

Glossary of Symbols

Compound Interest

- I/Y : Nominal interest rate (%)
- C/Y : frequency of compounding period per year
- P/Y : number of periodic payments per year; for compound interest problems it is equal to C/Y
- PMT : Periodic payment; for compound interest problems it is zero.
- i : Interest rate per compounding period (%)
- t : time period of a loan or investment term (in years)
- N : Number of compounding periods in the term
- PV : Principal (Present Value) (\$)
- FV : Future (maturity) value (\$)
- CI or I : Compound Interest Amount (\$)

Interest Conversion

- **NOM**: Nominal interest rate (%)
- **EFF**: Effective interest rate (%)

Date

- **DT1**: Beginning date
- **DT2**: End date
- **DBD**: The number of days between dates DT1 and DT2

PART III

ANNUITIES

Chapter Outline

This chapter delves into the topic of annuities, structured into two main parts. The first part addresses using a financial calculator for annuity calculations, while the second part delves into algebraic formulas for these calculations.

3.1 Introduction to Annuities: This initial section lays the groundwork by defining annuities, explaining essential terms, and categorizing different types of annuities.

Calculator Approach

Sections in the calculator approach are typically divided into subsections, focusing on an ordinary annuity or an annuity due.

3.2 Future Value (FV) of Annuities – Calculator Approach: This section teaches how to calculate the future value of annuities with a financial calculator.

3.3 Present Value (PV) of Annuities – Calculator Approach: Here, the focus is on determining the present value of annuities using a financial calculator.

3.4 Payment (PMT) of Annuities – Calculator Approach: This part covers the calculation of annuity payments with a financial calculator.

3.5 Number of Payments and Duration – Calculator Approach: This section demonstrates how to find out the number of payments and the term length for annuities with a financial calculator.

3.6 Nominal Interest Rate – Calculator Approach: This section explores calculating nominal and periodic interest rates for annuities with a financial calculator.

Formula Approach

Each section in the Formula Approach is organized into subsections based on four annuity types: ordinary simple annuity, ordinary general annuity, simple annuity due, and general annuity due.

3.7 Future Value (FV) of Annuities – Formula Approach: This section focuses on the future value of annuities calculation using algebraic formulas.

3.8 Present Value (PV) of Annuities – Formula Approach: This section discusses the present value of annuities calculation with algebraic formulas.

3.9 Payment (PMT) of Annuities – Formula Approach: This section covers how to compute annuity payments using algebraic formulas.

3.10 Number of Payments and Duration – Formula Approach: This section shows how to calculate the number of payments and the duration of annuities using algebraic formulas.

3.11 Nominal Interest Rate – Formula Approach: Calculating the periodic interest rate (i) algebraically using formulas is challenging because it is difficult to isolate the interest rate variable in either the present value or future value formulas. As a result, we focus on using a financial calculator to determine the nominal interest rate (I/Y). For instructions on the calculator approach, please refer to Section 3.6.

3.12 Terminology and Glossary of Symbols: A section dedicated to defining terms and symbols used throughout the chapter, aiding in understanding and reference.

3.13 Appendix: Offers proofs for some of the formulas discussed in the chapter.

3.1 INTRODUCTION TO ANNUITIES

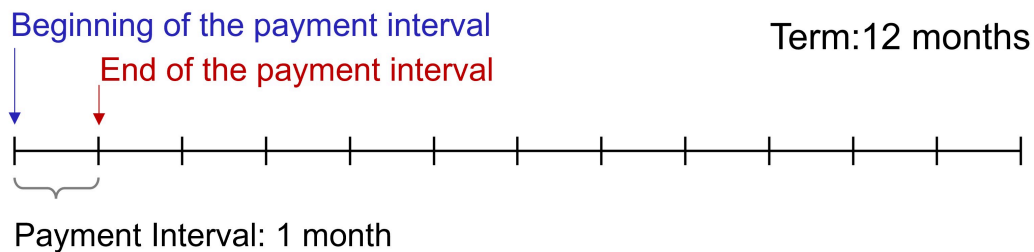
A. Definition of Annuity

An **annuity** is defined as a series of periodic payments, usually in equal size, made at regular time intervals. In annuities, the interest that is charged on money is usually compound interest. Mortgages and car loans or leases are examples of annuities. Typical payment intervals of annuities are monthly, quarterly, semi-annually, and annually, as summarized in Table 3.1.1.

Table 3.1.1 Number of payments per year for different payment intervals

| Payment Interval | Length of Payment Period | Number of Payments per Year (P/Y) |
|------------------|--------------------------|-----------------------------------|
| Annually | Every 12 months (1 year) | 1 |
| Semi-annually | Every 6 months | 2 |
| Quarterly | Every 3 months | 4 |
| Monthly | Every month | 12 |
| Bi-weekly | Every two weeks | 26 |
| Weekly | Every week | 52 |

Payments can be made at the beginning or end of payment intervals. For example, for monthly payments, the payment can be made on the first day of a month or the last day of a month.



Unlike compound interest problems where N represents the total number of compounding periods in the loan or investment term, in annuities, N represents the total number of payments in the term and can be obtained by

$$N = P/Y \cdot t \qquad \text{Formula 3.1a}$$

where P / Y is the number of periodic payments in a year, and t is the time period of the annuity in years.

When solving time-value-of-money problems, it is crucial first to identify the type of problem you are dealing with. Compound interest problems typically involve a single lump sum or irregular deposits or withdrawals occurring over time. In contrast, annuity problems involve a series of equal cash flows, either payments or withdrawals, that are made or received at regular intervals over a specified period.

To differentiate between compound interest and annuity problems, pay attention to certain keywords. Phrases like “every” or “each,” or time-related adverbs such as “weekly” and “monthly,” often indicate that the problem is related to annuities, involving regular, equal payments or withdrawals.

Example 1.3.1: Number of Payments in Annuities

For each given scenario, determine the number of payments per Year (P/Y), term in years (t), and the number of payments in the term (N).

- a) You deposit \$200 at the end of every 3 months for 5 years.
- b) Payments of \$1050 made at the end of each month for 4 years and 5 months.
- c) Monthly payment of \$150 for 4.5 years.

Show/Hide Solution

We first look for a keyword that indicates that the given payment is periodic, and thus the problem is indeed an annuity. In each example, the keyword is boldfaced. Once the number of payments in a year (P/Y) and the term duration (t) are determined, we use Formula 3.1a to compute the number of payments in the term (N).

- a) You deposit \$200 at the end of **every** 3 months for 5 years.
 - Number of payments in a year: Every 3 months so $P / Y = 4$
 - Term: $t = 5$ years
 - Number of payments in the term: $N = 4(5) = 20$
- b) Payments of \$1050 made at the end of **each** month for 4 years and 5 months.

- Number of payments in a year: Each month so $P/Y = 12$
- Term: $t = 4 \text{ years} + 5 \text{ months} = 4 \text{ years} + \frac{5}{12} \text{ years} = 4\frac{5}{12} \text{ years}$
- Number of payments in the term: $N = 12\left(4\frac{5}{12}\right) = 53$

c) **Weekly** payment of \$150 for 4.5 years.

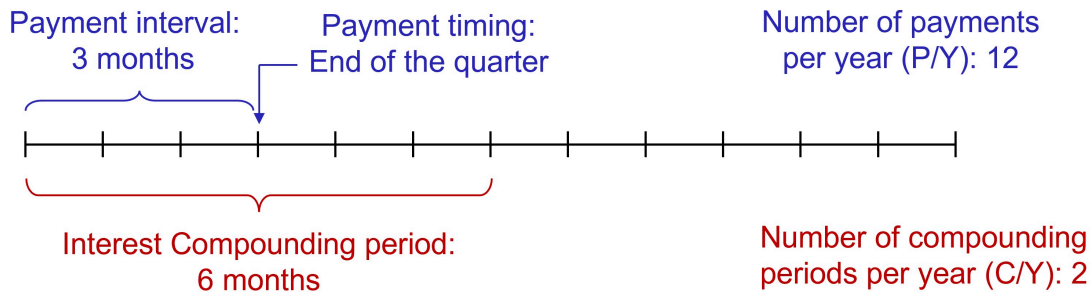
- Number of payments in a year: Weekly so $P/Y = 52$
- Term: $t = 4.5 \text{ years}$
- Number of payments in the term: $N = 52(4.5) = 234$

B. Components of Annuities

Each annuity problem has two pieces of information that you need to pay attention to:

- Information about the periodic payments
 - Size of the periodic payment (PMT)
 - Number of payments per year (P/Y)
 - The timing of payments (whether it is at the beginning or the end of the payment period)
- Information about the interest charged
 - Nominal interest rate (I/Y)
 - The number of interest compounding periods per year (C/Y)

For example, in the case of “payments of \$200 are made at the end of every quarter for 3 years into a savings account earning 6% compounded semi-annually”, the first part of the sentence provides information about the size, frequency, and timing of the payments, and the second part of the sentence gives information regarding the interest. Note that the phrase related to the interest is identical to compound interest problems.



C. Types of Annuities

Annuities can be classified in various ways, with a common approach based on the timing of payments in annuities and the relationship between frequencies of payments and interest-compounding periods. In this book, our focus is on immediate-start annuities that have predetermined maturity dates, aligning with these common classifications. We will not delve into deferred annuities, which begin payments at a later date, nor will we cover perpetuities, which do not have a set endpoint. Our discussion aims to provide a clear understanding of the most typical annuity structures encountered in financial planning:

- Classified in terms of the timing of payments whether it is at the beginning or the end of the payment period.
 - Ordinary Annuity (end)
 - Annuity due (beginning)
- Classified in terms of the frequencies of payments and interest-compounding periods
 - Simple Annuity (equal frequencies)
 - General Annuity (unequal frequencies)

The combination of those four classes gives four different types of annuities, as illustrated in Figure 3.1.1.

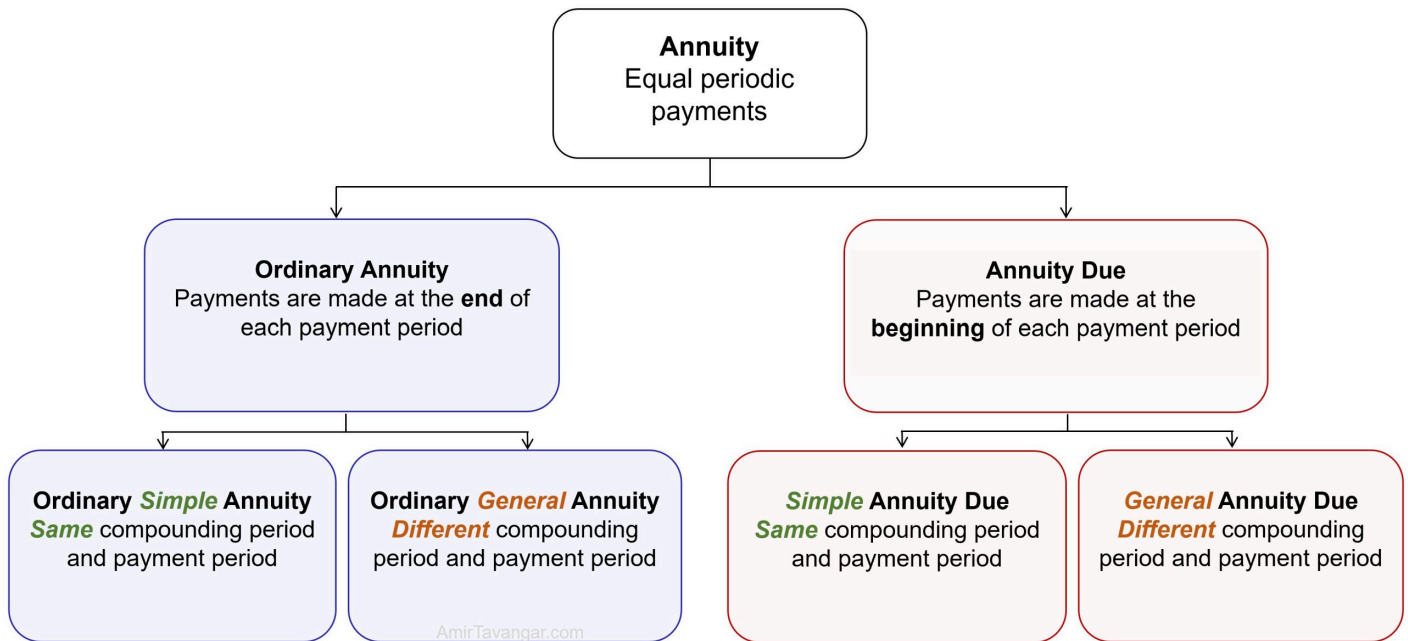
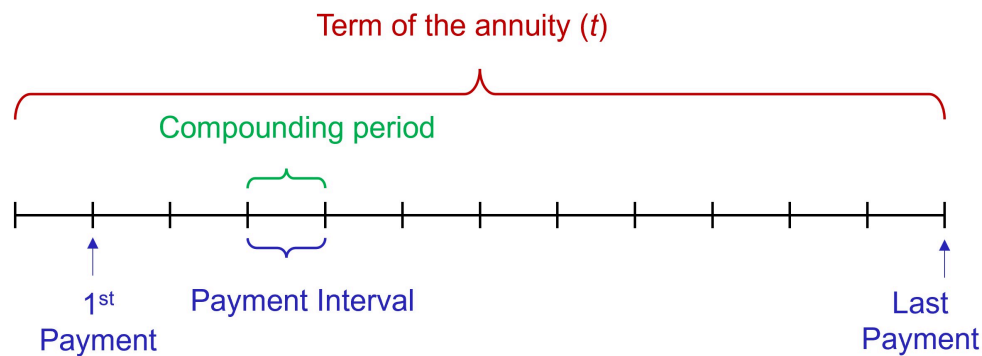
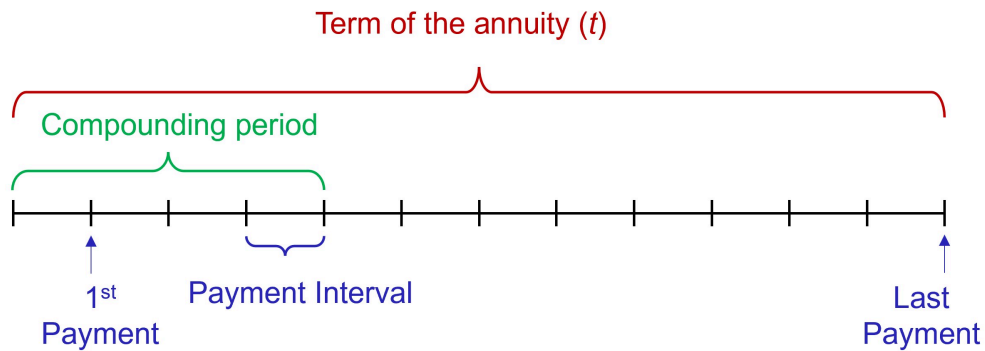


Figure 3.1.1 Types of Annuities

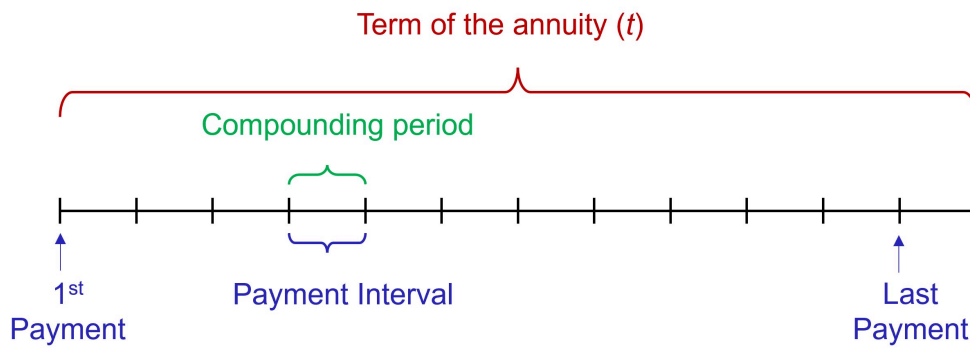
In an **Ordinary Simple Annuity**, the payment period and the interest compounding period are the **same** ($P/Y = C/Y$), and the payments are made at the **end** of the payment period.



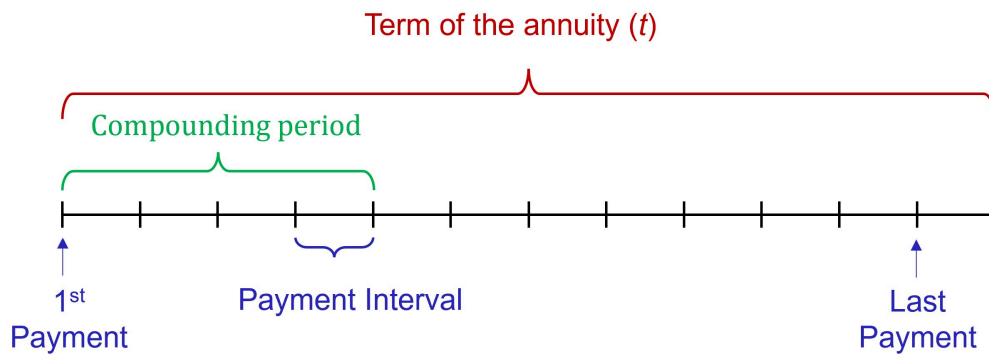
In an **Ordinary General Annuity**, the payment period and the interest compounding period are **NOT the same** ($P/Y \neq C/Y$) and the payments are made at the **end** of the payment period.



In a **Simple Annuity Due**, the payment period and the interest compounding period are the **same** ($P/Y = C/Y$), and the payments are made at the **beginning** of the payment period.



In a **General Annuity Due**, the payment period and the interest compounding period are **NOT the same** ($P/Y \neq C/Y$) and the payments are made at the **beginning** of the payment period.



Example 3.1.2: Type of Annuity

Identify the type of Annuities.

- a) Payments of \$540 at the end of every year for 6 years at 4.76% compounded annually.
- b) Payments of \$85 at the end of every month for 6 years at 8.15% compounded annually.
- c) Payments of \$540 at the beginning of every 3 months for 4.5 years at 5 % compounded quarterly.
- d) Payments of \$1200 at the beginning of every 6 months for 8 years at 3.40% compounded monthly.

Show/Hide Solution

- a) Payments of \$540 at the $\overbrace{\text{end}}^{\text{Ordinary}}$ of every $\overbrace{\text{year}}^{P/Y=1}$ for 6 years at 4.76% compounded $\overbrace{\text{annually}}^{C/Y=1}$.

Since $P/Y = C/Y$, the annuity is simple. Also, the payments are made at the END of the payment period, so the annuity is ordinary. Putting them together, the annuity is an **Ordinary Simple Annuity**.

- b) Payments of \$85 at the $\overbrace{\text{end}}^{\text{Ordinary}}$ of every $\overbrace{\text{month}}^{P/Y=12}$ for 6 years at 8.15% compounded $\overbrace{\text{annually}}^{C/Y=1}$.

Since $P/Y \neq C/Y$, the annuity is general. Also, the payments are made at the END of the payment period, so the annuity is ordinary. Putting them together, the annuity is an **Ordinary General Annuity**.

- c) Payments of \$540 at the $\overbrace{\text{beginning}}^{\text{Due}}$ of every $\overbrace{\text{3 months}}^{P/Y=4}$ for 4.5 years at 5 % compounded $\overbrace{\text{quarterly}}^{C/Y=4}$.

Since $P/Y = C/Y$, the annuity is simple. Also, the payments are made at the BEGINNING of the payment period, so the annuity is an annuity due. Putting them together, the annuity is a **Simple Annuity Due**.

- d) Payments of \$1200 at the $\overbrace{\text{beginning}}^{\text{Due}}$ of every $\overbrace{\text{6 months}}^{P/Y=2}$ for 8 years at 3.40% compounded $\overbrace{\text{monthly}}^{C/Y=12}$.

Since $P/Y \neq C/Y$, the annuity is general. Also, the payments are made at the BEGINNING of the payment period, so the annuity is an annuity due. Putting them together, the annuity is a **General Annuity Due**.

Try an Example



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Section 3.1 Exercises

1. Identify the type of annuity: Payments of \$300 are made at the beginning of every six months for 9 years at 8% compounded semi-annually.

Show/Hide Answer

Simple Annuity due

2. Identify the type of annuity: Payments of \$670 are made at the end of every year for 5 years at 3.7% compounded monthly.

Show/Hide Answer

Ordinary General annuity

3. Identify the type of annuity: Payments of \$510 are made at the beginning of every year for 6 years at 2.9% compounded quarterly.

Show/Hide Answer

General Annuity due

3.2 FV OF ANNUITIES: CALCULATOR APPROACH

A. Future Value of Annuities and Interest Amount

After covering the basics and types of annuities, we now focus on understanding and calculating the future value of annuities. This skill is crucial for financial planning, whether for retirement savings, education funds, or other long-term financial goals. Consider, for instance, a scenario where you're investing \$1,000 monthly into a retirement plan with a 10% annual interest rate compounded monthly over 30 years. To determine the amount you will have at retirement, you need to calculate the future value of this annuity. The future value here refers to the total accumulated value of all payments at the end of the annuity's term, including interest.

For solving annuity problems, just like with compound interest, we use the time-value-of-money (TVM) worksheet in financial calculators (refer to Figure 2.2.1 in Section 2.2, Chapter 2). It's also important to adhere to the cash flow sign convention (see Table 2.2.1 in Section 2.2) when inputting monetary values into the calculator. This ensures consistency and accuracy in our calculations.

It's important to first verify the payment timing setting on your calculator before solving annuity problems. The default setting is usually at 'END' (end of the payment period), which is appropriate for ordinary annuities. However, for annuities due, where payments are made at the beginning of each period, this setting needs to be adjusted to 'BGN' (beginning of the payment period).

To change the payment timing, use the '2ND' key followed by the 'PMT' key. This action accesses the secondary function 'BGN'. To switch between 'BGN' and 'END', press the "2ND" and "ENTER" keys. Figure 3.2.1 illustrates the process for adjusting the payment timing setting on your calculator.

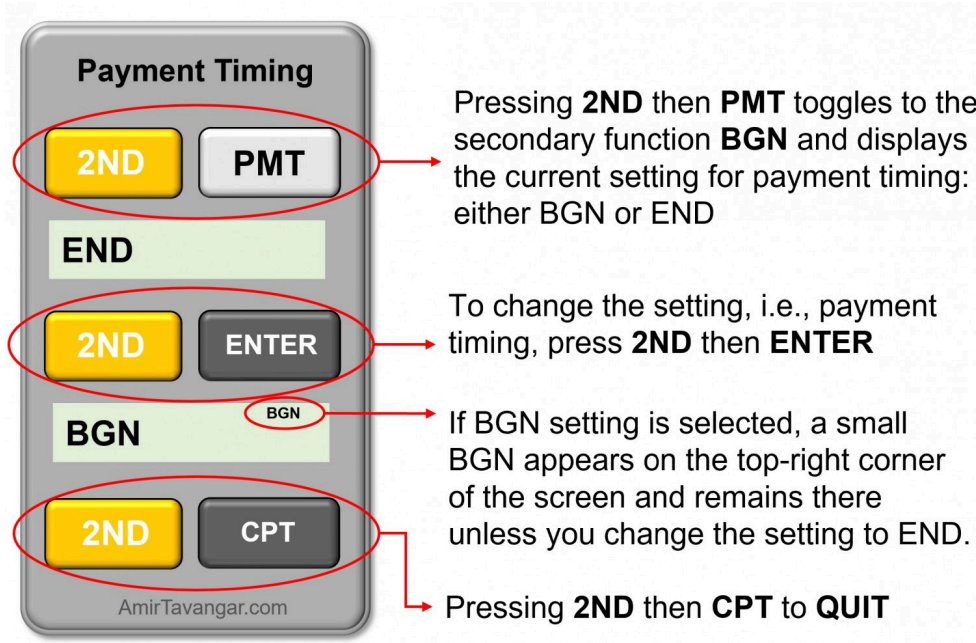


Figure 3.2.1 The process for adjusting the payment timing setting on a financial calculator

When the calculator is set to 'BGN' (annuity due), a small 'BGN' indicator will display in the top-right corner of the calculator screen, and it will remain until the setting is switched back to 'END'. Remember, this setting adjustment is crucial for accurate calculations in your annuity problems.

Computing Interest Amount

The future value of an annuity includes both the total payments made and interest earned over the term of the annuity. To calculate the total interest earned, you should deduct the sum of all payments (calculated as $PMT \cdot N$) from the future value. The amount of interest earned can be calculated using

$$I = FV - (N \cdot PMT) \quad \text{Formula 3.2}$$

If there is an initial investment at the start of the annuity (a nonzero Present Value, PV), you need to modify the calculation. In this scenario, the future value (FV) includes the initial investment, all subsequent payments, and interest earned over the term of the annuity. Therefore, the formula to calculate the interest earned is adjusted to

$$I = FV - (N \cdot PMT) - PV \quad \text{Formula 3.3}$$

In this formula, PMT is the regular payment amount, and N is the total number of payments, calculated as $N = P/Y \cdot t$ (Formula 3.1a).

How to Compute the Future Value of Annuities Using a Financial Calculator

1. **Clear the TVM Worksheet:** Before starting, clear your TVM worksheet to remove any previous values by pressing '2ND' followed by the 'FV' key.
2. **Identify the type of annuity:** Determine whether the annuity is an ordinary annuity (payments at the end of each period) or an annuity due (payments at the beginning of each period). Adjust your calculator's setting to 'END' for an ordinary annuity or to 'BGN' for an annuity due.
3. **Enter the values:** Input the given values for the number of periods (N), the nominal interest rate in percent (I/Y), and the periodic payment amount (PMT). Remember to follow the cash flow sign convention, typically entering the payment (PMT) as a negative value (outflow).
4. **Determine present value (PV):**
 - If you are starting the annuity without an initial lump sum investment, set PV to zero.
 - If there's an initial lump sum investment, input this value as PV, which is entered as a negative value if it is an outflow.
5. **Compute future value (FV):** With all other variables correctly inputted, proceed to calculate the FV. The calculator will display the future value of the annuity, which is the total value accumulated at the end of the term, including the compounded interest.
6. **Compute the amount of interest (I):** When required, apply Formula 3.2 to calculate the total amount of interest earned over the entire term of the annuity. However, if there is an initial investment at the start of the annuity (a nonzero Present Value, PV), you need to modify the calculation. In this scenario, the future value (FV) includes both the initial investment and all subsequent payments. Therefore, the formula to calculate the interest earned is adjusted to Formula 3.3.

It's important to note that when working with algebraic formulas, such as Formulas 3.2 or 3.3 for calculating the amount of interest, all values are treated as positive values. This approach differs from the cash flow sign convention used in financial calculators.

B. Future Value of Ordinary Annuity

We begin by examining the future value of ordinary annuities, where payments are made at the end of each period. It's important to first verify the payment timing setting on your calculator before solving annuity problems.

Example 3.2.1: Compute FV of Ordinary Simple Annuity

Lauren deposits \$360 at the end of each month for 12 years in her Registered Retirement Savings Plan (RRSP) account. The interest rate on the RRSP account is 7.5% compounded monthly. **a)** How much will be the accumulated value of her investment? **b)** How much will she have contributed to the account by the end of the term? **c)** How much interest will be earned on the account?

Show/Hide Solution

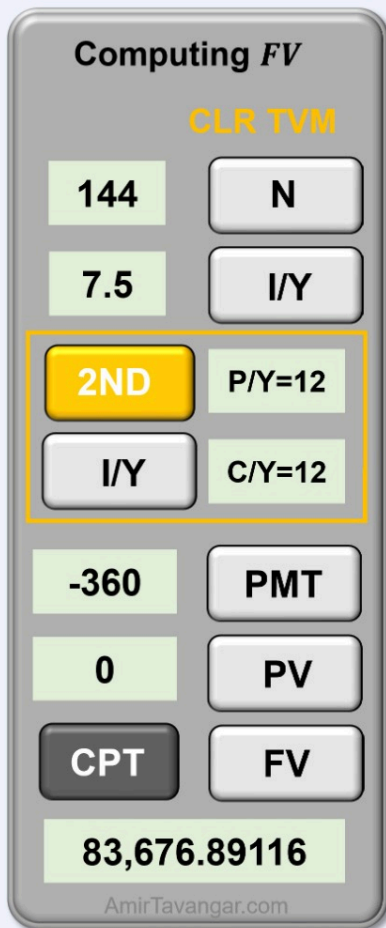
Given information:

- Payments are made at the **end** of each month, so it is an ordinary Annuity: **END**
- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of each month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Investment Term: $t = 12$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(12) = 144$
- Nominal interest rate: $I/Y = 7.5\%$
- In investments, the periodic payment is cash outflow, so $PMT = -\$360$
- The initial balance of the account is zero, so $PV = 0$

a) $FV = ?$



$$FV \approx \$83,676.89$$

Thus the accumulated value of her investment will be \$83,676.89.

b) Lauren's contribution to the account is through periodic payments. She will have contributed N payments of size PMT :

$$\begin{aligned} \text{Contribution through payments} &= N \cdot PMT \\ &= 144(360) \\ &= \$51,840 \end{aligned}$$

c) $I = ?$

The difference between FV and total payment (i.e., Lauren's contribution) is the amount of interest earned in the account. By Formula 3.2, we have

$$\begin{aligned}
 I &= FV - (N \cdot PMT) \\
 &= 83,676.89 - 51,840 \\
 &= \$31,836.89
 \end{aligned}$$

Try an Example



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Example 3.2.2: Compute FV of Ordinary General Annuity

Consider a scenario where you deposit \$250 at the end of every three months for 15 years into an account that offers a 6% interest rate compounded semi-annually. **a)** What will be the maturity value of these deposits at the end of the 15-year period? **b)** How much interest will be earned during the 15-year term?

Show/Hide Solution

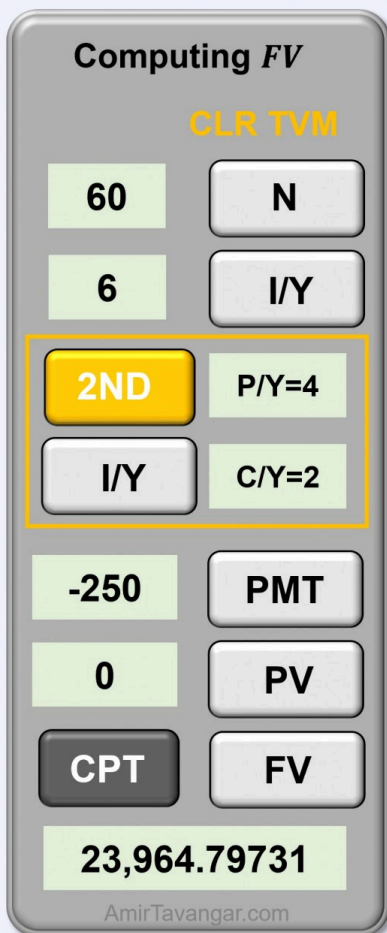
Given information:

- Interest is compounded semi-annually so $C/Y = 2$
- Deposits are made at the end of every three months so $P/Y = 4$

$C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity

- Investment Term: $t = 15$ years
- Number of payments in the term: $N = P/Y \cdot t = 4(15) = 60$
- Nominal interest rate: $I/Y = 6\%$
- The periodic payment is cash outflow, so $PMT = -\$250$
- The initial balance of the account is zero, so $PV = 0$

a) $FV = ?$



$FV \approx \$23,964.80$

Thus the maturity value will be \$23,964.80.

b) $I = ?$

The difference between FV and total deposits is the amount of interest earned in the account. Using Formula 3.2, we have

$$\begin{aligned} I &= FV - N \cdot PMT \\ I &= 23,964.80 - 60(250) \\ &= 23,964.80 - 15,000 \\ &= \$8,964.80 \end{aligned}$$

The below figure displays a donut chart with the Future Value (FV) of \$23,964.80 at its center, breaking down into components of interest (I) and total payments ($N \cdot PMT$) around the ring, illustrating how each contributes to the FV .



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C. Future Value of Annuity Due

In the upcoming examples, we will focus on calculating the future value of annuities due. In these annuities, since payments are made at the start of each payment period, it is necessary to change the payment timing setting on your calculator to 'BGN' (beginning of the payment period).

Example 3.2.3: Compute FV of Simple Annuity Due

Kian plans to start saving for a sabbatical trip he intends to take in 10 years. To finance his adventure, he decides to make monthly deposits of \$500 at the beginning of each month into a high-yield savings account that offers a 3.24% annual interest rate compounded monthly. Determine the future value of Kian's sabbatical fund at the end of 10 years if he starts to make the deposit at the beginning of the next month.

Show/Hide Solution

Given information:

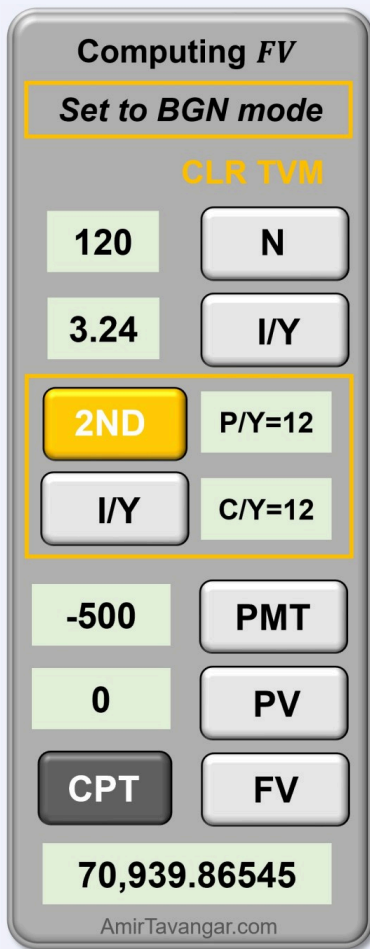
- Interest is compounded semi-annually so $C/Y = 12$
- Deposits are made at the beginning of every month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Simple Annuity Due}$$

- Investment term: $t = 10$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(10) = 120$
- Nominal interest rate: $I/Y = 3.24\%$
- For investments PMT is a cash outflow, so $PMT = -\$500$
- There is no initial lump sum in the fund at the start of the annuity, so $PV = 0$

a) $FV = ?$

Ensure to first change the payment timing to BGN for an annuity due before starting to enter values.



$FV \approx \$70,939.87$

The future value of the deposits in Kian's sabbatical fund will be \$70,8939.87 at the end of the 10-year term.

Try an Example



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Example 3.2.4: Compute FV of General Annuity Due

Adriel made deposits of \$250 in a fund at the beginning of every month for 18 years. The fund was earning interest of 6.12% compounded quarterly. **a)** What was the total amount Adriel deposited in the fund? **b)** What was the accumulated value of the investment? **c)** How much interest was earned in the fund during the term?

Show/Hide Solution

Given information:

- Interest is compounded quarterly so $C/Y = 4$
- Deposits are made at the beginning of every month so $P/Y = 12$

$$C/Y \neq P/Y \Rightarrow \text{General Annuity Due}$$

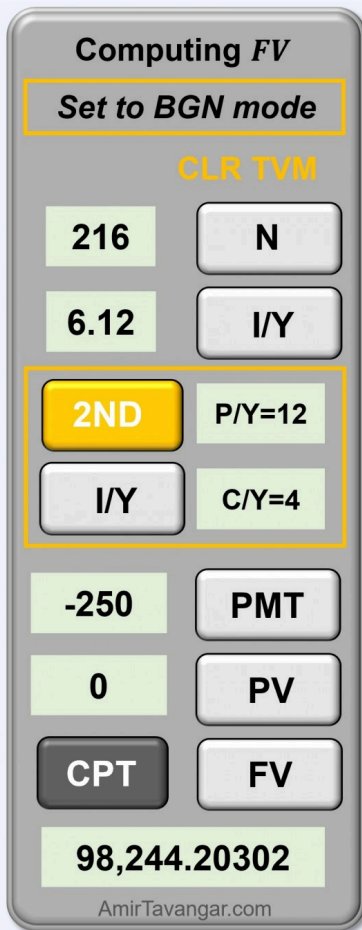
- Investment Term: $t = 18$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(18) = 216$
- Nominal interest rate: $I/Y = 6.12\%$
- For investments PMT is a cash outflow, so $PMT = -\$250$
- There is no initial lump sum in the fund at the start of the annuity, so $PV = 0$

a) Total amount Adriel deposited in the fund is N deposits of size PMT:

$$\begin{aligned} \text{Total deposits} &= N \cdot PMT \\ &= 216(250) \\ &= \$54,000 \end{aligned}$$

b) $FV = ?$

Ensure to first change the payment timing to BGN for an annuity due before starting to enter values.



$FV \approx \$98,244.20$

The accumulated value of the deposits in the fund will be \$98,244.20 at the end of the 18-year term.

c) $I = ?$

The difference between FV and total deposits is the amount of interest earned in the account. Using Formula 3.2, we have

$$\begin{aligned}
 I &= FV - (N \cdot PMT) \\
 I &= 98,244.20 - 216(250) \\
 &= 98,244.20 - 54,000 \\
 &= \$44,244.20
 \end{aligned}$$

The below figure displays a donut chart with the Future Value (FV) of \$98,244.20 at its center, breaking

down into components of interest (I) and total payments ($N \cdot PMT$) around the ring, illustrating how each contributes to the FV .



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D. Adjusting for Variable Changes Within the Term

It is important to note that if any key financial variables alter during the term, like the payment amount, the nominal interest rate, or the frequency of compounding, the term must be divided into separate time intervals at the point of each change. The upcoming example demonstrates the procedure for dealing with such adjustments.

Example 3.2.5: Compute FV of Annuity Combined with Compound Interest

\$2,650 was deposited at the end of every six months for 5 years into a fund earning 4.7% compounded semi-

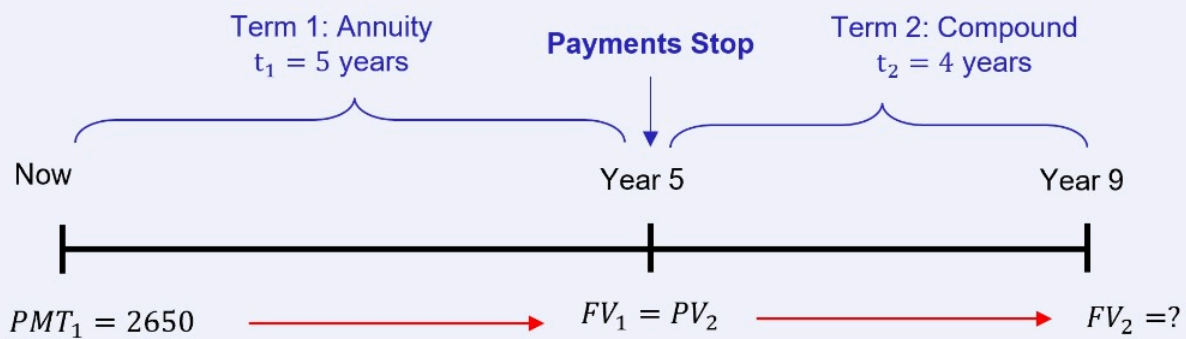
annually. After this period, no further deposit was made but the accumulated money was left in the account for another 4 years at the same interest rate.

- a) Calculate the accumulated amount at the end of the 9-year period.
- b) Calculate the total amount of interest earned during the 9-year period.

Show/Hide Solution

Given that payments stop midway through the term, we need to divide the investment term into two segments at the point of change, which occurs at the end of year 5. In the initial segment of five years, periodic deposits are made, classifying this period as an annuity term. In the subsequent four years, no additional deposits are made, and the account simply earns interest on the accumulated amount, which is a compound interest term. To approach this calculation:

1. First, calculate the future value of the annuity at the end of the first five-year segment.
2. The amount accumulated at the end of year 5 will then serve as the present value for the following four-year term, during which compound interest will be applied to this sum to determine the final future value at the end of year 9.



a)

Term 1: First 5 years – Ordinary Simple Annuity

Given Information

- Interest is compounded semi-annually so $C/Y = 2$
- Payments are made at the end of every six months so $P/Y = 2$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

◦

- Investment Term: $t_1 = 5$ years
- Number of payments in the term: $N_1 = P/Y \cdot t_1 = 2(5) = 10$
- Nominal interest rate: $I/Y = 4.7\%$
- The payment is cash outflow, so $PMT_1 = -\$2650$
- The initial balance of the account is zero, so $PV_1 = 0$
- $FV_1 = ?$

Term 2: Next 4 years – Compound Interest

Given Information

- In investments, the present value is cash outflow, so $PV_2 = -FV_1 = -\$29,485.420395\dots$
- Investment Term: $t_2 = 4$ years
- Interest is compounded semi-annually so $C/Y = 2$
- For compound interest problems, $P/Y = C/P$, so $P/Y = 2$
- Number of compounding periods in the term: $N_2 = C/Y \cdot t_2 = 2(4) = 8$
- Same interest rate
- No periodic payment, so $PMT = 0$
- $FV_2 = ?$

Computing FV_1

CLR TVM

10 N

4.7 I/Y

2ND P/Y=2

I/Y C/Y=2

0 PV

-2650 PMT

CPT FV

29,485.4204

AmirTavangar.com

Computing FV_2

CLR TVM

8 N

4.7 I/Y

2ND P/Y=2

I/Y C/Y=2

-29,485.4204 PV

0 PMT

CPT FV

35,506.6828

AmirTavangar.com

$FV_2 \approx \$35,506.68$

So the accumulated amount at the end of the 9-year period is \$35,506.68.

b) $I = ?$

The total interest earned in the account (I) is the sum of interest earned during the first term (I_1) and the interest earned during the second term (I_2). These interests can be computed individually and then added together.

Alternatively, given that the only additions to the account are the deposits of \$2,650, the total interest can also be determined by subtracting the total deposit amount from the final accumulated value (FV_2) of the account.

$$I = FV_2 - N_1 \cdot PMT$$

$$I = 35,506.68 - 10(2650)$$

$$= \$9,006.68$$

The below figure displays a donut chart with the final Future Value (FV_2) of \$35,506.68 at its center, breaking down into components of interest (I) and total payments ($N \cdot PMT$) around the ring, illustrating how each contributes to the final accumulated value. The interest is further expanded to show interest earned during the first term (I_1) and interest earned during the second term (I_2).



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Try an Example



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E. Annuities with Initial Lump Sum

In our earlier examples, we assumed that the annuities began without any initial investment, meaning the present value (PV) was zero. However, if an annuity starts with an initial lump sum investment, you must enter this amount as the present value (PV) in your calculations. Remember to input the PV as a negative number as it represents a cash outflow.

When the annuity calculation includes an initial lump sum (PV), the future value will include this initial investment, all the periodic payments made thereafter, and the interest that accrues over time. To calculate the total interest earned over the term of the annuity, you need to use Formula 3.3.

Example 3.2.6: Compute FV of an Annuity with Initial Lump Sum Investment (Nonzero PV)

Nancy is diligently preparing for her retirement and has already saved \$15,000 in her 401(k) retirement fund. She is currently 30 years old and aims to retire at 67. To supplement her savings, Nancy begins to contribute \$500 at the end of each month to her 401(k). The account is expected to earn an average interest rate of 7% per year compounded quarterly. **a)** Calculate the future value of Nancy's 401(k) when she retires. **b)** Calculate the total amount of interest that will have been earned on the account by the time Nancy reaches retirement age.

Show/Hide Solution

Given information:

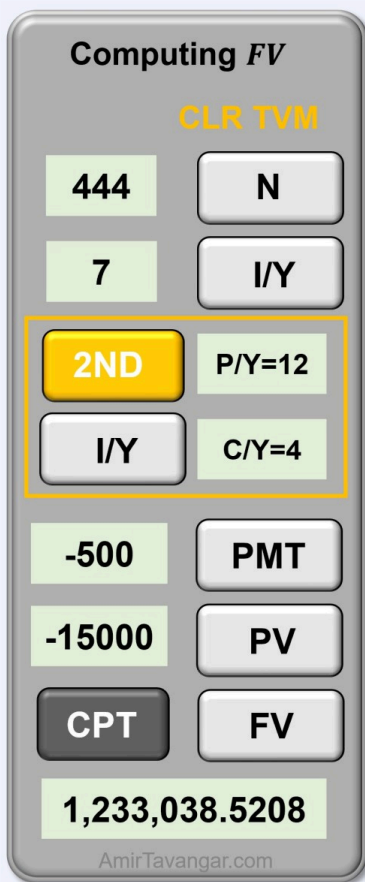
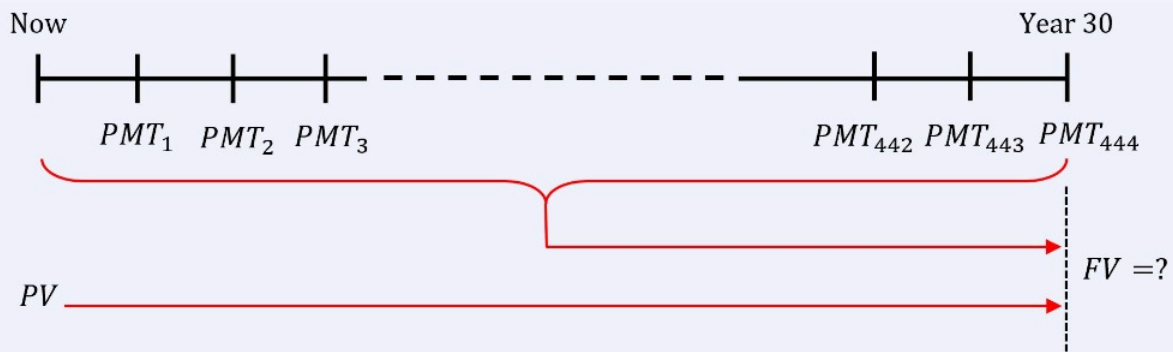
- Payments are made at the **end** of each month, so it is an ordinary annuity: **END**
- Interest is compounded quarterly so $C/Y = 4$
- Deposits are made at the end of every month so $P/Y = 12$

$$C/Y \neq P/Y \Rightarrow \text{Ordinary General Annuity}$$

- Nominal interest rate: $I/Y = 7\%$
- Investment Term: $t = 37$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(37) = 444$
- The periodic payments of \$500 are cash outflow, so $PMT = -\$500$
- The initial value of \$15,000 at the start of the annuity is also a cash outflow, so $PV = -\$15,000$
- The questions ask for FV and I.

a) $FV = ?$

The total value accumulated at the end of the term is composed of two parts: the sum of the future values of all periodic payments and the future value of the initial lump sum investment.



$$FV \approx \$1,233,038.52$$

Thus the future value of Nancy's 401(k) when she retires is \$1,233,038.52.

b) $I = ?$

Because the present value (PV) is not zero, to calculate the amount of interest earned over the 37-year term, we use the modified formula for interest amount (Formula 3.3):

$$I = FV - (N \cdot PMT) - PV$$

$$\begin{aligned}
 I &= 1,233,038.52 - (444 \times 500) - 15000 \\
 &= 1,233,038.52 - 222,000 - 15,000 \\
 &= \$996,038.52
 \end{aligned}$$

Try an Example



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Section 3.2 Exercises

- Thiago would like to save \$470 at the end of every month for the next 10 years in a savings account at 3.44% compounded annually. **a)** What would be the accumulated value of the investment at the end of the term? **b)** What would be the amount of interest earned?

Show/Hide Answer

a) $FV = \$67,015.84$

b) $I = \$10,615.84$

- \$4,800 was deposited at the end of every three months for 9 years into a fund earning 2.8% compounded quarterly. After this period, the accumulated money was left in the account for another 4.5 years at the same interest rate. **a)** Calculate the accumulated amount at the end of the 13.5-year term. **b)** Calculate the total amount of interest earned during the 13.5-year period.

Show/Hide Answer

a) $FV = \$221,936.95$

b) $I = \$49,136.95$

3. Erick made deposits of \$320.41 in a fund at the beginning of every quarter for 7 years. The fund was earning interest of 3.38% compounded semi-annually. **a)** What was the accumulated value of the investment? **b)** What was the total amount Erick deposited in the fund? **c)** How much interest was earned in the fund during the term?

Show/Hide Answer

a) $FV = \$10,153.73$

b) Total deposit = \$8,971.48

c) $I = \$1,182.25$

4. Carolyn is preparing for her retirement and has already saved \$26,500 in her 401(k) retirement fund. She is currently 30 years old and aims to retire at 61. To supplement her savings, Carolyn begins to contribute \$1,520 at the end of every quarter to her 401(k). The account is expected to earn an average interest rate of 5.46% compounded quarterly. **a)** Calculate the future value of the 401(k) fund when Carolyn retires. **b)** Calculate the total amount of interest that will have been earned on the account by the time Carolyn reaches retirement age.

Show/Hide Answer

a) $FV = \$629,167.72$

b) $I = \$414,187.72$

3.3 PV OF ANNUITIES: CALCULATOR APPROACH

A. Present Value of Annuities and Interest Amount

In this section, we now turn our attention to the present value of annuities, a key concept in valuing the current worth of future payments. The present value of an annuity is crucial in scenarios such as determining how much a series of future cash flows is worth today or evaluating the current worth of a long-term investment or loan. Consider a scenario where you have the option to receive a monthly payment of \$1,000 for the next 30 years from a retirement plan. This plan is expected to yield a 10% annual interest rate compounded monthly. To understand the current worth of these future payments, you need to calculate the present value of this annuity. The present value in this context refers to the total value in today's dollars of all the annuity payments you are set to receive over the 30-year term, taking into account the interest rate.

For computing the present value in annuity problems, similar to computing future values, we use the time-value-of-money (TVM) worksheet in financial calculators. It's also important to adhere to the cash flow sign convention (see Table 2.2.1 in Section 2.2) when inputting monetary values into the calculator to ensure consistency and accuracy in the calculations.

As we discussed in the previous section, it's important to first verify the payment timing setting on your calculator before solving annuity problems. The default setting is usually at 'END' (end of the payment period), which is appropriate for ordinary annuities. However, for annuities due, where payments are made at the beginning of each period, this setting needs to be adjusted to 'BGN' (beginning of the payment period). Refer to Figure 3.2.1 in Section 3.2, to learn how to change the payment timing on a financial calculator.

Computing Interest Amount

In problems where the present value of an annuity is known or is calculated (usually for loan scenarios), the periodic payments of the annuity include interest, and therefore, the amount of interest is obtained by

$$I = (N \cdot PMT) - PV \quad \text{Formula 3.4}$$

In this formula, PMT is the periodic payment amount, and N is the total number of payments, calculated as $N = P / Y \cdot t$ (Formula 3.1a).

How to Compute the Present Value of Annuities Using a Financial Calculator

1. **Clear the TVM Worksheet:** Before starting, clear your TVM worksheet to remove any previous values by pressing '2ND' followed by the 'FV' key.
2. **Identify the Type of Annuity:** Determine if it's an ordinary annuity (payments at the end of each period) or an annuity due (payments at the beginning of each period). Adjust your calculator's setting accordingly (END for ordinary annuities, BGN for annuities due).
3. **Enter Key Values:** Input the relevant values for the number of periods (N), the nominal interest rate in percent (I/Y), and the payment amount (PMT). Remember to adhere to the cash flow sign convention, entering payment (PMT) as a negative value (outflow) if you make payments and as a positive value (cash inflow) if you receive payments.
4. **Set Future Value (FV):**
 - If the annuity has no remaining balance after the final payment (e.g. a loan has been fully paid off, or a retirement saving has been fully withdrawn), set FV to zero.
 - If there is a lump sum to be paid or received at the end of the annuity term apart from the regular payments (e.g., a maturity value or a residual value), set FV to that amount.
5. **Calculate Present Value (PV):** With the other variables entered correctly, compute PV. The calculator will display the present value of the annuity, which is the current worth of all future payments, adjusted for the interest rate.
6. **Compute the amount of interest (I):** When required, apply Formula 3.4 to calculate the total amount of interest incurred over the entire term of the annuity.

B. Present Value of an Ordinary Annuity

We begin by examining the present value of ordinary annuities, where payments are made at the end of each period. It is important to first verify the payment timing setting on your calculator before solving annuity problems.

Example 3.3.1: Compute PV of Ordinary Simple Annuity

Considering a bank offering an investment opportunity. To provide annual payments of \$1200 at the end of each year for the next 7 years, at a nominal interest rate of 6% compounded annually, what is the required initial deposit that should be made by a customer today?

Show/Hide Solution

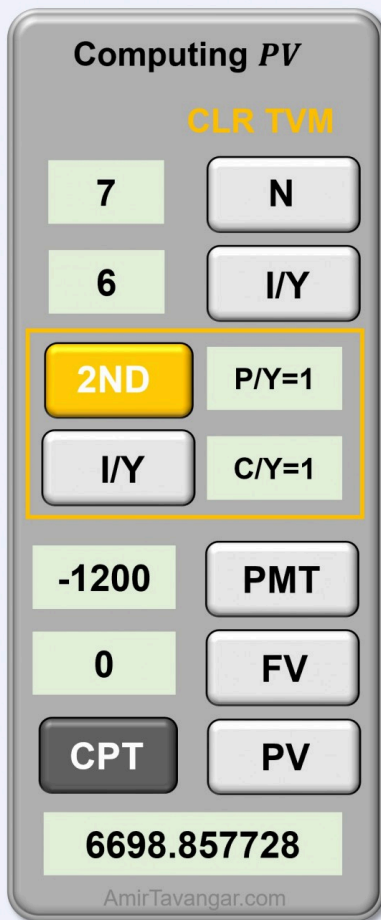
Given information

- Interest is compounded annually so $C/Y = 1$
- Payments are made at the end of every year so $P/Y = 1$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Investment Term: $t = 7$ years
- Number of payments in the term: $N = P/Y \cdot t = 1(7) = 7$
- Nominal interest rate: $I/Y = 6\%$
- The payments are made (not received), so they are cash outflow: $PMT = -\$1200$
- No remaining balance at the end of the annuity term, so $FV = 0$

Enter the given values into the calculator and compute PV:



$PV \approx \$6,698.86$

Therefore, the present value of the annuity is \$6,698.86.

Try an Example



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Example 3.3.2: Compute PV Ordinary General Annuity

A 25-year mortgage on a condominium requires payments of \$1000 at the end of each month. If interest is 4% compounded semi-annually, **a)** what was the mortgage principal? **b)** How much interest was charged on the mortgage?

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Show/Hide Solution

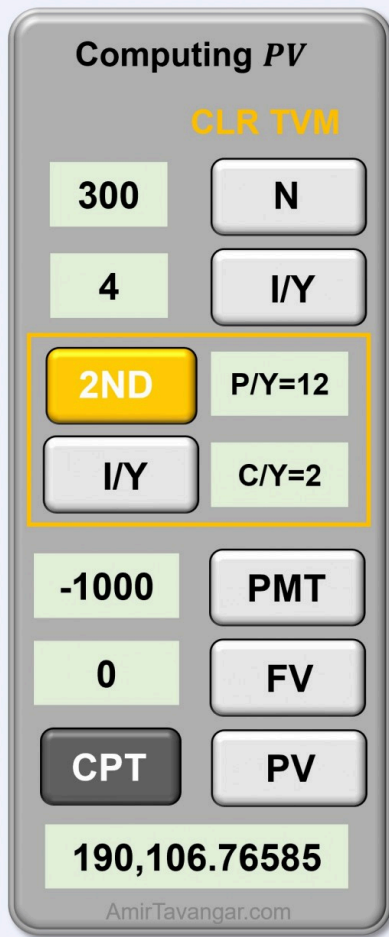
Given information

- Interest is compounded semi-annually so $C/Y = 2$
- Payments are made at the end of each month so $P/Y = 12$

$C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity

- Mortgage Term: $t = 25$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(25) = 300$
- Nominal interest rate: $I/Y = 4\%$
- For loans such as mortgages, payments are made, so they are cash outflow: $PMT = -\$1000$
- The mortgage is fully paid by the end of the term (i.e., no remaining balance), so $FV = 0$

a) $PV = ?$



$PV \approx \$190,106.77$

Thus, the mortgage principal was \$190,106.77.

b) $I = ?$

The amount of interest for loans is given by Formula 3.4.

$$\begin{aligned}
 I &= (N \cdot PMT) - PV \\
 &= 1000(300) - 190,106.77 \\
 &= 300,000 - 190,106.77 \\
 &= \$109,893.23
 \end{aligned}$$

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C. Present Value of Annuity Due

In the following examples, we will determine the present value of annuities due. For these types of annuities, where payments are made at the beginning of each payment period, it is essential to adjust the payment timing setting on your calculator to 'BGN', which stands for the beginning of the payment period.

Example 3.3.3: Compute PV of General Annuity Due

Shayan received an inheritance which was placed in a savings account. This account pays him \$1,000 at the beginning of each week for 30 years. The account accrues interest at a rate of 3% per year compounded monthly. What was the amount of the inheritance?

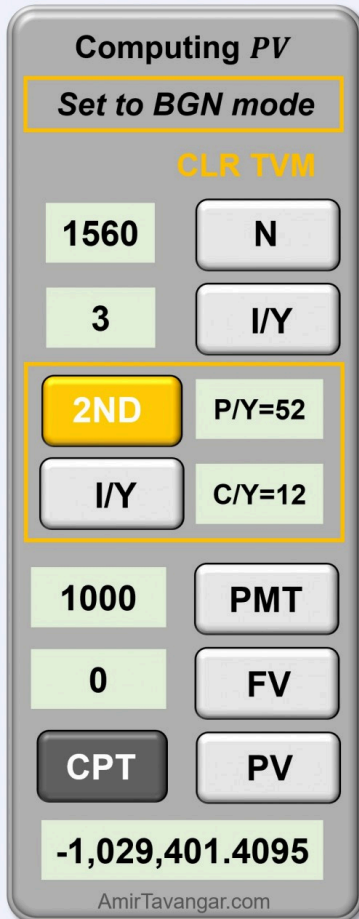
Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of each week so $P/Y = 52$

$$C/Y \neq P/Y \Rightarrow \text{General Annuity Due}$$

- Annuity Term: $t = 30$ years
- Number of payments in the term: $N = P/Y \cdot t = 52(30) = 1560$
- Nominal interest rate: $I/Y = 3\%$
- Shayan receives the payments, so they are cash inflow: $PMT = \$1000$
- Since the fund will be fully exhausted by the end of the term, $FV = 0$



$PV \approx \$1,029,401.41$

The amount of inheritance was \$1,029,401.41.

Try an Example



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D. Present Value Problems with Down Payment

A down payment refers to a relatively small initial payment made by a buyer at the time of purchasing a significant asset, such as real estate property. This payment is made upfront and the remaining balance of the purchase price is typically covered through a loan obtained from a financial institution. The amount of this loan corresponds to the present value (PV) of the future periodic payments that the buyer will make to pay off the loan. These periodic payments include not only the principal amount but also the interest charged on the loan. The relationship among the purchase price (also known as the cash price), the loan amount (PV), and the down payment can be expressed as follows:

$$\text{Purchase Price} = \text{Down Payment} + PV \qquad \text{Formula 3.5}$$

Example 3.3.4: Compute PV and Purchase Price with Known Down Payment

Andy made a down payment of \$40,000 on an apartment and secured a mortgage for the rest of the purchase price. He has agreed to repay this mortgage with end-of-month payments of \$1,580 for 30 years at a 3.45% annual interest rate compounded monthly.

- a) Calculate the original purchase price of the apartment.
- b) Determine the total amount Andy will have paid by the end of the 30-year mortgage term.
- c) Calculate the total amount of interest Andy will have paid on the mortgage over the 30 years.

Show/Hide Solution

Given information

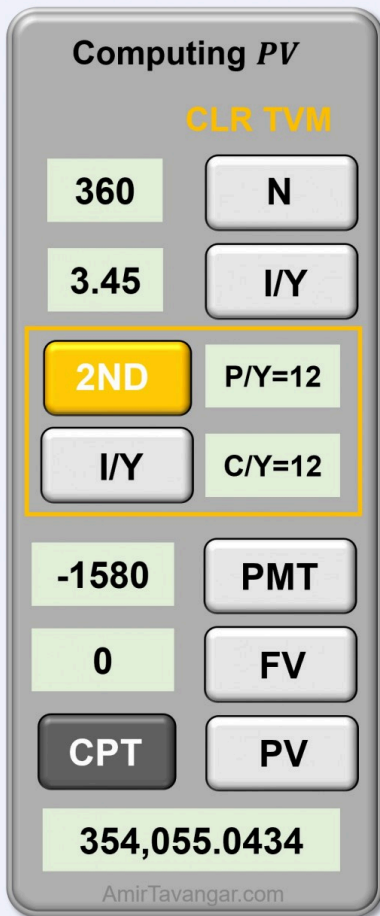
- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of every month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Mortgage term: $t = 30$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(30) = 360$
- Nominal interest rate: $I/Y = 3.45\%$
- For loans such as mortgages, payments are made, so they are cash outflow: $PMT = -\$1580$
- Down payment = \$40,000
- The mortgage is fully paid off by the end of the term, so $FV = 0$

a) To determine the purchase price of the apartment, we first need to calculate the mortgage amount. This is done by determining the present value of the monthly payments Andy is scheduled to make.

Computing PV



$PV \approx \$354,055.04$

Thus, the mortgage principal was \$354,055.04.

Calculating the purchase price

By substituting the values of the down payment and the present value (PV) into Formula 3.5, we can calculate the total purchase price.

$$\begin{aligned}
 \text{Purchase Price} &= \text{Down Payment} + PV \\
 &= 40,000 + 354,055.04 \\
 &= \$394,055.04
 \end{aligned}$$

Therefore, the purchase price of the apartment was \$394,055.04.

b) The total amount that Andy will pay over the 30-year term of the mortgage is equal to the number of payments (N) multiplied by the size of each payment (PMT).

$$\begin{aligned}\text{Total amount paid to repay loan} &= N \cdot PMT \\ &= 360(1580) \\ &= \$568,800\end{aligned}$$

c) The amount of interest for loans is given by Formula 3.4.

$$\begin{aligned}I &= (N \cdot PMT) - PV \\ &= 568,800 - 354,055.04 \\ &= \$214,744.96\end{aligned}$$

Note: When calculating the interest charged on the mortgage, we use the present value (PV) of the loan, not the purchase price of the property.

The below figure displays a donut chart with the total payments ($N \cdot PMT$) of \$741,000 at its center, breaking down into components of interest (I) and the loan principal (PV) around the ring.



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Example 3.3.5: Compute PV and Purchase Price with Unknown Down Payment

Bijan made a 20% down payment on the purchase of a vacation property and obtained a mortgage from a bank to cover the remaining cost. He has arranged to repay this mortgage with end-of-week payments of \$570 for 25 years. The interest charged on the mortgage is 3.05% compounded monthly.

- What was the purchase price of the property?
- What was the total amount paid over the 25-year term to repay the mortgage?
- How much interest was charged on the mortgage?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of the week so $P/Y = 52$

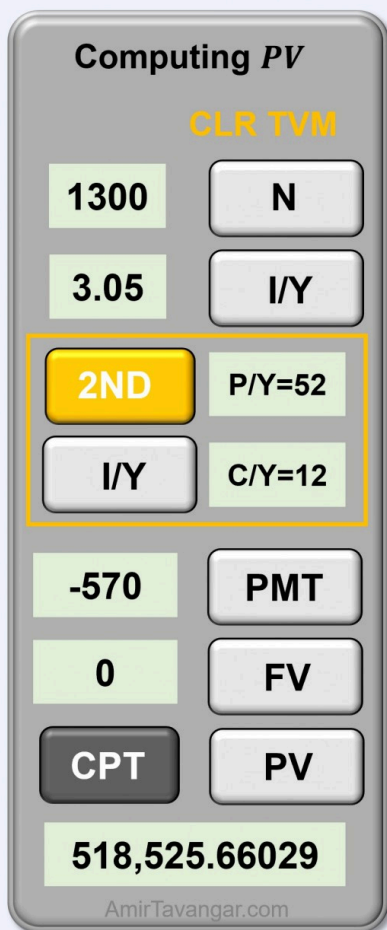
$C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity

- Mortgage term: $t = 25$ years
- Number of payments in the term: $N = P/Y \cdot t = 52(25) = 1300$
- Nominal interest rate: $I/Y = 3.05\%$
- For loans payments are made, so they are cash outflow: $PMT = -\$570$

- Down payment = 20% of the purchase price
- The mortgage is fully paid off by the end of the term, so $FV = 0$

a) To determine the purchase price, we first need to calculate the mortgage amount, which is done by determining the present value of the weekly payments Bijan is scheduled to make.

Computing PV



$PV \approx \$518,525.66$

Thus, the mortgage amount was \$518,525.66.

Calculating the purchase price

Since the down payment is given as a percentage of the unknown purchase price, we need to express the down

payment in terms of the purchase price when applying Formula 3.5. To simplify this, let's use 'X' to denote the purchase price. Thus, the down payment can be represented as 20% of X, or 0.2X.

$$\text{Purchase Price} = \text{Down Payment} + PV$$

$$X = 0.2X + 518,525.66$$

$$X - 0.2X = 518,525.66$$

$$0.8X = 518,525.66$$

$$X = \frac{518,525.66}{0.8}$$

$$= \$648,157.08$$

Therefore, the purchase price of the vacation property was \$648,157.08.

b) The total amount that Bijan will pay over the 25-year term of the mortgage is equal to the number of payments (N) multiplied by the size of each payment (PMT).

$$\text{The total amount paid to repay the loan} = N \cdot PMT$$

$$= 570(1300)$$

$$= \$741,000$$

c) The amount of interest for loans is given by Formula 3.4.

$$I = N \cdot PMT - PV$$

$$= 741,000 - 518,525.66$$

$$= \$222,474.34$$

The below figure displays a donut chart with the total payments ($N \cdot PMT$) of \$741,000 at its center, breaking down into components of interest charged (I) and the loan principal (PV) around the ring.



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E. Lease and Residual Value

A **lease** is a contractual agreement where one party, the lessee, pays the lessor (the owner) for the use of an asset over a specified period. Common in real estate, vehicles, and equipment, leases often include terms that specify payments, duration, and conditions of use. A key component of many lease agreements, particularly in auto leasing, is the **residual value**. This term refers to the projected value of the leased asset at the end of the lease term. It's an estimate of the asset's worth after depreciation over the lease period.

When calculating the present value (PV) of lease payments, the residual value plays a significant role. The present value of a lease is essentially the sum of the discounted values of all lease payments and the discounted residual value. Therefore, the calculation of PV in leasing scenarios must account for both the periodic payments and the residual value. This approach helps in comparing different lease options. For lease calculations, the residual value is often treated as a future value (FV). Since the residue value represents the value of the asset (such as a car) that needs to be returned to the owner at the end of the term, it is considered a cash outflow and should be entered as a negative value in the financial calculator.

Example 3.3.6: Compute PV of Annuity Due with Residual Value

Patricia's car lease agreement entails monthly payments of \$190.00, due at the start of each month for 3 years. At the end of the lease term, she can choose to either return the car or buy it for a residual value of \$14,600. The lease is subject to a 2.55% interest rate compounded monthly. Determine the original cash value of the car based on this lease agreement.

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of each month so $P/Y = 12$

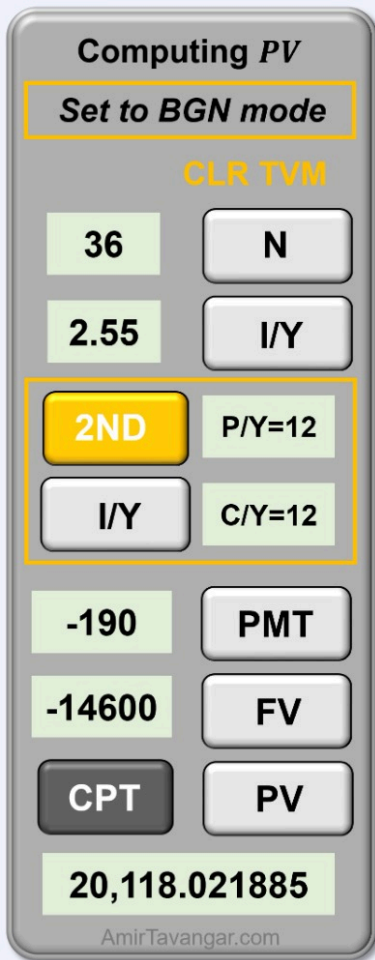
$$C/Y = P/Y \Rightarrow \text{Simple Annuity Due}$$

- Lease Term: $t = 3$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(3) = 36$
- Nominal interest rate: $I/Y = 2.55\%$
- The payments are made, so they are cash outflow: $PMT = -\$190$
- At the end of the lease, Patricia either returns the car valued at \$14,600 or purchases it by paying the amount. Either way, the value is considered a cash outflow: $FV = -\$14,600$

To determine the original cash value of the car, we need to calculate the present value (PV) of both the lease payments and the residual value. The car's cash value is equal to the combined present value of these amounts:

$$\text{Cash value} = PV_{PMT} + PV_{\text{Residual}}$$

Using a financial calculator simplifies this process, as it can compute the total present value in a single step. In the calculator, enter the residual value as the future value (FV) and the lease payment as the payment (PMT). Both FV and PMT should be input as negative values since they represent cash outflows.



