Applied Algebra

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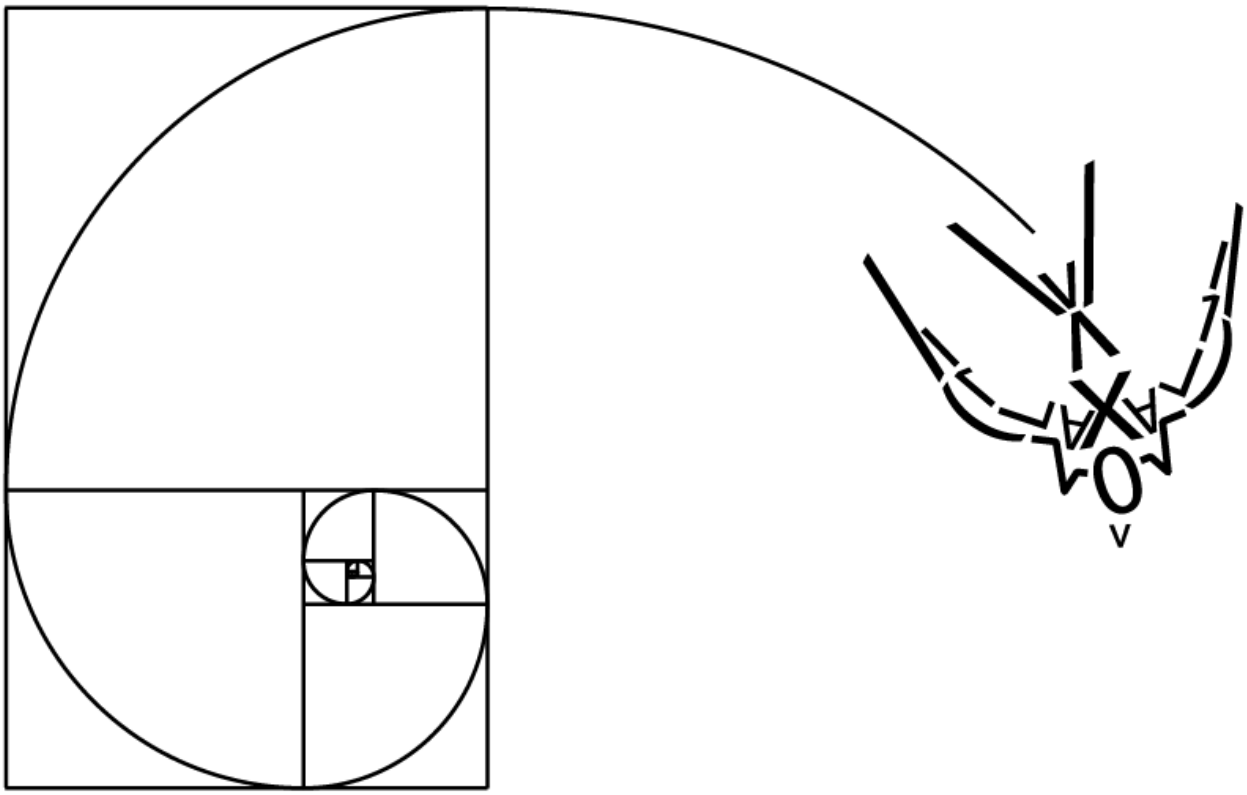
About This Book



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CHAPTER 1: INTRODUCTION TO NUMBERS

Chapter Outline

1.1 Real Numbers: Algebra Essentials

- Classifying a Real Number
- Irrational Numbers
- Real Numbers
- Sets of Numbers

1.2 Order of Operations

- Grouping Symbols
- Order of Operations

1.3 Fractions

- Reducing
- Multiplying and Dividing Fractions
- Adding and Subtracting Fractions

1.4 Exercises

- 1.1 Real Numbers: Algebra Essentials
- 1.2 Order of Operations
- 1.3 Fractions

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1.1 Real Numbers: Algebra Essentials

Learning Objectives

By the end of this section, you should be able to:

- Classify a real number as a natural, whole, integer, rational, or irrational number.

Classifying a Real Number

The numbers we use for counting or enumerating items are the **natural numbers**: $1, 2, 3, 4, 5,$ and so on. The natural numbers are also called the *counting numbers*. The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, \dots\}$.

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers.

$$\underset{\dots, -3, -2, -1,}{\text{negative integers}} \qquad \underset{0,}{\text{zero}} \qquad \underset{1, 2, 3, \dots}{\text{positive integers}}$$

The set of **rational numbers** is written as $\left\{ \frac{m}{n} \mid m \text{ and } n \text{ are integers and } n \neq 0 \right\}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Rational numbers can be written as a fraction or in decimal form as:

1. a terminating decimal: $\frac{15}{8} = 1.875,$ or
2. a repeating decimal: $\frac{4}{11} = 0.36363636\cdots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

Writing Integers as Rational Numbers

Example 1

Write each of the following as a rational number.

a. 7
b. 0
c. –8

Solution:

Write a fraction with the integer in the numerator and 1 in the denominator.

a. $7 = \frac{7}{1}$
b. $0 = \frac{0}{1}$
c. $-8 = -\frac{8}{1}$ (this can also be written as $-8 = \frac{-8}{1}$ or $-8 = \frac{8}{-1}$)

Identifying Rational Numbers

Example 2

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $-\frac{5}{7}$
b. $\frac{15}{5}$

c. $\frac{13}{25}$

Solution:

Write each fraction as a decimal by dividing the numerator by the denominator.

a. $-\frac{5}{7} = -0.\overline{714285},$ a repeating decimal

b. $\frac{15}{5} = 3$ (or 3.0), a terminating decimal

c. $\frac{13}{25} = 0.52,$ a terminating decimal

Try It

Activity Description
Drag the words into the correct boxes.

1. Integers can be rewritten as rational numbers.

   11 as a rational number is:
   3 as a rational number is:
   -4 as a rational number is:

   Choices:

   $\frac{3}{1}$ $\frac{-4}{1}$ $\frac{1}{3}$ $\frac{11}{1}$ $\frac{4}{1}$ $\frac{1}{11}$

2. Similarly, rational numbers can be rewritten as decimals with terminating or repeating decimals:

   $\frac{68}{17}$ as a decimal is:
   $\frac{8}{13}$ as a decimal is:
   This decimal is:
   $\frac{-17}{20}$ as a decimal is:
   This decimal is:

   Choices:

$0.\overline{615384}$ $-0.\overline{85}$ 4 -0.85 irrational 0.615384 $\frac{1}{4}$ repeating terminating

---

Answers:

$\frac{11}{1}$

$\frac{3}{1}$

$\frac{-4}{1}$

4

$0.\overline{615384}$

This decimal is: repeating.

-0.85

This decimal is: terminating.

Irrational Numbers

Irrational numbers

cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational.

Differentiating Rational and Irrational Numbers

Example 3

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\sqrt{25}$

b. $\frac{33}{9}$

c. $\sqrt{11}$

d. $\frac{17}{34}$

e. $0.3033033303333\ldots$

Solution:

a. $\sqrt{25}$: This can be simplified as $\sqrt{25} = 5$. Therefore, $\sqrt{25}$ is rational.

b. $\frac{33}{9}$: Because it is a fraction, $\frac{33}{9}$ is a rational number. Next, simplify and divide.

$$\frac{33}{9} = \frac{\overset{11}{\cancel{33}}}{\underset{3}{\cancel{9}}} = \frac{11}{3} = 3.\overline{6}$$

So, $\frac{33}{9}$, is rational and a repeating decimal.

c. $\sqrt{11}$: This cannot be simplified any further. Therefore, $\sqrt{11}$ is an irrational number.

d. $\frac{17}{34}$: Because it is a fraction, $\frac{17}{34}$ is a rational number. Simplify and divide.

$$\frac{17}{34} = \frac{\overset{1}{\cancel{17}}}{\underset{2}{\cancel{34}}} = \frac{1}{2} = 0.5$$

So, $\frac{17}{34}$ is rational and a terminating decimal.

e. $0.3033033303333\ldots$ is not a terminating decimal. Also, note that there is no repeating pattern because the group of 3s increases each time. Therefore, it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

Try It

Activity Description

Convert the numbers to decimals, then drag them to the correct category.

Rational		Irrational
Terminating	Repeating	

Numbers:

$$\frac{47}{19} \quad -\frac{\sqrt{5}}{2} \quad \sqrt{73} \quad -11.411411411\ldots \quad \frac{91}{13} \quad 4.270270027\ldots \quad \frac{7}{77} \quad \sqrt{81}$$

Answer:

Rational		Irrational
Terminating	Repeating	
$\frac{91}{13}$ $\sqrt{81}$	$\frac{47}{19}$ $-11.411411411\ldots$ $\frac{7}{77}$	$-\frac{\sqrt{5}}{2}$ $4.270270027\ldots$ $\sqrt{73}$

Real Numbers

Given any number n, we know that n is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of **real numbers**. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or –). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with a point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line. The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the **real number line**, as shown in Figure 1.1.1.

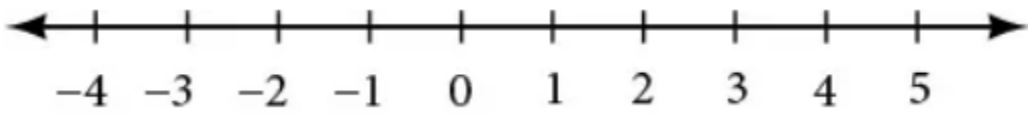

Figure 1.1.1. "The real number line" by OpenStax, CC BY 4.0.

Classifying Real Numbers

Example 4

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $-\frac{10}{3}$
b. $\sqrt{5}$
c. $-\sqrt{289}$
d. -6π
e. $0.615384615384\ldots$

Solution:

a. $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.
b. $\sqrt{5}$ is positive and irrational. It lies to the right of 0.
c. $-\sqrt{289} = -\sqrt{17^2} = -17$ is negative and rational. It lies to the left of 0.
d. -6π is negative and irrational. It lies to the left of 0.
e. $0.615384615384\ldots$ is a repeating decimal, so it is rational and positive. It lies to the right of 0.

Sets of Numbers as Subsets

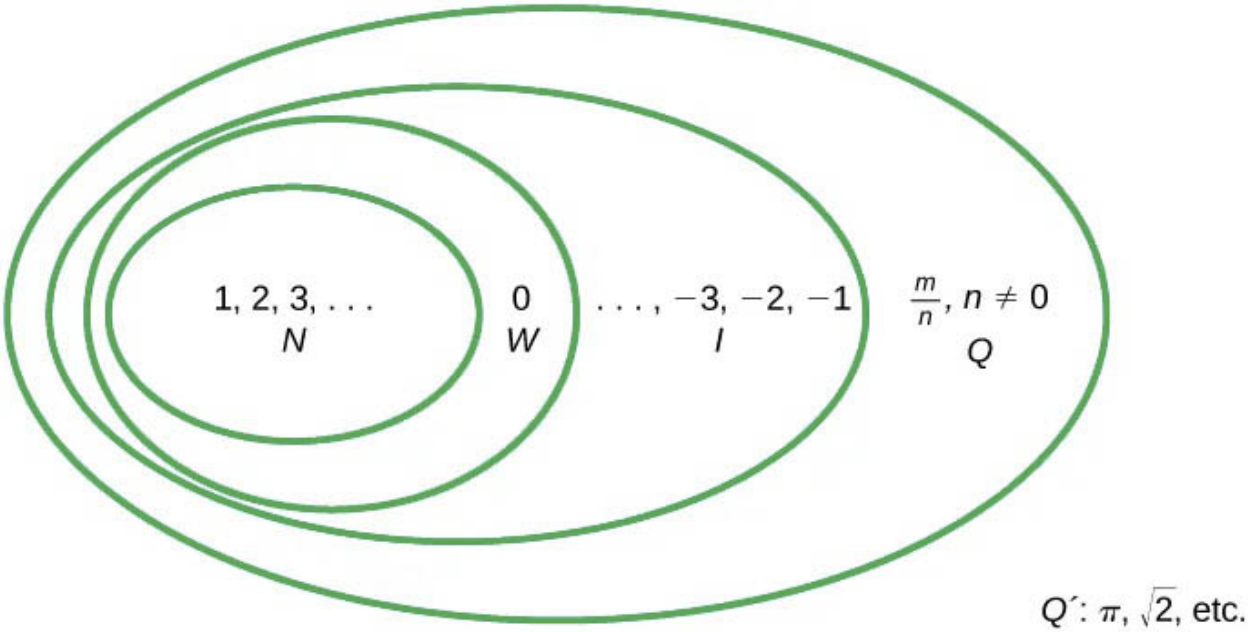

Figure 1.1.2. Sets of numbers: N: the set of natural numbers, W: the set of whole numbers, I: the set of integers, Q: the set of rational numbers, Q´: the set of irrational numbers.
Image by OpenStax, CC BY 4.0. Mods: recoloured.

Sets of Numbers

The set of natural numbers includes the numbers used for counting: $\{1, 2, 3, \ldots\}$.

The set of whole numbers is the set of natural numbers plus zero: $\{0, 1, 2, 3, \ldots\}$.

The set of integers adds the negative natural numbers to the set of whole numbers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The set of rational numbers includes fractions written as $\left\{\frac{m}{n} \text{ such that } m \text{ and } n \text{ are integers and } n \neq 0\right\}$.

The set of irrational numbers is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h \text{ such that } h \text{ is not a rational number}\}$.

Example 5

Classify each number as being a natural number (*N*), whole number (*W*), integer (*I*), rational number (*Q*), and/or irrational number (*Q'*).

a. $\sqrt{36}$
b. $\frac{8}{3}$
c. $\sqrt{73}$
d. -6
e. $3.2121121112\ldots$

Solution

Number	*N*	*W*	*I*	*Q*	*Q'*
a. $\sqrt{36} = 6$	X	X	X	X	
b. $\frac{8}{3} = 2.\overline{6}$				X	
c. $\sqrt{73}$					X
d. -6			X	X	
e. $3.2121121112\ldots$					X

Try It

Activity Description
For the following numbers, fit each one into the innermost category that is true.

Natural	Whole	Integers	Rational	Irrational

Numbers:

$$\frac{35}{7} \quad \sqrt{24} \quad 4.763763\ldots \quad -\frac{35}{7} \quad \sqrt{169} \quad 0$$

Answer:

Natural	Whole	Integers	Rational	Irrational
$\sqrt{169}$ $\frac{35}{7}$	0	$-\frac{35}{7}$	$4.763763\ldots$	$\sqrt{24}$

Key Takeaways

- Rational numbers may be written as fractions or terminating or repeating decimals.
- Determine whether a number is rational or irrational by writing it as a decimal.
- The rational numbers and irrational numbers make up the set of real numbers. A number can be classified as natural, whole, integer, rational, or irrational.

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1.2 Order of Operations

Learning Objectives

By the end of this section, you should be able to:

- Identify and calculate with grouping symbols.
- Explain the order of operations.
- Simplify using the order of operations.

Grouping Symbols

When more than one operation is involved, **grouping symbols** helps tell us which operations to perform first.

The grouping symbols commonly used in algebra are:

$$\begin{array}{ll} (\ ) & \text{Parentheses} \\ [\ ] & \text{Brackets} \\ \{ \ \} & \text{Braces} \\ - & \text{Fraction bar} \end{array}$$

All of the above grouping symbols, as well as **absolute value**, have the same **order of precedence**.

Perform operations inside the innermost grouping symbol or absolute value first and work outwards.

Example 1

Simplify:

$$5 - (4 - 12)$$

Solution:
Perform the operations within the parentheses first. In this case, subtract 12 from 4.

$$\begin{aligned} 5-(4-12) &= 5-(-8) \\ &= 5+8 \\ &= 13 \end{aligned}$$

Answer

13

Example 2

Simplify:

$$3\{-2[-(-3-1)]\}$$

Solution:
Work from the innermost grouping outward.

$$\begin{aligned} 3\{-2[-(-3-1)]\} &= 3\{-2[-(-4)]\} \\ &= 3\{-2[4]\} \\ &= 3(-8) \\ &= -24 \end{aligned}$$

Answer

-24

Example 3

Simplify:

$$\frac{5-|4-(-3)|}{|-3|-(5-7)}$$

Solution:

$$\begin{aligned}\frac{5-|4-(-3)|}{|-3|-(5-7)} &= \frac{5-|4+3|}{|-3|-(-2)} \\ &= \frac{5-|7|}{|-3|+2} \\ &= \frac{5-7}{3+2} \\ &= \frac{-2}{5} \\ &= -\frac{2}{5}\end{aligned}$$

Answer

$-\frac{2}{5}$

Try It

Simplify:

$$-[-3(2+3)]$$

Show Solution

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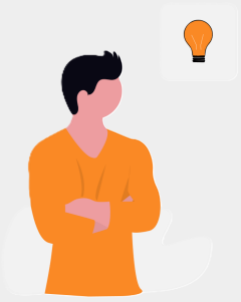
Order of Operations

When several operations are to be applied within a calculation, we must follow a specific order to ensure a single correct result.

1. Perform all calculations within the innermost parentheses or grouping symbols.
2. Evaluate all **exponents**.
3. Perform multiplication and division operations from left to right.
4. Finally, perform all remaining addition and subtraction operations from left to right.



Note

Multiplication and division operations ***must*** be worked from ***left to right***.

Visualize

Visit the Geogebra site to practice Correcting Errors in an Order of Operations. Link available on page 1.2 of the digital version of this text-book.

How To

Given a mathematical expression, simplify it using the order of operations.

1. Simplify any expressions within grouping symbols.
2. Simplify any expressions containing exponents or radicals.
3. Perform any multiplication and division in order, from left to right.
4. Perform any addition and subtraction in order, from left to right.

Example 4

Simplify:

$$5^2 - 4 \cdot 3 \div 12$$

Solution:

First evaluate 5^2, then perform multiplication and division from left to right.

$$\begin{aligned} 5^2 - 4 \cdot 3 \div 12 &= 25 - \underbrace{4 \cdot 3}\div 12 \\ &= 25 - \underbrace{12 \div 12} \\ &= \underbrace{25} - 1 \\ &= 24 \end{aligned}$$

Answer

24

Because multiplication and division operations should be worked from left to right, it is sometimes correct to perform division before multiplication.

Example 5

Simplify:

$$2^4 - 12 \div 3 \cdot 2 + 11$$

Solution:

Begin by evaluating the exponent, $2^4 = 16$.

$$\begin{aligned} 2^4 - 12 \div 3 \cdot 2 + 11 &= 16 - \underbrace{12 \div 3}\cdot 2 + 11 \\ &= 16 - \underbrace{4 \cdot 2} + 11 \\ &= \underbrace{16 - 8} + 11 \\ &= \underbrace{8 + 11} \\ &= 19 \end{aligned}$$

Multiplying first leads to an incorrect result.

$$\begin{aligned} 2^4 - 12 \div 3 \cdot 2 + 11 &= 16 - 12 \div \underbrace{3 \cdot 2}_{\text{Incorrect}} + 11 \\ &= 16 - \underbrace{12 \div 6} + 11 \\ &= \underbrace{16 - 2} + 11 \\ &= 25 \text{ Incorrect} \end{aligned}$$

Answer

19

Example 6

Simplify:

$$-3-5^2+(-7)^2$$

Solution:

Take care to correctly identify the base when squaring.

$$\begin{aligned} -3-5^2+(-7)^2 &= -3-25+49 \\ &= -28+49 \\ &= 21 \end{aligned}$$

Answer

21

Example 7

Simplify:

$$5-3[2^3-5+7(-3)]$$

Solution:

It is tempting to first subtract, but this will lead to an incorrect result. The order of operations requires us to simplify within the brackets first.

$$\begin{aligned} 5-3[2^3-5+7(-3)] &= 5-3[8-5-21] \\ &= 5-3[-18] \\ &= 5+54 \\ &= 59 \end{aligned}$$

Subtracting $5-3$ first leads to an incorrect result.

$$5 - 3[2^3 - 5 + 7(-3)] = \underbrace{5 - 3}_{\text{Incorrect}}[8 - 5 - 21]$$
$$= 2[-18]$$
$$= -36 \text{ Incorrect}$$

Answer

59

Example 8

Simplify:

$$-3^2 - [5 - (4^2 - 10)]$$

Solution:

Perform the operations within the innermost parentheses first.

$$-3^2 - [5 - (4^2 - 10)] = -9 - [5 - (16 - 10)]$$
$$= -9 - [5 - 6]$$
$$= -9 - [-1]$$
$$= -8$$

Answer

-8

Example 9

Simplify:

$$\left(-\frac{2}{3}\right)^2 \div \left[\frac{5}{3} - \left(-\frac{1}{2}\right)^3\right]$$

Solution:

$$\left(-\frac{2}{3}\right)^2 \div \left[\frac{5}{3} - \left(-\frac{1}{2}\right)^3\right] = \frac{4}{9} \div \left[\frac{5}{3} - \left(-\frac{1}{8}\right)\right]$$

$$= \frac{4}{9} \div \left[\frac{5}{3} + \frac{1}{8}\right]$$

$$= \frac{4}{9} \div \left[\frac{40}{24} + \frac{3}{24}\right]$$

$$= \frac{4}{9} \div \frac{43}{24}$$

$$= \frac{4}{\cancel{9}_{3}} \cdot \frac{\overset{8}{\cancel{24}}}{43}$$

$$= \frac{96}{387}$$

$$= \frac{32}{129}$$

Answer

$$\frac{32}{129}$$

We are less likely to make a mistake if we work one operation at a time. Some problems may involve an absolute value, in which case we assign it the same order of precedence as parentheses.

Example 10

Simplify:

$$2-4|-4-3|+(-2)^4$$

Solution:
Evaluate the absolute value and exponent first.

$$\begin{aligned} 2-4|-4-3|+(-2)^4 &= 2-4|-7|+16 \\ &= 2-4(7)+16 \\ &= 2-28+16 \\ &= -10 \end{aligned}$$

Answer

-10

Try It

Simplify:

$$10 \div 5 \cdot 2\,|(-4)+|-3||+(-3)^2$$

Show Solution

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Applied Example

You are preparing a solution of magnesium sulphate and need to determine the molecular mass (the bottle has no mass on it). Its formula is $MgSO_4 \cdot 7H_2O$.

Atomic masses rounded to the nearest whole number (in g/mol):

Mg=24, S=32, O=16, H=1

Use the order of operations to first determine the mass of each element present in the molecule, then sum the elements.

$$24 + 32 + 4(16) + 7\left[2(1) + 16\right] = 246 \text{ g/mol}$$

Try It

You are preparing a solution of washing soda and need to determine the molecular mass.

$$Na_2CO_3 \cdot 10H_2O$$

Molecular masses rounded to nearest whole number (in g/mol):

Na=23, C=12, O=16, H=1

Show Solution

$$2(23) + 12 + 3(16) + 10\left[2(1) + (16)\right] = 286\text{g/mol}$$

Key Takeaways

- Grouping symbols indicate which operations to perform first. We usually group mathematical operations with parentheses, brackets, braces, and the fraction bar.