$PV \approx \$20,118.02$

Therefore, the combined present value of the lease payments and the residual amounts to \$20,118.02, representing the cash value of the car.

Try an Example





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Example 3.3.7: Compute PV of Annuity Due with Residual Value and Down Payment

Slavica has a car lease agreement that requires an initial down payment of \$10,500 and lease payments of \$850 at the beginning of every month for 4 years. At the end of 4 years, she has the option to either return the car or purchase it for a residual value of \$21,000. If the interest charged on the lease is 3.4% compounded monthly, what was the cash value of the car?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of each month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Simple Annuity Due}$$

- Lease Term: $t = 4$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(4) = 48$
- Nominal interest rate: $I/Y = 3.4\%$
- The payments are made, so they are cash outflow: $PMT = -\$850$
- Down Payment = \$10,500
- At the end of the lease, you either return the car valued at \$21,000 or purchase it by paying the amount. Either way, the value is considered a cash outflow: $FV = -\$21,000$

To determine the original cash value of the car, we need to calculate the present value (PV) of both the lease

payments and the residual value. The car's cash value is equal to the combined present value of these amounts plus the down payment:

$$\text{Cash value} = \text{Down} + PV_{\text{PMT}} + PV_{\text{Residual}}$$

Using a financial calculator simplifies this process, as it can compute the total present value in a single step. In the calculator, enter the residual value as the future value (FV) and the lease payment as the payment (PMT). Both FV and PMT should be input as negative values since they represent cash outflows.

The calculator screen shows the following steps and values:

- Mode: **Computing PV**
- Mode selection: **Set to BGN mode**
- Mode reset: **CLR TVM**
- Number of periods: **48** (N)
- Interest rate per year: **3.4** (I/Y)
- Payments per year: **2ND P/Y=12**
- Compounding periods per year: **I/Y C/Y=12**
- Payment amount: **-850** (PMT)
- Future value: **-21000** (FV)
- Function: **CPT PV**
- Result: **56,538.01176**

$PV \approx \$56,538.01$

Therefore, the sum of the present value of the payments and the residual is \$56,538.01.

Substituting the down payment and PV yields

$$\begin{aligned} \text{Cash value} &= \text{Down} + PV_{\text{total}} \\ &= 10,500 + 56,538.01 \end{aligned}$$

$$= 67,038.01$$

Try an Example



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Section 3.3 Exercises

1. A mortgage with a 30-year term has monthly payments of \$5,896.97, due at the end of each month. The interest rate on this mortgage is 3.02%, compounded semi-annually. **a)** Calculate the original principal amount of the mortgage. **b)** Determine the total amount of interest that will be charged over the life of the mortgage.

Show/Hide Answer

a) $PV = \$1,398,488.92$

b) $I = \$724,420.28$

2. Trinity paid \$30,400 as a down payment toward a vacation property purchase and received a mortgage from a bank for the remaining amount. She agreed to pay end-of-month payments of \$3,005.14 for 15 years to repay the mortgage. The interest charged on the mortgage was 3.72% compounded annually. **a)** Calculate the amount of the mortgage. **b)** What was the purchase price of the property? **c)** What was the total amount paid over the 15-year term to repay the mortgage? **d)** How much interest was charged on the mortgage?

Show/Hide Answer

a) $PV = \$415,839.19$

b) Purchase price = $\$446,239.19$

c) Total amount paid = $\$540,925.20$

d) $I = \$125,086.01$

3. Carmen inherited a sum of money that was invested in a savings account providing her with $\$613.29$ at the beginning of each quarter for 12 years. The account earned interest of 5.46% compounded semi-annually. **a)** What was the amount of the inheritance? **b)** How much interest was earned in the account during the term?

Show/Hide Answer

a) $PV = \$21,827.09$

b) $I = \$7,610.83$

4. Cheyenne has a car lease agreement that requires an initial down payment of $\$8,900$ and lease payments of $\$260.00$ at the beginning of every month for 5 years. At the end of 5 years, she has the option to either return the car or purchase it for a residual value of $\$14,800$. If the interest charged on the lease is 2.65% compounded semi-annually, what was the cash value of the car?

Show/Hide Answer

Cash value of the car = $\$36,507.58$

3.4 PMT OF ANNUITIES: CALCULATOR APPROACH

A. Payment of Ordinary Annuity

In this section, we focus on calculating the periodic payment amount (PMT) required for an annuity. This payment is the amount that is either paid or received regularly throughout the term of the annuity. Depending on the scenario, whether it involves retirement plans, loans, or savings schemes, the calculation of PMT may be based on the Future Value (FV), the Present Value (PV) of the annuity, or a combination of both.

Similar to the previous sections, we use the time-value-of-money (TVM) worksheet in financial calculators for computing the amount of periodic payments in annuity problems. It's also important to adhere to the cash flow sign convention (see Table 2.2.1 in Section 2.2) when inputting monetary values into the calculator. This ensures consistency and accuracy in our calculations.

As discussed in the previous sections, it's crucial to first verify the payment timing setting on your calculator before solving annuity problems. The default setting is usually at 'END' (end of the payment period), which is appropriate for ordinary annuities. However, for annuities due, where payments are made at the beginning of each period, this setting needs to be adjusted to 'BGN' (beginning of the payment period). Refer to Figure 3.2.1 in Section 3.2 to learn how to change the payment timing on a financial calculator.

How to Compute the Size of Payment of Annuities Using a Financial Calculator

- 1. Clear the TVM Worksheet:** Before starting, clear your TVM worksheet to remove any previous values by pressing '2ND' followed by the 'FV' key.
- 2. Identify the Type of Annuity:** Determine if it's an ordinary annuity (payments at the end of each period) or an annuity due (payments at the beginning of each period). Adjust your calculator's setting accordingly (END for ordinary annuities, BGN for annuities due).
- 3. Input Known Values:**

- **Number of Periods (N):** Enter the total number of payment periods.
- **Interest Rate per Period (I/Y):** Input the nominal interest rate in percent without the percent symbol.
- **Present Value (PV):** For loans, enter the loan amount as a positive value (cash inflow). For investments, if there is an initial lump sum, enter it as a negative value (cash outflow). Set to zero if not applicable.
- **Future Value (FV):** For investments, enter the desired future value as a positive value (inflow). For loans, like lease agreements, input the residual value as the future value and enter it as a negative value (outflow). Set to zero if not applicable.

4. **Calculate PMT:** After entering the other values, compute PMT. The calculator will display PMT as a negative number for outflows (e.g., savings, and loan repayments) and a positive number for inflows (e.g., annuity payouts).

Example 3.4.1: Compute PMT Given FV of Ordinary Simple Annuity

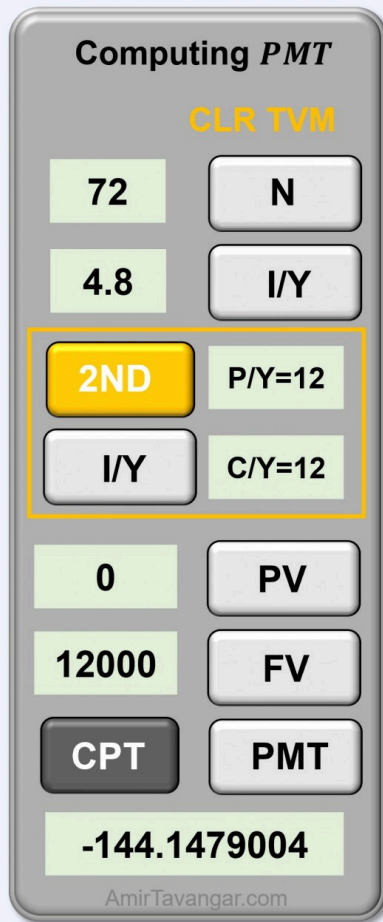
What deposit made at the end of each month will accumulate to \$12,000 in 6 years at 4.8% p.a. compounded monthly?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of each month so $P/Y = 12$
- $C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity
- Annuity term: $t = 6$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(6) = 72$
- Nominal interest rate: $I/Y = 4.8\%$
- The initial balance of the account is zero, so $PV = 0$
- Future value of investments is cash inflow: $FV = \$12,000$

$PMT = ?$



$$PMT \approx \$144.15$$

The size of the deposits made must be \$144.15.

Example 3.4.2: Compute PMT Given PV of Ordinary Simple Annuity

What payment is required at the end of each month for 8 years to repay a \$32,000 loan if the interest charged is 3.8% compounded monthly?

[Watch Video](#)



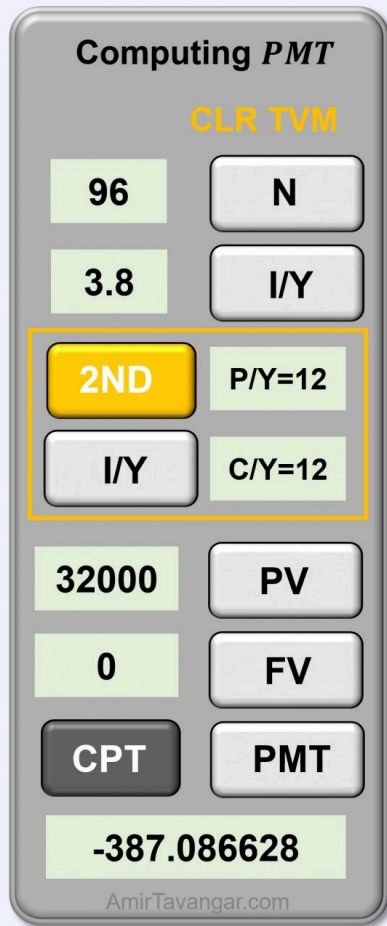
One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=547#oembed-1>

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of each month so $P/Y = 12$
- $C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity
- Annuity term: $t = 8$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(8) = 96$
- Nominal interest rate: $I/Y = 3.8\%$
- In loans, the present value (the loan principal) is cash inflow: $PV = \$32,000$
- No remaining balance at the end of the annuity term, so $FV = 0$

$$PMT = ?$$



$$PMT \approx \$387.09$$

Thus, the payments of \$387.09 are required to repay the loan.

Try an Example



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=547>

Example 3.4.3: Compute PMT Given FV of Ordinary General Annuity

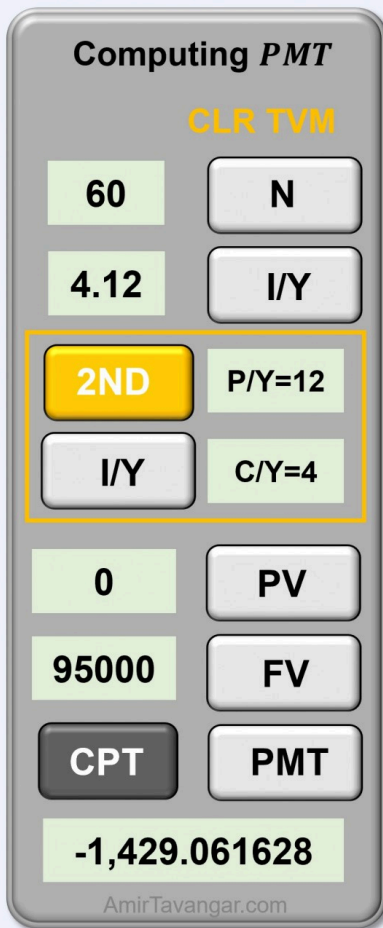
Dakota is planning for a renovation project that will cost \$95,000. To finance this, she intends to save money in her savings account earning an interest rate of 4.12% compounded quarterly. **a)** Determine the amount Dakota needs to deposit at the end of each month to reach her goal of \$95,000 in 5 years. **b)** Calculate the total amount Dakota will have deposited into the account over this 5-year period. **c)** Determine the total interest Dakota will earn on her savings account.

Show/Hide Solution

Given information

- Interest is compounded quarterly so $C/Y = 4$
- Payments are made at the end of each month so $P/Y = 12$
- $C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity
- Annuity term: $t = 5$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(5) = 60$
- Nominal interest rate: $I/Y = 4.12\%$
- The initial balance of the account is zero, so $PV = 0$
- Future value of investments is cash inflow: $FV = \$95,000$

a) $PMT = ?$



$$PMT \approx \$1429.06$$

Dakota will need to make monthly deposits of \$1429.06 to accumulate her desired amount.

b) The total amount that Dakota will have deposited over the 5-year term is equal to the number of payments (N) multiplied by the size of each payment (PMT).

$$\begin{aligned} \text{Total amount deposited} &= N \cdot PMT \\ &= 60(1429.06) \\ &= \$85,743.60 \end{aligned}$$

c) $I = ?$

Given this is an investment scenario where FV is known, the total interest amount earned in the savings account is given by Formula 3.2.

$$\begin{aligned}
 I &= FV - (N \cdot PMT) \\
 &= 95,000 - 85,743.60 \\
 &= \$9256.40
 \end{aligned}$$

Try an Example



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Example 3.4.4: Compute PMT Given Purchase Price

Jeff purchased a car listed for \$29,900. He paid 20% of the cost as a down payment and financed the balance amount at 5.1% compounded monthly for 12 years. **a)** What is the size of payment made at the end of every three months to settle the loan? **b)** How much was the amount of interest charged?

Show/Hide Solution

Given information

- Interest is compounded monthly, so $C/Y = 12$
- Payments are made at the end of every three months so $P/Y = 4$

$C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity

- Annuity term: $t = 12$ years
- Number of payments in the term: $N = P/Y \cdot t = 4(12) = 48$
- Nominal interest rate: $I/Y = 5.1\%$
- Purchase price = \$29,900
- Down payment = 20% of the purchase price
- The loan is fully settled, so $FV = 0$

a)

Finding PV of loan

First, we need to find the amount of the loan, which is the remaining balance after paying the down payment.

$$\text{Down payment} = 20\% (29,900) = \$5980$$

Using Formula 3.5, we have

$$\text{Purchase Price} = \text{Down payment} + PV$$

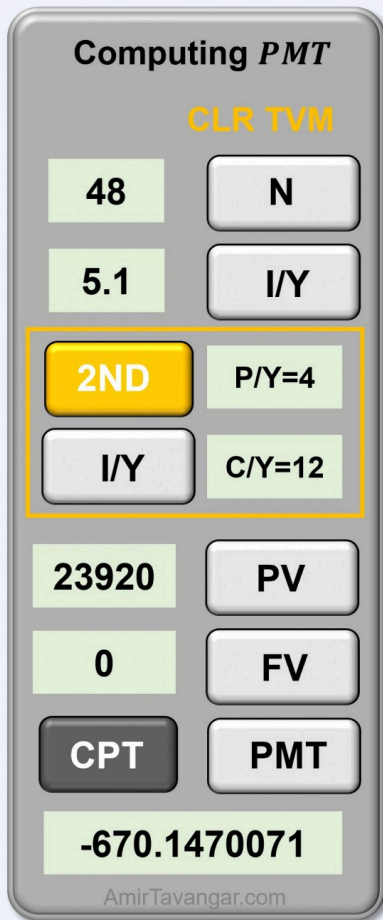
Rearranging the equation for PV gives

$$PV = \text{Purchase Price} - \text{Down payment}$$

$$PV = 29,900 - 5980$$

$$= \$23,920$$

Next, we use the PV and other given values to compute the size of the payments. We use the calculator approach.



$PMT \approx \$670.15$

The size of the quarterly payments is \$670.15.

b) The amount of interest for loans is given by Formula 3.4.

$$\begin{aligned}
 I &= N \cdot PMT - PV \\
 &= 48(670.15) - 23,920 \\
 &= 32,167.2 - 23,920 \\
 &= \$8247.20
 \end{aligned}$$

Example 3.4.5: Compute PMT Given Both FV and PV

Inesh has already saved \$11,600 in his savings account as of today, and he plans to contribute equal deposits at the end of each month for the next 10 years. What monthly deposit is required to accumulate \$105,000 in total at the end of the 10 years assuming the account earns 5.2% compounded annually?

Show/Hide Solution

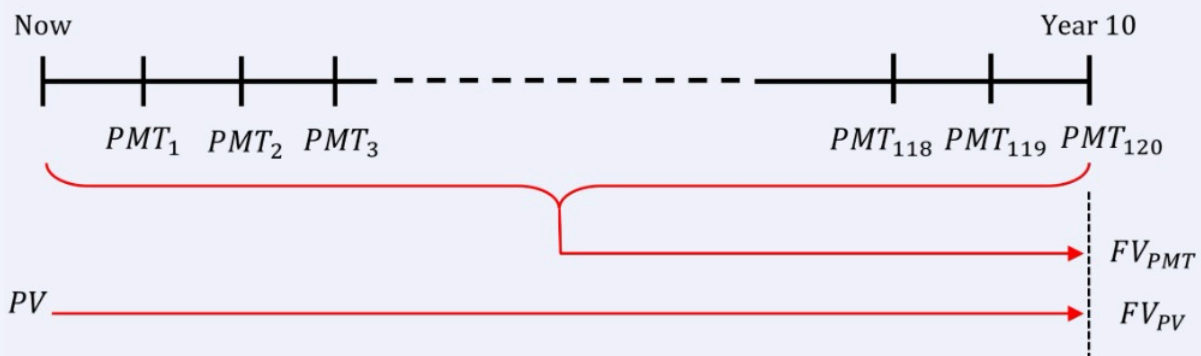
Given information

- Interest is compounded quarterly so $C/Y = 1$
- Payments are made at the end of each month so $P/Y = 12$
- $C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity
- Investment term: $t = 10$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(10) = 120$
- Nominal interest rate: $I/Y = 5.2\%$
- In investments, the future value is cash inflow (received), so $FV = \$105,000$
- In investments, the present value is cash outflow (invested), so $PV = -\$11,600$

$PMT = ?$

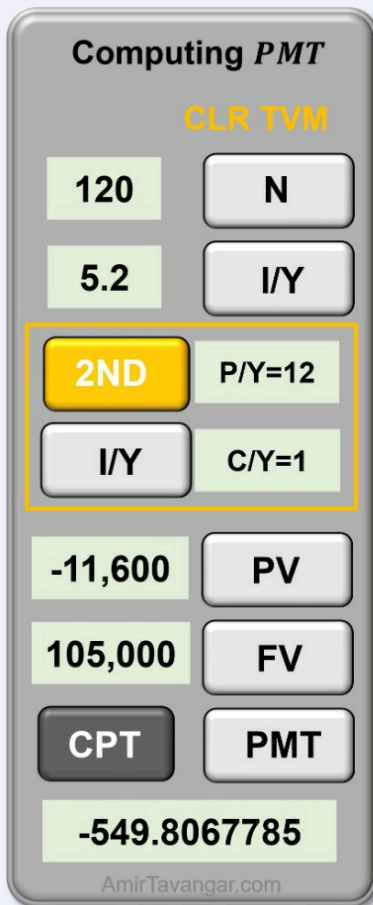
The accumulated value at the end of the term consists of two components: the future values of all the periodic payments (FV_{PMT}) and the future value of the initial lump sum investment (FV_{PV}).

$$FV = FV_{PMT} + FV_{PV}$$



To calculate the size of each periodic payment, we must first determine FV_{PMT} , the future value of these payments. Knowing that the combined future value of the periodic payments and the initial lump sum equals the target accumulation of \$105,000, our first step is to calculate the future value of the initial lump sum (FV_{PV}). This amount is then subtracted from the desired accumulated total of \$105,000.

Using a financial calculator simplifies this process, as it can compute the periodic payment in a single step.



$$PMT \approx \$549.81$$

Monthly deposits of \$549.81 are needed to accumulate to the desired amount.

Note that the present value (PV) should be input as a negative number, as it represents a cash outflow – money leaving Inesh’s possession and going into the account. Conversely, the future value (FV) is entered as a positive value, indicating a cash inflow, which is the amount Inesh will receive at the end of the term.

Try an Example



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B. Payment of Annuity Due

In the following examples, we will focus on calculating the periodic payments of annuities due. In these annuities, since payments are made at the start of each payment period, it is important to change the payment timing setting on your calculator to 'BGN' (beginning of the payment period).

Example 3.4.6: Compute PMT Given PV of Simple Annuity Due

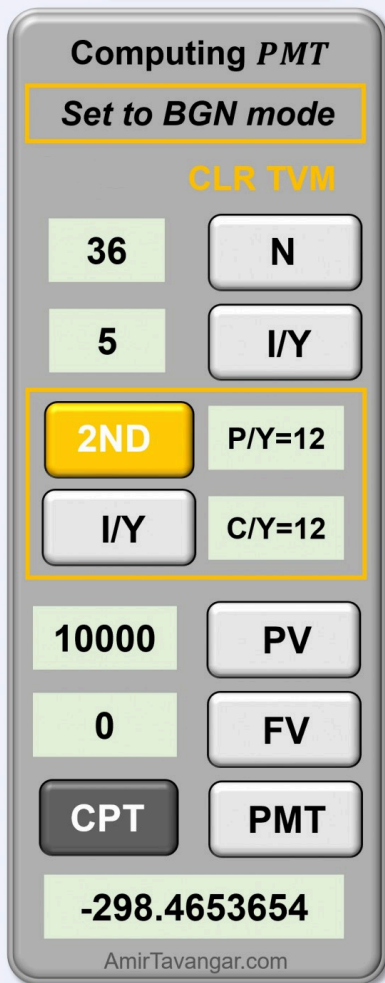
What monthly rent payment at the beginning of each month for three years is needed to fulfill a lease contract that is worth \$10,000 if money is worth 5% compounded monthly?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of each month so $P/Y = 12$
- $C/Y = P/Y \Rightarrow$ Simple Annuity Due
- Annuity term: $t = 3$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(3) = 36$
- Nominal interest rate: $I/Y = 5\%$
- The present value is received, so it is cash inflow: $PV = \$10,000$
- $FV = 0$

$PMT = ?$



$PMT \approx \$298.47$

Monthly rents of \$298.47 are required to fulfill the lease contract.

Try an Example



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Example 3.4.7: Compute PMT Given Both PV and FV

Colleen entered into a lease agreement for a car with a list price of \$42,000, which is expected to have a residual value of \$21,450 after four years. She agreed to make a down payment of 10% of the car's list price. Given an interest rate of 2.8% compounded monthly, calculate the amount of Colleen's lease payment that is due at the beginning of each week.

Note: The residual value of a leased vehicle represents an estimate of how much the car will be worth at the end of the lease term.

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of every week so $P/Y = 52$
- $C/Y \neq P/Y \Rightarrow$ General Annuity Due
- Annuity term: $t = 4$ years
- Number of payments in the term: $N = P/Y \cdot t = 52(4) = 208$
- Nominal interest rate: $I/Y = 2.8\%$
- List price = \$42,000
- Down payment = 10% of the list price
- Residual value = \$21,450

Finding PV of loan

The residual value of the car is the estimated value of the car at the end of the lease term, i.e., in four years. So it is considered a future value. We need to find the amount of the loan without considering the residual value, which is the remaining balance after paying the down payment. Using Formula 3.5, we have

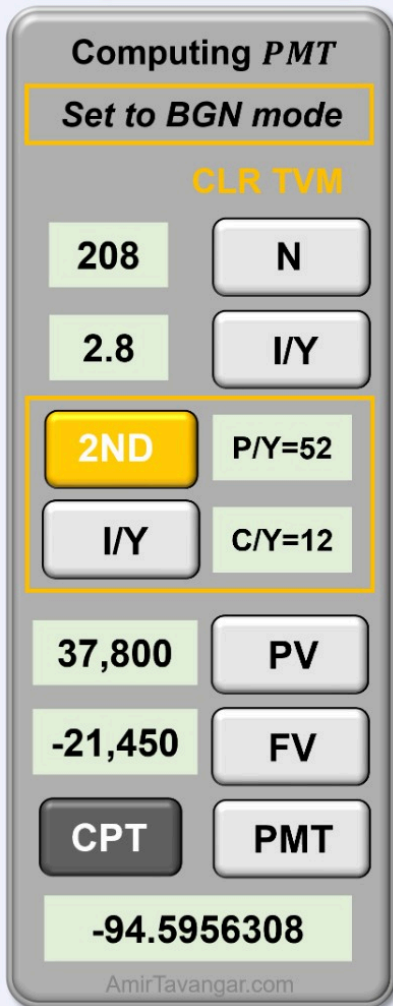
$$\text{List Price} = \text{Down payment} + PV$$

Rearranging the equation for PV gives

$$PV = \text{List Price} - \text{Down payment}$$

$$\begin{aligned}
 PV &= 42,000 - 10\% (42,000) \\
 &= \$37,800
 \end{aligned}$$

Next, we use the above PV, residual as FV, and other given values to compute the size of the payment.



$$PMT \approx \$94.60$$

The size of the weekly payments is \$94.60.

Notice that PV is entered as a positive value since it is considered a cash inflow (Colleen receives the lease loan), and FV is entered as a negative value since Colleen would have to return the car (equivalent to the residual value) or pay the residual value to own the car (if such an option is offered) at the end of the lease term (cash outflow).

Try an Example



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C. Payments in Scenarios with Multiple Annuities

The following example delves into calculating payments for a scenario involving a Registered Retirement Savings Plan (RRSP) and a Registered Retirement Income Fund (RRIF). Before we begin, it is beneficial to introduce these concepts.

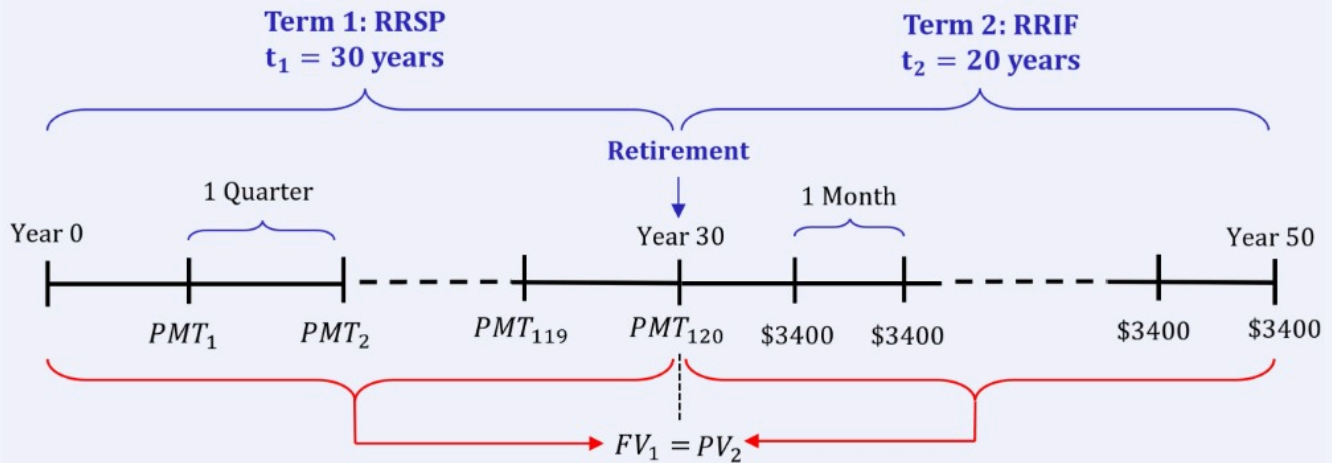
RRSP and RRIF are fundamental components of Canada's retirement savings system, designed to help individuals save for their future. The RRSP is a savings account with tax advantages to encourage saving for retirement. Contributions to an RRSP are tax-deductible, meaning they can reduce the amount of income tax you pay in the year you make the contribution. Once you retire, RRSP is converted into a RRIF which provides you with a regular income during retirement. Withdrawals from an RRIF are taxed as income at your marginal tax rate. The frequency of these withdrawals can be monthly, quarterly, semi-annually, or annually, offering flexibility in managing retirement income.

Example 3.4.8: Compute PMT of First Term Given PMT of the Second Term

Darius is planning for his retirement, which is 30 years away, by making end-of-quarter contributions to his Registered Retirement Savings Plan (RRSP), set to later convert into a Registered Retirement Income Fund (RRIF). Upon retirement, he intends to withdraw \$3,400 at the end of each month for 20 years. An annual interest rate of 4.8%, compounded monthly, applies to both the RRSP during the accumulation phase and the RRIF during the distribution phase.

- a) What is the total amount Darius must have in his RRIF at the start of retirement to support his goal of monthly withdrawals of \$3,400?
- b) How much must Darius deposit into his RRSP at the end of each quarter to ensure he accumulates the required amount in his RRIF by his retirement in 30 years?

Show/Hide Solution



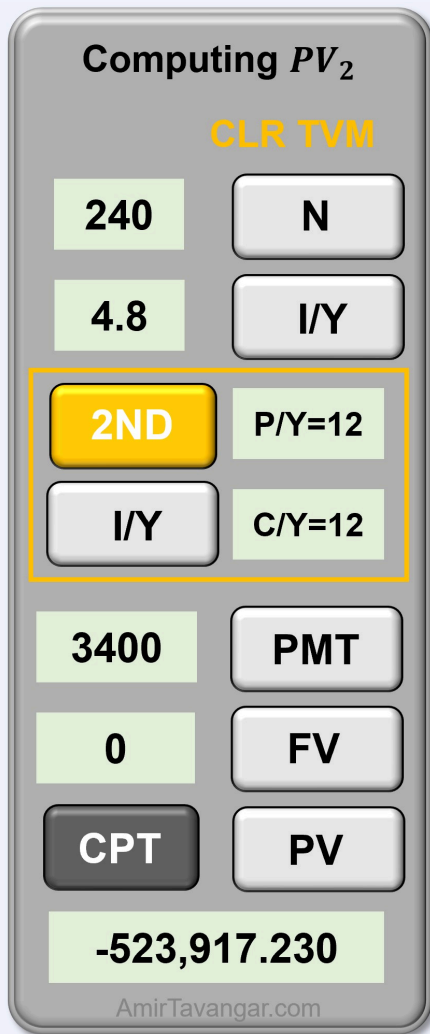
a) Term 2: RRIF – Ordinary Simple Annuity

To determine the total amount Darius needs in his RRIF at the start of retirement to afford monthly withdrawals of \$3,400 for 20 years, we calculate the present value of those payments at the time of retirement.

Given Information

- Interest is compounded monthly so $C/Y = 12$
- Withdrawals are made at the end of every month so $P/Y_2 = 12$
- $C/Y = P/Y_2 \Rightarrow$ Ordinary Simple Annuity
- RRIF term: $t_2 = 20$ years
- Number of payments in the term: $N_2 = P/Y_2 \cdot t_2 = 12(20) = 240$
- Nominal interest rate: $I/Y = 4.8\%$
- The payments are received and are cash inflow, so $PMT_2 = \$3400$
- The fund will be depleted by the end of the term, so $FV_2 = 0$

$$PV_2 = ?$$



$$PV_2 \approx \$523,917.23$$

Therefore, the total amount Darius must have in his RRIF at the start of retirement to support his goal is \$523,917.23.

b) Term 1: RRSP – Ordinary General Annuity

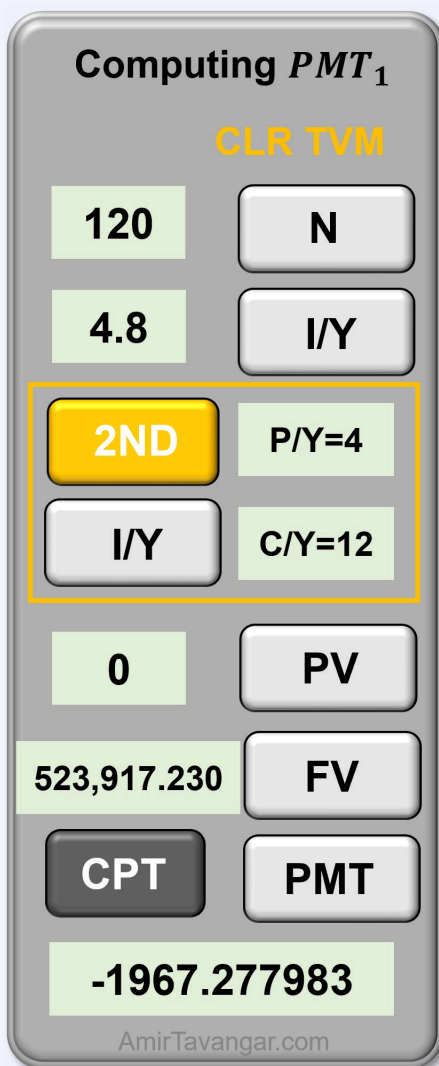
The value needed at retirement, determined in Part (a) becomes the target FV for the RRSP contributions. We then calculate the quarterly contribution Darius must make to his RRSP over 30 years to accumulate to that target FV at retirement.

Given Information

- Interest is compounded monthly so $C/Y = 12$

- Withdrawals are made at the end of every quarter so $P/Y_1 = 4$
- $C/Y \neq P/Y_1 \Rightarrow$ Ordinary General Annuity
- RRSP term: $t_1 = 30$ years
- Number of payments in the term: $N_1 = P/Y_1 \cdot t_1 = 4(30) = 120$
- Nominal interest rate: $I/Y = 4.8\%$
- No initial balance in RRSP, so $PV_1 = 0$
- In investments, the future value is cash inflow, so $FV_1 = PV_2 = \$523,917.230\dots$ (use unrounded value for accuracy)

$$PMT_1 = ?$$



$$PMT_1 \approx \$1967.28$$

Thus, Darius will need to make quarterly deposits of \$1967.28 to ensure he accumulates the required amount in his RRIF by his retirement.

Try an Example



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Section 3.4 Exercises

1. What payment is required at the end of every month for 5.5 years to repay a \$11,968 loan if interest is 4.4% compounded monthly?

Show/Hide Answer

$$\text{PMT} = \$204.49$$

2. Gregory purchased a car for \$53,500; he paid 5% of the cost as a down payment and financed the balance amount at 3.5% compounded monthly for 3 years. **a)** What is the size of payment made at the end of every month to settle the loan? **b)** What was the amount of interest charged?

Show/Hide Answer

a) $\text{PMT} = \$1,489.28$

b) $I = \$2,789.08$

3. To fulfill a lease contract valued at \$29,000, rent payments are required at the beginning of each month for 5 years. The interest rate on this lease is assumed to be 4.79%, compounded monthly. **a)** What is the size of the payments? **b)** How much interest was charged on the lease?

Show/Hide Answer

a) $PMT = \$542.32$

b) $I = \$3,539.20$

4. Dawn entered into a lease agreement for a car with a list price of \$56,460.00, which is expected to have a residual value of \$31,053 after 6 years. She agreed to make a down payment of 15% of the car's list price. Given an interest rate of 2.06% compounded semi-annually, calculate the amount of Dawn's lease payment that is due at the beginning of each month.

Show/Hide Answer

$PMT = \$302.79$

5. Kristina has already saved \$7,600 in her savings account as of today, and she plans to contribute equal deposits at the end of each year for the next 10 years. What deposit is required to accumulate \$91,500 in total at the end of the 10 years assuming the account earns 2.42% compounded quarterly?

Show/Hide Answer

$PMT = \$7323.15$

3.5 NUMBER OF PAYMENTS AND TIME: CALCULATOR APPROACH

A. Introduction

Suppose you are setting aside funds for a renovation project with a target of \$50,000. You plan to save \$100 every week and deposit it into your savings account. The key question is how long will it take to reach your \$50,000 savings goal. To find this out, you need to first calculate the number of weekly payments (N) you must make to accumulate the desired amount. Once you have determined N , you will know the exact duration required to achieve your savings target for the renovation project.

In this section, we turn our attention to computing the total number of payments in annuities (N) and understanding how long it takes for an annuity to achieve its financial objectives, whether it's accumulating a desired sum for retirement, paying off a loan, or reaching a specific investment goal.

When the time of the annuity term (t) is unknown, the initial step is to calculate N using either a formula or a financial calculator. Once N is known, it can then be used to work out the time. Depending on the scenario, whether it involves retirement plans, loans, or savings schemes, the calculation of N may be based on the Periodic Payment (PMT) and either the Future Value (FV), the Present Value (PV) of the annuity, or a combination of both.

For the calculator approach, we use the time-value-of-money (TVM) worksheet in financial calculators. It's also important to follow the cash flow sign convention (see Table 2.2.1 in Section 2.2) when inputting monetary values into the calculator. This ensures consistency and accuracy in our calculations.

It is important to note that in annuities calculations, unlike compound interest calculation where N is the number of compounding periods in the term, N represents the number of payments in the term. This value is always *rounded up* to the next whole number of payments, which may result in the last payment being smaller than the others.

Once N is computed, the annuity's term (t) can be determined by rearranging Formula 3.1a.

$$t = \frac{N}{P/Y} \quad \text{Formula 3.1b}$$

B. Payment of Ordinary Annuity

We first focus on calculating N and the time for ordinary annuities. In these annuities, since payments are made at the end of each payment period, it is important to change the payment timing setting on your calculator to 'END' (end of the payment period).

Example 3.5.1: Compute N and Time Given FV and PMT of Ordinary Annuity

Mahsa wants to save \$80,000 for her house down payment. Suppose she plans to make deposits of \$1,290 at the end of every three months into an account earning 4.7% compounded quarterly. **a)** How many deposits will she have to make to achieve her goal? **b)** How long will it take? Give the time in terms of years and months.

Show/Hide Solution

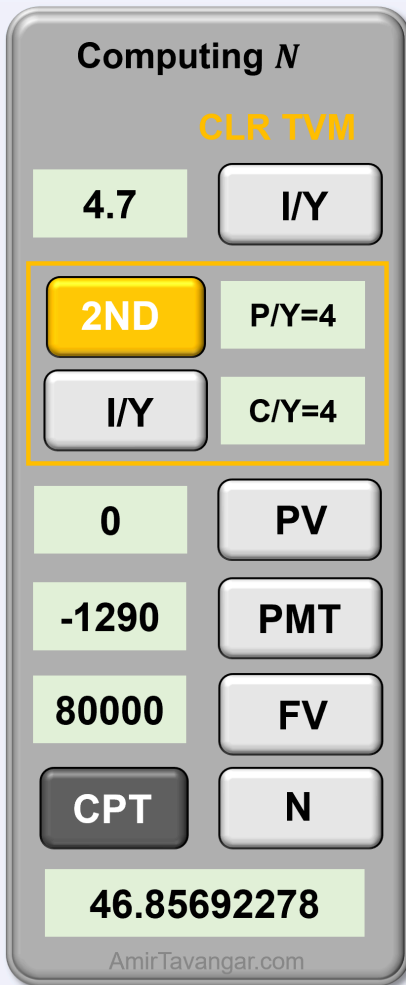
Given Information

- Interest is compounded quarterly so $C/Y = 4$
- Payments are made at the end of every three months so $P/Y = 4$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Nominal interest rate: $I/Y = 4.7\%$
- No initial investment: $PV = 0$
- The payments are made to a bank, so they are cash outflow: $PMT = -\$1,290$
- Future value is received, so it is a cash inflow: $FV = \$80,000$

a) $N = ?$



Rounded up

$$N = 47$$

The number of payments in the term, N , is rounded up to 47 deposits.

b) $t = ?$

To find the time period of the term, we substitute N and P/Y into Formula 3.1b:

$$t = \frac{N}{P/Y}$$

$$t = \frac{47}{4}$$

$$= 11.75 \text{ years}$$

$$= 11 \text{ years} + 0.75(12) \text{ months}$$

$$= 11 \text{ years and } 9 \text{ months}$$

Note: In annuities, N represents the number of payments in the term and its value is always *rounded up* to the next whole number of payments regardless of the decimal part of the calculated N .

Try an Example



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Example 3.5.2: Compute N and Time Given FV , PV , and PMT of Ordinary Annuity

Mahsa already has \$25,000 in her savings account, and she wants this to accumulate to \$80,000 for a house down payment. Suppose she plans to make deposits of \$1,290 at the end of every three months into the account earning 4.7% compounded quarterly. **a)** How many deposits will she have to make to achieve her goal? **b)** How long will it take? Give the time in terms of years and months.

Show/Hide Solution

Given Information

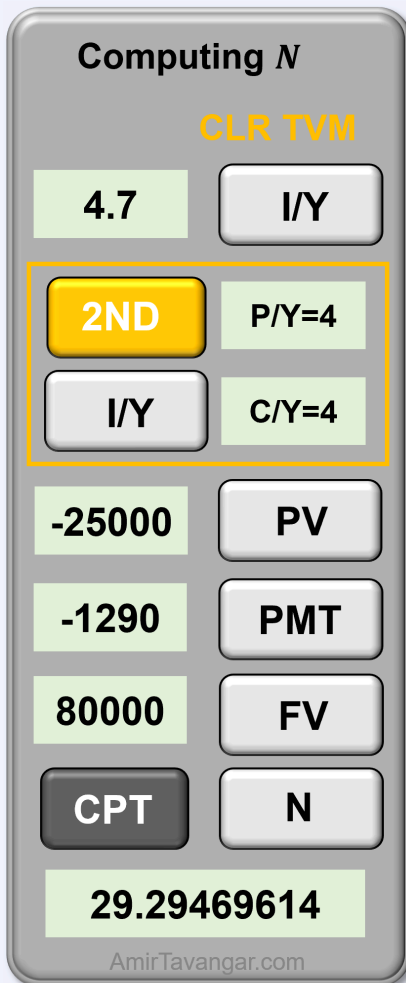
- Interest is compounded quarterly so $C / Y = 4$

- Payments are made at the end of every three months so $P/Y = 4$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Nominal interest rate: $I/Y = 4.7\%$
- The payments are made to a bank, so they are cash outflow: $PMT = -\$1,290$
- Present value is a lumpsum invested, so it is a cash outflow: $PV = -\$25,000$
- Future value is received, so it is a cash inflow: $FV = \$80,000$

a) $N = ?$



Rounded up

$$N = 30$$

The number of payments in the term, N , is rounded up to 30 deposits.

b) $t = ?$

To find the time period of the term, we substitute N and P/Y into Formula 3.1b:

$$t = \frac{N}{P/Y}$$

$$t = \frac{30}{4}$$

$$= 7.5 \text{ years}$$

$$= 7 \text{ years} + 0.5(12) \text{ months}$$

$$= 7 \text{ years and } 6 \text{ months}$$

Try an Example



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Example 3.5.3: Compute N and Time Given PV and PMT of Ordinary Annuity

Emanuel won a lottery prize of \$1,000,000 and invested it in a fund that earns 4% interest compounded monthly. He plans to withdraw \$1,000 at the end of each week. **a)** Determine the total number of withdrawals Emanuel will make. **b)** How long will the money last with these weekly withdrawals? Give the time in terms of years and weeks.

Show/Hide Solution

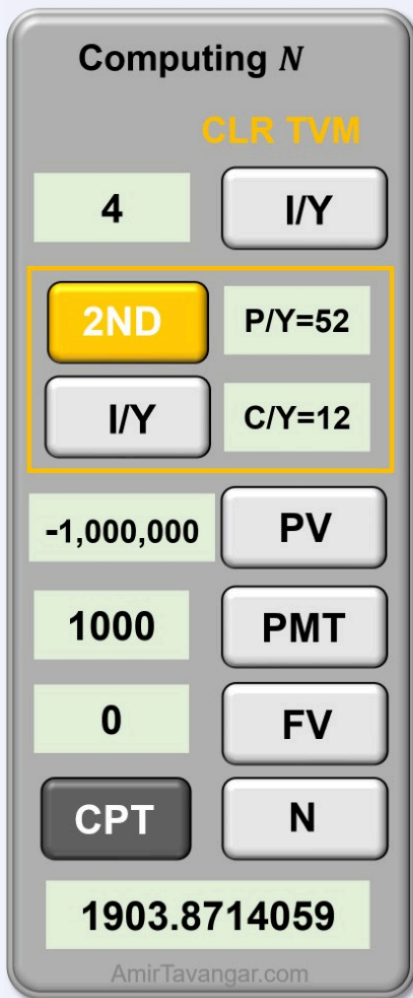
Given Information

- Interest is compounded annually so $C/Y = 12$
- Withdrawals are at the end of every week so $P/Y = 52$

$$C/Y \neq P/Y \Rightarrow \text{Ordinary General Annuity}$$

- Nominal interest rate: $I/Y = 4\%$
- Payments are received, so they are cash inflow: $PMT = \$1000$
- Present value is invested in a bank, so it is a cash outflow: $PV = -\$1,000,000$
- The account will be depleted at the end of the term: $FV = 0$

a) $N = ?$



Rounded up
 $N = 1904$

The number of payments in the term, N , is rounded up to 1904 withdrawals.

b) $t = ?$

To find the time period of the term, we substitute N and P/Y into Formula 3.1b:

$$t = \frac{N}{P/Y}$$

$$t = \frac{1904}{52}$$

$$= 36.6153... \text{ years}$$

$$= 36 \text{ years} + 0.6153... (52) \text{ weeks}$$

$$= 36 \text{ years and } 32 \text{ weeks}$$

C. Payment of Annuity Due

In the following examples, we will focus on calculating N and the time for annuities due. In these annuities, since payments are made at the start of each payment period, it is important to change the payment timing setting on your calculator to 'BGN' (beginning of the payment period).

Example 3.5.4: Compute N and Time Given FV , PMT of Annuity Due

Esmeralda wants to accumulate \$71,000 for her condominium down payment. She plans to make deposits of \$3,890 at the beginning of every three months into her savings account, which earns interest at 5.29% compounded quarterly. **a)** How many deposits will she have to make to achieve her goal? **b)** How long will it take? Give the time in terms of years and months.

Show/Hide Solution

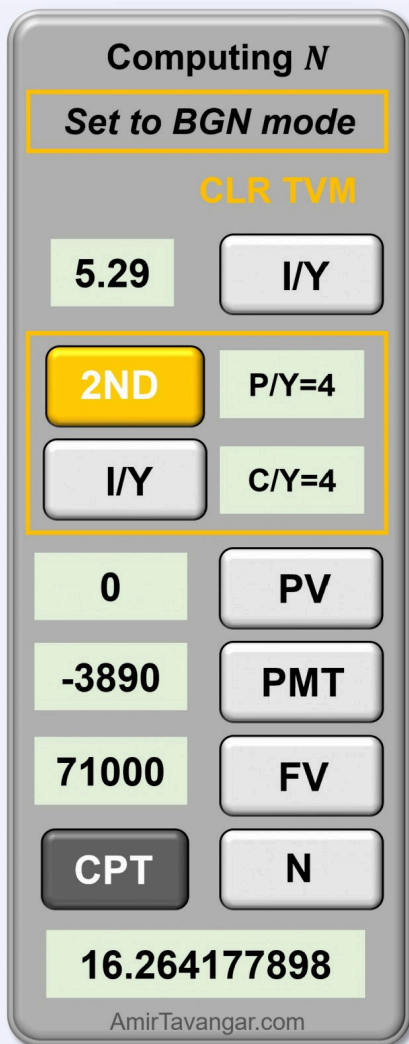
Given Information

- Annuity Due: Set your calculator to BGN mode
- Interest is compounded quarterly so $C/Y = 4$
- Payments are made at the beginning of every three months so $P/Y = 4$

$$C/Y = P/Y \Rightarrow \text{Simple Annuity Due}$$

- Nominal interest rate: $I/Y = 5.29\%$
- No initial balance: $PV = 0$
- Payments are made, so they are cash outflow: $PMT = -\$3,890$
- Future value is received, so it is a cash inflow: $FV = \$71,000$

a) $N = ?$



Rounded up

$$N = 17$$

The number of payments in the term, N , is rounded up to 17 deposits.

b) $t = ?$

To find the time period of the term, we substitute N and P/Y into Formula 3.1b:

$$t = \frac{N}{P/Y}$$

$$\begin{aligned}
 t &= \frac{17}{4} \\
 &= 4.25 \text{ years} \\
 &= 4 \text{ years} + 0.25(12) \text{ months} \\
 &= 4 \text{ years and } 3 \text{ months}
 \end{aligned}$$

Try an Example



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Example 3.5.5: Compute N and Time Given PV, PMT of Annuity Due

Dario recently retired and has \$500,000 in his retirement fund, which earns an annual interest rate of 8% compounded annually. He intends to withdraw \$4,000 at the beginning of every month. **a)** Calculate the total number of withdrawals Dario will make. **b)** How long will it take for the fund to be depleted?

Show/Hide Solution

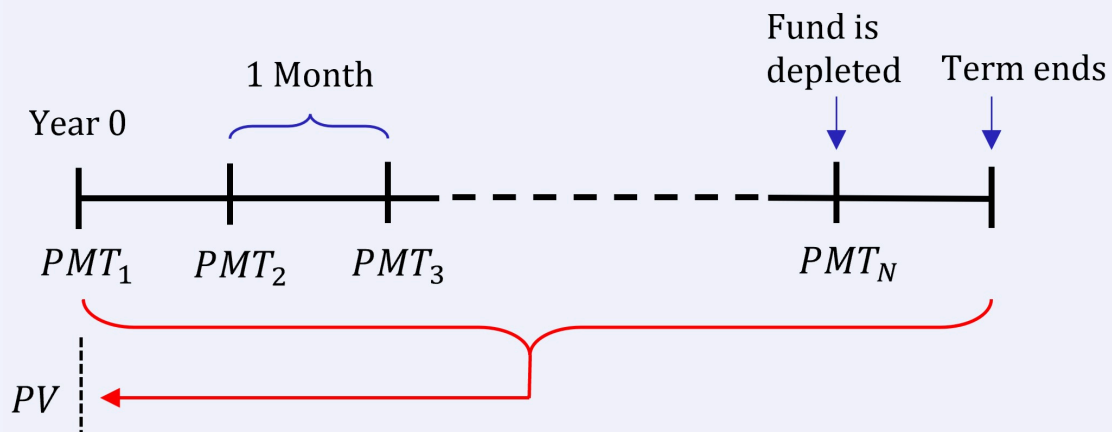
Given Information

- Annuity Due: Set your calculator to BGN mode

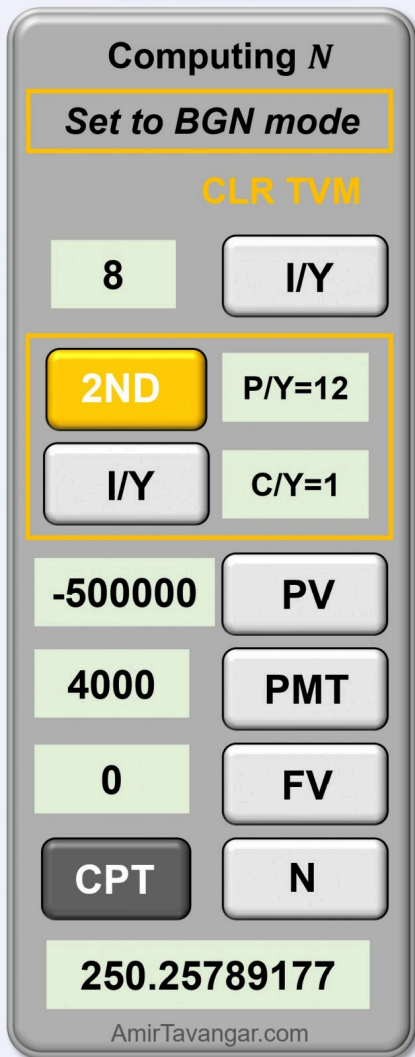
- Interest is compounded annually so $C/Y = 1$
- Payments are withdrawn at the beginning of every month so $P/Y = 12$

$C/Y \neq P/Y \Rightarrow$ General Annuity Due

- Nominal interest rate: $I/Y = 8\%$
- Payments are received, so they are cash inflow: $PMT = \$4,000$
- Present value is invested in a bank, so it is a cash outflow: $PV = -\$500,000$
- The fund balance at the end of the term will be zero, so $FV = 0$



a) $N = ?$



Rounded up

$$N = 251$$

The number of payments in the term, N , is rounded up to 251 withdrawals.

b) $t = ?$

To find the duration of the annuity term, we substitute N and P/Y into Formula 3.1b:

$$t = \frac{N}{P/Y}$$

$$t = \frac{251}{12}$$

$$\begin{aligned}
 &= 20.91\bar{6} \text{ years} \\
 &= 20 \text{ years} + 0.91\bar{6}(12) \text{ months} \\
 &= 20 \text{ years and } 11 \text{ months}
 \end{aligned}$$

As shown in the timeline, in an annuity due, the term extends beyond the last withdrawal by one period (in this case, one month). Because our focus is on the timeframe up to the final withdrawal (when the fund is depleted), we must deduct one period (one month) from the annuity due term.

Thus, the fund will be depleted in **20 years and 10 months**.

Section 3.5 Exercises

- Karina wants to accumulate \$52,500 for her condominium down payment. She plans to make deposits of \$1,330 at the beginning of every month into her savings account, which earns interest at 6.61% compounded monthly. **a)** How many deposits will Karina have to make to achieve her goal? **b)** How long will it take? Give the time in terms of years and months.

Show/Hide Answer

a) $N = 36$ deposits

b) $t = 3$ years and 0 months

- BoardWalk Corporation aims to save a total of \$641,000 over the coming years. They plan to start with an initial deposit of \$176,800 and then make deposits of \$5,200 at the end of each month. **(a)** Determine the number of additional monthly deposits and **(b)** the total time required for BoardWalk Corporation to accumulate \$641,000, considering an interest rate of 6.37% compounded semi-annually. Give the time in years and months.

Show/Hide Answer

a) $N = 64$ payments

b) $t = 5$ years and 4 months

3. Alo has saved \$2,500,000 in his retirement fund, which earns an annual interest rate of 1.5% compounded daily. He intends to withdraw \$10,000 at the beginning of every month. **a)** Calculate the total number of withdrawals Alo will make. **b)** Determine the duration for which Alo's retirement fund will sustain these monthly withdrawals. Give the time in years and months.

Show/Hide Answer

a) $N = 300$ withdrawals

b) $t = 25$ years

3.6 NOMINAL INTEREST RATE (I/Y): CALCULATOR APPROACH

A. Introduction

In this section, we learn how to calculate both nominal and periodic interest rates. Calculating the periodic interest rate (i) algebraically (formula approach) is difficult as i cannot be easily isolated in the PV or FV formula. Therefore, we only discuss the calculator approach to find the nominal interest rate (I/Y). Once I/Y is determined, we can easily calculate the periodic interest rate with Formula 2.1a:

$$i = \frac{I/Y}{C/Y}$$

We use the time-value-of-money (TVM) worksheet in financial calculators. It's also important to follow the cash flow sign convention (see Table 2.2.1 in Section 2.2) when inputting monetary values into the calculator. This ensures consistency and accuracy in our calculations.

Depending on the scenario, whether it involves retirement plans, loans, or savings schemes, the calculation of I/Y may be based on the Periodic Payment (PMT) and either the Future Value (FV), the Present Value (PV) of the annuity, or a combination of both. The following examples demonstrate determining the interest rate in different scenarios.

B. Nominal and Periodic Interest Rates of Ordinary Annuities

We first focus on calculating the interest rates of ordinary annuities. In these annuities, since payments are made at the end of each payment period, it is important to change the payment timing setting on your calculator to 'END' (end of the payment period).

Example 3.6.1: Compute Nominal and Periodic Interest Rates Given PV and PMT

Reza has \$500,000 in his registered retirement income fund (RRIF). He wants to withdraw \$4000 at the end of

each month for 20 years. **a)** What nominal annual interest rate compounded monthly is required? **b)** What is the periodic interest rate per month? Give your answers in percent rounded to two decimal places.

Show/Hide Solution

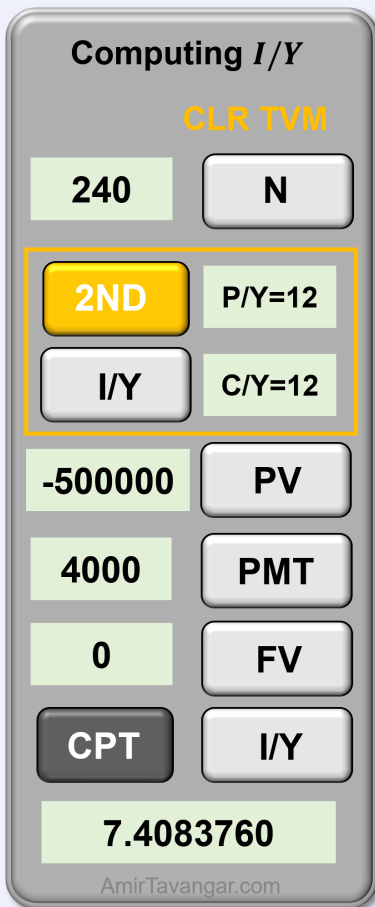
Given Information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of each month so $P/Y = 12$

$C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity

- Annuity term: $t = 20$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(20) = 240$
- Payments are received, so they are cash inflow: $PMT = \$4000$
- Present value is invested, so it's cash outflow: $PV = -\$500,000$
- The balance at the end of the term will be zero: $FV = 0$

a) $I/Y = ?$



$$I/Y \approx 7.41\%$$

The nominal interest rate, I/Y , should be 7.41% compounded monthly.

b) $i = ?$

The periodic interest rate then is given by Formula 2.1a:

$$i = \frac{I/Y}{C/Y}$$

$$i = \frac{7.41\%}{12}$$

$$= 0.6175\%$$

$$\approx 0.62\% \text{ per month}$$

Try an Example



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Example 3.6.2: Compute Nominal and Periodic Interest Rates Given, PV, PMT, and FV

Sydney currently has \$12,500 in her investment account. She intends to deposit an additional \$100 at the end of each month for the next 20 years. To meet her target of \$48,000 at the end of this 20-year period, what effective interest rate must her investment earn?

Show/Hide Solution

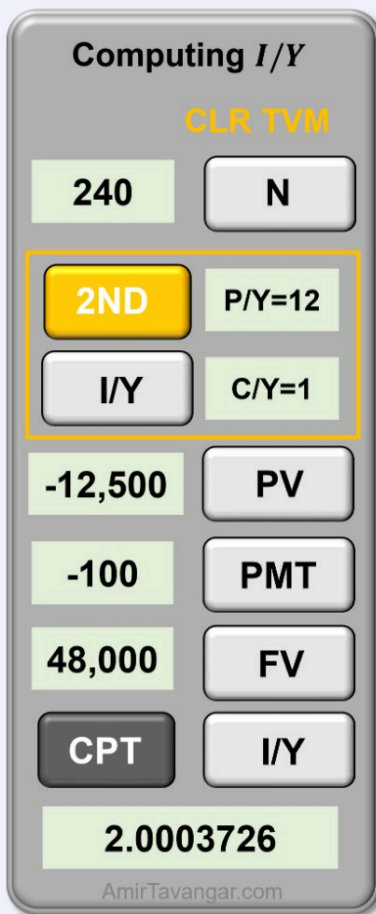
Given Information

- Effective interest rate is compounded annually so $C/Y = 1$
- Payments are made at the end of each month so $P/Y = 12$

$$C/Y \neq P/Y \Rightarrow \text{Ordinary General Annuity}$$

- Annuity term: $t = 20$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(20) = 240$
- Payments are made, so they are cash outflow: $PMT = -\$100$
- The initial investment is the present value and it is cash outflow: $PV = -\$12,500$
- The future value is received, so it is a cash inflow: $FV = \$48,000$

a) $I/Y = ?$



$$I/Y \approx 2.00\%$$

The effective interest rate, I/Y , is 2.00% compounded annually.

b) $i = ?$

Given the interest is compounded annually, the periodic interest rate is equal to the nominal interest rate.

$$i = 2.00\%$$

C. Nominal and Periodic Interest Rates of Annuities Due

In the following examples, we focus on calculating the interest rates of ordinary annuities. In these annuities, since payments are made at the beginning of each payment period, it is important to change the payment timing setting on your calculator to 'BGN' (beginning of the payment period).

Example 3.6.3: Compute Nominal Rate Given, PV and PMT of Annuity Due

Mona won a lottery prize of \$475,000 and invested it in a savings account. She intends to make monthly withdrawals of \$3,500 at the beginning of each month for 15 years. Determine the nominal interest rate compounded monthly, needed to sustain these withdrawals. Provide your answer as a percentage rounded to two decimal places.

Show/Hide Solution

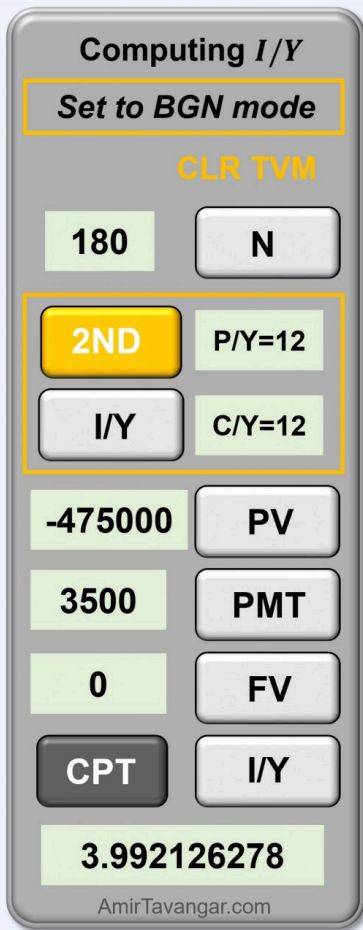
Given Information

- Annuity Due: Set your calculator to BGN mode
- Interest is compounded monthly so $C/Y = 12$
- Payments are withdrawn at the **beginning** of each month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Simple Annuity **Due**}$$

- Annuity term: $t = 15$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(15) = 180$
- Payments are received, so they are cash inflow: $PMT = \$3,500$
- Present value is invested in a bank, so it is a cash outflow: $PV = -\$475,000$
- The balance at the end of the term will be zero: $FV = 0$

$$I/Y = ?$$



$I/Y \approx 3.99\%$
Compounded monthly

The nominal interest rate, I/Y , should be 3.99% compounded monthly.

Try an Example



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Example 3.6.4: Compute Nominal Rate Given, PV, FV and PMT of Annuity Due

Sandeep leases a new car by making a \$5600 down payment and payments of \$520 at the beginning of every month for three years. The lease amount (list price) of the car is \$51,000, and the residual value of the car at the end of the lease term will be \$32,000. **a)** What nominal annual interest rate compounded quarterly is he being charged? **b)** What is the periodic interest rate per quarter? Give your answers in percent rounded to two decimal places.

Show/Hide Solution

Given Information

- Interest is compounded quarterly so $C/Y = 4$
- Payments are made at the **beginning** of every month so $P/Y = 12$

$$C/Y \neq P/Y \Rightarrow \text{General Annuity Due}$$

- Lease term: $t = 3$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(3) = 36$
- Payments are made, so they are cash outflow: $PMT = -\$520$
- List price = \$51,000
- Down payment = \$5600
- Residual value (FV) is a cash outflow = $-\$21,450$

a) $I/Y = ?$

First, we need to find the amount of PV (the loan on the lease), which is the remaining balance after paying the down payment.

$$\text{Lease Amount} = \text{Down payment} + PV$$

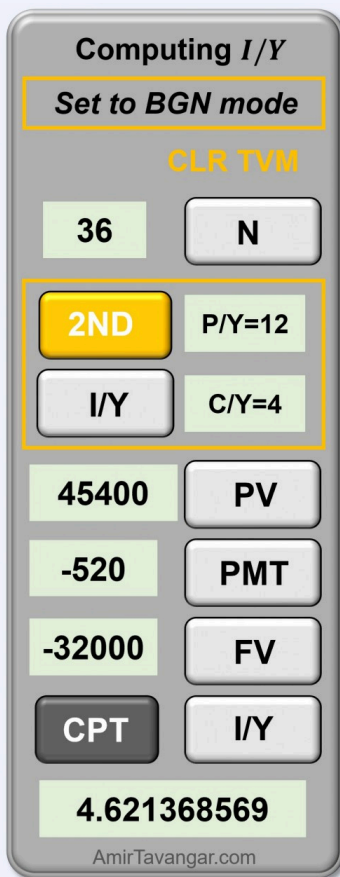
Rearranging the equation for PV gives

$$PV = \text{Lease Amount} - \text{Down payment}$$

$$PV = 51,000 - 5600$$

$$= \$45,400$$

Next, we compute the nominal interest rate using the above PV, residual as FV, PMT, and N. We use the calculator approach. Note that PV is entered as a positive value since it is considered a cash inflow (Sandeep receives the lease loan), and FV is entered as a negative value since Sandeep must either return the car (equivalent to the residual value) or pay the residual value to own the car (if such an option is offered) at the end of the lease term (cash outflow). The periodic payments are also cash outflow.



$I/Y \approx 4.62\%$
Compounded quarterly

The nominal interest rate, I/Y , being charged is 4.62% compounded quarterly.

b) $i = ?$

The periodic interest rate is then given by Formula 2.1a.

$$i = \frac{I/Y}{C/Y}$$

$$i = \frac{4.62\%}{4}$$

$$= 1.155\%$$

$$\approx 1.16\% \text{ per quarter}$$

Try an Example



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Section 3.6 Exercises

- Blake has saved \$79,500 in his registered retirement income fund (RRIF). He plans to make withdrawals of \$4,222.32 every six months for a period of 12 years. **a)** Determine the nominal annual interest rate required for his plan, assuming it is compounded semi-annually. **b)** Calculate the periodic interest rate for each six-month interval. Give your answers in percent rounded to two decimal places.

Show/Hide Answer

a) $I/Y = 4.08\%$ compounded semi-annually

b) $i = 2.04\%$ per six months

2. Bisa won a lottery prize of \$29,000 and invested it in a savings account. She intends to make monthly withdrawals of \$1,597.64 at the beginning of each quarter for 5 years. What nominal annual interest rate compounded quarterly is required to sustain these withdrawals? Give your answer in percent rounded to two decimal places.

Show/Hide Answer

I/Y = 4.18% compounded quarterly

3. Tom leases a new car by making a \$2,444 down payment and payments of \$452 at the beginning of every month for 6 years. The lease amount (list price) of the car is \$48,870, and the residual value of the car at the end of the lease term will be \$20,525. What nominal annual interest rate compounded semi-annually is Tom being charged? Give your answer in percent rounded to two decimal places.

Show/Hide Answer

I/Y = 3.31% compounded semi-annually

3.7 FV OF ANNUITIES: FORMULA APPROACH

A. Future Value of Ordinary Simple Annuity

After covering the basics and types of annuities, we now focus on understanding and calculating the future value of annuities. This skill is crucial for financial planning, whether it is for retirement savings, education funds, or other long-term financial goals. We begin by examining the future value of ordinary annuities, where payments are made at the end of each period.

In an Ordinary Simple Annuity, the payment period and the interest compounding period are the same ($P/Y = C/Y$), and the payments are made at the end of the payment period.

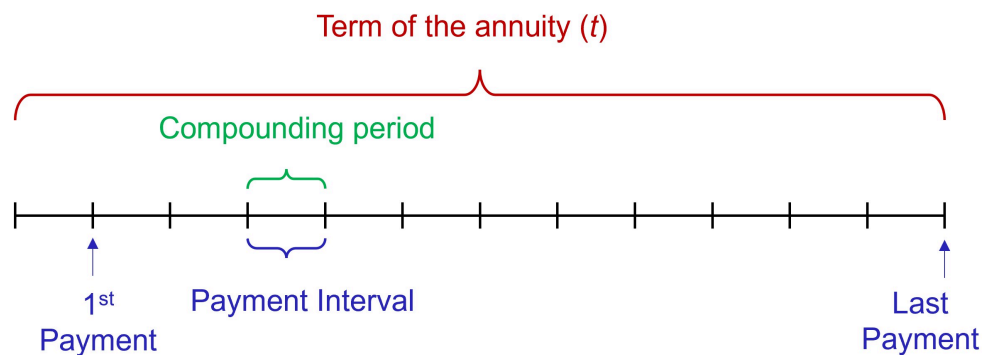


Figure 3.7.1 A timeline for an ordinary simple annuity

Consider a scenario where you invest \$1,000 at the end of every year into a savings account that offers a 10% annual interest rate compounded annually, over five years. To find out the total amount in your account at the end of these five years, you need to calculate the future value of this annuity. Here, the future value represents the total value accumulated from all your annual payments, including the interest earned, by the end of the five-year term.

To calculate the total accumulated value, we can calculate the future value of each payment using the formula for the future value of compound interest, $FV = PV(1 + i)^N$ (Formula 2.4a). The timeline for this calculation is depicted in Figure 3.7.2.

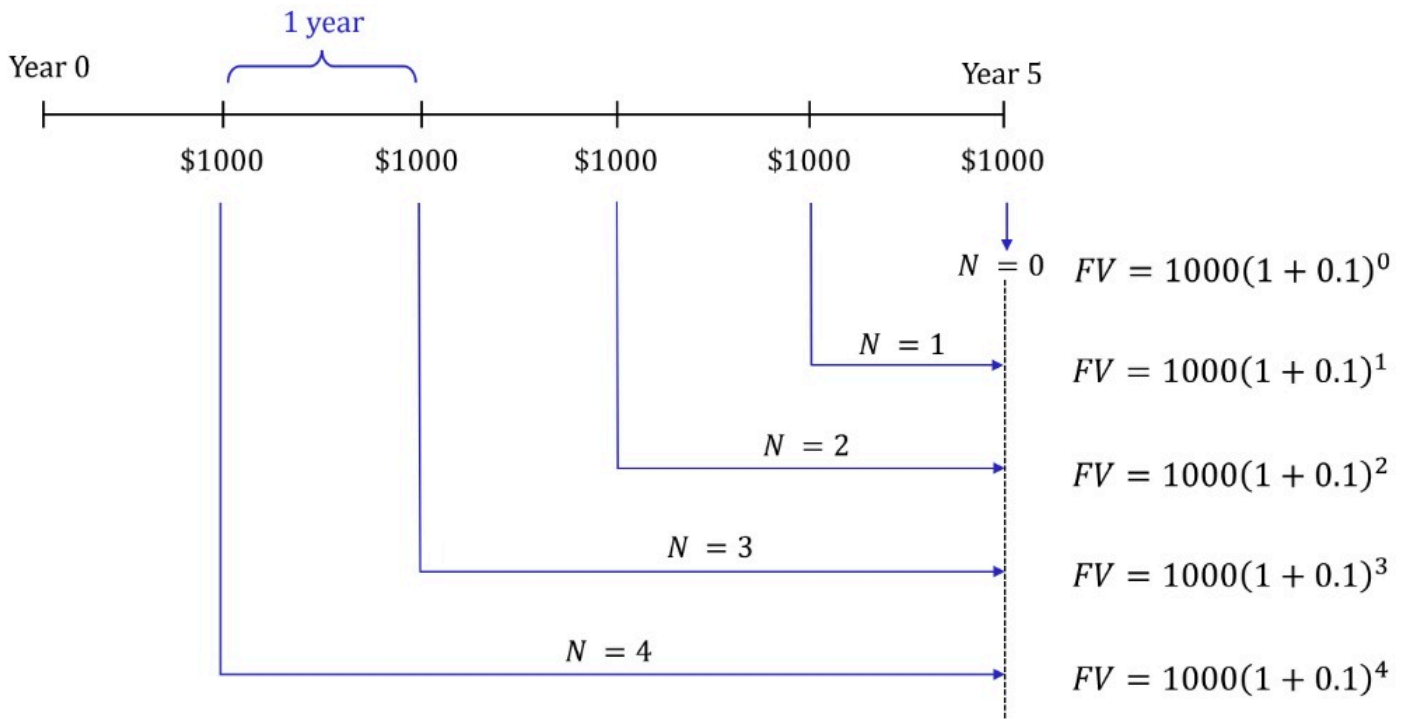


Figure 3.7.2 Future value of Ordinary simple Annuity

In our example, since the interest is compounded annually ($C/Y = 1$), the periodic interest rate (i) is 10%, and the number of compounding periods (N) for each payment is the number of years from the payment date until the end of the fifth year. The present value (PV) for each payment is \$1,000.

$$\begin{aligned}
 \text{Total FV} &= 1000(1 + 0.1)^0 + 1000(1 + 0.1)^1 + 1000(1 + 0.1)^2 \\
 &\quad + 1000(1 + 0.1)^3 + 1000(1 + 0.1)^4 \\
 &= 1000 + 1100 + 1210 + 1331 + 1464.10 \\
 &= \$6105.10
 \end{aligned}$$

Therefore, the total accumulated value from investing \$1,000 at the end of each year for five years amounts to \$6,105.10.

Now, consider a different scenario where you deposit \$1,000 monthly for 30 years. This would result in 360 payments, and calculating the future value for each payment, as done in the first example, would be impractical due to its time-consuming nature. For such cases, we need a more straightforward method to compute the future value for annuities. The uniformity and periodic nature of the payments enable us to use a simplified formula for this calculation.

$$FV = PMT \left[\frac{(1 + i)^N - 1}{i} \right] \quad \text{Formula 3.5a}$$

(See how the formula is derived)

where PMT represents the size of the periodic payment, i is the periodic interest rate (calculated as $i = \frac{I/Y}{C/Y}$, Formula 2.1a), and N is the number of payments in the annuity term (computed as $N = P/Y \cdot t$, Formula 3.1a).

B. Computing Interest Amount

When the future value of an annuity is known or needs to be calculated, particularly in investment contexts, it includes both the total payments made and interest earned over the term of the annuity. To calculate the total interest earned, you should deduct the sum of all payments (calculated as $PMT \cdot N$) from the future value. As previously introduced in Section 3.2, the amount of interest earned can be calculated using

$$I = FV - (N \cdot PMT) \quad \text{Formula 3.2}$$

If there is an initial investment at the start of the annuity (a nonzero Present Value, PV), you need to modify the calculation. In this scenario, the future value (FV) includes the initial investment, all subsequent payments, and interest earned over the term of the annuity. Therefore, the formula to calculate the interest earned is adjusted to

$$I = FV - (N \cdot PMT) - PV \quad \text{Formula 3.3}$$

Example 3.7.1: Compute FV of Ordinary Simple Annuity

Lauren deposits \$360 at the end of each month for 12 years in her Registered Retirement Savings Plan (RRSP) account. The interest rate on the RRSP account is 7.5% compounded monthly. **a)** How much will be the accumulated value of her investment? **b)** How much will she have contributed to the account by the end of the term? **c)** How much interest will be earned on the account?

Show/Hide Solution

Given information:

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of each month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Investment Term: $t = 12$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(12) = 144$
- Nominal interest rate: $I/Y = 7.5\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{7.5\%}{12} = 0.625\%$
- Periodic payment: $PMT = \$360$
- The initial balance of the account is zero, so $PV = 0$

a) $FV = ?$

Substituting the values into Formula 3.5a gives

$$\begin{aligned} FV &= PMT \left[\frac{(1+i)^N - 1}{i} \right] \\ FV &= 360 \left[\frac{(1+0.625\%)^{144} - 1}{0.625\%} \right] \\ &= 83,676.891\dots \\ &\approx \$83,676.89 \end{aligned}$$

Thus the accumulated value of her investment will be \$83,676.89.

b) Lauren's contribution to the account is through periodic payments. She will have contributed N payments of size PMT :

$$\text{Contribution through payments} = N \cdot PMT$$

$$= 144(360)$$

$$= \$51,840$$

c) $I = ?$

The difference between FV and total payment (i.e., Lauren's contribution) is the amount of interest earned in the account. By Formula 3.2, we have

$$I = FV - (N \cdot PMT)$$

$$= 83,676.89 - 51,840$$

$$= \$31,836.89$$

Try an Example



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C. Future Value of Ordinary General Annuity

In an Ordinary General Annuity, the payment period and the interest compounding period are NOT the same ($P/Y \neq C/Y$) and the payments are made at the end of the payment period.

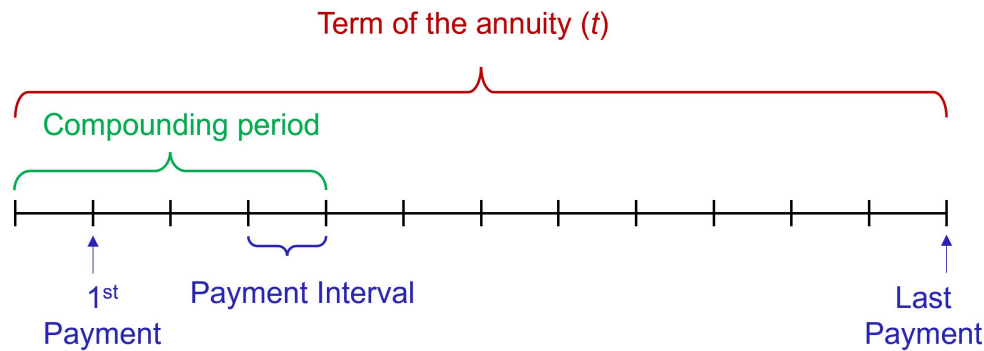


Figure 3.7.3 A timeline for an ordinary general annuity

To calculate the future value of an ordinary general annuity, we can adapt the formula originally developed for the future value of an ordinary simple annuity. However, a minor adjustment is needed before applying this formula. In an ordinary simple annuity, the periodic interest rate corresponds to the interest rate per compounding period, which is the same as the payment period. But for an ordinary general annuity, it's necessary to determine the interest rate per payment period and then incorporate this rate into the future value formula.

The periodic interest rate per payment period is denoted by i_2 and obtained by

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

Then, in the future value formula for an ordinary simple annuity, replace i with i_2 to adjust for an ordinary general annuity.

$$FV = PMT \left[\frac{(1 + i_2)^N - 1}{i_2} \right] \quad \text{Formula 3.6a}$$

Note that if the number of compounding periods per year (C/Y) is equal to the number of payment periods per year (P/Y), then i_2 will be the same as i . In such a case, Formula 3.6 for an ordinary general annuity will be identical to Formula 3.5a for an ordinary simple annuity.

Example 3.7.2: Compute FV of Ordinary General Annuity

Consider a scenario where you deposit \$250 at the end of every three months for 15 years into an account that

offers a 6% interest rate compounded semi-annually. **a)** What will be the maturity value of these deposits at the end of the 15-year period? **b)** How much interest will be earned during the 15-year term?

Show/Hide Solution

Given information:

- Interest is compounded semi-annually so $C/Y = 2$
- Deposits are made at the end of every three months so $P/Y = 4$

$$C/Y \neq P/Y \Rightarrow \text{Ordinary General Annuity}$$

- Investment Term: $t = 15$ years
- Number of payments in the term: $N = P/Y \cdot t = 4(15) = 60$
- Nominal interest rate: $I/Y = 6\%$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{6\%}{2} = 3\%$
- Periodic payment: $PMT = \$250$
- The initial balance of the account is zero, so $PV = 0$

a) $FV = ?$

First, we need to find the interest rate per payment period i_2 :

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$i_2 = (1 + 3\%)^{\frac{2}{4}} - 1$$

$$= 0.0148891\dots$$

Then, by plugging the values into Formula 3.6a, we get the following calculation:

$$FV = PMT \left[\frac{(1 + i_2)^N - 1}{i_2} \right]$$

$$\begin{aligned}
 FV &= 250 \left[\frac{(1 + 0.0148891\dots)^{60} - 1}{0.0148891\dots} \right] \\
 &= 23,964.797\dots \\
 &\approx \$23,964.80
 \end{aligned}$$

Thus the maturity value will be \$23,964.80.

b) $I = ?$

The difference between FV and total deposits is the amount of interest earned in the account. By Formula 3.2, we have

$$\begin{aligned}
 I &= FV - N \cdot PMT \\
 I &= 23,964.80 - 60(250) \\
 &= 23,964.80 - 15,000 \\
 &= \$8,964.80
 \end{aligned}$$

The below figure displays a donut chart with the Future Value (FV) of \$23,964.80 at its center, breaking down into components of interest (I) and total payments ($N \cdot PMT$) around the ring, illustrating how each contributes to the FV .



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Try an Example



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D. Future Value of Simple Annuity Due

In a Simple Annuity Due, the payment period and the interest compounding period are the same ($P/Y = C/Y$) and the payments are made at the beginning of the payment period.

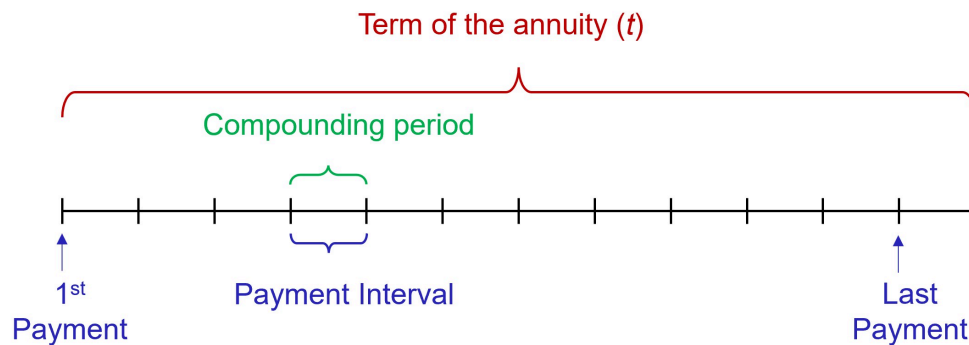


Figure 3.7.4 A timeline for a Simple annuity due.

Consider a scenario we used at the start of this section for an ordinary simple annuity. Now consider this time you invest \$1,000 at the beginning of every year into a savings account that offers a 10% annual interest rate compounded annually over five years. To find out the total amount in your account at the end of these five years, you need to calculate the future value of this annuity. Similar to the previous scenario, to calculate the total accumulated value, we calculate the future value of each payment using the formula for the future value of compound interest (Formula 2.4a). The timeline for this calculation is depicted in Figure 3.7.5.

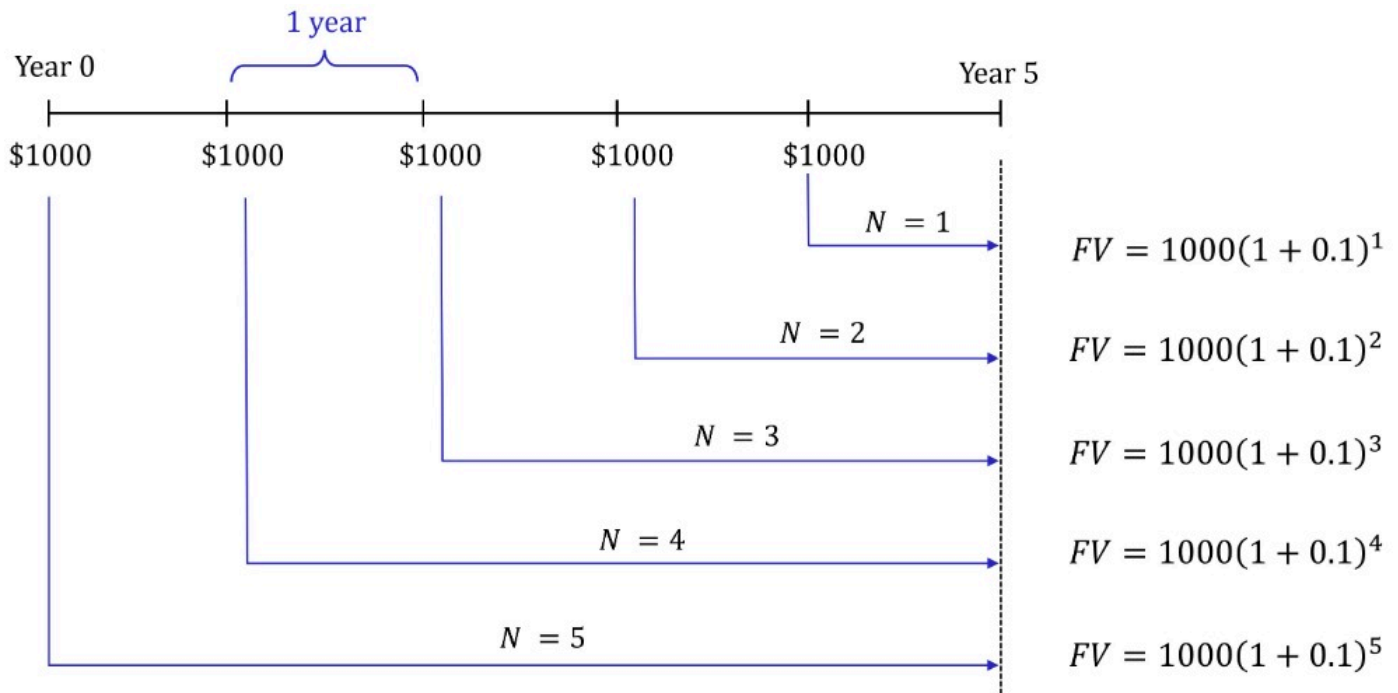


Figure 3.7.5 Future Value of Simple Annuity Due

$$FV = 1000(1 + 0.1)^1 + 1000(1 + 0.1)^2 + 1000(1 + 0.1)^3 + 1000(1 + 0.1)^4 + 1000(1 + 0.1)^5$$

If we factor out $(1 + 0.1)$ from the right-hand side of the equation, we arrive at

$$FV = (1 + 0.1) \left[1000(1 + 0.1)^0 + 1000(1 + 0.1)^1 + 1000(1 + 0.1)^2 + 1000(1 + 0.1)^3 + 1000(1 + 0.1)^4 \right]$$

It is important to note that the expression in the square brackets matches the Future Value (FV) calculated for an ordinary simple annuity in our earlier example. Therefore

$$\begin{aligned} FV &= (1.1)[\$6105.10] \\ &= \$6715.61 \end{aligned}$$

Generally, the future value of a simple annuity due is equal to the future value of an ordinary simple annuity multiplied by a factor of $(1 + i)$. Therefore, the formula for the future value of an annuity due can be expressed as $FV_{\text{due}} = FV_{\text{Ordinary}}(1 + i)$ and is given by

$$FV = PMT \left[\frac{(1 + i)^N - 1}{i} \right] (1 + i)$$

Formula 3.7a

(See how the formula is derived)

Example 3.7.3: Compute FV of Simple Annuity Due

Kian plans to start saving for a sabbatical trip he intends to take in 10 years. To finance his adventure, he decides to make monthly deposits of \$500 at the beginning of each month into a high-yield savings account that offers a 3.24% annual interest rate compounded monthly. Determine the future value of Kian's sabbatical fund at the end of 10 years if he starts to make the deposit at the beginning of the next month.

Show/Hide Solution

Given information:

- Interest is compounded monthly so $C/Y = 12$
- Deposits are made at the beginning of every month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Simple Annuity Due}$$

- Investment term: $t = 10$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(10) = 120$
- Nominal interest rate: $I/Y = 3.24\%$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{3.24\%}{12} = 0.27\% = 0.0027$
- The periodic payment: $PMT = \$500$
- There is no initial lump sum in the fund at the start of the annuity, so $PV = 0$

$$FV = ?$$

Plugging the values into Formula 3.7a results in

$$\begin{aligned}
 FV &= PMT \left[\frac{(1+i)^N - 1}{i} \right] (1+i) \\
 FV &= 500 \left[\frac{(1+0.0027)^{120} - 1}{0.0027} \right] (1+0.0027) \\
 &= 70,939.865\dots \\
 &\approx \$70,939.87
 \end{aligned}$$

The future value of the deposits in Kian's sabbatical fund will be \$70,939.87 at the end of the 10-year term.

Try an Example



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E. Future Value of General Annuity Due

In a General Annuity Due, the payment period and the interest compounding period are different ($P/Y \neq C/Y$) and the payments are made at the beginning of the payment period.

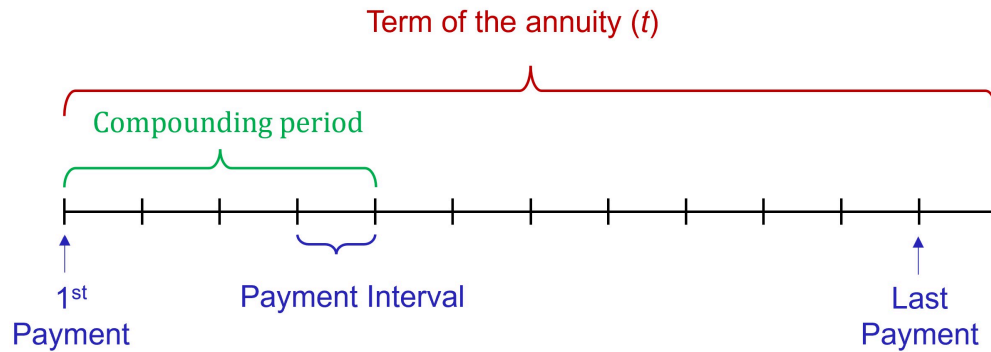


Figure 3.7.6 The timeline and timing of the payments in a general Annuity Due

Just as with the future value of ordinary annuities, the formula initially developed for the future value of a simple annuity due can be modified by including the interest rate per payment period, i_2 , into the future value calculation. Therefore, in the future value formula for the simple annuity due, substitute i with i_2 to make it suitable for calculating the future value of a general annuity due.

$$FV = PMT \left[\frac{(1 + i_2)^N - 1}{i_2} \right] (1 + i_2) \quad \text{Formula 3.8a}$$

where the periodic interest rate per payment period is obtained by

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

Example 3.7.4: Compute FV of General Annuity Due

Adriel made deposits of \$250 in a fund at the beginning of every month for 18 years. The fund was earning interest of 6.12% compounded quarterly. **a)** What was the total amount Adriel deposited in the fund? **b)** What was the accumulated value of the investment? **c)** How much interest was earned in the fund during the term?

Show/Hide Solution

Given information:

- Interest is compounded quarterly so $C/Y = 4$
- Deposits are made at the beginning of every month so $P/Y = 12$

$$C/Y \neq P/Y \Rightarrow \text{General Annuity Due}$$

- Investment Term: $t = 18$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(18) = 216$
- Nominal interest rate: $I/Y = 6.12\%$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{6.12\%}{4} = 1.53\%$
- Periodic payment: $PMT = \$250$
- There is no initial lump sum in the fund at the start of the annuity, so $PV = 0$

a) Total amount Adriel deposited in the fund is N deposits of size PMT :

$$\begin{aligned} \text{Total deposits} &= N \cdot PMT \\ &= 216(250) \\ &= \$54,000 \end{aligned}$$

b) $FV = ?$

First, we need to find the interest rate per payment period i_2 :

$$\begin{aligned} i_2 &= (1 + i)^{\frac{C/Y}{P/Y}} - 1 \\ i_2 &= (1 + 1.53\%)^{\frac{4}{12}} - 1 \\ &= 0.00507420\dots \end{aligned}$$

Then substituting the values into Formula 3.8a gives

$$FV = PMT \left[\frac{(1 + i_2)^N - 1}{i_2} \right]$$

$$\begin{aligned}
 FV &= 250 \left[\frac{(1 + 0.00507420\dots)^{216} - 1}{0.00507420\dots} \right] (1 + 0.00507420\dots) \\
 &= 98,244.204\dots \\
 &\approx \$98,244.20
 \end{aligned}$$

The accumulated value of the deposits in the fund will be \$98,244.20 at the end of the 18-year term.

c) $I = ?$

The difference between FV and total deposits is the amount of interest earned in the account. Using Formula 3.2, we have

$$\begin{aligned}
 I &= FV - N \cdot PMT \\
 I &= 98,244.20 - 216(250) \\
 &= 98,244.20 - 54,000 \\
 &= \$44,244.20
 \end{aligned}$$

The below figure displays a donut chart with the Future Value (FV) of \$98,244.20 at its center, breaking down into components of interest (I) and total payments ($N \cdot PMT$) around the ring, illustrating how each contributes to the FV .



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F. Adjusting for Variable Changes Within the Term

It is important to note that if any key financial variables alter during the term, like the payment amount, the nominal interest rate, or the frequency of compounding, the term must be divided into separate time intervals at the point of each change. The upcoming example demonstrates the procedure for dealing with such adjustments.

Example 3.7.5: Compute FV of Annuity Combined with Compound Interest

\$2,650 was deposited at the end of every six months for 5 years into a fund earning 4.7% compounded semi-annually. After this period, no further deposit was made but the accumulated money was left in the account for another 4 years at the same interest rate.

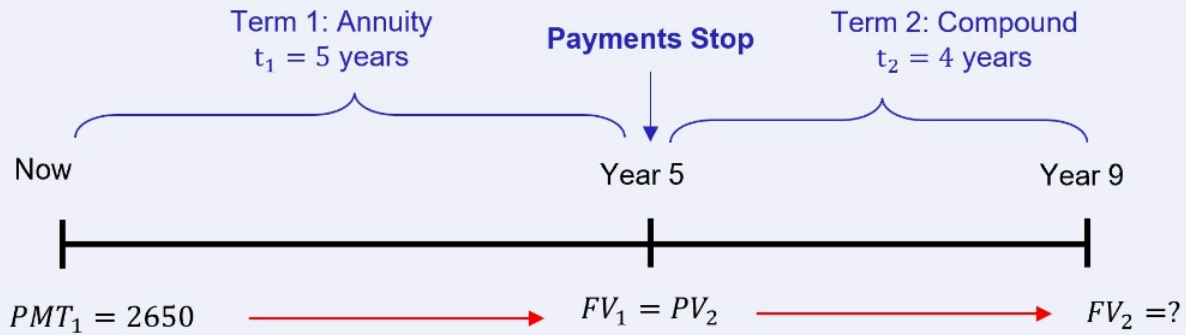
- a) Calculate the accumulated amount at the end of the 9-year period.
- b) Calculate the total amount of interest earned during the 9-year period.

Show/Hide Solution

Given that payments stop midway through the term, we need to divide the investment term into two segments at the point of change, which occurs in year 5. In the initial segment of five years, periodic deposits are made, classifying this period as an annuity term. In the subsequent four years, no additional deposits are made, and the account simply earns interest on the accumulated amount, which is a compound interest term. To approach this calculation:

1. First, calculate the future value of the annuity at the end of the first five-year segment.

2. The amount accumulated at the end of year 5 will then serve as the present value for the following four-year term, during which compound interest will be applied to this sum to determine the final future value at the end of year 9.



a)

Term 1: First 5 years – Ordinary Simple Annuity

Given Information

- Interest is compounded semi-annually so $C/Y = 2$
- Payments are made at the end of every six months so $P/Y = 2$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Investment Term: $t_1 = 5$ years
- Number of payments in the term: $N_1 = P/Y \cdot t_1 = 2(5) = 10$
- Nominal interest rate: $I/Y = 4.7\%$
- The annuity payment: $PMT_1 = \$2650$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{4.7\%}{2} = 2.35\% = 0.0235$

$$FV_1 = ?$$

Substituting the values into Formula 3.5a gives

$$FV_1 = PMT_1 \left[\frac{(1+i)^{N_1} - 1}{i} \right]$$

$$\begin{aligned}
 FV_1 &= 2650 \left[\frac{(1 + 0.0235)^{10} - 1}{0.0235} \right] \\
 &= 29,485.420\dots \text{ (Do not round)}
 \end{aligned}$$

Term 2: Next 4 years – Compound Interest

Given Information

- Present value: $PV_2 = FV_1 = \$29,485.420\dots$
- Investment Term: $t_2 = 4$ years
- Interest is compounded semi-annually so $C/Y = 2$
- Number of compounding periods in the term: $N_2 = C/Y \cdot t_2 = 2(4) = 8$
- Same interest rate
- No periodic payment, so $PMT_2 = 0$

$$FV_2 = ?$$

Substituting the values into the future value formula of compound interest (Formula 2.4a) gives

$$\begin{aligned}
 FV_2 &= PV_2(1 + i)^{N_2} \\
 FV_2 &= 29,485.420\dots(1 + 0.0235)^8 \\
 &= 35,506.682\dots \\
 &\approx \$35,506.68 \text{ (Rounded to the nearest cents)}
 \end{aligned}$$

Thus the accumulated amount at the end of the 9-year period is \$35,506.68.

b) $I = ?$

The total interest earned in the account (I) is the sum of interest earned during the first term (I_1) and the interest earned during the second term (I_2). These interests can be computed individually and then added together.

Alternatively, given that the only additions to the account are the deposits of \$2,650, the total interest can

also be determined by subtracting the total deposit amount from the final accumulated value (FV_2) of the account.

$$I = FV_2 - N_1 \cdot PMT$$

$$I = 35,506.68 - 10(2650)$$

$$= \$9,006.68$$

The below figure displays a donut chart with the final Future Value (FV_2) of \$35,506.68 at its center, breaking down into components of interest (I) and total payments ($N \cdot PMT$) around the ring, illustrating how each contributes to the final accumulated value. The interest is further expanded to show interest earned during the first term (I_1) and interest earned during the second term (I_2).



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G. Annuities with Initial Lump Sum

In our earlier examples, we assumed that the annuities began without any initial investment, meaning the present value (PV) was zero. However, if an annuity starts with an initial lump sum investment, you must enter this amount as the present value (PV) in your calculations. Remember to input the PV as a negative number as it represents a cash outflow.

When the annuity calculation includes an initial lump sum (PV), the future value will include this initial investment, all the periodic payments made thereafter, and the interest that accrues over time. To calculate the total interest earned over the term of the annuity, you need to use Formula 3.3.

Example 3.7.6: Compute FV of an Annuity with Initial Lump Sum Investment (Nonzero PV)

Nancy is diligently preparing for her retirement and has already saved \$15,000 in her 401(k) retirement fund. She is currently 30 years old and aims to retire at 67. To supplement her savings, Nancy begins to contribute \$500 at the end of each month to her 401(k). The account is expected to earn an average interest rate of 7% per year compounded quarterly. **a)** Calculate the future value of Nancy's 401(k) when she retires. **b)** Calculate the total amount of interest that will have been earned on the account by the time Nancy reaches retirement age.

Show/Hide Solution

Given information:

- Payments are made at the **end** of each month, so it is an ordinary annuity: **END**
- Interest is compounded quarterly so $C/Y = 4$
- Deposits are made at the end of every month so $P/Y = 12$

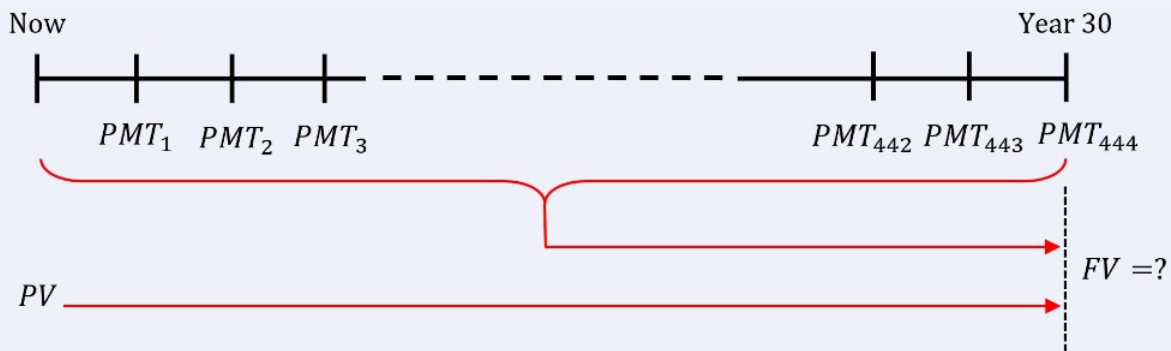
$$C/Y \neq P/Y \Rightarrow \text{Ordinary General Annuity}$$

- Nominal interest rate: $I/Y = 7\%$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{7\%}{4} = 1.75\% = 0.0175$

- Investment Term: $t = 37$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(37) = 444$
- Periodic payment: $PMT = \$500$
- The initial value of \$15,000 at the start of the annuity is the present value: $PV = \$15,000$
- The questions ask for FV and I.

a) $FV = ?$

The total value accumulated at the end of the term is composed of two parts: the sum of the future values of all periodic payments and the future value of the initial lump sum investment. To calculate this, we use the future value formula for an ordinary general annuity for the periodic payments, and the formula for the future value of compound interest for the initial lump sum.



1. *Calculating the future value of payments:*

First, we need to find the interest rate per payment period i_2 :

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$i_2 = (1 + 0.0175)^{\frac{4}{12}} - 1$$

$$= 0.00579963\dots$$

Then, by plugging the values into Formula 3.6a, we get the following calculation:

$$\begin{aligned}
 FV_1 &= PMT \left[\frac{(1 + i_2)^N - 1}{i_2} \right] \\
 FV_1 &= 500 \left[\frac{(1 + 0.00579963\dots)^{444} - 1}{0.00579963\dots} \right] \\
 &= 1,037,521.258\dots
 \end{aligned}$$

2. Calculating the future value of the initial lump sum:

For compound interest, N represents the number of compounding periods in the term. Thus, we need to calculate N before using the compound interest future value formula (Formula 2.4a). Also, for compound interest problems, we use the periodic interest rate per compounding period (i).

$$N = C/Y \cdot t = 4(37) = 148$$

$$\begin{aligned}
 FV_2 &= PV(1 + i)^N \\
 &= 15000(1 + 0.0175)^{148} \\
 &= 195,517.262\dots
 \end{aligned}$$

The future value of Nancy's 401(k) when she retires is obtained by summing the above future values:

$$\begin{aligned}
 FV &= FV_1 + FV_2 \\
 &= 1,037,521.258\dots + 195,517.262\dots \\
 &= 1,233,038.520\dots \\
 &= \$1,233,038.52
 \end{aligned}$$

b) $I = ?$

Because the present value (PV) is not zero, to calculate the amount of interest earned over the 37-year term, we use the modified formula for interest amount (Formula 3.3):

$$I = FV - (N \cdot PMT) - PV$$

$$\begin{aligned} I &= 1,233,038.52 - (444 \cdot 500) - 15000 \\ &= 1,233,038.52 - 222,000 - 15,000 \\ &= \$996,038.52 \end{aligned}$$

Try an Example



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H. Summary of Formulas

If we use the formula for the future value of an ordinary simple annuity as the base, the future value formulas for other types of annuities can be derived with slight modifications to this primary formula as follows:

The FV of Ordinary Simple Annuity:

$$FV = PMT \left[\frac{(1 + i)^N - 1}{i} \right]$$

For FV of Ordinary General Annuity,
replace i with i_2 :

For FV of Simple Annuity Due, multiply
by $(i + 1)$:

For FV of General Annuity Due, multiply
by $(i + 1)$ and replace i with i_2 :

$$FV = PMT \left[\frac{(1 + i_2)^N - 1}{i_2} \right]$$

$$FV = PMT \left[\frac{(1 + i)^N - 1}{i} \right] (i + 1)$$

$$FV = PMT \left[\frac{(1 + i_2)^N - 1}{i_2} \right] (i_2 + 1)$$

Where

$$N = P/Y \cdot t$$

$$i = \frac{I/Y}{C/Y}$$

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

Section 3.7 Exercises

1. Thiago would like to save \$470 at the end of every month for the next 10 years in a savings account at 3.44% compounded annually. **a)** What would be the accumulated value of the investment at the end of the term? **b)** What would be the amount of interest earned?

Show/Hide Answer

a) $FV = \$67,015.84$

b) $I = \$10,615.84$

2. \$4,800 was deposited at the end of every three months for 9 years into a fund earning 2.8% compounded quarterly. After this period, the accumulated money was left in the account for another 4.5 years at the same

interest rate. **a)** Calculate the accumulated amount at the end of the 13.5-year term. **b)** Calculate the total amount of interest earned during the 13.5-year period.

Show/Hide Answer

a) $FV = \$221,936.95$

b) $I = \$49,136.95$

3. Erick made deposits of \$320.41 in a fund at the beginning of every quarter for 7 years. The fund was earning interest of 3.38% compounded semi-annually. **a)** What was the accumulated value of the investment? **b)** What was the total amount Erick deposited in the fund? **c)** How much interest was earned in the fund during the term?

Show/Hide Answer

a) $FV = \$10,153.73$

b) Total deposit = \$8,971.48

c) $I = \$1,182.25$

4. Carolyn is preparing for her retirement and has already saved \$26,500 in her 401(k) retirement fund. She is currently 30 years old and aims to retire at 61. To supplement her savings, Carolyn begins to contribute \$1,520 at the end of every quarter to her 401(k). The account is expected to earn an average interest rate of 5.46% compounded quarterly. **a)** Calculate the future value of the 401(k) fund when Carolyn retires. **b)** Calculate the total amount of interest that will have been earned on the account by the time Carolyn reaches retirement age.

Show/Hide Answer

a) $FV = \$629,167.72$

b) $I = \$414,187.72$

3.8 PV OF ANNUITIES: FORMULA APPROACH

A. Present Value of Ordinary Simple Annuity

In this section, we turn our attention to the present value of annuities, a key concept in valuing the current worth of future payments. The present value of an annuity is crucial in scenarios such as determining how much a series of future cash flows is worth today or evaluating the current worth of a long-term investment or loan. We begin by examining the present value of ordinary annuities, where payments are made at the end of the payment period. In an Ordinary Simple Annuity, the payment period and the interest compounding period are the same ($P/Y = C/Y$), and the payments are made at the end of the payment period.

Consider a scenario where you have the option to receive an end-of-year payment of \$1,000 for the next 5 years from a fund. This fund is expected to yield a 10% annual interest rate compounded annually. To understand the current worth of these future payments, you need to calculate the present value of this annuity. The present value in this context refers to the total value in today's dollars of all the annuity payments you are set to receive over the 5-year term, taking into account the interest rate.

One way to calculate the present value is to calculate the present value of each payment using the formula for the present value of compound interest, $PV = FV(1 + i)^{-N}$ (Formula 2.4c). The timeline for this calculation is depicted in Figure 3.8.1.

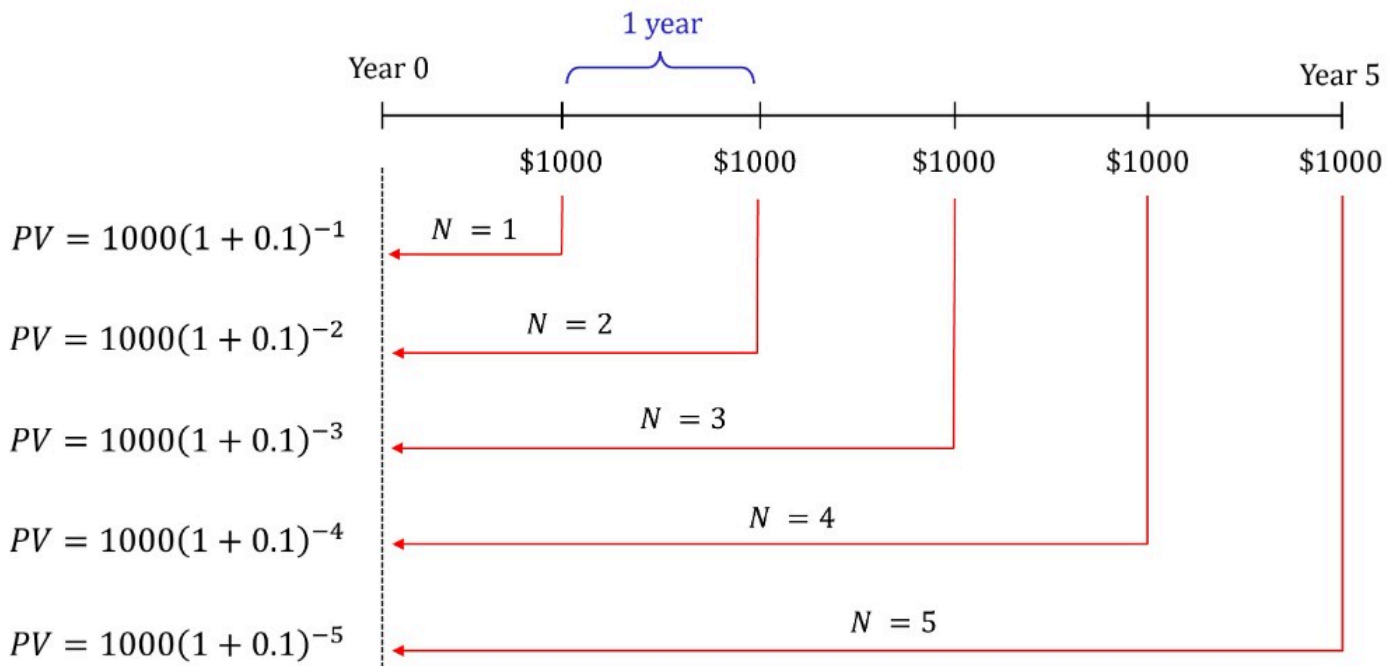


Figure 3.8.1 A Timeline for the Present Value of Ordinary Simple Annuity

In our example, since the interest is compounded annually ($C/Y = 1$), the periodic interest rate (i) is 10%, and the number of compounding periods (N) for each payment equals the number of years from the payment date until the present (i.e., the beginning of the year 0). The future value (FV) for each payment is \$1,000.

$$\begin{aligned}
 PV &= 1000(1 + 0.1)^{-1} + 1000(1 + 0.1)^{-2} + \\
 &1000(1 + 0.1)^{-3} + 1000(1 + 0.1)^{-4} + 1000(1 + 0.1)^{-5} \\
 &= 909.09 + 826.45 + 751.31 + 683.01 + 620.92 \\
 &= \$3790.79
 \end{aligned}$$

Therefore, the required present value in the fund to sustain payments of \$1,000 at the end of each year for five years is \$3790.79.

As noted earlier in the section on future value, calculating the present value for each payment individually, as demonstrated in this example, is impractical due to its time-consuming nature. Therefore, for an ordinary simple annuity, we utilize a more efficient and straightforward method to compute the overall present value.

$$PV = PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] \quad \text{Formula 3.9a}$$

(See how the formula is derived)

where PMT represents the size of the periodic payment, i is the periodic interest rate (calculated as $i = \frac{I/Y}{C/Y}$, Formula 2.1a), and N is the number of payments in the annuity term (computed as $N = P/Y \cdot t$, Formula 3.1a).

B. Computing Interest Amount

In problems where the present value of an annuity is known or is calculated (usually for loan scenarios), the periodic payments of the annuity include interest. Therefore, the amount of interest is obtained by deducting the present value from the total payment amount.

$$I = (N \cdot PMT) - PV \quad \text{Formula 3.4}$$

In this formula, initially introduced in Section 3.3, PV is the present value of the annuity, PMT is the periodic payment amount, and N is the total number of payments, calculated as $N = P/Y \cdot t$ (Formula 3.1a).

Example 3.8.1: Compute PV of Ordinary Simple Annuity

Considering a bank offering an investment opportunity. To provide annual payments of \$1200 at the end of each year for the next 7 years, at a nominal interest rate of 6% compounded annually, what is the required initial deposit that should be made by a customer today?

Show/Hide Solution

Given information

- Interest is compounded annually so $C/Y = 1$
- Payments are made at the end of every year so $P/Y = 1$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Investment Term: $t = 7$ years
- Number of payments in the term: $N = P/Y \cdot t = 1(7) = 7$

- Nominal interest rate: $I/Y = 6\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{6\%}{1} = 6\%$
- Periodic payments: $PMT = \$1200$
- No remaining balance at the end of the annuity term, so $FV = 0$
- $PV = ?$

Substituting the values into Formula 3.9a gives

$$PV = PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right]$$

$$PV = 1200 \left[\frac{1 - (1 + 0.06)^{-7}}{0.06} \right]$$

$$= 6,698.857\dots$$

$$\approx \$6,698.86$$

Therefore, the present value of the annuity is \$6,698.86.

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C. Present Value of Ordinary General Annuity

In an Ordinary General Annuity, the payment period and the interest compounding period are NOT the same ($P/Y \neq C/Y$) and the payments are made at the end of the payment period.

Just as we did with the future value of an ordinary general annuity in the previous section, we can modify the formula for the present value of an ordinary simple annuity by replacing the periodic interest rate per compounding period (i) with the periodic interest rate per payment period (i_2). The periodic interest rate per payment period can be calculated as follows:

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

Therefore, the formula for calculating the present value of an ordinary general annuity is obtained by

$$PV = PMT \left[\frac{1 - (1 + i_2)^{-N}}{i_2} \right] \quad \text{Formula 3.10a}$$

Example 3.8.2: Compute PV Ordinary General Annuity

A 25-year mortgage on a condominium requires payments of \$1000 at the end of each month. If interest is 4% compounded semi-annually, **a)** what was the mortgage principal? **b)** How much interest was charged on the mortgage?

Show/Hide Solution

Given information

- Interest is compounded semi-annually so $C/Y = 2$
- Payments are made at the end of each month so $P/Y = 12$

$C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity

- Mortgage Term: $t = 25$ years

- Number of payments in the term: $N = P/Y \cdot t = 12(25) = 300$
- Nominal interest rate: $I/Y = 4\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{4\%}{2} = 2\%$
- Periodic payments: $PMT = \$1000$
- The mortgage is fully paid by the end of the term (i.e., no remaining balance), so $FV = 0$

a) $PV = ?$

First, we need to find the interest rate per payment period i_2 :

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$i_2 = (1 + 0.02)^{\frac{2}{12}} - 1$$

$$= 0.003305890\dots$$

Substituting the values into Formula 3.10a gives

$$PV = PMT \left[\frac{1 - (1 + i_2)^{-N}}{i_2} \right]$$

$$PV = 1000 \left[\frac{1 - (1 + 0.003305890)^{-300}}{0.003305890} \right]$$

$$= 190,106.773\dots$$

$$\approx \$190,106.77$$

Thus, the mortgage principal was \$190,106.77.

b) $I = ?$

The amount of interest for loans is given by Formula 3.4.

$$\begin{aligned}
 I &= (N \cdot PMT) - PV \\
 &= 1000(300) - 190,106.77 \\
 &= 300,000 - 190,106.77 \\
 &= \$109,893.23
 \end{aligned}$$

Try an Example



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D. Present Value of Simple Annuity Due

In a Simple Annuity Due, the payment period and the interest compounding period are the same ($P/Y = C/Y$), and the payments are made at the beginning of the payment period.

Consider a scenario we used at the start of this section for an ordinary simple annuity. This time, you receive payments of \$1,000 from a fund at the beginning of every year for the next 5 years. This fund is expected to yield a 10% annual interest rate compounded annually. To understand the current worth of these future payments, you need to calculate the present value of this annuity. Similar to the previous scenario, to calculate the total present value, we calculate the present value of each payment using the formula for the present value of compound interest (Formula 2.4c). The timeline for this calculation is shown in Figure 3.8.2.

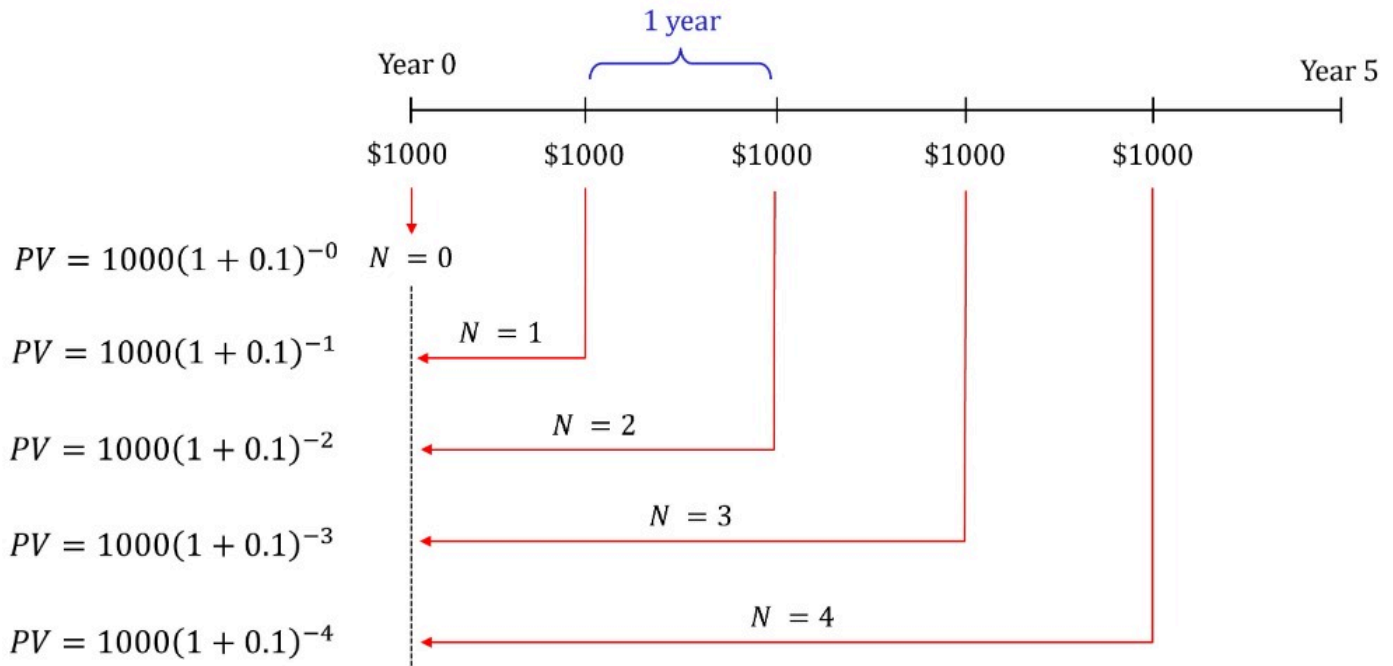


Figure 3.8.2 A Timeline for the Present Value of Simple Annuity Due

$$PV = 1000(1 + 0.1)^0 + 1000(1 + 0.1)^{-1} + 1000(1 + 0.1)^{-2} + 1000(1 + 0.1)^{-3} + 1000(1 + 0.1)^{-4}$$

If we factor out $(1 + 0.1)$ from the right-hand side of the equation, we obtain

$$PV = (1 + 0.1) \left[1000(1 + 0.1)^{-1} + 1000(1 + 0.1)^{-2} + 1000(1 + 0.1)^{-3} + 1000(1 + 0.1)^{-4} + 1000(1 + 0.1)^{-5} \right]$$

Notice that the expression in the square brackets matches the Present Value (PV) calculated for an ordinary simple annuity in our previous example. Thus,

$$\begin{aligned} PV &= (1.1)[\$3790.79] \\ &= \$4169.87 \end{aligned}$$

Similar to the case with future values, the present value of a simple annuity due equals the present value of an ordinary simple annuity multiplied by a factor of $(1 + i)$. Therefore, the formula for the future value of an annuity due can be expressed as $PV_{\text{due}} = PV_{\text{Ordinary}}(1 + i)$. This formula is defined as follows

$$PV = PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] (1 + i)$$

Formula 3.11a

(See how the formula is derived)

Example 3.8.3: Compute PV of Simple Annuity Due

Cai bought an appliance for a new home and opted for a payment plan. Under this plan, he will pay \$490 at the beginning of each month for 3 years. The interest rate on this payment plan is 6.67%, compounded monthly. **a)** Determine the cash value of the appliance. **b)** Calculate the total amount of interest charged to Cai throughout the payment plan.

Show/Hide Solution

Given information

- Interest is compounded monthly, so $C/Y = 12$
- Payments are made at the beginning of every month, so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Simple Annuity due}$$

- Time period: $t = 3$ years
- Number of payments in the term: $N = (P/Y) \cdot t = 12(3) = 36$
- Nominal Interest rate: $I/Y = 6.67\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{6.67\%}{12} = 0.005558\bar{3}$
- Size of the periodic payment: $PMT = \$490$

a) $PV = ?$

Substituting the values into Formula 3.11a gives

$$PV = PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] (1 + i)$$

$$PV = 490 \left[\frac{1 - (1 + 0.005558\bar{3})^{-36}}{0.005558\bar{3}} \right] (1 + 0.005558\bar{3})$$

$$\approx \$16,035.82$$

Therefore, the cash value of the appliance is \$16,035.82.

b) $I = ?$

For loans, PMT includes interest, and the amount of interest is given by Formula 3.4.

$$I = (PMT \cdot N) - PV$$

$$I = (490 \times 36) - 16,035.82$$

$$= 17,640.00 - 16,035.82$$

$$= \$1,604.18$$

Try an Example



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E. Present Value of General Annuity Due

In a General Annuity Due, the payment period and the interest compounding period are different ($P/Y \neq C/Y$) and the payments are made at the beginning of the payment period.

Just as with the present value of ordinary annuities, the formula initially developed for the present value of a simple annuity due can be modified by including the interest rate per payment period, i_2 , into the present value calculation. Thus, in the present value formula for the simple annuity due, replace i with i_2 to make it applicable for calculating the present value of a general annuity due.

$$PV = PMT \left[\frac{1 - (1 + i_2)^{-N}}{i_2} \right] (1 + i_2) \quad \text{Formula 3.12a}$$

Example 3.8.4: Compute PV of General Annuity Due

Shayan received an inheritance which was placed in a savings account. This account pays him \$1,000 at the beginning of each week for 30 years. The account accrues interest at a rate of 3% per year compounded monthly. What was the amount of the inheritance?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of each week so $P/Y = 52$

$C/Y \neq P/Y \Rightarrow$ General Annuity Due

- Annuity Term: $t = 30$ years
- Number of payments in the term: $N = P/Y \cdot t = 52(30) = 1560$
- Nominal interest rate: $I/Y = 3\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{3\%}{12} = 0.25\%$

- Periodic payments: $PMT = \$1000$
- Since the fund will be fully exhausted by the end of the term, $FV = 0$

a) $PV = ?$

First, we need to find the interest rate per payment period i_2 :

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$i_2 = (1 + 0.0025)^{\frac{12}{52}} - 1$$

$$= 0.000576369\dots$$

Substituting the values into Formula 3.12a gives

$$PV = PMT \left[\frac{1 - (1 + i_2)^{-N}}{i_2} \right] (1 + i_2)$$

$$PV = 1000 \left[\frac{1 - (1 + 0.000576369\dots)^{-1560}}{0.000576369\dots} \right] (1 + 0.000576369\dots)$$

$$= 1,029,401.416\dots$$

$$\approx \$1,029,401.42$$

The amount of inheritance was \$1,029,401.42.

Try an Example



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F. Present Value Problems with Down Payment

A down payment refers to a relatively small initial payment made by a buyer at the time of purchasing a significant asset, such as real estate property. This payment is made upfront and the remaining balance of the purchase price is typically covered through a loan obtained from a financial institution. The amount of this loan corresponds to the present value (PV) of the future periodic payments that the buyer will make to pay off the loan. These periodic payments include not only the principal amount but also the interest charged on the loan. The relationship among the purchase price (also known as the cash price), the loan amount (PV), and the down payment can be expressed as follows:

$$\text{Purchase Price} = \text{Down Payment} + PV \qquad \text{Formula 3.5}$$

It is worth noting that this formula was first introduced in Section 3.3.

Example 3.8.5: Compute PV and Purchase Price with Known Down Payment

Andy made a down payment of \$40,000 on an apartment and secured a mortgage for the rest of the purchase price. He has agreed to repay this mortgage with end-of-month payments of \$1,580 for 30 years at a 3.45% annual interest rate compounded monthly.

- a) Calculate the original purchase price of the apartment.
- b) Determine the total amount Andy will have paid by the end of the 30-year mortgage term.
- c) Calculate the total amount of interest Andy will have paid on the mortgage over the 30 years.

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of every month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Mortgage term: $t = 30$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(30) = 360$
- Nominal interest rate: $I/Y = 3.45\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{3.45\%}{12} = 0.2875\%$
- Periodic payments: $PMT = \$1580$
- Down payment = \$40,000
- The mortgage is fully paid off by the end of the term, so $FV = 0$

a) To determine the purchase price of the apartment, we first need to calculate the mortgage amount. This is done by determining the present value of the monthly payments Andy is scheduled to make.

Computing PV

Substituting the values into Formula 3.9a gives

$$\begin{aligned} PV &= PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] \\ PV &= 1580 \left[\frac{1 - (1 + 0.2875\%)^{-360}}{0.2875\%} \right] \\ &= 354,055.043... \\ &\approx \$354,055.04 \end{aligned}$$

Thus, the mortgage principal was \$354,055.04.

Calculating the purchase price

By substituting the values of the down payment and the present value (PV) into Formula 3.5, we can calculate the total purchase price.

$$\begin{aligned}\text{Purchase Price} &= \text{Down Payment} + PV \\ &= 40,000 + 354,055.04 \\ &= \$394,055.04\end{aligned}$$

Therefore, the purchase price of the apartment was \$394,055.04.

b) The total amount that Andy will pay over the 30-year term of the mortgage is equal to the number of payments (N) multiplied by the size of each payment (PMT).

$$\begin{aligned}\text{Total amount paid to repay loan} &= N \cdot PMT \\ &= 360(1580) \\ &= \$568,800\end{aligned}$$

c) The amount of interest for loans is given by Formula 3.4.

$$\begin{aligned}I &= (N \cdot PMT) - PV \\ &= 568,800 - 354,055.04 \\ &= \$214,744.96\end{aligned}$$

Note: When calculating the interest charged on the mortgage, we use the present value (PV) of the loan, not the purchase price of the property.

The below figure displays a donut chart with the total payments ($N \cdot PMT$) of \$741,000 at its center, breaking down into components of interest (I) and the loan principal (PV) around the ring.



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Example 3.8.6: Compute PV and Purchase Price with Unknown Down Payment

Bijan made a 20% down payment on the purchase of a vacation property and obtained a mortgage from a bank to cover the remaining cost. He has arranged to repay this mortgage with end-of-week payments of \$570 for 25 years. The interest charged on the mortgage is 3.05% compounded monthly.

- a) What was the purchase price of the property?
- b) What was the total amount paid over the 25-year term to repay the mortgage?
- c) How much interest was charged on the mortgage?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of the week so $P/Y = 52$

$$C/Y \neq P/Y \Rightarrow \text{Ordinary General Annuity}$$

- Mortgage term: $t = 25$ years
- Number of payments in the term: $N = P/Y \cdot t = 52(25) = 1300$
- Nominal interest rate: $I/Y = 3.05\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{3.05\%}{12} = 0.2541\bar{6}\%$
- Periodic payments: $PMT = \$570$
- Down payment = 20% of the purchase price
- The mortgage is fully paid off by the end of the term, so $FV = 0$

a) To determine the purchase price, we first need to calculate the mortgage amount, which is done by determining the present value of the weekly payments Bijan is scheduled to make.

Computing PV

First, we need to find the interest rate per payment period i_2 :

$$\begin{aligned} i_2 &= (1 + i)^{\frac{C/Y}{P/Y}} - 1 \\ i_2 &= (1 + 0.002541\bar{6})^{\frac{12}{52}} - 1 \\ &= 0.000585965\dots \end{aligned}$$

Substituting the values into Formula 3.10a gives

$$PV = PMT \left[\frac{1 - (1 + i_2)^{-N}}{i_2} \right]$$

$$\begin{aligned}
 PV &= 570 \left[\frac{1 - (1 + 0.000585965\dots)^{-1300}}{0.000585965} \dots \right] \\
 &= 518,525.660\dots \\
 &\approx \$518,525.66
 \end{aligned}$$

Thus, the mortgage amount was \$518,525.66.

Calculating the purchase price

Since the down payment is given as a percentage of the unknown purchase price, we need to express the down payment in terms of the purchase price when applying Formula 3.5. To simplify this, let's use 'X' to denote the purchase price. Thus, the down payment can be represented as 20% of X, or 0.2X.

$$\text{Purchase Price} = \text{Down Payment} + PV$$

$$X = 0.2X + 518,525.66$$

$$X - 0.2X = 518,525.66$$

$$0.8X = 518,525.66$$

$$X = \frac{518,525.66}{0.8}$$

$$= \$648,157.08$$

Therefore, the purchase price of the vacation property was \$648,157.08.

b) The total amount that Bijan will pay over the 25-year term of the mortgage is equal to the number of payments (N) multiplied by the size of each payment (PMT).

$$\text{The total amount paid to repay the loan} = N \cdot PMT$$

$$= 570(1300)$$

$$= \$741,000$$

c) The amount of interest for loans is given by Formula 3.4.

$$\begin{aligned} I &= N \cdot PMT - PV \\ &= 741,000 - 518,525.66 \\ &= \$222,474.34 \end{aligned}$$

The below figure displays a donut chart with the total payments ($N \cdot PMT$) of \$741,000 at its center, breaking down into components of interest charged (I) and the loan principal (PV) around the ring.



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G. Lease and Residual Value

A **lease** is a contractual agreement where one party, the lessee, pays the lessor (the owner) for the use of an asset over a specified period. Common in real estate, vehicles, and equipment, leases often include terms that specify payments, duration, and conditions of use. A key component of many lease agreements, particularly in auto leasing, is the

residual value. This term refers to the projected value of the leased asset at the end of the lease term. It's an estimate of the asset's worth after depreciation over the lease period.

When calculating the present value (PV) of lease payments, the residual value plays a significant role. The present value of a lease is essentially the sum of the discounted values of all lease payments and the discounted residual value. Therefore, the calculation of PV in leasing scenarios must account for both the periodic payments and the residual value. This approach helps in comparing different lease options. For lease calculations, the residual value is often treated as a future value (FV). Since the residue value represents the value of the asset (such as a car) that needs to be returned to the owner at the end of the term, it is considered a cash outflow and should be entered as a negative value in the financial calculator.

Example 3.8.7: Compute PV of Annuity Due with Residual Value

Patricia's car lease agreement entails monthly payments of \$190.00, due at the start of each month for 3 years. At the end of the lease term, she can choose to either return the car or buy it for a residual value of \$14,600. The lease is subject to a 2.55% interest rate compounded monthly. Determine the original cash value of the car based on this lease agreement.

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of each month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Simple Annuity Due}$$

- Lease Term: $t = 3$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(3) = 36$
- Nominal interest rate: $I/Y = 2.55\%$
- Periodic interest rate = $i = \frac{I/Y}{C/Y} = \frac{2.55\%}{12} = 0.2125\%$
- Periodic payments: $PMT = \$190$
- The residual value: $FV = \$14,600$

To determine the original cash value of the car, we need to calculate the present value (PV) of both the lease payments and the residual value. The car's cash value is equal to the combined present value of these amounts:

$$\text{Cash value} = PV_{\text{PMT}} + PV_{\text{Residual}}$$

Finding PV_{PMT} :

Substituting the values into Formula 3.11a gives

$$PV = PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] (1 + i)$$

$$PV = 190 \left[\frac{1 - (1 + 0.002125)^{-36}}{0.002125} \right] (1 + 0.002125)$$

$$\approx 6,592.171$$

Finding PV_{Residual} (Compound Interest):

In compound interest problems, N represents the number of compounding periods within the term, calculated as $N = C/Y \cdot t = 12(3) = 36$. In this particular problem, N coincidentally matches the N used for the Present Value (PV) of an annuity. This is because the annuity is a simple annuity where the number of compounding periods per year (C/Y) equals the number of payment periods per year (P/Y). Also for compound problems, P/Y should be set to the value of C/Y .

Substituting the values into Formula 2.4c yields

$$PV = FV(1 + i)^{-N}$$

$$PV = 14,600(1 + 0.002125)^{-36}$$

$$\approx 13,525.850$$

Therefore, the cash value of the car will be the sum of those present values:

$$\begin{aligned}
 \text{Cash value} &= PV_{\text{PMT}} + PV_{\text{Residual}} \\
 &= 6,592.171 + 13,525.850 \\
 &= \$20,118.02
 \end{aligned}$$

Try an Example



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Example 3.8.8: Compute PV of Annuity Due with Residual Value and Down Payment

Slavica has a car lease agreement that requires an initial down payment of \$10,500 and lease payments of \$850 at the beginning of every month for 4 years. At the end of 4 years, she has the option to either return the car or purchase it for a residual value of \$21,000. If the interest charged on the lease is 3.4% compounded monthly, what was the cash value of the car?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of each month so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Simple Annuity Due}$$

- Lease Term: $t = 4$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(4) = 48$
- Nominal interest rate: $I/Y = 3.4\%$
- Periodic interest rate = $i = \frac{I/Y}{C/Y} = \frac{3.4\%}{12} = 0.28\bar{3}\%$
- Periodic payments: $PMT = \$850$
- Down Payment = \$10,500
- Residual value: $FV = \$21,000$

To determine the original cash value of the car, we need to calculate the present value (PV) of both the lease payments and the residual value. The car's cash value is equal to the combined present value of these amounts plus the down payment:

$$\text{Cash value} = \text{Down} + PV_{PMT} + PV_{\text{Residual}}$$

Finding PV_{PMT} :

Substituting the values into Formula 3.11a gives

$$\begin{aligned}
 PV &= PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] (1 + i) \\
 PV &= 850 \left[\frac{1 - (1 + 0.0028\bar{3})^{-48}}{0.0028\bar{3}} \right] (1 + 0.0028\bar{3}) \\
 &\approx 38,204.791
 \end{aligned}$$

Finding PV_{Residual} (Compound Interest):

- $N = C/Y \cdot t = 12(4) = 48$
- $P/Y = C/Y = 12$

Substituting the values into Formula 2.4c yields

$$\begin{aligned}
 PV &= FV(1 + i)^{-N} \\
 PV &= 21,000(1 + 0.0028\bar{3})^{-48} \\
 &\approx 18,333.223
 \end{aligned}$$

Substituting the down payment and PVs yields

$$\begin{aligned}
 \text{Cash value} &= \text{Down} + PV_{\text{PMT}} + PV_{\text{Residual}} \\
 &= 10,500 + 38,204.791 + 18,333.223 \\
 &= 67,038.01
 \end{aligned}$$

Try an Example



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Section 3.8 Exercises

1. A mortgage with a 30-year term has monthly payments of \$5,896.97, due at the end of each month. The interest rate on this mortgage is 3.02%, compounded semi-annually. **a)** Calculate the original principal amount of the mortgage. **b)** Determine the total amount of interest that will be charged over the life of the mortgage.

Show/Hide Answer

a) $PV = \$1,398,488.92$

b) $I = \$724,420.28$

2. Trinity paid \$30,400 as a down payment toward a vacation property purchase and received a mortgage from a bank for the remaining amount. She agreed to pay end-of-month payments of \$3,005.14 for 15 years to repay the mortgage. The interest charged on the mortgage was 3.72% compounded annually. **a)** Calculate the amount of the mortgage. **b)** What was the purchase price of the property? **c)** What was the total amount paid over the 15-year term to repay the mortgage? **d)** How much interest was charged on the mortgage?

Show/Hide Answer

a) $PV = \$415,839.19$

b) Purchase price = \$446,239.19

c) Total amount paid = \$540,925.20

d) $I = \$125,086.01$

3. Carmen inherited a sum of money that was invested in a savings account providing her with \$613.29 at the beginning of each quarter for 12 years. The account earned interest of 5.46% compounded semi-annually. **a)** What was the amount of the inheritance? **b)** How much interest was earned in the account during the term?

Show/Hide Answer

a) $PV = \$21,827.09$

b) $I = \$7,610.83$

4. Cheyenne has a car lease agreement that requires an initial down payment of \$8,900 and lease payments of \$260.00 at the beginning of every month for 5 years. At the end of 5 years, she has the option to either return the car or purchase it for a residual value of \$14,800. If the interest charged on the lease is 2.65% compounded semi-annually, what was the cash value of the car?

Show/Hide Answer

Cash value of the car = \$36,507.58

3.9 PMT OF ANNUITIES: FORMULA APPROACH

A. Payment of Ordinary Simple Annuity

In this section, we focus on calculating the periodic payment amount (PMT) required for an annuity. This payment is the amount that is either paid or received regularly throughout the term of the annuity. Depending on the scenario, whether it involves retirement plans, loans, or savings schemes, the calculation of PMT may be based on the Future Value (FV), the Present Value (PV) of the annuity, or a combination of both.

If we know the future value of an annuity, the number of payment periods, and the periodic interest rate, we can apply the future value of the ordinary simple annuity formula (Formula 3.5a) to determine the unknown periodic payment (PMT).

$$PMT = \frac{FV \cdot i}{(1 + i)^N - 1} \quad \text{Formula 3.5b}$$

Likewise, when the present value of an annuity, the number of payment periods, and the periodic interest rate are known, the present value formula (Formula 3.9a) can be rearranged to solve for the unknown periodic payment PMT.

$$PMT = \frac{PV \cdot i}{1 - (1 + i)^{-N}} \quad \text{Formula 3.9b}$$

Example 3.9.1: Compute PMT Given FV of Ordinary Simple Annuity

What deposit made at the end of each month will accumulate to \$12,000 in 6 years at 4.8% p.a. compounded monthly?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of each month so $P/Y = 12$
- $C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity
- Annuity term: $t = 6$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(6) = 72$
- Nominal interest rate: $I/Y = 4.8\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{4.8\%}{12} = 0.4\%$
- The initial balance of the account is zero, so $PV = 0$
- Future value of investments: $FV = \$12,000$

$PMT = ?$

Given the future value is known, we use Formula 3.5b to calculate the size of the payments:

$$PMT = \frac{FV \cdot i}{(1 + i)^N - 1}$$

$$PMT = \frac{12,000(0.004)}{(1 + 0.004)^{72} - 1}$$

$$= 144.147\dots$$

$$\approx \$144.15$$

The size of the deposits made must be \$144.15.