We also group operations within absolute values. All groupings have the same order of precedence: the operations within the innermost grouping are performed first.

- When applying operations within a calculation, follow the order of operations to ensure a single correct result.
    - Address innermost parentheses or groupings first.
    - Simplify all exponents.
    - Perform multiplication and division operations from left to right.
    - Finally, perform addition and subtraction operations from left to right.
- It is important to highlight the fact that multiplication and division operations should be applied as they appear from left to right.
- It is a common mistake to always perform multiplication before division, which, as we have seen, in some cases produces incorrect results.

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1.3 Fractions

Learning Objectives

By the end of this section, you should be able to:

- Reduce a fraction to its lowest terms.
- Multiply and divide fractions.
- Add and subtract fractions.

Reducing

A **fraction** is a real number written as a quotient, or ratio, of two integers a and b, where $b \neq 0$.

$$\begin{aligned} \text{Numerator} &\rightarrow a \\ \text{Denominator} &\rightarrow b \end{aligned}$$

The integer above the fraction bar is called the **numerator**, and the integer below is called the **denominator**. The numerator is often called the "part," and the denominator is often called the "whole." **Equivalent fractions** are two equal ratios expressed using different numerators and denominators. For example,

$$\frac{50}{100} = \frac{1}{2}$$

Fifty parts out of 100 is the same ratio as 1 part out of 2 and represents the same real number. Consider the following factorizations of 50 and 100:

$$\begin{aligned} 50 &= 2 \cdot 25 \\ 100 &= 4 \cdot 25 \end{aligned}$$

The numbers 50 and 100 share the factor 25. A shared factor is called a common factor. We can rewrite the ratio $\frac{50}{100}$ as follows:

$$\frac{50}{100} = \frac{2 \cdot 25}{4 \cdot 25}$$

Given that anything multiplied by 1 equals itself and the fact that $\frac{25}{25} = 1$, we have

$$\frac{50}{100} = \frac{2 \cdot \cancel{25}}{4 \cdot \cancel{25}} = \frac{2}{4} \cdot 1 = \frac{2}{4}$$

Dividing $\frac{25}{25}$ and replacing this factor with a 1 is called *canceling*. In the previous example, you cancelled 25 from both the numerator and the denominator. Together, these basic steps for finding equivalent fractions define the process of *reducing*. Since factors divide their product evenly, we achieve the same result by dividing both the numerator and denominator by 25 as follows:

$$\frac{50 \div 25}{100 \div 25} = \frac{2}{4}$$

Finding equivalent fractions where the numerator and denominator have no common factor other than 1 is called **reducing to lowest terms**. When learning how to reduce to lowest terms, it is helpful to first rewrite the numerator and denominator as a product of primes and then cancel. For example,

$$\begin{aligned} \frac{50}{100} &= \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{5}}}{2 \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{5}}} \\ &= \frac{1}{2} \end{aligned}$$

We achieve the same result by dividing the numerator and denominator by the **greatest common factor** (GCF). The GCF is the largest number that divides both the numerator and denominator evenly. One way to find the GCF of 50 and 100 is to list all the factors of each and identify the largest number that appears in both lists. Remember, each number is also a factor of itself.

$$\begin{aligned} \{1, 2, 5, 10, 25, 50\} &\quad \text{Factors of } 50 \\ \{1, 2, 4, 5, 10, 20, 25, 50, 100\} &\quad \text{Factors of } 100 \end{aligned}$$

Common factors are listed in bold, and we see that the greatest common factor is 50. We use the following notation to indicate the GCF of two numbers: $\text{GCF}(50, 100) = 50$. After determining the GCF, reduce by dividing both the numerator and the denominator as follows:

$$\frac{50 \div 50}{100 \div 50} = \frac{1}{2}$$

Visualize

Visit the Geogebra site to practice Expanding & Reducing Fractions.
Link available on page 1.3 of the digital version of this textbook.

How To

Simplify a Fraction.

1. Rewrite the numerator and denominator to show the common factors.
   If needed, factor the numerator and denominator into prime numbers first.
2. Simplify using the Equivalent Fractions Property by dividing out common factors.
3. Multiply any remaining factors.

Example 1

Simplify:

$$\frac{105}{300}$$

Solution

Rewrite the numerator and denominator as a product of primes and then cancel.

$$\begin{aligned}\frac{105}{300} &= \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{5}} \cdot 7}{2 \cdot 2 \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{5}} \cdot 5} \\ &= \frac{7}{2 \cdot 2 \cdot 5} \\ &= \frac{7}{20}\end{aligned}$$

Alternatively, we achieve the same result if we divide both the numerator and denominator by the GCF $(105, 300)$.

A quick way to find the GCF of the two numbers requires us to first write each as a product of primes.

The GCF is the product of all the common prime factors.

$$\left.\begin{aligned}105 &= 3 \cdot 5 \cdot 7 \\ 300 &= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5\end{aligned}\right\} \text{GCF}(105, 300) = 3 \cdot 5 = 15$$

In this case, the common prime factors are 3 and 5 and the greatest common factor of 105 and 300 is 15.

$$\frac{105}{300} = \frac{105 \div 15}{300 \div 15} = \frac{7}{20}$$

Answer

$$\frac{7}{20}$$

Example 2

Reduce to lowest terms:

$$\frac{32}{96}$$

Solution

$$\begin{aligned} \frac{32}{96} &= \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} \\ &= \frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 3} \\ &= \frac{1}{3} \end{aligned}$$

Answer

$$\frac{1}{3}$$

An **improper fraction** is one where the numerator is larger than the denominator. A **mixed number** is a number that represents the sum of a whole number and a fraction. For example, $5\frac{1}{2}$ is a mixed number that represents the sum $5 + \frac{1}{2}$. Use long division to convert an improper fraction to a mixed number; the remainder is the numerator of the fractional part.

Example 3

Write:

$\frac{23}{5}$ as a mixed number.

Solution

Notice that 5 divides into 23 four times with a remainder of 3.

$$\begin{array}{r} 4 \\ 5\overline{)23} \\ 20 \\ 3 \end{array}$$

$$\begin{aligned} \frac{23}{5} &= 4 + \frac{3}{5} \\ &= 4\frac{3}{5} \end{aligned}$$

Note that the denominator of the fractional part of the mixed number remains the same as the denominator of the original fraction.

Answer

$4\frac{3}{5}$

How To

To convert mixed numbers to improper fractions, multiply the whole number by the denominator and then add the numerator; write this result over the original denominator.

Example 4

Write:

$3\frac{5}{7}$ as an improper fraction.

Solution

Obtain the numerator by multiplying 7 times 3 and then add 5.

$$\begin{aligned} 3\frac{5}{7} &= \frac{7 \cdot 3 + 5}{7} \\ &= \frac{21 + 5}{7} \\ &= \frac{26}{7} \end{aligned}$$

Answer

$\frac{26}{7}$

It is important to note that converting to a mixed number is not part of the reducing process. We consider improper fractions, such as $\frac{26}{7}$, to be reduced to lowest terms. In algebra, it is often preferable to work with improper fractions, although in some applications, mixed numbers are more appropriate.

Visualize

Visit the Geogebra site to practice Converting Between Improper Fractions & Mixed Numbers. Link available on page 1.3 of the digital version of this textbook.

Try It

Convert $10\frac{1}{2}$ to an improper fraction.

Show Solution

$$\frac{21}{2}$$

Science Application

Methane is CH_4. What fraction of methane's mass is hydrogen?

Atomic Masses in g/mol: Carbon (C) = 12, Hydrogen (H) = 1

We need to determine the ratio of hydrogen to total mass in methane.

First, calculate the molar mass for methane. This will be the denominator in our ratio:

$$12 + (1 \times 4) = 16\text{g/mol}$$

Now, determine the mass of hydrogen in a mole of methane. This will be the numerator in our ratio:

Mass of Hydrogen:

$$1 \times 4 = 4\text{g/mol}$$

The proportional mass of hydrogen in a mole of methane is:

$$\frac{4}{16} \text{ or in lowest terms } \frac{1}{4}$$

In chemistry class, you would convert this to a decimal or percentage by dividing the numerator into the denominator:

$$\frac{1}{4} = 0.25$$

One quarter, or $\frac{1}{4}$ or 25% of the mass of methane is from hydrogen.

Try It

Carbon Dioxide is CO_2. What fraction of carbon dioxide's mass is carbon?

Atomic Masses in g/mol:

Carbon (C) = 12, Oxygen (O) = 16

Show Solution

Solution: $\frac{12}{44}$ or in lowest terms $\frac{3}{11}$.

As a percent, this is $27.\overline{27}\%$.

Multiplying and Dividing Fractions

In this section, assume that $a, b, c,$ and d are all nonzero integers. The product of two fractions is the fraction formed by the product of the numerators and the product of the denominators. In other words, to multiply fractions, multiply the numerators and multiply the denominators:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example 5

Multiply:

$$\frac{2}{3} \cdot \frac{5}{7}$$

Solution

Multiply the numerators and multiply the denominators.

$$\begin{aligned} \frac{2}{3} \cdot \frac{5}{7} &= \frac{2 \cdot 5}{3 \cdot 7} \\ &= \frac{10}{21} \end{aligned}$$

Answer

$$\frac{10}{21}$$

Example 6

Multiply:

$$\frac{5}{9}\left(-\frac{1}{4}\right)$$

Solution

Recall that the product of a positive number and a negative number is negative.

$$\frac{5}{9}\left(-\frac{1}{4}\right) = -\frac{5 \cdot 1}{9 \cdot 4}$$
$$= -\frac{5}{36}$$

Answer

$-\frac{5}{36}$

Example 7

Multiply:

$$\frac{2}{3} \cdot 5\frac{3}{4}$$

Solution

Begin by converting $5\frac{3}{4}$ to an improper fraction.

$$\frac{2}{3} \cdot 5\frac{3}{4} = \frac{2}{3} \cdot \frac{23}{4}$$

$$= \frac{\overset{1}{\cancel{2}}}{3} \cdot \frac{23}{\underset{2}{\cancel{4}}}$$

$$= \frac{23}{6}$$

$$= 3\frac{5}{6}$$

In this example, we noticed that we could reduce before we multiplied the numerators and the denominators.

Reducing in this way is called **cross-cancelling**, and can save time when multiplying fractions.

Answer

$3\frac{5}{6}$

Visualize

Visit the Geogebra site to practice Multiplying & Dividing Fractions.
Link available on page 1.3 of the digital version of this textbook.

Two real numbers whose product is 1 are called **reciprocals**. Therefore, $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals because $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1$. For example,

$$\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1$$

Because their product is 1, $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals. Some other reciprocals are listed below:

$$\frac{5}{8} \text{ and } \frac{8}{5} \qquad 7 \text{ and } \frac{1}{7} \qquad -\frac{4}{5} \text{ and } -\frac{5}{4}$$

This definition is important because dividing fractions requires that you multiply the dividend by the reciprocal of the divisor.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example 8

Divide:

$$\frac{2}{3} \div \frac{5}{7}$$

Solution

Multiply $\frac{2}{3}$ by the reciprocal of $\frac{5}{7}$.

$$\begin{aligned} \frac{2}{3} \div \frac{5}{7} &= \frac{2}{3} \cdot \frac{7}{5} \\ &= \frac{2 \cdot 7}{3 \cdot 5} \\ &= \frac{14}{15} \end{aligned}$$

Answer

$$\frac{14}{15}$$

You also need to be aware of other forms of notation that indicate division: / and —. For example,

$$5/(1/2) = 5 * (2/1) = (5/1) * (2/1) = 10/1 = 10$$

Or

$$\frac{\frac{7}{8}}{\frac{2}{3}} = \frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{3}{2} = \frac{21}{16}$$

A **complex fraction**, has a numerator, denominator, or both that are fractions.

Note

Students often ask why dividing is equivalent to multiplying by the reciprocal of the divisor.

A mathematical explanation comes from the fact that the product of reciprocals is 1.

If we apply the multiplicative identity property and multiply the numerator and denominator by the reciprocal of the denominator, then we obtain the following:

$$\frac{\frac{7}{8}}{\frac{2}{3}} = \frac{\frac{7}{8}}{\frac{2}{3}} \cdot 1 = \frac{\frac{7}{8}}{\frac{2}{3}} \cdot \frac{\frac{3}{2}}{\frac{3}{2}} = \frac{\frac{7}{8} \cdot \frac{3}{2}}{1} = \frac{7}{8} \cdot \frac{3}{2}$$

Before multiplying, look for common factors to cancel; this eliminates the need to reduce the end result.

Example 9

Divide:

$$\frac{\frac{5}{2}}{\frac{7}{4}}$$

Solution

$$\frac{\frac{5}{2}}{\frac{7}{4}} = \frac{5}{2} \div \frac{7}{4}$$

$$= \frac{5}{2} \cdot \frac{4}{7}$$

$$= \frac{5}{\cancel{2}_{1}} \cdot \frac{\cancel{4}^{2}}{7}$$

$$= \frac{5 \cdot 2}{1 \cdot 7}$$

$$= \frac{10}{7}$$

Answer

$\frac{10}{7}$

When dividing by an integer, it is helpful to rewrite it as a fraction over 1.

Example 10

Divide:

$$\frac{2}{3} \div 6$$

Solution

Rewrite 6 as $\frac{6}{1}$ and multiply by its reciprocal.

$$\frac{2}{3} \div 6 = \frac{2}{3} \div \frac{6}{1}$$

$$= \frac{2}{3} \cdot \frac{1}{6}$$

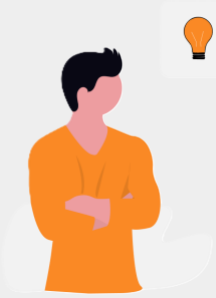
$$= \frac{\overset{1}{\cancel{2}}}{3} \cdot \frac{1}{\underset{3}{\cancel{6}}}$$

$$= \frac{1 \cdot 1}{3 \cdot 3}$$

$$= \frac{1}{9}$$

Answer

$\frac{1}{9}$



Note

We only cancel when working with multiplication. Rewrite any division problem as a product *before* cancelling.

Visualize

Visit the Geogebra site to practice Dividing Fractions Using Visual Models. Link available on page 1.3 of the digital version of this textbook.

Try It

Divide: $5 \div 2\frac{3}{5}$

Show Solution

$1\frac{12}{13}$

Science Application

Calcium Carbonate $CaCO_3$ is a solid found in rocks like limestone and in chalk. You have a solid chunk of it that weighs 500 grams. Find the Mass of Calcium in the 500g Sample.

Atomic masses (in g/mol):

Calcium (Ca): 40, Carbon (C): 12, Oxygen (O): 16

First, you have to find the molar mass of calcium carbonate using the atomic masses

$$\text{Total Mass} = 40 + 12 + (16 \times 3)$$
$$\text{Total Mass} = 40 + 12 + 48 = 100\text{g/mol}$$

Now, write the fraction of calcium in the total molecular mass:

The mass of Calcium is 40 out of the total 100: $\frac{40}{100}$

(If you were in the chemistry lab, you would convert to a decimal at this point. Let's keep working with fractions for now.)

Simplify the fraction.

Both can be divided by 20.

$$\frac{40 \div 20}{100 \div 20} = \frac{2}{5}$$

Calculate the mass in the 500g sample:

$$500 \times \frac{2}{5} = \frac{1,000}{5} = 200 \text{ grams of Calcium}$$

Important! Take a step back and ask if this mass of calcium makes sense. You had 500 g of rock, and 40% (or a bit less than half) was calcium. A bit less than half of 500 g is 200 g. Makes sense!

Try it

Iron(III) Oxide Fe_2O_3 is the solid chemical name for common rust. You have a rusty iron plate where the rust itself weighs 160 grams. Find the mass of iron (III) in the rust.

Atomic masses (in g/mol):

Iron (Fe): 56, Oxygen (O): 16,

Show Solution

112 g of iron (III).

Adding and Subtracting Fractions

Negative fractions are indicated with the negative sign in front of the fraction bar, in the numerator, or in the denominator. All such forms are equivalent and interchangeable.

$$\frac{-3}{4} = -\frac{3}{4} = \frac{3}{-4}$$

Adding or subtracting fractions requires a **common denominator**. In this section, assume the common denominator c is a nonzero integer.

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$$

It is good practice to use positive common denominators by expressing negative fractions with negative numerators. In short, avoid negative denominators.

Example 11

Subtract:

$$\frac{12}{15} - \frac{3}{15}$$

Solution

The two fractions have a common denominator 15. Therefore, subtract the numerators and write the result over the common denominator:

$$\begin{aligned} \frac{12}{15} - \frac{3}{15} &= \frac{12-3}{15} && \text{Subtract the numerators.} \\ &= \frac{9}{15} \\ &= \frac{9 \div 3}{15 \div 3} && \text{Reduce.} \\ &= \frac{3}{5} \end{aligned}$$

Answer

$$\frac{3}{5}$$

Most problems that you are likely to encounter will have **unlike denominators**. In this case, first find equivalent fractions with a common denominator before adding or subtracting the numerators. One way to obtain equivalent fractions is to divide the numerator and the denominator by the same number. We can find equivalent fractions by multiplying the numerator and the denominator by the same number.

It should be clear that $5/5$ is equal to 1 and that 1 multiplied times any number is that number:

$$\frac{1}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10}$$

We have equivalent fractions $\frac{1}{2} = \frac{5}{10}$.

Use this idea to find equivalent fractions with a common denominator to add or subtract fractions. The steps are outlined in the following example.

Example 12

Simplify:

$$\frac{7}{15} - \frac{3}{10}$$

Solution:

Step 1: Determine a common denominator. To do this, use the **least common multiple** (LCM) of the given denominators. The LCM of 15 and 10 is indicated by LCM$(15, 10)$. Try to think of the smallest number that both denominators divide into evenly.

List the multiples of each number:

$$\{10, 20, \mathbf{30}, 40, 50, \mathbf{60}, \ldots\} \text{ Multiples of } 10$$
$$\{15, \mathbf{30}, 45, \mathbf{60}, 75, 90, \ldots\} \text{ Multiples of } 15$$

Common multiples are listed in bold, and the least common multiple is 30.

$$\text{LCM}(10, 15) = 30$$

Step 2: Multiply the numerator and the denominator of each fraction by values that result in equivalent fractions with the determined common denominator.

$$\frac{7}{15} - \frac{3}{10} = \frac{7 \cdot 2}{15 \cdot 2} - \frac{3 \cdot 3}{10 \cdot 3}$$

$$= \frac{14}{30} - \frac{9}{30}$$

Step 3: Add or subtract the numerators, write the result over the common denominator and then reduce if possible.

$$\frac{14}{30} - \frac{9}{30} = \frac{14-9}{30}$$

$$= \frac{5}{30}$$

$$= \frac{5 \div 5}{30 \div 5}$$

$$= \frac{1}{6}$$

Answer

$\frac{1}{6}$

The least common multiple of the denominators is called the **least common denominator** (LCD). Finding the LCD is often the difficult step. It is worth finding because if any common multiple other than the least is used, then there will be more steps involved when reducing.

Example 13

Simplify:

$$\frac{5}{10} + \frac{1}{18}$$

Solution

First, determine that the LCM$(10, 18)$ is 90 and then find equivalent fractions with 90 as the denominator.

$$\frac{5}{10} + \frac{1}{18} = \frac{5 \cdot 9}{10 \cdot 9} + \frac{1 \cdot 5}{18 \cdot 5}$$

$$= \frac{45}{90} + \frac{5}{90}$$

$$= \frac{45 + 5}{90}$$

$$= \frac{50}{90}$$

$$= \frac{50 \div 10}{90 \div 10}$$

$$= \frac{5}{9}$$

Answer

$\frac{5}{9}$

Try It

Add: $\frac{2}{30} + \frac{5}{21}$

Show Solution

$\frac{32}{105}$

Example 14

Simplify:

$$2\frac{1}{3}+\frac{3}{5}-\frac{1}{2}$$

Solution

Begin by converting $2\frac{1}{3}$ to an improper fraction.

$$2\frac{1}{3}+\frac{3}{5}-\frac{1}{2}=\frac{7}{3}+\frac{3}{5}-\frac{1}{2}$$ Convert to improper fractions

$$=\frac{7\cdot 10}{3\cdot 10}+\frac{3\cdot 6}{5\cdot 6}-\frac{1\cdot 15}{2\cdot 15}$$ $\text{LCM}(3,5,2)=30$

$$=\frac{70}{30}+\frac{18}{30}-\frac{15}{30}$$ Equivalent fractions with a common denominator

$$=\frac{70+18-15}{30}$$

$$=\frac{73}{30}$$

$$=2\frac{13}{30}$$

Answer

$$2\frac{13}{30}$$

In general, it is preferable to work with improper fractions. However, when the original problem involves mixed numbers, if appropriate, present your answers as mixed numbers. Also, mixed numbers are often preferred when working with numbers on a number line and with real-world applications.

How To

Add or Subtract Fractions.

1. **Step 1**. Do they have a common denominator?
    - Yes—go to step 2.
    - No—rewrite each fraction with the LCD (least common denominator).
        - Find the LCD.
        - Change each fraction into an equivalent fraction with the LCD as its denominator.
2. **Step 2**. Add or subtract the fractions.
3. **Step 3**. Simplify, if possible.

Try It

Simplify: $\frac{5}{7} - 2\frac{1}{7}$

Show Solution

$-1\frac{3}{7}$

Example 15

How many $\frac{1}{2}$ inch thick paperback books can be stacked to fit on a shelf that is $1\frac{1}{2}$ feet in height?

Solution

First, determine the height of the shelf in inches.

Conversion to inches:
To do this, use the fact that there are 12 inches in 1 foot and multiply as follows:

$$\begin{aligned} 1\frac{1}{2}\text{ft} &= 1\frac{1}{2}\text{ft} \cdot \left(\frac{12\text{in}}{1\text{ft}}\right) \\ &= \frac{3}{\cancel{2}_{1}} \cdot \frac{\cancel{12}^{6}}{1}\text{in} \\ &= 18\text{in} \end{aligned}$$

Next, determine how many notebooks will fit by dividing the height of the shelf by the thickness of each book.

Division to find the number of books:

$$\begin{aligned} 18\text{in} \div \frac{1}{2}\text{in} &= 18 \cdot \frac{2}{1} \\ &= 18 \cdot 2 \\ &= 36 \end{aligned}$$

Answer

36 books can be stacked on the shelf.