Try an Example



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Example 3.9.2: Compute PMT Given PV of Ordinary Simple Annuity

What payment is required at the end of each month for 8 years to repay a \$32,000 loan if the interest charged is 3.8% compounded monthly?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the end of each month so $P/Y = 12$
- $C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity
- Annuity term: $t = 8$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(8) = 96$
- Nominal interest rate: $I/Y = 3.8\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{3.8\%}{12} = 0.31\bar{6}\%$
- Loan principal: $PV = \$32,000$
- No remaining balance at the end of the annuity term, so $FV = 0$

$PMT = ?$

Considering the present value is known, we use Formula 3.9b to find the size of the payments:

$$PMT = \frac{PV \cdot i}{1 - (1 + i)^{-N}}$$

$$\begin{aligned}
 PMT &= \frac{32,000(0.0031\bar{6})}{1 - (1 + 0.0031\bar{6})^{-96}} \\
 &= 387.086... \\
 &\approx \$387.09
 \end{aligned}$$

Thus, the payments of \$387.09 are required to fully repay the loan.

Try an Example



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B. Payment of Ordinary General Annuity

As we did with the ordinary simple annuity, if the future value of an annuity, the number of payment periods, and the periodic interest rate are known, we can use the future value formula of an ordinary general annuity (Formula 3.6a) to calculate the unknown periodic payment, PMT.

$$PMT = \frac{FV \cdot i_2}{(1 + i_2)^N - 1} \quad \text{Formula 3.6b}$$

Similarly, if we have information about the present value of an annuity, along with the number of payment periods and the periodic interest rate, we can rearrange the present value formula (Formula 3.10a) to find the unknown periodic payment PMT.

$$PMT = \frac{PV \cdot i_2}{1 - (1 + i_2)^{-N}} \quad \text{Formula 3.10b}$$

Here i_2 is the interest rate per payment period

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

Example 3.9.3: Compute PMT Given FV of Ordinary General Annuity

Dakota is planning for a renovation project that will cost \$95,000. To finance this, she intends to save money in her savings account earning an interest rate of 4.12% compounded quarterly. **a)** Determine the amount Dakota needs to deposit at the end of each month to reach her goal of \$95,000 in 5 years. **b)** Calculate the total amount Dakota will have deposited into the account over this 5-year period. **c)** Determine the total interest Dakota will earn on her savings account.

Show/Hide Solution

Given information

- Interest is compounded quarterly so $C/Y = 4$
- Payments are made at the end of each month so $P/Y = 12$
- $C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity
- Annuity term: $t = 5$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(5) = 60$
- Nominal interest rate: $I/Y = 4.12\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{4.12\%}{4} = 1.03\%$
- The initial balance of the account is zero, so $PV = 0$
- Future value of investment: $FV = \$95,000$

a) $PMT = ?$

First, we need to find the interest rate per payment period i_2 :

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$\begin{aligned}
 i_2 &= (1 + 0.0103)^{\frac{4}{12}} - 1 \\
 &= 0.00342161\dots
 \end{aligned}$$

Given the future value is known, we then use Formula 3.6b to calculate the size of the payments:

$$\begin{aligned}
 PMT &= \frac{FV \cdot i_2}{(1 + i_2)^N - 1} \\
 PMT &= \frac{95,000(0.00342161\dots)}{(1 + 0.00342161\dots)^{60} - 1} \\
 &= 1429.061\dots \\
 &\approx \$1429.06
 \end{aligned}$$

Dakota will need to make monthly deposits of \$1429.06 to accumulate her desired amount.

b) The total amount that Dakota will have deposited over the 5-year term is equal to the number of payments (N) multiplied by the size of each payment (PMT).

$$\begin{aligned}
 \text{Total amount deposited} &= N \cdot PMT \\
 &= 60(1429.06) \\
 &= \$85,743.60
 \end{aligned}$$

c) $I = ?$

Given this is an investment scenario where FV is known, the total interest amount earned in the savings account is given by Formula 3.2.

$$\begin{aligned}
 I &= FV - (N \cdot PMT) \\
 &= 95,000 - 85,743.60
 \end{aligned}$$

$$= \$9256.40$$

Try an Example



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=558>

Example 3.9.4: Compute PMT Given Purchase Price

Jeff purchased a car listed for \$29,900. He paid 20% of the cost as a down payment and financed the balance amount at 5.1% compounded monthly for 12 years. **a)** What is the size of payment made at the end of every three months to settle the loan? **b)** How much was the amount of interest charged?

Show/Hide Solution

Given information

- Interest is compounded monthly, so $C/Y = 12$
- Payments are made at the end of every three months so $P/Y = 4$
- $C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity
- Annuity term: $t = 12$ years
- Number of payments in the term: $N = P/Y \cdot t = 4(12) = 48$
- Nominal interest rate: $I/Y = 5.1\%$

- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{5.1\%}{12} = 0.425\%$
- Purchase price = \$29,900
- Down payment = 20% of the purchase price
- The loan is fully settled, so $FV = 0$

a)

Finding PV of loan

First, we need to find the amount of the loan, which is the remaining balance after paying the down payment.

$$\text{Down payment} = 20\% (29,900) = \$5980$$

Using Formula 3.5, we have

$$\text{Purchase Price} = \text{Down payment} + PV$$

Rearranging the equation for PV gives

$$PV = \text{Purchase Price} - \text{Down payment}$$

$$PV = 29,900 - 5980$$

$$= \$23,920$$

Next, we use the PV and other given values to compute the size of the payments. We first need to find the interest rate per payment period i_2 :

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$i_2 = (1 + 0.00425)^{\frac{12}{4}} - 1$$

$$= 0.0128042\dots$$

Substituting the values into Formula 3.10b, we obtain

$$\begin{aligned}
 PMT &= \frac{PV \cdot i_2}{1 - (1 + i_2)^{-N}} \\
 PMT &= \frac{23,920(0.0128042\dots)}{1 - (1 + 0.0128042\dots)^{-48}} \\
 &= 670.146\dots \\
 &\approx \$670.15
 \end{aligned}$$

The size of the quarterly payments is \$670.15.

b) The amount of interest for loans is given by Formula 3.4:

$$\begin{aligned}
 I &= N \cdot PMT - PV \\
 &= 48(670.15) - 23,920 \\
 &= 32,167.2 - 23,920 \\
 &= \$8,247.20
 \end{aligned}$$

Example 3.9.5: Compute PMT Given Both FV and PV

Inesh has already saved \$11,600 in his savings account as of today, and he plans to contribute equal deposits at the end of each month for the next 10 years. What monthly deposit is required to accumulate \$105,000 in total at the end of the 10 years assuming the account earns 5.2% compounded annually?

Show/Hide Solution

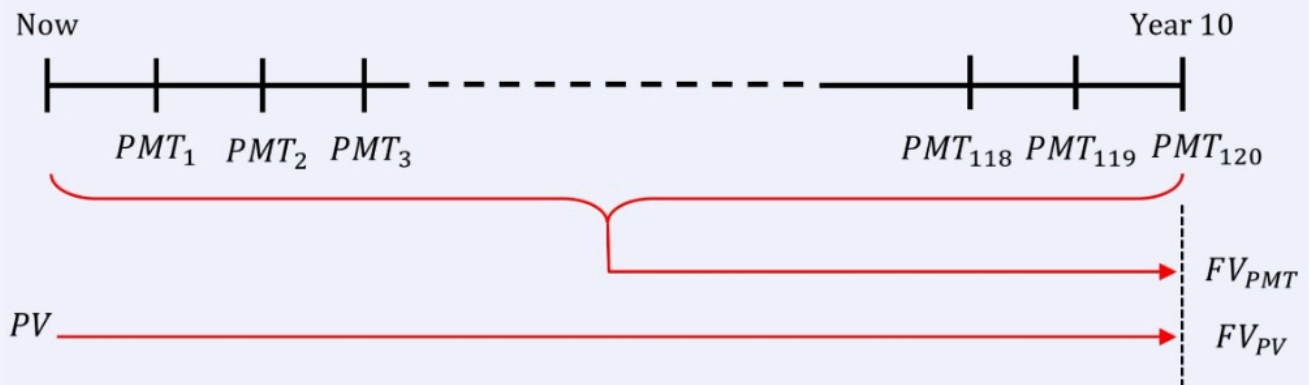
Given information

- Interest is compounded quarterly so $C/Y = 1$
- Payments are made at the end of each month so $P/Y = 12$
- $C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity
- Investment term: $t = 10$ years
- Number of payments in the term: $N_1 = P/Y \cdot t = 12(10) = 120$
- Nominal interest rate: $I/Y = 5.2\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{5.2\%}{1} = 5.2\%$
- The desired accumulated value: $FV = \$105,000$
- The initial lump sum investment: $PV = \$11,600$

$PMT = ?$

The accumulated value at the end of the term consists of two components: the future values of all the periodic payments (FV_{PMT}) and the future value of the initial lump sum investment (FV_{PV}).

$$FV = FV_{PMT} + FV_{PV}$$



To calculate the size of each periodic payment, we must first determine FV_{PMT} , the future value of these payments. Knowing that the combined future value of the periodic payments and the initial lump sum equals the target accumulation of \$105,000, our first step is to calculate the future value of the initial lump sum (FV_{PV}). This amount is then subtracted from the desired accumulated total of \$105,000.

1. Calculating the future value of the initial lump sum (FV_{PV}):

For compound interest, N_2 represents the number of compounding periods in the term. So we need to calculate N before using the compound interest future value formula. Additionally, for compound interest problems, the calculation should be based on the periodic interest rate per compounding period (i).

$$N_2 = C/Y \cdot t = 1(10) = 10$$

$$\begin{aligned} FV_{PV} &= PV(1 + i)^{N_2} \\ &= 11,600(1 + 0.052)^{10} \\ &\approx 19,258.186 \quad (\text{Do not round yet}) \end{aligned}$$

2. Finding the future value of the payments (FV_{PMT}):

$$\begin{aligned} 105,000 &= FV_{PMT} + FV_{PV} \\ FV_{PMT} &= 105,000 - 19,258.186 \\ &= 85,741.814 \end{aligned}$$

3. Find the size of the periodic payment given the future value:

First, we need to find the interest rate per payment period i_2 .

$$\begin{aligned} i_2 &= (1 + i)^{\frac{C/Y}{P/Y}} - 1 \\ i_2 &= (1 + 0.052)^{\frac{1}{12}} - 1 \\ &0.00423336... \end{aligned}$$

Plugging the values into Formula 3.6b, we get

$$PMT = \frac{FV \cdot i_2}{(1 + i_2)^{N_1} - 1}$$

$$\begin{aligned}
 PMT &= \frac{85,741.814(0.00423336\dots)}{(1 + 0.00423336\dots)^{120} - 1} \\
 &= 549.806\dots \\
 &\approx \$549.81
 \end{aligned}$$

Monthly deposits of \$549.81 are needed to accumulate to the desired amount.

Try an Example



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C. Payment of Simple Annuity Due

If we know the future value of a simple annuity due, the number of payment periods, and the periodic interest rate, we can apply the future value formula (Formula 3.7a) to determine the unknown periodic payment (PMT).

$$PMT = \frac{FV \cdot i}{\left[(1 + i)^N - 1 \right] (1 + i)} \qquad \text{Formula 3.7b}$$

Similarly, when the present value of a simple annuity due, the number of payments per year, and the periodic interest rate are known, the present value formula (Formula 3.11a) can be rearranged to solve for the unknown periodic payment PMT.

$$PMT = \frac{PV \cdot i}{\left[1 - (1 + i)^{-N}\right](1 + i)} \quad \text{Formula 3.11b}$$

Example 3.9.6: Compute PMT Given PV of Simple Annuity Due

What monthly rent payment at the beginning of each month for three years is needed to fulfill a lease contract that is worth \$10,000 if money is worth 5% compounded monthly?

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of each month so $P/Y = 12$
- $C/Y = P/Y \Rightarrow$ Simple Annuity Due
- Annuity term: $t = 3$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(3) = 36$
- Nominal interest rate: $I/Y = 5\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{5\%}{12} = 0.41\bar{6}\%$
- The present value: $PV = \$10,000$
- $FV = 0$

$PMT = ?$

Considering the present value is known, we use Formula 3.11b to find the size of the payments:

$$\begin{aligned}
 PMT &= \frac{PV \cdot i}{\left[1 - (1 + i)^{-N}\right](1 + i)} \\
 PMT &= \frac{10,000(0.0041\bar{6})}{\left[1 - (1 + 0.0041\bar{6})^{-36}\right](1 + 0.0041\bar{6})} \\
 &= 298.465\dots
 \end{aligned}$$

$$\approx \$298.47$$

Monthly rents of \$298.47 are required to fulfill the lease contract.

Try an Example



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D. Payment of General Annuity Due

If the future value of a general annuity due, along with the number of payment periods and the periodic interest rate, are known, we can use the future value formula (Formula 3.8a) to calculate the unknown periodic payment (PMT).

$$PMT = \frac{FV \cdot i_2}{\left[(1 + i_2)^N - 1 \right] (1 + i_2)} \quad \text{Formula 3.8b}$$

Similarly, if we have information about the present value of a general annuity due, as well as the number of payment periods and the periodic interest rate, we can use the present value formula (Formula 3.12a) to find the unknown periodic payment PMT.

$$PMT = \frac{PV \cdot i_2}{\left[1 - (1 + i_2)^{-N} \right] (1 + i_2)} \quad \text{Formula 3.12b}$$

Example 3.9.7: Compute PMT Given Both PV and FV of general Annuity Due

Colleen entered into a lease agreement for a car with a list price of \$42,000, which is expected to have a residual value of \$21,450 after four years. She agreed to make a down payment of 10% of the car's list price. Given an interest rate of 2.8% compounded monthly, calculate the amount of Colleen's lease payment that is due at the beginning of each week.

Note: The residual value of a leased vehicle represents an estimate of how much the car will be worth at the end of the lease term.

Show/Hide Solution

Given information

- Interest is compounded monthly so $C/Y = 12$
- Payments are made at the beginning of every week so $P/Y = 52$
- $C/Y \neq P/Y \Rightarrow$ General Annuity Due
- Annuity term: $t = 4$ years
- Number of payments in the term: $N_1 = P/Y \cdot t = 52(4) = 208$
- Nominal interest rate: $I/Y = 2.8\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{2.8\%}{12} = 0.2\bar{3}\%$
- List price (cash value) = \$42,000
- Down payment = 10% of the list price
- Residual value = \$21,450

To determine the size of the payment, we need to have the present value (PV) of the lease payments. The car's cash value is equal to the combined present value of the lease payment and the residual plus the down payment:

$$\text{Cash value} = \text{Down} + PV_{\text{PMT}} + PV_{\text{Residual}}$$

To find the present value of the payments (PV_{PMT}), we first need to calculate the present value of the residual (PV_{Residual}).

1. Calculating the present value of the residual (PV_{Residual}):

For compound interest, we need to calculate N_2 before using the compound interest future value formula. Additionally, for compound interest problems, the calculation should be based on the periodic interest rate per compounding period (i).

$$N_2 = C / Y \cdot t = 12(4) = 48$$

$$\begin{aligned} PV_{\text{Residual}} &= FV(1 + i)^{-N_2} \\ &= 21,450(1 + 0.002\bar{3})^{-48} \\ &\approx 19,179.751\dots \text{ (Do not round yet)} \end{aligned}$$

2. Finding the present value (loan amount) of the payments (PV_{PMT}):

$$\text{Cash value} = \text{Down} + PV_{\text{PMT}} + PV_{\text{Residual}}$$

Rearranging the equation for PV_{PMT} gives

$$\begin{aligned} PV_{\text{PMT}} &= \text{Cash value} - \text{Down} - PV_{\text{Residual}} \\ &= 42,000 - 10\% (42,000) - 19,179.751\dots \\ &= 18,620.249\dots \text{ (Do not round yet)} \end{aligned}$$

3. Find the size of the periodic payment given the present value:

We should first find the interest rate per payment period i_2 .

$$\begin{aligned} i_2 &= (1 + i)^{\frac{C/Y}{P/Y}} - 1 \\ i_2 &= (1 + 0.002\bar{3})^{\frac{12}{52}} - 1 \\ &= 0.000537978\dots \end{aligned}$$

Plugging the values into Formula 3.12b, we get

$$\begin{aligned}
 PMT &= \frac{PV \cdot i_2}{\left[1 - (1 + i_2)^{-N_1}\right] (1 + i_2)} \\
 PMT &= \frac{(18,620.249\dots)(0.000537978\dots)}{\left[1 - (1 + 0.000537978\dots)^{-208}\right] (1 + 0.000537978\dots)} \\
 &= 94.595\dots \\
 &\approx \$94.60
 \end{aligned}$$

The size of the weekly payments is \$94.60.

Try an Example



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E. Payments in Scenarios with Multiple Annuities

The following example delves into calculating payments for a scenario involving a Registered Retirement Savings Plan (RRSP) and a Registered Retirement Income Fund (RRIF). Before we begin, it is beneficial to introduce these concepts.

RRSP and RRIF are fundamental components of Canada's retirement savings system, designed to help individuals save for their future. The RRSP is a savings account with tax advantages to encourage saving for retirement.

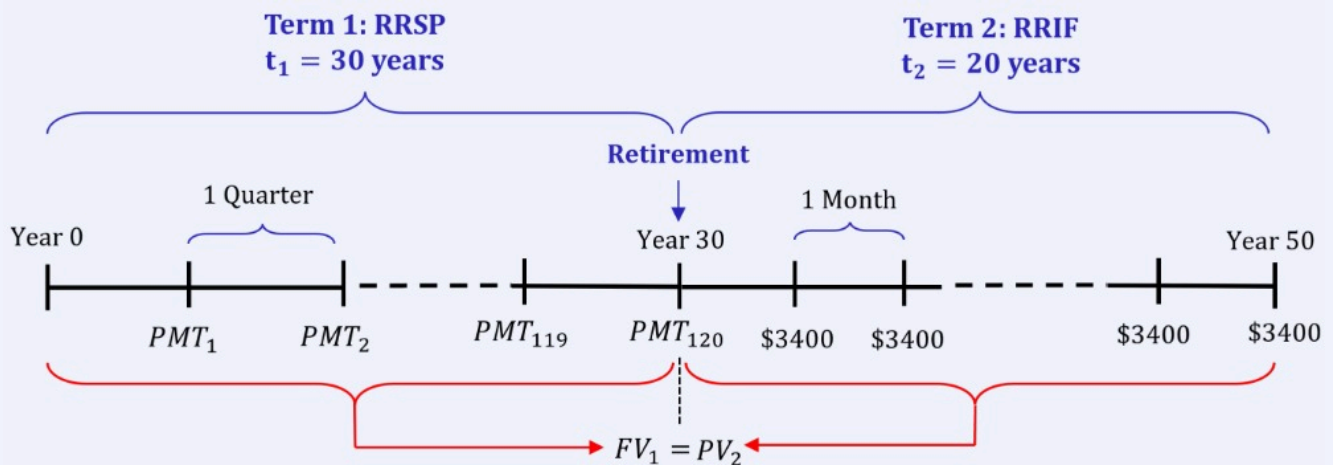
Contributions to an RRSP are tax-deductible, meaning they can reduce the amount of income tax you pay in the year you make the contribution. Once you retire, RRSP is converted into a RRIF which provides you with a regular income during retirement. Withdrawals from a RRIF are taxed as income at your marginal tax rate. The frequency of these withdrawals can be monthly, quarterly, semi-annually, or annually, offering flexibility in managing retirement income.

Example 3.9.8: Compute PMT of First Term Given PMT of the Second Term

Darius is planning for his retirement, which is 30 years away, by making end-of-quarter contributions to his Registered Retirement Savings Plan (RRSP), set to later convert into a Registered Retirement Income Fund (RRIF). Upon retirement, he intends to withdraw \$3,400 at the end of each month for 20 years. An annual interest rate of 4.8%, compounded monthly, applies to both the RRSP during the accumulation phase and the RRIF during the distribution phase.

- What is the total amount Darius must have in his RRIF at the start of retirement to support his goal of monthly withdrawals of \$3,400?
- How much must Darius deposit into his RRSP at the end of each quarter to ensure he accumulates the required amount in his RRIF by his retirement in 30 years?

Show/Hide Solution



a) Term 2: RRIF – Ordinary Simple Annuity

To determine the total amount Darius needs in his RRIF at the start of retirement to afford monthly withdrawals of \$3,400 for 20 years, we calculate the present value of those payments at the time of retirement.

Given Information

- Interest is compounded monthly so $C/Y = 12$
- Withdrawals are made at the end of every month so $P/Y_2 = 12$
- $C/Y = P/Y_2 \Rightarrow$ Ordinary Simple Annuity
- RRIF term: $t_2 = 20$ years
- Number of payments in the term: $N_2 = P/Y_2 \cdot t_2 = 12(20) = 240$
- Nominal interest rate: $I/Y = 4.8\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{4.8\%}{12} = 0.4\%$
- Periodic payments: $PMT_2 = \$3400$
- The fund will be depleted by the end of the second term, so $FV_2 = 0$

$PV_2 = ?$

To compute the present value of the withdrawals, we apply Formula 3.9a.

$$\begin{aligned}
 PV &= PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] \\
 PV_2 &= 3400 \left[\frac{1 - (1 + 0.004)^{-240}}{0.004} \right] \\
 &= 523,917.230\dots \\
 &\approx \$523,917.23
 \end{aligned}$$

Therefore, the total amount Darius must have in his RRIF at the start of retirement to support his goal is \$523,917.23.

b) Term 1: RRSP – Ordinary General Annuity

The value needed at retirement, determined in Part (a) becomes the target FV for the RRSP contributions. We then calculate the quarterly contribution Darius must make to his RRSP over 30 years to accumulate to that target FV at retirement.

Given Information

- Interest is compounded monthly so $C/Y = 12$
- Withdrawals are made at the end of every quarter so $P/Y_1 = 4$
- $C/Y \neq P/Y_1 \Rightarrow$ Ordinary General Annuity
- RRSP term: $t_1 = 30$ years
- Number of payments in the term: $N_1 = P/Y_1 \cdot t_1 = 4(30) = 120$
- Nominal interest rate: $I/Y = 4.8\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{4.8\%}{12} = 0.4\%$
- No initial balance in RRSP, so $PV_1 = 0$
- Future value: $FV_1 = PV_2 = \$523,917.230\dots$ (use unrounded value for accuracy)

$$PMT_1 = ?$$

First, we need to find the interest rate per payment period i_2 :

$$\begin{aligned} i_2 &= (1 + i)^{\frac{C/Y}{P/Y}} - 1 \\ i_2 &= (1 + 0.004)^{\frac{12}{4}} - 1 \\ &= 0.0120480\dots \end{aligned}$$

Given the future value is known, we use Formula 3.6b to calculate the size of the payments:

$$\begin{aligned} PMT &= \frac{FV \cdot i_2}{(1 + i_2)^N - 1} \\ PMT_1 &= \frac{523,917.230\dots(0.0120480\dots)}{(1 + 0.0120480\dots)^{120} - 1} \\ &= 1967.277\dots \end{aligned}$$

$$\approx \$1967.28$$

Darius will need to make quarterly deposits of \$1967.28 to ensure he accumulates the required amount in his RRIF by his retirement.

Try an Example



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Section 3.9 Exercises

1. What payment is required at the end of every month for 5.5 years to repay a \$11,968 loan if interest is 4.4% compounded monthly?

Show/Hide Answer

$$\text{PMT} = \$204.49$$

2. Gregory purchased a car for \$53,500; he paid 5% of the cost as a down payment and financed the balance amount at 3.5% compounded monthly for 3 years. **a)** What is the size of payment made at the end of every month to settle the loan? **b)** What was the amount of interest charged?

Show/Hide Answer

$$\text{a) PMT} = \$1,489.28$$

$$\text{b) } I = \$2,789.08$$

3. To fulfill a lease contract valued at \$29,000, rent payments are required at the beginning of each month for 5 years. The interest rate on this lease is assumed to be 4.79%, compounded monthly. **a)** What is the size of the payments? **b)** How much interest was charged on the lease?

Show/Hide Answer

$$\text{a) } PMT = \$542.32$$

$$\text{b) } I = \$3,539.20$$

4. Dawn entered into a lease agreement for a car with a list price of \$56,460.00, which is expected to have a residual value of \$31,053 after 6 years. She agreed to make a down payment of 15% of the car's list price. Given an interest rate of 2.06% compounded semi-annually, calculate the amount of Dawn's lease payment that is due at the beginning of each month.

Show/Hide Answer

$$PMT = \$302.79$$

5. Kristina has already saved \$7,600 in her savings account as of today, and she plans to contribute equal deposits at the end of each year for the next 10 years. What deposit is required to accumulate \$91,500 in total at the end of the 10 years assuming the account earns 2.42% compounded quarterly?

Show/Hide Answer

$$PMT = \$7323.15$$

3.10 NUMBER OF PAYMENTS AND TIME: FORMULA APPROACH

A. Number of Payments and Time of Annuities

Suppose you are setting aside funds for a renovation project with a target of \$50,000. You plan to save \$100 every week and deposit it into your savings account. The key question is how long will it take to reach your \$50,000 savings goal. To find this out, you need to first calculate the number of weekly payments (N) you must make to accumulate the desired amount. Once you have determined N , you will know the exact duration required to achieve your savings target for the renovation project.

In this section, we turn our attention to computing the total number of payments in annuities (N) and understanding how long it takes for an annuity to achieve its financial objectives, whether it's accumulating a desired sum for retirement, paying off a loan, or reaching a specific investment goal.

When the time of the annuity term (t) is unknown, the initial step is to calculate N using either a formula or a financial calculator. Once N is known, it can then be used to work out the time. Depending on the scenario, whether it involves retirement plans, loans, or savings schemes, the calculation of N may be based on the Periodic Payment (PMT) and either the Future Value (FV), the Present Value (PV) of the annuity, or a combination of both.

It is important to note that in annuities calculations, unlike compound interest calculation where N is the number of compounding periods in the term, N represents the number of payments in the term. This value is always *rounded up* to the next whole number of payments, which may result in the last payment being smaller than the others.

Once N is computed, the annuity's term (t) can be determined by rearranging Formula 3.1a.

$$t = \frac{N}{P/Y} \quad \text{Formula 3.1b}$$

B. Number of Payments of Ordinary Simple Annuity

If FV is known, we can rearrange the FV formula for an ordinary simple annuity (Formula 3.5a) to solve for the unknown number of payments in the term, N .

$$N = \frac{\ln\left[1 + \frac{FV \cdot i}{PMT}\right]}{\ln(1 + i)} \quad \text{Formula 3.5c}$$

Similarly, if the present value (PV) is known, we can adjust the PV formula (Formula 3.9a) to determine the unknown number of payments in the term, N.

$$N = - \frac{\ln\left[1 - \frac{PV \cdot i}{PMT}\right]}{\ln(1 + i)} \quad \text{Formula 3.9c}$$

Example 3.10.1: Compute N and Time Given FV and PMT of Ordinary Simple Annuity

Mahsa wants to save \$80,000 for her house down payment. Suppose she plans to make deposits of \$1,290 at the end of every three months into an account earning 4.7% compounded quarterly. **a)** How many deposits will she have to make to achieve her goal? **b)** How long will it take? Give the time in terms of years and months.

Show/Hide Solution

Given Information

- Interest is compounded quarterly so $C/Y = 4$
- Payments are made at the end of every three months so $P/Y = 4$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Nominal interest rate: $I/Y = 4.7\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{4.7\%}{4} = 1.175\%$
- Periodic payments: $PMT = \$1,290$
- The initial balance of the account is zero, so $PV = 0$
- Future value: $FV = \$80,000$

a) $N = ?$

Given the future value is known, we use Formula 3.5c to calculate the number of payments:

$$N = \frac{\ln\left[1 + \frac{FV \cdot i}{PMT}\right]}{\ln(1 + i)}$$

$$N = \frac{\ln\left[1 + \frac{80,000(0.01175)}{1290}\right]}{\ln(1 + 0.01175)}$$

$$= 46.856\dots$$

$$= 47$$

The number of payments in the term, N , is rounded up to 47 deposits.

b) $t = ?$

To find the time period of the term, we substitute N and P/Y into Formula 3.1b:

$$t = \frac{N}{P/Y}$$

$$t = \frac{47}{4}$$

$$= 11.75 \text{ years}$$

$$= 11 \text{ years} + 0.75(12) \text{ months}$$

$$= 11 \text{ years and } 9 \text{ months}$$

Note: In annuities, N represents the number of payments in the term and its value is always **rounded up** to the next whole number of payments regardless of the decimal part of the calculated N .

Try an Example



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Example 3.10.2: Compute N and Time Given FV , PV , and PMT of Ordinary Simple Annuity

Mahsa already has \$25,000 in her savings account, and she wants this to accumulate to \$80,000 for a house down payment. Suppose she plans to make deposits of \$1,290 at the end of every three months into the account earning 4.7% compounded quarterly. **a)** How many deposits will she have to make to achieve her goal? **b)** How long will it take? Give the time in terms of years and months.

Show/Hide Solution

Given Information

- Interest is compounded quarterly so $C/Y = 4$
- Payments are made at the end of every three months so $P/Y = 4$

$C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity

- Nominal interest rate: $I/Y = 4.7\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{4.7\%}{4} = 1.175\%$
- Periodic payments: $PMT = \$1,290$
- Present value is the initial lumpsum invested: $PV = \$25,000$
- Future value: $FV = \$80,000$

a) $N = ?$

The accumulated value of \$80,000 is the sum of the future value of the initial investment of \$25,000 (FV_{PV}) and the future value of the payments (FV_{PMT}). We can plug in the formulas for the future value of the ordinary simple annuity and the compound interest into the following equation and solve for N .

It is crucial to recognize that in the case of a simple annuity, the number of payments and the number of compounding periods within the term are identical. As a result, the N variable used in both formulas is the same. However, if we were dealing with a general annuity, it would be necessary to express each N in terms of time and solve for the time first.

$$FV = FV_{PV} + FV_{PMT}$$

$$FV = PV(1 + i)^N + PMT \left[\frac{(1 + i)^N - 1}{i} \right]$$

We can rearrange the equation for N :

$$\begin{aligned} N &= \frac{\ln\left(\frac{FV \cdot i + PMT}{PV \cdot i + PMT}\right)}{\ln(1 + i)} \\ N &= \frac{\ln\left(\frac{80,000(0.01175) + 1290}{25,000(0.01175) + 1290}\right)}{\ln(1 + 0.01175)} \\ &= 29.294... \\ &= 30 \text{ (Rounded up)} \end{aligned}$$

The number of payments in the term, N , is rounded up to 30 deposits.

b) $t = ?$

To find the time period of the term, we substitute N and P/Y into Formula 3.1b:

$$t = \frac{N}{P/Y}$$

$$t = \frac{30}{4}$$

$$= 7.5 \text{ years}$$

$$= 7 \text{ years} + 0.5(12) \text{ months}$$

$$= 7 \text{ years and } 6 \text{ months}$$

Try an Example



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=560>

C. Number of Payments of Ordinary General Annuity

Like an ordinary simple annuity, when FV is known, we can rearrange the FV formula for an ordinary general annuity (Formula 3.6a) to solve for the unknown number of payments in the term, N.

$$N = \frac{\ln\left[1 + \frac{FV \cdot i_2}{PMT}\right]}{\ln(1 + i_2)} \quad \text{Formula 3.6c}$$

Likewise, if the present value (PV) is known, we can use the PV formula (Formula 3.10a) to determine the unknown number of payments in the term, N.

$$N = - \frac{\ln \left[1 - \frac{PV \cdot i_2}{PMT} \right]}{\ln(1 + i_2)}$$

Where i_2 is the interest rate per payment period

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

Example 3.10.3: Compute N and Time Given PV and PMT of Ordinary General Annuity

Emanuel won a lottery prize of \$1,000,000 and invested it in a fund that earns 4% interest compounded monthly. He plans to withdraw \$1,000 at the end of each week. **a)** Determine the total number of withdrawals Emanuel will make. **b)** How long will the money last with these weekly withdrawals? Give the time in terms of years and weeks.

Show/Hide Solution

Given Information

- Interest is compounded annually so $C/Y = 12$
- Withdrawals are at the end of every week so $P/Y = 52$

$$C/Y \neq P/Y \Rightarrow \text{Ordinary General Annuity}$$

- Nominal interest rate: $I/Y = 4\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{4\%}{12} = 0.\bar{3}\%$
- Periodic payments: $PMT = \$1000$
- Present value: $PV = \$1,000,000$
- The fund will be fully depleted at the end of the term, so $FV = 0$

a) $N = ?$

First, we should find the interest rate per payment period i_2 :

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$i_2 = (1 + 0.003)^{\frac{12}{52}} - 1$$

$$= 0.000768246\dots$$

Given the present value is known, we then use Formula 3.10c to calculate the number of payments:

$$N = - \frac{\ln \left[1 - \frac{PV \cdot i_2}{PMT} \right]}{\ln(1 + i_2)}$$

$$N = - \frac{\ln \left[1 - \frac{1,000,000(0.000768246\dots)}{1000} \right]}{\ln(1 + 0.000768246\dots)}$$

$$= 1903.8\dots$$

$$= 1904 \text{ (Rounded up)}$$

The number of payments in the term, N , is rounded up to 1904 withdrawals.

b) $t = ?$

To find the time period of the term, we substitute N and P/Y into Formula 3.1b:

$$t = \frac{N}{P/Y}$$

$$t = \frac{1904}{52}$$

$$= 36.6153\dots \text{ years}$$

$$= 36 \text{ years} + 0.6153\dots(52) \text{ weeks}$$

$$= 36 \text{ years and } 32 \text{ weeks}$$

D. Number of Payments of Simple Annuity Due

If the FV of a simple annuity due is known, we can rearrange the FV formula for such an annuity (Formula 3.7a) to solve for the unknown number of payments in the term, N .

$$N = \frac{\ln\left[1 + \frac{FV \cdot i}{PMT \cdot (1+i)}\right]}{\ln(1+i)} \quad \text{Formula 3.7c}$$

Similarly, if the present value (PV) is known, we can adjust the PV formula (Formula 3.11a) to determine the unknown number of payments in the term, N .

$$N = - \frac{\ln\left[1 - \frac{PV \cdot i}{PMT \cdot (1+i)}\right]}{\ln(1+i)} \quad \text{Formula 3.11c}$$

Example 3.10.4: Compute N and Time Given FV, PMT of Simple Annuity Due

Esmeralda wants to accumulate \$71,000 for her condominium down payment. She plans to make deposits of \$3,890 at the beginning of every three months into her savings account, which earns interest at 5.29% compounded quarterly. **a)** How many deposits will she have to make to achieve her goal? **b)** How long will it take? Give the time in terms of years and months.

Show/Hide Solution

Given Information

- Interest is compounded quarterly so $C/Y = 4$
- Payments are made at the beginning of every three months so $P/Y = 4$

$$C/Y = P/Y \Rightarrow \text{Simple Annuity Due}$$

- Nominal interest rate: $I/Y = 5.29\%$

- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{5.29\%}{4} = 1.3225\%$
- Periodic payments: $PMT = \$3,890$
- Initial account balance is zero, so $PV = 0$
- Accumulated value: $FV = \$71,000$

a) $N = ?$

Given the future value is known, we use Formula 3.7c to calculate the number of payments:

$$N = \frac{\ln\left[1 + \frac{FV \cdot i}{PMT \cdot (1+i)}\right]}{\ln(1+i)}$$

$$N = \frac{\ln\left[1 + \frac{71,000(0.013225)}{3890(1+0.013225)}\right]}{\ln(1+0.013225)}$$

$$= 16.26\dots$$

$$= 17 \text{ (Rounded up)}$$

The number of payments in the term, N , is rounded up to 17 deposits.

b) $t = ?$

To find the time period of the term, we substitute N and P/Y into Formula 3.1b:

$$t = \frac{N}{P/Y}$$

$$t = \frac{17}{4}$$

$$= 4.25 \text{ years}$$

$$= 4 \text{ years} + 0.25(12) \text{ months}$$

= 4 years and 3 months

Try an Example



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=560>

E. Number of Payments of General Annuity Due

When the FV of a general annuity due is known, we can rearrange the FV formula for such an annuity (Formula 3.8a) to solve for the unknown number of payments in the term, N .

$$N = \frac{\ln \left[1 + \frac{FV \cdot i_2}{PMT \cdot (1 + i_2)} \right]}{\ln(1 + i_2)} \quad \text{Formula 3.8c}$$

Likewise, when the present value (PV) is known, we can rearrange the PV formula (Formula 3.12a) to determine the unknown number of payments in the term, N .

$$N = - \frac{\ln \left[1 - \frac{PV \cdot i_2}{PMT \cdot (1 + i_2)} \right]}{\ln(1 + i_2)} \quad \text{Formula 3.12c}$$

Where i_2 is the interest rate per payment period

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

Example 3.10.5: Compute N and Time Given PV , PMT of General Annuity Due

Dario recently retired and has \$500,000 in his retirement fund, which earns an annual interest rate of 8% compounded annually. He intends to withdraw \$4,000 at the beginning of every month. **a)** Calculate the total number of withdrawals Dario will make. **b)** How long will it take for the fund to be depleted?

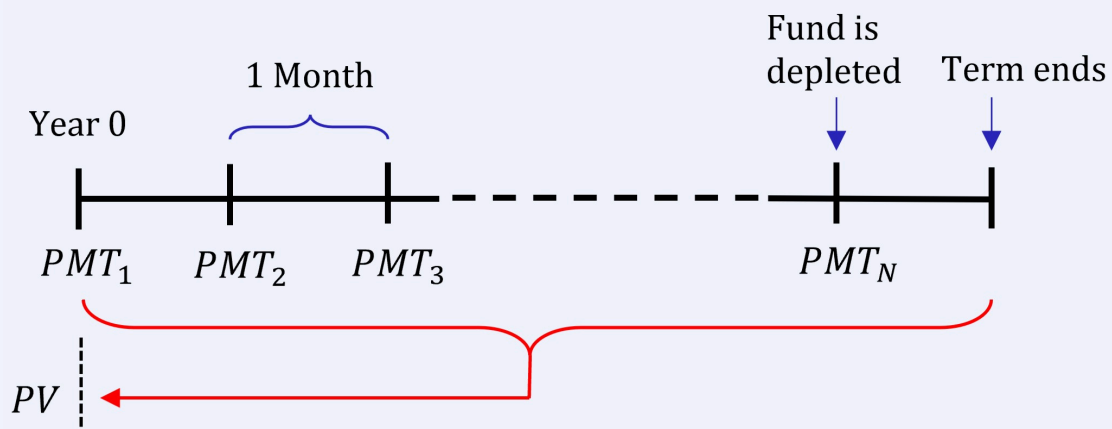
Show/Hide Solution

Given Information

- Interest is compounded annually so $C/Y = 1$
- Payments are withdrawn at the beginning of every month so $P/Y = 12$

$C/Y \neq P/Y \Rightarrow$ General Annuity Due

- Nominal interest rate: $I/Y = 8\%$
- Periodic interest rate: $i = \frac{I/Y}{C/Y} = \frac{8\%}{1} = 8\%$
- Periodic payments: $PMT = \$4,000$
- Present value: $PV = \$500,000$
- The fund balance at the end of the term will be zero, so $FV = 0$



a) $N = ?$

We should first find the interest rate per payment period i_2 :

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$\begin{aligned} i_2 &= (1 + 0.08)^{\frac{1}{12}} - 1 \\ &= 0.00643403\dots \end{aligned}$$

Given the present value is known, we use Formula 3.12c to find the number of payments.

$$\begin{aligned} N &= - \frac{\ln \left[1 - \frac{PV \cdot i_2}{PMT \cdot (1 + i_2)} \right]}{\ln(1 + i_2)} \\ N &= - \frac{\ln \left[1 - \frac{500,000(0.00643403\dots)}{4000(1 + 0.00643403\dots)} \right]}{\ln(1 + 0.00643403\dots)} \\ &= 250.2\dots \\ &= 251 \text{ (Rounded up)} \end{aligned}$$

The number of payments in the term, N , is rounded up to 251 withdrawals.

b) $t = ?$

To find the duration of the annuity term, we substitute N and P/Y into Formula 3.1b:

$$\begin{aligned} t &= \frac{N}{P/Y} \\ t &= \frac{251}{12} \\ &= 20.91\bar{6} \text{ years} \\ &= 20 \text{ years} + 0.91\bar{6}(12) \text{ months} \\ &= 20 \text{ years and } 11 \text{ months} \end{aligned}$$

As shown in the timeline, in an annuity due, the term extends beyond the last withdrawal by one period (in this case, one month). Because our focus is on the timeframe up to the final withdrawal (when the fund is depleted), we must deduct one period (one month) from the annuity due term.

Thus, the fund will be depleted in **20 years and 10 months**.

Section 3.10 Exercises

1. Karina wants to accumulate \$52,500 for her condominium down payment. She plans to make deposits of \$1,330 at the beginning of every month into her savings account, which earns interest at 6.61% compounded monthly. **a)** How many deposits will Karina have to make to achieve her goal? **b)** How long will it take? Give the time in terms of years and months.

Show/Hide Answer

a) $N = 36$ deposits

b) $t = 3$ years and 0 months

2. BoardWalk Corporation aims to save a total of \$641,000 over the coming years. They plan to start with an initial deposit of \$176,800 and then make deposits of \$5,200 at the end of each month. **(a)** Determine the number of additional monthly deposits and **(b)** the total time required for BoardWalk Corporation to accumulate \$641,000, considering an interest rate of 6.37% compounded semi-annually. Give the time in years and months.

Show/Hide Answer

a) $N = 64$ payments

b) $t = 5$ years and 4 months

3. Alo has saved \$2,500,000 in his retirement fund, which earns an annual interest rate of 1.5% compounded daily. He intends to withdraw \$10,000 at the beginning of every month. **a)** Calculate the total number of withdrawals Alo will make. **b)** Determine the duration for which Alo's retirement fund will sustain these monthly withdrawals. Give the time in years and months.

Show/Hide Answer

a) $N = 300$ withdrawals

b) $t = 25$ years

3.11 NOMINAL INTEREST RATE (I/Y): FORMULA APPROACH

Calculating the periodic interest rate (i) algebraically using formulas is challenging because it is difficult to isolate the interest rate variable in either the Present Value (PV) or Future Value (FV) formulas. As a result, we focus on using a financial calculator to determine the nominal interest rate (I/Y). This rate can then be applied to calculate the periodic payment. For instructions on the calculator approach, please refer to Section 3.6.

3.12 TERMINOLOGY AND GLOSSARY OF SYMBOLS

Terminology

- **Annuity:** a series of payments, usually in equal amounts, made at periodic time intervals.
- **Payment (PMT):** the amount of the periodic payment
- **Payment Interval (payment period):** the time between the successive payments
- **Future value (accumulated or maturity value) (FV):** the accumulated value of the series payments at the end of the term.
- **Nominal rate (I/Y):** the yearly or annual rate of interest
- **Present value (principal or discounted value) (PV):** The discounted value of the series of payments at the beginning of the term.
- **Term (t):** in this module refers to the term of an annuity, which is the time period (in years) from the beginning of the first payment interval to the end of the last payment interval.

Glossary of the Symbols

- I/Y Nominal interest rate per year (%)
- C/Y : Frequency of interest compounding period per year
- i : Interest rate per compounding period (%)
- t : Term of annuity (in years)
- N : Total number of periodic payments in the annuity term
- P/Y : Number of payments per year
- PMT : Amount of periodic payment (\$)
- PV : Present value of an annuity (\$)
- FV : Future (maturity) value of an annuity (\$)
- I : Interest Amount (\$)

3.13 APPENDIX

Deriving Annuity Formulas

FV Formula – Simple Annuity

Before we delve into deriving the future value formula for annuities, it's essential to understand geometric sequences. A geometric sequence is a sequence of numbers where each term after the first is found by multiplying the previous term by a fixed, non-zero number called the common ratio and is represented by r . The first term of a geometric sequence is denoted by a . An example of a geometric sequence is $\{3, 6, 12, 24, 48, 96\}$ where each term is obtained by multiplying the previous term by 2. Here, the first term is $a = 3$ and the common ratio is $r = 2$.

In general, a geometric sequence with n terms is written as $\{a, ar, ar^2, ar^3, \dots, ar^{n-1}\}$. A partial sum of a geometric sequence, denoted as S_n , is the sum of the first n terms of the sequence. This sum can be calculated using the formula:

$$S_n = a \left(\frac{1 - r^n}{1 - r} \right)$$

where n is the number of terms to be summed.

Returning to our main topic, we aim to find a simple formula for calculating the future value of annuities. We begin with an ordinary simple annuity, where payments are made at the end of each payment period. The total value accumulated at the end of the term is the sum of the future values of each individual payment using the formula for the future value of compound interest.

$$\begin{aligned} FV &= PMT(1 + i)^0 + PMT(1 + i)^1 + \dots + PMT(1 + i)^{N-2} + PMT(1 + i)^{N-1} \\ &= PMT \underbrace{\left[(1 + i)^0 + (1 + i)^1 + \dots + (1 + i)^{N-2} + (1 + i)^{N-1} \right]}_{S_n} \end{aligned}$$

After factoring out PMT from this summation, we observe that the expression inside the square brackets is the partial sum of N terms of a geometric sequence. This sequence starts with a first term of $(1 + i)^0 = 1$ and has a common ratio of $(1 + i)$, meaning each term is derived by multiplying the previous term by $(1 + i)$.

Therefore, the partial sum of this geometric sequence with N terms, the first term $a = 1$, and the common ratio $r = 1 + i$ can be determined by

$$\begin{aligned} S_n &= a \left(\frac{1 - r^n}{1 - r} \right) \\ S_n &= 1 \left[\frac{1 - (1 + i)^N}{1 - (1 + i)} \right] \\ &= \left[\frac{1 - (1 + i)^N}{-i} \right] \\ &= \left[\frac{(1 + i)^N - 1}{i} \right] \end{aligned}$$

Thus, the future value of all the payments in an ordinary simple annuity is calculated as follows

$$\begin{aligned} FV &= PMT(S_n) \\ FV &= PMT \left[\frac{(1 + i)^N - 1}{i} \right] \end{aligned}$$

FV Formula – Simple Annuity Due

Recall that a geometric sequence with n terms is written as $\{a, ar, ar^2, ar^3, \dots, ar^{n-1}\}$. A partial sum of a geometric sequence, denoted as S_n , is the sum of the first n terms of the sequence. This sum can be calculated using the formula

$$S_n = a \left(\frac{1 - r^n}{1 - r} \right)$$

where n is the number of terms to be summed.

We aim to find a simple formula for calculating the future value of a simple annuity due, where payments are made at the beginning of each payment period. The total value accumulated at the end of the term is the sum of the future values of each individual payment using the formula for the future value of compound interest.

$$\begin{aligned}
 FV &= PMT(1+i)^1 + PMT(1+i)^2 + \dots + PMT(1+i)^{N-1} + PMT(1+i)^N \\
 &= PMT \underbrace{\left[(1+i)^1 + (1+i)^2 + \dots + (1+i)^{N-1} + (1+i)^N \right]}_{S_n}
 \end{aligned}$$

After factoring out PMT from this summation, we observe that the expression inside the square brackets is the partial sum of N terms of a geometric sequence. This sequence starts with a first term of $(1+i)^1$ and has a common ratio of $(1+i)$, meaning each term is derived by multiplying the previous term by $(1+i)$.

Therefore, the partial sum of this geometric sequence with N terms, the first term $a = 1+i$, and the common ratio $r = 1+i$ can be determined by

$$\begin{aligned}
 S_n &= a \left(\frac{1-r^n}{1-r} \right) \\
 S_n &= (1+i) \left[\frac{1-(1+i)^N}{1-(1+i)} \right] \\
 &= (1+i) \left[\frac{1-(1+i)^N}{-i} \right] \\
 &= (1+i) \left[\frac{(1+i)^N - 1}{i} \right]
 \end{aligned}$$

Thus, the future value of all the payments in simple annuity due is calculated as follows

$$\begin{aligned}
 FV &= PMT(S_n) \\
 FV &= PMT \left[\frac{(1+i)^N - 1}{i} \right] (1+i)
 \end{aligned}$$

PV Formula – Ordinary Simple annuity

Recall that a geometric sequence with n terms is written as $\{a, ar, ar^2, ar^3, \dots, ar^{n-1}\}$. A partial sum of a geometric sequence, denoted as S_n , is the sum of the first n terms of the sequence. This sum can be calculated using the formula

$$S_n = a \left(\frac{1 - r^n}{1 - r} \right)$$

where n is the number of terms to be summed.

We aim to find a simple formula for calculating the present value of an ordinary simple annuity, where payments are made at the end of each payment period. The total present value is the sum of the present values of each individual payment using the formula for the present value of compound interest.

$$\begin{aligned} PV &= PMT(1 + i)^{-1} + PMT(1 + i)^{-2} + \dots + PMT(1 + i)^{-(N-1)} + PMT(1 + i)^{-N} \\ &= PMT \underbrace{\left[(1 + i)^{-1} + (1 + i)^{-2} + \dots + (1 + i)^{-(N-1)} + (1 + i)^{-N} \right]}_{S_n} \end{aligned}$$

After factoring out PMT from this summation, we observe that the expression inside the square brackets is the partial sum of N terms of a geometric sequence. This sequence starts with a first term of $(1 + i)^{-1}$ and has a common ratio of $(1 + i)^{-1}$, meaning each term is derived by multiplying the previous term by $(1 + i)^{-1}$.

Therefore, the partial sum of this geometric sequence with N terms, the first term $a = (1 + i)^{-1}$, and the common ratio $r = (1 + i)^{-1}$ can be determined by

$$\begin{aligned} S_n &= a \left(\frac{1 - r^n}{1 - r} \right) \\ S_n &= (1 + i)^{-1} \left[\frac{1 - \left((1 + i)^{-1} \right)^N}{1 - (1 + i)^{-1}} \right] \end{aligned}$$

Applying the product of powers property of exponents, which states that $(b^x)^y = b^{x \cdot y}$, we can express $\left((1 + i)^{-1} \right)^N$ as $(1 + i)^{-N}$, thereby simplifying the numerator. Additionally, considering that $(1 + i)^{-1}$ equals $\frac{1}{1 + i}$, we can multiply the term $(1 + i)$ by the denominator.

$$S_n = \frac{1}{1 + i} \left[\frac{1 - (1 + i)^{-N}}{1 - (1 + i)^{-1}} \right]$$

When you distribute $(1 + i)$ across the terms in the denominator and recognize that multiplying a term by its reciprocal results in 1, it follows that $(1 + i) \cdot (1 + i)^{-1} = 1$.

$$\begin{aligned} S_n &= \left[\frac{1 - (1 + i)^{-N}}{(1 + i) - 1} \right] \\ &= \left[\frac{1 - (1 + i)^{-N}}{i} \right] \end{aligned}$$

Therefore, the present value of all payments in an ordinary simple annuity can be calculated using the following formula:

$$\begin{aligned} PV &= PMT(S_n) \\ PV &= PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] \quad \text{Formula 3.9} \end{aligned}$$

PV Formula – Simple Annuity Due

Recall that a geometric sequence with n terms is written as $\{a, ar, ar^2, ar^3, \dots, ar^{n-1}\}$. A partial sum of a geometric sequence, denoted as S_n , is the sum of the first n terms of the sequence. This sum can be calculated using the formula

$$S_n = a \left(\frac{1 - r^n}{1 - r} \right)$$

where n is the number of terms to be summed.

We want to find a simple formula for calculating the present value of a simple annuity due, where payments are made at the beginning of each payment period. The total present value is the sum of the present values of each payment using the formula for the present value of compound interest.

$$\begin{aligned} PV &= PMT(1 + i)^0 + PMT(1 + i)^{-1} + \dots + PMT(1 + i)^{-(N-2)} + PMT(1 + i)^{-(N-1)} \\ &= PMT \underbrace{\left[(1 + i)^0 + (1 + i)^{-1} + \dots + (1 + i)^{-(N-2)} + (1 + i)^{-(N-1)} \right]}_{S_n} \end{aligned}$$

After factoring out PMT from this summation, we notice that the expression inside the square brackets is the partial sum of N terms of a geometric sequence. This sequence starts with a first term of $(1 + i)^0$ and has a common ratio of $(1 + i)^{-1}$, meaning each term is derived by multiplying the previous term by $(1 + i)^{-1}$.

Therefore, the partial sum of this geometric sequence with N terms, the first term $a = (1 + i)^0 = 1$, and the common ratio $r = (1 + i)^{-1}$ can be determined by

$$S_n = a \left(\frac{1 - r^n}{1 - r} \right)$$

$$S_n = \left[\frac{1 - \left((1 + i)^{-1} \right)^N}{1 - (1 + i)^{-1}} \right]$$

Applying the product of powers property of exponents, which states that $(b^x)^y = b^{x \cdot y}$, we can express $\left((1 + i)^{-1} \right)^N$ as $(1 + i)^{-N}$, thus simplifying the numerator. To simplify the denominator, we multiply both the numerator and the denominator of the expression on the right-hand side by the factor $(1 + i)$:

$$S_n = \left[\frac{1 - (1 + i)^{-N}}{1 - (1 + i)^{-1}} \right] \times \frac{1 + i}{1 + i}$$

Next, we distribute the factor $(1 + i)$ across the terms in the denominator, recognizing that multiplying a term by its reciprocal equals 1, we see that $(1 + i) \cdot (1 + i)^{-1} = 1$.

$$S_n = \left[\frac{1 - (1 + i)^{-N}}{(1 + i) - 1} \right] (1 + i)$$

$$= \left[\frac{1 - (1 + i)^{-N}}{i} \right] (1 + i)$$

Therefore, the present value of all payments in a simple annuity due can be calculated using the following formula:

$$PV = PMT(S_n)$$

$$PV = PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] (1 + i) \quad \text{Formula 3.11}$$

PART IV

LOAN AMORTIZATION

Chapter Outline

This chapter explores loan amortization and mortgages, focusing on the financial calculator and formula calculation methods.

Calculator Approach

4.1 Loan Amortization – Calculator Approach: This section covers definitions, amortization schedules, and calculations involving interest, principal, and loan balances using financial calculators.

4.2 Mortgages – Calculator Approach: In this part, we delve into mortgage definitions, types, and the calculation of payments and term renewals using financial calculators.

Formula Approach

4.3 Loan Amortization – Formula Approach: Here, we examine loan amortization definitions, schedules, and calculations related to interest, principal, and loan balances using mathematical formulas.

4.4 Mortgages – Formula Approach: This section discusses mortgage definitions, types, and calculations for payments and term renewals using mathematical formulas.

4.5 Terminology and Glossary of Symbols

4.1 LOAN AMORTIZATION: CALCULATOR APPROACH

A. Amortization Definitions

Amortization of a loan is the process of the gradual reduction in the loan amount through periodic repayments, usually of equal size, over a predetermined length of time. The **amortization period** is the time period set to repay the loan. An **amortization schedule** is a detailed table, similar to what is shown in Table 4.1.1, that breaks down each loan payment amount into its interest (INT) and principal portions (PRN). It also shows the outstanding principal balance (BAL) after each payment is made.

Table 4.1.1 An Amortization Schedule

| Payment Number | Payment Amount PMT (\$) | Interest Portion INT (\$) | Principal Portion PRN (\$) | Loan Balance BAL (\$) |
|----------------|-------------------------|---------------------------|----------------------------|-----------------------|
| 0 | n/a | n/a | n/a | Loan Amount (PV) |
| 1 | PMT | INT_1 | PRN_1 | BAL_1 |
| 2 | PMT | INT_2 | PRN_2 | BAL_2 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| N-1 | PMT | INT_{N-1} | PRN_{N-1} | BAL_{N-1} |
| N | PMT_N | INT_N | PRN_N | 0 |
| Totals | Total Amount Paid | Total Interest Portion | Total Principal Portion | n/a |

Note that the size of payment (PMT) is the same for every period except the last payment as it needs to be adjusted to fully repay the loan. In banking, all actual monetary values are rounded to the nearest cent (two decimal places). Because of this rounding, the final payment needs to be adjusted and thus is almost always different from other payments.

B. Amortization Schedules for Loans

For each payment (row) in the schedule, you can use the Amortization (AMORT) Worksheet in a financial calculator

to calculate the principal portion (PRN), interest portion (INT), and the balance (BAL) after that payment. Given all the other information about the loan has already been entered, set P1 and P2 to that payment number to obtain the principal portion and interest portion of a particular payment number. For example, when P1 = 5 and P2 = 5, PRN displays the principal portion of the 5th payment and INT displays the interest portion of the 5th payment. BAL displays the principal balance after the 5th payment.

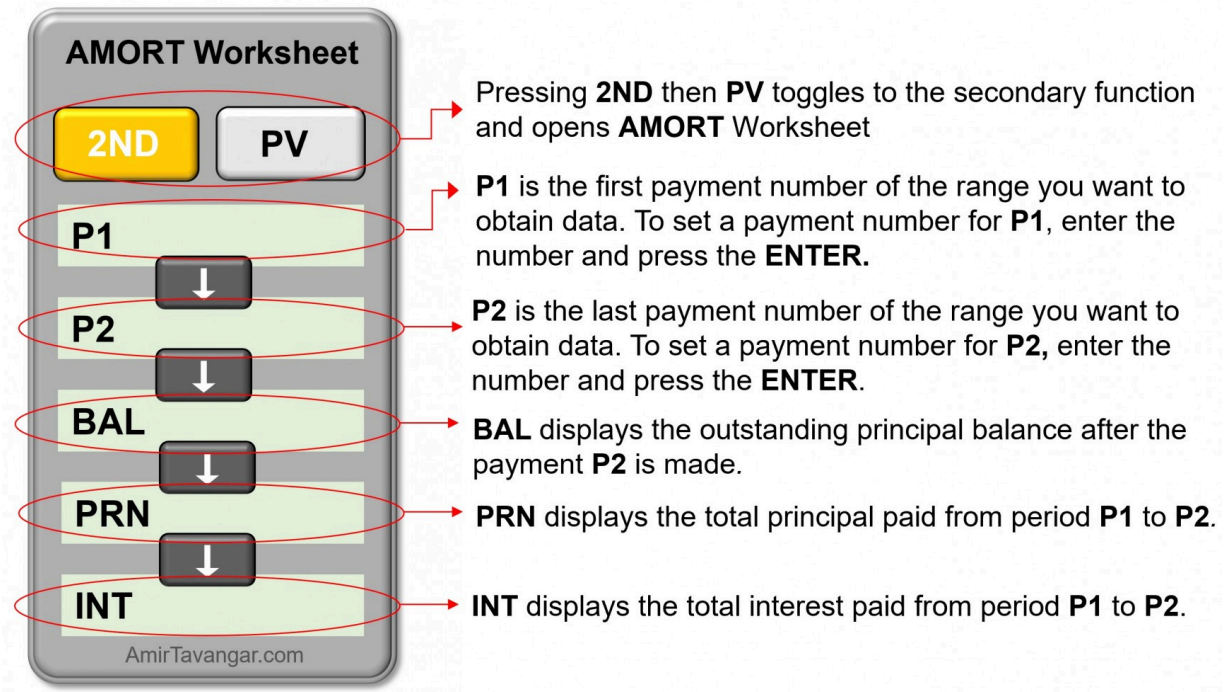


Figure 4.1.1 The keys on the AMORT worksheet on a financial calculator

How to Construct an Amortization Schedule: Calculator Approach

1. Enter the given information about the loan annuity in the TVM worksheet and compute the unknown variable, which is usually either the size of payment (PMT) or the number of payments in the term (N).

- **Unknown PMT**: compute PMT first using the TVM worksheet, round it off to the nearest cent (if needed), re-enter the rounded value (ensure to keep the same sign) to the PMT button in the calculator, and then use the Amortization Worksheet.
- **Unknown N**: compute N first using the TVM worksheet, round it up to the nearest whole payment, re-enter it to the N button in the calculator (optional), and then use the Amortization Worksheet.

2. In the first row of the schedule, enter zero for the payment number and the loan amount (PV) under the balance.

To construct the other rows except for the last two rows, complete steps 3-7:

3. Enter the size of payment (PMT) against its respective payment number. The periodic payment amount is usually a rounded amount (to the nearest cent).

4. In the AMORT worksheet, set both P1 and P2 to the payment number to obtain the principal portion and interest portion of that particular payment number. Use the down arrow key to move to BAL.

5. Enter the outstanding principal balance (BAL) in the 5th column of the schedule. Use the down arrow key to move to PRN.

6. Enter the principal portion (PRN) of the payment in the 4th column. Use the down arrow key to move to INT.

7. Enter the interest portion (INT) of the payment in the 3rd column.

To construct the row of last payment, complete steps 8-11:

8. For the final payment, we only take the interest portion from the AMORT worksheet, and we will calculate the BAL and the size of the payment as below.

9. In the row of the final payment number, the principal balance will be zero because the loan will be fully paid off.

10. For the final payment, the principal portion equals the remaining balance before the last payment ($PRN_N = BAL_{N-1}$)

11. Since PMT or N might be rounded, the final payment may be different than PMT . The final payment amount has to be calculated using any of the two methods:

- $PMT_N = PRN_N + INT_N$ where $PRN_N = BAL_{N-1}$
- $PMT_N = PMT - \text{Overpayment}$ where the overpayment is the absolute value of BAL on the last payment (BAL_N).

12. The last row lists all the totals, which can be used to cross-check the calculations:

- Total Principal Portion = Original loan amount
- Total Amount Paid = Total Interest Portion + Total Principal Portion

Note that for clear presentation all values in the table should be entered as nonnegative values.

Example 4.1.1: Construct an Amortization Schedule With Unknown PMT

Pearline took out a loan of \$10,000 from TD Bank to buy office supplies. The loan has an annual interest rate of 10%, compounded annually, and is to be repaid over four years. **a)** Determine the amount of her payments due at the end of each year. **b)** Create an amortization schedule for her loan.

Show/Hide Solution

Given information:

- Interest is compounded annually so $C/Y = 1$
- Deposits are made at the end of every year so $P/Y = 1$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Loan amortization period: $t = 4$ years
- Number of payments in the term: $N = P/Y \cdot t = 1(4) = 4$
- Nominal interest rate: $I/Y = 10\%$
- Loan amount is cash inflow: $PV = \$10,000$
- Loan is fully repaid, so $FV = 0$

a) First, we use the TVM worksheet to compute PMT.

Computing *PMT*

CLR TVM

4 N

10 I/Y

2ND P/Y=1

I/Y C/Y=1

10000 PV

0 FV

CPT PMT

-3154.708037

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Re-entering *PMT*

-3154.71 PMT

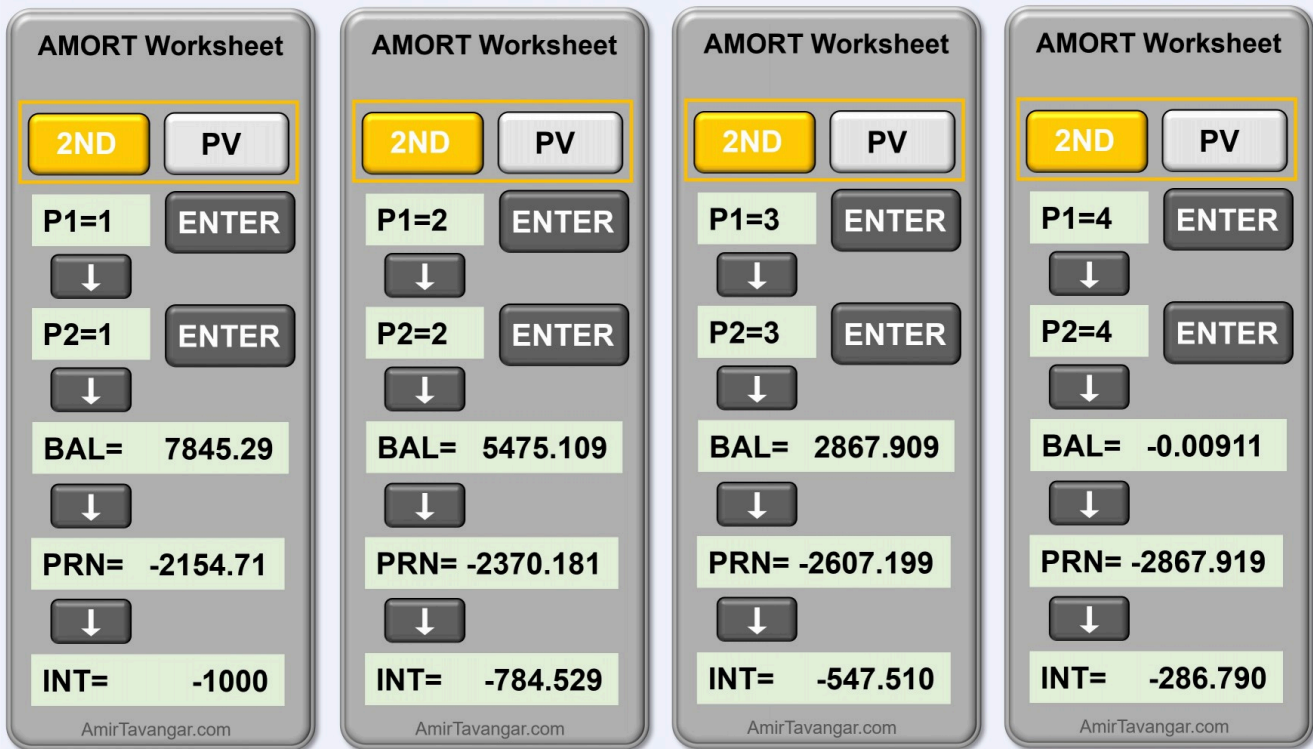
AmirTavangar.com

$PMT \approx 3154.71$

Thus, the size of the payment is \$3,154.71.

Note that the PMT value should be rounded to the nearest cent. After rounding, this adjusted value should then be re-entered into the PMT key with the same negative sign.

b) Next, we follow the steps given in the “How To – Calculator Approach” to fill in the amortization schedule. Steps 1 to 3 are already done. We now use the AMORT worksheet to obtain BAL, PRN, and INT for each payment.



It is important to note that when entering values into the amortization schedule, they should be rounded to two decimal places. Also, notice that the balance after payment number 4 is negative, which indicates overpayment.

Therefore, we need to compute the final PMT separately (Steps 8-11):

Method 1:

$$PMT_N = PRN_N + INT_N \text{ where } PRN_N = BAL_{N-1}$$

$$PMT_4 = BAL_3 + INT_4 = 2867.91 + 286.79 = \$3154.70$$

Method 2:

$$PMT_N = PMT - \text{Overpayment}$$

Here the overpayment is the absolute value of BAL on the last payment (BAL_4).

$$PMT_4 = 3154.71 - 0.01 = \$3154.70$$

Step 12: Finally, we check that all totals add up as we did before using the formula approach.

| Payment Number | Payment Amount PMT (\$) | Interest Portion INT (\$) | Principal Portion PRN (\$) | Loan Balance BAL (\$) |
|----------------|-------------------------|---------------------------|----------------------------|-----------------------|
| 0 | n/a | n/a | n/a | 10,000 |
| 1 | 3154.71 | 1000 | 2154.71 | 7845.29 |
| 2 | 3154.71 | 784.53 | 2370.18 | 5475.11 |
| 3 | 3154.71 | 547.51 | 2607.20 | 2867.91 |
| 4 | 3154.70 | 286.79 | 2867.91 | 0 |
| Totals | \$12,618.83 | \$2618.83 | \$10,000 | n/a |



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=213>

The figure above features a bar chart that shows the interest and principal portions of each payment for this example. Additionally, it includes a line chart that tracks the remaining principal balance after each payment is made. An important observation is that as the outstanding balance reduces over time, the interest portion of each payment also decreases. Since the payment amount (PMT) is constant for each period (except for the last payment), the diminishing interest portion results in an increased proportion of each payment being allocated to reduce the loan principal.

Try an Example



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Example 4.1.2: Construct an Amortization Schedule With Unknown Term

A \$15,000 loan is settled by end-of-quarter payments of \$4500. The interest is 6.8% compounded semi-annually.

a) Find the number of payments needed to settle the loan. **b)** construct an amortization schedule for the loan.

Watch Video



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Show/Hide Solution

Given information:

- Interest is compounded semi-annually so $C/Y = 2$
- Payments are made at the end of every quarter so $P/Y = 4$

$C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity

- Nominal interest rate: $I/Y = 6.8\%$

- Loan principal is cash inflow: $PV = \$15,000$
- Periodic payments are cash outflow: $PMT = -\$4500$
- The loan is fully paid: $FV = 0$

a) $N = ?$

First, we use the TVM worksheet to compute N .

Computing N

CLR TVM

6.8 I/Y

2ND P/Y=4

I/Y C/Y=2

15000 PV

-4500 PMT

0 FV

CPT N

3.459457942

AmirTavangar.com

Re-entering N

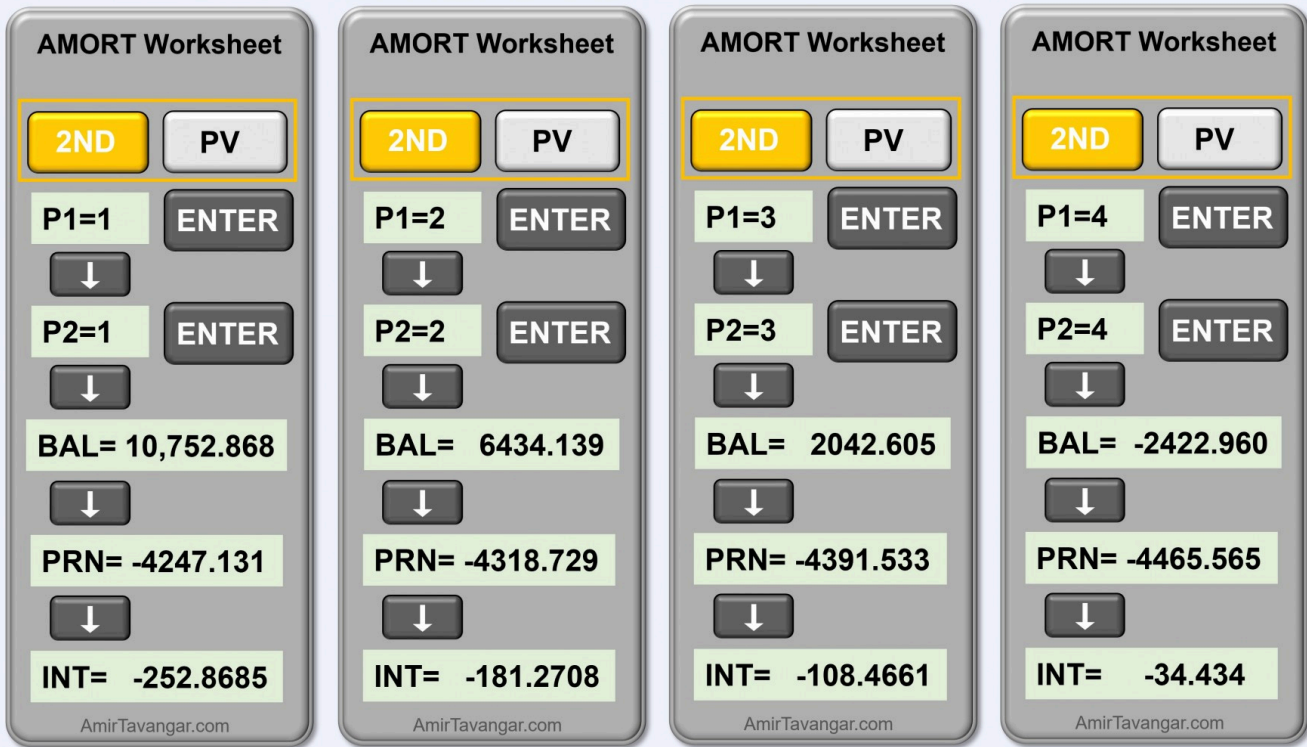
4 N

AmirTavangar.com

$N = 4$
Rounded up

Note that N is rounded up to 4 payments. The rounded value is then re-entered in the N key.

b) Next, we follow the steps given in the “How To – Calculator Approach” to fill in the amortization schedule. We now use the AMORT worksheet to obtain BAL, PRN, and INT for each payment.



Note that when entering values into the amortization schedule, they should be rounded to two decimal places. Additionally, observe that the balance after payment number 4 is negative, which indicates overpayment. Therefore, we need to adjust the final payment using the methods discussed in Steps 8-11:

Method 1:

$$PMT_N = PRN_N + INT_N \text{ where } PRN_N = BAL_{N-1}$$

$$PMT_4 = BAL_3 + INT_4 = 2042.61 + 34.43 = \$2077.04$$

Method 2:

$$PMT_N = PMT - \text{Overpayment}$$

Here the overpayment is the absolute value of BAL on the last payment (BAL_4).

$$PMT_4 = 4500 - 2422.96 = \$2077.04$$

Step 12: Finally, we check that all totals add up:

- *Total Principal Portion = Original loan amount*
- *Total Amount Paid = Total Interest Portion + Total Principal Portion*

| Payment Number | Payment Amount PMT (\$) | Interest Portion INT (\$) | Principal Portion PRN (\$) | Loan Balance BAL (\$) |
|----------------|-------------------------|---------------------------|----------------------------|-----------------------|
| 0 | n/a | n/a | n/a | 15,000 |
| 1 | 4500 | 252.87 | 4247.13 | 10,752.87 |
| 2 | 4500 | 181.27 | 4318.73 | 6434.14 |
| 3 | 4500 | 108.47 | 4391.53 | 2042.61 |
| 4 | 2077.04 | 34.43 | 2042.61 | 0 |
| Totals | \$15,577.04 | \$577.04 | \$15,000 | n/a |

The below figure features a bar chart that shows the interest and principal portions of each payment for this example. It also includes a line chart that depicts the remaining principal balance after each payment is made. It is important to note that as the outstanding balance reduces over time, the interest portion of each payment also decreases. Since the payment amount (PMT) is constant for each period (except for the last payment), the decreasing interest portion leads to an increased proportion of each payment being allocated to reduce the loan principal.



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Try an Example



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C. Calculating The Principal Balance

At times, we may need to determine the principal balance of a loan before the loan term ends, such as when considering an early settlement or making a partial repayment. This can be achieved by constructing an amortization schedule to find the outstanding principal at the desired time, or by utilizing the calculator approach outlined below.

Calculator Approach:

1. Enter the loan details into the Time Value of Money (TVM) worksheet of the financial calculator.
2. Calculate any variable (if any) that is unknown, for example, the payment size (PMT) or the number of payments (N).
3. Access the Amortization (AMORT) worksheet on the calculator. Input the range of payment periods for which the amortization needs to be calculated. For instance, to determine the balance at the end of the first year of a loan with monthly payments, set P2 to 12. P1 can be any value less than or equal to 12 since the calculator primarily considers P2 to calculate the remaining balance.
4. The calculator will provide the balance at the end of the specified range, which represents the principal balance at that date.

Example 4.1.3: Compute the Principal Balance

Megan took out a \$20,000 loan at an interest rate of 4% compounded quarterly. The loan is scheduled to be fully repaid over 8 years, with payments due at the end of each quarter. Calculate the remaining balance on the loan after the first year.

Show/Hide Solution

Given information:

- Interest is compounded annually so $C/Y = 4$
- Deposits are made at the end of every year so $P/Y = 4$

$C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity

- Loan amortization period: $t = 8$ years
- Number of payments in the term: $N = P/Y \cdot t = 4(8) = 32$
- Nominal interest rate: $I/Y = 4\%$
- Loan principal is cash inflow: $PV = \$20,000$
- Loan is fully repaid, so $FV = 0$

a) $PMT = ?$

First, we need to compute the size of the payment (PMT) using the TVM worksheet.

Computing PMT

CLR TVM

32 N

4 I/Y

2ND P/Y=4

I/Y C/Y=4

20,000 PV

0 FV

CPT PMT

-733.4177149

AmirTavangar.com

Re-entering PMT

-733.42 PMT

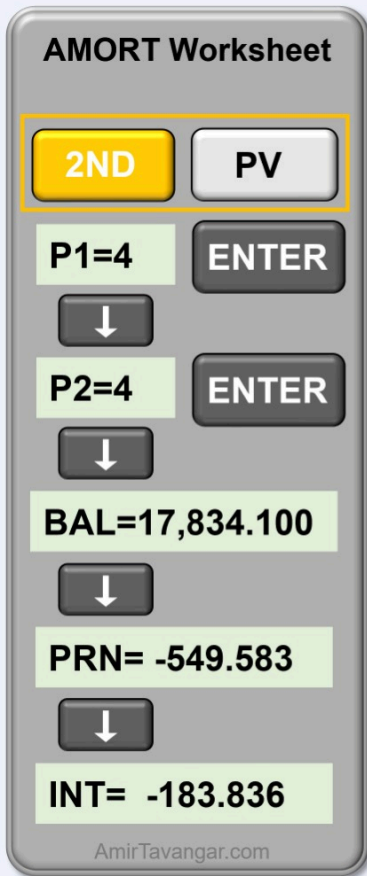
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$PMT \approx 733.42$

Thus, the size of the payment is \$733.42. Note that PMT is rounded to the nearest cents, and then the rounded value is re-entered in the PMT key.

b) The number of payments up to the focal date of $t = 1$ year is $N = 4(1) = 4$.

To find the outstanding balance at the end of the first year, we use the AMORT worksheet and set $P1 = P2 = 4$ ($P1$ can be any value less than or equal to 4 but make sure $P2 = 4$).



Therefore, the outstanding principal balance at the end of the first year is \$17,834.10.

Try an Example



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D. Calculating the Interest and Principal Portions

Sometimes, it may be necessary to determine the interest or principal portion of a specific payment or a specific period of time within the loan term.

For a Specific Payment: To find the interest and principal portion of a specific payment, in the AMORT worksheet, set both P1 and P2 to that specific payment number. Then, review the details provided to see the breakdown of interest and principal for that specific payment.

For a Specific Period: To determine the total principal or interest paid during a certain period of a loan, such as the 5th year, first, identify the payment numbers that correspond to the start and end of this period. In the AMORT worksheet, set P1 to the payment number marking the start of the period and P2 to the payment number at the end of the period. The calculator will then display the cumulative interest and principal portions paid between these two points.

Example 4.1.4: Compute the Interest and Principal Portion of a Payment

Samuel took out a \$308,000 mortgage to buy an apartment. The mortgage is structured to be repaid with monthly payments of \$2,375.11 at the end of each month. The interest rate on the mortgage is 4.62%, compounded monthly, and the loan is amortized over 15 years. **a)** Calculate the interest portion of the 21st payment. **b)** Calculate the principal portion of the 21st payment.

Show/Hide Solution

Given information:

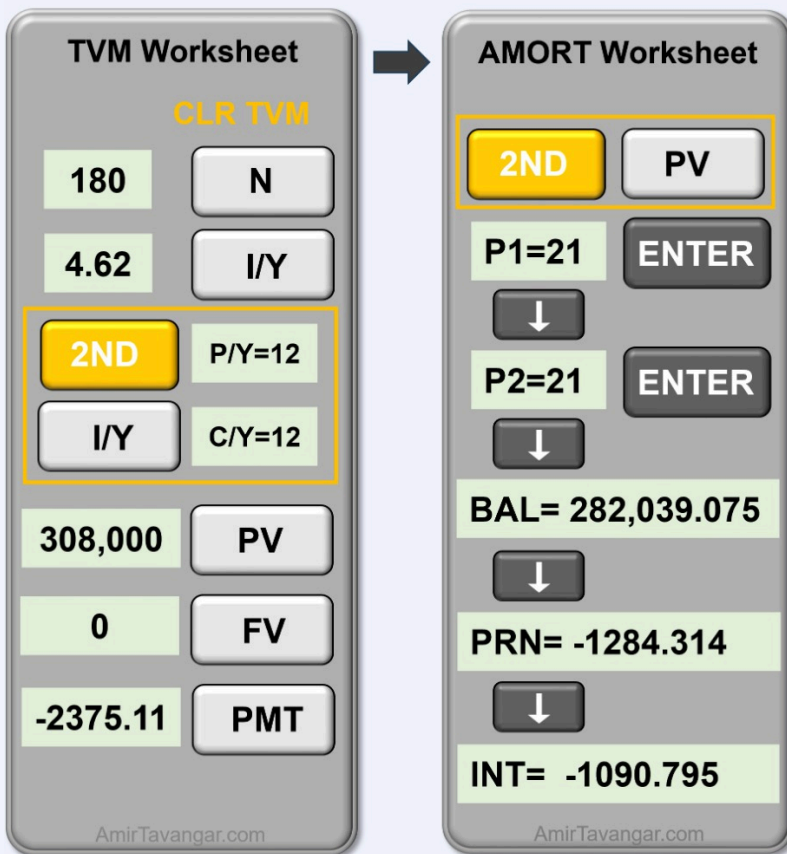
- Interest is compounded monthly so $C/Y = 12$
- Deposits are made at the end of every year so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Loan amortization period: $t = 15$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(15) = 180$
- Nominal interest rate: $I/Y = 4.62\%$

- Loan principal is cash inflow: $PV = \$308,000$
- Periodic payments are cash outflow: $PMT = -\$2,375.11$
- Loan is fully repaid, so $FV = 0$

To find the interest and principal portions of the 21st payment, start by entering the loan information into the TVM worksheet. Next, access the AMORT worksheet and set both P1 and P2 to 21, which represents the 21st payment. Then, review the details provided to see the breakdown of interest and principal for that specific payment.



a) Therefore, the interest portion of the 21st payment is \$1090.80.

b) The principal portion of the 21st payment is \$1284.31.

Try an Example



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Example 4.1.5: Compute the Interest and Principal Portion of a Time Period

Samuel took out a \$308,000 mortgage to buy an apartment. The mortgage is structured to be repaid with monthly payments of \$2,375.11 at the end of each month. The interest rate on the mortgage is 4.62%, compounded monthly, and the loan is amortized over 15 years. **a)** Calculate the total principal amount repaid in the 8th year. **b)** Calculate the total amount of interest paid in the 8th year.

Show/Hide Solution

Given information:

- Interest is compounded monthly so $C/Y = 12$
- Deposits are made at the end of every year so $P/Y = 12$

$C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity

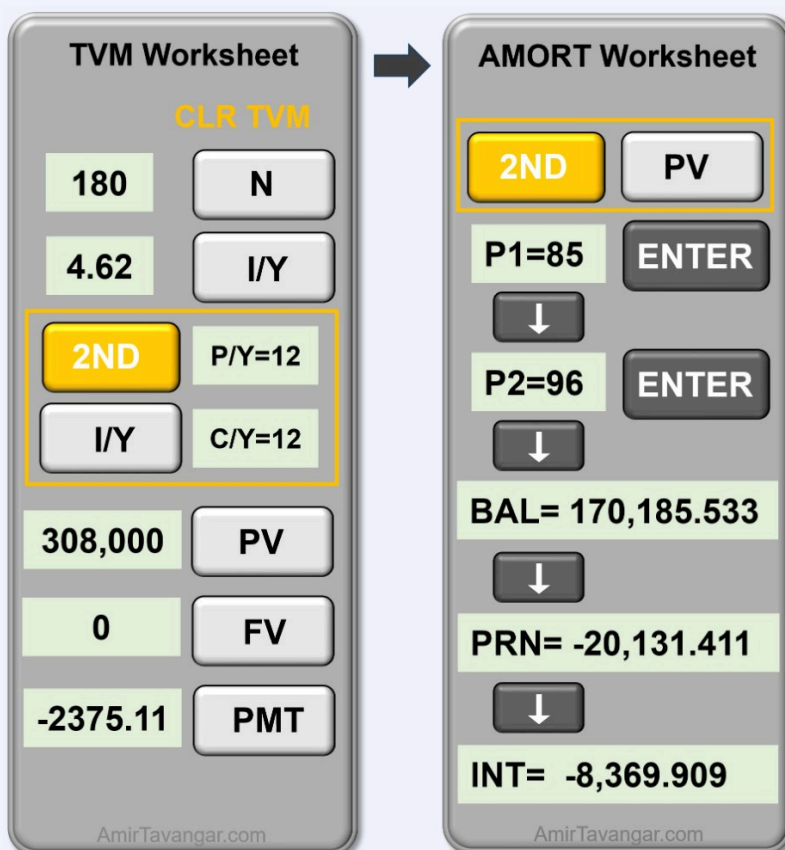
- Loan amortization period: $t = 15$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(15) = 180$
- Nominal interest rate: $I/Y = 4.62\%$
- Loan principal is cash inflow: $PV = \$308,000$
- Periodic payments are cash outflow: $PMT = -\$2,375.11$
- Loan is fully repaid, so $FV = 0$

To determine the total principal or interest paid during a certain year of a loan, such as the 8th year, we need to identify the specific payment numbers that mark the beginning and end of that year. We then use the AMORT worksheet on a financial calculator to calculate those amounts:

i) To find the first payment of the 8th year, we calculate the number of payments made by the end of the 7th year, which is 12 payments per year times 7 years, equaling 84 payments. Therefore, the first payment of the 8th year is payment number 85 (84 from the previous years plus 1).

ii) The last payment of the 8th year is determined by the total number of payments made in 8 years, which is 12 payments per year times 8 years, equaling 96 payments.

iii) To calculate the interest and principal portions for the 8th year, you input the loan details into the TVM worksheet. Then, in the AMORT worksheet, set P1 to 85 and P2 to 96. The calculator will then display the cumulative interest and principal portions paid between these two points in time.



a) Therefore, the total principal amount repaid in the 8th year is \$20,131.41.

b) The total interest amount paid in the 8th year is \$8369.91.

Try an Example



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Section 4.1 Exercises

1. Martina took out a \$84,000 loan at an interest rate of 5.88% compounded semi-annually. The loan is scheduled to be fully repaid over 12 years, with payments due at the end of each month. **a)** Calculate the size of month-end payments. **b)** Calculate the principal balance at the end of year 2.

Show/Hide Answer

a) $PMT = \$811.45$

b) $BAL = \$73,724.15$

2. Erika took out a \$32,600 loan at an interest rate of 4.83% compounded monthly. The loan is scheduled to be fully repaid over 9 years, with payments due at the end of each month. **a)** Calculate the size of month-end payments. **b)** Calculate the total interest amount paid in the 4th year. **b)** Calculate the total principal amount repaid in the 4th year.

Show/Hide Answer

a) $PMT = \$372.80$

b) $INT = \$1,048.36$

$$\text{c) PRN} = \$3,425.24$$

3. Johnetta took out a \$20,200 loan at an interest rate of 3.53% compounded monthly. The loan is scheduled to be fully repaid over 8 years, with payments due at the end of each month. **a)** Calculate the size of month-end payments. **b)** Calculate the interest portion of the 60th payment. **c)** Calculate the principal portion of the 60th payment.

Show/Hide Answer

$$\text{a) PMT} = \$241.83$$

$$\text{b) INT} = \$24.91$$

$$\text{c) PRN} = \$216.92$$

4.2 MORTGAGES: CALCULATOR APPROACH

A. Mortgage Definition and Types

A **mortgage** is a loan that is issued by a financial institution to a borrower to purchase real estate properties (business or residential properties) at a specific interest rate for a specific time period. The **mortgage amortization period** is the length of time it takes to pay off the entire amount borrowed based on the original mortgage contract. A **mortgage term** is the length of time for which the mortgage agreement (a specific interest rate) will be in effect. Mortgage agreements are generally negotiated several times during the amortization period. At the end of each mortgage term, the borrower renegotiates and renews for another term. The interest rate, the size of the payment, and payment and compounding frequencies can change each term. Figure 4.2.1 depicts an example of the amortization period and typical renewal terms for a 25-year mortgage.

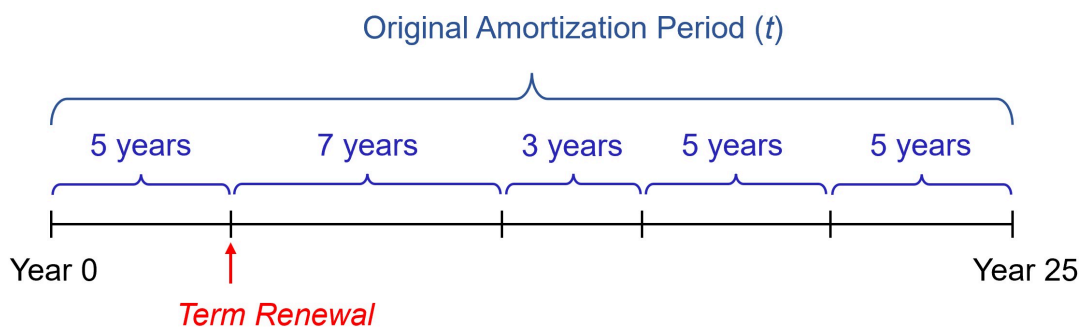


Figure 4.2.1 An example of the amortization period and typical renewal terms for a 25-year mortgage

Financial institutions usually offer two types of mortgages: fixed-rate mortgages and variable-rate mortgages. **Fixed-rate mortgages** have a fixed interest rate for a specific period of time (mortgage term). Once the term is completed, the mortgage is renewed. The new term may have a different interest rate, size of the payment, and payment frequency. **Variable-rate Mortgages** are the ones in which the interest rate fluctuates depending on changes in the money market. This rate change is usually related to the lender's prime rate.

Both mortgage types can be either open or closed. **Closed Mortgages** have a specific term and borrowers would have to pay a penalty for prepaying a large amount, closing the loan, or transferring to another lender. In contrast, **open Mortgages** allow borrowers to prepay any amount at any time, close, or transfer from one lender to another without

being penalized. Because of the flexibility, interest rates of open mortgages are usually higher than those of closed mortgages. Figure 13.2 summarizes the types of mortgages.

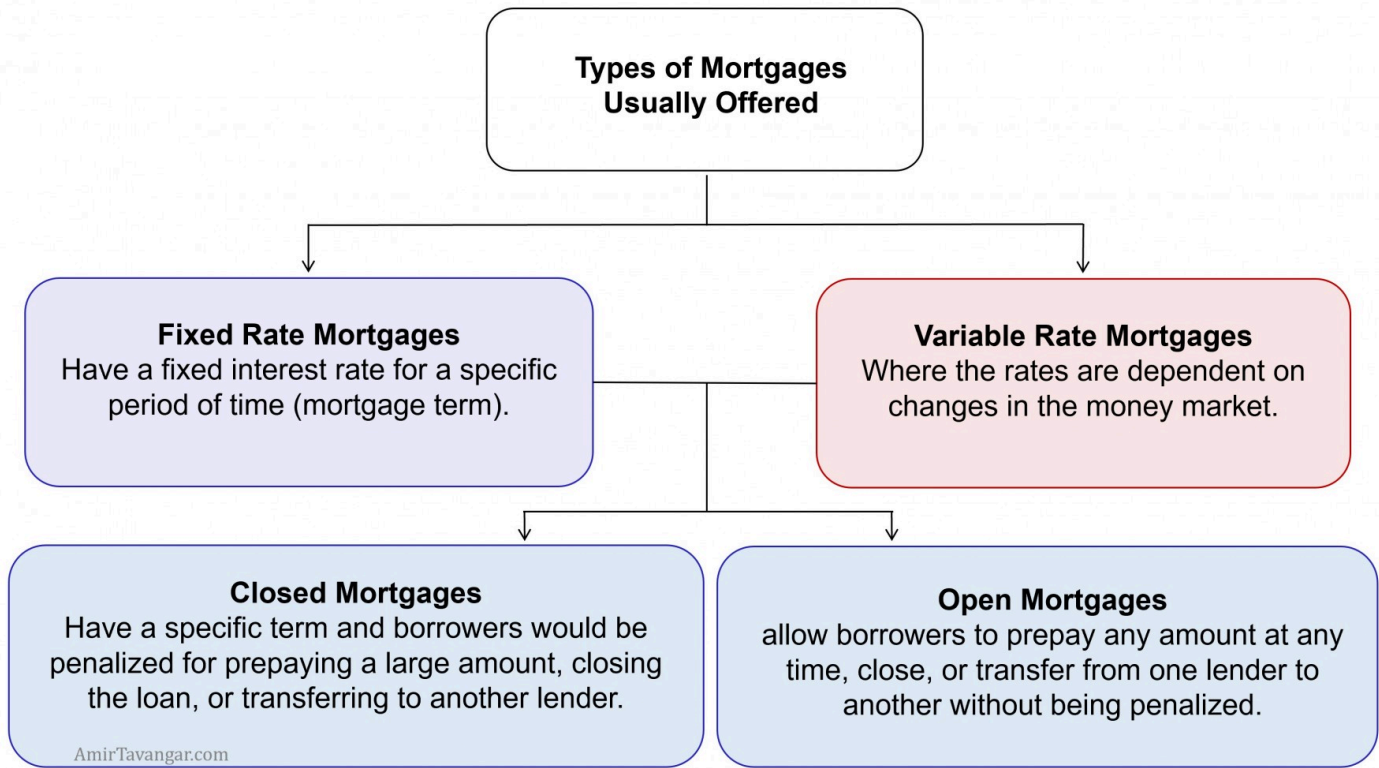


Figure 4.2.2 Types of Mortgages

B. Payment Amounts and the Outstanding Principal Balance on Mortgages

Calculations of payment amounts and the outstanding principal balance on mortgages are similar to what was discussed previously for all loan types. The next example illustrates.

Example 4.2.1: Compute Mortgage Payments for the First and Renewal Terms

John purchased a house for \$350,000. He made a down payment of 15% of the value of the house and received a mortgage for the rest of the amount amortized over 20 years. He negotiated a fixed interest rate of 3.8% compounded semi-annually for a 3-year term. **a)** Calculate the end-of-quarter payment size. **b)** Find the

outstanding principal balance at the end of the 3-year term. **c)** Calculate the new end-of-quarter payment amount if the mortgage was renewed for another 3 years at 2.5% compounded semi-annually.

Watch Video



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Show/Hide Solution

Given information:

- Interest is compounded semi-annually so $C/Y = 2$
- Payments are made at the end of every quarter so $P/Y = 4$

$C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity

- Amortization period: $t = 20$ years
- Number of payments in the amortization period: $N = P/Y \cdot t = 4(20) = 80$
- Nominal interest rate for Term 1: $I/Y_1 = 3.8\%$
- Nominal interest rate for Term 2: $I/Y_2 = 2.5\%$
- Purchase price = \$350,000
- Down payment = $15\% (350,000) = \$52,500$

First, we need to find the amount of the mortgage, which is the remaining balance of the house price after paying the down payment.

$$PV = \text{Purchase Price} - \text{Down}$$

$$PV = 350,000 - 52,500 = \$297,500$$

a) We use the TVM worksheet to compute the PMT of the first 3-year term. Mortgage principal is cash inflow, so PV is input as a positive value.

The image shows two calculator screens. The left screen, titled "Computing PMT", shows the following inputs: N=80, I/Y=3.8, P/Y=4, C/Y=2, PV=297500, FV=0. The PMT key is highlighted, and the result is -5,317.616287. The right screen, titled "Re-entering PMT", shows the value -5317.62 being entered into the PMT field. A red arrow points from the PMT result on the left screen to the PMT input on the right screen.

Computing PMT

CLR TVM

80 N

3.8 I/Y

2ND P/Y=4

I/Y C/Y=2

297500 PV

0 FV

CPT PMT

-5,317.616287

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Re-entering PMT

-5317.62 PMT

AmirTavangar.com

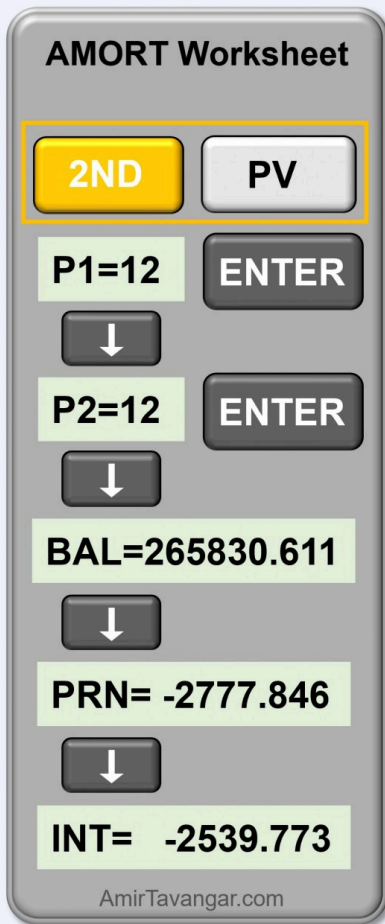
$PMT \approx 5317.62$

The size of the quarterly payments is \$5317.62. Note that PMT is rounded to the nearest cent, and then the rounded value is re-entered in the PMT key.

b) The number of payments by the end of the 3-year term will be

$$\text{Number of payments in Term 1} = P/Y \cdot \text{Number of years in Term 1} = 4(3) = 12$$

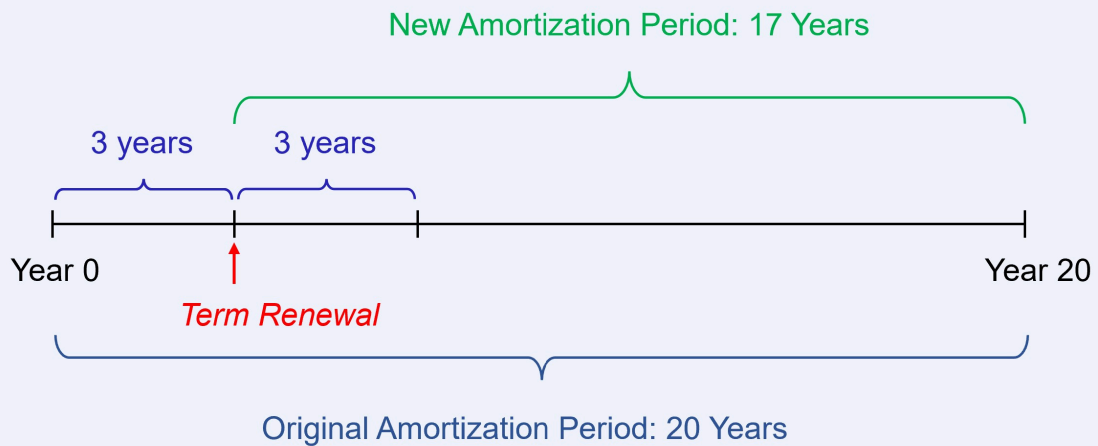
To find the outstanding balance at the end of the first 3-year term, we use the AMORT worksheet and set $P1 = P2 = 12$ ($P1$ can be any value less than and equal to 12 but make sure $P2 = 12$).



***BAL* ≈ 265,830.61**

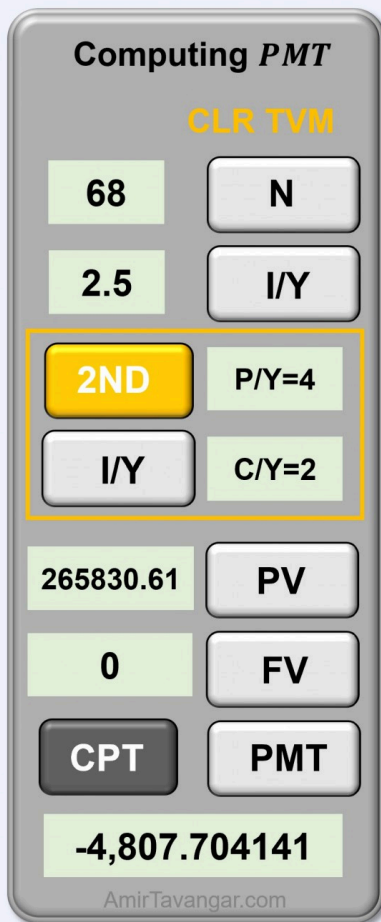
Therefore, the outstanding principal balance at the end of the first 3-year term is \$265,830.61.

c) The renewed term will have a new PMT as the nominal interest rate has reduced. To compute the new PMT, we use the TVM worksheet and the information for Term 2:



New amortization period = Original Amortization period – the first term

- New amortization period: $t_2 = 20 \text{ years} - 3 \text{ years} = 17 \text{ years}$
- The present value is the balance at the end of the first term: $PV = \$265,830.61$
- Number of payments in the new amortization period: $N = P/Y(t_2) = 4(17) = 68$
- New nominal interest rate $I/Y_2 = 2.5\%$



$$PMT \approx 4807.70$$

Therefore, the size of the quarterly payments in the renewed term is \$4807.70.



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The above figure includes a bar chart displaying the interest and principal portions of both the payments already made and those projected until the end of the amortization period. Also, it features line charts depicting the outstanding principal balance for Term 1 (covering the first 12 payments) and Term 2 (encompassing payments 13 to 24). It is noteworthy that due to the lower interest rate in the second term, the size of the payments and the interest portions are smaller compared to those in the first term. The slider tool allows you to zoom in and focus on a specific period within the chart.

Try an Example



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Section 4.2 Exercises

- Sheniqua purchased a house for \$930,000. He made a down payment of 16% of the value of the house and received a mortgage for the rest of the amount amortized over 25 years. She negotiated a fixed interest rate of 3.56% compounded semi-annually for a 5-year term. **a)** Calculate the size of month-end payments. **b)** Calculate the principal balance at the end of the 5-year term. **c)** Calculate the size of month-end payments if the mortgage was renewed for another 5-year term at a fixed rate of 2.97% compounded semi-annually.

Show/Hide Answer

a) $PMT = \$3,925.08$

b) $BAL = \$674,757.75$

c) $\text{New } PMT = \$3,725.93$

- Sarah purchased a house for \$1,770,000. She made a down payment of 15% of the value of the house and received a mortgage for the rest of the amount amortized over 25 years. She negotiated a fixed interest rate of 3.2% compounded semi-annually for a 4-year term. **a)** Calculate the size of month-end payments. **b)** Calculate the principal balance at the end of the 4-year term. **c)** Calculate the size of month-end payments if the mortgage was renewed for another 4-year term at a fixed rate of 2.01% compounded semi-annually.

Show/Hide Answer

a) $PMT = \$7,275.27$

b) $BAL = \$1,336,349.88$

c) $\text{New } PMT = \$6,499.72$

3. Arman bought an apartment for \$850,000. He made a down payment of 26% of the value of the apartment and received a mortgage for the rest of the amount amortized over 25 years. He negotiated a fixed interest rate of 3.96% compounded semi-annually for a 7-year term. **a)** What was the size of month-end payments? **b)** Calculate the principal balance at the end of the 7-year term. **c)** What is the size of month-end payments if the mortgage is renewed for another 7-year term at a fixed rate of 3.9% compounded semi-annually?

Show/Hide Answer

a) $PMT = \$3,295.04$

b) $BAL = \$509,698.20$

c) $\text{New } PMT = \$3,279.57$

4.3 LOAN AMORTIZATION: FORMULA APPROACH

A. Amortization Definitions

Amortization of a loan is the process of the gradual reduction in the loan amount through periodic repayments, usually of equal size, over a predetermined length of time. The **amortization period** is the time period set to repay the loan. An **amortization schedule** is a detailed table, similar to what is shown in Table 4.3.1, that breaks down each loan payment amount into its interest (INT) and principal portions (PRN). It also shows the outstanding principal balance (BAL) after each payment is made.

Table 4.3.1 An Amortization Schedule

| Payment Number | Payment Amount PMT (\$) | Interest Portion INT (\$) | Principal Portion PRN (\$) | Loan Balance BAL (\$) |
|----------------|----------------------------|------------------------------|-------------------------------|-----------------------|
| 0 | n/a | n/a | n/a | Loan Amount (PV) |
| 1 | PMT | INT_1 | PRN_1 | BAL_1 |
| 2 | PMT | INT_2 | PRN_2 | BAL_2 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| N-1 | PMT | INT_{N-1} | PRN_{N-1} | BAL_{N-1} |
| N | PMT_N | INT_N | PRN_N | 0 |
| Totals | Total Amount Paid | Total Interest Portion | Total Principal Portion | n/a |

Note that the size of payment (PMT) is the same for every period except the last payment as it needs to be adjusted to fully repay the loan. In banking, all actual monetary values are rounded to the nearest cent (two decimal places). Because of this rounding, the final payment needs to be adjusted and thus is almost always different from other payments.

B. Amortization Schedules for Loans

How to Construct an Amortization Schedule: Algebraic (Formula) Approach

1. Calculate the periodic interest rate per payment period.

- For simple annuities, it will be $i = \frac{I/Y}{C/Y}$
- For general Annuities, it will be $i_2 = (1 + i)^c - 1$ where $c = \frac{C/Y}{P/Y}$.

2. In the first row, enter zero for the payment number and the loan amount (PV) under the balance.

3. In the second row, enter the size of payment (PMT) against its respective payment number. The periodic payment amount is usually a rounded amount (to the nearest cent).

4. Enter the interest portion (INT) of the payment in the 3rd column. This is the previous principal balance multiplied by the periodic interest rate:

- $INT_1 = BAL_0 \times i$ for simple annuities or
- $INT_1 = BAL_0 \times i_2$ for general annuities.

5. Enter the principal portion (PRN) of the payment in the 4th column. This is the difference between the periodic payment amount and the interest portion: $PRN_1 = PMT - INT_1$

6. Enter the outstanding principal balance (BAL) in the 5th column. This is the difference between the previous principal balance and the principal portion: $BAL_1 = BAL_0 - PRN_1$

7. Construct the remaining rows of the schedule by doing the same calculations in Steps 3 – 6 until the final payment number, where the principal balance will be zero.

8. The final payment amount is calculated by adding the previous principal balance and the interest charged on the previous principal balance. $PMT_N = PRN_N + INT_N$ where $PRN_N = BAL_{N-1}$ and

- $INT_N = PRN_{N-1} \times i$ for simple annuities
- $INT_N = PRN_{N-1} \times i_2$ for general annuities

9. The last row lists all the totals, which can be used to cross-check the calculations:

- Total Principal Portion = Original loan amount
- Total Amount Paid = Total Interest Portion + Total Principal Portion

Example 4.3.1: Construct an Amortization Schedule With Unknown PMT

Pearline took out a loan of \$10,000 from TD Bank to buy office supplies. The loan has an annual interest rate of 10%, compounded annually, and is to be repaid over four years. **a)** Determine the amount of her payments due at the end of each year. **b)** Create an amortization schedule for her loan.

Show/Hide Solution

Given information:

- Interest is compounded annually so $C/Y = 1$
- Deposits are made at the end of every year so $P/Y = 1$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Loan amortization period: $t = 4$ years
- Number of payments in the term: $N = P/Y \cdot t = 1(4) = 4$
- Nominal interest rate: $I/Y = 10\%$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{10\%}{1} = 10\%$
- Loan amount: $PV = \$10,000$
- Loan is fully repaid, so $FV = 0$

a) First, we need to find the size of the payment (PMT). Substituting the values into Formula 3.9b gives

$$PMT = \frac{PV \cdot i}{1 - (1 + i)^{-N}}$$

$$PMT = \frac{10000(10\%)}{1 - (1 + 10\%)^{-4}}$$

$$= 3154.708\dots$$

$$\approx \$3154.71$$

b) Next, we follow the steps given in the “How To – Formula Approach” to fill in the amortization schedule. For simple annuities, the interest per payment period is equal to the interest per compounding period, so we use $i = 10\%$ for the payment interest portion (INT) calculation.

| Payment Number | Payment Amount PMT (\$) | Interest Portion INT (\$) | Principal Portion PRN (\$) | Loan Balance BAL (\$) |
|----------------|-------------------------------|---------------------------|-------------------------------|--------------------------------|
| 0 | n/a | n/a | n/a | 10,000 |
| 1 | 3154.71 | 10,000(10%) = 1000 | 3154.71 – 1000 = 2154.71 | 10,000 – 2154.71 = 7845.29 |
| 2 | 3154.71 | 7845.29(10%) = 784.53 | 3154.71 – 784.53 = 2370.18 | 7845.29 – 2370.18 = 5475.11 |
| 3 | 3154.71 | 5475.11(10%) = 547.51 | 3154.71 – 547.51 = 2607.20 | 5475.11 – 2607.20 = 2867.91 |
| 4 | 286.79 + 2867.91 = 3154.70 | 2867.91(10%) = 286.79 | 2867.91 | 0 |
| Totals | \$12,618.83 | \$2618.83 | \$10,000 | n/a |

Step 9: Once the schedule is filled, we compute the totals and check that

- *Total Principal Portion = Original loan amount*
- *Total Amount Paid = Total Interest Portion + Total Principal Portion*

which are correct for this schedule.



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The figure above features a bar chart that shows the interest and principal portions of each payment for this example. Additionally, it includes a line chart that tracks the remaining principal balance after each payment is made. An important observation is that as the outstanding balance reduces over time, the interest portion of each payment also decreases. Since the payment amount (PMT) is constant for each period (except for the last payment), the diminishing interest portion results in an increased proportion of each payment being allocated to reduce the loan principal.

Try an Example



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Example 4.3.2: Construct an Amortization Schedule With Unknown Term

A \$15,000 loan is settled by end-of-quarter payments of \$4500. The interest is 6.8% compounded semi-annually.
a) Find the number of payments needed to settle the loan. **b)** construct an amortization schedule for the loan.

Show/Hide Solution,

Given information:

- Interest is compounded semi-annually so $C/Y = 2$
- Payments are made at the end of every quarter so $P/Y = 4$

$C/Y \neq P/Y \Rightarrow$ Ordinary General Annuity

- Nominal interest rate: $I/Y = 6.8\%$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{6.8\%}{2} = 3.4\%$
- Loan principal: $PV = \$15,000$
- Size of payments: $PMT = \$4500$
- The loan is fully paid: $FV = 0$

a) $N = ?$

The term duration and thus the number of payments in the term are unknown. Since the present value of the annuity is known, we can apply Formula 3.10c to determine the number of payments, N , for an ordinary general annuity. The first step in this process is to calculate the interest rate per payment period, i_2 .

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$i_2 = (1 + 0.034)^{\frac{2}{4}} - 1$$

$$= 0.016857\dots$$

Applying Formula 3.10c yields

$$N = - \frac{\ln \left[1 - \frac{PV \cdot i_2}{PMT} \right]}{\ln(1 + i_2)}$$

$$N = - \frac{\ln \left[1 - \frac{15,000(0.016857...)}{4500} \right]}{\ln(1 + 0.016857...)}$$

$$= 3.4...$$

$$= 4 \text{ (Rounded up)}$$

The number of payments in the term, N , is rounded up to 4 payments.

b)

Next, we follow the steps given in the “How To – Formula Approach” to fill in the amortization schedule. For general annuities, the interest per payment period, $i_2 = 0.016857...$, should be used for the payment interest portion (INT) calculation.

| Payment Number | Payment Amount PMT (\$) | Interest Portion INT (\$) | Principal Portion PRN (\$) | Loan Balance BAL (\$) |
|----------------|----------------------------------|--|--------------------------------|--------------------------------------|
| 0 | n/a | n/a | n/a | 15,000 |
| 1 | 4500 | $15,000(0.016857\dots)$ $= 252.87$ | $4500 - 252.87$ $= 4247.13$ | $15,000 - 4247.13$ $= 10,752.87$ |
| 2 | 4500 | $10,752.87(0.016857\dots)$ $= 181.27$ | $4500 - 181.27$ $= 4318.73$ | $10,752.87 - 4318.73$ $= 6434.14$ |
| 3 | 4500 | $6434.14(0.016857\dots)$ $= 108.47$ | $4500 - 108.47$ $= 4391.53$ | $6434.14 - 4391.53$ $= 2042.61$ |
| 4 | $34.43 + 2042.61$ $= 2077.04$ | $2042.61(0.016857\dots)$ $= 34.43$ | 2042.61 | 0 |
| Totals | \$15,577.04 | \$577.04 | \$15,000 | n/a |

The below figure features a bar chart that shows the interest and principal portions of each payment for this example. It also includes a line chart that depicts the remaining principal balance after each payment is made. It is important to note that as the outstanding balance reduces over time, the interest portion of each payment also decreases. Since the payment amount (PMT) is constant for each period (except for the last payment), the decreasing interest portion leads to an increased proportion of each payment being allocated to reduce the loan principal.



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Try an Example



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C. Calculating The Principal Balance

At times, we may need to determine the principal balance of a loan before the loan term ends, such as when considering an early settlement or making a partial repayment. This can be achieved by constructing an amortization schedule to find the outstanding principal at the desired time, or by utilizing the formula approach outlined below.

The retrospective method can be used to calculate the principal balance at a specific focal point in time. This method involves the calculation of the future values of both the payments made and the loan principal up to that focal point. The principal balance at the focal date is then found by taking the difference between these future values.

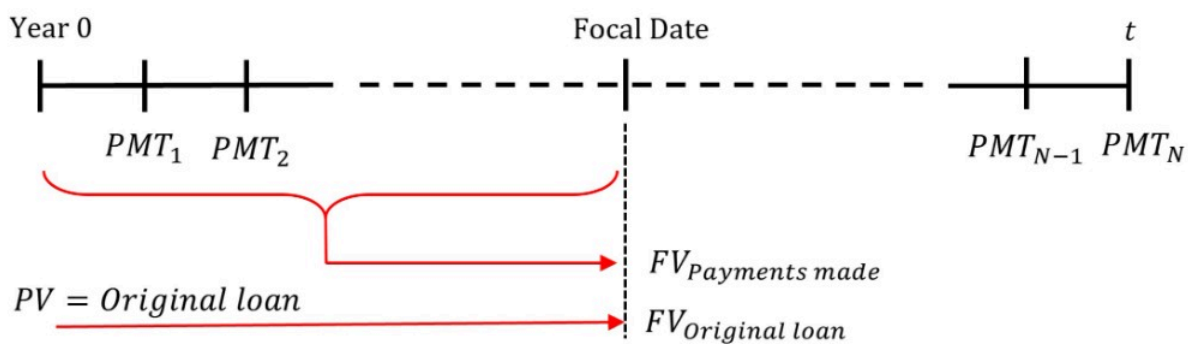


Figure 4.3.2 Timeline for Retrospective method of calculating the balance at a specific focal date

$$BAL_{\text{Focal date}} = FV_{\text{Original loan}} - FV_{\text{Payments up to focal date}} \quad \text{Formula 4.1}$$

Example 4.3.3: Compute the Principal Balance

Megan took out a \$20,000 loan at an interest rate of 4% compounded quarterly. The loan is scheduled to be fully repaid over 8 years, with payments due at the end of each quarter. Calculate the remaining balance on the loan after the first year.

Show/Hide Solution

Given information:

- Interest is compounded annually so $C/Y = 4$
- Deposits are made at the end of every year so $P/Y = 4$

$C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity

- Loan amortization period: $t = 8$ years
- Number of payments in the term: $N = P/Y \cdot t = 4(8) = 32$
- Nominal interest rate: $I/Y = 4\%$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{4\%}{4} = 1\%$
- Loan principal: $PV = \$20,000$
- Loan is fully repaid, so $FV = 0$

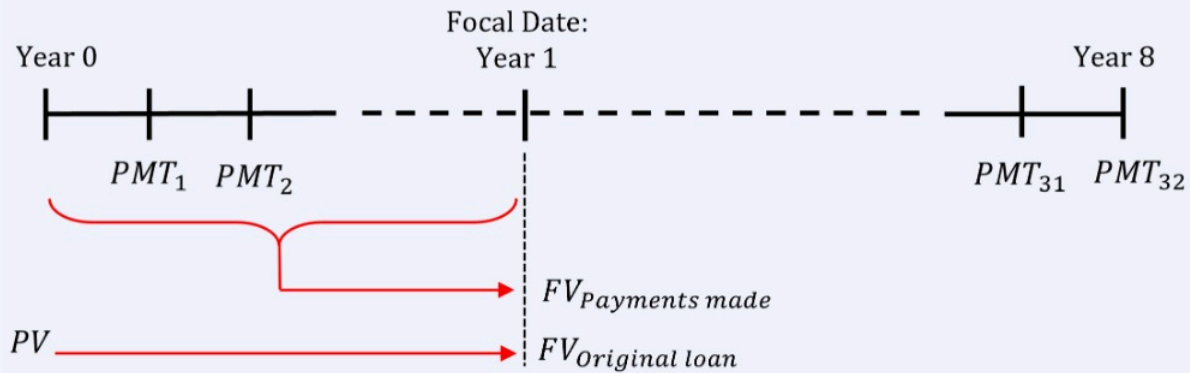
i) $PMT = ?$

First, we need to find the size of the payment (PMT). Substituting the values into Formula 3.9b gives

$$\begin{aligned}
 PMT &= \frac{PV \cdot i}{1 - (1 + i)^{-N}} \\
 PMT &= \frac{20,000(1\%)}{1 - (1 + 1\%)^{-32}} \\
 &= 733.417... \\
 &\approx \$733.42
 \end{aligned}$$

Thus, the size of the payment is \$733.42.

ii) Next, we use Formula 4.1 to calculate the balance at the end of the first year (focal date). By the end of the first year, four payments will have been made.



$$BAL_{\text{Focal date}} = FV_{\text{Original loan}} - FV_{\text{PMTs made up to focal date}}$$

To find the future value of the original loan, we use the future value formula of compound interest (Formula 2.4a), and to calculate the future value of the payments, we apply the future value formula of the ordinary simple annuity (Formula 3.5a).

The number of payments up to the focal date of $t = 1$ year is $N = 4(1) = 4$.

$$BAL_{\text{Focal date}} = FV_{\text{Original loan}} - FV_{\text{PMTs made up to focal date}}$$

$$BAL_{\text{Focal date}} = PV(1 + i)^N - PMT \left[\frac{(1 + i)^N - 1}{i} \right]$$

$$= 20,000(1 + 0.01)^4 - 733.42 \left[\frac{(1 + 0.01)^4 - 1}{0.01} \right]$$

$$= 20,812.08 - 2977.98$$

$$= \$17,834.10$$

Try an Example



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D. Calculating the Interest and Principal Portions

Sometimes, it may be necessary to determine the interest or principal portion of a specific payment within the loan term. The interest portion for any payment is calculated by multiplying the principal balance outstanding after the previous payment and the periodic interest rate per payment period. After calculating the interest portion, the principal portion of the payment can be found by deducting the interest portion from the payment amount.

Example 4.3.4: Compute the Interest and Principal Portion of a Payment

Samuel took out a \$308,000 mortgage to buy an apartment. The mortgage is structured to be repaid with monthly payments of \$2,375.11 at the end of each month. The interest rate on the mortgage is 4.62%, compounded monthly, and the loan is amortized over 15 years. **a)** Calculate the interest portion of the 21st payment. **b)** Calculate the principal portion of the 21st payment.

Show/Hide Solution

Given information:

- Interest is compounded monthly so $C/Y = 12$
- Deposits are made at the end of every year so $P/Y = 12$

$$C/Y = P/Y \Rightarrow \text{Ordinary Simple Annuity}$$

- Loan amortization period: $t = 15$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(15) = 180$
- Nominal interest rate: $I/Y = 4.62\%$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{4.62\%}{12} = 0.385\%$
- Loan principal: $PV = \$308,000$
- Periodic payments: $PMT = \$2,375.11$
- Loan is fully repaid, so $FV = 0$

To find the interest portion of the 21st payment, we need to find the principal balance outstanding after the previous payment (20th payment) is made.

i) Finding the previous outstanding balance

We use Formula 4.1 to calculate the balance after 20 payments (focal date).

$$BAL_{\text{Focal date}} = FV_{\text{Original loan}} - FV_{\text{PMTs made up to focal date}}$$

To find the future value of the original loan, we use the future value formula of compound interest (Formula 2.4a), and to calculate the future value of the payments, we apply the future value formula of the ordinary simple annuity (Formula 3.5a). Note that since the annuity is simple, the number of compounding periods and payments are equal for any time period, so in both formulas $N = 20$.

$$BAL_{\text{Focal date}} = FV_{\text{Original loan}} - FV_{\text{PMTs made up to focal date}}$$

$$\begin{aligned} BAL_{\text{Focal date}} &= PV(1+i)^N - PMT \left[\frac{(1+i)^N - 1}{i} \right] \\ &= 308,000(1 + 0.00385)^{20} - 2375.11 \left[\frac{(1 + 0.00385)^{20} - 1}{0.00385} \right] \\ &= 332,603.78 - 49,280.39 \\ &= \$283,323.39 \end{aligned}$$

ii) Calculating the interest portion of the 21st payment

a) The interest portion on a payment is the product of the outstanding balance and the interest rate per payment period (Step 4 of the “How to Construct an Amortization Schedule”). For simple annuities, the interest per payment period (i_2) equals the interest rate per compounding period (i). Thus the interest portion on the 21st payment is obtained by

$$\begin{aligned} INT_{21} &= BAL_{20} \cdot i \\ &= 283,323.39(0.00385) \\ &\approx \$1090.80 \end{aligned}$$

iii) Calculating the principal portion of the 21st payment

b) The principal portion of a payment is the difference between the amount of payment and the interest portion of that payment.

$$\begin{aligned} PRN_{21} &= PMT - INT_{21} \\ &= 2375.11 - 1090.80 \\ &= \$1284.31 \end{aligned}$$

Try an Example



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Example 4.3.5: Compute the Interest and Principal Portion of a Time Period

Samuel took out a \$308,000 mortgage to buy an apartment. The mortgage is structured to be repaid with monthly payments of \$2,375.11 at the end of each month. The interest rate on the mortgage is 4.62%, compounded monthly, and the loan is amortized over 15 years. **a)** Calculate the total principal amount repaid in the 8th year. **b)** Calculate the total amount of interest paid in the 8th year.

Show/Hide Solution

Given information:

- Interest is compounded monthly so $C/Y = 12$
- Deposits are made at the end of every year so $P/Y = 12$

$C/Y = P/Y \Rightarrow$ Ordinary Simple Annuity

- Loan amortization period: $t = 15$ years
- Number of payments in the term: $N = P/Y \cdot t = 12(15) = 180$
- Nominal interest rate: $I/Y = 4.62\%$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{4.62\%}{12} = 0.385\%$
- Loan principal: $PV = \$308,000$
- Periodic payments: $PMT = \$2,375.11$
- Loan is fully repaid, so $FV = 0$

The total amount of principal repaid over a specific time period is calculated by subtracting the principal balance at the end of the period from the principal balance at the beginning of that period.

$$\text{Principal repaid in year 8} = BAL_{7\text{th year}} - BAL_{8\text{th year}}$$

i) Finding the principal balance at the end of the 7th year (same as the beginning of the 8th year):

The number of payments up to the focal date of $t = 7$ year is $N = 12(7) = 84$.

$$BAL_{7\text{th year}} = FV_{\text{Original loan}} - FV_{\text{payments made}}$$

$$\begin{aligned}
 BAL_{7\text{th year}} &= PV(1+i)^N - PMT \left[\frac{(1+i)^N - 1}{i} \right] \\
 &= 308,000(1+0.00385)^{84} - 2375.11 \left[\frac{(1+0.00385)^{84} - 1}{0.00385} \right] \\
 &= 425,335.739 - 235,018.795 \\
 &\approx \$190,316.94
 \end{aligned}$$

ii) Finding the principal balance at the end of the 8th year:

The number of payments up to the focal date of $t = 8$ year is $N = 12(8) = 96$.

$$\begin{aligned}
 BAL_{8\text{th year}} &= FV_{\text{Original loan}} - FV_{\text{payments made}} \\
 BAL_{8\text{th year}} &= PV(1+i)^N - PMT \left[\frac{(1+i)^N - 1}{i} \right] \\
 &= 308,000(1+0.00385)^{96} - 2375.11 \left[\frac{(1+0.00385)^{96} - 1}{0.00385} \right] \\
 &= 445,407.736 - 275,222.203 \\
 &\approx \$170,185.53
 \end{aligned}$$

iii) principal repaid in the 8th year:

a)

$$\begin{aligned}
 \text{Principal repaid in year 8} &= BAL_{7\text{th year}} - BAL_{8\text{th year}} \\
 &= 190,316.94 - 170,185.53 \\
 &= \$20,131.41
 \end{aligned}$$

b) The total interest paid in the 8th year of a loan is calculated by subtracting the amount of principal repaid during that year from the total payments made in the 8th year.

$$\begin{aligned}
 \text{Interest paid in the 8th year} &= \text{Total paid in the 8th year} - \text{Principal repaid in the 8th year} \\
 &= (12 \text{ payments} \times \$2375.11) - \$20,131.41 \\
 &= 28,501.32 - 20,131.41 \\
 &= \$8369.91
 \end{aligned}$$

Try an Example



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Section 4.3 Exercises

- Martina took out a \$84,000 loan at an interest rate of 5.88% compounded semi-annually. The loan is scheduled to be fully repaid over 12 years, with payments due at the end of each month. **a)** Calculate the size of month-end payments. **b)** Calculate the principal balance at the end of year 2.

Show/Hide Answer

a) $PMT = \$811.45$

b) $BAL = \$73,724.15$

- Erika took out a \$32,600 loan at an interest rate of 4.83% compounded monthly. The loan is scheduled to be fully repaid over 9 years, with payments due at the end of each month. **a)** Calculate the size of month-end payments. **b)** Calculate the total interest amount paid in the 4th year. **b)** Calculate the total principal amount

repaid in the 4th year.

Show/Hide Answer

a) $PMT = \$372.80$

b) $INT = \$1,048.37$

c) $PRN = \$3,425.23$

3. Johnetta took out a \$20,200 loan at an interest rate of 3.53% compounded monthly. The loan is scheduled to be fully repaid over 8 years, with payments due at the end of each month. **a)** Calculate the size of month-end payments. **b)** Calculate the interest portion of the 60th payment. **c)** Calculate the principal portion of the 60th payment.

Show/Hide Answer

a) $PMT = \$241.83$

b) $INT = \$24.91$

c) $PRN = \$216.92$

4.4 MORTGAGES: FORMULA APPROACH

A. Mortgage Definition and Types

A **mortgage** is a loan that is issued by a financial institution to a borrower to purchase real estate properties (business or residential properties) at a specific interest rate for a specific time period. The **mortgage amortization period** is the length of time it takes to pay off the entire amount borrowed based on the original mortgage contract. A **mortgage term** is the length of time for which the mortgage agreement (a specific interest rate) will be in effect. Mortgage agreements are generally negotiated several times during the amortization period. Mortgage agreements are generally negotiated several times during the amortization period. The interest rate, the size of the payment, and payment and compounding frequencies can change each term. Figure 4.4.1 depicts an example of the amortization period and typical renewal terms for a 25-year mortgage.

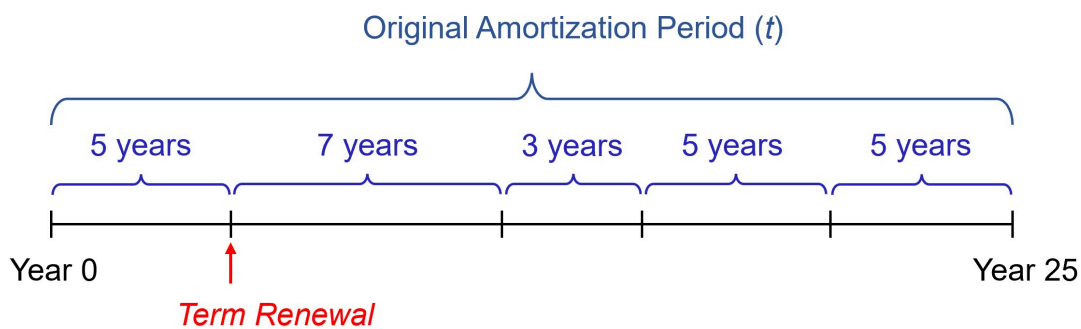


Figure 4.4.1 An example of the amortization period and typical renewal terms for a 25-year mortgage

Financial institutions usually offer two types of mortgages: fixed-rate mortgages and variable-rate mortgages. **Fixed-rate mortgages** have a fixed interest rate for a specific period of time (mortgage term). Once the term is completed, the mortgage is renewed. The new term may have a different interest rate, size of the payment, and payment frequency. **Variable-rate Mortgages** are the ones in which the interest rate fluctuates depending on changes in the money market. This rate change is usually related to the lender's prime rate.

Both mortgage types can be either open or closed. **Closed Mortgages** have a specific term and borrowers would have to pay a penalty for prepaying a large amount, closing the loan, or transferring to another lender. In contrast, **open Mortgages** allow borrowers to prepay any amount at any time, close, or transfer from one lender to another without

being penalized. Because of the flexibility, interest rates of open mortgages are usually higher than those of closed mortgages. Figure 4.4.2 summarizes the types of mortgages.

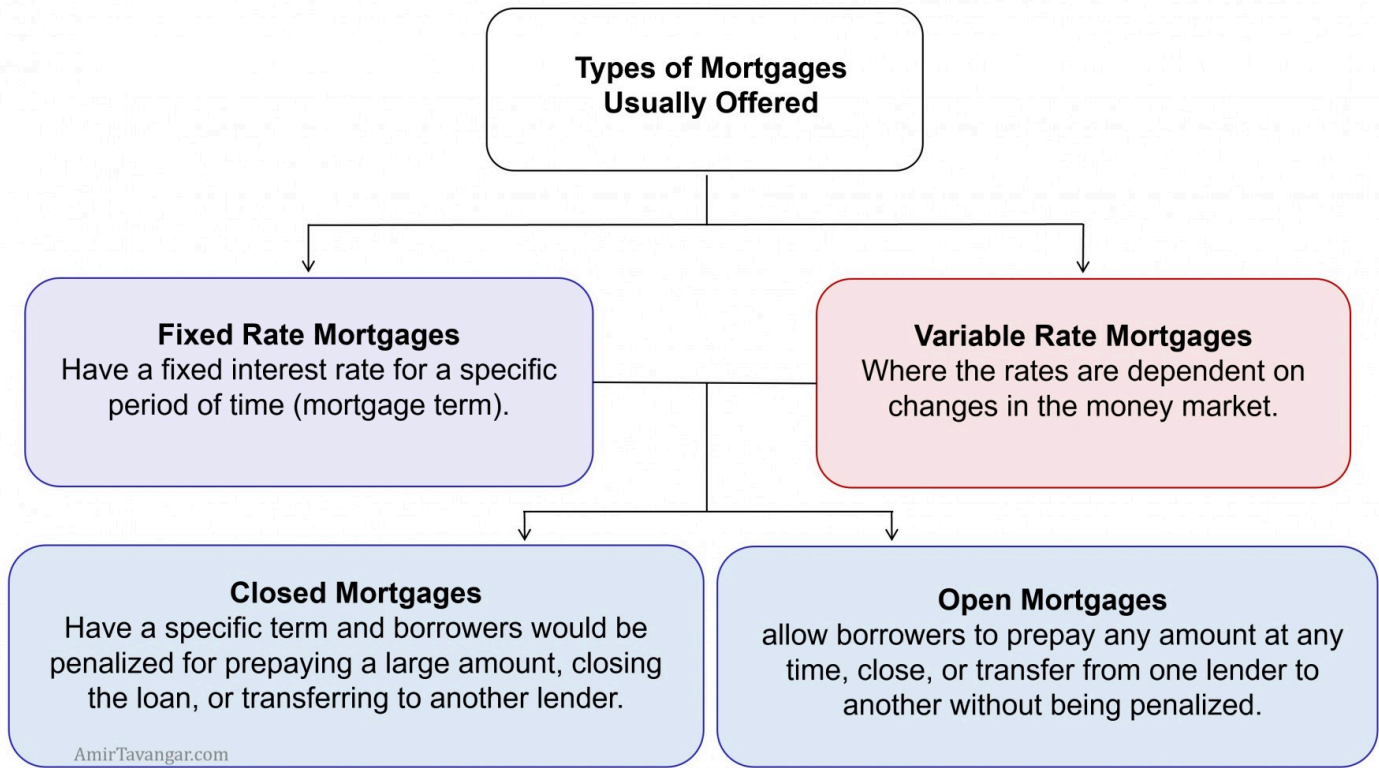


Figure 4.4.2 Types of Mortgages

B. Payment Amounts and the Outstanding Principal Balance on Mortgages

Calculations of payment amounts and the outstanding principal balance on mortgages are similar to what was discussed previously for all loan types. The next example illustrates.

Example 4.4.1: Compute Mortgage Payments for the First and Renewal Terms

John purchased a house for \$350,000. He made a down payment of 15% of the value of the house and received a mortgage for the rest of the amount amortized over 20 years. He negotiated a fixed interest rate of 3.8% compounded semi-annually for a 3-year term. **a)** Calculate the end-of-quarter payment size. **b)** Find the

outstanding principal balance at the end of the 3-year term. **c)** Calculate the new end-of-quarter payment amount if the mortgage was renewed for another 3 years at 2.5% compounded semi-annually.

Show/Hide Solution

Given information:

- Interest is compounded semi-annually so $C/Y = 2$
- Payments are made at the end of every quarter so $P/Y = 4$

$$C/Y \neq P/Y \Rightarrow \text{Ordinary General Annuity}$$

- Amortization period: $t = 20$ years
- Number of payments in the amortization period: $N = P/Y \cdot t = 4(20) = 80$
- Nominal interest rate for Term 1: $I/Y_1 = 3.8\%$
- Periodic interest rate for Term 1: $i = \frac{I/Y_1}{C/Y} = \frac{3.8\%}{2} = 1.9\%$
- Nominal interest rate for Term 2: $I/Y_2 = 2.5\%$
- Periodic interest rate for Term 2: $i = \frac{I/Y_2}{C/Y} = \frac{2.5\%}{2} = 1.25\%$
- Purchase price = \$350,000
- Down payment = 15% (350,000) = \$52,500

First, we need to find the amount of the mortgage, which is the remaining balance of the house price after paying the down payment.

$$\begin{aligned} PV &= \text{Purchase Price} - \text{Down} \\ PV &= 350,000 - 52,500 = \$297,500 \end{aligned}$$

a) Next, we use the PV and other given values to compute the size of the payments for the first 3-year term. We first need to find the interest rate per payment period i_2 :

$$\begin{aligned} i_2 &= (1 + i)^{\frac{C/Y}{P/Y}} - 1 \\ i_2 &= (1 + 0.019)^{\frac{2}{4}} - 1 \end{aligned}$$

$$= 0.00945529\dots$$

Substituting the values into Formula 3.10b, we obtain

$$\begin{aligned} PMT &= \frac{PV \cdot i_2}{1 - (1 + i_2)^{-N}} \\ PMT &= \frac{297,500(0.00945529\dots)}{1 - (1 + 0.00945529\dots)^{-80}} \\ &= 5317.616\dots \\ &\approx \$5317.62 \end{aligned}$$

The size of the quarterly payments is \$5317.62.

b) We use Formula 4.1 to calculate the balance at the end of the first term with the focal date in the 3rd year. To find the future value of the original loan, we use the future value formula of compound interest (Formula 2.4a), and to calculate the future value of the payments, we apply the future value formula of the ordinary general annuity (Formula 3.6a). Since the annuity is general, the number of compounding periods and payments in the term are not equal. therefore, we need to calculate them separately.

$$\text{Number of payments in Term 1: } N_p = P/Y \cdot \text{Number of years in Term 1} = 4(3) = 12$$

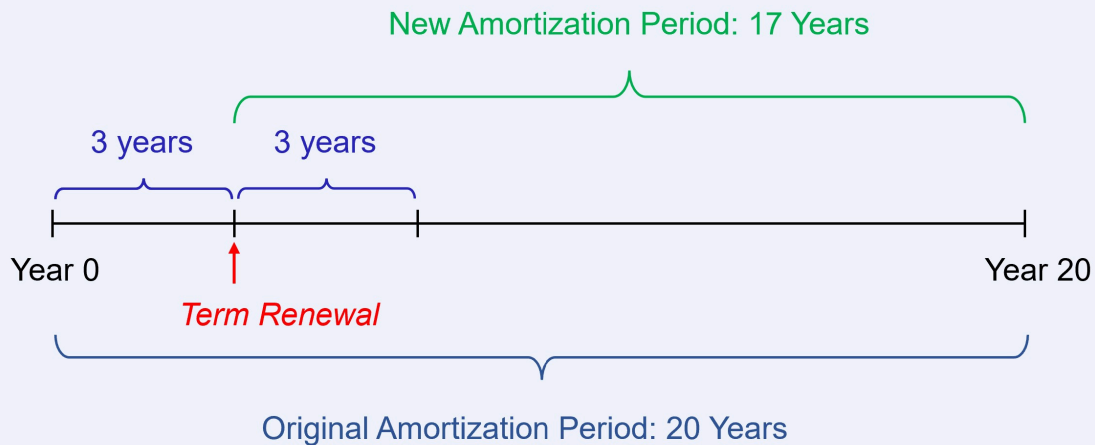
$$\begin{aligned} \text{Number of compounding periods in Term 1: } N_c &= C/Y \cdot \text{Number of years in Term 1} \\ &= 2(3) = 6 \end{aligned}$$

$$BAL_{\text{Focal date}} = FV_{\text{Original loan}} - FV_{\text{PMTs made up to focal date}}$$

$$\begin{aligned} BAL_{\text{Focal date}} &= PV(1 + i)^{N_c} - PMT \left[\frac{(1 + i_2)^{N_p} - 1}{i_2} \right] \\ &= 297,500(1 + 0.019)^6 - 5317.62 \left[\frac{(1 + 0.00945529\dots)^{12} - 1}{0.00945529\dots} \right] \\ &\approx 333,067.36 - 67,236.75 \end{aligned}$$

$$= \$265,830.61$$

c) The renewed term will have a new PMT as the nominal interest rate has reduced. To compute the new PMT, we use the outstanding balance at the end of the first term as the new PV, the remaining number of years as the new amortization period, and the new interest rate I / Y_2 .