Key Takeaways

- Fractions are not unique; there are many ways to express the same ratio. Find equivalent fractions

by multiplying or dividing the numerator and the denominator by the same real number.

- Equivalent fractions in lowest terms are generally preferred. It is a good practice to always reduce.
- In algebra, improper fractions are generally preferred. However, in real-life applications, mixed number equivalents are often preferred. We may present answers as improper fractions unless the original question contains mixed numbers, or it is an answer to a real-world or geometric application.
- Multiplying fractions does not require a common denominator; multiply the numerators and multiply the denominators to obtain the product. It is a best practice to cancel any common factors in the numerator and the denominator before multiplying.
- Reciprocals are rational numbers whose product is equal to 1. Given a fraction $\frac{a}{b}$, its reciprocal is $\frac{b}{a}$.
- Divide fractions by multiplying the dividend by the reciprocal of the divisor.
- Rewrite any division problem as a product *before* cancelling.
- Adding or subtracting fractions requires a common denominator. When the denominators of any number of fractions are the same, simply add or subtract the numerators and write the result over the common denominator.
- Before adding or subtracting fractions, ensure that the denominators are the same by finding equivalent fractions with a common denominator. Multiply the numerator and the denominator of each fraction by the appropriate value to find the equivalent fractions.
- Typically, it is best to convert all mixed numbers to improper fractions before beginning the process of adding, subtracting, multiplying, or dividing.

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1.4 Exercises

1.1 Real Numbers: Algebra Essentials

1. Is $\sqrt{2}$ an example of a rational terminating, rational repeating, or irrational number? Tell why it fits that category.
2. What is the order of operations? What acronym is used to describe the order of operations, and what does it stand for?
3. What do the Associative Properties allow us to do when following the order of operations? Explain your answer.

For the following exercises, simplify the given expression.

4. $10 + 2 \times (5 - 3)$
5. $6 \div 2 - \left(81 \div 3^2\right)$
6. $18 + (6 - 8)^3$
7. $-2 \times \left[16 \div (8 - 4)^2\right]^2$
8. $4 - 6 + 2 \times 7$
9. $3(5 - 8)$
10. $4 + 6 - 10 \div 2$
11. $12 \div (36 \div 9) + 6$
12. $(4 + 5)^2 \div 3$
13. $3 - 12 \times 2 + 19$
14. $2 + 8 \times 7 \div 4$

15. $5+(6+4)-11$

16. $9-18 \div 3^2$

17. $14 \times 3 \div 7-6$

18. $9-(3+11) \times 2$

19. $6+2 \times 2-1$

20. $64 \div(8+4 \times 2)$

21. $9+4\left(2^2\right)$

22. $(12 \div 3 \times 3)^2$

23. $25 \div 5^2-7$

24. $(15-7) \times(3-7)$

25. $2 \times 4-9(-1)$

26. $4^2-25 \times \frac{1}{5}$

27. $12(3-1) \div 6$

28. If a whole number is not a natural number, what must the number be?

29. Determine whether the statement is true or false: The multiplicative inverse of a rational number is also rational.

30. Determine whether the statement is true or false: The product of a rational and irrational number is always irrational.

31. Determine whether the simplified expression is rational or irrational: $\sqrt{-18-4(5)(-1)}$

32. Determine whether the simplified expression is rational or irrational: $\sqrt{-16+4(5)+5}$

33. The division of two whole numbers will always result in what type of number?

34. What property of real numbers would simplify the following expression: $4+7(x-1)$?

Show Solutions

1. Irrational number. The square root of two does not terminate, and it does not repeat a pattern. It cannot be written as a quotient of two integers, so it is irrational.

2. The order of operations is parentheses, exponents, multiplication and division from left to right, and addition and subtraction from left to right. The acronym PEMDAS stands for Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction.

3. The Associative Properties state that the sum or product of multiple numbers can be grouped differently without affecting the result. Associative Properties allow regrouping of sums or products without changing the value.

4. 14

5. -6

6. 10

7. -2

8. 12

9. -9

10. 5

11. 9

12. 27

13. -2

14. 16

15. 4

16. 7

17. 0

18. -19

19. 9
20. 4
21. 25
22. 144
23. -6
24. -32
25. 17
26. 11
27. 4
28. The number must be 0.
29. True
30. False. If the rational number is 0, the product is 0, which is rational.
31. Irrational
32. Rational
33. Rational number
34. The Distributive Property

1.2 Order of Operations

Simplify.

1. $-7-3\cdot 5$
2. $3+2\cdot 3$
3. $-3(2)-6^2$
4. $2(-3)^2+5(-4)$
5. $\dfrac{6}{3}\cdot 2$
6. $\dfrac{6}{3\cdot 2}$
7. $-\dfrac{1}{2}-\dfrac{3}{5}\cdot\dfrac{2}{3}$
8. $\dfrac{5}{8}\div\dfrac{1}{2}-\dfrac{5}{6}$
9. $3.2^2-6.9\div 2.3$
10. $8.2-3\div 1.2\cdot 2.1$
11. $2+3(-2)-7$
12. $8\div 2-3\cdot 2$
13. $3+6^2\div 12$
14. $5-4^2\div(-8)$
15. $-9-3\cdot 2\div 3(-2)$
16. $-2-3^2+(-2)^2$
17. $12\div 6\cdot 2-2^2$
18. $4\cdot 3\div 12\cdot 2-(-2)^2$
19. $(-5)^2-2(5)^2\div 10$
20. $-3(4-7)+2$

21. $(-2+7)^2-10^2$

22. $10-7(3+2)+7^2$

23. $-7-3(4-2\cdot 8)$

24. $5-3[6-(2+7)]$

25. $1+2[(-2)^3-(-3)^2]$

26. $-3[2(7-5)\div 4\cdot(-2)+(-3)^3]$

27. $-7^2-[-20-(-3)^2]-(-10)$

28. $4.7-3.2(4-1.2^3)$

29. $-5.4(6.1-3.1\div 0.1)-8.2^2$

30. $-7.3^2+(-9.3)^2-37.8\div 1.8$

31. $2-7(3^2-3+4\cdot 3)$

32. $\left(\frac{1}{2}\right)^2-\left(-\frac{2}{3}\right)^2$

33. $\left(\frac{1}{2}\right)^3+(-2)^3$

34. $\left(-\frac{1}{3}\right)^2-\left(-\frac{2}{3}\right)^3$

35. $\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{5}$

36. $\frac{5}{8}\div\frac{3}{2}\cdot\frac{14}{15}$

37. $5\cdot\frac{2}{15}-\left(\frac{1}{2}\right)^3$

38. $\frac{5}{17}\left(\frac{3}{5}-\frac{4}{35}\right)$

39. $\frac{3}{16} \div \left(\frac{5}{12}-\frac{1}{2}+\frac{2}{3}\right) \cdot 4$

40. $\left(\frac{2}{3}\right)^2-\left(\frac{1}{2}\right)^2$

41. $\frac{1}{2}\left[\frac{3}{4} \cdot (-4)^2-2\right]^2$

42. $6 \cdot \left[\left(\frac{2}{3}\right)^2-\left(\frac{1}{2}\right)^2\right] \div (-2)^2$

43. $(-5)^2+\frac{3}{2}-\frac{4}{2}+2 \cdot 7$

44. $(-3.2-3.3)(8.7-4.7)(-4.7+3.9+2.1)$

45. $2-\left[3-(5-7)^2\right] \cdot 3(6-32)$

46. $\frac{2+3 \cdot 6-4 \cdot 32}{2-32}$

47. $(2+7) \cdot 2-\frac{23}{10}+\frac{9}{2}+3^3$

48. $\frac{(-1-3)^2-1}{5-3 \cdot (-7+22)-5}$

49. $\frac{(7+4(-2))}{(-3+(-2)^2)}$

50. $4+3 \cdot \frac{(-3)^3+5^2}{6}-2^2$

51. Mary purchased 14 bottles of water at $\$0.75$ per bottle, 4 pounds of assorted candy at $\$3.50$ per pound, and 16 packages of microwave popcorn costing $\$0.50$ each for her party. What was her total bill?

52. Joe bought four 8-foot 2-by-4 boards for $\$24.00$. How much did he spend per linear foot?

53. Margaret bought two cases of soda at the local discount store for $\$23.52$. If each case contained 24 bottles, how much did she spend per bottle?

54. Billy earns $\$12.00$ per hour and "time and a half" for every hour he works over 40 hours a week. What is his pay for 47 hours of work this week?

55. Audry bought 4 bags of marbles each containing 15 assorted marbles. If she wishes to divide them up evenly between her 3 children, how many will each child receive?

56. Mark and Janet carpooled home from college for the Thanksgiving holiday. They shared the driving, but Mark drove twice as far as Janet. If Janet drove 135 miles, then how many miles was the entire trip?

Show Solutions

1. -22
2. 9
3. -42
4. -2
5. 4
6. 1
7. $-\frac{9}{10}$
8. $\frac{5}{12}$
9. 7.24
10. 2.95
11. -11
12. -2

13. 6
14. 7
15. -5
16. -7
17. 0
18. -2
19. 20
20. 11
21. -75
22. 24
23. 29
24. 14
25. -33
26. 87
27. -10
28. -2.5704
29. 67.22
30. 12.2
31. -124
32. $-\frac{7}{36}$

33. $-\frac{63}{8}$
34. $\frac{11}{27}$
35. $\frac{7}{30}$
36. $\frac{7}{18}$
37. $\frac{13}{24}$
38. $\frac{1}{7}$
39. $\frac{9}{7}$
40. $\frac{7}{36}$
41. 50
42. $\frac{7}{24}$
43. 38.5
44. -33.8
45. -76
46. $\frac{18}{5}$
47. 47.2
48. $-\frac{1}{3}$

49. -1
50. -1
51. $\$32.50$
52. $\$0.75$
53. $\$0.49$
54. $\$606.00$
55. 20 marbles
56. 405 miles

1.3 Fractions

Reduce each fraction to its lowest terms.

1. $\frac{5}{30}$
2. $\frac{6}{24}$
3. $\frac{30}{70}$
4. $\frac{18}{27}$
5. $\frac{44}{84}$
6. $\frac{54}{90}$

7. $\frac{135}{30}$
8. $\frac{105}{300}$
9. $\frac{18}{6}$
10. $\frac{256}{16}$
11. $\frac{126}{45}$
12. $\frac{52}{234}$
13. $\frac{54}{162}$
14. $\frac{2000}{3000}$
15. $\frac{270}{360}$

Rewrite as an improper fraction.

16. $2\frac{1}{2}$
17. $4\frac{3}{4}$
18. $1\frac{1}{2}$
19. $5\frac{7}{15}$
20. $1\frac{3}{4}$

21. $3\frac{5}{8}$

22. $-1\frac{3}{4}$

23. $-2\frac{1}{2}$

Rewrite as a mixed number.

24. $\frac{15}{2}$

25. $\frac{9}{2}$

26. $\frac{40}{13}$

27. $\frac{103}{25}$

28. $\frac{73}{10}$

29. $\frac{-52}{7}$

30. $\frac{-59}{6}$

Multiply and reduce to lowest terms.

31. $\frac{2}{3} \cdot \frac{5}{7}$

32. $\frac{1}{5} \cdot \frac{4}{8}$

33. $\frac{1}{2} \cdot \frac{1}{3}$

34. $\frac{3}{4} \cdot \frac{20}{9}$

35. $\frac{5}{7} \cdot \frac{49}{10}$

36. $\frac{2}{3} \cdot \frac{9}{12}$

37. $\frac{6}{14} \cdot \frac{21}{12}$

38. $\frac{44}{15} \cdot \frac{15}{11}$

39. $3\frac{3}{4} \cdot 2\frac{1}{3}$

40. $2\frac{7}{10} \cdot 5\frac{5}{6}$

41. $\frac{3}{11}\left(-\frac{5}{2}\right)$

42. $-\frac{4}{5}\left(\frac{9}{5}\right)$

43. $\left(-\frac{9}{5}\right)\left(-\frac{3}{10}\right)$

44. $\frac{6}{7}\left(-\frac{14}{3}\right)$

45. $\left(-\frac{9}{12}\right)\left(-\frac{4}{8}\right)$

46. $\left(-\frac{3}{8}\right)\left(-\frac{4}{15}\right)$

47. $\frac{1}{7} \cdot \frac{1}{2} \cdot \frac{1}{3}$

48. $\frac{3}{5} \cdot \frac{15}{21} \cdot \frac{7}{27}$

49. $\frac{2}{5} \cdot 3\frac{1}{8} \cdot \frac{4}{5}$

50. $2\frac{4}{9} \cdot \frac{2}{5} \cdot 2\frac{5}{11}$

Determine the reciprocal of the following numbers.

51. $\frac{1}{2}$

52. $\frac{8}{5}$

53. $-\frac{2}{3}$

54. $-\frac{4}{3}$

55. 10

56. -4

57. $2\frac{1}{3}$

58. $1\frac{5}{8}$

Divide and reduce to lowest terms.

59. $\frac{1}{2} \div \frac{2}{3}$

60. $\frac{5}{9} \div \frac{1}{3}$

61. $\frac{5}{8} \div \left(-\frac{4}{5}\right)$

62. $\left(-\frac{2}{5}\right) \div \frac{15}{3}$

63. $\dfrac{-\frac{6}{7}}{-\frac{6}{7}}$

64. $\dfrac{-\frac{1}{2}}{\frac{1}{4}}$

65. $\dfrac{-\frac{10}{3}}{-\frac{5}{20}}$

66. $\dfrac{\frac{2}{3}}{\frac{9}{2}}$

67. $\dfrac{\frac{30}{50}}{\frac{5}{3}}$

68. $\dfrac{\frac{1}{2}}{2}$

69. $\dfrac{5}{\frac{2}{5}}$

70. $\dfrac{-6}{\frac{5}{4}}$

71. $2\dfrac{1}{2} \div \dfrac{5}{3}$

72. $4\dfrac{2}{3} \div 3\dfrac{1}{2}$

73. $5 \div 2\dfrac{3}{5}$

74. $4\dfrac{3}{5} \div 23$

Add or subtract and reduce to lowest terms.

75. $\dfrac{17}{20} - \dfrac{5}{20}$

76. $\frac{4}{9} - \frac{13}{9}$

77. $\frac{3}{5} + \frac{1}{5}$

78. $\frac{11}{15} + \frac{9}{15}$

79. $\frac{5}{7} - 2\frac{1}{7}$

80. $5\frac{1}{8} - 1\frac{1}{8}$

81. $\frac{1}{2} + \frac{1}{3}$

82. $\frac{1}{5} - \frac{1}{4}$

83. $\frac{3}{4} - \frac{5}{2}$

84. $\frac{3}{8} + \frac{7}{16}$

85. $\frac{7}{15} - \frac{3}{10}$

86. $\frac{3}{10} + \frac{2}{14}$

87. $\frac{2}{30} + \frac{5}{21}$

88. $\frac{3}{18} - \frac{1}{24}$

89. $5\frac{1}{2} + 2\frac{1}{3}$

90. $1\frac{3}{4} + 2\frac{1}{10}$

91. $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$

92. $\frac{2}{3}+\frac{3}{5}-\frac{2}{9}$

93. $\frac{7}{3}-\frac{3}{2}+\frac{2}{15}$

94. $\frac{9}{4}-\frac{3}{2}+\frac{3}{8}$

95. $1\frac{1}{3}+2\frac{2}{5}-1\frac{1}{15}$

96. $\frac{2}{3}-4\frac{1}{2}+3\frac{1}{6}$

97. $1-\frac{6}{16}+\frac{3}{18}$

98. $3-\frac{1}{21}-\frac{1}{15}$

Perform the operations. Reduce answers to lowest terms.

99. $\frac{3}{14}\cdot\frac{7}{3}\div\frac{1}{8}$

100. $\frac{1}{2}\cdot\left(-\frac{4}{5}\right)\div\frac{14}{15}$

101. $\frac{1}{2}\div\frac{3}{4}\cdot\frac{1}{5}$

102. $-\frac{5}{9}\div\frac{5}{3}\cdot\frac{5}{2}$

103. $\frac{9}{21}+\frac{3}{9}$

104. $\frac{4}{5}\div 4\cdot\frac{1}{2}$

105. $\frac{5}{3} \div 15 \cdot \frac{2}{3}$

106. In a mathematics class of 44 students, one-quarter of the students signed up for a special Saturday study session. How many students signed up?

107. Determine the length of fencing needed to enclose a rectangular pen with dimensions $35\frac{1}{2}$ feet by $20\frac{2}{3}$ m.

108. Each lap around the track measures $\frac{1}{4}$ mile. How many laps are required to complete a $2\frac{1}{2}$ mile run?

109. A retiree earned a pension that consists of three-fourths of his regular monthly salary. If his regular monthly salary was $\$5,200$, then what monthly payment can the retiree expect from the pension plan?

Show Solutions

1. $\frac{1}{6}$
2. $\frac{1}{4}$
3. $\frac{3}{7}$
4. $\frac{2}{3}$
5. $\frac{11}{21}$
6. $\frac{3}{5}$
7. $\frac{9}{2}$

8. $\frac{7}{20}$
9. 3
10. 16
11. $\frac{14}{5}$
12. $\frac{2}{9}$
13. $\frac{1}{3}$
14. $\frac{2}{3}$
15. $\frac{3}{4}$
16. $\frac{5}{2}$
17. $\frac{19}{4}$
18. $\frac{3}{2}$
19. $\frac{82}{15}$
20. $\frac{7}{4}$
21. $\frac{29}{8}$
22. $-\frac{7}{4}$

23. $-\frac{5}{2}$
24. $7\frac{1}{2}$
25. $4\frac{1}{2}$
26. $3\frac{1}{13}$
27. $4\frac{3}{25}$
28. $7\frac{3}{10}$
29. $-7\frac{3}{7}$
30. $-9\frac{5}{6}$
31. $\frac{10}{21}$
32. $\frac{1}{10}$
33. $\frac{1}{6}$
34. $\frac{5}{3}$
35. $\frac{7}{2}$
36. $\frac{1}{2}$

37. $\frac{3}{4}$

38. 4

39. $8\frac{3}{4}$

40. $15\frac{3}{4}$

41. $\frac{-15}{22}$

42. $\frac{-36}{25}$

43. $\frac{27}{50}$

44. -4

45. $\frac{3}{8}$

46. $\frac{1}{10}$

47. $\frac{1}{42}$

48. $\frac{1}{9}$

49. 1

50. $\frac{12}{5}$

51. 2

52. $\frac{5}{8}$

53. $-\frac{3}{2}$
54. $-\frac{3}{4}$
55. $\frac{1}{10}$
56. $-\frac{1}{4}$
57. $\frac{3}{7}$
58. $\frac{8}{13}$
59. $\frac{3}{4}$
60. $\frac{5}{3}$
61. $\frac{-25}{32}$
62. $-\frac{2}{25}$
63. 1
64. -2
65. $\frac{40}{3}$
66. $\frac{4}{27}$
67. $\frac{9}{25}$

68. $\frac{1}{4}$

69. $\frac{25}{2}$

70. $-\frac{24}{5}$

71. $1\frac{1}{2}$

72. $1\frac{1}{3}$

73. $\frac{25}{13}$

74. $\frac{1}{5}$

75. $\frac{3}{5}$

76. -1

77. $\frac{4}{5}$

78. $\frac{4}{3}$

79. $-1\frac{3}{7}$

80. 4

81. $\frac{5}{6}$

82. $-\frac{1}{20}$

83. $-\frac{7}{4}$
84. $\frac{13}{16}$
85. $\frac{1}{6}$
86. $\frac{31}{70}$
87. $\frac{32}{105}$
88. $\frac{1}{8}$
89. $7\frac{5}{6}$
90. $3\frac{17}{20}$
91. 1
92. $1\frac{2}{45}$
93. $\frac{29}{30}$
94. $1\frac{1}{8}$
95. $2\frac{2}{3}$
96. $-\frac{2}{3}$
97. $\frac{19}{24}$

98. $2\frac{31}{35}$
99. 4
100. $-\frac{3}{7}$
101. $\frac{2}{15}$
102. $-\frac{5}{6}$
103. $\frac{16}{21}$
104. $\frac{1}{10}$
105. $\frac{2}{27}$
106. 11 students
107. $112\frac{1}{3}$ m
108. 10 laps
109. $\$3,900$

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CHAPTER 2: MEASUREMENTS AND RADICALS

Chapter Outline

2.1 Units of Measure

- Units of Length
- Units of Mass
- Units of Volume
- Units of Substance
- Derived Units
- American Units to Metric Units and Vice Versa

2.2 Exponents and Scientific Notation

- Exponential Notation and Positive Integer Exponents
- Using the Product Rule of Exponents
- Using the Quotient Rule of Exponents
- Using the Power Rule of Exponents
- Using the Zero Exponent Rule of Exponents
- Using the Negative Rule of Exponents
- Scientific Notation
- Key Equations

2.3 Roots and Radicals

- Square Roots
- Cube Roots
- n^th Roots
- Simplifying Using the Product and Quotient Rule for Radicals

2.4 Exercises

- 2.1 Units of Measure
- 2.2 Exponents and Scientific Notation
- 2.3 Roots and Radicals

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2.1 Units of Measure

Learning Objectives

By the end of this section, you should be able to:

- Convert between different systems of measurement used in technical fields.

The **metric system** of units is the standard system of units preferred by scientists. It is based on the **base ten number system** and its decimal format is more friendly to users of this system. There is a common set of prefixes adopted by the metric system to indicate a **power** of ten to apply to the **base** unit.

Table 2.1.1: Metric Units

Measurement	Unit Name	Symbol	What it measures
Length	meter	m	Distance or height.
Mass	kilogram	kg	How much matter is in an object.
Time	second	s	Duration or how long something takes.
Amount of substance	mole	mol	Number of particles (atoms, molecules).
Temperature	kelvin	K	How hot or cold something is.
Electric Current	ampere	A	The flow of electricity.
Luminous Intensity	candela	cd	The brightness of a light source.

All other units that you know are **derived**, meaning they are from combinations of units through multiplication, division and exponents. Think of area and volume, which are distances multiplied by distances; speed is distance divided by time; concentration is mass divided by volume or mass and moles divided by volume.

Units of Length

The standard measure of length in the metric system is the meter. Historically, the meter was defined by the French Academy of Sciences as the length between two marks on a platinum-iridium bar, which was designed to represent 110,000,000 of the distance from the equator to the north pole through Paris. In 1983, it was redefined by the International Bureau of Weights and Measures (BIPM) as the distance travelled by light in free space in 299,792,458 of a second ("History of the the metre", 2025).