The new amortization period equals the original amortization period minus the number of years in the first term of the mortgage.

- New amortization period: $t_2 = 20 \text{ years} - 3 \text{ years} = 17 \text{ years}$
- $N = P / Y(t_2) = 4(17) = 68$
- $i = 1.25 \%$
- The present value is the balance at the end of the first term: $PV = \$265,830.61$

Since the interest rate has changed, we need to recalculate the interest rate per payment period i_2 :

$$i_2 = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

$$i_2 = (1 + 0.0125)^{\frac{2}{4}} - 1$$

$$= 0.00623058\dots$$

Substituting the values into Formula 3.10b, we obtain

$$PMT = \frac{PV \cdot i_2}{1 - (1 + i_2)^{-N}}$$

$$PMT = \frac{265,830.61(0.00623058\dots)}{1 - (1 + 0.00623058\dots)^{-68}}$$

$$= 4807.704\dots$$

$$\approx \$4807.70$$

Therefore, the size of the quarterly payments in the renewed term is \$4807.70.



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The above figure includes a bar chart displaying the interest and principal portions of both the payments already made and those projected until the end of the amortization period. Also, it features line charts depicting the outstanding principal balance for Term 1 (covering the first 12 payments) and Term 2 (encompassing payments 13 to 24). It is noteworthy that due to the lower interest rate in the second term, the size of the payments and the interest portions are smaller compared to those in the first term. The slider tool allows you to zoom in and focus on a specific period within the chart.

Try an Example



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=219>

Section 4.4 Exercises

1. Sheniqua purchased a house for \$930,000. He made a down payment of 16% of the value of the house and received a mortgage for the rest of the amount amortized over 25 years. She negotiated a fixed interest rate of 3.56% compounded semi-annually for a 5-year term. **a)** Calculate the size of month-end payments. **b)** Calculate the principal balance at the end of the 5-year term. **c)** Calculate the size of month-end payments if the mortgage was renewed for another 5-year term at a fixed rate of 2.97% compounded semi-annually.

Show/Hide Answer

a) $PMT = \$3,925.08$

b) $BAL = \$674,757.75$

c) $\text{New } PMT = \$3,725.93$

2. Joey purchased a house for \$1,770,000. He made a down payment of 15% of the value of the house and received a mortgage for the rest of the amount amortized over 25 years. He negotiated a fixed interest rate of 3.2% compounded semi-annually for a 4-year term. **a)** Calculate the size of month-end payments. **b)** Calculate the principal balance at the end of the 5-year term. **c)** Calculate the size of month-end payments if the mortgage was renewed for another 4-year term at a fixed rate of 2.01% compounded semi-annually.

Show/Hide Answer

a) $PMT = \$7,275.27$

b) $BAL = \$1,336,349.88$

c) $\text{New } PMT = \$6,499.72$

3. Arman bought an apartment for \$850,000. He made a down payment of 26% of the value of the apartment and received a mortgage for the rest of the amount amortized over 25 years. He negotiated a fixed interest rate of 3.96% compounded semi-annually for a 7-year term. **a)** What was the size of month-end payments? **b)** Calculate the principal balance at the end of the 7-year term. **c)** What is the size of month-end payments if the mortgage is renewed for another 7-year term at a fixed rate of 3.9% compounded semi-annually?

Show/Hide Answer

a) $PMT = \$3,295.04$

b) $BAL = \$509,698.20$

c) $\text{New } PMT = \$3,279.57$

4.5 TERMINOLOGY AND GLOSSARY OF THE SYMBOLS

Terminology

- **Loan amortization:** The gradual reduction in the loan amount through periodic repayments, usually of equal amounts, over a predetermined length of time.
- **Amortization period:** The time period set to repay the loan.
- **Amortization schedule:** A detailed table that breaks down each loan repayment amount into its interest portion (INT) and principal portion (PRN)
- **Interest portion (INT):** The interest portion of a payment amount (PMT)
- **Principal portion (PRN):** The principal portion of a payment amount (PMT)
- **Principal balance (BAL):** The loan balance at a given time
- **Term (t):** In this unit refers to the term of a loan (i.e., an annuity), which is the time period (in years) from the beginning of the first payment interval to the end of the last payment interval.

Glossary of the Symbols

Annuities

- I/Y : Nominal interest rate per year (%)
- C/Y : Frequency of interest compounding period per year
- i : Interest rate per compounding period (%)
- i_2 : Equivalent interest rate per payment period (%)
- c : Number of compounding periods in each payment period
- t : Term of the annuity (in years)
- N : Total number of periodic payments in the annuity term
- P/Y : Number of payments per year
- PMT : Amount of periodic payment (\$)
- PV : Present value of an annuity (\$)
- FV : Future (maturity) value of an annuity (\$)
- I : Interest Amount (\$)

Amortization

- ***PMT***: the amount of periodic payment (\$)
- ***INT***: the interest portion of a payment amount (\$)
- ***PRN***: the principal portion of a payment amount (\$)
- ***BAL***: The principal balance of a loan at a given time (\$)

PART V

BUSINESS INVESTMENT DECISIONS

Chapter Outline

This chapter focuses on the evaluation of business investment decisions using both calculator and formula approaches. It's designed to guide readers through the processes of Discounted Cash Flow (DCF) and Net Present Value (NPV) calculations, fundamental concepts in assessing the value of potential investments. The chapter is divided into sections that address these calculations through financial calculators and theoretical formulas.

Introduction to Investment Decision Concepts: Before diving into specific methodologies, this section provides an overview of the principles underlying business investment decisions, emphasizing the importance of DCF and NPV calculations in determining investment viability.

Calculator Approach

5.1 Discounted Cash Flow (DCF) – Calculator Approach: This part provides an overview of the principles underlying business investment decisions and demonstrates how to calculate the DCF of potential investments using a financial calculator.

5.2 Net Present Value (NPV) – Calculator Approach: This section focuses on calculating the NPV of investments using a financial calculator.

Formula Approach

5.3 Discounted Cash Flow (DCF) – Formula Approach: This section explains DCF calculation using compound and annuity formulas.

5.4 Net Present Value (NPV) – Formula Approach: This section discusses NPV calculation, offering a comprehensive look at how future cash flows are discounted to their present value using compound and annuity formulas to assess investment opportunities.

5.5 Terminology and Glossary of Symbols: Concluding the chapter, this section provides definitions and explanations of key terms and symbols used throughout the chapter.

5.1 DISCOUNTED CASH FLOW (DCF): CALCULATOR APPROACH

A. Introduction

Investment decisions are among the most critical choices facing business managers and investors because they determine the future profitability and sustainability of projects, assets, or entire firms. The right investments can lead to substantial growth and profits, while poor choices can lead to losses and strategic setbacks. Given the significant amounts of money and the long-term implications involved, these decisions require careful analysis and consideration.

When making investment decisions, it is crucial to consider the time value of money, as cash flows from an investment occur at different times in the future and cannot be directly compared due to their differing time periods. Money today has a different value than money tomorrow or years down the line, primarily due to interest. To address this, the concept of present value, discussed in previous chapters, becomes indispensable. By discounting future cash flows back to their present value using a suitable discount rate, we can accurately assess and compare the worth of these cash flows as if they occurred today. This process of discounting ensures that the value of money is standardized to the present moment, allowing for a fair and direct comparison between different investments.

To navigate business investment decisions, several financial appraisal methods have been developed. Each offers a unique perspective on the value and potential of an investment opportunity. This chapter introduces and explores two methods widely used in business and finance: Discounted Cash Flow (DCF) and Net Present Value (NPV).

Cash flows in business investments refer to the streams of cash that are expected to be received or paid out over the lifetime of the investment. They are the lifeblood of any financial analysis and the primary focus of valuation techniques such as NPV and DCF.

Cash Inflows are the amounts of money expected to be received from the investment. For a business, this could include revenue from sales of goods or services, returns on other investments, sale of assets, or any form of income or capital gains.

Cash Outflows represent the expenditure or costs associated with maintaining and operating the business or making an investment. This can include initial capital expenditures, ongoing operational costs, taxes, maintenance expenses, loan repayments, and any other forms of spending necessary to sustain the business or investment.

B. Discounted Cash Flow (DCF) Method

The DCF method estimates an investment's value by discounting each of its expected future cash flows back to its present value at a focal date (usually 'now') using a discount rate, known as the **required rate of return**. This rate reflects the risk and the time value of money — essentially, the return that could be earned if the money was invested elsewhere with a similar risk profile. The sum of these present values provides the total value of the cash flows, which is considered the intrinsic value of the investment.

$$PV_{All\ Cash\ Flows} = PV_{Cash\ flow\ 1} + PV_{Cash\ flow\ 2} + \dots \quad \text{Formula 5.1}$$

If the DCF value is greater than the initial cost of the investment, the investment is considered acceptable. This means the investment is expected to generate a return above the discount rate, covering the cost of capital and providing additional value.

It is important to note that when working with algebraic formulas, such as Formula 5.1 for calculating DCF, all the present values are treated as positive values.

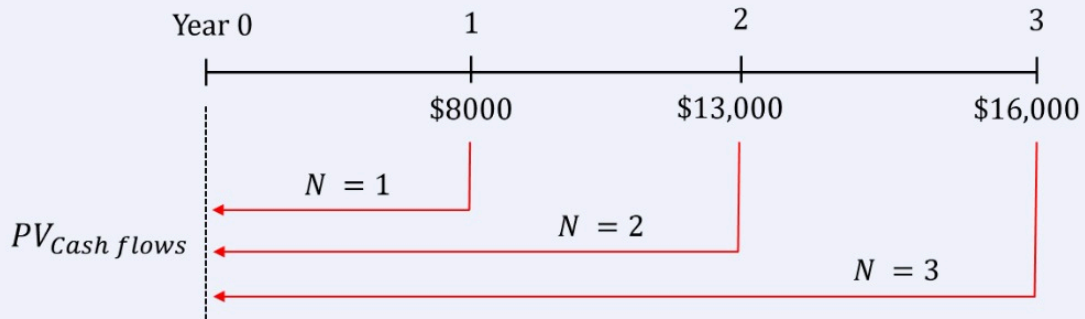
Example 5.1.1: Calculate the DCF Value of a Project

A delivery company is considering investing \$30,000 in a project. The expected returns from this investment are projected to be \$8,000 in one year, \$13,000 in two years, and \$16,000 in three years. The company's required rate of return for investments is 8%. Calculate the discounted cash flow and determine whether this is a worthwhile investment.

Show/Hide Solution

- Required rate of return: $I/Y = 8\%$
- The frequency of compounding periods is not provided, so it is assumed annually: $C/Y = 1$
- For compound problems $P/Y = C/Y$, so $P/Y = 1$
- Cost of capital = \$30,000
- Cash flow 1: $FV_1 = 8000$
- cash flow 2: $FV_2 = 13,000$
- Cash flow 3: $FV_3 = 16,000$
- Cash flow 1 time: $t_1 = 1$ year; $N_1 = C/Y \cdot t_1 = 1$

- Cash flow 2 time: $t_2 = 2$ years; $N_2 = 2$
- Cash flow 3 time: $t_3 = 3$ years; $N_3 = 3$



It is important to note that all the cash flows in this scenario are returns on investment, meaning they are cash inflows. Given the need to make an immediate decision on the investment opportunity, we use 'now' as the focal date to determine the present values of the future cash flows. We use the TVM worksheet to calculate the present value of each expected cash inflow from the investment.

Computing Cash Flow 1

CLR TVM

1 N

8 I/Y

2ND P/Y=1

I/Y C/Y=1

0 PMT

8000 FV

CPT PV

-7407.407407

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Computing Cash Flow 2

CLR TVM

2 N

8 I/Y

2ND P/Y=1

I/Y C/Y=1

0 PMT

13,000 FV

CPT PV

-11,145.404664

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Computing Cash Flow 1

CLR TVM

3 N

8 I/Y

2ND P/Y=1

I/Y C/Y=1

0 PMT

16,000 FV

CPT PV

-12,701.315856

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We then sum all the present values (Formula 5.1).

$$\begin{aligned}
 PV_{All\ Cash\ Flows} &= PV_{Cash\ flow\ 1} + PV_{Cash\ flow\ 2} + PV_{Cash\ flow\ 3} \\
 &= 7407.407... + 11,145.404... + 12,701.315... \\
 &\approx \$31,254.13
 \end{aligned}$$

Therefore, the discounted cash flow of the project is \$31,254.13, which is greater than the cost of capital of \$30,000. The returns are sufficient to cover the cost of capital and provide an additional value, indicating the company should proceed with the investment project under the given financial criteria.

Try an Example



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Example 5.1.2: Comparing Two Investment Option Using the DCF Values

The CEO of an ice cream company is choosing between two investment options: Project A and Project B. Both projects require the same initial investment. Project A is expected to yield returns of \$45,000 in one year and another \$45,000 in two years. In contrast, Project B is anticipated to bring in \$10,000 in one year and \$80,000 in two years. With the company's required rate of return set at 12%, which project should the CEO choose based on the financial viability and the expected returns of each option?

Show/Hide Solution

- Required rate of return: $I / Y = 12\%$
- The frequency of compounding periods is not provided, so it is assumed annually: $C / Y = 1$

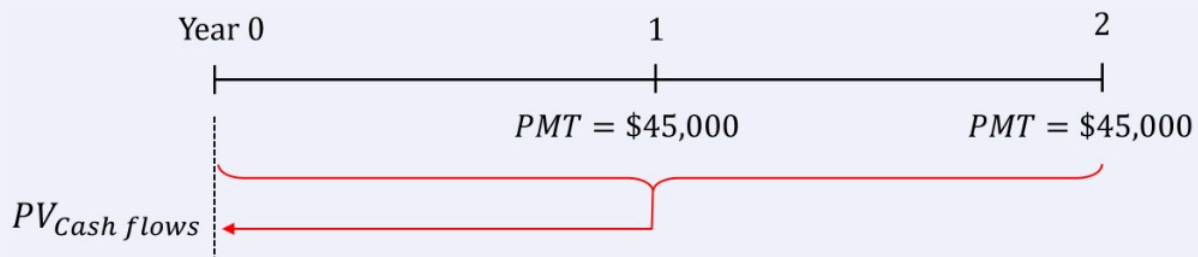
Project A:

For Project A, both anticipated cash flows are returns on the investment, categorizing them as cash inflows. To determine the present values of these future cash inflows, we have two methods:

–**Using the concept of Compound Interest:** We can calculate the present value of each expected cash inflow individually treating each cash inflow as a future value of compound interest.

–**Using the concept of Ordinary Simple Annuity:** Alternatively, as both cash inflows from Project A are of equal amounts and are expected to occur at regular intervals (one year apart), they can be treated as payments of an ordinary simple annuity.

In this case, we can calculate the combined present value of both cash inflows in a single step. We use the second method here.

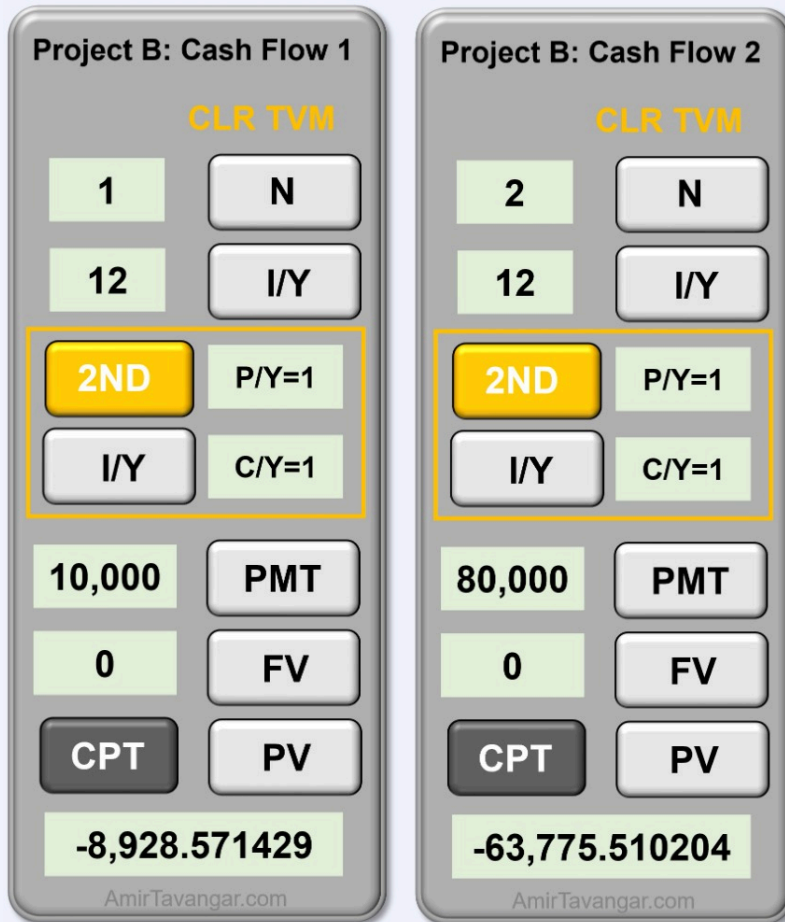


- Cash inflows 1,2: $PMT = 45,000$
- Annuity term: $t = 2$ years
- One payment per year, so $P / Y = 1$
- Number of payments in the term: $N = P / Y \cdot t = 1(2) = 2$
- No additional inflows: $FV = 0$

Compute PV :

- Cash flow 1: $FV_1 = 10,000$
- cash flow 2: $FV_2 = 80,000$
- Cash flow 1 time: $t_1 = 1$ year; $N_1 = C/Y \cdot t_1 = 1$
- Cash flow 2 time: $t_2 = 2$ years; $N_2 = 2$

Compute PV :



$$\begin{aligned}
 PV_{All\ Cash\ Flows} &= PV_{Cash\ flow\ 1} + PV_{Cash\ flow\ 2} \\
 &= 8928.571\dots + 63,775.510\dots \\
 &\approx \$72,704.08
 \end{aligned}$$

Therefore, the discounted cash flow of Project B is \$72,704.08.

Make a Decision:

The CEO should choose the project with the higher discounted cash flow value because it represents a greater return on investment after considering the time value of money at the company's required rate of return. In this case, since Project A has a higher DCF value than Project B, the CEO should opt for Project A as the investment choice.

Try an Example



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Section 5.1 Exercises

1. A delivery company is considering investing \$26,260 in a project. The expected returns from this investment are projected to be \$6,000 in 1 year, \$10,000 in 5 years, and \$17,500 in 8 years. The company's required rate of return for investments is 4.5%. **a)** Calculate the discounted cash flow. **b)** Is the investment worthwhile?

Show/Hide Answer

a) DCF = \$26,071.88

b) No, it is not since DCF is less than the investment capital.

2. The CFO of a company is choosing between two investment options: Project A and Project B. Both projects require the same initial investment. Project A is expected to yield returns of \$35,000 in one year and another \$35,000 in two years. In contrast, Project B is anticipated to bring in \$29,000 in one year and \$39,000 in two years. The company's required rate of return is set at 8%. **a)** Calculate the discounted cash flow of Project A. **b)** Calculate the discounted cash flow of Project B. **c)** Which project should the CFO choose based on the

discounted returns of each option?

Show/Hide Answer

a) Project A: DCF = \$62,414.27

b) Project B: DCF = \$60,288.07

c) Project A should be chosen because of the higher DCF.

3. A company is considering two investment opportunities. Alternative 1 will result in payments of \$9,000 in two years and \$22,000 in four years. Alternative 2 will result in annual payments of \$4,500 at the end of each year for a total of five years. The company's required rate of return is 6.58%. **a)** Calculate the DCF of Alternative 1. **b)** Calculate the DCF of Alternative 2. **c)** Which investment alternative should the company choose to maximize its financial benefit? *Round your answers to the nearest dollar.*

Show/Hide Answer

a) Alternative 1: DCF = \$24,973

b) Alternative 2: DCF = \$18,660

c) Alternative 1 should be chosen because of the higher DCF.

5.2 NET PRESENT VALUE (NPV): CALCULATOR APPROACH

Net Present Value (NPV) is a financial metric used to evaluate the profitability of an investment or project. It represents the difference between the present value of cash inflows and the present value of cash outflows (also known as outlays) over the investment's lifetime. The NPV is calculated by discounting all expected future cash flows to their present values using a specific discount rate, often the required rate of return. By using a required rate of return in NPV calculations, investors and managers can factor in the risk profile of the investment and the opportunity cost of capital. This rate typically reflects the return that could be earned on alternative investments of similar risk.

$$NPV = PV_{\text{Cash inflows}} - PV_{\text{Cash outflows}} \quad \text{Formula 5.2}$$

A positive NPV indicates that the project is expected to generate more value than the cost, considering the time value of money and the investment's risk. Conversely, a negative NPV suggests that the project is likely to diminish value. An NPV of zero means that the investment is expected to yield a return equal to the minimum acceptable rate of return.

The NPV decision criterion is as follows:

- If $NPV > 0$: Accept the Project
 - A positive NPV indicates that the present value of the project's cash inflows exceeds the present value of its cash outflows. This means the investment is expected to generate a return greater than the required rate of return. In other words, the project adds value to the firm or investor and is considered financially viable and profitable.
- If $NPV = 0$: Indifference
 - An NPV of zero suggests the project's returns are exactly sufficient to cover the cost of capital and meet the required rate of return. The project is breaking even in present value terms. While financially neutral, the decision might still consider strategic, operational, or social factors.
- If $NPV < 0$: Reject the Project
 - A negative NPV indicates that the present value of the project's cash inflows is less than the present value of its cash outflows. This means the project is expected to yield a return less than the required rate of return. The project is not adding value and is considered unprofitable and financially unviable.

When comparing multiple projects using Net Present Value (NPV), the decision-making process involves evaluating the NPV of each project and then selecting the one(s) that provide the greatest financial return.

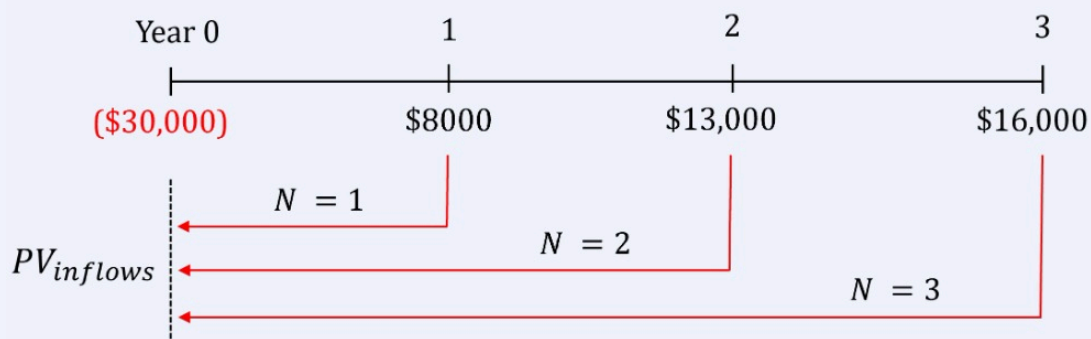
It is important to note that when working with algebraic formulas, such as Formula 5.2 for calculating NPV, both the present value of inflows and the present value of outflows are treated as positive values.

Example 5.2.1: Calculate NPV of a Project with Future Cash Inflows

The management team of a company is evaluating a project that requires a \$30,000 investment today. The expected returns from this investment are projected to be \$8,000 in one year, \$13,000 in two years, and \$16,000 in three years. The company's required rate of return for investments is 8%. Calculate the net present value of the project and determine whether they should accept the project.

Show/Hide Solution

- Required rate of return: $I / Y = 8 \%$
- The frequency of compounding periods is not provided, so it is assumed annually: $C / Y = 1$
- For compound problems $P / Y = C / Y$, so $P / Y = 1$
- Cash outflow = \$30,000
- Cash inflow 1: $FV_1 = 8000$
- cash inflow 2: $FV_2 = 13,000$
- Cash inflow 3: $FV_3 = 16,000$
- Cash inflow 1 time: $t_1 = 1$ year; $N_1 = C / Y \cdot t_1 = 1$
- Cash inflow 2 time: $t_2 = 2$ years; $N_2 = 2$
- Cash inflow 3 time: $t_3 = 3$ years; $N_3 = 3$



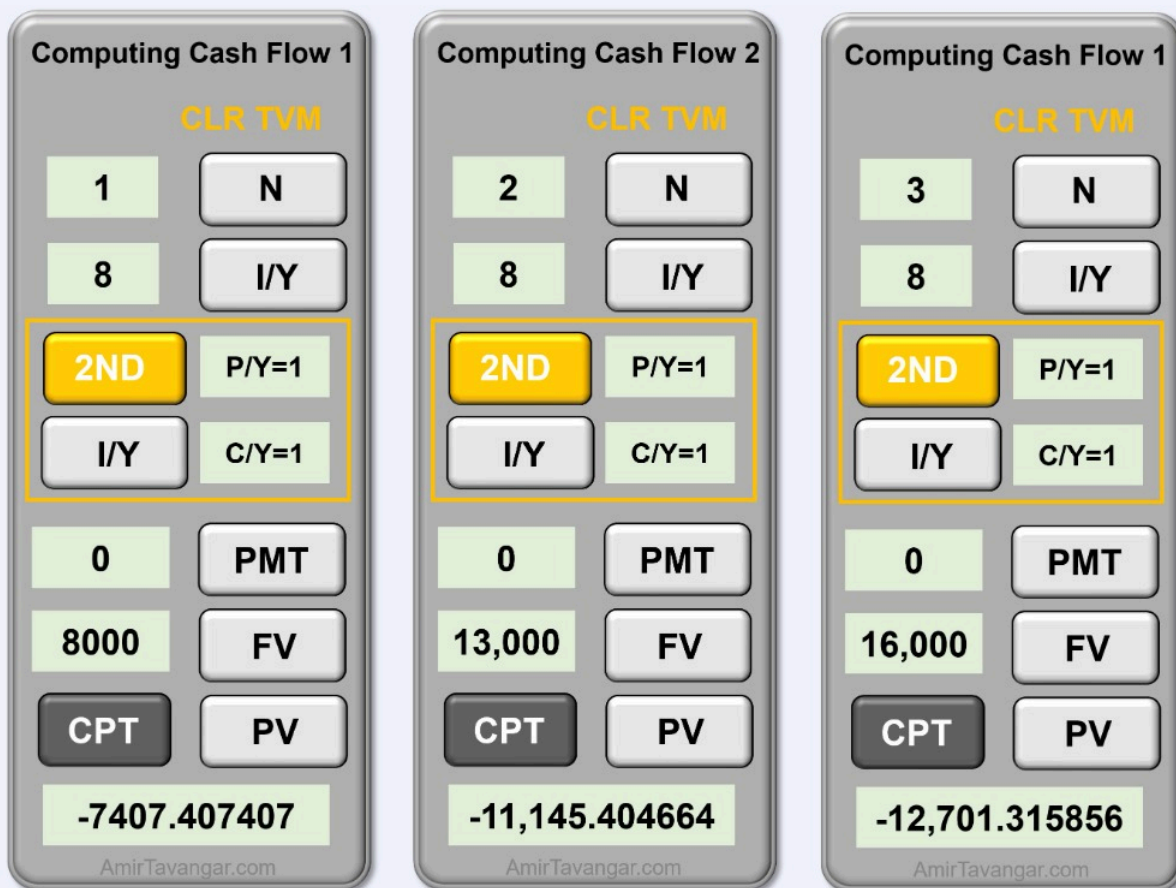
Cash Outflows:

As the initial investment of \$30,000 must be paid immediately, there is no need to calculate its present value. It is already in present-day terms.

$$PV_{\text{outflows}} = \$30,000$$

Cash Inflows:

It is important to note that all the future cash flows of the project are returns on investment, meaning they are cash inflows. Given the need to make an immediate decision on the investment opportunity, we use ‘now’ as the focal date to determine the present values of the future cash inflows. We use the TVM worksheet to calculate the present value of each expected cash inflow from the investment.



We then sum all the present values.

$$PV_{\text{inflow}} = PV_{\text{Cash flow 1}} + PV_{\text{Cash flow 2}} + PV_{\text{Cash flow 3}}$$

$$= 7407.407... + 11,145.404... + 12,701.315...$$

$$\approx \$31,254.13$$

Determine NPV:

Using Formula 5.2, the NPV is

$$NPV = PV_{\text{inflows}} - PV_{\text{outflows}}$$

$$NPV = 31,254.13 - 30,000$$

$$= 1254.13$$

Make a Decision:

The NPV of the project is more than zero, which means the returns are sufficient to cover the cost of capital and provide an additional value, indicating the company should accept the project under the given financial criteria.

Try an Example



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Example 5.2.2: Calculate NPV of a Project with Annuity Cash Inflows and Residual

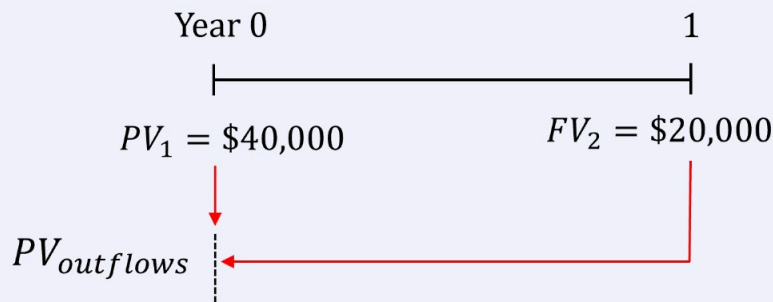
A company is evaluating the feasibility of investing in machinery to expand its manufacturing plant. It would need to invest \$40,000 today and \$20,000 in one year. The machinery is expected to generate a net return of \$6,295 at the end of every year for 11 years. At the end of the 11 years, the machinery is expected to have a salvage (residual) value of \$10,200, which can be considered a final cash inflow. The company's required rate of return for investments is 5.4%. **a)** Calculate the Net Present Value (NPV) of the investment. **b)** Determine whether the purchase of this machinery is a financially sound decision according to the NPV criterion.

Show/Hide Solution

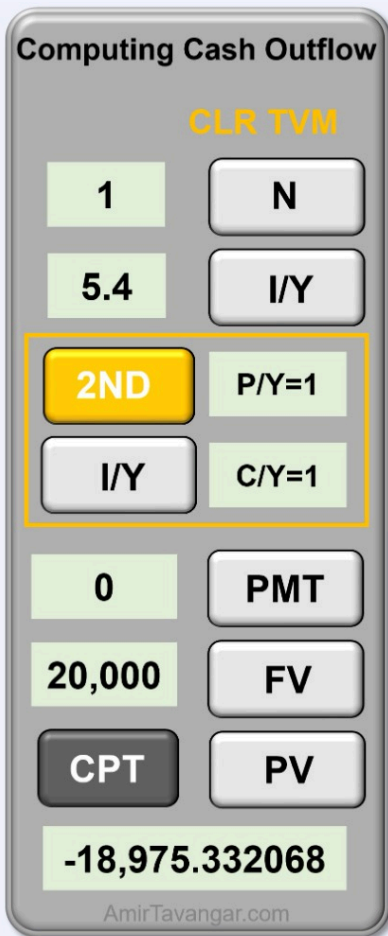
- Required rate of return: $I / Y = 5.4 \%$
- The frequency of compounding periods is not provided, so it is assumed annually: $C / Y = 1$
- For compound problems $P / Y = C / Y$, so $P / Y = 1$

Cash Outflows:

- Cash outflow 1: $PV_1 = \$40,000$
- Cash outflow 2: $FV_2 = 20,000$
- Cash inflow 1 time: $t_1 = 0$ year
- Cash inflow 2 time: $t_2 = 1$ years; $N_2 = C / Y \cdot t_2 = 1$



As the initial investment of \$40,000 must be paid immediately, there is no need to calculate its present value. It is already in present-day terms. For the second investment of \$20,000, we utilize the TVM worksheet to calculate the present value.



$$\begin{aligned}
 PV_{\text{outflows}} &= PV_{\text{outflow 1}} + PV_{\text{outflow 2}} \\
 &= 40,000 + 18,975.332\dots \\
 &\approx 58,975.33
 \end{aligned}$$

Cash Inflows:

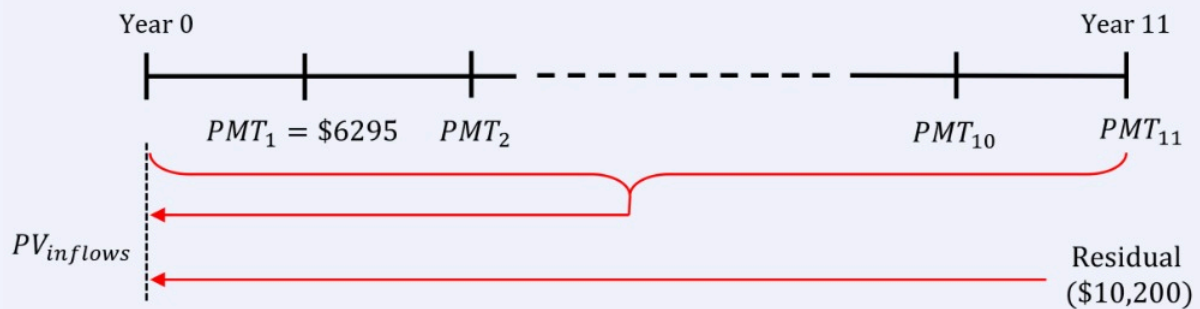
There are two types of cash inflows:

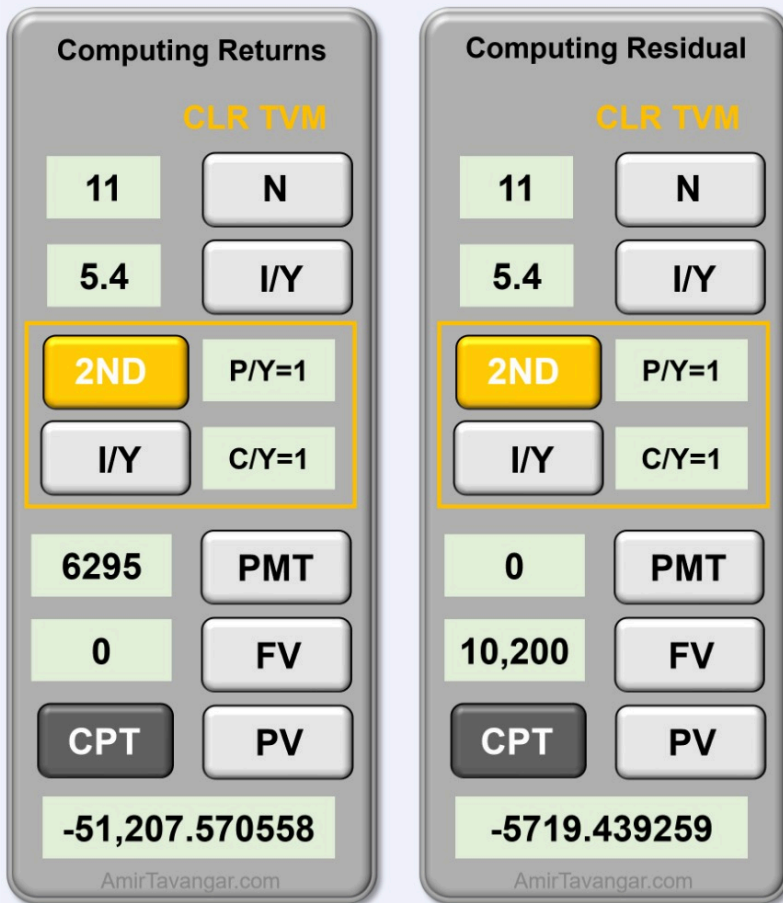
- Annual investment return of \$6295: as all these cash inflows are of equal amounts and are expected to occur at regular intervals (one year apart), they can be treated as payments of an ordinary simple annuity. In this case, we can use the TVM worksheet to calculate the combined present value of both cash inflows in a single step.

- Returns: $PMT = 6295$
- Annuity term: $t = 11$ years
- One payment per year, so $P/Y = 1$
- Number of payments in the term: $N = P/Y \cdot t = 1(11) = 11$

– The machinery’s salvage value of \$10,200: The company can sell the machinery at the end of the 11th year. This salvage value is considered a cash inflow, and its present value can be calculated using the TVM worksheet.

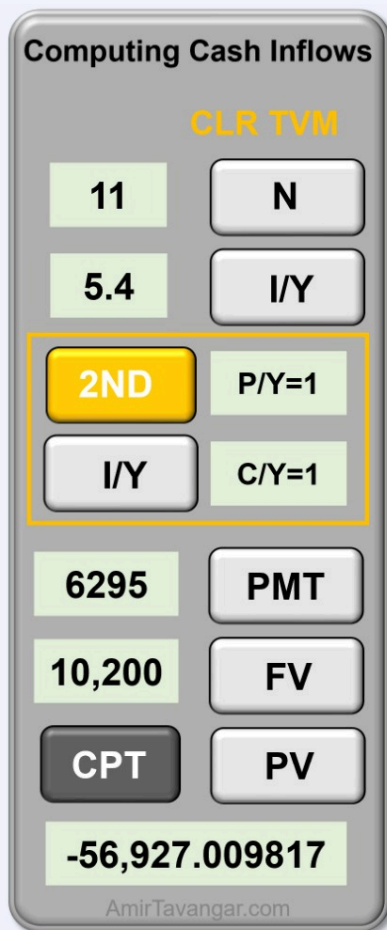
- Salvage value: $FV_s = 10,200$
- Compound term: $t = 11$ years
- $P/Y = C/Y = 1$
- Number of compound periods in the term: $N = C/Y \cdot t = 1(11) = 11$





$$\begin{aligned}
 PV_{\text{inflows}} &= PV_{\text{returns}} + PV_{\text{residual}} \\
 &= 51,207.570\dots + 5719.439\dots \\
 &\approx \$56,927.01
 \end{aligned}$$

Alternatively, when considering an investment where the number of payments (N) and the compounding periods for both the investment returns and the residual value are the same, we can simplify the calculation. By entering the regular returns as the payment (PMT) and the residual value as the future value (FV) in the Time Value of Money (TVM) worksheet, we can calculate their combined present value in a single step. It is important to note that since both the payments and the residual are cash inflow, they should be entered as positive values (or at least with the same sign)



Using this approach, the present value of all inflows is calculated as \$56,927.01, which matches the value obtained previously through separate calculations.

Determine NPV:

Using Formula 5.2, the NPV is

$$\begin{aligned}
 NPV &= PV_{\text{inflows}} - PV_{\text{outflows}} \\
 NPV &= 56,927.01 - 58,975.33 \\
 &= -\$2048.32
 \end{aligned}$$

Make a Decision:

The NPV of the project is negative, which means the investment does not generate enough return to cover its initial cost plus the minimum required return. Therefore, the company should reject the project under the given financial criteria.

Try an Example



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Example 5.2.3: Evaluate NPV of Two Projects with Annuity Cash Inflows and Residual

Riverflow Utilities is faced with two different strategies for enhancing the environmental sustainability of one of its water treatment facilities to meet increased regulatory standards. Each option involves different cost structures and timelines, and the decision must be made using the Net Present Value (NPV) method, considering the required rate of return of 10%.

Strategy A: Upgrade with Advanced Technology

- Annual Costs: \$50,000 paid at the end of each year for 9 years.
- Residual Value: At the end of 9 years, Riverflow Utilities expects to sell the specialized technology components for \$80,000.

Strategy B: Outsourcing to Environmental Specialists

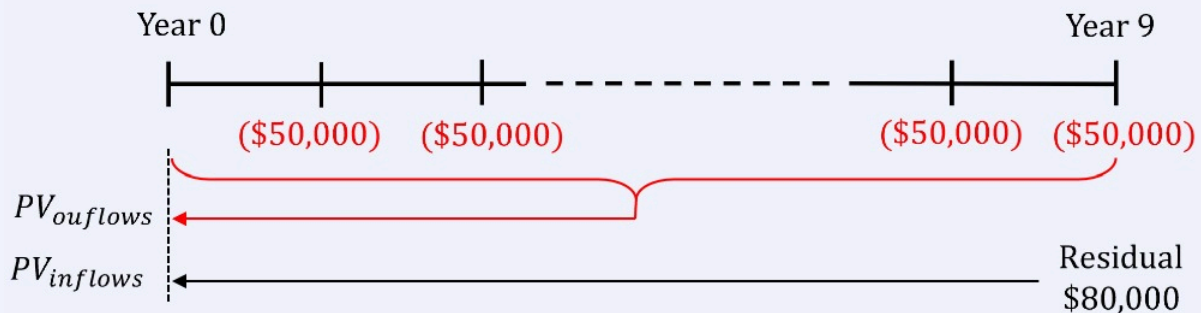
- Initial Cost: A one-time payment of \$120,000 upfront.
- Annual Costs: \$25,000 paid at the end of each year for 9 years.

a) Calculate the Net Present Value (NPV) of Strategy A. b) Calculate the Net Present Value (NPV) of Strategy B. c) Determine which strategy Riverflow Utilities should implement based on the NPV criterion.

Show/Hide Solution

- Required rate of return: $I / Y = 10\%$
- The frequency of compounding periods is not provided, so it is assumed annually: $C / Y = 1$

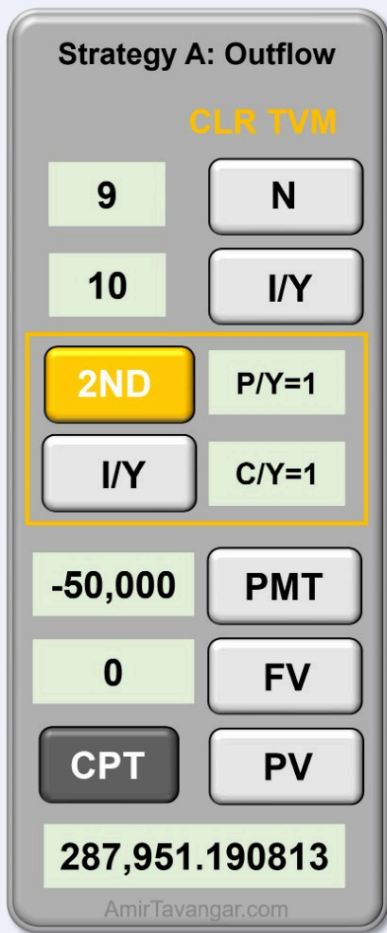
a) Strategy A:



Cash outflow:

The annual costs of \$50,000 are the cash outflows. They are of equal amounts and are expected to occur at regular intervals (one year apart), they can be treated as payments of an ordinary simple annuity. In this case, we use the TVM worksheet to calculate the present value of the payments.

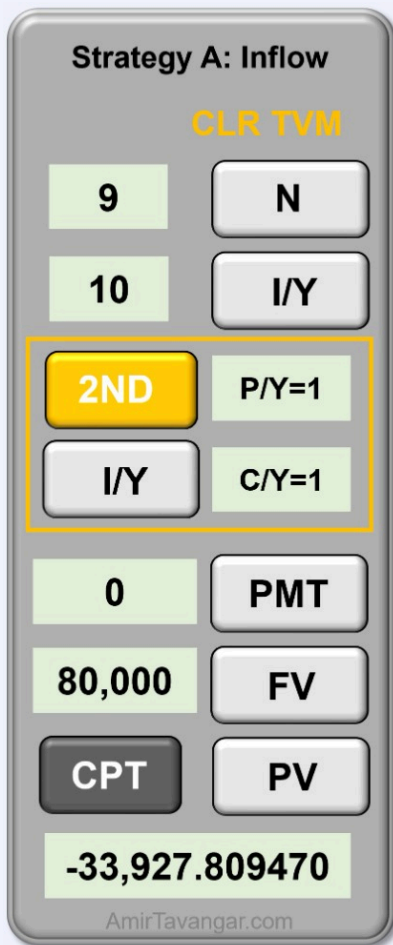
- Frequency of payments per year: $P / Y = 1$
- Time period: $t = 9$ years
- Number of payments in the term: $N = P / Y \cdot t = 1(9) = 9$
- Payments are cash outflow: $PMT = -50,000$



Cash inflow:

The company can sell the component at the of the 9th year. This residual value is considered a cash inflow, and its present value can be calculated using the TVM worksheet.

- Residual value: $FV = 80,000$
- Compound term: $t = 9$ years
- $P/Y = C/Y = 1$
- Number of compound periods in the term: $N = C/Y \cdot t = 1(9) = 9$



Thus the present value of the \$80,000 is \$33,927.81.

The net present value of Strategy A (NPV_A):

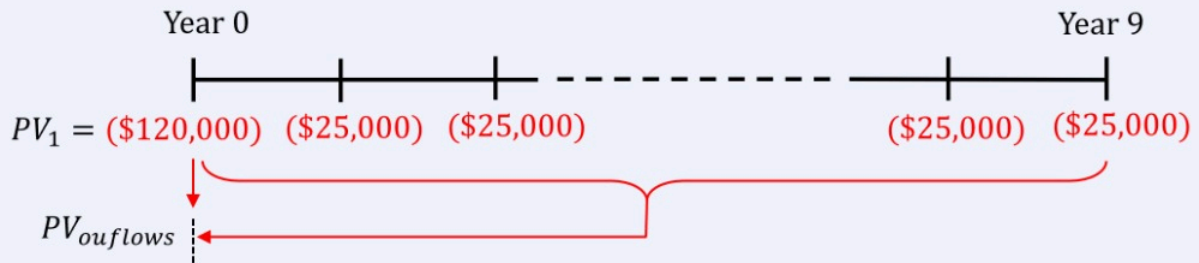
Using Formula 5.2, the NPV is

$$NPV = PV_{\text{inflows}} - PV_{\text{outflows}}$$

$$NPV_A = 33,927.81 - 287,951.19$$

$$= -\$254,023.38$$

b) Strategy B:

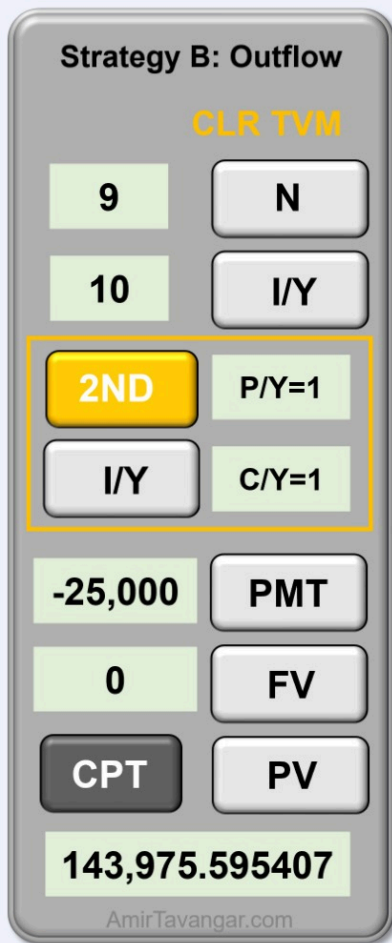


Cash outflow:

As the initial cost of \$120,000 must be paid immediately, there is no need to calculate its present value. It is already in present-day terms.

The annual costs of \$25,000 are of equal amounts and are expected to occur at regular intervals (one year apart), they can be treated as payments of an ordinary simple annuity. In this case, we can use the TVM worksheet to calculate the present value of the payments.

- Frequency of payments per year: $P/Y = 1$
- Time period: $t = 9$ years
- Number of payments in the term: $N = P/Y \cdot t = 1(9) = 9$
- Payments: $PMT = -25,000$



Thus the present value of the \$25,000 payments is \$143,975.60.

$$PV_{\text{outflows}} = \text{Initial pay} + PV_{\text{Annuity}}$$

$$\begin{aligned} PV_{\text{outflows}} &= 120,000 + 143,975.60 \\ &= \$263,975.60 \end{aligned}$$

Cash inflow:

There is no cash inflow in the second strategy, so $PV_{\text{inflows}} = 0$.

The net present value of Strategy B (NPV_B):

Using Formula 5.2, the NPV is

$$NPV = PV_{\text{inflows}} - PV_{\text{outflows}}$$

$$NPV_B = 0 - 263,975.60$$

$$= - \$263,975.60$$

c) Make a decision:

Since the net present values for both strategies are negative, the company should opt for the strategy with the lesser cost or larger net present value. Therefore, the company should choose Strategy A, as it has the lower negative net present value and thus represents the less costly option, which is consistent with the NPV acceptance criteria as $NPV_A > NPV_B$.

Try an Example



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Section 5.2 Exercises

1. The management team of a company is evaluating a project that requires a \$17,099 investment today. The expected returns from this investment are projected to be \$6,000 in 1 year, \$12,000 in 7 years, and \$13,500 in 8 years. The company's required rate of return for investments is 7%. **a)** Calculate the net present value of the project. **b)** Should the management accept the project?

Show/Hide Answer

a) $NPV = \$3,839.00$

b) Yes, since $NPV > 0$.

2. A company is evaluating the feasibility of investing in machinery to expand its manufacturing plant. It would need to invest \$53,300 today and \$54,400 in 3 years. The machinery is expected to generate a net return of \$9,500 at the end of every year for 12 years. At the end of the 12 years, the machinery is expected to have a residual value of \$16,500. The company's required rate of return for investments is 3%. **a)** Calculate the Net Present Value (NPV) of the investment. **b)** Is the investment worthwhile?

Show/Hide Answer

a) $NPV = \$3,052.10$

b) Yes, since $NPV > 0$

3. VitalFlow Systems is faced with two different strategies for enhancing the environmental sustainability of one of its water treatment facilities to meet increased regulatory standards. Each option involves different cost structures and timelines, and the decision must be made using the Net Present Value (NPV) method, considering the required rate of return of 7%. **a)** Calculate the net present value of Strategy A. **b)** Calculate the discounted cash flow of Strategy B. **c)** Which strategy should VitalFlow Systems choose based on the NPV of each option? *Strategy A: Upgrade with Advanced Technology*

- Annual Costs: \$48,000 paid at the end of each year for 9 years.
- Residual Value: At the end of 9 years, VitalFlow Systems expects to sell the specialized technology components for \$64,800.

Strategy B: Outsourcing to Environmental Specialists

- Initial Cost: A one-time payment of \$61,000 upfront.
- Annual Costs: \$32,200 paid at the end of each year for 9 years.

Show/Hide Answer

a) Strategy A: NPV = -\$277,484.24

b) Strategy B: NPV = -270,790.48

c) Strategy B since it costs less.

5.3 DISCOUNTED CASH FLOW (DCF): FORMULA APPROACH

A. Introduction

Investment decisions are among the most critical choices facing business managers and investors because they determine the future profitability and sustainability of projects, assets, or entire firms. The right investments can lead to substantial growth and profits, while poor choices can lead to losses and strategic setbacks. Given the significant amounts of money and the long-term implications involved, these decisions require careful analysis and consideration.

When making investment decisions, it is crucial to consider the time value of money, as cash flows from an investment occur at different times in the future and cannot be directly compared due to their differing time periods. Money today has a different value than money tomorrow or years down the line, primarily due to interest. To address this, the concept of present value, discussed in previous chapters, becomes indispensable. By discounting future cash flows back to their present value using a suitable discount rate, we can accurately assess and compare the worth of these cash flows as if they occurred today. This process of discounting ensures that the value of money is standardized to the present moment, allowing for a fair and direct comparison between different investments.

To navigate business investment decisions, several financial appraisal methods have been developed. Each offers a unique perspective on the value and potential of an investment opportunity. This chapter introduces and explores two methods widely used in business and finance: Discounted Cash Flow (DCF) and Net Present Value (NPV).

Cash flows in business investments refer to the streams of cash that are expected to be received or paid out over the lifetime of the investment. They are the lifeblood of any financial analysis and the primary focus of valuation techniques such as NPV and DCF.

Cash Inflows are the amounts of money expected to be received from the investment. For a business, this could include revenue from sales of goods or services, returns on other investments, sale of assets, or any form of income or capital gains.

Cash Outflows represent the expenditure or costs associated with maintaining and operating the business or making an investment. This can include initial capital expenditures, ongoing operational costs, taxes, maintenance expenses, loan repayments, and any other forms of spending necessary to sustain the business or investment.

B. Discounted Cash Flow (DCF) Method

The DCF method estimates an investment's value by discounting each of its expected future cash flows back to its present value at a focal date (usually 'now') using a discount rate, known as the **required rate of return**. This rate reflects the risk and the time value of money — essentially, the return that could be earned if the money was invested elsewhere with a similar risk profile. The sum of these present values provides the total value of the cash flows, which is considered the intrinsic value of the investment.

$$PV_{All\ Cash\ Flows} = PV_{Cash\ flow\ 1} + PV_{Cash\ flow\ 2} + \dots \quad \text{Formula 5.1}$$

If the DCF value is greater than the initial cost of the investment, the investment is considered acceptable. This means the investment is expected to generate a return above the discount rate, covering the cost of capital and providing additional value.

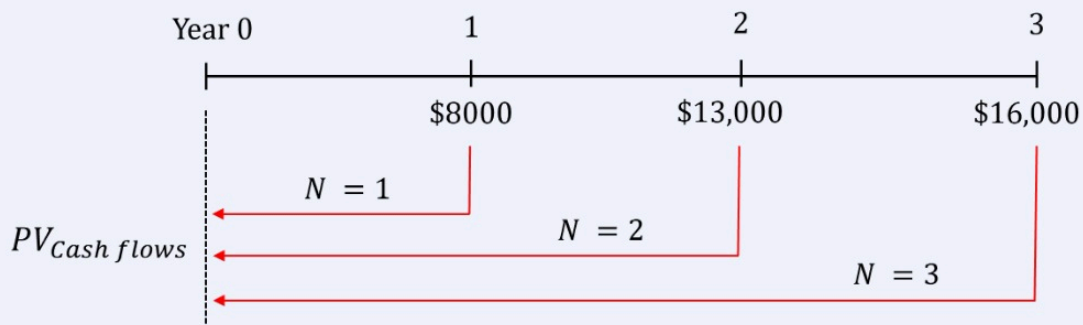
Example 5.3.1: Calculate the DCF Value of a Project

A delivery company is considering investing \$30,000 in a project. The expected returns from this investment are projected to be \$8,000 in one year, \$13,000 in two years, and \$16,000 in three years. The company's required rate of return for investments is 8%. Calculate the discounted cash flow and determine whether this is a worthwhile investment.

Show/Hide Solution

- Required rate of return: $I/Y = 8\%$
- The frequency of compounding periods is not provided, so it is assumed annually:
 $P/Y = C/Y = 1$
- Interest rate per compounding period: $i = \frac{I/Y}{C/Y} = \frac{8\%}{1} = 8\% = 0.08$
- Cost of capital = 30,000
- Cash flow 1: $FV_1 = 8000$
- cash flow 2: $FV_2 = 13,000$
- Cash flow 3: $FV_3 = 16,000$
- Cash flow 1 time: $t_1 = 1$ year; $N_1 = C/Y \cdot t_1 = 1$
- Cash flow 2 time: $t_2 = 2$ years; $N_2 = 2$

- Cash flow 3 time: $t_3 = 3$ years; $N_3 = 3$



It is important to note that all the cash flows in this scenario are returns on investment, meaning they are cash inflows. Given the need to make an immediate decision on the investment opportunity, we use ‘now’ as the focal date to determine the present values of the future cash flows. Starting with Formula 5.1, we will apply the present value formula from compound interest (Formula 2.4c) to calculate the present value of each expected cash inflow from the investment.

$$\begin{aligned}
 PV_{All\ Cash\ Flows} &= PV_{Cash\ flow\ 1} + PV_{Cash\ flow\ 2} + PV_{Cash\ flow\ 3} \\
 &= FV_1(1 + i)^{-N_1} + FV_2(1 + i)^{-N_2} + FV_3(1 + i)^{-N_3} \\
 &= 8000(1 + 0.08)^{-1} + 13,000(1 + 0.08)^{-2} + 16,000(1 + 0.08)^{-3} \\
 &= 7407.407... + 11,145.405... + 12,701.314... \\
 &\approx \$31,254.13
 \end{aligned}$$

Therefore, the discounted cash flow of the project is \$31,254.13, which is greater than the cost of capital of \$30,000. The returns are sufficient to cover the cost of capital and provide an additional value, indicating the company should proceed with the investment project under the given financial criteria.

Try an Example



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Example 5.3.2: Comparing Two Investment Option Using the DCF Values

The CEO of an ice cream company is choosing between two investment options: Project A and Project B. Both projects require the same initial investment. Project A is expected to yield returns of \$45,000 in one year and another \$45,000 in two years. In contrast, Project B is anticipated to bring in \$10,000 in one year and \$80,000 in two years. With the company's required rate of return set at 12%, which project should the CEO choose based on the financial viability and the expected returns of each option?

Show/Hide Solution

- Required rate of return: $I / Y = 12 \%$
- The frequency of compounding periods is not provided, so it is assumed annually: $C / Y = 1$
- Interest rate per compounding period: $i = \frac{I / Y}{C / Y} = \frac{12 \%}{1} = 12 \% = 0.12$

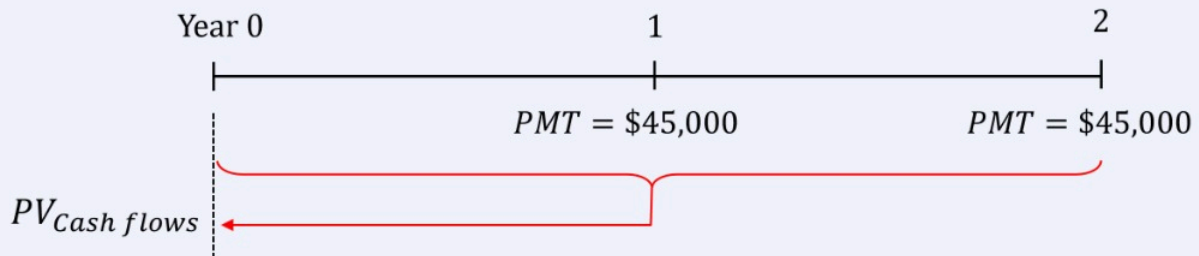
Project A:

For Project A, both anticipated cash flows are returns on the investment, categorizing them as cash inflows. To determine the present values of these future cash inflows, we have two methods:

Using the Present Value Formula for Compound Interest: We can calculate the present value of each expected cash inflow individually by applying the formula for compound interest (Formula 2.4c). This involves discounting each future cash inflow back to its present value based on the specified discount rate.

Using the Present Value Formula for an Ordinary Simple Annuity: Alternatively, as both cash inflows from Project A are of equal amounts and are expected to occur at regular intervals (one year apart), they can

be treated as payments of an ordinary simple annuity. In this case, we can apply the present value formula for an ordinary simple annuity (Formula 3.9a) to calculate the combined present value of both cash inflows in a single step. We use the second method here.



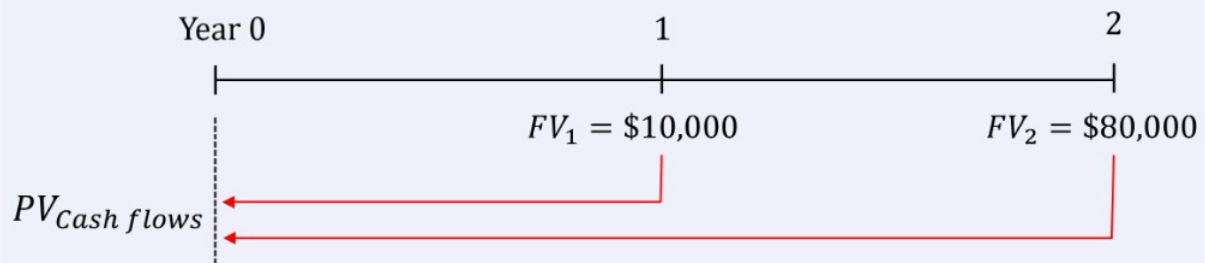
- Cash inflows 1,2: $PMT = 45,000$
- Annuity term: $t = 2$ years
- One payment per year, so $P/Y = 1$
- Number of payments in the term: $N = P/Y \cdot t = 1(2) = 2$

$$\begin{aligned}
 PV_{All\ Cash\ Flows} &= PV_{Annuity} \\
 &= PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] \\
 &= 45,000 \left[\frac{1 - (1 + 0.12)^{-2}}{0.12} \right] \\
 &\approx \$76,052.30
 \end{aligned}$$

Therefore, the discounted cash flow of Project A is \$76,052.30.

Project B:

Both cash flows in Project B are also returns on investment, meaning they are cash inflows. We apply the present value formula from compound interest (Formula 2.3c) to calculate the present value of each expected cash inflow from the investment.



- Cash flow 1: $FV_1 = 10,000$
- cash flow 2: $FV_2 = 80,000$
- Cash flow 1 time: $t_1 = 1$ year; $N_1 = C/Y \cdot t_1 = 1$
- Cash flow 2 time: $t_2 = 2$ years; $N_2 = 2$

Using Formula 5.1, we have

$$\begin{aligned}
 PV_{All\ Cash\ Flows} &= PV_{Cash\ flow\ 1} + PV_{Cash\ flow\ 2} \\
 &= FV_1(1 + i)^{-N_1} + FV_2(1 + i)^{-N_2} \\
 &= 10,000(1 + 0.12)^{-1} + 80,000(1 + 0.12)^{-2} \\
 &= 8928.571\dots + 63,775.510\dots \\
 &\approx \$72,704.08
 \end{aligned}$$

Therefore, the discounted cash flow of Project B is \$72,704.08.

Make a Decision:

The CEO should choose the project with the higher discounted cash flow value because it represents a greater return on investment after considering the time value of money at the company's required rate of return. In this case, since Project A has a higher DCF value than Project B, the CEO should opt for Project A as the investment choice.

Try an Example



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Section 5.3 Exercises

1. A delivery company is considering investing \$26,260 in a project. The expected returns from this investment are projected to be \$6,000 in 1 year, \$10,000 in 5 years, and \$17,500 in 8 years. The company's required rate of return for investments is 4.5%. **a)** Calculate the discounted cash flow. **b)** Is the investment worthwhile?

Show/Hide Answer

a) DCF = \$26,071.88

b) No, it is not since DCF is less than the investment capital.

2. The CFO of a company is choosing between two investment options: Project A and Project B. Both projects require the same initial investment. Project A is expected to yield returns of \$35,000 in one year and another \$35,000 in two years. In contrast, Project B is anticipated to bring in \$29,000 in one year and \$39,000 in two years. The company's required rate of return is set at 8%. **a)** Calculate the discounted cash flow of Project A. **b)** Calculate the discounted cash flow of Project B. **c)** Which project should the CFO choose based on the discounted returns of each option?

Show/Hide Answer

a) Project A: DCF = \$62,414.27

b) Project B: DCF = \$60,288.07

- c) Project A should be chosen because of the higher DCF.
3. A company is considering two investment opportunities. Alternative 1 will result in payments of \$9,000 in two years and \$22,000 in four years. Alternative 2 will result in annual payments of \$4,500 at the end of each year for a total of five years. The company's required rate of return is 6.58%. **a)** Calculate the DCF of Alternative 1. **b)** Calculate the DCF of Alternative 2. **c)** Which investment alternative should the company choose to maximize its financial benefit? *Round your answers to the nearest dollar.*

Show/Hide Answer

- a) Alternative 1: DCF = \$24,973
- b) Alternative 2: DCF = \$18,660
- c) Alternative 1 should be chosen because of the higher DCF.

5.4 NET PRESENT VALUE (NPV): FORMULA APPROACH

Net Present Value (NPV) is a financial metric used to evaluate the profitability of an investment or project. It represents the difference between the present value of cash inflows and the present value of cash outflows (also known as outlays) over the investment's lifetime. The NPV is calculated by discounting all expected future cash flows to their present values using a specific discount rate, often the required rate of return. By using a required rate of return in NPV calculations, investors and managers can factor in the risk profile of the investment and the opportunity cost of capital. This rate typically reflects the return that could be earned on alternative investments of similar risk.

$$NPV = PV_{\text{Cash inflows}} - PV_{\text{Cash outflows}} \quad \text{Formula 5.2}$$

A positive NPV indicates that the project is expected to generate more value than the cost, considering the time value of money and the investment's risk. Conversely, a negative NPV suggests that the project is likely to diminish value. An NPV of zero means that the investment is expected to yield a return equal to the minimum acceptable rate of return.

The NPV decision criterion is as follows:

- If $NPV > 0$: Accept the Project
 - A positive NPV indicates that the present value of the project's cash inflows exceeds the present value of its cash outflows. This means the investment is expected to generate a return greater than the required rate of return. In other words, the project adds value to the firm or investor and is considered financially viable and profitable.
- If $NPV = 0$: Indifference
 - An NPV of zero suggests the project's returns are exactly sufficient to cover the cost of capital and meet the required rate of return. The project is breaking even in present value terms. While financially neutral, the decision might still consider strategic, operational, or social factors.
- If $NPV < 0$: Reject the Project
 - A negative NPV indicates that the present value of the project's cash inflows is less than the present value of its cash outflows. This means the project is expected to yield a return less than the required rate of return. The project is not adding value and is considered unprofitable and financially unviable.

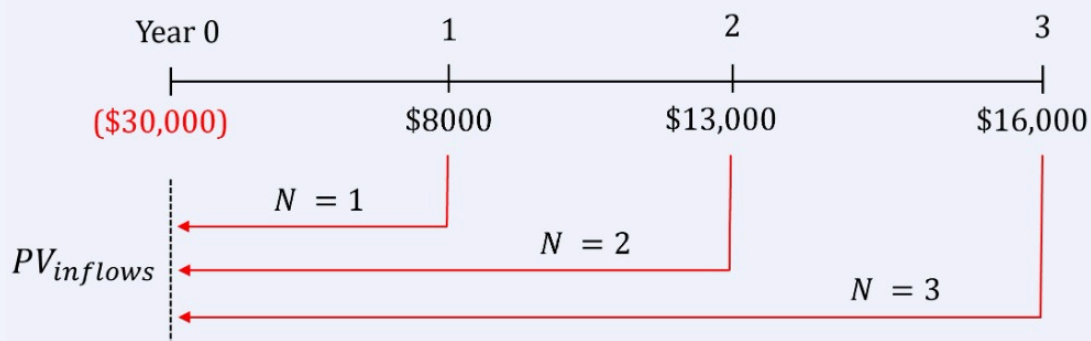
When comparing multiple projects using Net Present Value (NPV), the decision-making process involves evaluating the NPV of each project and then selecting the one(s) that provide the greatest financial return.

Example 5.4.1: Calculate NPV of a Project with Future Cash Inflows

The management team of a company is evaluating a project that requires a \$30,000 investment today. The expected returns from this investment are projected to be \$8,000 in one year, \$13,000 in two years, and \$16,000 in three years. The company's required rate of return for investments is 8%. Calculate the net present value of the project and determine whether they should accept the project.

Show/Hide Solution

- Required rate of return: $I / Y = 8 \%$
- The frequency of compounding periods is not provided, so it is assumed annually: $C / Y = 1$
- Interest rate per compounding period: $i = \frac{I / Y}{C / Y} = \frac{8 \%}{1} = 8 \%$
- Cash outflow = \$30,000
- Cash inflow 1: $FV_1 = 8000$
- Cash inflow 2: $FV_2 = 13,000$
- Cash inflow 3: $FV_3 = 16,000$
- Cash inflow 1 time: $t_1 = 1$ year; $N_1 = C / Y \cdot t_1 = 1$
- Cash inflow 2 time: $t_2 = 2$ years; $N_2 = 2$
- Cash inflow 3 time: $t_3 = 3$ years; $N_3 = 3$



Cash Outflows:

As the initial investment of \$30,000 must be paid immediately, there is no need to calculate its present value. It is already in present-day terms.

$$PV_{\text{outflows}} = \$30,000$$

Cash Inflows:

It is important to note that all the future cash flows of the project are returns on investment, meaning they are cash inflows. Given the need to make an immediate decision on the investment opportunity, we use 'now' as the focal date to determine the present values of the future cash inflows. We will apply the present value formula from compound interest (Formula 2.3c).

$$\begin{aligned} PV_{\text{inflow}} &= PV_{\text{Cash flow 1}} + PV_{\text{Cash flow 2}} + PV_{\text{Cash flow 3}} \\ &= FV_1(1+i)^{-N_1} + FV_2(1+i)^{-N_2} + FV_3(1+i)^{-N_3} \\ &= 8000(1+0.08)^{-1} + 13,000(1+0.08)^{-2} + 16,000(1+0.08)^{-3} \\ &= 7407.407\dots + 11,145.405\dots + 12,701.314\dots \\ &\approx \$31,254.13 \end{aligned}$$

Determine NPV:

Using Formula 5.2, the NPV is

$$\begin{aligned} NPV &= PV_{\text{inflows}} - PV_{\text{outflows}} \\ NPV &= 31,254.13 - 30,000 \\ &= 1254.13 \end{aligned}$$

Make a Decision:

The NPV of the project is greater than zero, which means the returns are sufficient to cover the cost of capital

and provide an additional value, indicating the company should accept the project under the given financial criteria.

Try an Example



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Example 5.4.2: Calculate NPV of a Project with Annuity Cash Inflows and Residual

A company is evaluating the feasibility of investing in machinery to expand its manufacturing plant. It would need to invest \$40,000 today and \$20,000 in one year. The machinery is expected to generate a net return of \$6,295 at the end of every year for 11 years. At the end of the 11 years, the machinery is expected to have a salvage (residual) value of \$10,200, which can be considered a final cash inflow. The company's required rate of return for investments is 5.4%. **a)** Calculate the Net Present Value (NPV) of the investment. **b)** Determine whether the purchase of this machinery is a financially sound decision according to the NPV criterion.

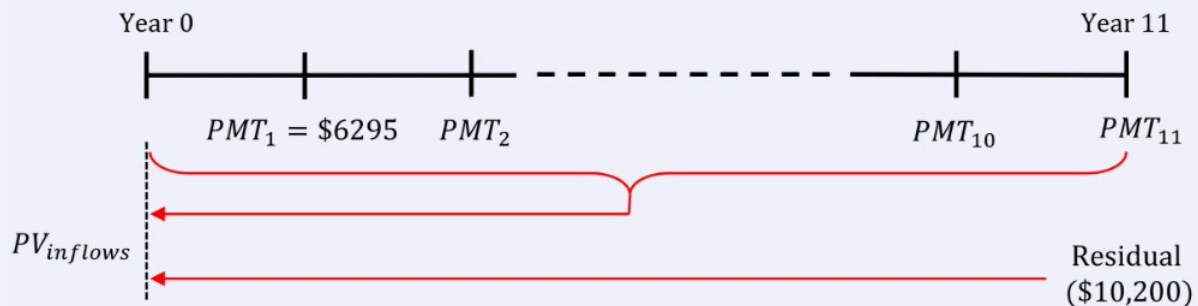
Show/Hide Solution

- Required rate of return: $I / Y = 5.4 \%$
- The frequency of compounding periods is not provided, so it is assumed annually: $C / Y = 1$
- Interest rate per compounding period: $i = \frac{I / Y}{C / Y} = \frac{5.4 \%}{1} = 5.4 \%$

- Returns: $PMT = 6295$
- Annuity term: $t = 11$ years
- One payment per year, so $P/Y = 1$
- Number of payments in the term: $N = P/Y \cdot t = 1(11) = 11$

-The machinery's salvage value of \$10,200: The company can sell the machinery at the of the 11th year. This salvage value is considered a cash inflow, and its present value can be calculated using the present formula of compound interest (Formula 2.3c).

- Residual (salvage) value: $FV_s = 10,200$
- Compound term: $t = 11$ years
- $P/Y = C/Y = 1$
- Number of compound periods in the term: $N = C/Y \cdot t = 1(11) = 11$



$$\begin{aligned}
 PV_{\text{inflows}} &= PV_{\text{returns}} + PV_{\text{residual}} \\
 &= PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right] + FV_s (1 + i)^{-N} \\
 &= 6295 \left[\frac{1 - (1 + 0.054)^{-11}}{0.054} \right] + 10,200(1 + 0.054)^{-11} \\
 &= 51,207.570... + 5719.439... \\
 &\approx \$56,927.01
 \end{aligned}$$

Determine NPV:

Using Formula 5.2, the NPV is

$$\begin{aligned}
 NPV &= PV_{\text{inflows}} - PV_{\text{outflows}} \\
 NPV &= 56,927.01 - 58,975.33 \\
 &= -\$2048.32
 \end{aligned}$$

b)

Make a Decision:

The NPV of the project is negative, which means the investment does not generate enough return to cover its initial cost plus the minimum required return. Therefore, the company should reject the project under the given financial criteria.

Try an Example



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=515>

Example 5.4.3: Evaluate NPV of Two Projects with Annuity Cash Inflows and Residual

Riverflow Utilities is faced with two different strategies for enhancing the environmental sustainability of one of its water treatment facilities to meet increased regulatory standards. Each option involves different cost structures

and timelines, and the decision must be made using the Net Present Value (NPV) method, considering the required rate of return of 10%.

Strategy A: Upgrade with Advanced Technology

- Annual Costs: \$50,000 paid at the end of each year for 9 years.
- Residual Value: At the end of 9 years, Riverflow Utilities expects to sell the specialized technology components for \$80,000.

Strategy B: Outsourcing to Environmental Specialists

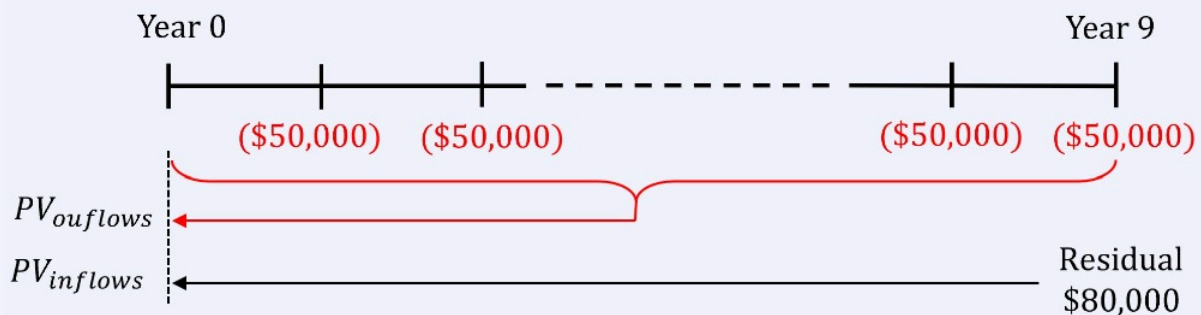
- Initial Cost: A one-time payment of \$120,000 upfront.
- Annual Costs: \$25,000 paid at the end of each year for 9 years.

a) Calculate the Net Present Value (NPV) of Strategy A. **b)** Calculate the Net Present Value (NPV) of Strategy B. **c)** Determine which strategy Riverflow Utilities should implement based on the NPV criterion.

Show/Hide Solution

- Required rate of return: $I / Y = 10 \%$
- The frequency of compounding periods is not provided, so it is assumed annually: $C / Y = 1$
- Interest rate per compounding period: $i = \frac{I / Y}{C / Y} = \frac{10 \%}{1} = 10 \%$

a) Strategy A:



Cash outflow:

The annual costs of \$50,000 are the cash outflows. They are of equal amounts and are expected to occur at regular intervals (one year apart), they can be treated as payments of an ordinary simple annuity. In this case, we can apply the present value formula for an ordinary simple annuity (Formula 3.9) to calculate the combined present value of cash outflows in a single step.

- Frequency of payments per year: $P/Y = 1$
- Time period: $t = 9$ years
- Number of payments in the term: $N = P/Y \cdot t = 1(9) = 9$
- Payments: $PMT = 50,000$

$$PV_{\text{outflows}} = PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right]$$

$$PV_{\text{outflows}} = 50,000 \left[\frac{1 - (1 + 0.1)^{-9}}{0.1} \right]$$

$$\approx \$287,951.19$$

Cash inflow:

The company can sell the component at the of the 9th year. This residual value is considered a cash inflow, and its present value can be calculated using the present formula of compound interest (Formula 2.3c).

- Residual value: $FV = 80,000$
- Compound term: $t = 9$ years
- $C/Y = 1$
- Number of compound periods in the term: $N = C/Y \cdot t = 1(9) = 9$

$$PV_{\text{inflows}} = PV_{\text{residual}}$$

$$PV_{\text{inflows}} = FV(1 + i)^{-N}$$

$$= 80,000(1 + 0.1)^{-9}$$

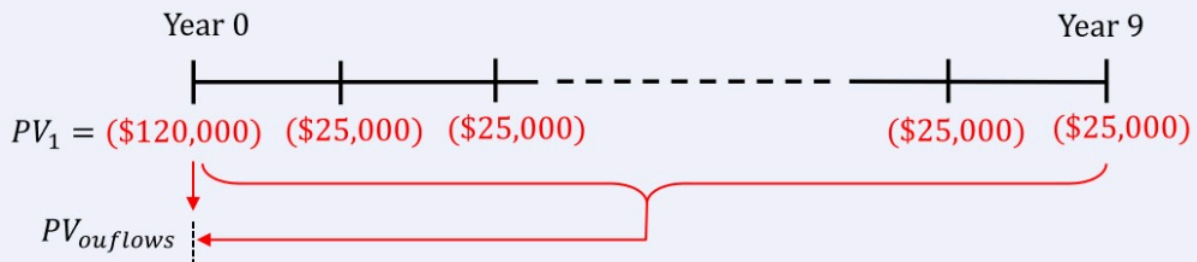
$$\approx 33,927.81$$

The net present value of Strategy A (NPV_A):

Using Formula 5.2, the NPV is

$$\begin{aligned} NPV &= PV_{\text{inflows}} - PV_{\text{outflows}} \\ NPV_A &= 33,927.81 - 287,951.19 \\ &= -\$254,023.38 \end{aligned}$$

b) Strategy B:



Cash outflow:

As the initial cost of \$120,000 must be paid immediately, there is no need to calculate its present value. It is already in present-day terms.

The annual costs of \$25,000 are of equal amounts and are expected to occur at regular intervals (one year apart), they can be treated as payments of an ordinary simple annuity. In this case, we can apply the present value formula for an ordinary simple annuity (Formula 3.9) to calculate the present value.

- Frequency of payments per year: $P/Y = 1$
- Time period: $t = 9$ years
- Number of payments in the term: $N = P/Y \cdot t = 1(9) = 9$
- Payments: $PMT = 25,000$

$$PV_{\text{outflows}} = \text{Initial pay} + PMT \left[\frac{1 - (1 + i)^{-N}}{i} \right]$$

$$\begin{aligned}
 PV_{\text{outflows}} &= 120,000 + 25,000 \left[\frac{1 - (1 + 0.1)^{-9}}{0.1} \right] \\
 &= 120,000 + 143,975.595\dots \\
 &\approx \$263,975.60
 \end{aligned}$$

Cash inflow:

There is no cash inflow in the second strategy, so $PV_{\text{inflows}} = 0$.

The net present value of Strategy B (NPV_B):

Using Formula 5.2, the NPV is

$$\begin{aligned}
 NPV &= PV_{\text{inflows}} - PV_{\text{outflows}} \\
 NPV_B &= 0 - 263,975.60 \\
 &= -\$263,975.60
 \end{aligned}$$

c) Making a decision:

Since the net present values for both strategies are negative, the company should opt for the strategy with the lesser cost or larger net present value. Therefore, the company should choose Strategy A, as it has the lower negative net present value and thus represents the less costly option, which is consistent with the NPV acceptance criteria as $NPV_A > NPV_B$.

Try an Example



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Section 5.4 Exercises

1. The management team of a company is evaluating a project that requires a \$17,099 investment today. The expected returns from this investment are projected to be \$6,000 in 1 year, \$12,000 in 7 years, and \$13,500 in 8 years. The company's required rate of return for investments is 7%. **a)** Calculate the net present value of the project. **b)** Should the management accept the project?

Show/Hide Answer

a) $NPV = \$3,839.00$

b) Yes, since $NPV > 0$.

2. A company is evaluating the feasibility of investing in machinery to expand its manufacturing plant. It would need to invest \$53,300 today and \$54,400 in 3 years. The machinery is expected to generate a net return of \$9,500 at the end of every year for 12 years. At the end of the 12 years, the machinery is expected to have a residual value of \$16,500. The company's required rate of return for investments is 3%. **a)** Calculate the Net Present Value (NPV) of the investment. **b)** Is the investment worthwhile?

Show/Hide Answer

a) $NPV = \$3,052.10$

b) Yes, since $NPV > 0$

3. VitalFlow Systems is faced with two different strategies for enhancing the environmental sustainability of one of its water treatment facilities to meet increased regulatory standards. Each option involves different cost structures and timelines, and the decision must be made using the Net Present Value (NPV) method,

considering the required rate of return of 7%. **a)** Calculate the net present value of Strategy A. **b)** Calculate the discounted cash flow of Strategy B. **c)** Which strategy should VitalFlow Systems choose based on the NPV of each option?*Strategy A: Upgrade with Advanced Technology*

- Annual Costs: \$48,000 paid at the end of each year for 9 years.
- Residual Value: At the end of 9 years, VitalFlow Systems expects to sell the specialized technology components for \$64,800.

Strategy B: Outsourcing to Environmental Specialists

- Initial Cost: A one-time payment of \$61,000 upfront.
- Annual Costs: \$32,200 paid at the end of each year for 9 years.

Show/Hide Answer

a) Strategy A: NPV = -\$277,484.24

b) Strategy B: NPV = -270,790.48

c) Strategy B since it costs less.

5.5 TERMINOLOGY AND GLOSSARY OF SYMBOLS

Terminology

- **Cash Flows:** In business investments, this refers to the streams of cash that are expected to be received or paid out over the lifetime of the investment.
- **Cash Inflow:** This is the money expected to be received from the investment.
- **Cash Outflow:** This represents the expenditures or costs associated with maintaining and operating the business or making an investment.
- **Discounted Cash Flow (DCF):** a method to find the discounted value of the expected future cash flows from investing in a project to determine their present value on a focal date to evaluate the investment option.
- **Net Present Value (NPV):** a method of evaluating the feasibility of an investment decision by determining the difference between the present value of all cash inflows (returns) and the present value of all cash outflows (investment outlays).
- **Required Rate of Return:** This rate represents the expected yield on an investment, taking into account both the risk involved and the time value of money.

Glossary of Symbols

Annuity and Compound

- **I/Y:** Nominal interest Rate per year (%)
- **C/Y:** Frequency of compounding period per year
- **i:** Interest rate per compounding period (%)
- **t:** Term of annuity (in years)
- **N:** Total number of payments/compounding periods in the term
- **PMT:** the amount of periodic payment (\$)
- **P/Y:** Number of payments per year
- **PV:** Present value of an annuity (\$)
- **FV:** Future (maturity) value of an annuity (\$)
- **I:** Interest Amount (\$)

PART VI

CVP AND BREAK-EVEN ANALYSES

Chapter Outline

This chapter provides a comprehensive exploration of Cost-Volume-Profit (CVP) and Break-Even analyses, essential tools for understanding the relationship between costs, sales volume, and profits. Each section is designed to equip readers with the knowledge and skills to effectively perform these analyses, whether through algebraic formulas, graphical representations, or using a financial calculator.

6.1 Break-Even Analysis: This section introduces the concept of break-even analysis, a fundamental tool for determining the point at which a business neither makes a profit nor incurs a loss. It covers how to calculate the break-even point in units sold and in sales dollars.

6.2 Contribution Margin Approach: This section introduces the contribution margin, a key metric in CVP analysis that represents the portion of the selling price that contributes to the coverage of fixed costs. This part explains how to calculate the break-even point using the contribution margin approach.

6.3 Cost-Volume-Profit (CVP) Analysis: This part delivers an in-depth look at CVP analysis, which examines how changes in costs (both fixed and variable), sales volume, and price affect a company's profit.

6.4 Break-Even Chart: This section explains how to visually represent break-even analysis using a break-even chart. It demonstrates how the chart is constructed and how it can be used to visually assess how changes in costs and sales levels impact profitability.

6.5 CVP Analysis Using a Financial Calculator: This section is tailored for readers who prefer to use technological tools, this part shows how to conduct CVP analysis efficiently with a financial calculator.

6.6 Terminology and Glossary of Symbols: This section concludes the chapter with a glossary of terms and symbols used throughout the chapter.

6.1 BREAK-EVEN ANALYSIS

A. Introduction

One of the primary goals of many businesses is to make a profit. Achieving this goal requires that total revenue surpasses the total costs incurred. As such, it becomes essential for business owners to carefully examine variables like the selling price per unit, the quantity of products sold, and the relationship between total revenue and costs. A powerful tool to aid this understanding is the **Cost-Volume-Profit (CVP) analysis**.

A part of CVP analysis looks at the 'break-even point'. This represents the sales volume at which total revenue exactly equals total costs, resulting in a balance where neither profit nor loss is realized. This type of analysis is known as **break-even analysis**.

While the break-even point serves as a foundational benchmark, most businesses want to do better than just breaking even. They want to earn a profit. Therefore, understanding the impact of sales on earnings is important. Furthermore, to be competitive in the market, businesses try to set competitive selling prices and look into ways to lower their costs to increase profitability. This chapter will delve into these critical aspects of business decision-making, looking at how CVP and break-even analyses can help businesses make these decisions.

B. Compute Break-even Point Using Break-even Analysis

Imagine Sarah, who has started a small business selling hand-made candles. Each candle costs her \$5 to make (materials), and she sells them for \$25 each. Sarah also has fixed costs, which include her monthly rent for the workshop, totaling \$2000 a month. Before she can make any profit, Sarah needs to know how many candles she must sell to cover her costs, or in other words, to break even. Table 5.1.1 summarizes the revenue, cost, and profit for Sarah's business based on the different numbers of candles sold in a given month.

Table 5.1.1: Revenue, cost, and profit for different numbers of candles sold in a given month

| | | | | |
|-------------------------|----------------|------------------------|----------------|------------------|
| Number of Candles | 0 | 10 | 100 | 200 |
| Total Revenue (TR) | 0 | 10(\$25) = \$250 | \$2500 | \$5000 |
| Cost of Materials (TVC) | 0 | 10(\$5) = \$50 | \$500 | \$1000 |
| Rent (FC) | \$2000 | \$2000 | \$2000 | \$2000 |
| Total Costs (TC) | \$2000 | \$50 + \$2000 = \$2050 | \$2500 | \$3000 |
| Net Income (NI) | -\$2000 (Loss) | -\$1800 (Loss) | 0 (Break-even) | +\$2000 (Profit) |

Note that at the point where 100 candles are sold, the net income is zero. This is recognized as the break-even point, a point where total revenue precisely matches total costs, resulting in a net income of zero. Selling fewer than 100 units (e.g., 0 and 10 candle cases) results in a financial loss, whereas selling more than the break-even point of 100 units (such as the 200 candle case) ensures a profit.

C. CVP Assumptions

To streamline our discussion on CVP and break-even analyses, let us establish some foundational assumptions:

- a. **Selling Price:** The price at which we sell each unit remains the same, so as we sell more, our total revenue increases at a consistent rate.
- b. **Costs categories and behavior:** Costs are linear and can be classified as either fixed or variable cost
 - **Fixed costs** remain constant over a given time period. They do not depend on volume and remain unchanged throughout a specific time period. Examples include rent, mortgage, property taxes, and salaries.
 - **Variable costs** are constant per unit regardless of volume. Some of the examples are direct material, labor costs, and sales commissions.
- c. **Inventory:** All units produced in a particular operating period are sold, i.e., there is no inventory.
- d. **Multiple products:** When multiple products are produced within a business, the ratio of various products remains constant.

Note that in reality, fixed costs don't always remain the same across different levels of production; they can change. Similarly, the unit variable cost can vary, often due to economies of scale. Furthermore, it's not always clear-cut to label costs as either fixed or variable. Some costs have elements of both and are termed semi-fixed costs or mixed costs.

D. CVP Formulas

The CVP analysis focuses on profitability. In its simplest form, the income statement shows that profit is the result of subtracting total costs (TC) from total revenue (TR). This profit is commonly known as net income (NI) or operating income.

$$\text{Net Income} = \text{Total Revenue} - \text{Total Costs}$$

Or

$$NI = TR - TC \quad \text{Formula 6.1}$$

Assuming the business only sells one product, the total revenue is the product of the unit selling price and the number of units sold or volume. Given SP represents the selling price and X the volume (quantity), the total revenue can be written as

$$TR = SP \cdot X \quad \text{Formula 6.2}$$

Formula 6.2 is referred to as the **total revenue function**.

The total costs are the sum of the total variable costs (TVC) and the fixed costs (FC).

$$TC = TVC + FC \quad \text{Formula 6.3}$$

The total variable costs can be written in terms of the variable cost per unit (VC) and the volume (X).

$$TVC = VC \cdot X \quad \text{Formula 6.4}$$

Substituting Formula 6.4 into Formula 6.3 yields

$$TC = VC \cdot X + FC \quad \text{Formula 6.5}$$

Formula 6.5 is called the **total cost function**. Now plugging Formulas 6.2 and 6.5 into Formula 6.1 gives

$$NI = (SP \cdot X) - (VC \cdot X + FC) \quad \text{Formula 6.6a}$$

Formula 6.6a is referred to as the **net income function** and it shows the relationship among net income (NI), the unit selling price (SP), unit variable cost (VC), fixed costs (FC), and volume (X).

E. Compute Break-even Point Using Break-even Analysis

The breakeven point (BP) is the point at which a business neither makes a profit nor incurs a loss. Therefore, to find out the quantity needed to break even, you can use Formula 6.6a and set the net income (NI) to zero.

$$0 = SP \cdot X - (VC \cdot X + FC) \quad \text{Formula 6.6b}$$

Alternatively, we can use the fact that at break-even total costs are equal to the total revenue.

$$TR = TC$$

$$SP \cdot X = VC \cdot X + FC \quad \text{Formula 6.6c}$$

Both Formulas 6.6b and 6.6c give the quantity at the breakeven point. The break-even point may be expressed in various ways such as

- number of units
- total revenue at break-even units, or
- as a percent of capacity.

When Unit Selling Price And Variable Cost Are Known

In most cases, the unit selling price and the unit variable cost are known. Therefore, the values are directly input in Formula 6.6b and the volume is solved for.

Example 6.1.1: Compute Break-even Point with Known Unit Selling Price and Variable Cost

Margaret runs a business that makes custom-printed shirts. It will cost her \$7 each to purchase and print on shirts, and she will have to pay a rent of \$1,596 per month for her workshop. Based on market research, Margaret estimates that she can sell custom shirts for \$26 each. **a)** Find the revenue function. **b)** Find the cost function. **c)** Calculate the number of shirts she needs to sell per month to break even. **d)** Calculate the break-even in dollars (round off to the nearest cent).

Show/Hide Solution

The selling price is \$26, so $SP = \$26$

The variable cost per t-shirt is \$7, so $VC = \$7$

The only fixed cost is the workshop rent, so $FC = \$1596$

a) Total revenue function is given by Formula 6.2:

$$TR = SP \cdot X$$

$$TR = 26X$$

b) Total cost function is given by Formula 6.5:

$$TC = VC \cdot X + FC$$

$$TC = 7X + 1596$$

c) To find the quantity at break-even, we can use Formula 6.6b or use the fact that at break-even total costs equals the total revenue (Formula 6.6c) and solve for X :

$$TR = TC$$

$$26X = 7X + 1596$$

$$(26 - 7)X = 1596$$

$$19X = 1596$$

$$X = \frac{1596}{19}$$

$$X = 84$$

Therefore, Margaret needs to sell 84 T-shirts to break even.

d) Break-even in dollars is the total revenue at break-even quantity ($X = 84$)

$$TR = 26 \cdot X$$

$$TR = 26(84)$$

$$= \$2184$$

Try an Example



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Example 6.1.2: Compute Break-even Point with Known Unit Selling Price and Variable Cost

A product can be sold at \$50 per unit. Cost analysis provides the following information:

- Rent per period = \$6640
- Utilities per period = \$1100
- Insurance per period = \$900
- Manufacturing cost per unit = \$12
- Labour costs per unit = \$18
- Production capacity per period = 900 units

Perform a break-even analysis. Provide:

- (a) algebraic functions of the total revenue, the total cost, and net income,
- (b) computation of the break-even point in units,
- (c) computation of break-even point in dollars, and
- (d) computation of break-even point as a percent of capacity.

Show/Hide Solution

The selling price is \$50, so $SP = \$50$

Costs provided on a per-unit basis represent variable costs. The variable cost per unit consists of the manufacturing cost per unit and the labor cost per unit, so $VC = \$12 + \$18 = \$30$

Costs stated on a per-period basis are classified as fixed costs. The fixed costs are the sum of rent, utilities, and insurance, so $FC = \$6640 + \$1100 + \$900 = \8640

a) The total revenue function is given by Formula 6.2:

$$TR = SP \cdot X$$

$$TR = 50X$$

The total cost function is given by Formula 6.5:

$$TC = VC \cdot X + FC$$

$$TC = 30X + 8640$$

The net income function is given by Formula 6.6a:

$$NI = SP \cdot X - (VC \cdot X + FC)$$

$$NI = 50X - (30X + 8640)$$

$$NI = 50X - 30X - 8640$$

$$NI = 20X - 8640$$

b) To find the quantity at break-even, we can use Formula 6.6b or Formula 6.6c. We can also use the net income function, set the net income to zero, and solve for X :

$$NI = 20X - 8640$$

$$0 = 20X - 8640$$

$$20X = 8640$$

$$X = \frac{8640}{20}$$

$$X = 432$$

Therefore, the break-even volume is 432 units.

c) Break-even in dollars is the total revenue at break-even quantity ($X = 432$)

$$TR = 50X$$

$$TR = 50(432)$$

$$= \$21,600$$

d) To find the break-even point as a percent capacity, divide the quantity at the break-even point by the quantity at the capacity and convert the quotient to percent.

$$\text{Break-even as a percent of capacity} = \frac{432}{900}$$

$$= 0.48$$

$$= 48\%$$

When Unit Selling Price And Variable Cost Are Unknown

The relationship between total variable costs (TVC) and total revenue (TR) can be expressed as a constant ratio, equivalent to the ratio of unit variable costs (VC) to the selling price (SP). This is represented as:

$$\frac{TVC}{TR} = \frac{VC \cdot X}{SP \cdot X}$$

Simplifying this, we get:

$$\frac{TVC}{TR} = \frac{VC}{SP}$$

In cases where we only have the total dollar values without specific unit selling prices or variable costs, we can still determine the break-even point in dollar terms (i.e., the total revenue at the break-even point) rather than in units. By assuming a unit selling price (SP) of \$1, we can then calculate the unit variable cost (VC) as follows:

$$\frac{TVC}{TR} = \frac{VC}{1}$$

$$VC = \frac{TVC}{TR}$$

Using the assumed SP of \$1 and the calculated VC value, we can apply Formula 6.6c to find 'X', where 'X' now represents the total revenue at the break-even point, rather than the number of units sold.

Example 6.1.3: Compute Break-even Point with Unknown Unit Selling Price and Variable Cost

Comfort Furnishings Inc. had a total revenue of \$146,400 last year. Its total variable costs and fixed costs for the period were \$25,150 and \$31,500, respectively. Compute the break-even point in sales dollars.

Show/Hide Solution

Total revenue is \$146,400, so $TR = \$146,400$

Total variable costs are \$25,150 so $TVC = \$25,150$

Fixed costs are \$31,500 so $FC = \$31,500$

Since unit selling price and variable cost are unknown, we assume $SP = \$1$ and

$$VC = \frac{TVC}{TR} = \frac{25150}{146400} \approx \$0.1718$$

Plugging SP , VC , and FC into Formula 6.6b gives

$$\$1X = \$0.1718X + 31,500$$

$$1X - 0.1718X = 31,500$$

$$(1 - 0.1718)X = 31,500$$

$$0.8282X = 31,500$$

$$X = \frac{31,500}{0.8282}$$

$$X = \$38,034.291\dots$$

$$\approx \$38,034.29 \text{ Rounded to the nearest cent}$$

Try an Example



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Section 6.1 Exercises

1. A company sells a product for \$76 each. The variable costs are \$23 per unit and fixed costs are \$34,768 per month. a) Find the revenue function. b) Find the cost function. c) Calculate the number of units needed to be

sold per month to break even. d) Calculate the revenue at the break-even point.

Show/Hide Answer

a) $TR = 76x$

b) $TC = 23x + 34,768$

c) $x = 656$ units

d) $TR = \$49,856$

2. Femto Laser Tech. had a total revenue of \$103,050 last year. Its total variable costs and fixed costs for the period were \$25,300 and \$17,150, respectively. Compute the break-even point in sales dollars.

Show/Hide Answer

$$TR = \$22,730.65$$

3. Wendy runs a business that makes custom bags. It will cost her \$9 each to purchase the needed materials, and she will have to pay a rent of \$1,188 per month for her workshop. Based on market research, Wendy estimates that she can sell each bag for \$21. a) Calculate the number of bags she needs to sell per month to break even. b) Calculate the break-even point in dollars.

Show/Hide Answer

a) $x = 99$ bags

b) $TR = \$2,079$

6.2 CONTRIBUTION MARGIN

The difference between unit selling price and variable cost is called the **contribution margin per unit**, denoted by CM .

$$CM = SP - VC \quad \text{Formula 6.7}$$

The contribution margin represents the amount that contributes to covering fixed costs and generating profit. For each unit sold beyond the break-even point, profit increases by the amount of the unit contribution margin. On the other hand, for each unit sold less than the break-even quantity, there is a loss equivalent to the unit contribution margin.

If we collect the X terms in Formula 6a, we obtain

$$NI = (SP - VC) \cdot X - FC$$

Substituting CM into Formula 6.7 and rearranging for X , we get

$$NI = CM \cdot X - FC$$

$$X = \frac{NI + FC}{CM} \quad \text{Formula 6.8a}$$

Formula 6.8a can be used to find the quantity (volume) for a desired net income. To find the quantity at break-even (BE), we let $NI = 0$ and the formula becomes

$$X_{BE} = \frac{FC}{CM} \quad \text{Formula 6.8b}$$

Another helpful variable is the **contribution margin rate**. Expressed as a percentage, the contribution margin rate provides insights into the profitability of products and the ability to cover fixed costs. For example, if the contribution margin rate is 40%, this means that 40 cents of every sales dollar is available to cover fixed costs and contribute to profit, after accounting for variable costs.

$$CM(\%) = \frac{CM}{SP} \quad \text{Formula 6.9a}$$

If only the total revenue and total variables are known, the contribution margin rate can be found by

$$CM(\%) = \frac{TCM}{TR} \quad \text{Formula 6.9b}$$

Here TCM is the total contribution margin, which is the difference between the total revenue and total variable costs.

$$TCM = TR - TVC$$

Example 6.2.1: Compute Break-even Using Contribution Margin Approach

Future Technology can make a maximum of 2957 webcams per month and sell them for \$134 each. The company's fixed costs per month are \$111,690, and the variable costs are \$32 per unit. **a)** Find the contribution margin per unit. **b)** Find the contribution margin rate. **c)** Calculate the number of webcams the company needs to sell per month to break even. **d)** Calculate the number of webcams the company needs to sell per month to achieve a profit of \$36,210.

Show/Hide Solution

- The selling price is \$134, so $SP = \$134$
- The variable cost per unit is \$32, so $VC = \$32$
- The fixed costs are \$111,690, so $FC = \$111,690$
- The capacity is 2957.

a) Contribution margin:

$$\begin{aligned} CM &= SP - VC \\ &= 134 - 32 \\ &= \$102 \end{aligned}$$

b) The contribution margin rate:

$$\begin{aligned} CM(\%) &= \frac{CM}{SP} \\ &= \frac{102}{134} \times 100\% \end{aligned}$$

$$\approx 76 \%$$

c) The quantity at break-even is given by Formula 6.8b:

$$\begin{aligned} X_{BE} &= \frac{FC}{CM} \\ &= \frac{111,690}{102} \\ &= 1095 \text{ Webcams} \end{aligned}$$


d) The quantity at any given profit is given by Formula 6.8a:

$$NI = \$36,210$$

$$\begin{aligned} X &= \frac{NI + FC}{CM} \\ &= \frac{36,210 + 111,690}{102} \\ &= \frac{147,900}{102} \\ &= 1450 \text{ Webcams} \end{aligned}$$

Try an Example



 One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=467>

If we only know the total dollar values and not the specific unit selling price or variable costs, we can determine the break-even point in terms of dollars instead of units using the contribution margin rate.

$$TR = \frac{FC}{CM(\%)}$$

Formula 6.10

Example 6.2.3: Compute Break-even Point with Unknown Unit Selling Price and Variable Cost

BestDeal Inc. had a total revenue of \$146,400 last month. Its total variable costs and fixed costs for the period were \$25,150 and \$31,500, respectively. Compute its break-even point in sales dollars.

Show/Hide Solution

- Total revenue is \$146,400, so $TR = \$146,400$
- Total variable costs are \$25,150 so $TVC = \$25,150$
- Fixed costs are \$31,500 so $FC = \$31,500$

Since unit selling price and variable cost are unknown, we find the contribution margin rate from Formula 6.8b and use Formula 6.10 to find the revenue at break-even.

$$\begin{aligned} TCM &= TR - TVC \\ &= 146,400 - 25,150 \end{aligned}$$

$$= \$121,250$$

So the contribution margin rate is

$$\begin{aligned} CM(\%) &= \frac{TCM}{TR} \\ &= \frac{121,250}{146,400} \times 100\% \\ &\approx 82.82\% \end{aligned}$$

Plugging CM(%) and FC into Formula 6.10 yields

$$\begin{aligned} TR &= \frac{FC}{CM(\%)} \\ &= \frac{31,500}{82.82\%} \\ &\approx \$38,034.29 \text{ Rounded to the nearest cents} \end{aligned}$$

Try an Example



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Section 6.2 Exercises

1. Qual Electronics can make a maximum of 2636 tablets per month and sell them for \$121 each. The company's

fixed costs per month are \$90,942, and the variable costs are \$52 per unit. a) Compute the contribution margin per unit. b) Compute the contribution margin rate. c) Calculate the number of tablets the company needs to sell per month to break even.

Show/Hide Answer

a) $CM = \$69$

b) $CM \text{ Rate} = 57\%$

c) $X = 1318 \text{ Tablets}$

2. A manufacturing plant had a total revenue of \$143,800 last year. Its total variable costs and fixed costs for the period were \$23,700 and \$30,900, respectively. Compute the break-even point in sales dollars.

Show/Hide Answer

$TR = \$36,997.66$

3. Future Technology sells webcams for \$134 each. The company's fixed costs per month are \$111,690, and the variable costs are \$32 per unit. a) Compute the contribution margin per unit. b) Compute the contribution margin rate. c) Calculate the total revenue at the break-even point.

Show/Hide Answer

a) $CM = \$102$

b) $CM \text{ Rate} = 76\%$

c) $TR = \$146,730.00$

6.3 COST-VOLUME-PROFIT (CVP) ANALYSIS

The cost-volume-profit formula (Formula 6.6a) we introduced in the first section shows how cost, volume, and profit are connected. You can use it to solve for any of the variables or determine changes to any of the variables in the formula. The latter is referred to as what-if analysis or sensitivity analysis. Sensitivity analysis involves adjusting these variables to see how changes impact the break-even point, overall profit, and other financial outcomes.

Example 6.3.1: CVP Analysis

Best Tech. can produce a maximum of 4305 tablets per month and sell them for \$220 each. The company's fixed costs per month are \$193,110, and the variable costs are \$63 per unit. **a)** Calculate the number of tablets the company needs to sell per month to break even. **b)** If the company wants to make a profit of \$60,288 per month, how many tablets does it need to sell? **c)** If the company sold 3800 tablets in a month, how much would its net income be? **d)** If the company were to spend an additional \$25,120 in a month for advertising, how much would it need to sell to break even?

Show/Hide Solution

- The selling price is \$220, so $SP = \$220$
- The variable cost per unit is \$63, so $VC = \$63$
- The fixed costs are \$193,110 per month, so $FC = \$193,110$
- The capacity is 4305.

a) Contribution margin approach:

$$CM = SP - VC = 220 - 63 = \$157$$

The quantity at break-even is given by Formula 6.8b:

$$X_{BE} = \frac{FC}{CM}$$

$$= \frac{193,110}{157}$$

$$= 1230 \text{ Tablets}$$

b) The quantity at any given profit is given by Formula 6.8a:

$$NI = \$60,288$$

$$X = \frac{NI + FC}{CM}$$

$$= \frac{60,288 + 193,110}{157}$$

$$= \frac{253,398}{157}$$

$$= 1614 \text{ Tablets}$$

c) We are looking for NI given $X = 3800$ Tablets. We use Formula 6.6a to calculate NI:

$$NI = SP \cdot X - (VC \cdot X + FC)$$

$$= 220(3800) - (63(3800) + 193,110)$$

$$= 836,000 - (239,400 + 193,110)$$

$$= 836,000 - 432,510$$

$$= \$403,490 \text{ Profit}$$

d) The advertising cost is considered a fixed cost as it is per time period and independent of the quantity. So the fixed costs are increased by \$25,120.

$$FC_{New} = 193,110 + \$25,120 = \$218,230$$

The new quantity at break-even is given by Formula 6.8b:

$$X_{BE} = \frac{FC_{New}}{CM}$$

$$= \frac{218,230}{157}$$

$$= 1390 \text{ Tablets}$$

Try an Example



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Example 6.3.2: CVP Analysis

Sparkling Shoes is a small company that sells designer shoes. Its fixed costs, including rent, salaries, and utilities, are \$10,000 per month. The variable costs (like materials, labor, and packaging) are \$40 per pair. Each pair of shoes is sold for \$100. Answer the following *independent* parts.

- a) How many pairs of shoes does the company need to sell to break even?
- b) What is the break-even point in sales dollars?
- c) If Sparkling Shoes wants to achieve a monthly profit of \$5,000, how many pairs of shoes should they sell?
- d) If Sparkling Shoes decides to reduce the selling price to \$90 per pair, how many shoes do they need to sell to break even?
- e) If the fixed costs increase by \$1,900 due to a rent hike, how will that impact the break-even point in units?

Show/Hide Solution

- The selling price per pair is \$100, so $SP = \$100$
- The variable cost per pair is \$40, so $VC = \$40$
- The fixed costs are \$10,000 per month, so $FC = \$10,000$

a) Contribution margin approach:

$$CM = SP - VC = 100 - 40 = \$60$$

The quantity at break-even is given by Formula 6.8b:

$$\begin{aligned} X_{BE} &= \frac{FC}{CM} \\ &= \frac{10,000}{60} \\ &= 166.66... \\ &\approx 167 \text{ Pairs (Rounded up to the nearest whole number)} \end{aligned}$$

b) The Break-even point in dollars is the total revenue at the break-even point in units:

$$\begin{aligned} TR &= SP \cdot X_{BE} \\ &= 100(167) \\ &= \$16,700 \end{aligned}$$

c) The quantity at any given profit is given by Formula 6.8a:

$$NI = \$5,000$$

$$\begin{aligned} X &= \frac{NI + FC}{CM} \\ &= \frac{5000 + 10,000}{60} \end{aligned}$$

$$= \frac{15,000}{60}$$

$$= 250 \text{ Pairs}$$

d) The new selling price is \$90, so $SP_{New} = \$90$

The new contribution margin: $CM_{New} = 90 - 40 = \$50$

The quantity at break-even is given by Formula 6.8b:

$$X_{BE} = \frac{FC}{CM_{New}}$$

$$= \frac{10,000}{50}$$

$$= 200 \text{ Pairs}$$

e) If FC increases by \$1900, $FC_{New} = 10,000 + \$1900 = \$11,900$. Note that the part questions are independent, so we use the original selling price to answer Part (e).

The quantity at break-even is given by Formula 6.8b:

$$X_{BE} = \frac{FC}{CM}$$

$$= \frac{11,900}{60}$$

$$= 198.33... \text{ Pairs}$$

$$\approx 199 \text{ Pairs (Rounded up to the nearest whole number)}$$

Note: When calculating the number of units, it is important to always round up to the nearest whole number, irrespective of the decimal value. This ensures that the business remains on the profitable side of the break-even point.

Try an Example



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Section 6.3 Exercises

- GreatSound Technology manufactures and sells headsets for \$84 each. Annually, the company incurs fixed costs totaling \$54,234. Additionally, each headset incurs variable costs of \$20 for manufacturing and \$18 for labor. a) Determine the yearly sales volume of headsets required for GreatSound Technology to break even. b) If GreatSound Technology produced and sold 2245 headsets in a year, calculate the company's profit or loss for that year. c) To achieve an annual profit of \$143,658, how many headsets must the company sell?

Show/Hide Answer

a) $X = 1179$ Headsets

b) NI = \$49,036; Profit

c) $X = 4302$ Headsets

- Wearit is a small company that sells designer shoes. Its fixed costs, including rent, salaries, and utilities, are \$22,045.40 per month. The variable costs (like materials, labor, and packaging) are \$85.4 per pair. Each pair of shoes is sold for \$244. a) How many pairs of shoes does Wearit need to sell each month to break even? b) If Wearit wants to achieve a monthly profit of \$13,322.40, how many pairs of shoes should they sell? c) If Wearit decides to reduce the selling price to \$234 per pair, how many shoes do they need to sell to break even? d) If the fixed costs increase by \$6,613.62 due to a rent hike, how will that impact the break-even point in units? Each part is independent.

Show/Hide Answer

a) $X = 139$ Pairs

b) $X = 223$ Pairs

c) $X = 149$ Pairs

d) $X = 181$ Pairs

3. XYZ Electronics is a company that manufactures and sells wireless speakers. The selling Price per Speaker is \$150. The variable costs per speaker are \$90, which includes costs of materials, labor, and variable overhead. The company's fixed costs include rent, salaries of permanent staff, and fixed overhead totaling \$300,000. a) Calculate the break-even point in units and in total sales dollars. b) If XYZ Electronics aims to achieve a profit of \$200,000 for the year, how many speakers must they sell? c) How would an increase in variable costs to \$100 per speaker affect the break-even point? d) If fixed costs increase by 10%, how many units need to be sold to achieve the original profit target of \$200,000? Each part is independent.

Show/Hide Answer

a) $X = 5000$ Speakers; TR = \$750,000

b) $X = 8334$ Speakers

c) $X = 6000$ Speakers

d) $X = 8834$ Speakers

6.4 BREAK-EVEN CHART

A break-even chart, also known as a break-even analysis or graph, visually represents the relationship between costs, revenue, and profit levels at various levels of sales volume. The chart displays fixed costs, variable costs per unit, total costs, total revenue, and the break-even point (where total revenue equals total costs) on a graph. It helps businesses determine the minimum level of sales needed to cover all expenses and start generating profit. This analysis is crucial for decision-making regarding pricing strategies, production volume, and overall business planning.

Example 6.4.1: Creating a Break-Even Chart

Jina's small business has the capacity to make and sell 200 bags per month. Each bag is sold for \$100. The fixed costs are \$7000 and the variable cost per bag is \$30.

- a) Draw a break-even chart showing the total revenue, total costs, and fixed costs lines.
- b) Determine the break-even point in volume and sales dollars.
- c) Calculate the break-even as a percent of capacity.

Show/Hide Solution

a)

Step 1: Let X represent the number of units, and Y represent dollar values. Draw and label the X- and Y-axis. We only need the positive sides of the axes. The intersection of the axes is the origin. The capacity or the maximum quantity is 200 units. Thus, the X-axis should extend to at least 200. The revenue at capacity is $\$100 \times 200 = \$20,000$. So the Y-axis should extend to at least 20,000.

Step 2: Draw the fixed costs, which should be a horizontal line parallel to the X-axis starting from point (0, 7000) on the Y-axis.

Step 3: Draw the total revenue and total cost lines.

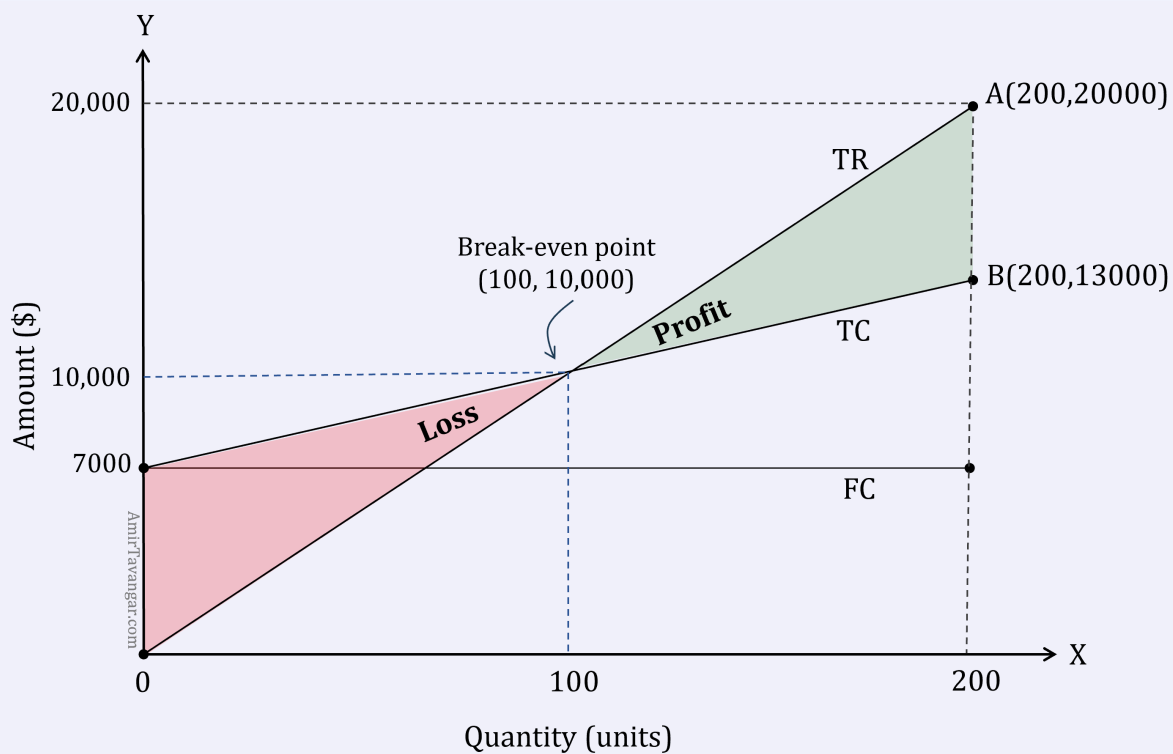
- The total revenue function is $TR = SP \cdot X = 100X$
- The total cost function is $TC = VC \cdot X + FC = 30X + 7000$

We create a table of values for the function at zero and maximum quantities.

| X | TR | TC |
|-----|--------|--------|
| 0 | 0 | 7000 |
| 200 | 20,000 | 13,000 |

- Locate points (0,0) and A(200,20000) and connect them to form a straight line. That will be the total revenue line.
- Locate points (0,7000) and B(200,13000) and connect them to form a straight line. That will be the total cost line.

Step 4: Mark the point where the total revenue (TR) and total cost (TC) lines intersect. That is the break-even point.



b) The X-coordinate of the break-even point is the break-even volume, which is 100 units. The Y-coordinate of the break-even point is the break-even revenue, which is \$10,000.

$$\begin{aligned} \text{c) Break-even as a percent of capacity} &= \frac{\text{Break-even volume}}{\text{Capacity}} \times 100 \% \\ &= \frac{100}{200} \times 100 \% \\ &= 50 \% \end{aligned}$$

Key observations from the Break-even Chart:

- If the number of units sold is below the break-even point (100 units), Total Costs (TC) exceed Total Revenue (TR), as indicated by the TC line being above the TR line. This scenario results in a loss.
- Conversely, when the number of units sold surpasses the break-even point (100 units), Total Revenue (TR) is higher than Total Costs (TC). This is shown by the TR line being above the TC line, indicating a profit.

Try an Example



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As we saw in the previous example, a Break-even Chart uses straight lines to demonstrate the relationship between

total revenue, total costs, and the fixed cost functions, offering an easy-to-understand visual representation of how these elements interact. Figure 6.4.1 illustrates a typical Break-even Chart that includes all of these components.

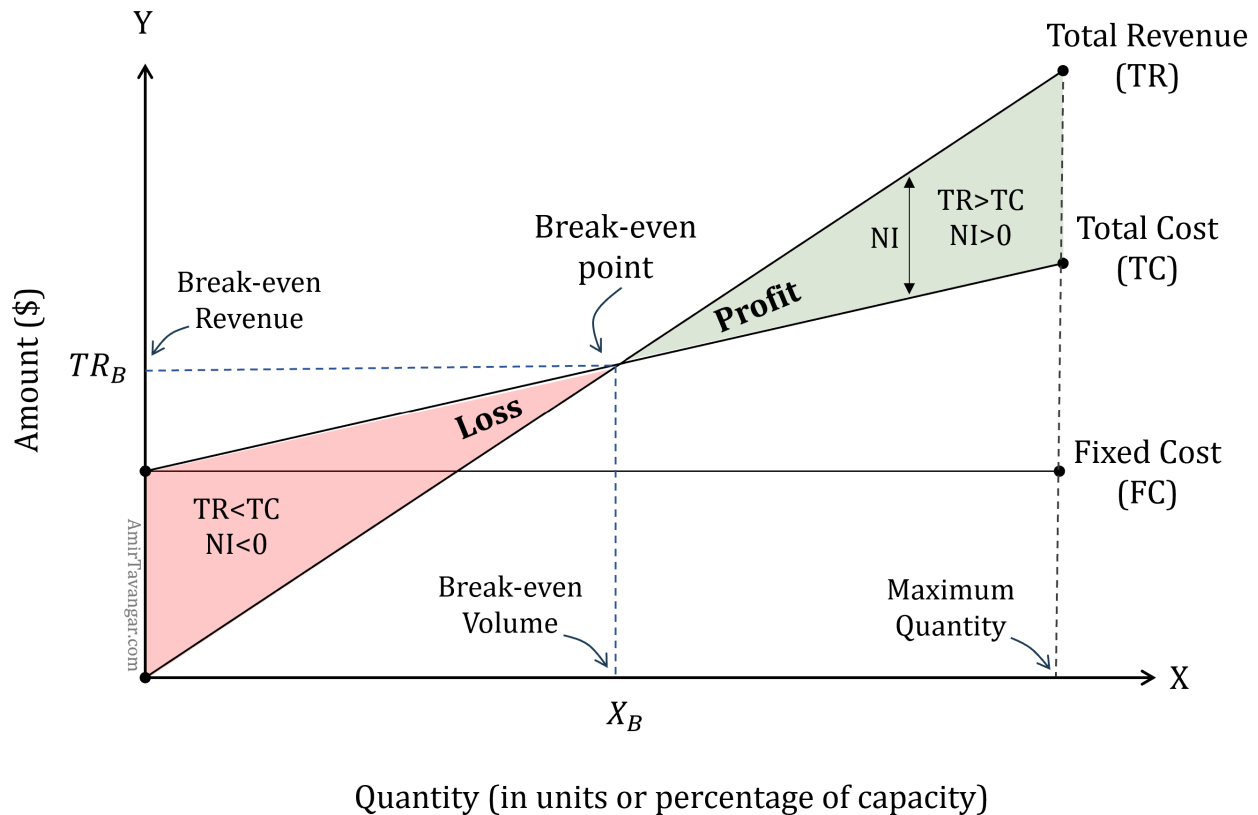


Figure 6.4.1: Break-even Chart

Here is a breakdown of the key components of a Break-even Chart:

- **X-axis (Horizontal):** Represents the number of units produced and sold (sales volume).
- **Y-axis (Vertical):** Represents the dollar amount of revenue and costs.
- **Total Revenue Line:** Starts from the origin and slopes upward. Its slope represents the selling price per unit (SP). As sales volume increases, the total revenue increases linearly.
- **Fixed Costs Line:** This is a straight, horizontal line that runs parallel to the X-axis because fixed costs stay the same regardless of the quantity produced.
- **Total Cost Line:** This begins at a point equivalent to the fixed costs (since fixed costs are incurred even when no units are produced) and slopes upward. Its slope represents the variable costs per unit (VC), and its y-intercept equals the fixed costs. The increase in the production volume leads to an increase in total cost.

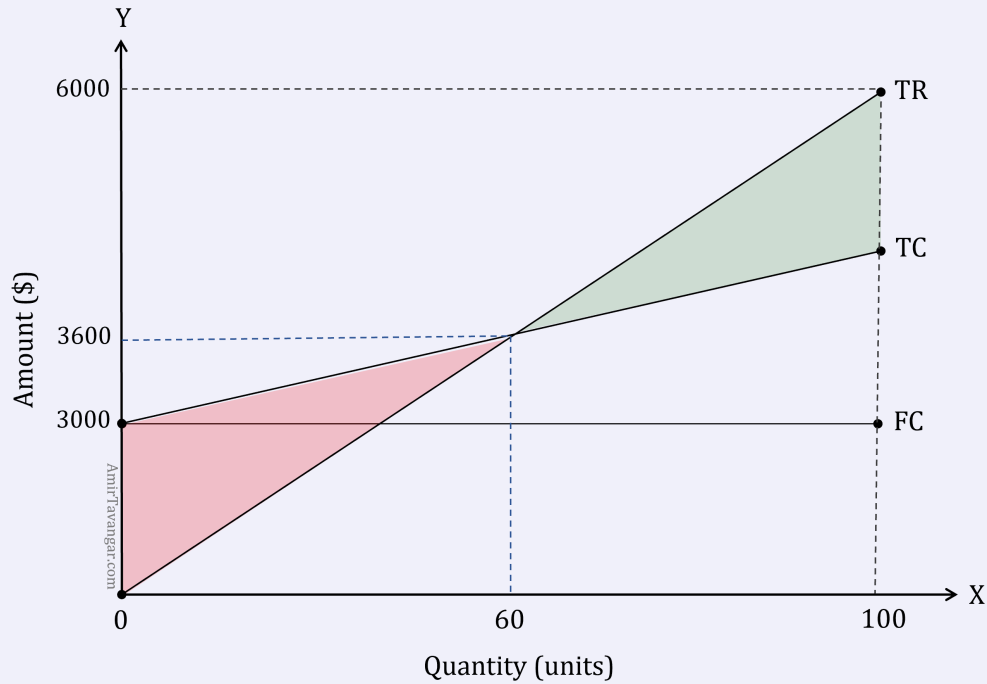
- **Break-even Point:** The point where the Total Revenue Line intersects the Total Cost Line. At this point, the business neither makes a profit nor incurs a loss. The X-coordinate of this point gives the number of units that need to be sold to break even (X_B), while the Y-coordinate gives the corresponding dollar amount of sales or revenue (TR_B).
- **Profit Area:** The space between the Total Revenue Line and the Total Cost Line above the break-even point; when the number of items produced exceeds the break-even volume, the total revenue surpasses total costs, leading to a profit.
- **Loss Area:** The space between the Total Revenue Line and the Total Cost Line below the break-even point; when the number of items produced is below the break-even volume, the total revenue falls short of total costs, causing a loss.

A break-even chart provides a visual tool for businesses to quickly assess the impact of changes in costs, prices, and other variables on their profitability. It's particularly useful for understanding how variations in sales volume can affect profit or loss situations.

Example 6.4.2: CVP Analysis Given the Break-Even Chart

A break-even chart of a certain product is given below. Use the chart to answer the following questions.

- What is the break-even volume?
- What is the revenue at break-even?
- How much are the fixed costs?
- What is the unit selling price?
- What is the unit variable cost?
- What is the amount of the profit or loss if 80 units are produced and sold?



Show/Hide Solution

- a)** The break-even point is the intersection of the total revenue (TR) and the total costs (TC) lines. This point has the coordinates of (60,3600). The X-coordinate of the break-even point is the volume at break-even. So $X_B = 60$ units.
- b)** The Y-coordinate of the break-even point is the revenue or sales at break-even. So $TR_B = \$3600$.
- c)** The fixed costs amount is the y-intercept of the total costs or the fixed costs line, which is 3000. So $FC = \$3000$.
- d)** Given that the total revenue (TR) line is represented by the equation $TR = SP \cdot X$, and knowing that the break-even point (60,3600) lies on this TR line, we can plug in the coordinates of the break-even point into the TR equation and solve for unit selling price (SP).

$$TR = SP \cdot X$$

$$3600 = SP(60)$$

$$SP = \frac{3600}{60}$$

$$SP = \$60$$

e) Given that the total costs (TC) line is represented by the equation $TC = VC \cdot X + FC$, and knowing that the break-even point (60,3600) also lies on the TC line, we can use the fixed costs amount (FC = \$3000) and the coordinates of the break-even point to solve for the unit variable cost (VC).

$$TC = VC \cdot X + FC$$

$$3600 = VC(60) + 3000$$

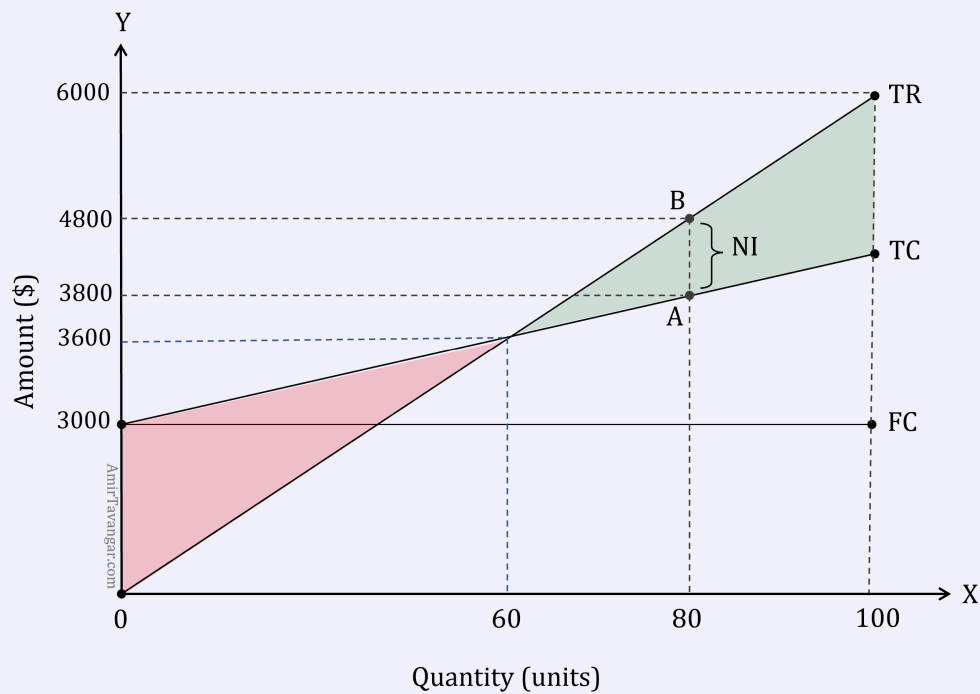
$$3600 - 3000 = 60VC$$

$$600 = 60VC$$

$$VC = \$10$$

f)

Graphically: To determine the net income when $X = 80$ units, draw a vertical line at the 80-unit mark. Where this line intersects the TC (Point A) and TR (Point B) lines will give us the corresponding revenues and costs. The Y-value at the intersection with the TR line shows the total revenue is \$4800 for 80 units. The Y-value at the intersection with the TC line indicates the total cost is \$3800 for 80 units. The net income is the difference between these two values.



$$\begin{aligned}
 NI &= TR - TC \\
 &= 4800 - 3800 \\
 &= \$1000 \text{ Profit}
 \end{aligned}$$

Algebraically: Given we have the values for unit selling price ($SP = \$60$), variable cost ($VC = \10), fixed costs ($FC = \$3000$), and quantity ($X = 80$), we can use Formula 6.6a to find the net income (NI):

$$\begin{aligned}
 NI &= SP \cdot X - (VC \cdot X + FC) \\
 &= 60(80) - (10(80) + 3000) \\
 &= 4800 - (800 + 3000) \\
 &= \$1000 \text{ Profit}
 \end{aligned}$$

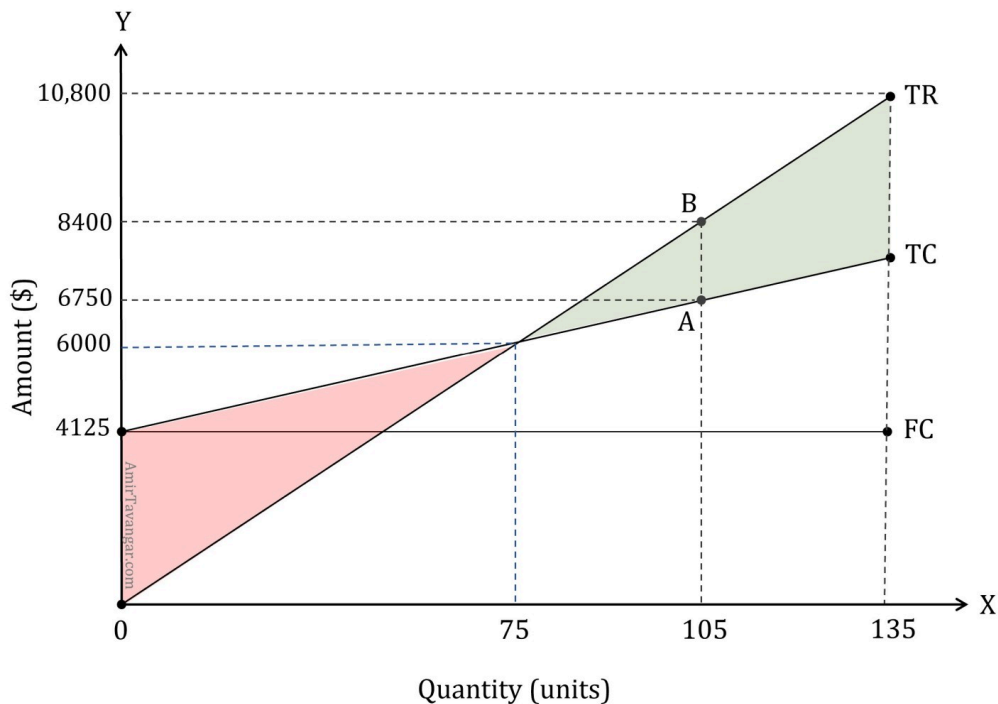
Try an Example



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Section 6.4 Exercises

1. break-even chart of a certain product is given below. Use the chart to answer the following questions. **a)** What is the break-even volume? **b)** What is the revenue at break-even? **c)** How much are the fixed costs? **d)** What is the unit selling price? **e)** What is the unit variable cost? **f)** What is the net income if 105 units are produced and



sold?

Show/Hide Answer

a) $X = 75$ Units

b) $TR = \$6000$

c) $FC = \$4125$

d) $SP = \$80$

e) $VC = \$25$

f) $NI = \$1650$

6.5 CVP ANALYSIS USING A FINANCIAL CALCULATOR

In this section, we'll explore how to use the Texas Instruments BA II Plus financial calculator for conducting break-even and Cost-Volume-Profit (CVP) analyses. We will specifically look at the break-even worksheet, a useful feature for these types of analyses.

The break-even worksheet on the BA II Plus, shown in Figure 6.5.1, can be accessed by using the secondary function of the “6” key. To do this, press the “2ND” key followed by “6”. This worksheet includes five variables: Fixed Costs, Variable Costs per Unit, Selling Price, Net Income, and Quantity.

When using the calculator's worksheet:

- **FC** represents the fixed costs.
- **VC** represents the variable costs per unit.
- **P** represents the selling price per unit in the calculator, which differs from the algebraic SP notation.
- **PFT** stands for profit and denotes the net income in the calculator, which contrasts with the algebraic NI for net income.
- **Q** represents quantity, volume, or the number of units in the calculator, instead of the algebraic x notation.

Understanding how these variables align with the algebraic approach and how to input them into the calculator is crucial for accurate break-even and CVP analysis.

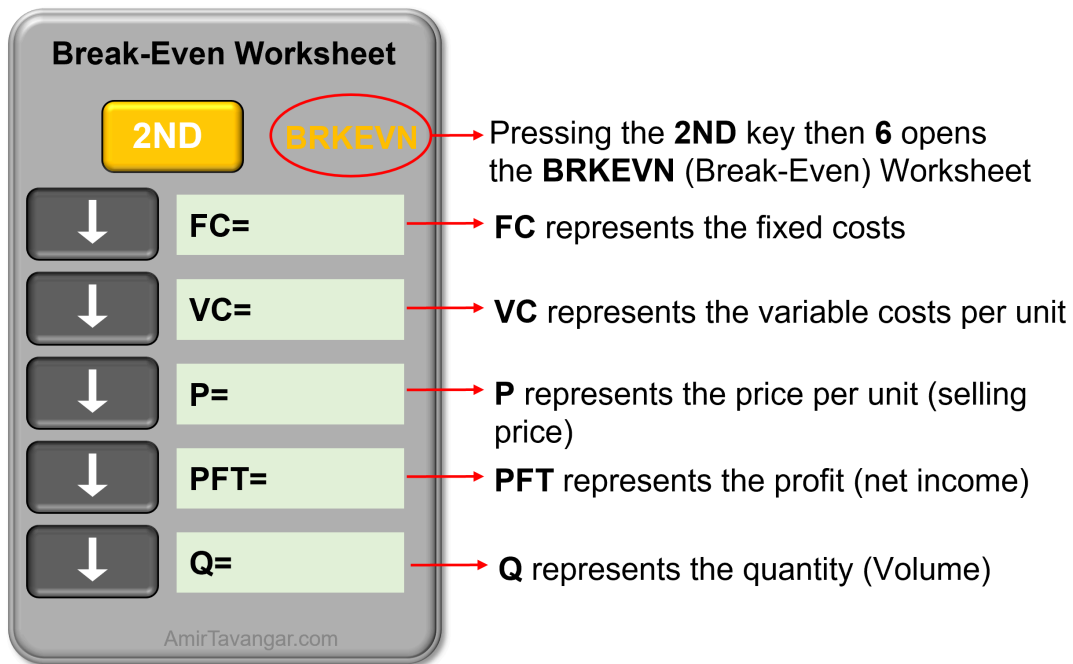


Figure 6.5.1: The Break-Even Worksheet and its variables

How to use the Break-Even Worksheet

1. Activate the Break-Even Worksheet by pressing the “2ND” function key followed by “6”.
2. Clear the worksheet (reset all variables to zero) by pressing the “2ND” function key followed by the “CE|C” key.
3. Press the up and down arrows to access the different variables.
4. Input the known values to their corresponding variables and then press “ENTER”.
5. Navigate to the variable you want to compute and press the “CPT” (compute) button to get the result.

Example 6.5.1: Compute Break-even Point with Known Unit Selling Price and Variable Cost

A company sells webcams for \$85 each. Its fixed costs are \$18,500 per month, and the variable costs per unit are \$35. **a)** Calculate the number of webcams the company needs to sell each month to break even. **b)** Calculate the break-even in dollars.