We can apply the standard prefixes to get the following result.

Metric Units of Length

These units of length are used in the metric system.

Table 2.1.2: Length Units and Abbreviations

Unit Length	Unit Abbreviation
1 kilometer = 1000 meters	km
1 hectometer = 100 meters	hm
1 dekameter = 10 meters	dam
1 meter	m
1 decimeter = $\frac{1}{10}$ meter	dm
1 centimeter = $\frac{1}{100}$ meter	cm
1 millimeter = $\frac{1}{1000}$ meter	mm
1 micrometer= $\frac{1}{1000000}$ meter	μm

We can use these facts to build **conversion factors**. For example, because

$$1\text{km} = 1000\text{m}$$

We can divide both sides by 1000 m to produce the conversion factor

$$1 = \frac{1\text{ km}}{1000\text{ m}}$$

This conversion factor can help change meters into kilometres.

Before using this conversion factor in an example, we repeat here the rules for multiplying and dividing by powers of ten. We will be making heavy use of these rules in this section.

Multiplying and Dividing by Powers of Ten

Multiplying a decimal number by 10^n will move the decimal point n places to the right. For example,

$3.2567 \cdot 10^2 = 3.2567 \cdot 100 = 325.67$

Dividing a decimal number by 10^n will move the decimal point n places to the left. For example,

$3.2567 \div 10^2 = 3.2567 \div 100 = 0.032567$

And now some examples.

Example 1

Change:

2,326 meters to kilometres.

Solution:
Multiply by the conversion factor 1 km/1000 m.

$$2326\ \text{m} = 2326\ \text{m} \cdot \frac{1\ \text{km}}{1000\ \text{m}} \qquad \text{Apply conversion factor.}$$

$$= 2326\ \cancel{\text{m}} \cdot \frac{1\ \text{km}}{1000\ \cancel{\text{m}}} \qquad \text{Cancel common units.}$$

$$= \frac{2326 \cdot 1}{1000}\ \text{km} \qquad \text{Multiply fractions.}$$

$$= 2.326\ \text{km} \qquad \text{Simplify.}$$

In the last step, note that dividing by 1000 moves the decimal point three places to the left. Thus, 2326 meters is equal to 2.326 kilometres.

Alternate Solution:
A second solution depends upon the fact that multiplying or dividing by a power of ten will move the decimal point right or left a number of places equal to the number of zeros present in the multiplier or divisor. Thus, as we saw above, dividing by 1000 moved the decimal point 3 places to the left. Suppose that we arrange the metric units of length in order, from largest to smallest, as shown below.

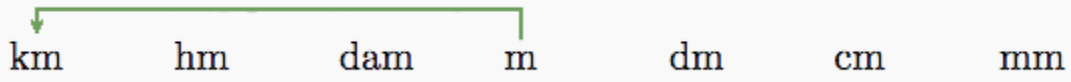

Figure 2.1.1.

Note that we must move 3 places left to move from the meters (m) abbreviation to the kilometres (km) abbreviation. In like manner, if we write 2,326 meters as 2,326.0 meters, then we can convert to kilometres by moving the decimal 3 places to the left.

$$2,326.0\,\text{m} = 2.236\,0\,\text{km} = 2.326\,\text{km}$$

Figure 2.1.2.

Answer

2.326km

Example 2

Change:

537 centimetres to meters

Solution:

$$1\text{ cm} = \frac{1}{100}\text{ m}$$

Or multiplying both sides of this result by 100,

$$100\text{ cm} = 1\text{ m}$$

Dividing both sides of this last result by 100 cm, we obtain the conversion factor 1 m/100 cm.

$$\begin{aligned} 537\text{ cm} &= 537\text{ cm} \cdot \frac{1\text{ m}}{100\text{ cm}} && \text{Apply conversion factor.} \\ &= 537\ \cancel{\text{cm}} \cdot \frac{1\text{ m}}{100\ \cancel{\text{cm}}} && \text{Cancel common units.} \\ &= \frac{537 \cdot 1}{100}\text{ m} && \text{Multiply fractions.} \\ &= 5.37\text{ m} \end{aligned}$$

In the last step, note that dividing by 100 moves the decimal point two places to the left. Alternatively, we can set up our ordered list of units.

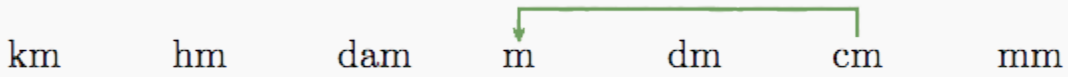

Figure 2.1.3.

Note that we must move 2 places left to move from the centimetres (cm) abbreviation to the meters (m) abbreviation. In like manner, if we write 537 centimetres as 537.0 centimetres, then we can convert to meters by moving the decimal 2 places to the left.

$$537.0\,\text{cm} = 5.37\,0\,\text{m} = 5.37\,\text{m}$$

Figure 2.1.4.

Sometimes more than one conversion factor is needed.

Answer

5.37m

Try It

Change 276 centimetres to meters.

Show Solution

2.76 meters

Units of Mass

The fundamental unit of mass in the metric system is called a gram. Originally, it was defined to be equal to one cubic centimetre of water measured at the temperature of melting ice. Now it is simply defined as 1/1000 of a kilogram, which is defined by a physical prototype preserved by the International Bureau of Weights and Measures ("Gram", 2025). The mass of an object is not the same as an object's weight. Rather mass is a resistance to motion when an external force is applied.

The same metric system prefixes apply.

Metric Units of Mass

These units of mass are commonly used in the metric system.

Table 2.1.3: Mass Units and Abbreviations

Units of Mass	Unit Abbreviation
1 kilogram = 1000 grams	kg
1 gram	g
1 milligram = $\frac{1}{1000}$ gram	mg
1 microgram= $\frac{1}{10^6}$	µg

Example 3

Convert:

0.025 dekagrams to milligrams.

Solution:
We'll use two conversion factors:

- 1 dag = 10 g, which yields the conversion factor 10 g/1 dag.
- 1 mg = (1/1000) g, which yields the conversion factor 1000 mg/1 g.

$$\begin{aligned} 0.025\ \text{dag} &= 0.025\ \text{dag} \cdot \frac{10\ \text{g}}{1\ \text{dag}} \cdot \frac{1000\ \text{mg}}{1\ \text{g}} \\ &= 0.025\ \cancel{\text{dag}} \cdot \frac{10\ \cancel{\text{g}}}{1\ \cancel{\text{dag}}} \cdot \frac{1000\ \text{mg}}{1\ \cancel{\text{g}}} \\ &= (0.025 \cdot 10 \cdot 1000)\ \text{mg} \\ &= 250\ \text{mg} \end{aligned}$$

Alternatively, we can set up our ordered list of units.

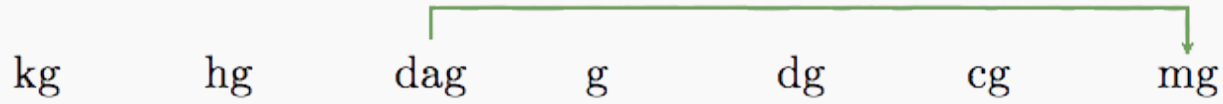

Figure 2.1.5.

Note that we must move 4 places right to move from the dekagrams (dag) abbreviation to the milligrams (mg) abbreviation. In like manner, we can convert 0.025 dekagrams to milligrams by moving the decimal 4 places to the right.

$$0.0250\ \text{dag} = 0\ 0250.\ \text{mg} = 250\ \text{mg}$$

Figure 2.1.6.

Answer

250mg

Try It

Convert 0.05 dekagrams to milligrams.

Show Solutions

500 milligrams

Units of Volume

The fundamental unit of volume in the metric system is called a litre. Originally, one litre was defined as the volume of one kilogram of water measured at 4°C at 760 millimetres of mercury ("Litre", 2026). Currently, 1 litre is defined as 1 cubic decimeter (imagine a cube with each edge 1/10 of a meter).

The same metric system prefixes apply.

Metric Units of Volume

These metric units of volume are commonly used in the lab.

Table 2.1.4: Volume Units and Abbreviations

Unit of Volume	Unit Abbreviation
1 kilolitre = 1000 litres	kL
1 litre	L
1 decilitre = $\frac{1}{10}$ litre	dL
1 millilitre = $\frac{1}{1000}$ litre	mL
1 microlitre= $\frac{1}{10^6}$ litre	µl

Example 4

Convert:

11,725 millilitres to dekalitres.

Solution:
We'll use two conversion factors:

- 1 daL = 10 L, which yields the conversion factor 1 daL/10 L.
- 1 mL = (1/1000) L, which yields the conversion factor 1 L/1000 mL.

$$\begin{aligned} 11{,}725\ \text{mL} &= 11{,}725\ \text{mL} \cdot \frac{1\ \text{L}}{1000\ \text{mL}} \cdot \frac{1\ \text{daL}}{10\ \text{L}} \\ &= 11{,}725\ \cancel{\text{mL}} \cdot \frac{1\ \cancel{\text{L}}}{1000\ \cancel{\text{mL}}} \cdot \frac{1\ \text{daL}}{10\ \cancel{\text{L}}} \\ &= \frac{11{,}725}{1000 \cdot 10}\ \text{daL} \\ &= 1.1725\ \text{daL} \end{aligned}$$

Alternatively, we can set up our ordered list of units.

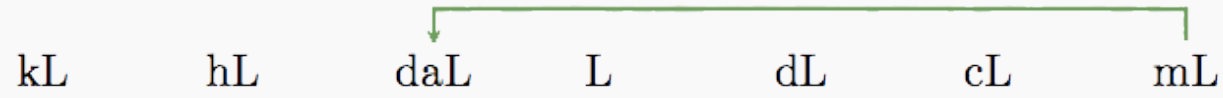

Figure 2.1.7.

Note that we must move 4 places left to move from the millilitres (mL) abbreviation to the dekalitres (daL) abbreviation. In like manner, we can convert 11,725 millilitres to dekalitres by moving the decimal 4 places to the left.

$$11,725.0\ \text{mL} = 1.1725\ 0\ \text{daL} = 1.1725\ \text{daL}$$

Figure 2.1.8.

Answer

1.1725daL

Example 5

Convert:

5,763 millilitres to dekalitres.

Answer

0.5763dekalitres

Try It

Convert 5,763 millilitres to dekalitres.

Show Solution

$0.5763\ \text{dekalitres}$

Units of Substance

Moles are the base unit for the entities in a substance (atoms, molecules, electrons, etc.). A mole is exactly $6.02214076 \times 10^{23}$ entities.

Table 2.1.5

Units of Substance	Unit Abbreviation
1 mole	mol
1 millimole= $\frac{1}{10^3}$ mole	mmol
1 micromole= $\frac{1}{10^6}$	umol
1 nanomole= $\frac{1}{10^9}$	nmol

Derived Units

Converting between **derived units** means changing from one combination of SI base units to another—often by using **conversion factors**, which are ratios equal to 1.

Here's a clear, structured explanation with examples you can use in chemistry, physics, or math.

How to Convert Between Derived Units

The process is always the same:

Step 1 — Write the original value with its unit.

Step 2 — Multiply by conversion factors.

A **conversion factor** is a ratio equal to 1, such as:

- $\frac{1000\ \text{mL}}{1\ \text{L}} = 1$
- $\frac{1\ \text{m}}{100\ \text{cm}} = 1$

Step 3 — Cancel units like algebraic variables.

Units behave exactly like numbers: they multiply, divide, and cancel. Set up the ratios so that the original units will cancel and leave only the new units in the answer.

Step 4 — Simplify the number and write the new unit.

Science Application

Molarity is a derived unit as it combines moles and volume. $\text{Molarity} = \dfrac{\text{moles}}{\text{litre}}$

Convert 0.25M to μM.

Conversion:

$$1\ \text{M} = 1{,}000{,}000\ \mu\text{M} = 10^6\ \mu\text{M}$$

To convert $\text{M} \rightarrow \mu\text{M}$, multiply by **1,000,000** (or 10^6).

$$0.25\ \text{M} \times \frac{10^6 \mu\text{M}}{\text{M}} = 250{,}000\ \mu\text{M}$$

Try It

Convert $3.6 \times 10^{-3}\text{M}$ to mM.

Show Solution

$$3.6\text{M} \times \frac{10^{-3}\ \text{mM}}{M} = 3.6\ \text{mM}$$

Example 6

Convert speed (km/h → m/s):

What is 90 km/h in m/s?

Solution:

Conversion factors:

1 km = 1000 m
1 hour = 3600 s

$$\frac{90\text{km}}{h} \times \frac{1000\text{ m}}{1\text{ km}} \times \frac{1\text{ h}}{3600\text{ s}} = 25\ \frac{\text{m}}{\text{s}}$$

Answer

$90\text{km/h} = 25\text{m/s}$

American Units to Metric Units and Vice Versa

We often need to convert from the American system of units to the metric system of units or vice versa. That will be our focus in this section.

Converting Units of Length

One meter is slightly longer than one yard. Indeed,

$$1\text{ m} \approx 1.0936\text{ yd}$$

If we divide both sides of this equation by 1.0936, then

$$\frac{1\text{ m}}{1.0936} \approx \frac{1.0936\text{ yd}}{1.0936}$$
$$0.9144\text{ m} \approx 1\text{ yd}$$

Further conversions can be made. For example, to change meters to feet, we make the following conversions.

$$\begin{aligned} 1\text{ m} &\approx 1\text{ m} \cdot 1\frac{\text{yd}}{0.9144\text{ m}} \cdot \frac{3\text{ ft}}{1\text{ yd}} && \text{Apply conversion factors.} \\ &\approx 1\,\cancel{\text{m}} \cdot \frac{1\ \cancel{\text{yd}}}{0.9144\ \cancel{\text{m}}} \cdot \frac{3\text{ ft}}{1\ \cancel{\text{yd}}} && \text{Cancel common units.} \\ &\approx \frac{1 \cdot 1 \cdot 3}{0.9144 \cdot 1}\text{ ft} && \text{Multiply fractions.} \\ &\approx 3.2808\text{ ft} \end{aligned}$$

Table 2.1.6 shows some of the most common conversions between American units of length and metric units of length.

Table 2.1.6: Length conversions: American — Metric.

American to Metric	Metric to American
1 in ≈ 2.54 cm	1 cm ≈ 0.3937 in
1 ft ≈ 0.3048 m	1 m ≈ 3.2808 ft
1 yd ≈ 0.9144 m	1 m ≈ 1.0936 yd
1 mi ≈ 1.6093 km	1 km ≈ 0.6214 mi

Example 7

A car's speedometer shows that a family has currently travelled 154 miles en route to their vacation destination. Convert this distance to kilometres.

Solution:

Choose 1 mi = 1.6093 km from Table 2.1.6.

$$\begin{aligned} 154\text{ mi} &\approx 154\text{ mi} \cdot \frac{1.6093\text{ km}}{1\text{ mi}} && \text{Apply conversion factor.} \\ &\approx 154\ \cancel{\text{mi}} \cdot \frac{1.6093\text{ km}}{1\ \cancel{\text{mi}}} && \text{Cancel common units.} \\ &\approx \frac{154 \cdot 1.6093}{1}\text{ km} && \text{Multiply fractions.} \\ &\approx 247.8\text{ km} \end{aligned}$$

Hence, 154 miles is approximately 247.8 kilometres.

Alternate Solution:

Note that this would work equally well if we chose 1 km ≈ 0.6214 mi from Table 2.1.2.

$$
\begin{aligned}
154\ \text{mi} &\approx 154\ \text{mi} \cdot \frac{1\ \text{km}}{0.6214\ \text{mi}} && \text{Apply conversion factor.} \\
&\approx 154\ \cancel{\text{mi}} \cdot \frac{1\ \text{km}}{0.6214\ \cancel{\text{mi}}} && \text{Cancel common units.} \\
&\approx \frac{154 \cdot 1}{0.6214}\ \text{km} && \text{Multiply fractions.} \\
&\approx 247.8\ \text{km}
\end{aligned}
$$

Answer

247.8 km

Try It

Change 227 miles to kilometres. Round to the nearest tenth of a kilometre.

Show Solution

365.3 kilometres

Converting Units of Weight and Mass

It is known that

$$1\,\text{kg} \approx 2.2\,\text{lb}$$

Dividing both sides of this equation by 2.2,

$$\frac{1\ \text{kg}}{2.2} \approx \frac{2.2\ \text{lb}}{2.2}$$

$$0.454\ \text{kg} \approx 1\text{lb}$$

A summary of the more common conversion factors regarding mass and weight are given in Table 2.1.7.

Table 2.1.7: Mass—Weight conversions: American — Metric.

American to Metric	Metric to American
1 oz ≈ 28.35 g	1 g ≈ 0.035 oz
1 lb ≈ 0.454 kg	1 kg ≈ 2.2 lb

Example 8

Change:

2.3 kilograms to ounces.

Solution:
One kilogram weighs 2.2 pounds, and there are 16 ounces in a pound.

$$2.3\ \text{kg} \approx 2.3\ \text{kg} \cdot \frac{2.2\ \text{lb}}{1\ \text{kg}} \cdot \frac{16\ \text{oz}}{1\ \text{lb}} \quad \text{Apply conversion factors.}$$

$$\approx 2.3\ \cancel{\text{kg}} \cdot \frac{2.2\ \cancel{\text{lb}}}{1\ \cancel{\text{kg}}} \cdot \frac{16\ \text{oz}}{1\ \cancel{\text{lb}}} \quad \text{Cancel common units.}$$

$$\approx \frac{2.3 \cdot 2.2 \cdot 16}{1 \cdot 1}\ \text{oz} \quad \text{Multiply fractions.}$$

$$\approx 80.96\ \text{oz}$$

Hence, 2.3 kilograms weighs 80.96 ounces.

Alternate Solution:
Another approach uses the facts that 1 kilogram equals 1000 grams and 1 ounce equals 28.35 grams.

$$2.3\ \text{kg} \approx 2.3\ \text{kg} \cdot \frac{1000\ \text{g}}{1\ \text{kg}} \cdot \frac{1\ \text{oz}}{28.35\ \text{g}} \quad \text{Apply conversion factors.}$$

$$\approx 2.3\ \cancel{\text{kg}} \cdot \frac{1000\ \cancel{\text{g}}}{1\ \cancel{\text{kg}}} \cdot \frac{1\ \text{oz}}{28.35\ \cancel{\text{g}}} \quad \text{Cancel common units.}$$

$$\approx \frac{2.3 \cdot 1000 \cdot 1}{1 \cdot 28.35}\ \text{oz} \quad \text{Multiply fractions.}$$

$$\approx 81.13\ \text{oz}$$

Roundoff Error

Why the discrepancy in answers? This difference in approximations is due to something called *round-off error*. Indeed, in the first calculation, we used a conversion factor that is rounded to the nearest tenth; i.e., 1 kg ≈ 2.2 lb. If we use a more accurate conversion factor to change kilograms to pounds, namely 1 kg ≈ 2.2046 lb, we get the following result.

$$
\begin{aligned}
2.3\text{ kg} &\approx 2.3\text{ kg} \cdot \frac{2.2046\text{ lb}}{1\text{ kg}} \cdot \frac{16\text{ oz}}{1\text{ lb}} && \text{Apply conversion factors.}\\
&\approx 2.3\ \cancel{\text{kg}} \cdot \frac{2.2046\ \cancel{\text{lb}}}{1\ \cancel{\text{kg}}} \cdot \frac{16\text{oz}}{1\ \cancel{\text{lb}}} && \text{Cancel common units.}\\
&\approx \frac{2.3 \cdot 2.2046 \cdot 16}{1 \cdot 1}\text{ oz} && \text{Multiply fractions.}\\
&\approx 81.13\text{ oz}
\end{aligned}
$$

Note that this result is in better agreement with the second result above.

Answer

80.96 ounces

Try It

Change 5.7 kilograms to ounces. Round to the nearest ounce.

Show Solution

201 ounces

Important Observation

To obtain better approximations, you need to use a conversion factor that is more accurate. Any time you need more accuracy, you might try an online conversion utility.

Converting Units of Volume

It is known that $1\ \text{qt} \approx 0.946\ \text{L}$.

Dividing both sides of this equation by 0.946, $1\ \text{L} \approx 1.06\ \text{qt}$.

Again, these conversion factors for volume have been well worked out by scientists. The more common conversion factors for volume are shown in Table 2.1.8.

Table 2.1.8: Volume conversions: American — Metric.

American to Metric	Metric to American
1 fl oz ≈ 0.030 L	1 L ≈ 33.8 fl oz
1 pt ≈ 0.473 L	1 L ≈ 2.1 pt
1 qt ≈ 0.946 L	1 L ≈ 1.06 qt
1 gal ≈ 3.785 L	1 L ≈ 0.264 gal

Example 9

Change:

2.5 dekalitres to gallons.

Solution:

Recall that 1 daL = 10 L and 1 L = 0.264 gal.

$$
\begin{aligned}
2.5\ \text{daL} &\approx 2.5\ \text{daL} \cdot \frac{10\ \text{L}}{1\ \text{daL}} \cdot \frac{0.264\ \text{gal}}{1\ \text{L}} && \text{Apply conversion factors.} \\
&\approx 2.5\ \cancel{\text{daL}} \cdot \frac{10\ \cancel{\text{L}}}{1\ \cancel{\text{daL}}} \cdot \frac{0.264\ \text{gal}}{1\ \cancel{\text{L}}} && \text{Cancel common units.} \\
&\approx \frac{2.5 \cdot 10 \cdot 0.264}{1 \cdot 1}\ \text{gal} && \text{Multiply fractions.} \\
&\approx 6.6\ \text{gal}
\end{aligned}
$$

Hence, 2.5 dekalitres is approximately equal to 6.6 gallons.

Try It

Change 3.2 dekalitres to gallons. Round your answer to the nearest tenth of a gallon.

Show Solution

8.4 gallons

Converting Units of Speed

Modern speedometers often show a car's speed in both miles per hour and kilometres per hour.

Example 10

Solve:

A car's speedometer is showing it speeding along at 60 kilometres per hour. How fast is it travelling in miles per hour? Round your answer to the nearest mile per hour.

Solution:

From Table 2.1.6, 1 km ≈ 0.6214 mi.

$$\begin{aligned} 60\frac{\text{km}}{\text{h}} &\approx 60\frac{\text{km}}{\text{h}} \cdot \frac{0.6214\ \text{mi}}{1\ \text{km}} && \text{Apply conversion factor.} \\ &\approx 60\frac{\cancel{\text{km}}}{\text{h}} \cdot \frac{0.6214\ \text{mi}}{1\ \cancel{\text{km}}} && \text{Cancel common units.} \\ &\approx \frac{60 \cdot 0.6214}{1}\frac{\text{mi}}{\text{h}} && \text{Multiply fractions.} \\ &\approx 37.284\frac{\text{mi}}{\text{h}} \end{aligned}$$

To round to the nearest mile per hour, identify the rounding and test digits.