Watch Video



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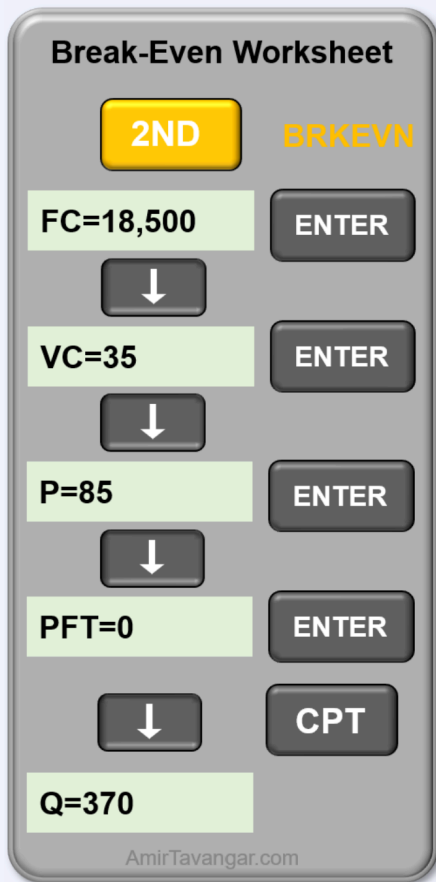
Show/Hide Solution

The selling price is \$85, so $SP = \$85$

The variable costs per webcam are \$35, so $VC = \$35$

The fixed costs are \$18,500, so $FC = \$18,500$

a) To find the quantity at break-even, we input the given values in the break-even worksheet to their corresponding variables and then compute Q (quantity). Note that at breakeven, net income is zero, so we enter zero for PFT:



Therefore, the company needs to sell 370 webcams each month to break even.

b) Break-even in dollars is the total revenue at break-even quantity ($X_B = 370$).

$$TR = SP \cdot X$$

$$TR = 85 \cdot X$$

$$TR_B = 85(370)$$

$$= \$31,450$$

Example 6.5.2: Compute Break-even Point with Known Unit Selling Price and Variable Cost

Vida runs a business that makes custom-printed shirts. She must make 84 T-shirts each month to break even. It will cost her \$5 to purchase and \$2 to print each T-shirt. She will have to pay a rent of \$1,596 per month for her workshop. **a)** What is the selling price per T-shirt to break even? **b)** What is the total revenue needed to break even? **C)** In February, she sold 60 T-shirts. Did she make a profit or incur a loss and by how much?

Show/Hide Solution

The number of T-shirts at the break-even point is 84, so quantity $Q = 84$

The variable cost per T-shirt is the sum of the cost of purchasing and printing, so $VC = \$5 + \$2 = \$7$

The only fixed cost is the workshop rent, so $FC = \$1596$

a) To find the selling price (P) at break-even, we input the given values in the break-even worksheet to their corresponding variables and then compute P. Note that at break-even, net income is zero, so we enter zero for PFT:

Break-Even Worksheet

2ND BRKEVN

FC=1596 ENTER

↓

VC=7 ENTER

↓

PFT=0 ENTER

↓

Q=84 ENTER

↓

P=26 CPT

AmirTavangar.com

Therefore, Margaret needs to sell each T-shirt at \$26 to break even.

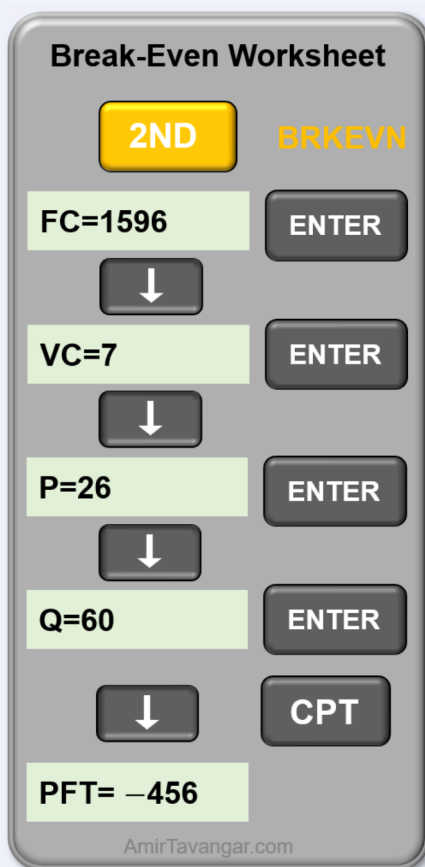
b) Break-even in dollars is the total revenue at break-even quantity ($X_B = 84$)

$$TR = 26 \cdot X$$

$$TR_B = 26(84)$$

$$= \$2184$$

c) Since the quantity of 60 is less than the quantity at break-even (84), she incurred a loss in February. To determine the extent of this loss, we input 60 for the quantity (Q) and calculate the net income (PFT).



The profit is calculated to be — \$456, where the negative sign indicates a loss, and the amount represents the extent of that loss.

Try an Example



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Section 6.5 Exercises

1. On March 9, 2020, Jack borrowed \$62,200 from a bank at 8.62% p.a. If he repaid the amount on July 20, 2020, calculate the amount of interest charged on the loan.

Show/Hide Answer

$$I = \$1953.69$$

2. Kyrie has to pay \$290.36 as an interest payment on a loan he has taken for a period of 6 months. If the interest rate charged on the loan is 7.6% p.a., what is the principal amount of the loan?

Show/Hide Answer

$$P = \$7641.05$$

3. Luke received a loan of \$18,540 from his friend. If he was charged the interest amount of \$994 at the end of 15 months, calculate the annual rate of simple interest on the loan.

Show/Hide Answer

$$r = 4.29\% \text{ p.a.}$$

6.6 TERMINOLOGY AND GLOSSARY OF SYMBOLS

Terminology

- **Total Revenue (TR):** This is the income earned by a company from its primary business activities, typically through the sale of goods and services to customers.
- **Net Income (NI):** This represents the profit a company makes after subtracting all its expenses from its total revenue over a specific period. Net income is calculated as Total Revenue minus Total Expenses.
- **Total Costs (TC):** These are the aggregate expenses a business incurs, comprising both variable and fixed costs, during its operation.
- **Total Variable Costs (TVC):** Variable costs are expenses that fluctuate with the level of output or sales volume. They include costs directly associated with production and sales, such as materials, labor, and sales commissions.
- **Fixed Costs (FC):** These costs remain constant regardless of the company's output or sales volume within a certain period. Examples include rent, salaries for management, and property taxes.
- **Contribution Margin (CM) per Unit:** This metric indicates the amount by which the selling price of a product exceeds its variable costs per unit. It is a crucial figure for determining how much revenue contributes towards covering fixed costs and generating profit.
- **Break-Even Point (BP):** The break-even point is achieved when a company's total revenues exactly match its total costs. At this point, the company neither makes a profit nor incurs a loss.
- **Capacity:** This term refers to the maximum output—either products or services—a business can produce in a given period without compromising quality or efficiency.

Glossary of Symbols

Per unit

- **SP:** Selling Price
- **VC:** Variable costs per unit
- **CM:** Contribution margin per unit
- **X:** Volume (the number of units sold)

Totals

- **TR:** Total Revenue
- **TC:** Total Costs
- **TVC:** Total Variable Costs
- **FC:** Fixed Costs
- **TCM:** Total Contribution Margin
- **NI:** Net Income

Others

- **BP:** Breakeven Point
- **CVP Analysis:** Cost-Volume-Profit Analysis

PART VII

MATHEMATICS OF MERCHANDISING

Chapter Outline

This chapter provides a comprehensive exploration of the mathematical concepts in merchandising operations. From trade and cash discounts to markup and markdown strategies, readers will gain insights into essential calculations that drive profitability and pricing decisions in retail and wholesale environments.

7.1 Trade Discounts: This section introduces the concept of trade discounts, which are reductions from the list price granted by sellers to buyers. The calculation methods for single trade discounts and discount series are discussed.

7.2 Cash Discounts: This part delves into cash discounts, focusing on the incentives offered for early payment. It explores various terms of cash discounts, including ordinary dating, end-of-month dating, and receipt of goods dating.

7.3 Markup: This section examines the process of applying a percentage increase to the cost of goods to determine the selling price. It covers the rationale behind markup strategies, how to calculate markups and their impact on pricing and profitability.

7.4 Markdown: This part addresses the reduction of prices from their original selling price. The section discusses how to calculate markdowns and sale prices.

7.5 Integrated Problem: This section presents comprehensive problems that integrate the concepts of trade discounts, cash discounts, markup, and markdown. It aims to apply the mathematical principles discussed throughout the chapter to a real-world scenario, enhancing the reader's understanding and application of merchandising mathematics.

7.6 Terminology and Glossary of Symbols: Concluding the chapter, this section provides definitions and explanations of key terms and symbols used throughout the discussion.

7.1 TRADE DISCOUNTS

A. Introduction to Mathematics of Merchandising

Imagine walking into a shopping mall during the holiday season. Every store displays an array of products, each tagged with a price, and every discount sign represents a carefully calculated decision. This is where the unseen world of merchandising mathematics plays a pivotal role.

In this chapter, we will explore the fundamental mathematical concepts essential to successful merchandising. We start by understanding the supply chain and the relationships between its members. Moving forward, we will delve into the calculations of trade discounts, both single and series, and understand the significance of cash discounts in financial transactions.

The art of pricing, through markup and markdown strategies, will be our next focus, uncovering how businesses balance profitability with market competitiveness. We will then study integrated real-world problems, providing a practical perspective on how mathematics is applied in various merchandising scenarios.

Merchandising is the strategic promotion and sale of products to consumers. It involves a range of activities including product selection, pricing, display, and advertising. The primary goal of merchandising is to attract customers and motivate purchases, thus driving sales and profitability for businesses. It is a key component in retail, but it also extends to various other sectors where goods are sold, be it physical products or online services.

The **supply chain** encompasses the entire process of producing and delivering a product or service, from the sourcing of raw materials to the delivery of the final product to the consumer. The supply chain is concerned with the flow of materials, information, and finances as they move from manufacturer to wholesaler to retailer to consumer.

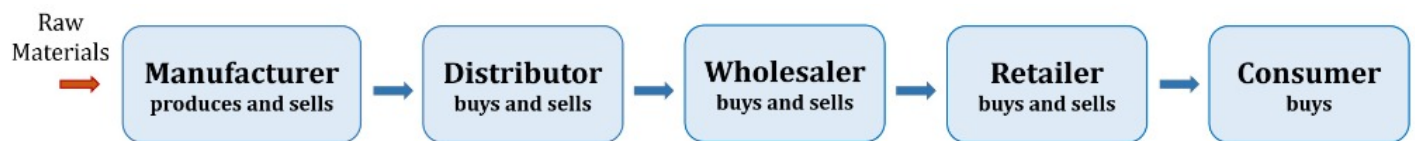


Figure 7.1.1 Members of a typical supply chain

As shown in Figure 7.1.1, the typical members of a supply chain include:

- **Manufacturers/Producers:** They create the product, sourcing raw materials and turning them into finished goods.

- Distributors/Wholesalers: They buy in bulk from manufacturers and sell in smaller quantities to retailers or sometimes directly to consumers.
- Retailers: They sell products to the end consumer, either from physical stores or online platforms.
- Consumers: The end-users who purchase and use the products.

Some organizations may bypass one or more stages of the traditional supply chain. For instance, a manufacturer might also act as a retailer, selling directly to consumers, Apple is a notable example of this. Similarly, a wholesaler can sometimes also be a retailer, as is the case with Costco.

Each member of the supply chain adds costs to the product, from production, handling, and transportation to storage and marketing. As a product moves through the supply chain, these costs accumulate, and each member, except the consumer, adds a markup to cover their costs and earn a profit. The cumulative effect of these markups contributes to the final price paid by the consumer.

B. Trade Discounts

Manufacturers usually recommend the price a product should be sold to the consumer. This price is referred to as the Manufacturer's Suggested Retail Price (**MSRP**), catalog price, or **list price**. It represents the intended final sale price before any discounts are applied. When manufacturers set the MSRP, they often take into account the potential discounts they will offer to supply chain members. When a manufacturer sells a product to the members of the supply chain, they deduct a certain amount from their list price, usually a percentage of the list price. This is known as a **trade discount**. They are often used in business-to-business transactions such as the name 'trade' discount. Trade discounts can serve other functions:

- Volume Incentives: Larger orders are often incentivized with bigger discounts, encouraging bulk purchases and ensuring more significant sales for manufacturers.
- Promoting New Products: Discounts can be used to encourage retailers to stock and promote new or less popular products.
- Maintaining Relationships: Offering discounts helps in building and maintaining long-term business relationships between various supply chain members.
- Competitive Strategy: In highly competitive markets, manufacturers might offer trade discounts to make their products more attractive to retailers compared to rival products.

Each member of the supply chain offers successive buyer trade discounts. The final price that a buyer pays for a product after all trade discounts have been applied is called the **net price**. If only one trade discount is offered to a buyer, it is called a **single trade discount**. If a buyer is qualified for more than one trade discount, the discount is referred to as a **trade discount series**.

C. Single Trade Discount

A single trade discount involves only one discount percentage being applied to the list price of a product. The amount of trade discount, denoted as 'A', is calculated by multiplying the discount rate, represented by 'd', by the list price, denoted as 'L'.

$$A = L \cdot d \quad \text{Formula 7.1a}$$

The net price, denoted as 'N', is determined by subtracting the discount amount from the list price.

$$N = L - A \quad \text{Formula 7.2}$$

By inserting the discount amount from Formula 7.1a into Formula 7.2, we establish a relationship between the net price and the list price, expressed as

$$\begin{aligned} N &= L - L \cdot d \\ N &= L(1 - d) \end{aligned} \quad \text{Formula 7.3a}$$

The term $(1 - d)$, known as the net price factor (NPF), yields the net price when multiplied by the list price, similar to how the discount rate determines the discount amount. The discount rate and the net price factor are complementary to each other, and they are typically expressed as percentages. Formula 7.3a can also be expressed in terms of the net price factor.

$$N = L \cdot NPF$$

where $NPF = 1 - d$.

Example 7.1.1: Calculate the Amount of Single Trade Discount and Net Price

A bicycle has a list price of \$800.00 and qualifies for a trade discount of 20%. **a)** Determine the amount of trade discount. **b)** Calculate the net price of the bicycle after applying the trade discount.

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Show/Hide Solution

- The list price is \$800: $L = 800$
- Trade discount rate: $d = 20\% = 0.2$

a) $A = ?$

Substituting the given values into Formula 7.1a, we obtain

$$\begin{aligned} A &= L \cdot d \\ A &= 800(20\%) \\ &= \$160 \end{aligned}$$

b) $N = ?$

Since the amount of discount is now known, we can use Formula 7.2 to find the net price.

$$\begin{aligned} N &= L - A \\ N &= 800 - 160 \\ &= \$640 \end{aligned}$$

Alternatively, we can apply Formula 7.3a to calculate the net price directly.

$$\begin{aligned} N &= L(1 - d) \\ N &= 800(1 - 0.2) \end{aligned}$$

$$= 800(0.8)$$

$$= \$640$$

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Sometimes, we may know the discount amount and need to find the original list price or the discount rate. In these instances, we can adjust Formula 7.1a to isolate and solve for the unknown variable.

- If looking for the list price: $L = \frac{A}{d}$ Formula 7.1b
- If looking for the trade discount rate: $d = \frac{A}{L}$ Formula 7.1c

When we know the net price and need to find the original list price or the discount rate, we can rearrange Formula 7.3a to isolate and solve for the unknown variable.

- If looking for the list price: $L = \frac{N}{1 - d}$ Formula 7.3b
- If looking for the trade discount rate: $d = 1 - \frac{N}{L}$ Formula 7.3c

Example 7.1.2: Calculate the List Price Given the Amount of Single Trade Discount

A retailer is offered a 12% trade discount on a product, and this discount amounts to \$30. Calculate the list price of the product.

Show/Hide Solution

- Trade discount rate: $d = 12\% = 0.12$
- The discount amount: $A = \$30$

$L = ?$

Since the list price is asked for given the amount of discount, we apply Formula 7.1b.

$$L = \frac{A}{d}$$

$$L = \frac{30}{0.12}$$

$$= \$250$$

Try an Example



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Example 7.1.3: Calculate the List Price Given the Net Price

If an item has a net price of \$174 after a 25% trade discount is applied, **a)** what was the original list price, and **b)** how much was the trade discount amount?

Show/Hide Solution

- Trade discount rate: $d = 25\% = 0.25$
- The net price: $N = \$174$

a) $L = ?$

Since the list price is asked for given the net price, we apply Formula 7.3b.

$$\begin{aligned} L &= \frac{N}{1 - d} \\ L &= \frac{174}{1 - 0.25} \\ &= \frac{174}{0.75} \\ &= \$232 \end{aligned}$$

b) $A = ?$

We can apply either Formula 7.1a or 7.2. Let us use Formula 7.2:

$$\begin{aligned} N &= L - A \\ 174 &= 232 - A \\ A &= 232 - 174 \\ &= \$58 \end{aligned}$$

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Example 7.1.4: Calculate the Trade Discount Rate Given the List price and Net Price

The list price of an item is \$200 and the net price after applying a trade discount is \$150. **a)** What is the net price factor? **b)** What is the trade discount rate?

Show/Hide Solution

- List price: $L = \$200$
- Net price: $N = \$150$

a) $NPF = ?$

The relation between the list price and the net price is given by

$$N = L \cdot NPF$$

We can rearrange the formula for NPF .

$$NPF = \frac{N}{L}$$

$$NPF = \frac{150}{200}$$

$$= 0.75$$

$$= 75\%$$

b) $d = ?$

We can find the discount rate as

$$\begin{aligned} d &= 1 - NPF \\ &= 1 - 0.75 \\ &= 0.25 \\ &= 25\% \end{aligned}$$

Alternatively, since the list price and the net price are given, we can apply Formula 7.3c to determine the discount rate.

$$\begin{aligned} d &= 1 - \frac{N}{L} \\ d &= 1 - \frac{150}{200} \\ &= 1 - 0.75 \\ &= 0.25 \\ &= 25\% \end{aligned}$$

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D. Trade Discount Series

As previously mentioned, there are various reasons why trade discounts are offered. Occasionally, a buyer might qualify for multiple discounts, such as one for placing a large order and another for a marketing incentive. When a buyer is offered multiple discount rates, this set of discounts is known as a **trade discount series**. In such cases, when calculating the net price it is important to apply each discount rate in the series sequentially, rather than combining them into a single rate. For instance, if an item's list price is \$100 and the trade discount series offered is 10%, 20%, and 30%, the calculation should proceed in stages:

- Net price after the first discount rate = $100(1 - 10\%) = \$90$
- Net price after the second discount rate = $90(1 - 20\%) = \$72$
- Net price after the third discount rate = $72(1 - 30\%) = \$50.40$



Figure 7.1.2: Example of sequential application of discount series

To simplify the calculation of the final net price with multiple trade discounts, we can extend the Net Price Factor (NPF) method to a discount series. This approach consolidates all the calculations into one step. Here, the NPF for each discount rate is first multiplied together, and then this cumulative NPF is applied to the list price. This method is more efficient than applying each discount rate one at a time. For instance:

With a single discount rate, the net price formula is: $N = L(1 - d)$

For two discount rates, it becomes: $N = L(1 - d_1)(1 - d_2)$

For three discount rates, the formula is: $N = L(1 - d_1)(1 - d_2)(1 - d_3)$

In general, if there are ' n ' discount rates, the net price is calculated using the formula:

$$N = L(1 - d_1)(1 - d_2)\dots(1 - d_n)$$

Formula 7.4

Where the product of the individual net price factor is the overall net price factor:

$$NPF = \underbrace{(1 - d_1)}_{NPF_1} \underbrace{(1 - d_2)}_{NPF_2} \dots \underbrace{(1 - d_n)}_{NPF_n}$$



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Example 7.1.5: Calculate the List Price Given Trade Discount Series

Determine the net price of a product with a list price of \$1400, considering it is subject to a trade discount series of 25%, 10%, and 5%.

Show/Hide Solution

- List price: $L = \$1400$
- Trade discount rate 1: $d_1 = 25\% = 0.25$
- Trade discount rate 2: $d_2 = 10\% = 0.1$
- Trade discount rate 3: $d_3 = 5\% = 0.05$

$N = ?$

Substituting the given values into Formula 7.4 gives

$$\begin{aligned} N &= L(1 - d_1)(1 - d_2)(1 - d_3) \\ N &= 1400(1 - 0.25)(1 - 0.1)(1 - 0.05) \\ &= 1400(0.75)(0.90)(0.95) \\ &= \$897.75 \end{aligned}$$

Notice that the overall net price factor here is

$$\begin{aligned}
 NPF &= (1 - d_1)(1 - d_2)(1 - d_3) \\
 &= (0.75)(0.90)(0.95) \\
 &= 0.64125
 \end{aligned}$$

Try an Example



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E. Single Equivalent Trade Discount for a Discount Series

When dealing with multiple discounts, it can be complex and time-consuming to understand the total impact. A single equivalent rate simplifies this by providing a clear, concise figure that represents the overall effect of all the discounts combined. This makes it easier to compare different discount strategies or options. The equivalent rate of discount for a discount series is a single percentage that represents the cumulative effect of multiple trade discounts applied in sequence. It is a way to summarize the overall impact of a discount series as if it were a single discount rate.

For a single trade discount, the discount rate d is calculated as $d = 1 - NPF$. Similarly, for a series of discounts, the single equivalent discount rate d_{eq} is determined as one minus the overall net price factor. Therefore, you can obtain d_{eq} by using the formula

$$\begin{aligned}
 d_{eq} &= 1 - NPF \\
 d_{eq} &= 1 - [(1 - d_1)(1 - d_2)\dots(1 - d_n)]
 \end{aligned}
 \tag{Formula 7.5}$$

This single equivalent discount rate, when applied to the same list price, results in the same net price as that achieved through the series of discount rates.



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Example 7.1.6: Calculate the List Price and Equivalent Discount Rate for Trade Discount Series

A manufacturer offers a TV for sale at \$1080 with a trade discount series of 30%, 15%, and 20%. Calculate the following: **a)** The Net Price after applying all discounts, **b)** The Total Amount of Discount given, and **c)** The Single Equivalent Rate of Discount for the entire series.

Show/Hide Solution

- List price: $L = \$1080$
- Trade discount rate 1: $d_1 = 30\% = 0.30$
- Trade discount rate 2: $d_2 = 15\% = 0.15$
- Trade discount rate 3: $d_3 = 20\% = 0.20$

a) $N = ?$

Substituting the given values into Formula 7.4 gives

$$\begin{aligned}
 N &= L(1 - d_1)(1 - d_2)(1 - d_3) \\
 N &= 1080(1 - 0.30)(1 - 0.15)(1 - 0.20) \\
 &= 1080(0.70)(0.85)(0.80) \\
 &= \underbrace{1080(0.476)}_{NPF} \\
 &= \$514.08
 \end{aligned}$$

$$\text{b) } A_{\text{total}} = ?$$

To find the total amount of discount, we can apply Formula 7.2.

$$\begin{aligned} N &= L - A_{\text{total}} \\ 514.08 &= 1080 - A_{\text{total}} \\ A_{\text{total}} &= 1080 - 514.08 \\ &= \$565.92 \end{aligned}$$

$$\text{c) } d_{eq} = ?$$

Approach 1:

The single equivalent discount rate is calculated by Formula 7.5.

$$\begin{aligned} d_{eq} &= 1 - [(1 - 0.30)(1 - 0.15)(1 - 0.20)] \\ d_{eq} &= 1 - \underbrace{0.476}_{NPF} \\ &= 0.524 \\ &= 52.4\% \end{aligned}$$

Approach 2:

The single equivalent discount rate can also be calculated using Formula 7.1c given the total amount of discount is known.

$$\begin{aligned} d_{eq} &= \frac{A_{\text{total}}}{L} \\ d_{eq} &= \frac{565.92}{1080} \\ &= 0.524 \\ &= 52.4\% \end{aligned}$$

Try an Example



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Section 7.1 Exercises

1. A laptop is listed for \$380 and is subject to a trade discount of 19%. **a)** Calculate the amount of trade discount. **b)** Calculate the net price.

Show/Hide Answer

a) $A = \$72.20$

b) $N = \$307.80$

2. After a 28% trade discount, the net price of an item is \$455. **a)** Calculate the list price. **b)** How much is the amount of discount?

Show/Hide Answer

a) $L = \$631.94$

b) $A = \$176.94$

3. The list price of an item is \$314.92 and the net price after applying a trade discount is \$285. **a)** What is the net price factor? **b)** What is the trade discount rate? Give your answers in percent rounded to two decimal places.

Show/Hide Answer

a) $\text{NPF} = 90.5\%$

b) $d = 9.5\%$

4. A wholesaler sells a phone for \$900 less trade discount series of 20%, 14.4%, 5.7%. **a)** Find the net price. **b)** Find the amount of discount. **c)** Determine the single equivalent rate of discount. Round your answers to two decimal places if required.

Show/Hide Answer

a) $N = \$581.19$

b) $A = 318.81$

c) $d = 35.42\%$

5. BestStuff offers an item for \$560 with three trade discounts of 29%, 16%, and 4.7%. QualStuff offers the same model for \$627.20 with two trade discounts of 27.5% and 23%. **a)** Which offer is cheaper? **b)** and by how much?

Show/Hide Answer

a) BestStuff offer is cheaper

b) by \$31.84

7.2 CASH DISCOUNTS

Payment Terms and Cash Discount

When businesses, including those within the supply chain, sell goods and services, they often do so on credit. This practice allows buyers to receive goods immediately but pay for them later. The total amount due for these purchases is indicated on the invoice and typically has a set credit period before payment is required, commonly 30 or 60 days after the invoice date.

To incentivize quicker payment, sellers may offer a cash discount. This discount is a reduction of the invoice's net amount and is only available within a specified time frame, shorter than the full credit period. The discount is calculated as a percentage of the invoice's net amount. For example, terms such as "2/10, net 30" imply a 2% discount is available if payment is made within 10 days; otherwise, the full net amount is payable within 30 days.

If the buyer does not take advantage of the cash discount by paying within the allotted time frame, the full net amount becomes due at the end of the credit period. This system of cash discounts benefits both the seller, by improving cash flow, and the buyer, by reducing the cost of their purchases.

Figure 7.2.1 illustrates a sample invoice with a description of the payment terms. In the invoice, the payment term is indicated as "3/10, n/30" implying a 2% discount is available if payment is made within 10 days from the invoice date of 4/2/2020; otherwise, the full net amount is payable within 30 days.

| Item # | Description | Qty | Unit List Price | Trade Discount | Net Amount |
|------------------|-------------|-----|-----------------|----------------|--------------------|
| Z4568 | Item 1 | 40 | \$ 40.00 | 5% | \$ 1,520.00 |
| Z4569 | Item 2 | 30 | \$ 56.00 | 15%, 5% | \$ 1,356.60 |
| Z4570 | Item 3 | 40 | \$ 32.00 | 10%, 5% | \$ 1,094.40 |
| Z4571 | Item 4 | 10 | \$ 10.00 | 8% | \$ 92.00 |
| Invoice Subtotal | | | | | \$ 4,063.00 |
| HST (13%) | | | | | \$ 528.19 |
| Other | | | | | \$ 0.00 |
| Deposit Received | | | | | \$ 0.00 |
| TOTAL | | | | | \$ 4,591.19 |

| | |
|----------------------|-------------------|
| INVOICE # | 3-456-2 |
| Invoice Date: | 4/2/2020 |
| Payment Terms | 3/10, n/30 |

Figure 7.2.1 A sample invoice with a description of the payment terms

There are several types of cash discounts commonly used in business transactions.

1. Ordinary Dating:

- This is the most common type of cash discount, where the discount and credit terms are calculated from the date of the invoice. For example, “2/10, net 30” means the buyer can take a 2% discount if the invoice is paid within 10 days from the invoice date; otherwise, the full amount is due in 30 days.

2. End of Month (EOM) Dating:

- The discount and credit periods start from the end of the month in which the invoice is issued. If the terms are “2/10 EOM,” it means the buyer has until the 10th day of the following month to pay the invoice and receive a 2% discount.

3. Receipt of Goods (ROG) Dating:

- The discount period begins from the day the buyer receives the goods. This is used when delivery times are uncertain. Terms might be “2/10 ROG,” allowing a 2% discount if payment is made within 10 days of receiving the goods.

How to Apply the Cash Discounts

1. Identify the Dating Method:

- Ordinary dating assumes terms are calculated from the invoice date unless specified as EOM (End of Month) or ROG (Receipt of Goods).
- Terms without EOM or ROG indicate ordinary dating.

2. Interpret the Payment Terms:

- Ordinary Dating: For “2/10, n/30,” a 2% discount applies if paid within 10 days from the invoice date.
- End-of-the-Month (EOM): “2/10 EOM” indicates a 2% discount if payment is within 10 days after the invoice month’s end.
- Receipt of Goods (ROG): “2/10 ROG” allows a 2% discount if payment is within 10 days of goods receipt.

3. Determine the Discount Period:

- Ordinary Dating: Add the discount period (e.g., 10 days) to the invoice date.
- EOM: The discount period starts from the first day of the month following the invoice.
- ROG: The discount period starts from the goods receipt date.

The Date Worksheet on financial calculators can be used to determine the discount period.

For staggered discounts like “5/10, 3/20, n/60,” a 5% discount is available within 10 days, 3% within 20 days, and the full amount is due within 60 days from the invoice date.

4. Calculate the Discount: If paying within the discount period, apply the discount rate to the invoice amount. You can

- Subtract the calculated discount from the invoice amount for the net price.
- Alternatively, use the net price factor (NPF), where the invoice amount is multiplied by the NPF (1 minus the discount rate).

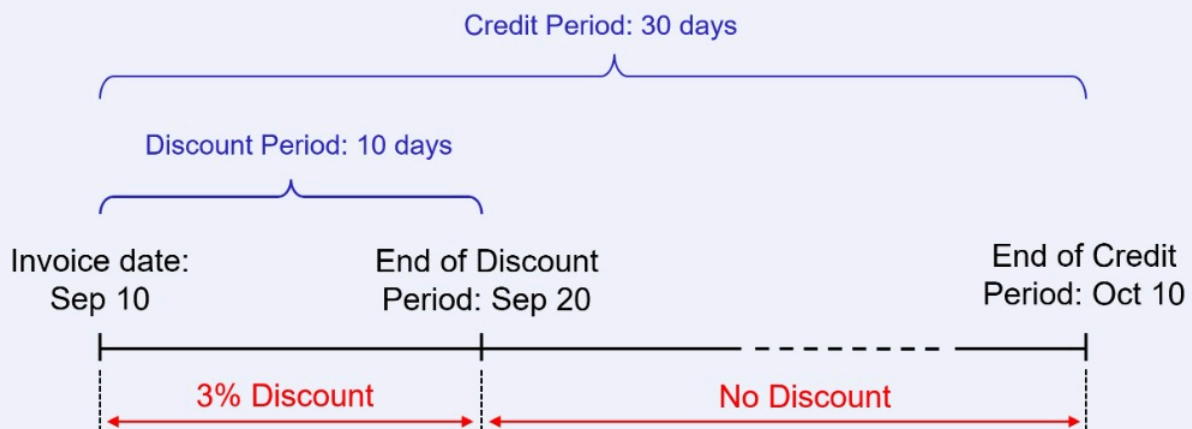
If payment is made after the discount window, the full invoice amount is payable. It is important to adjust the discount period based on the specific dating method for accurate calculations.

Example 7.2.1: Ordinary Dating with a Single Cash Discount Rate

K2 Distributors has received an invoice of \$10,000 dated September 10, terms 3/10, n/30. **a)** What is the last day for taking the cash discount? **b)** How much is to be paid if the discount is taken?

Show/Hide Solution

a) Given that it’s an ordinary dating term of “3/10“, we calculate the final day to take the cash discount by adding 10 days to the invoice date, excluding the day the invoice is issued. This places the deadline for the cash discount on September 20th. The credit term “n/30” indicates that the full payment is due 30 days following the invoice date, which would be October 10th.



b) Given the term “3/10”, the cash discount rate is 3%. To find the net price, we can apply the net price factor approach:

$$\begin{aligned}
 \text{Net amount} &= 10,000(1 - 3\%) \\
 &= 10,000(0.97) \\
 &= \$9700
 \end{aligned}$$

Therefore, \$9700 should be paid if the payment is made within the cash discount period.

Try an Example



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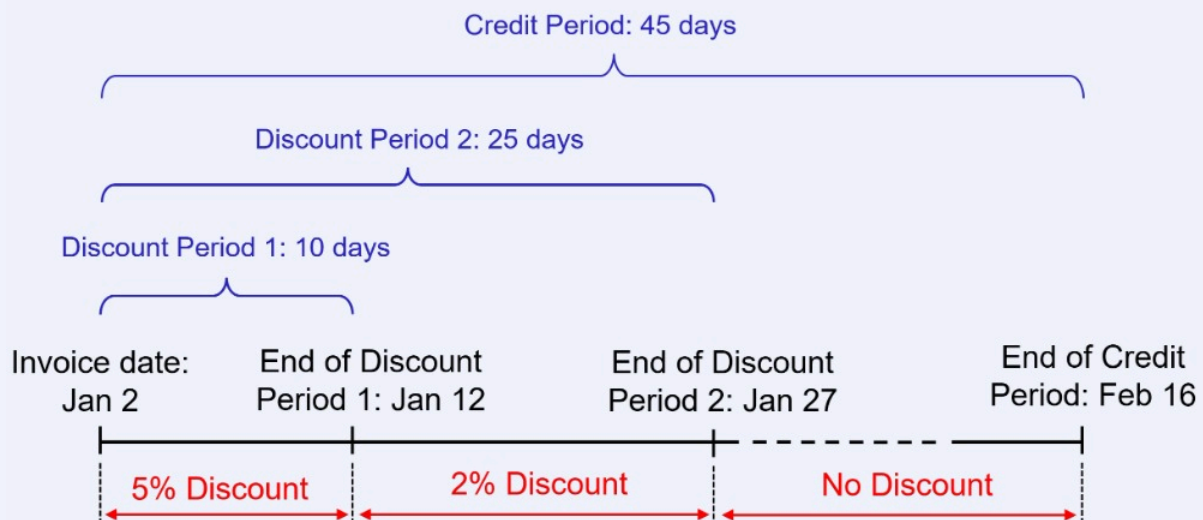
Example 7.2.2: Ordinary Dating with Staggered Cash Discount Rate

A retailer received an invoice for \$3000 issued on January 2, 2023, with the following payment terms: 5/10, 2/25, n/45. What will settle the invoice if it is paid on **a)** January 10th, **b)** January 25th, or **c)** January 30th.

Show/Hide Solution

The terms “5/10, 2/25, n/45” mean:

- A 5% discount is available if the invoice is paid within 10 days (by January 12th).
- A 2% discount is available if the invoice is paid within 25 days (by January 27th).
- The full amount is due without any discount if the invoice is paid within 45 days (by February 16th).



a) The retailer can take advantage of the 5% discount if they pay the invoice amount on January 10th.

$$\begin{aligned}
 \text{Net amount} &= 3000(1 - 5\%) \\
 &= 3,000(0.95) \\
 &= \$2850
 \end{aligned}$$

b) The retailer receives a 2% discount if they pay the invoice amount on January 25th.

$$\text{Net amount} = 3000(1 - 2\%)$$

$$= 3,000(0.98)$$

$$= \$2940$$

c) No discount applies, but the payment of \$3000 is still within the credit period if the invoice is paid on January 30th.

Try an Example



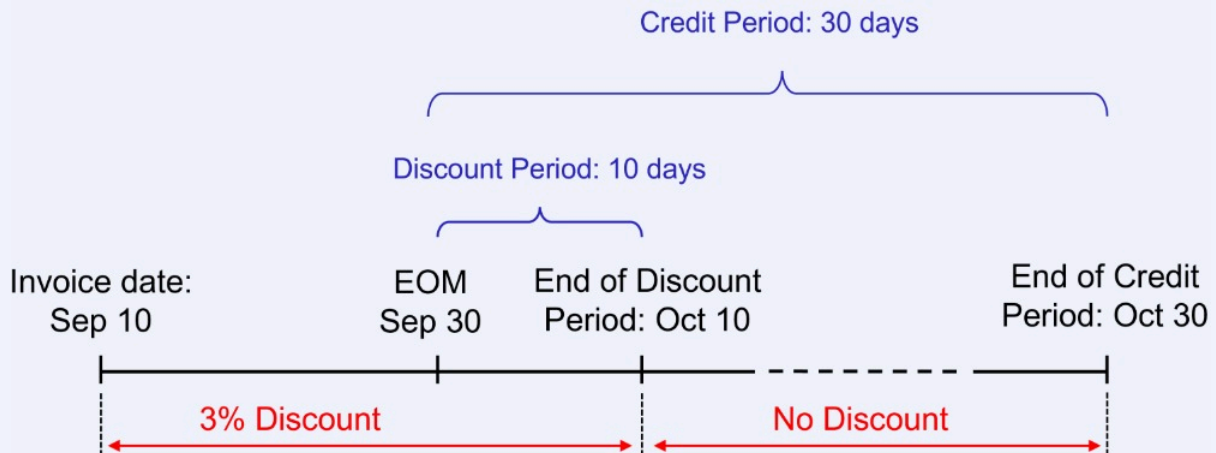
One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=407>

Example 7.2.3: EOM Dating with a Single Cash Discount Rate

K2 Distributors has received an invoice of \$10,000 dated September 10, terms 3/10, n/30 EOM. **a)** What is the last day for taking the cash discount? **b)** How much is to be paid if the discount is taken?

Show/Hide Solution

a) The terms “3/10, n/30 EOM” mean that a 3% cash discount is available if the payment is made within **10** days from the **end of the month** in which the invoice is dated. Since the invoice is dated September 10, the end of September is September 30. Therefore, the last day for taking the cash discount is 10 days after September 30, which is October 10. The credit term “n/30” indicates that the full payment is due **30** days from the end of the month in which the invoice is dated, which would be October 30th.



Note that if the payment is made before the start of the cash discount period, for example, September 15th, it is qualified for the 3% discount.

b) Given the term “3/10”, the cash discount rate is 3%. To find the net price, we can apply the net price factor approach:

$$\begin{aligned}
 \text{Net amount} &= 10,000(1 - 3\%) \\
 &= 10,000(0.97) \\
 &= \$9700
 \end{aligned}$$

Therefore, \$9700 should be paid if the payment is made within the cash discount period.

Try an Example



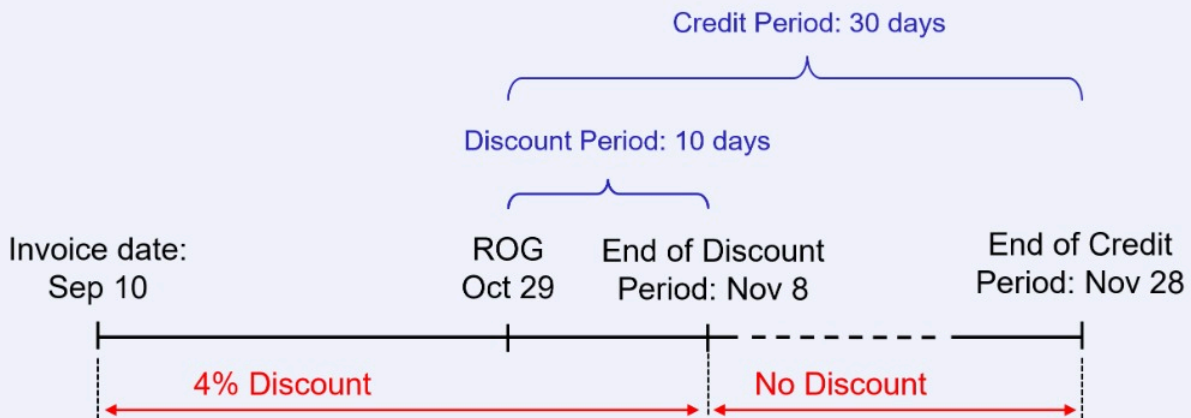
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Example 7.2.4: ROG Dating with a Single Cash Discount Rate

K2 Distributors has received an invoice of \$10,000.00 dated September 10, terms 4/10, n/30 **ROG**, for a shipment of lamps that arrived on October 29. **a)** What is the last day for taking the cash discount? **b)** How much is to be paid if the discount is taken?

Show/Hide Solution

a) The terms “4/10, n/30 ROG” indicate that a 4% cash discount is available if payment is made within 10 days of the date that goods have been received. Since the shipment of lamps arrived on October 29, the last day to take the cash discount is 10 days later, which is November 8.



b) Given the term “4/10”, the cash discount rate is 4%. To find the net price, we can apply the net price factor approach:

$$\begin{aligned}
 \text{Net amount} &= 10,000(1 - 4\%) \\
 &= 10,000(0.96) \\
 &= \$9600
 \end{aligned}$$

Therefore, \$9600 should be paid if the payment is made within the cash discount period.

Try an Example



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Section 7.2 Exercises

1. A wholesaler has received an invoice of \$19,180 dated May 30, terms 3/20, n/30, for a shipment of coffee makers that arrived on June 24. **a)** What is the last day for taking the cash discount? **b)** How much is to be paid if the discount is taken?

Show/Hide Answer

a) June 19

b) \$18,604.60

2. A retailer received an invoice for \$9,500 issued on November 1, with the following payment terms: 4/20, 3/25, n/60. What will settle the invoice if it is paid on November 26?

Show/Hide Answer

\$9,215.00

3. A retailer has received an invoice of \$20,190 dated June 22, terms 6/15, n/30 EOM, for a shipment of leaf blowers that arrived on July 28. **a)** What is the last day for taking the cash discount? **b)** How much is to be paid if the discount is taken?

Show/Hide Answer

a) July 15

b) \$18,978.60

7.3 MARKUP

A. Introduction to Markup

Each member of the supply chain adds costs to the product, from production, handling, and transportation to storage and marketing. As a product moves through the supply chain, these costs accumulate, and each member, except the consumer, adds a markup to cover their costs and earn a profit. The markup amount must be carefully calculated to be high enough to cover all these expenses while remaining competitive in the market. In this section, we will delve deeper into the nuances of markup, explore how it's calculated, and understand its critical role in merchandising.

B. Markup Calculation

Let's consider an example to understand markup calculation. A retailer buys a USB flash drive from a manufacturer for \$5. The retailer incurs an operating expense of \$3 on each USB and aims to make a profit of \$2 per unit. Consequently, the retailer sets the selling price at \$10. Here, the markup of \$5 is the sum of the operating expenses and the desired profit added to the cost price to arrive at the selling price of \$10.



Figure 7.3.1: The relationship among selling price, markup, expenses, and profit

Markup is essentially the amount added to the cost price of a product to establish its selling price. This can be represented by

$$S = C + M$$

Formula 7.6

Where S is the selling price, C is the cost of purchasing or producing a product, and M represents the amount of markup.

The markup includes overhead expenses or operating expenses (E), which are expenses associated with running a business, and operating profit or simply profit (P).

$$M = E + P \quad \text{Formula 7.7}$$

Combining Formulas 7.6 and 7.7, the selling price can be expressed as the sum of cost, expenses, and profit.

$$S = C + E + P \quad \text{Formula 7.8}$$

Example 7.3.1: Compute Markup Given cost, Expenses, and Profit

A retailer purchases printers for \$110 each. The overhead expenses on each printer are \$85 and the profit on each printer is \$130 **a)** Calculate the amount of markup. **b)** Determine the selling price.

Show/Hide Solution

- The cost of purchasing each camera is \$110, so $C = 110$
- The operating expenses on each camera are \$85, so $E = 85$
- The operating profit is \$130 so $P = 130$

a) $M = ?$

The amount of markup is the sum of expenses and profit.

$$M = E + P$$

$$M = 85 + 130$$

$$= \$215$$

b) $S = ?$

The selling price can be found using either Formula 7.6 or 7.8:

$$S = C + M$$

$$\begin{aligned}
 S &= 110 + 215 \\
 &= \$325
 \end{aligned}$$

Try an Example



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C. Markup Rate

Markup can also be expressed in percentage terms, either as a percentage of the cost (Markup on Cost) or as a percentage of the selling price (Markup on Selling Price):

$$\text{Rate of Markup on Cost}(MoC) = \frac{M}{C} \times 100 \% \quad \text{Formula 7.9}$$

and

$$\text{Rate of Markup on Selling Price}(MoS) = \frac{M}{S} \times 100 \% \quad \text{Formula 7.10}$$

Example 7.3.2: Compute Markup Rate Given cost, Expenses, and Profit

A retailer purchases cameras for \$1020 each and wants to sell them each for a profit of \$300. The overhead expenses on each camera are \$250. **a)** Calculate the amount of markup. **b)** Determine the selling price. **c)** What

is the rate of markup on cost? **d)** What is the rate of markup on selling price? *Round your answers to two decimal places where needed.*

Show/Hide Solution

- The cost of purchasing each camera is \$1020, so $C = 1020$
- The operating expenses on each camera is \$250, so $E = 250$
- The operating profit is \$300, so $P = 300$

a) $M = ?$

The amount of markup is the sum of expenses and profit.

$$M = E + P$$

$$M = 250 + 300$$

$$= \$550$$

b) $S = ?$

The selling price can be found using either Formula 7.6 or 7.8:

$$S = C + M$$

$$S = 1020 + 550$$

$$= \$1570$$

c) MoC Rate = ?

The rate of markup on cost can be determined using Formula 7.9:

$$\text{MoC Rate} = \frac{M}{C} \times 100\%$$

$$\text{MoC Rate} = \frac{550}{1020} \times 100\%$$

$$\approx 53.92\%$$

d) MoS Rate = ?

The rate of markup on the selling price can be calculated using Formula 7.10:

$$\text{MoS Rate} = \frac{M}{S} \times 100\%$$

$$\text{MoS Rate} = \frac{550}{1570} \times 100\%$$

$$\approx 35.03\%$$

Try an Example



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Example 7.3.3: Compute Markup Given Markup Rate and Expenses on Cost

A wholesaler purchases snowboards for \$130 each and has a markup rate of 20% on the cost. The wholesaler's overhead expenses are 10% of the cost. **a)** Calculate the amount of markup. **b)** Determine the selling price. **c)** What is the rate of markup on cost? **d)** What is the rate of markup on selling price? **e)** What are the operating expenses? **f)** What is the operating profit? *Round your answers to two decimal places where needed.*

Show/Hide Solution

- The cost of each snowboard is \$130, so $C = 130$
- The markup is 20% of the cost, so $M = 20 \% \cdot C$
- The operating expenses are 10% of the cost, so $E = 10 \% \cdot C$

a) $M = ?$

$$\begin{aligned} M &= 20 \% \cdot C \\ &= 20 \% (130) \\ &= \$26 \end{aligned}$$

b) $S = ?$

The selling price can be found using either Formula 7.6:

$$\begin{aligned} S &= C + M \\ S &= 130 + 26 \\ &= \$156 \end{aligned}$$

c) MoC Rate = ?

The rate of markup on cost can be determined using Formula 7.9:

$$\begin{aligned} \text{MoC Rate} &= \frac{M}{C} \times 100 \% \\ \text{MoC Rate} &= \frac{26}{130} \times 100 \% \\ &= 20 \% \end{aligned}$$

d) MoS Rate = ?

The rate of markup on selling price can be calculated using Formula 7.10:

$$\text{MoS Rate} = \frac{M}{S} \times 100 \%$$

$$\text{MoS Rate} = \frac{26}{156} \times 100\%$$

$$\approx 16.67\%$$

e) $E = ?$

$$E = 10\% \cdot C$$

$$= 10\% (130)$$

$$= \$13$$

f) $P = ?$

To find the profit, we can rearrange Formula 7.7 for P :

$$M = E + P$$

$$\rightarrow P = M - E$$

$$P = 26 - 13$$

$$= \$13$$

Try an Example



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Example 7.3.4: Compute Markup and Selling Price Given Cost, Expenses, and Profit on Selling Price

A store purchases a set of cookware for \$60. The overhead expenses for each set amount to 15% of its selling price, and the store aims for a profit of 10% of the selling price on each set. **a)** What should the selling price of the cookware be? **b)** How much profit does the store make per set of cookware? **c)** What are the overhead expenses for each set of cookware? **d)** What is the amount of markup on each set?

Show/Hide Solution

- The cost of each cookware is \$60, so $C = 60$
- The operating expenses are 15% of the selling price, so $E = 15\% \cdot S$
- The operating profit is 10% of the selling price, so $P = 10\% \cdot S$

a) $S = ?$

Since we know the percentages for expenses and profit relative to the selling price for each set of cookware, but not their exact amounts, we can apply Formula 7.8 to calculate the selling price. However, as the actual values of expenses and profit are not provided, we need to express them in terms of the unknown selling price. By substituting these expressions for expenses and profit into the formula, we can solve for the selling price algebraically.

$$S = C + E + P$$

$$1S = 60 + 0.15S + 0.10S$$

$$1S - 0.15S - 0.10S = 60$$

$$(1 - 0.15 - 0.10)S = 60$$

$$0.75S = 60$$

$$S = \frac{60}{0.75}$$

$$= \$80$$

b) $P = ?$

Now that the selling price is known, we can calculate the profit and expenses.

$$\begin{aligned} P &= 10\% S \\ &= 10\% (80) \\ &= \$8 \end{aligned}$$

c) $E = ?$

$$\begin{aligned} E &= 15\% S \\ &= 15\% (80) \\ &= \$12 \end{aligned}$$

d) $M = ?$

The amount of markup can be calculated using Formula 7.7:

$$\begin{aligned} M &= E + P \\ &= 12 + 8 \\ &= \$20 \end{aligned}$$

Try an Example



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D. Break-even Selling Price

The break-even selling price is the price point where a product's selling price covers exactly its cost (C) and overhead expenses (E), without generating any profit ($P = 0$). This means at the break-even price, a business neither makes a profit nor incurs a loss. The formula for calculating the break-even selling price is

$$S_{BE} = C + E \quad \text{Formula 7.11}$$

where S_{BE} represents the break-even selling price. Note that Formula 7.11 is derived by setting the profit to zero in Formula 7.8.

Example 7.3.5: Compute Break-Even Selling Price

Furry Friends Boutique purchased a cat tree for \$80. The overhead expenses are 25% of the cost, and the desired profit is 15% of the cost. **a)** What is the regular selling price of the cat tree? **b)** During a clearance event, the cat tree was sold at the break-even price. What is the break-even selling price?

Show/Hide Solution

- Cost: $C = 80$
- The operating expenses are 25% of the cost, so $E = 25\% \cdot C$
- The profit is 15% of the cost, so $P = 15\% \cdot C$

a) $S = ?$

$$\begin{aligned} S &= C + E + P \\ S &= 80 + 25\% (80) + 15\% (80) \\ &= 80 + 20 + 12 \\ &= \$112 \end{aligned}$$

b) $S_{BE} = ?$

The break-even selling price is given by Formula 7.11:

$$\begin{aligned}
 S_{BE} &= C + E \\
 &= 80 + 20 \\
 &= \$100
 \end{aligned}$$

At a selling price of \$100, the business covers just its costs and expenses, resulting in neither a profit nor a loss.

Try an Example



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Section 7.3 Exercises

1. A retailer purchases cameras for \$210 each and wants to sell them each for a profit of \$210. The overhead expenses on each camera are \$25. **a)** Calculate the amount of markup. **b)** Determine the selling price.

Show/Hide Answer

a) $M = \$235$

b) $S = \$445$

2. A store pays \$225 for a set of cookware. Overhead expense on each cookware is 13% of the selling price and the profit is 18% of the selling price. **a)** What is the selling price? **b)** What is the amount of markup? **c)** What is the rate of markup on cost? **d)** What is the rate of markup on the selling price?

Show/Hide Answer

a) $S = \$326.09$

b) $M = \$101.09$

c) MoC Rate = 44.93%

d) MoS Rate = 31%

3. A store purchased an item for \$130 and planned to sell it for \$234.00 so that their profit would be 40% of their cost. If they were unable to sell it for this amount, what minimum selling price would allow them to break even?

Show/Hide Answer

$S = \$182$

4. A Retailer purchased headphones for \$150 per unit. The overhead expenses are 25% of the cost, and the desired profit is 15% of the cost. **a)** What is the regular selling price of each headphone? **b)** During a clearance event, the headphone was sold at the break-even price. What is the break-even selling price?

Show/Hide Answer

a) $S = \$210$

b) $S = \$187.50$

7.4 MARKDOWN

Returning to the previous example, consider a retailer who purchases USB flash drives from a manufacturer at \$5 each. The retailer's operating expenses amount to \$3 for each USB, aiming for a profit of \$2 per unit, leading to an initial selling price of \$10. However, suppose market demand decreases, prompting the retailer to lower the USBs' selling price by \$4. This adjustment results in a reduced selling price of \$6. Retailers sometimes reduce their selling price to clear old or excess stock, make room for new inventory, respond to reduced demand, or compete with other retailers' pricing strategies. This reduction in the original selling price of goods is referred to as **Markdown (MD)**.

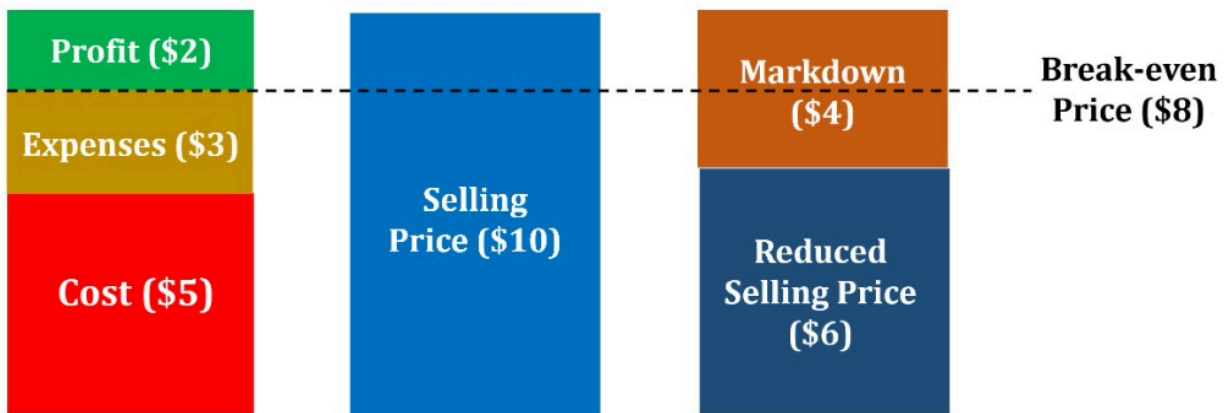


Figure 7.4.1: The relationship among selling price, markdown, and reduced selling price

$$\text{Markdown Amount} = \text{Original Price} - \text{Reduced Price}$$

$$MD = S - S_{Red} \quad \text{Formula 7.12}$$

where S_{Red} represents the reduced selling price or sale price.

The markdown is often expressed as a percentage of the original price.

$$\text{Markdown (MD) Rate} = \frac{MD}{S} \times 100\% \quad \text{Formula 7.13}$$

Reduced selling price can be directly obtained using the net price factor.

$$S_{Red} = S(1 - \text{MD Rate}) \quad \text{Formula 7.14}$$

Example 7.4.1: Compute Markdown Rate

A boutique initially offers a dress for \$120, but during a seasonal clearance event, the price is marked down to \$72. Calculate the rate of markdown.

Show/Hide Solution

- Regular selling price: $S = 120$
- Reduced selling price: $S_{Red} = 72$

$MD\ Rate = ?$

We first need to find the amount of markdown using Formula 7.12.

$$MD = S - S_{Red}$$

$$\begin{aligned} MD &= 120 - 72 \\ &= \$48 \end{aligned}$$

The rate of markdown is then given by Formula 7.13.

$$MD\ Rate = \frac{MD}{S} \times 100\%$$

$$\begin{aligned} MD\ Rate &= \frac{48}{120} \times 100\% \\ &= 40\% \end{aligned}$$

Try an Example



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Example 7.4.2: Compute Markdown and Reduced Selling Price

A retailer purchased a product for \$110 and had operating expenses of 10% of the cost and operating profit of 40% of the cost of each product. During a seasonal sale, the product was marked down by 25%. **a)** What was the regular selling price? **b)** What was the amount of markdown? **c)** What was the reduced selling price during the seasonal sale? **d)** What was the profit or loss at the sale price?

Show/Hide Solution

- Cost: $C = 110$
- Expense: $E = 10\% \cdot C$
- Profit: $P = 40\% \cdot C$
- Markdown rate is 25% of the regular selling price: MD Rate = 25%

a) $S = ?$

Applying Formula 7.8 gives

$$\begin{aligned} S &= C + E + P \\ S &= 110 + 10\% (110) + 40\% (110) \\ &= 110 + 11 + 44 \\ &= \$165 \end{aligned}$$

b) $MD = ?$

$$\begin{aligned}
 MD &= 25 \% \cdot S \\
 &= 25 \% (165) \\
 &= \$41.25
 \end{aligned}$$

c) $S_{red} = ?$

The reduced selling price is the difference between the regular selling price and the amount of markdown:

$$\begin{aligned}
 S_{red} &= S - MD \\
 S_{red} &= 165 - 41.25 \\
 &= \$123.75
 \end{aligned}$$

Alternatively, you can calculate the reduced selling price directly from the markdown rate using Formula 7.14:

$$\begin{aligned}
 S_{red} &= S(1 - \text{MD Rate}) \\
 S_{red} &= 165(1 - 25 \%) \\
 &= 165(0.75) \\
 &= \$123.75
 \end{aligned}$$

d) $P_{Red} = ?$

Since the cost of the product and overhead expenses are fixed, reducing the selling price will also decrease the operating profit. Consequently, we can use Formula 7.8 with the reduced selling price to calculate the profit during the sale period.

$$\begin{aligned}
 S_{Red} &= C + E + P_{Red} \\
 P_{Red} &= S_{Red} - C - E \\
 &= 123.75 - 110 - 11 \\
 &= \$2.75
 \end{aligned}$$

Since the profit during the sale is a positive value, the retailer made a profit of \$2.75.

Try an Example



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Section 7.4 Exercises

1. A boutique initially offers a dress for \$119, but during a seasonal clearance event, the price is marked down to \$97.58. Calculate the rate of markdown.

Show/Hide Answer

MD Rate = 18%

2. A retailer purchased a product for \$65 and had operating expenses of 35% of the cost and operating profit of 50% of the cost of each product. During a seasonal sale, the product was marked down by 35%. **a)** What was the regular selling price? **b)** What was the sale price? **c)** What was the profit or loss at the sale price?

Show/Hide Answer

a) S = \$120.25

b) S = \$78.16

c) Loss of \$9.59

3. A retailer applied a markup of 48% on selling price on a product that cost \$145 per unit. During a sale, the store marked the items down by 35%. **a)** Calculate the regular selling price of the product. **b)** Calculate the reduced selling price of the product.

Show/Hide Answer

a) $S = \$278.85$

b) $S = \$181.25$

7.5 INTEGRATED PROBLEMS

Integrated problems often involve calculations that include trade discounts, markups, and markdowns. When tackling such problems, it is beneficial to approach the problem from the perspective of a business owner, focusing on determining costs and selling prices.

As these calculations are often sequential, with one building upon the next, it is important to break down the problem into manageable steps:

1. **Determine the Purchase Cost (C):** Start by calculating the cost of purchase if it's not directly provided. This may involve applying trade discounts to the list price to find the actual cost.
2. **Establish Selling Price (S):** Use the information given about markup (M), expenses (E), and desired profit (P) and the proper formula that connects these elements. This will help you determine the regular selling price.
3. **Calculate the Reduced Selling Price (S_{Red}):** If the problem includes a markdown, apply the markdown rate to the original selling price to find the sale price or reduced selling price.

Example 7.5.1: Compute Selling Price, Sale price Given List Price, Trade Discount, Markup on Selling Price

A store purchased drawers for \$90 less 10% trade discount and marked them up 40% of the selling price. Operating expenses for each drawer accounted for 15% of the selling price. Later, when the drawer was discontinued, the store marked it down by 30%. **a)** What was the cost of each drawer? **b)** What was the regular selling price? **c)** What was the break-even price? **d)** What was the sale price? **e)** What was the profit or loss at the sale price?

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Show/Hide Solution

- List price: $L = 90$
- Trade discount: $d = 10\%$
- Markup amount: $M = 40\% \cdot S$
- Expenses are 12% of the selling price: $E = 15\% \cdot S$
- Markdown rate: $MD \text{ Rate} = 30\%$

a) $C = ?$

The purchase cost for each drawer is the net price that the retailer pays after applying the trade discount to the list price. Using Formula 7.3a, we can find the net price:

$$\begin{aligned} C &= L(1 - d) \\ C &= 90(1 - 0.1) \\ &= \$81 \end{aligned}$$

b) $S = ?$

Since we know the percentages for markup relative to the selling price for each drawer (but not the markup exact amount), we can apply Formula 7.6 to calculate the selling price. However, as the actual value of the markup is not provided, we need to express it in terms of the unknown selling price. By substituting the markup into the formula, we can solve for the selling price algebraically.

$$\begin{aligned} S &= C + M \\ 1S &= 81 + 0.40S \\ 1S - 0.40S &= 81 \end{aligned}$$

$$(1 - 0.40)S = 81$$

$$0.60S = 81$$

$$S = \frac{81}{0.60}$$

$$= \$135$$

c) $S_{BE} = ?$

To find the break-even selling price we need to know the expenses.

$$E = 15\% \cdot S = 15\% (135) = \$20.25$$

The break-even selling price is given by Formula 7.11:

$$S_{BE} = C + E$$

$$= 81 + 20.25$$

$$= \$101.25$$

d) $S_{Red} = ?$

To directly find the reduced selling price, we can apply Formula 7.14:

$$S_{Red} = S(1 - \text{MD Rate})$$

$$S_{Red} = 135(1 - 0.30)$$

$$= 135(0.70)$$

$$= \$94.50$$

e) $P_{Red} = ?$

Approach 1:

Since the cost of the product and overhead expenses are fixed, reducing the selling price will also decrease the

operating profit. Consequently, we can use Formula 7.8 with the reduced selling price to calculate the profit during the sale period.

$$\begin{aligned}
 P_{Red} &= S_{Red} - C - E \\
 P_{Red} &= 94.50 - 81 - 20.25 \\
 &= - \$6.75 \text{ Loss}
 \end{aligned}$$

Approach 2:

Once the break-even price is known, the profit or loss during the sale can be determined by comparing the sale price to the break-even price. If the sale price exceeds the break-even price, the retailer realizes a profit; conversely, if the sale price is below the break-even price, the retailer incurs a loss.

$$\begin{aligned}
 P_{Red} &= S_{Red} - S_{BE} \\
 P_{Red} &= 94.50 - 101.25 \\
 &= - \$6.75 \text{ Loss}
 \end{aligned}$$

Try an Example



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Example 7.5.2: Compute Selling Price, Sale price Given List Price, Trade Discount

Series, Markup On Cost

A retailer purchased tablets at a list price of \$420.00 each less trade discounts of 35% and 22%. The retailer's overhead expenses amounted to 20% of the cost, and they had a profit of 30% of the cost of each tablet. During a holiday sale, the tablets were offered at a 25% markdown. **a)** What was the cost of each tablet? **b)** What was the regular selling price? **c)** What was the sale price? **d)** What was the profit or loss at the sale price?

Show/Hide Solution

- Tablet's list price: $L = 420$
- Trade discount rate 1: $d_1 = 35\%$
- Trade discount rate 2: $d_2 = 22\%$
- Expense: $E = 20\% \cdot C$
- Profit: $P = 30\% \cdot C$
- Markdown rate was 25% of the regular selling price: MD Rate = 25%

a) $C = ?$

The purchase cost for each tablet is the net price that the retailer pays after applying the trade discount series to the list price. Using Formula 7.4, we can find the net price:

$$\begin{aligned} C &= L(1 - d_1)(1 - d_2) \\ C &= 420(1 - 0.35)(1 - 0.22) \\ &= 420(0.65)(0.78) \\ &= \$212.94 \end{aligned}$$

b) $S = ?$

Applying Formula 7.8 gives

$$\begin{aligned}
 S &= C + E + P \\
 S &= 1C + 0.2C + 0.3C \\
 &= 1.5C \\
 &= (1.5)(212.94) \\
 &= \$319.41
 \end{aligned}$$

c) $S_{Red} = ?$

To find the reduced selling price, we can use Formula 7.14:

$$\begin{aligned}
 S_{Red} &= S(1 - \text{MD Rate}) \\
 S_{Red} &= 319.41(1 - 0.25) \\
 &= 319.41(0.75) \\
 &= \$239.56
 \end{aligned}$$

d) $P_{Red} = ?$

To find the profit during the sale, we need to know the expenses.

$$E = 20\% \cdot C = 20\% (212.94) = \$42.59$$

Since the cost of the product and overhead expenses are fixed, reducing the selling price will also decrease the operating profit. Rearranging Formula 7.8 and plugging in the reduced selling price gives the profit during the sale:

$$\begin{aligned}
 P_{Red} &= S_{Red} - C - E \\
 P_{Red} &= 239.56 - 212.94 - 42.59 \\
 &= -\$15.97 \text{ Loss}
 \end{aligned}$$

Since P_{Red} is a negative value, the retailer incurred a loss of \$15.97 on each tablet.

Try an Example



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://ecampusontario.pressbooks.pub/financemath/?p=413>

Section 7.5 Exercises

1. A store purchased laptops for \$1,050 less 15% trade discount and marked them up 50% of the selling price. Operating expenses for each laptop accounted for 10% of the selling price. Later, when the laptop was discontinued, the store marked it down by 10%. **a)** What is the cost of purchasing each laptop? **b)** What was the regular selling price? **c)** What was the break-even price? **d)** What was the sale price? **e)** What was the profit or loss at the sale price?

Show/Hide Answer

a) $C = \$892.50$

b) $S = \$1785$

c) $BE = \$1071$

d) $\text{Sale} = \$1606.50$

e) $\text{Profit of } \$535.50$

2. A phone costs a retailer \$1,400 less 29%, 14%. The retailer's overhead expenses are 35% of the cost and its profit is 25% of the cost. During a sale, the phone is marked down 20%. **a)** What is the cost of purchasing each phone? **b)** What was the regular selling price? **c)** What was the amount of markdown? **d)** What was the sale price? **e)** What was the profit or loss at the sale price?

Show/Hide Answer

a) $C = 854.84$

b) $S = \$1367.74$

c) $MD = \$273.55$

d) $\text{Sale} = \$1094.19$

e) $\text{Loss of } \$59.84$

7.6 TERMINOLOGY AND GLOSSARY OF SYMBOLS

Terminology

- **Break-even selling price:** This is the price at which a product's selling price precisely covers both its cost (C) and overhead expenses (E), resulting in no profit.
- **(Trade) Discount Series:** When the list price is subject to more than one trade discount, it is referred to as a discount series.
- **List Price (L):** Also known as the sticker price or manufacturer's suggested retail price (MSRP), it is the price at which the manufacturer recommends selling a product to the consumer.
- **Markdown (MD):** The amount by which the regular selling price (S) of a product is reduced to determine the sale price or reduced selling price (S_{Red}), usually from the retailer to the consumer.
- **Markup (M):** The additional amount a business adds to the cost of purchasing a product (C) to determine the selling price (S).
- **Net Price (N):** The reduced price the buyer pays for a product after receiving the trade discount.
- **Net Price Factor (NPF):** The percentage of the list price that a business pays for a product after applying a trade discount.
- **Overhead expenses (E):** Also known as expenses, these are ongoing costs associated with operating a business.
- **Single equivalent trade discount rate (d_{eq}):** The single rate of discount that represents a series of trade discounts, yielding the same net price as a series of trade discount rates.
- **Single trade Discount (d):** This represents a reduction of the list price offered between traders in the supply chain.
- **Supply (merchandising) chain:** The network or stages through which a product passes as it transforms from a raw material into a finished product delivered to the consumer.

Glossary of Symbols

Trade discounts

- L: List Price (\$)
- A: Amount of trade discount (\$)
- d: Trade discount rate (%)

- N: Net Price (\$)
- NPF: Net Price Factor (%)
- d_{eq} : Equivalent rate of discount (%)

Markup

- S: Selling Price (\$)
- M: Markup (\$)
- P: Profit (operating profit) (\$)
- Rate of MO_C : Rate of markup on cost (%)
- Rate of MO_S : Rate of markup on selling price (%)

Markdown

- MD: Amount of markdown (\$)
- Rate of MD: Rate of markdown (%)
- S_{Red} : Reduced selling price during the sale period (\$)
- P_{Red} : Profit during the sale period (\$)