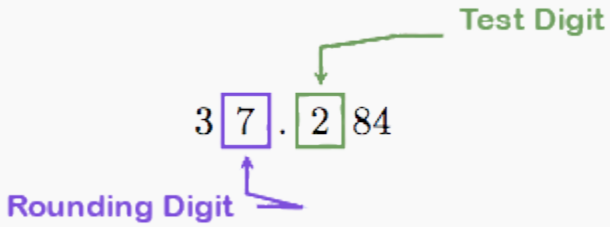

Figure 2.1.9.

Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, to the nearest mile per hour, the speed is approximately 37 miles per hour.

Try It

A car's speedometer registers 45 kilometres per hour. Change this speed to miles per hour. Round your answer to the nearest mile per hour.

Show Solution

28 miles per hour

Key Takeaways

- The metric system is base-10, so conversions are usually simple powers of ten.
- SI base units are the foundation used to build all other measurements.
- Many common measurements are derived units made by combining base units (multiply/divide/exponents).
- Metric prefixes (kilo-, centi-, milli-, micro-, etc.) indicate exact powers of ten.
- Conversions are done using conversion factors, which are ratios equal to 1.
- Units cancel like algebra, so cancellation confirms the setup is correct.
- Metric-to-metric conversions can often be done by shifting the decimal point.
- Some conversions require multiple conversion factors to reach the target unit.
- Mass measures matter/inertia, while weight relates to gravitational force.
- American-to-metric conversions use approximate factors, so more precision reduces round-off error.
- The same conversion method works for length, mass, volume, speed, and concentration.

OpenAI. (2026, January 22). ChatGPT. [Large language model]. https://chat.openai.com/chat

Prompt: Create some key takeaways for this content.

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where otherwise noted. Modifications: Added metric units table, micro and nano to standard prefixes; Removed exercises 1,3, example 3; Added Science Application example & Key Takeaways, Reformatted.

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2.2 Exponents and Scientific Notation

Learning Objectives

By the end of this section, you should be able to:

- Interpret exponential notation with positive integer exponents.
- Calculate the n^{th} power of a real number.
- Apply the rules of exponentiation and scientific notation while solving equations

We commonly encounter very large and very small numbers in science. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is $2,048$ pixels by $1,536$ pixels. It can also perceive a colour depth (gradations in colours) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour ($3,600$-second) digital film is then an extremely large number.

Using a calculator, we enter $2,048 \times 1,536 \times 48 \times 24 \times 3,600$ and press ENTER. The calculator displays $1.304596316E13$. What does this mean? The "$E13$" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this section, we review the rules of exponents first and then apply them to calculations involving very large or small numbers.

Exponential Notation and Positive Integer Exponents

If a number is repeated as a factor numerous times, then we can write the product in a more compact form using **exponential notation**. For example,

$$5 \cdot 5 \cdot 5 \cdot 5 = 5^4$$

The base is the factor, and the positive integer exponent indicates the number of times the base is repeated as a factor. In the above example, the base is 5 and the exponent is 4. In general, if a is the base that is repeated as a factor n times, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a}$$

When the exponent is 2, we call the result a **square**. For example,

$$3^2 = 3 \cdot 3 = 9$$

The number 3 is the base and the integer 2 is the exponent. The notation 3^2 can be read two ways: "three squared" or "3 raised to the second power." The base can be any real number.

$$(-7)^2 = (-7) \cdot (-7) = 49 \qquad \text{and} \qquad -5^2 = -(5^2) = -1 \cdot 5 \cdot 5 = -25$$

It is important to study the difference between the ways the last two examples are calculated. In the example $(-7)^2$, the base is -7 as indicated by the parentheses. In the example -5^2, the base is 5, not -5, so only the 5 is squared and the result remains negative. This subtle distinction is very important because it determines the sign of the result.

The textual notation for exponents is usually denoted using the **caret** $(\wedge)$ symbol as follows:

$$8^2 = 8 \wedge 2 = 8 \cdot 8 = 64$$
$$-5.1^2 = -\left(5.1^2\right) = -(5.1 \cdot 5.1) = -26.01$$

The square of an integer is called a **perfect square**. The ability to recognize perfect squares is useful in our study of algebra. The squares of the integers from 1 to 15 should be memorized. A partial list of perfect squares follows:

$$\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, \ldots\}$$

Visualize

Visit the Geogebra site to practice Squares & Perfect Squares. Link available on page 2.2 of the digital version of this textbook.

Try It

Simplify: $(-12)^2$

Show Solution

144

When the exponent is 3, we call the result a **cube**. For example,

$$3^3 = 3 \cdot 3 \cdot 3 = 27$$

The notation 3^3 can be read two ways: "three cubed" or "3 raised to the third power." As before, the base can be any real number.

$$-4^3 = -(4^3) = -4 \cdot 4 \cdot 4 = -64$$

Note that the result of cubing a negative number is negative. The cube of an integer is called a **perfect cube**. The ability to recognize perfect cubes is useful in our study of algebra. The cubes of the integers from 1 to 10 should be memorized. A partial list of perfect cubes follows:

$$\{0, 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, \ldots\}$$

Visualize

Visit the Geogebra site to practice Squares & Cubes. Link available on page 2.2 of the digital version of this textbook.

Try It

Simplify $(-2)^3$

Show Solution

-8

If the exponent is greater than 3, then the notation a^n is read "a raised to the n^{th} power."

$$10^6 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1,000,000$$

$$(-1)^4 = (-1)(-1)(-1)(-1) = 1$$

$$\left(\frac{1}{3}\right)^5 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{243}$$

Notice that the result of a negative base with an even exponent is positive. The result of a negative base with an odd exponent is negative. These facts are often confused when negative numbers are involved. Study the following four examples carefully:

Table 2.2.1 Negative Bases and Negated Exponents

The base is (-2)	The base is 2
$(-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = +16$	$-2^4 = -(2^4) = -(2 \cdot 2 \cdot 2 \cdot 2) = -16$
$(-2)^3 = (-2) \cdot (-2) \cdot (-2) = -8$	$-2^3 = -(2^3) = -(2 \cdot 2 \cdot 2) = -8$

The parentheses indicate that the negative number is to be used as the base.

Example 1

Calculate:

a. $\left(-\frac{1}{3}\right)^3$

b. $\left(-\frac{1}{3}\right)^4$

Solution:

The base is $-\frac{1}{3}$ for both problems.

a. Use the base as a factor three times.

$$\left(-\frac{1}{3}\right)^3 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)$$
$$= -\frac{1}{27}$$

b. Use the base as a factor four times.

$$\left(-\frac{1}{3}\right)^4 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)$$
$$= +\frac{1}{81}$$

Answer

a. $-\frac{1}{27}$; b. $\frac{1}{81}$

Example 2

Simplify:

$$-10^4 \text{ and } (-10)^4$$

Answer

$-10{,}000$ and $10{,}000$

Using the Product Rule of Exponents

Consider the product $x^3 \times x^4$. Both terms have the same base, x, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$\begin{aligned} x^3 \times x^4 &= \overbrace{x \times x \times x}^{3 \text{ factors}} \times \overbrace{x \times x \times x \times x}^{4 \text{ factors}} \\ &= \overbrace{x \times x \times x \times x \times x \times x \times x}^{7 \text{ factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \times x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the product rule of exponents.

$$a^m \times a^n = a^{m+n}$$

Now consider an example with real numbers.

$$2^3 \times 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product 8×16 equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

The Product Rule of Exponents

For any real number a and natural numbers m and n, the product rule of exponents states that

$$a^m \times a^n = a^{m+n}$$

Example 3

Write each of the following products with a single base. Do not simplify further.

a. $t^5 \times t^3$
b. $(-3)^5 \times (-3)$
c. $x^2 \times x^5 \times x^3$

Solution:
Use the product rule ($a^m \times a^n = a^{m+n}$) to simplify each expression.

a. $t^5 \times t^3 = t^{5+3} = t^8$
b. $(-3)^5 \times (-3) = (-3)^5 \times (-3)^1 = (-3)^{5+1} = (-3)^6$
c. $x^2 \times x^5 \times x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \times x^5 \times x^3 = (x^2 \times x^5) \times x^3 = (x^{2+5}) \times x^3 = x^7 \times x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \times x^5 \times x^3 = x^{2+5+3} = x^{10}$$

Answer

a. t^8
b. $(-3)^6$
c. x^10

Try It

Write each of the following products with a single base. Do not simplify further.

a. $k^6 \times k^9$

b. $\left(\frac{2}{y}\right)^4 \times \left(\frac{2}{y}\right)$

c. $t^3 \times t^6 \times t^5$

Show Solution

a. k^{15}

b. $\left(\frac{2}{y}\right)^5$

c. t^{14}

Using the Quotient Rule of Exponents

The quotient rule of exponents allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$, where $m > n$.

Consider the example $\frac{y^9}{y^5}$. Perform the division by cancelling common factors.

$$\begin{aligned}\frac{y^9}{y^5} &= \frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\ &= \frac{y \cdot y \cdot y \cdot y}{1} \\ &= y^4\end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

$$\frac{a^m}{a^n} = a^{m-n}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m > n$. Otherwise, the difference $m - n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

The Quotient Rule of Exponents

For any real number a and natural numbers m and n, such that $m > n$, the quotient rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 4

Write each of the following quotients with a single base. Do not simplify further.

a. $\dfrac{(-2)^{14}}{(-2)^{9}}$

b. $\dfrac{t^{23}}{t^{15}}$

c. $\dfrac{(z\sqrt{2})^{5}}{z\sqrt{2}}$

Solution:

Use the quotient rule to simplify each expression.

a. $\dfrac{(-2)^{14}}{(-2)^{9}} = (-2)^{14-9} = (-2)^{5}$

b. $\dfrac{t^{23}}{t^{15}} = t^{23-15} = t^{8}$

c. $\dfrac{(z\sqrt{2})^{5}}{z\sqrt{2}} = (z\sqrt{2})^{5-1} = (z\sqrt{2})^{4}$

Answer

a. $(-2)^{5}$
b. t^{8}
c. $(z\sqrt{2}^{4}$

Try It

Write each of the following quotients with a single base. Do not simplify further.

a. $\dfrac{s^{75}}{s^{68}}$

b. $\dfrac{(-3)^6}{-3}$

c. $\dfrac{(ef^2)^5}{(ef^2)^3}$

Show Solution

a. s^7

b. $(-3)^5$

c. $(ef^2)^2$

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the power rule of exponents. Consider the expression $\left(x^2\right)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

$$\begin{aligned}\left(x^2\right)^3 &= \left(x^2\right) \times \left(x^2\right) \times \left(x^2\right) \\ &= x \times x \times x \times x \times x \times x \\ &= x^6\end{aligned}$$

The exponent of the answer is the product of the exponents: $\left(x^2\right)^3 = x^{2 \cdot 3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

$$\left(a^m\right)^n = a^{m \cdot n}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule,

different terms with the same base are raised to exponents (add the exponents). When using the power rule, a term in exponential notation is raised to a power (multiply the exponents).

Table 2.2.2: Product Rule vs. Power Rule

Product Rule	Power Rule
$5^3 \times 5^4 = 5^{3+4} = 5^7$	$(5^3)^4 = 5^{3\times 4} = 5^{12}$
$x^5 \times x^2 = x^{5+2} = x^7$	$(x^5)^2 = x^{5\times 2} = x^{10}$
$(3a)^7 \times (3a)^{10} = (3a)^{17}$	$((3a)^7)^{10} = (3a)^{70}$

The Power Rule of Exponents

For any real number a and positive integers m and n, the power rule of exponents states that

$$(a^m)^n = a^{m \cdot n}$$

Example 5

Write each of the following expressions with a single base. Do not simplify further.

a. $(x^2)^7$
b. $((2t)^5)^3$
c. $((-3)^5)^{11}$

Solution:
Use the power rule to simplify each expression.

a. $(x^2)^7 = x^{2 \cdot 7} = x^{14}$
b. $((2t)^5)^3 = (2t)^{5 \cdot 3} = (2t)^{15}$
c. $((-3)^5)^{11} = (-3)^{5 \cdot 11} = (-3)^{55}$

Answer

1. x^{14}

2. $(2t)^{15}$
3. $(-3)^{55}$

Example 6

Write each of the following expressions with a single base. Do not simplify further.

a. $((3y)^8)^3$
b. $(t^5)^7$
c. $((-g)^4)^4$

Solution:

a. $\left((3y)^8\right)^3 = (3y)^{8 \cdot 3} = (3y)^{24}$
b. $(t^5)^7 = t^{5 \cdot 7} = t^{35}$
c. $\left((-g)^4\right)^4 = (-g)^{4 \cdot 4} = (-g)^{16}$

Answer

a. $(3y)^{24}$
b. t^{35}
c. $= (-g)^{16}$

Try it

Activity Description
Identify the correct answer for each blank.

a. $5^4 \times 5^6 =$_____

b. $\frac{4^6}{4^2} =$_____

c. $(7^6)^2 =$_____

Choices:

49^{12} 25^{10} 1^2 4^4 7^{12} 5^10 14^8 7^8

Answers:

a. 5^10
b. 4^4
c. 7^{12}

Try it

Activity Description
Identify the correct answer for each blank.

a. $(2 \times 10^3) \times (2 \times 10^4) =$_____

b. $\frac{6 \times 10^7}{2 \times 10^3} =$_____

c. $(3 \times 10^7)^3 =$_____

Choices:

2×10^7 4×10^7 3×10^4 3×10^{21} 3×10^{-4} 27×10^{21} 3×30^{21}

Answers:

a. 4×10^7
b. 3×10^4
c. 27×10^{21}

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m > n$ so that the difference $m - n$ would never be zero or negative. What would happen if $m = n$? In this case, we would use the zero exponent rule of exponents to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number or any variable representing a real number.

$$a^0 = 1$$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for now, we will consider the value to be undefined.

The Zero Exponent Rule of Exponents

For any nonzero real number a, the zero exponent rule of exponents states that $a^0 = 1$.

Example 7

Simplify each expression using the zero exponent rule of exponents.

a. $\dfrac{c^3}{c^3}$

b. $\dfrac{-3x^5}{x^5}$

c. $\dfrac{(j^2k)^4}{(j^2k) \times (j^2k)^3}$

d. $\dfrac{5(rs^2)^2}{(rs^2)^2}$

Solution:

Use the zero exponent and other rules to simplify each expression.

a. $c^3c^3 = c^{3-3} = c^0 = 1$

b. $\dfrac{-3x^5}{x^5} = -3 \cdot \dfrac{x^5}{x^5} = -3 \cdot x^{5-5} = -3 \cdot x^0 = -3$

c.

$$\begin{aligned}
\frac{(j^2k)^4}{(j^2k) \times (j^2k)^3} &= \frac{(j^2k)^4}{(j^2k)^{1+3}} && \text{Use the product rule in the denominator} \\
&= \frac{(j^2k)^4}{(j^2k)^4} && \text{Simplify} \\
&= (j^2k)^{4-4} && \text{Use the quotient rule} \\
&= (j^2k)^0 && \text{Simplify} \\
&= 1
\end{aligned}$$

d.

$$\begin{aligned}
\frac{5(rs^2)^2}{(rs^2)^2} &= 5(rs^2)^{2-2} && \text{Use the quotient rule} \\
&= 5(rs^2)^0 && \text{Simplify} \\
&= 5 \times 1 && \text{Use the zero exponent rule} \\
&= 5 && \text{Simplify}
\end{aligned}$$

Answer

a. 1
b. -3
c. 1
d. 5

Try It

Simplify each expression using the zero exponent rule of exponents.

a. $\dfrac{t^7}{t^7}$

b. $\dfrac{(de^2)^{11}}{2(de^2)^{11}}$

c. $\dfrac{w^4 \times w^2}{w^6}$

d. $\dfrac{t^3 \times t^4}{t^2 \times t^5}$

Show Solution

a. 1

b. $\dfrac{1}{2}$

c. 1

d. 1

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m > n$ in the quotient rule even further. For example, can we simplify $\frac{t^3}{t^5}$? When $m < n$—that is, where the difference $m - n$ is negative—we can use the negative rule of exponents to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{t^3}{t^5}$.

$$\begin{aligned}\frac{t^3}{t^5} &= \frac{t \times t \times t}{t \times t \times t \times t \times t} \\ &= \frac{1}{t \times t} \\ &= \frac{1}{t^2}\end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\begin{aligned}\frac{t^3}{t^5} &= t^{3-5} \\ &= t^{-2}\end{aligned}$$

Putting the answers together, we have $t^{-2} = \frac{1}{t^2}$. This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression a^n is defined when n is a natural number, 0, or the negative of a natural number. That means that a^n is defined for any integer n. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n.

The Negative Rule of Exponents

For any nonzero real number a and natural number n, the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n}$$

Visualize

Visit the Geogebra site to practice the Law of Exponents. Link available on page 2.2 of the digital version of this textbook.

Scientific Notation

is a way to write very large or very small numbers so they are easier to read and work with. It turns giant or microscopic numbers into a simple multiplication problem.

Every number in scientific notation looks like this:

$$a \times 10^n$$

a (The Coefficient): This number must be between 1 and 10 (it can be 1, but it must be less than 10).

10 (The Base): We always use base 10 because our number system is based on tens.

n (The Exponent): This tells you how many places the decimal point moved.

Large Numbers (Positive Exponents)

When you have a big number, like the distance to the sun, the exponent will be positive.

Example 8

Write:

$93,000,000$ in scientific notation

Solution

1. Find the decimal: In a whole number, it's at the very end.

$$93,000,000$$

2. Move the decimal: Slide it to the left until you have a number between 1 and 10.

9.3 is between 1 and 10.

3. Count the jumps: We moved the decimal 7 places.
4. Write it: 9.3×10^7

Answer

9.3×10^7

Small Numbers (Negative Exponents)

When you have a tiny decimal, like the width of a human hair, the exponent will be negative.

Example 9

Write:

0.000005 in scientific notation

Solution:

1. Move the decimal: Slide it to the right until you have a number between 1 and 10.
   5.0 (or just 5) is between 1 and 10.
2. Count the jumps: We moved the decimal 6 places.

3. Write it: 5×10^{-6}

Tip: Remember the negative exponent rule we discussed? 10^{-6} means the 5 is actually being divided by 1,000,000!

Answer

5×10^{-6}

How To

To convert a decimal to scientific notation.

1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
2. Count the number of decimal places, n, that the decimal point was moved.
3. Write the number as a product with a power of 10. If the original number is.
    - Greater than 1, the power of 10 will be 10^n.
    - Between 0 and 1, the power of 10 will be 10^{-n}.
4. Step 4. Check.

As a reminder for how to convert to standard form refer back to the section *Multiplying and Dividing By Powers of Ten*.

How To

Convert scientific notation to decimal form.

1. Determine the exponent, n, on the factor 10.
2. Move the decimal n places, adding zeros if needed.
    - If the exponent is positive, move the decimal point n places to the right.

- If the exponent is negative, move the decimal point |◈| places to the left.

3. Check.

Key Equations

Table 2.2.3: Rules of Exponents (for nonzero real numbers a and b and integers m and n)

Rule	Formula
Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \frac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Key Takeaways

- When using exponential notation , the base a is used as a factor n times.
- When the exponent is 2, the result is called a square. When the exponent is 3, the result is called a cube.
- Memorize the squares and the cubes of small integers to ease the difficulty of calculations.
- When negative numbers are involved, take care to associate the exponent with the correct base. Parentheses group a negative number raised to some power.
- A negative base raised to an even power is positive.
- A negative base raised to an odd power is negative.
- Products of exponential expressions with the same base can be simplified by adding exponents.
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents.
- Powers of exponential expressions with the same base can be simplified by multiplying exponents.
- An expression with an exponent of zero is defined as 1.
- An expression with a negative exponent is defined as a reciprocal.
- The power of a product of factors is the same as the product of the powers of the same factors.
- The power of a quotient of factors is the same as the quotient of the powers of the same factors.
- The rules for exponential expressions can be combined to simplify more complicated expressions.
- Scientific notation uses powers of 10 to simplify very large or very small numbers.
- Scientific notation may be used to simplify calculations with very large or very small numbers.

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2.3 Roots and Radicals

Learning Objectives

By the end of this section, you should be able to:

- Find square roots.
- Find cube roots.
- Find n^{th} roots.
- Simplify expressions using the product and quotient rules for radicals.

Square Roots

The **square root** of a number is that number that, when multiplied by itself, yields the original number. For example, 4 is a square root of 16, because $4^2 = 16$. Since $(-4)^2 = 16$, we can say that -4 is a square root of 16 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the **radical** sign $\sqrt{\ }$ to denote the **principal (nonnegative) square root** and a negative sign in front of the radical $-\sqrt{\ }$ to denote the negative square root.

$$\sqrt{16} = 4 \qquad \text{and} \qquad -\sqrt{16} = -4$$

Zero is the only real number with one square root.

$$\sqrt{0} = 0 \quad \text{because} \quad 0^2 = 0$$

If the **radicand**, the number inside the radical sign, is nonnegative and can be factored as the square of another nonnegative number, then the square root of the number is apparent. In this case, we have the following property:

$$\sqrt{a^2} = a, \quad \text{if} \quad a \geq 0$$

Example 1

Find the square root.

a. $\sqrt{36}$
b. $\sqrt{144}$
c. $\sqrt{0.04}$
d. $\sqrt{\frac{1}{9}}$

Solution:

a. $\sqrt{36} = \sqrt{6^2} = 6$
b. $\sqrt{144} = \sqrt{12^2} = 12$
c. $\sqrt{0.04} = \sqrt{0.2^2} = 0.2$
d. $\sqrt{\frac{1}{9}} = \sqrt{\left(\frac{1}{3}\right)^2} = \frac{1}{3}$

Answer

a. 6
b. 12
c. 0.2
d. $\frac{1}{3}$

Example 2

Find the negative square root.

a. $-\sqrt{4}$
b. $-\sqrt{1}$

Solution:

a. $-\sqrt{4} = -\sqrt{2^2} = -2$
b. $-\sqrt{1} = -\sqrt{1^2} = -1$

Answer

a. -2
b. -1

The radicand may not always be a perfect square. If a positive integer is not a perfect square, then its square root will be irrational. For example, $\sqrt{2}$ is an irrational number and can be approximated on most calculators using the square root button.

$$\sqrt{2} \approx 1.414 \quad \text{because} \quad 1.414^2 \approx 2$$

Next, consider the square root of a negative number. To determine the square root of -9, you must find a number that when squared results in -9:

$$\sqrt{-9} = ? \quad \text{or} \quad (?)^2 = -9$$

However, any real number squared always results in a positive number:

$$3^2 = 9 \quad \text{and} \quad (-3)^2 = 9$$

The square root of a negative number is currently left undefined. For now, we will state that $\sqrt{-9}$ is not a real number.

Science Applications

Period $T = 2\pi\sqrt{\frac{m}{k}}$: This tells you the time T it takes for a weight to go all the way down and all the way back up once on a spring.

m is the mass of the object.

k is the spring constant (how stiff the spring is).

Simplify the expression if $m = 8$ and $k = 1$.

$$T = 2\pi\sqrt{8}$$
$$T = 2\pi\sqrt{4}\sqrt{2}$$
$$T = 2\pi(2)\sqrt{2}$$
$$T = 4\pi\sqrt{2}$$

Try it

Simplify the expression if $m = 27$ and $k = 3$

Show Solution

$$T = 2\pi\sqrt{\frac{27}{3}}$$
$$T = 6\pi$$

Cube Roots

The **cube root** of a number is that number that, when multiplied by itself three times, yields the original number.

Furthermore, we denote a cube root using the symbol $\sqrt[3]{}$, where 3 is called the **index**. For example,

$$\sqrt[3]{125} = 5, \quad \text{because} \quad 5^3 = 125$$

The product of three equal **factors** will be positive if the factor is positive and negative if the factor is negative.

For this reason, any real number will have only one real cube root. Hence, the technicalities associated with the principal root do not apply. For example,

$$\sqrt[3]{-125} = -5, \quad \text{because} \quad (-5)^3 = -125$$

In general, given any real number a, we have the following property:

$$\sqrt[3]{a^3} = a$$

When simplifying cube roots, look for factors that are perfect cubes.

Example 3

Find the cube root.

a. $\sqrt[3]{27}$

b. $\sqrt[3]{64}$

c. $\sqrt[3]{0}$

d. $\sqrt[3]{\frac{1}{8}}$

Solution:

a. $\sqrt[3]{27} = \sqrt[3]{3^3} = 3$

b. $\sqrt[3]{64} = \sqrt[3]{4^3} = 4$

c. $\sqrt[3]{0} = \sqrt[3]{0^3} = 0$

d. $\sqrt[3]{\frac{1}{8}} = \sqrt[3]{\left(\frac{1}{2}\right)^3} = \frac{1}{2}$

Answer

a. 3
b. 4
c. 0
d. $\frac{1}{2}$

Example 4

Find the cube root.

a. $\sqrt[3]{-8}$

b. $\sqrt[3]{-1}$

c. $\sqrt[3]{-\frac{1}{27}}$

Solution:

a. $\sqrt[3]{-8} = \sqrt[3]{(-2)^3} = -2$

b. $\sqrt[3]{-1} = \sqrt[3]{(-1)^3} = -1$

c. $\sqrt[3]{-\frac{1}{27}} = \sqrt[3]{\left(-\frac{1}{3}\right)^3} = -\frac{1}{3}$

Answer

a. -2
b. -1
c. $-\dfrac{1}{3}$

It may be the case that the radicand is not a perfect cube. If an integer is not a perfect cube, then its cube root will be irrational. For example, $\sqrt[3]{2}$ is an irrational number which can be approximated on most calculators using the root button. Depending on the calculator, we typically type in the index prior to pushing the button and then the radicand. Therefore, we have:

$$\sqrt[3]{2} \approx 1.260, \quad \text{because} \quad 1.260^3 \approx 2$$

n^{th} Roots

For any integer $n \geq 2$, we define the n^{th} root of a positive real number as that number that when raised to the n^{th} power yields the original number. Given any nonnegative real number a, we have the following property:

$$\sqrt[n]{a^n} = a, \quad \text{if} \quad a \geq 0$$

Here n is called the **index** and a^n is called the **radicand**. Furthermore, we can refer to the entire expression $\sqrt[n]{a}$ as a radical. When the index is an integer greater than 3, we say “fourth root”, “fifth root”, and so on. The n^{th} root of any number is apparent if we can write the radicand with an exponent equal to the index.

How To

Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

Visualize

Visit the Geogebra site to practice How to do Radical Roots. Link available on page 2.3 of the digital version of this textbook.

Example 5

Find the n^{th} root.

a. $\sqrt[4]{81}$

b. $\sqrt[5]{32}$

c. $\sqrt[7]{1}$

d. $\sqrt[4]{\frac{1}{16}}$

Solution:

a. $\sqrt[4]{81} = \sqrt[4]{3^4} = 3$

b. $\sqrt[5]{32} = \sqrt[5]{2^5} = 2$

c. $\sqrt[7]{1} = \sqrt[7]{1^7} = 1$

d. $\sqrt[4]{\frac{1}{16}} = \sqrt[4]{\left(\frac{1}{2}\right)^4} = \frac{1}{2}$

Answer

a. 3
b. 2
c. 1
d. $\frac{1}{2}$

If the index is $n = 2$, then the radical indicates a square root, and it is customary to write the radical without the index:

$$\sqrt[2]{a} = \sqrt{a}$$

We have already taken care to define the principal square root of a number. At this point, we extend this idea to n^{th} roots when n is even. For example, 3 is a fourth root of 81, because $3^4 = 81$. And since $(-3)^4 = 81$, we can say that -3 is a fourth root of 81 as well. Hence, we use the radical sign $\sqrt[n]{}$ to denote the principal (nonnegative) n^{th} root when n is even. In this case, for any real number a, we use the following property:

$$\sqrt[n]{a^n} = |a| \qquad \text{When } n \text{ is even}$$

$$\sqrt[4]{81} = \sqrt[4]{3^4} = |3| = 3$$
$$\sqrt[4]{81} = \sqrt[4]{(-3)^4} = |-3| = 3$$

The negative n^{th} root, when n is even, will be denoted using a negative sign in front of the radical $-\sqrt[n]{}$.

$$-\sqrt[4]{81} = -\sqrt[4]{3^4} = -3$$

We have seen that the square root of a negative number is not real because any real number, when squared, will result in a positive number. In fact, a similar problem arises for any even index:

$$\sqrt[4]{-81} = ? \quad \text{or} \quad (?)^4 = -81$$

Here, the fourth root of -81 is not a real number because the fourth power of any real number is always positive.

$$\sqrt{-4} \quad \text{and} \quad \sqrt[4]{-81} \qquad \text{These radicals are not real numbers.}$$

Example 6

Simplify

a. $\sqrt[4]{-16}$
b. $-\sqrt[4]{16}$

Solution:

a. The radicand is negative, and the index is even. Therefore, there is no real number that, when raised to the fourth power, is -16.

$$\sqrt[4]{-16} \qquad \text{Not a real number}$$

b. Here, the radicand is positive. Furthermore, $16 = 2^4$, and we can simplify as follows:

$$-\sqrt[4]{16} = -\sqrt[4]{2^4} = -2$$

Answer

a. Not a real number
b. -2

When n is odd, the same problems do not occur. The product of an odd number of positive factors is positive, and the product of an odd number of negative factors is negative. Hence, when the index n is odd, there is only one real n^{th} root for any real number a. And we have the following property:

$$\sqrt[n]{a^n} = a \qquad \text{When } n \text{ is odd}$$

Example 7

Find the n^{th} root.

a. $\sqrt[5]{-32}$
b. $\sqrt[7]{-1}$

Solution:

a. $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$
b. $\sqrt[7]{-1} = \sqrt[7]{(-1)^7} = -1$

Answer

a. -2
b. -1

Try It

Find the fourth root:

$$\sqrt[4]{625}$$

Show Solution

5

Summary

When n ***is odd***, the n^{th} root is ***positive or negative*** depending on the sign of the radicand.

When ***n is even***, the n^{th} root is ***positive or not real*** depending on the sign of the radicand.

$$\sqrt[4]{16} = \sqrt[4]{(-2)^4} = |-2| = 2$$

$\sqrt[4]{-16}$ The radical is not a real number.

Simplifying Using the Product and Quotient Rule for Radicals

It will not always be the case that the radicand is a perfect power of the given index. If not, we use the following two properties to simplify them. If a and b represent positive real numbers, then we have:

Product rule for radicals:

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

Quotient rule for radicals:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

A radical is simplified if it does not contain any factor that can be written as a perfect power of the index.

Visualize

Visit the Geogebra site to practice Radical Simplification. Link available on page 2.3 of the digital version of this textbook.

Example 8

Simplify:

$$\sqrt{12}$$

Solution:
Here 12 can be written as $4 \cdot 3$, where 4 is a perfect square.

$$\begin{aligned} \sqrt{12} &= \sqrt{4 \cdot 3} && \text{Apply the product rule for radicals.} \\ &= \sqrt{4} \cdot \sqrt{3} && \text{Simplify.} \\ &= 2\sqrt{3} \end{aligned}$$

We can verify our answer on a calculator:

$$\sqrt{12} \approx 3.46 \quad \text{and} \quad 2\sqrt{3} \approx 3.46$$

Answer

$2\sqrt{3}$

Example 9

Simplify:

$$\sqrt{135}$$

Solution:
Begin by finding the largest perfect square factor of 135.

$$\begin{aligned} 135 &= 3^3 \cdot 5 \\ &= 3^2 \cdot 3 \cdot 5 \\ &= 9 \cdot 15 \end{aligned}$$

Therefore,

$$\begin{aligned} \sqrt{135} &= \sqrt{9 \cdot 15} && \text{Apply the product rule for radicals.} \\ &= \sqrt{9} \cdot \sqrt{15} && \text{Simplify.} \\ &= 3\sqrt{15} \end{aligned}$$

Answer

$3\sqrt{15}$

Example 10

Simplify:

$$\sqrt{\frac{50}{121}}$$

Solution:
Begin by finding the prime factorizations of both 50 and 121.

$$\begin{aligned} 50 &= 5^2 \cdot 2 \\ 121 &= 11^2 \end{aligned}$$

Therefore,

$$\begin{aligned}\sqrt{\frac{50}{121}} &= \sqrt{\frac{5^2 \cdot 2}{11^2}} && \text{Apply product and quotient rules.}\\ &= \frac{\sqrt{5^2} \cdot \sqrt{2}}{\sqrt{11^2}} && \text{Simplify.}\\ &= \frac{5\sqrt{2}}{11}\end{aligned}$$

Answer

$$\frac{5\sqrt{2}}{11}$$

Example 11

Simplify:

$$\sqrt[3]{162}$$

Solution:

Use the prime factorization of 162 to find the largest perfect cube factor:

$$\begin{aligned}162 &= 3^4 \cdot 2\\ &= 3^3 \cdot 3 \cdot 2\end{aligned}$$

Replace the radicand with this factorization and then apply the product rule for radicals.

$$\begin{aligned}\sqrt[3]{162} &= \sqrt[3]{3^3 \cdot 3 \cdot 2} && \text{Apply the product rule.}\\ &= \sqrt[3]{3^3} \cdot \sqrt[3]{3 \cdot 2} && \text{Simplify.}\\ &= 3\sqrt[3]{6}\end{aligned}$$

We can verify our answer on a calculator.

$$\sqrt[3]{162} \approx 5.451 \quad \text{and} \quad 3\sqrt[3]{6} \approx 5.451$$

Answer

$3\sqrt[3]{6}$

Try It

Simplify:

$$2\sqrt[3]{96}$$

Show Solution

$4\sqrt[3]{12}$

Example 12

Simplify:

$$\sqrt[5]{-96}$$

Solution:
The index is odd, and the radicand is negative; hence, the result will be negative.

Factor the radicand: $-96 = (-2)^5 \cdot 3$.

$$\begin{aligned}\sqrt[5]{-96} &= \sqrt[5]{(-2)^5 \cdot 3} && \text{Apply the product rule.}\\ &= \sqrt[5]{(-2)^5} \cdot \sqrt[5]{3} && \text{Simplify.}\\ &= -2\sqrt[5]{3}\end{aligned}$$

Answer

$-2\sqrt[5]{3}$

Example 13

Simplify:

$$\sqrt[3]{-\frac{8}{64}}$$

Solution:

Consider the equivalent fraction with $-8 = (-2)^3$ in the numerator and then simplify.

$$\begin{aligned}\sqrt[3]{-\frac{8}{64}} &= \sqrt[3]{\frac{-8}{64}} && \text{Apply the quotient rule.}\\ &= \frac{\sqrt[3]{(-2)^3}}{\sqrt[3]{4^3}} && \text{Simplify.}\\ &= \frac{-2}{4}\\ &= -\frac{1}{2}\end{aligned}$$

Answer

$-\frac{1}{2}$

Try It

Simplify:

$$\sqrt[3]{-108}$$

Show Solution

$$-3\sqrt[3]{4}$$

Key Takeaways

- The square root of a number is that number that, when multiplied by itself, yields the original number. When the radicand a is positive and $a \geq 0$, $\sqrt{a^2} = a$. When the radicand is negative, the result is not a real number.
- The cube root of a number is that number that, when used as a factor with itself three times, yields the original number. The cube root may be positive or negative depending on the sign of the radicand. Therefore, for any real number a, $\sqrt[3]{a^3} = a$.
- When working with n^{th} roots, n determines the definition that applies. We use $\sqrt[n]{a^n} = a$ when n is odd and $\sqrt[n]{a^n} = |a|$ when n is even. When n is even, the negative nth root is denoted with a negative sign in front of the radical sign.
- To simplify square roots, look for the largest perfect square factor of the radicand and then apply the product or quotient rule for radicals.
- To simplify cube roots, look for the largest perfect cube factor of the radicand and then apply the product or quotient rule for radicals.
- To simplify n^{th} roots, look for factors that have a power equal to the index n and then apply the product or quotient rule for radicals. The process is often streamlined by using the prime factorization of the radicand.

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How To: “1.3 Radicals and Rational Exponents” from College Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Used How To box “Given a square root radical expression, use the product rule to simplify it”; Reformatted.

2.4 Exercises

2.1 Units of Measure

1. What is the meaning of the metric system prefix centi?
2. What is the meaning of the metric system prefix deka?
3. What is the meaning of the metric system prefix hecto?
4. What is the meaning of the metric system prefix kilo?
5. What is the meaning of the metric system prefix deci?
6. What is the meaning of the metric system prefix milli?
7. What is the meaning of the metric system abbreviation mg
8. What is the meaning of the metric system abbreviation g?
9. What is the meaning of the metric system abbreviation m?
10. What is the meaning of the metric system abbreviation km?
11. What is the meaning of the metric system abbreviation kL?
12. What is the meaning of the metric system abbreviation daL?
13. What is the meaning of the metric system abbreviation hm?
14. What is the meaning of the metric system abbreviation dm?
15. What is the meaning of the metric system abbreviation dam?
16. What is the meaning of the metric system abbreviation cm?
17. What is the meaning of the metric system abbreviation dL?
18. What is the meaning of the metric system abbreviation L?
19. What is the meaning of the metric system abbreviation hg?
20. What is the meaning of the metric system abbreviation kg?
21. What is the meaning of the metric system abbreviation dg?
22. What is the meaning of the metric system abbreviation dag?
23. What is the meaning of the metric system abbreviation hL?
24. What is the meaning of the metric system abbreviation cL?
25. Change 5,490 millimetres to meters.
26. Change 8,528 millimetres to meters.
27. Change 64 meters to millimetres.
28. Change 65 meters to millimetres.
29. Change 4,571 millimetres to meters.
30. Change 8,209 millimetres to meters.
31. Change 15 meters to centimetres.
32. Change 12 meters to centimetres.
33. Change 569 centimetres to meters.
34. Change 380 centimetres to meters.
35. Change 79 meters to centimetres.
36. Change 60 meters to centimetres.

37. Change 7.6 kilometres to meters.
38. Change 4.9 kilometres to meters.
39. Change 861 centimetres to meters.
40. Change 427 centimetres to meters.
41. Change 4,826 meters to kilometres.
42. Change 1,929 meters to kilometres.
43. Change 4,724 meters to kilometres.
44. Change 1,629 meters to kilometres.
45. Change 6.5 kilometres to meters.
46. Change 7.9 kilometres to meters.
47. Change 17 meters to millimetres.
48. Change 53 meters to millimetres.
49. Change 512 milligrams to centigrams.
50. Change 516 milligrams to centigrams.
51. Change 541 milligrams to centigrams.
52. Change 223 milligrams to centigrams.
53. Change 70 grams to centigrams.
54. Change 76 grams to centigrams.
55. Change 53 centigrams to milligrams.
56. Change 30 centigrams to milligrams.
57. Change 83 kilograms to grams.
58. Change 70 kilograms to grams.
59. Change 8,196 grams to kilograms.
60. Change 6,693 grams to kilograms.
61. Change 564 centigrams to grams.
62. Change 884 centigrams to grams.
63. Change 38 grams to centigrams.
64. Change 88 grams to centigrams.
65. Change 77 centigrams to milligrams.
66. Change 61 centigrams to milligrams.
67. Change 5,337 grams to kilograms.
68. Change 4,002 grams to kilograms.
69. Change 15 kilograms to grams.
70. Change 45 kilograms to grams.
71. Change 833 centigrams to grams.
72. Change 247 centigrams to grams.
73. Change 619,560 centilitres to kilolitres.
74. Change 678,962 centilitres to kilolitres.
75. Change 15.2 litres to millilitres.
76. Change 9.7 litres to millilitres.
77. Change 10,850 centilitres to litres.
78. Change 15,198 centilitres to litres.
79. Change 10.7 litres to millilitres.
80. Change 17.3 litres to millilitres.
81. Change 15,665 millilitres to litres.
82. Change 12,157 millilitres to litres.

83. Change 6.3 kilolitres to centilitres.
84. Change 8.3 kilolitres to centilitres.
85. Change 4.5 kilolitres to centilitres.
86. Change 6.2 kilolitres to centilitres.
87. Change 10.6 litres to centilitres.
88. Change 16.6 litres to centilitres.
89. Change 14,383 centilitres to litres.
90. Change 11,557 centilitres to litres.
91. Change 9.9 litres to centilitres.
92. Change 19.5 litres to centilitres.
93. Change 407,331 centilitres to kilolitres.
94. Change 827,348 centilitres to kilolitres.
95. Change 14,968 millilitres to litres.
96. Change 18,439 millilitres to litres.
97. Use the conversion 1 in = 2.54 cm to convert 68 inches to centimetres, rounded to the nearest tenth of a centimetre.
98. Use the conversion 1 in = 2.54 cm to convert 42 inches to centimetres, rounded to the nearest tenth of a centimetre.
99. Use the conversion 1 in = 2.54 cm to convert 44 centimetres to inches, rounded to the nearest tenth of an inch.
100. Use the conversion 1 in = 2.54 cm to convert 22 centimetres to inches, rounded to the nearest tenth of an inch.
101. Use the conversion 1 mi = 1.6093 km to convert 79 miles to kilometres, rounded to the nearest tenth of a kilometre.
102. Use the conversion 1 mi = 1.6093 km to convert 39 miles to kilometres, rounded to the nearest tenth of a kilometre.
103. Use the conversion 1 yd = 0.9144 m to convert 1489 centimetres to yards, rounded to the nearest tenth of a yard.
104. Use the conversion 1 yd = 0.9144 m to convert 1522 centimetres to yards, rounded to the nearest tenth of a yard.
105. Use the conversion 1 yd = 0.9144 m to convert 28 yards to centimetres, rounded to the nearest tenth of a centimetre.
106. Use the conversion 1 yd = 0.9144 m to convert 34 yards to centimetres, rounded to the nearest tenth of a centimetre.
107. Use the conversion 1 m = 3.2808 ft to convert 8.6 meters to inches, rounded to the nearest tenth of an inch.
108. Use the conversion 1 m = 3.2808 ft to convert 8.3 meters to inches, rounded to the nearest tenth of an inch.
109. Use the conversion 1 in = 2.54 cm to convert 60 inches to centimetres, rounded to the nearest tenth of a centimetre.
110. Use the conversion 1 in = 2.54 cm to convert 75 inches to centimetres, rounded to the nearest tenth of a centimetre.
111. Use the conversion 1 m = 3.2808 ft to convert 208 inches to meters, rounded to the nearest tenth of a meter.
112. Use the conversion 1 m = 3.2808 ft to convert 228 inches to meters, rounded to the nearest tenth of a meter.

113. Use the conversion 1 m = 1.0936 yd to convert 20 yards to meters, rounded to the nearest tenth of a meter.
114. Use the conversion 1 m = 1.0936 yd to convert 44 yards to meters, rounded to the nearest tenth of a meter.
115. Use the conversion 1 mi = 1.6093 km to convert 29 miles to kilometres, rounded to the nearest tenth of a kilometre.
116. Use the conversion 1 mi = 1.6093 km to convert 15 miles to kilometres, rounded to the nearest tenth of a kilometre.
117. Use the conversion 1 m = 1.0936 yd to convert 8.2 meters to yards, rounded to the nearest tenth of a yard.
118. Use the conversion 1 m = 1.0936 yd to convert 6.9 meters to yards, rounded to the nearest tenth of a yard.
119. Use the conversion 1 mi = 1.6093 km to convert 4.9 kilometres to miles, rounded to the nearest tenth of a mile.
120. Use the conversion 1 mi = 1.6093 km to convert 4.2 kilometres to miles, rounded to the nearest tenth of a mile.
121. Use the conversion 1 m = 1.0936 yd to convert 25 yards to meters, rounded to the nearest tenth of a meter.
122. Use the conversion 1 m = 1.0936 yd to convert 2 yards to meters, rounded to the nearest tenth of a meter.
123. Use the conversion 1 in = 2.54 cm to convert 47 centimetres to inches, rounded to the nearest tenth of an inch.
124. Use the conversion 1 in = 2.54 cm to convert 19 centimetres to inches, rounded to the nearest tenth of an inch.
125. Use the conversion 1 mi = 1.6093 km to convert 8.3 kilometres to miles, rounded to the nearest tenth of a mile.
126. Use the conversion 1 mi = 1.6093 km to convert 4.8 kilometres to miles, rounded to the nearest tenth of a mile.
127. Use the conversion 1 yd = 0.9144 m to convert 41 yards to centimetres, rounded to the nearest tenth of a centimetre.
128. Use the conversion 1 yd = 0.9144 m to convert 20 yards to centimetres, rounded to the nearest tenth of a centimetre.
129. Use the conversion 1 m = 3.2808 ft to convert 3.7 meters to inches, rounded to the nearest tenth of an inch.
130. Use the conversion 1 m = 3.2808 ft to convert 7.9 meters to inches, rounded to the nearest tenth of an inch.
131. Use the conversion 1 yd = 0.9144 m to convert 1323 centimetres to yards, rounded to the nearest tenth of a yard.
132. Use the conversion 1 yd = 0.9144 m to convert 1715 centimetres to yards, rounded to the nearest tenth of a yard.
133. Use the conversion 1 m = 1.0936 yd to convert 8.4 meters to yards, rounded to the nearest tenth of a yard.
134. Use the conversion 1 m = 1.0936 yd to convert 7.3 meters to yards, rounded to the nearest tenth of a yard.
135. Use the conversion 1 m = 3.2808 ft to convert 289 inches to meters, rounded to the nearest tenth of a meter.

136. Use the conversion 1 m = 3.2808 ft to convert 251 inches to meters, rounded to the nearest tenth of a meter.
137. Use the conversion 1 kg = 2.2 lb to convert 15.8 kilograms to pounds, rounded to the nearest tenth of a pound.
138. Use the conversion 1 kg = 2.2 lb to convert 19.4 kilograms to pounds, rounded to the nearest tenth of a pound.
139. Use the conversion 1 oz = 28.35 g to convert 35 ounces to grams, rounded to the nearest tenth of a gram.
140. Use the conversion 1 oz = 28.35 g to convert 33 ounces to grams, rounded to the nearest tenth of a gram.
141. Use the conversion 1 kg = 2.2 lb to convert 2.48 kilograms to ounces, rounded to the nearest tenth of an ounce.
142. Use the conversion 1 kg = 2.2 lb to convert 3.74 kilograms to ounces, rounded to the nearest tenth of an ounce.
143. Use the conversion 1 kg = 2.2 lb to convert 2.35 kilograms to ounces, rounded to the nearest tenth of an ounce.
144. Use the conversion 1 kg = 2.2 lb to convert 3.57 kilograms to ounces, rounded to the nearest tenth of an ounce.
145. Use the conversion 1 kg = 2.2 lb to convert 15 pounds to kilograms, rounded to the nearest tenth of a kilogram.
146. Use the conversion 1 kg = 2.2 lb to convert 27 pounds to kilograms, rounded to the nearest tenth of a kilogram.
147. Use the conversion 1 kg = 2.2 lb to convert 10.4 kilograms to pounds, rounded to the nearest tenth of a pound.
148. Use the conversion 1 kg = 2.2 lb to convert 17.7 kilograms to pounds, rounded to the nearest tenth of a pound.
149. Use the conversion 1 kg = 2.2 lb to convert 352 ounces to kilograms, rounded to the nearest tenth of a kilogram.
150. Use the conversion 1 kg = 2.2 lb to convert 326 ounces to kilograms, rounded to the nearest tenth of a kilogram.
151. Use the conversion 1 oz = 28.35 g to convert 96 grams to ounces, rounded to the nearest tenth of an ounce.
152. Use the conversion 1 oz = 28.35 g to convert 100 grams to ounces, rounded to the nearest tenth of an ounce.
153. Use the conversion 1 oz = 28.35 g to convert 14 ounces to grams, rounded to the nearest tenth of a gram.
154. Use the conversion 1 oz = 28.35 g to convert 29 ounces to grams, rounded to the nearest tenth of a gram.
155. Use the conversion 1 kg = 2.2 lb to convert 54 pounds to kilograms, rounded to the nearest tenth of a kilogram.
156. Use the conversion 1 kg = 2.2 lb to convert 82 pounds to kilograms, rounded to the nearest tenth of a kilogram.
157. Use the conversion 1 oz = 28.35 g to convert 92 grams to ounces, rounded to the nearest tenth of an ounce.
158. Use the conversion 1 oz = 28.35 g to convert 103 grams to ounces, rounded to the nearest tenth of an ounce.

159. Use the conversion 1 kg = 2.2 lb to convert 388 ounces to kilograms, rounded to the nearest tenth of a kilogram.
160. Use the conversion 1 kg = 2.2 lb to convert 395 ounces to kilograms, rounded to the nearest tenth of a kilogram.
161. Use the conversion 1 qt = 0.946 L to convert 55.1 litres to quarts, rounded to the nearest tenth of a quart.
162. Use the conversion 1 qt = 0.946 L to convert 50.3 litres to quarts, rounded to the nearest tenth of a quart.
163. Use the conversion 1 L = 33.8 fl oz to convert 72073 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.
164. Use the conversion 1 L = 33.8 fl oz to convert 56279 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.
165. Use the conversion 1 L = 33.8 fl oz to convert 2.5 kilolitres to fluid ounces.
166. Use the conversion 1 L = 33.8 fl oz to convert 4.5 kilolitres to fluid ounces.
167. Use the conversion 1 qt = 0.946 L to convert 24 quarts to litres, rounded to the nearest tenth of a litre.
168. Use the conversion 1 qt = 0.946 L to convert 21 quarts to litres, rounded to the nearest tenth of a litre.
169. Use the conversion 1 qt = 0.946 L to convert 30 quarts to litres, rounded to the nearest tenth of a litre.
170. Use the conversion 1 qt = 0.946 L to convert 22 quarts to litres, rounded to the nearest tenth of a litre.
171. Use the conversion 1 gal = 3.785 L to convert 11.8 gallons to litres, rounded to the nearest tenth of a litre.
172. Use the conversion 1 gal = 3.785 L to convert 13.5 gallons to litres, rounded to the nearest tenth of a litre.
173. Use the conversion 1 gal = 3.785 L to convert 50.5 litres to gallons, rounded to the nearest tenth of a gallon.
174. Use the conversion 1 gal = 3.785 L to convert 55.9 litres to gallons, rounded to the nearest tenth of a gallon.
175. Use the conversion 1 L = 33.8 fl oz to convert 8.3 kilolitres to fluid ounces.
176. Use the conversion 1 L = 33.8 fl oz to convert 5.3 kilolitres to fluid ounces.
177. Use the conversion 1 qt = 0.946 L to convert 42.4 litres to quarts, rounded to the nearest tenth of a quart.
178. Use the conversion 1 qt = 0.946 L to convert 55.4 litres to quarts, rounded to the nearest tenth of a quart.
179. Use the conversion 1 gal = 3.785 L to convert 17.2 gallons to litres, rounded to the nearest tenth of a litre.
180. Use the conversion 1 gal = 3.785 L to convert 19.6 gallons to litres, rounded to the nearest tenth of a litre.
181. Use the conversion 1 L = 33.8 fl oz to convert 51274 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.
182. Use the conversion 1 L = 33.8 fl oz to convert 82164 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.
183. Use the conversion 1 gal = 3.785 L to convert 55.6 litres to gallons, rounded to the nearest tenth of a gallon.
184. Use the conversion 1 gal = 3.785 L to convert 59.2 litres to gallons, rounded to the nearest tenth of a gallon.
185. Change 60 miles per hour to kilometres per hour. Round your answer to the nearest kilometre per hour.

186. Change 56 miles per hour to kilometres per hour. Round your answer to the nearest kilometre per hour.
187. Change 77 miles per hour to kilometres per hour. Round your answer to the nearest kilometre per hour.
188. Change 57 miles per hour to kilometres per hour. Round your answer to the nearest kilometre per hour.
189. Change 42 kilometres per hour to miles per hour. Round your answer to the nearest mile per hour.
190. Change 56 kilometres per hour to miles per hour. Round your answer to the nearest mile per hour.
191. Change 62 kilometres per hour to miles per hour. Round your answer to the nearest mile per hour.
192. Change 63 kilometres per hour to miles per hour. Round your answer to the nearest mile per hour.
193. *Tallest tower*. The world's tallest tower in Dubai has 160 floors at a height of 2,717 feet. Convert the height of the tower to the nearest tenth of a meter.
194. *High peaks*. Convert the highest of the Himalayan peaks (26,246 feet) to the nearest tenth of a meter.
195. Ancient find. Convert the 4-meter statue of Thoth to the nearest tenth of a foot.
196. *Arctic wind*. Convert 80 miles per hour to the nearest kilometre per hour.
197. *Solar plane*. Convert 28 miles per hour to the nearest kilometre per hour.

Show Solutions

1. $\frac{1}{100}$
2. 10
3. 100
4. 1,000
5. $\frac{1}{10}$
6. $\frac{1}{1000}$
7. milligram
8. gram
9. meter
10. kilometer

11. kilolitre or kiloliter
12. dekalitre or dekaliter
13. hectometer
14. decimeter
15. dekameter
16. centimeter
17. decilitre or deciliter
18. litre or liter
19. hectogram
20. kilogram
21. decigram
22. dekagram
23. hectolitre or hectoliter
24. centilitre or centilitre
25. 5.49 meters
26. 8.528 meters
27. 64,000 millimetres
28. 65,000 millimetres
29. 4.571 meters
30. 8.209 meters
31. 1,500 centimetres

32. 1,200 centimetres
33. 5.69 meters
34. 3.8 meters
35. 7,900 centimetres
36. 6,000 centimetres
37. 7,600 meters
38. 4,900 meters
39. 8.61 meters
40. 4.27 meters
41. 4.826 kilometres
42. 1.929 kilometres
43. 4.724 kilometres
44. 1.629 kilometres
45. 6,500 meters
46. 7,900 meters
47. 17,000 millimetres
48. 53,000 millimetres
49. 51.2 centigrams
50. 51.6 centigrams
51. 54.1 centigrams
52. 22.3 centigrams

53. 7,000 centigrams
54. 7,600 centigrams
55. 530 milligrams
56. 300 milligrams
57. 83,000 grams
58. 70,000 grams
59. 8.196 kilograms
60. 6.693 kilograms
61. 5.64 grams
62. 8.84 grams
63. 3,800 centigrams
64. 8,800 centigrams
65. 770 milligrams
66. 610 milligrams
67. 5.337 kilograms
68. 4.002 kilograms
69. 15,000 grams
70. 45,000 grams
71. 8.33 grams
72. 2.47 grams
73. 6.1956 kilolitres

74. 6.78962 kilolitres
75. 15,200 millilitres
76. 9,700 millilitres
77. 108.5 litres
78. 151.98 litres
79. 10,700 millilitres
80. 17,300 millilitres
81. 15.665 litres
82. 12.157 litres
83. 630,000 centilitres
84. 830,000 centilitres
85. 450,000 centilitres
86. 620,000 centilitres
87. 1,060 centilitres
88. 1,660 centilitres
89. 143.83 litres
90. 115.57 litres
91. 990 centilitres
92. 1,950 centilitres
93. 4.07331 kilolitres
94. 8.27348 kilolitres

95. 14.968 litres
96. 18.439 litres
97. 172.7 centimetres
98. 106.7 centimetres
99. 17.3 inches
100. 8.7 inches
101. 127.1 kilometres
102. 62.8 kilometres
103. 16.3 yards
104. 16.6 yards
105. 2,560.3 centimetres
106. 3,109.0 centimetres
107. 338.6 inches
108. 326.8 inches
109. 152.4 centimetres
110. 190.5 centimetres
111. 5.3 meters
112. 5.8 meters
113. 18.3 meters
114. 40.2 meters
115. 46.7 kilometres

116. 24.1 kilometres
117. 9.0 yards
118. 7.5 yards
119. 3.0 miles
120. 2.6 miles
121. 22.9 meters
122. 1.8 meters
123. 18.5 inches
124. 7.5 inches
125. 5.2 miles
126. 3.0 miles
127. 3,749.0 centimetres
128. 1,828.8 centimetres
129. 145.7 inches
130. 311.0 inches
131. 14.5 yards
132. 18.8 yards
133. 9.2 yards
134. 8.0 yards
135. 7.3 meters
136. 6.4 meters

137. 34.8 pounds
138. 42.7 pounds
139. 992.3 grams
140. 935.6 grams
141. 87.3 ounces
142. 131.6 ounces
143. 82.7 ounces
144. 125.7 ounces
145. 6.8 kilograms
146. 12.3 kilograms
147. 22.9 pounds
148. 38.9 pounds
149. 10.0 kilograms
150. 9.3 kilograms
151. 3.4 ounces
152. 3.5 ounces
153. 396.9 grams
154. 822.2 grams
155. 24.5 kilograms
156. 37.3 kilograms
157. 3.2 ounces

158. 3.6 ounces
159. 11.0 kilograms
160. 11.2 kilograms
161. 58.2 quarts
162. 53.2 quarts
163. 2.1 kilolitres
164. 1.7 kilolitres
165. 84,500 fluid ounces
166. 152,100 fluid ounces
167. 22.7 litres
168. 19.9 litres
169. 28.4 litres
170. 20.8 litres
171. 44.7 litres
172. 51.1 litres
173. 13.3 gallons
174. 14.8 gallons
175. 280,540 fluid ounces
176. 179,140 fluid ounces
177. 44.8 quarts
178. 58.6 quarts

179. 65.1 litres

180. 74.2 litres

181. 1.5 kilolitres

182. 2.4 kilolitres

183. 14.7 gallons

184. 15.6 gallons

185. 97 km/hr

186. 90 km/hr

187. 124 km/hr

188. 92 km/hr

189. 26 mi/hr

190. 35 mi/hr

191. 39 mi/hr

192. 39 mi/hr

193. 828.2 meters

194. 7,999.9 meters

195. 13.1 feet

196. 129 km/hr

197. 45 km/hr

2.2 Real Numbers: Algebra Essentials

Simplify.

1. 10^2
2. 12^2
3. $(-9)^2$
4. -12^2
5. 11^2
6. $(-20)^2$
7. 0^2
8. 1^2
9. $-(-8)^2$
10. $(-13)^2$
11. $\left(\frac{1}{2}\right)^2$
12. $\left(-\frac{2}{3}\right)^2$
13. 0.5^2
14. 1.25^2
15. $(-2.6)^2$
16. $-(-5.1)^2$

17. $\left(2\frac{1}{3}\right)^2$

18. $\left(5\frac{3}{4}\right)^2$

19. 5^3

20. 2^6

21. $(-1)^4$

22. $(-3)^3$

23. -1^4

24. $(-2)^4$

25. -7^3

26. $(-7)^3$

27. $-(-3)^3$

28. $-(-10)^4$

29. $(-1)^{20}$

30. $(-1)^{21}$

31. $(-6)^3$

32. -3^4

33. 1^{100}

34. 0^{100}

35. $-\left(\frac{1}{2}\right)^3$

36. $\left(\frac{1}{2}\right)^6$

37. $\left(\frac{5}{2}\right)^3$

38. $\left(-\frac{3}{4}\right)^4$

39. List all the cubes of the integers −5 through 5.

40. List all the cubes of the integers from −10 to 0.

41. List all the cubes of the rational numbers in the set $\left\{-\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}\right\}$.

42. List all the cubes of the rational numbers in the set $\left\{-\frac{3}{7}, -\frac{1}{7}, 0, \frac{1}{7}, \frac{3}{7}\right\}$.

Show Solutions

1. 100
2. 144
3. 81
4. -144
5. 121
6. 400
7. 0
8. 1
9. -64
10. 169

11. $\frac{1}{4}$
12. $\frac{4}{9}$
13. 0.25
14. 1.5625
15. 6.76
16. -26.01
17. $5\frac{4}{9}$
18. $33\frac{1}{16}$
19. 125
20. 64
21. 1
22. -27
23. -1
24. 16
25. -343
26. -343
27. 27
28. -10000
29. 1

30. -1

31. -216

32. -81

33. 1

34. 0

35. $-\frac{1}{8}$

36. $\frac{1}{64}$

37. $\frac{125}{8}$

38. $\frac{81}{256}$

39. $\{-125, -64, -27, -8, -1, 0, 1, 8, 27, 64, 125\}$

40. $\{-1000, -729, -512, -343, -216, -125, -64, -27, -8, -1, 0\}$

41. $\left\{-\frac{8}{27}, -\frac{1}{27}, 0, \frac{1}{27}, \frac{8}{27}\right\}$

42. $\left\{-\frac{27}{343}, -\frac{1}{343}, 0, \frac{1}{343}, \frac{27}{343}\right\}$

2.3 Real Numbers: Algebra Essentials

Simplify.

1. $\sqrt{81}$
2. $\sqrt{100}$
3. $\sqrt{64}$
4. $\sqrt{121}$
5. $\sqrt{0}$
6. $\sqrt{1}$
7. $\sqrt{0.25}$
8. $\sqrt{0.01}$
9. $\sqrt{1.21}$
10. $\sqrt{2.25}$
11. $\sqrt{14}$
12. $\sqrt{136}$
13. $\sqrt{\frac{25}{16}}$
14. $\sqrt{\frac{9}{25}}$
15. $\sqrt{-25}$
16. $\sqrt{-9}$
17. $-\sqrt{36}$
18. $-\sqrt{81}$
19. $-\sqrt{100}$
20. $-\sqrt{1}$

21. $\sqrt[3]{27}$

22. $\sqrt[3]{125}$

23. $\sqrt[3]{64}$

24. $\sqrt[3]{8}$

25. $\sqrt[3]{\frac{1}{1}}$

26. $\sqrt[3]{\frac{1}{64}}$

27. $\sqrt[3]{\frac{8}{27}}$

28. $\sqrt[3]{\frac{64}{125}}$

29. $\sqrt[3]{0.001}$

30. $\sqrt[3]{1{,}000}$

31. $\sqrt[3]{-1}$

32. $\sqrt[3]{-8}$

33. $\sqrt[3]{-27}$

34. $\sqrt[3]{-64}$

35. $\sqrt[3]{-18}$

36. $-\sqrt[3]{\frac{27}{64}}$

37. $-\sqrt[3]{\frac{8}{27}}$

38. $-\sqrt[3]{\frac{1}{125}}$

39. $\sqrt[4]{81}$

40. $\sqrt[4]{625}$

41. $\sqrt[4]{16}$

42. $\sqrt[4]{10{,}000}$

43. $\sqrt[5]{32}$

44. $\sqrt[5]{1}$

45. $\sqrt[5]{243}$

46. $\sqrt[5]{100{,}000}$

47. $-\sqrt[4]{16}$

48. $-\sqrt[6]{1}$

49. $\sqrt[5]{-32}$

50. $\sqrt[5]{-1}$

51. $\sqrt{-1}$

52. $\sqrt[4]{-16}$

53. $-5\sqrt[3]{-27}$

54. $-2\sqrt[3]{-8}$

55. $5\sqrt[3]{-1{,}000}$

56. $3\sqrt[5]{-243}$

57. $10\sqrt[4]{-16}$

58. $2\sqrt[6]{-64}$

59. $\sqrt{325}$

60. $\sqrt{64}$

61. $2\sqrt[3]{27}$

62. $8\sqrt[3]{243}$

63. $-7\sqrt[3]{8}$

64. $-4\sqrt[4]{625}$

65. $6\sqrt[5]{100{,}000}$

66. $5\sqrt[7]{128}$

67. $\sqrt{32}$

68. $\sqrt{250}$

69. $\sqrt{80}$

70. $\sqrt{150}$

71. $\sqrt{160}$

72. $\sqrt{60}$

73. $\sqrt{175}$

74. $\sqrt{216}$

75. $\sqrt{5112}$

76. $\sqrt{10135}$

77. $\sqrt{\dfrac{50}{49}}$

78. $-\sqrt{2120}$

79. $-3\sqrt{162}$

80. $\sqrt{89}$

81. $\sqrt{\frac{45}{121}}$

82. $\sqrt{\frac{9}{681}}$

83. $\sqrt[3]{54}$

84. $\sqrt[3]{24}$

85. $\sqrt[3]{48}$

86. $\sqrt[3]{81}$

87. $\sqrt[3]{\frac{40}{343}}$

88. $\sqrt[3]{120}$

89. $\sqrt[3]{162}$

90. $\sqrt[3]{500}$

91. $\sqrt[3]{\frac{54}{125}}$

92. $\sqrt[3]{\frac{40}{343}}$

93. $5\sqrt[3]{-48}$

94. $2\sqrt[3]{-108}$

95. $8\sqrt[4]{96}$

96. $7\sqrt[4]{162}$

97. $\sqrt[5]{160}$

98. $\sqrt[5]{486}$

99. $\sqrt[5]{\frac{224}{243}}$

100. $\sqrt[5]{532}$

Simplify. Give the exact answer and the approximate answer rounded to the nearest hundredth.

101. $\sqrt{8}$

102. $\sqrt{200}$

103. $\sqrt{45}$

104. $\sqrt{72}$

105. $3\sqrt{4}$

106. $\sqrt{59}$

107. $\sqrt{3225}$

108. $\sqrt{4849}$

109. $\sqrt[3]{80}$

110. $\sqrt[3]{320}$

111. $\sqrt[3]{48}$

112. $\sqrt[3]{270}$

Show Solutions

1. 9
2. 10
3. 8

4. 11
5. 0
6. 1
7. 0.5
8. 0.1
9. 1.1
10. 1.5
11. $\sqrt{14}$
12. $2\sqrt{34}$
13. $\frac{5}{4}$
14. $\frac{3}{5}$
15. Not a real number
16. Not a real number
17. -6
18. -9
19. -10
20. -1
21. 3
22. 5
23. 4

24. 2
25. 1
26. $\frac{1}{4}$
27. $\frac{2}{3}$
28. $\frac{4}{5}$
29. 0.1
30. 10
31. -1
32. -2
33. -3
34. -4
35. $-\sqrt[3]{18}$
36. $-\frac{3}{4}$
37. $-\frac{2}{3}$
38. $-\frac{1}{5}$
39. 3
40. 5
41. 2

42. 10
43. 2
44. 1
45. 3
46. 10
47. -2
48. -1
49. -2
50. -1
51. Not a real number
52. Not a real number
53. 15
54. 4
55. -50
56. -9
57. Not a real number
58. Not a real number
59. $5\sqrt{13}$
60. 8
61. 6
62. $24\sqrt[3]{9}$

63. -14

64. -20

65. 60

66. 10

67. $4\sqrt{2}$

68. $5\sqrt{10}$

69. $4\sqrt{5}$

70. $5\sqrt{6}$

71. $4\sqrt{10}$

72. $2\sqrt{15}$

73. $5\sqrt{7}$

74. $6\sqrt{6}$

75. $6\sqrt{142}$

76. $\sqrt{10135}$

77. $\dfrac{5\sqrt{2}}{7}$

78. $-2\sqrt{530}$

79. $-27\sqrt{2}$

80. $\sqrt{89}$

81. $\dfrac{3\sqrt{5}}{11}$

82. $\frac{\sqrt{681}}{227}$

83. $3\sqrt[3]{2}$

84. $2\sqrt[3]{3}$

85. $2\sqrt[3]{6}$

86. $3\sqrt[3]{3}$

87. $\frac{2\sqrt[3]{5}}{7}$

88. $2\sqrt[3]{15}$

89. $3\sqrt[3]{6}$

90. $5\sqrt[3]{4}$

91. $\frac{3\sqrt[3]{2}}{5}$

92. $\frac{2\sqrt[3]{5}}{7}$

93. $-10\sqrt[3]{6}$

94. $-6\sqrt[3]{4}$

95. $16\sqrt[4]{6}$

96. $21\sqrt[4]{2}$

97. $2\sqrt[5]{5}$

98. $3\sqrt[5]{2}$

99. $\frac{2\sqrt[5]{7}}{3}$

100. $\sqrt[5]{532}$

101. $2\sqrt{2} \approx 2.83$

102. $10\sqrt{2} \approx 14.14$

103. $3\sqrt{5} \approx 6.71$

104. $6\sqrt{2} \approx 8.49$

105. $6 \approx 6.00$

106. $\sqrt{59} \approx 7.68$

107. $5\sqrt{129} \approx 56.79$

108. $\sqrt{4849} \approx 69.63$

109. $2\sqrt[3]{10} \approx 4.31$

110. $4\sqrt[3]{5} \approx 6.84$

111. $2\sqrt[3]{6} \approx 3.63$

112. $3\sqrt[3]{10} \approx 6.46$

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CHAPTER 3: ALGEBRAIC EXPRESSIONS

Chapter Outline

3.1 Polynomials

- Definitions
- Evaluating Polynomials

3.2 Adding and Subtracting Polynomials

- Adding Polynomials
- Subtracting Polynomials

3.3 Multiplying Polynomials

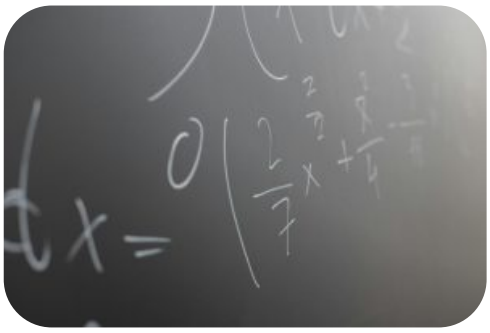
- Multiplying by a Monomial
- Multiplying by a Binomial
- Product of Polynomials

3.4 Dividing Polynomials

- Dividing by a Monomial
- Dividing by a Polynomial

3.5 Exercises

- 3.1 Polynomials
- 3.2 Adding and Subtracting Polynomials
- 3.3 Multiplying Polynomials
- 3.4 Dividing Polynomials



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3.1 Polynomials

Learning Objectives

By the end of this section, you should be able to:

- Identify a polynomial and determine its degree.
- Evaluate a polynomial for given values of the variables.

Definitions

A **polynomial** is a special algebraic expression with terms that consist of real number coefficients and variable factors with whole number exponents. Polynomials *do not have variables in the denominator* of any term.

Try it

What is a polynomial? Sort the expressions below based on the criteria above this box.

Activity Description
Identify each expression as *Polynomial* or *Not a Polynomial*.

$$5x^2 - 3x^{-2} + 7 \qquad 5\sqrt{x} + 5 \qquad 6x^2y - 4xy^3 + 7 \qquad 7xy + 5$$

$$\frac{2x^2}{y} \qquad 3x^2 \qquad \frac{2}{x} - \frac{5}{y} = 3 \qquad 3x^2 \qquad \frac{3}{2}x^3 + 3x^2 - \frac{1}{2}x + 1$$

Answers:

Polynomials

$7xy + 5$

Not a Polynomial

$5x^2 - 3x^{-2} + 7$

$\frac{3}{2}x^3 + 3x^2 - \frac{1}{2}x + 1$

$6x^2y - 4xy^3 + 7$

$3x^2$

$\frac{2x^2}{y}$

$\frac{2}{x} - \frac{5}{y} = 3$

$5\sqrt{x} + 5$

The **degree of a term** in a polynomial is defined to be the exponent of the variable, or if there is more than one variable in the term, the degree is the *sum of their exponents*. Recall that $x^0 = 1$; any **constant term** can be written as a product of x^0 and itself. Hence, the degree of a constant term is 0.

How To

Given a polynomial expression, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Try it

What is the degree of the term? Try the exercise below.

This is important to know because once you've simplified an expression, it's standard to arrange the terms from the highest degree to the lowest.

Activity Description
Fill in the blank with the degree of the term.

$3x^2$ has a degree of _____.

$6x^2y$ has a degree of _____.

$7a^2b^3$ has a degree of _____.

8 has a degree of _____ because $8 = 8x^0$.

$2x$ has a degree of _____ because $x = x^1$.

Options:

$$6 \quad 2 \quad 4 \quad 5 \quad 1 \quad 8 \quad 3 \quad 7 \quad 0$$

Answers:

$3x^2$ has a degree of 2.

$6x^2y$ has a degree of 3.

$7a^2b^3$ has a degree of 5.

8 has a degree of 0 because $8 = 8x^0$.

$2x$ has a degree of 1 because $x = x^1$.

Table 3.1.1.

Term	Degree
$3x^2$	2
$6x^2y$	$2 + 1 = 3$
$7a^2b^3$	$2 + 3 = 5$
8	0, since $8 = 8x^0$
$2x$	1, since $x = x^1$

The **degree of a polynomial** is the largest degree of all of its terms.

Table 3.1.2.

Polynomial	Degree
$4x^5 - 3x^3 + 2x - 1$	5
$6x^2y - 5xy^3 + 7$	4, because $5xy^3$ has degree 4.
$12x + 54$	1, because $x = x^1$

We classify polynomials by the number of terms and the degree as follows:

Table 3.1.3.

Expression	Classification	Degree
$5x^7$	**Monomial** (one term)	7
$8x^6 - 1$	**Binomial** (two terms)	6
$-3x^2 + x - 1$	**Trinomial** (three terms)	2
$5x^3 - 2x^2 + 3x - 6$	Polynomial (many terms)	3

In this text, we will call polynomials with four or more terms simply polynomials.

Example 1

Classify and state the degree:

$$7x^2 - 4x^5 - 1$$

Solution:

Here, there are three terms. The highest variable exponent is 5. Therefore, this is a trinomial of degree 5.

Answer

Trinomial; degree 5

Example 2

Classify and state the degree:

$$12a^5bc^3$$

Solution:

Since the expression consists of only multiplication, it is one term, a **monomial**. The variable part can be written as $a^5b^1c^3$; hence its degree is $5 + 1 + 3 = 9$.

Answer

Monomial; degree 9

Example 3

Classify and state the degree:

$$4x^2y - 6xy^4 + 5x^3y^3 + 4$$

Solution:
The term $4x^2y$ has degree 3; $-6xy^4$ has degree 5; $5x^3y^3$ has degree 6; and the constant term 4 has degree 0. Therefore, the polynomial has 4 terms with degree 6.

Answer

Polynomial; degree 6

Of particular interest are polynomials with one variable, where each term is of the form a_nx^n. Here a_n is any real number and n is any whole number. Such polynomials have the **standard form**

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

Typically, we arrange terms of polynomials in descending order based on the degree of each term. The **leading coefficient** is the coefficient of the variable with the highest power, in this case, a_n.

Example 4

Write in standard form:

$$3x - 4x^2 + 5x^3 + 7 - 2x^4$$

Solution:
Since terms are separated by addition, rewrite subtraction as addition of negatives, then reorder by descending degree:

$$\begin{aligned} 3x - 4x^2 + 5x^3 + 7 - 2x^4 &= 3x + (-4x^2) + 5x^3 + 7 + (-2x^4) \\ &= (-2x^4) + 5x^3 + (-4x^2) + 3x + 7 \\ &= -2x^4 + 5x^3 - 4x^2 + 3x + 7 \end{aligned}$$

Answer

$-2x^4 + 5x^3 - 4x^2 + 3x + 7$

We can further classify polynomials with one variable by their degree as follows:

Table 3.1.4.

Polynomial	Name
5	**Constant** (degree 0)
$2x + 1$	**Linear** (degree 1)
$3x^2 + 5x - 3$	**Quadratic** (degree 2)
$x^3 + x^2 + x + 1$	**Cubic** (degree 3)
$7x^4 + 3x^3 - 7x + 8$	Fourth-degree polynomial

In this text, we call any polynomial of degree $n \geq 4$ an nth-degree polynomial. In other words, if the degree is 4, we call the polynomial a fourth-degree polynomial. If the degree is 5, we call it a fifth-degree polynomial, and so on.

Visualize It

These expressions can be used to describe the chemical, biological and physical world around us. Variables represent real values. What does it mean to perform these expressions? It's important to be able to translate the words into math. Give this a try:

Visualize

Visit the Geogebra site to practice Algebraic Expression Matching. Link available on page 3.1 of the digital version of this textbook.

Visualize

Visit the Geogebra site to practice Verbal to Algebraic: Translating. Link available on page 3.1 of the digital version of this textbook.

Evaluating Polynomials

Given the values for the variables in a polynomial, we can substitute and simplify using the order of operations.

Science Application

A chemical plume is detected 15 meters (x_0) away from a monitoring station. The groundwater is carrying the plume at an initial velocity v_0 of 2 meters per day. Due to a change in the water table slope, the plume is accelerating a at a rate of 0.8 meters per day squared.

Calculate the distance of the plume from the station after 10 days t.

Substitute the values into the formula:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$x = 15 + (2)(10) + \frac{1}{2}(0.8)(10^2)$$

Use the order of operations to evaluate:

Calculate the term with the exponent first.

$$x = 15 + (20) + \frac{1}{2}(0.8)(100)$$

Multiply the factors in the third term:

$$(0.8) \times 100 = 80$$

$$\frac{1}{2} \text{of } 80 = 40$$

So the equation becomes:

$$x = 15 + 20 + 40$$

Then sum the terms

The chemical plume will be:

$$x = 75 \text{ meters away}$$

Example 5

Evaluate:

$$3x - 1, \text{ where } x = -\frac{3}{2}$$

Solution:

First, replace the variable with parentheses and then substitute the given value.

$$\begin{aligned} 3x - 1 &= 3\left(-\frac{3}{2}\right) - 1 \\ &= -\frac{9}{2} - 1 \\ &= -\frac{9}{2} - \frac{2}{2} \\ &= -\frac{11}{2} \end{aligned}$$

Answer

$$-\frac{11}{2}$$

Example 6

Evaluate:

$$3x^2 + 2x - 1, \text{ where } x = -1$$

Solution:

$$\begin{aligned} 3x^2 + 2x - 1 &= 3(-1)^2 + 2(-1) - 1 \\ &= 3 \cdot 1 - 2 - 1 \\ &= 0 \end{aligned}$$

Answer

0

Example 7

Evaluate:

$$-2a^2b + ab^2 - 7, \text{ where } a = 3 \text{ and } b = -2$$

Solution:

$$\begin{aligned} -2a^2b + ab^2 - 7 &= -2(3)^2(-2) + (3)(-2)^2 - 7 \\ &= -2 \cdot 9(-2) + 3 \cdot 4 - 7 \\ &= 36 + 12 - 7 \\ &= 41 \end{aligned}$$

Answer

41

Example 8

The volume of a sphere in cubic units is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius. Calculate the volume of a sphere with radius $r = \frac{3}{2}$ meters.

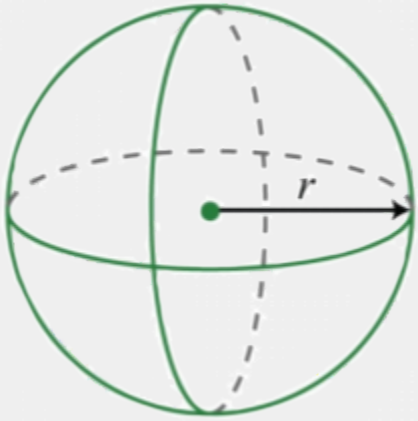

Figure 3.1.1.

Solution:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi\left(\frac{3}{2}\right)^3 \\ &= \frac{4}{3}\pi\frac{3^3}{2^3} \\ &= \frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{3}}}\pi\frac{\overset{9}{\cancel{27}}}{\underset{2}{\cancel{8}}} \\ &= \frac{9}{2}\pi \end{aligned}$$

Answer

$\frac{9}{2}\pi$ cubic meters

Try It

$$x^3 - x^2 + 4x - 2, \text{ where } x = -3$$

Show Solution

-50

Key Takeaways

- Polynomials are special algebraic expressions where the terms are the products of real numbers and variables with whole-number exponents.
- The degree of a polynomial with one variable is the largest exponent of the variable found in any term.
- The terms of a polynomial are typically arranged in descending order based on the degree of each term.
- When evaluating a polynomial, it is a good practice to replace all variables with parentheses and then substitute the appropriate values.

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How To: "1.4 Polynomials" from College Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Used How To box "Given a polynomial expression, identify the degree and leading coefficient"; Reformatted.

3.2 Adding and Subtracting Polynomials

Learning Objectives

By the end of this section, you should be able to:

- Add polynomials.
- Subtract polynomials.

Adding Polynomials

Recall that we combine like terms (terms with the same variable part) to simplify expressions. To do this, add the coefficients of the terms to obtain a single term with the same variable part. For example,

$$5x^2 + 8x^2 = 13x^2$$

Notice that the variable part, x^2, does not change.

Example 1

Add:

$$3x + (4x - 5)$$

Solution:
The property $+(a + b) = a + b$, which was derived using the **distributive property**, allows us to remove the parentheses so that we can add like terms.

$$\begin{aligned} 3x + (4x - 5) &= 3x + 4x - 5 \\ &= 7x - 5 \end{aligned}$$

Answer

$7x - 5$

Example 2

Add:

$$(3x^2 + 3x + 5) + (2x^2 - x - 2)$$

Solution:
Remove the parentheses and then combine like terms. Remember, the variable needs to have the same exponent to be "like".

$$\begin{aligned}(3x^2 + 3x + 5) + (2x^2 - x - 2) &= \underline{3x^2} + \underline{\underline{3x}} + 5 + \underline{2x^2} - \underline{\underline{x}} - 2 \\ &= 5x^2 + 2x + 3\end{aligned}$$

Answer

$5x^2 + 2x + 3$

Example 3

Add:

$$(-5x^2y - 2xy^2 + 7xy) + (4x^2y + 7xy^2 - 3xy)$$

Solution:
Remember that the variable parts have to be exactly the same before we can add the coefficients.

$$\begin{aligned}&(-5x^2y - 2xy^2 + 7xy) + (4x^2y + 7xy^2 - 3xy)\\ &= -5x^2y - 2xy^2 + 7xy + 4x^2y + 7xy^2 - 3xy\\ &= -x^2y + 5xy^2 + 4xy\end{aligned}$$

Answer

$-x^2y + 5xy^2 + 4xy$

We typically present polynomials in standard form, with terms in order from highest to lowest degree.

Example 4

Add:

$$(a - 4a^3 + a^5 - 8) + (-9a^5 + a^4 - 7a + 5 + a^3)$$

Solution:

$$\begin{aligned}(a - 4a^3 + a^5 - 8) + (-9a^5 + a^4 - 7a + 5 + a^3) &= a - 4a^3 + a^5 - 8 - 9a^5 + a^4 - 7a + 5 + a^3\\ &= -8a^5 + a^4 - 3a^3 - 6a - 3\end{aligned}$$

Answer

$-8a^5 + a^4 - 3a^3 - 6a - 3$

Try It

Add:

$$(6 - 5x^3 + x^2 - x) + (x^2 + x + 6x^3 - 1)$$

Show Solution

$x^3 + 2x^2 + 5$

Subtracting Polynomials

When subtracting polynomials, parentheses become very important. Recall that the distributive property allowed us to derive the following:

$$-(a + b) = -a - b$$

In order to remove the parentheses, we need to multiply each term inside by -1 (or, in other words, distribute the -1 over each term inside).

How To

Given multiple polynomials, add or subtract them to simplify the expressions.

1. Combine like terms.
2. Simplify and write in standard form.

Example 5

Subtract:

$$10x - (3x + 5)$$

Solution:
Subtract each term within the parentheses and then combine like terms.

$$10x - (3x + 5) = 10x - 3x - 5$$
$$= 7x - 5$$

Answer

$7x - 5$

Try these problems to see if you are understanding:

Visualize

Visit the Geogebra site to practice Operations of Polynomials. Link available on page 3.2 of the digital version of this textbook.

Subtracting a quantity is equivalent to multiplying it by -1.

Example 6

Subtract:

$$(3x^2 + 3x + 5) - (2x^2 - x - 2).$$

Solution:
Distribute the -1, remove the parentheses, and then combine like terms.

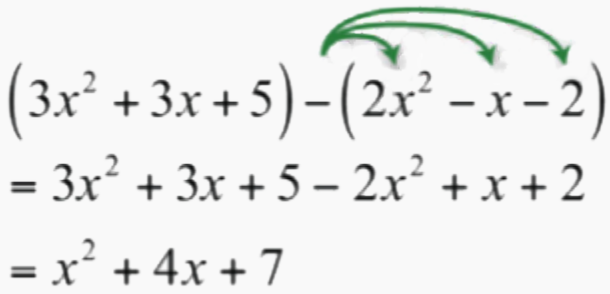

Figure 3.2.1.

Image Description

The top of the image displays green arrows with a singular point starting at the negative symbol pointing to each term in the second set of brackets, showing the distribution of the negative symbol

$$(3x^2+3x+5)-(2x^2-x-2)$$
$$=3x^2+3x+5-2x^2+x+2$$
$$=x^2+4x+7$$

Answer

x^2+4x+7

Multiplying the terms of a polynomial by -1 changes all the signs.

Example 7

Subtract:

$$(-5x^3-2x^2+7)-(4x^3+7x^2-3x+2)$$

Solution:
Distribute the -1, remove the parentheses, and then combine like terms.

$$\begin{aligned}(-5x^3 - 2x^2 + 7) - (4x^3 + 7x^2 - 3x + 2) &= -5x^3 - 2x^2 + 7 - 4x^3 - 7x^2 + 3x - 2 \\ &= -9x^3 - 9x^2 + 3x + 5\end{aligned}$$

Answer

$-9x^3 - 9x^2 + 3x + 5$

Example 8

Subtract:

$$6x^2 - 3x - 1 \text{ from } 2x^2 + 5x - 2$$

Solution

Since order matters in subtraction, we must take care to set up the difference correctly. First, write the quantity $(2x^2 + 5x - 2)$; from this, subtract the quantity $(6x^2 - 3x - 1)$.

$$\begin{aligned}(2x^2 + 5x - 2) - (6x^2 - 3x - 1) &= 2x^2 + 5x - 2 - 6x^2 + 3x + 1 \\ &= -4x^2 + 8x - 1\end{aligned}$$

Answer

$-4x^2 + 8x - 1$

Example 9

Simplify:

$$(2x^2 - 3x + 5) - (x^2 - 3x + 1) + (5x^2 - 4x - 8)$$

Solution:
Multiply -1 by every term inside the second set of parentheses, then remove the parentheses, and then combine like terms.

$$\begin{aligned}(2x^2 - 3x + 5) - (x^2 - 3x + 1) + (5x^2 - 4x - 8) &= 2x^2 - 3x + 5 - x^2 + 3x - 1 + 5x^2 - 4x - 8 \\ &= 6x^2 - 4x - 4\end{aligned}$$

Answer

$6x^2 - 4x - 4$

Try It

Subtract:

$$(8x^2y - 5xy^2 + 6) - (x^2y + 2xy^2 - 1)$$

Show Solution

$7x^2y - 7xy^2 + 7$

Key Takeaways

- When adding polynomials, remove the associated parentheses and then combine like terms.
- When subtracting polynomials, distribute the -1 and subtract all the terms before removing the parentheses and combining like terms.

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3.3 Multiplying Polynomials

Learning Objectives

By the end of this section, you should be able to:

- Multiply a polynomial by a monomial.
- Multiply a polynomial by a binomial.
- Multiply a polynomial by any size polynomial.
- Recognize and calculate special products.

Multiplying by a Monomial

Recall the **product rule for exponents**: if m and n are positive integers, then

$$x^m \cdot x^n = x^{m+n}$$

In other words, when multiplying two expressions with the same base, add the exponents. This rule applies when multiplying a monomial by a monomial. To find the product of monomials, multiply the coefficients and add the exponents of variable factors with the same base. For example,

$$\begin{aligned} 3x \cdot 5x^2 &= 3 \cdot 5 \cdot x^1 \cdot x^2 && \text{Commutative property} \\ &= 15x^{1+2} && \text{Product rule for exponents} \\ &= 15x^3 \end{aligned}$$

To multiply a polynomial by a monomial, apply the distributive property and then simplify each term.

Having trouble understanding what these expressions mean?

Visualize

Visit the Geogebra site to practice Multiplying Monomials with Polynomials Using Area Models. Link available on page 3.3 of the digital version of this textbook.

Example 1

Multiply:

$$-5x(4x-2)$$

Solution:

Multiply the monomial $-5x$ by each term in the binomial $4x-2$.

$$\begin{aligned} &-5x(4x-2) \\ &= (-5x)\cdot 4x + (-5x)\cdot(-2) \\ &= -20x^2 + 10x \end{aligned}$$

Answer

$-20x^2 + 10x$

Example 2

Multiply:

$$2x^2(3x^2-5x+1)$$

Solution:

Apply the distributive property and then simplify.

$$\begin{aligned} &2x^2(3x^3-5x+1) \\ &= 2x^2\cdot 3x^2 + 2x^2\cdot 5x + 2x^2\cdot 1 \\ &= 6x^4 - 10x^3 + 2x^2 \end{aligned}$$

Answer

$6x^4 - 10x^3 + 2x^2$

Example 3

Multiply:

$$-3ab^2(a^2b^3 + 2a^3b - 6ab - 4)$$

Solution:

Distribute $-3ab^2$ to each term and use the product rule for exponents.

$$\begin{aligned} -3ab^2(a^2b^3 + 2a^3b - 6ab - 4) &= -3ab^2 \cdot a^2b^3 - 3ab^2 \cdot 2a^3b + (-3ab^2)(-6ab) + (-3ab^2)(-4) \\ &= -3a^3b^5 - 6a^4b^3 + 18a^2b^3 + 12ab^2 \end{aligned}$$

Answer

$$-3a^3b^5 - 6a^4b^3 + 18a^2b^3 + 12ab^2$$

To summarize, multiplying a polynomial by a monomial involves the distributive property and the product rule for exponents. Multiply all of the terms of the polynomial by the monomial. For each term, multiply the coefficients and add exponents of variables where the bases are the same.

How To

Given the multiplication of two polynomials, use the distributive property to simplify the expression.

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

Try It

Multiply:

$$-5x^2y(2xy^2 - 3xy + 6x^2y - 1)$$

Show Solution

$$-10x^3y^3 + 15x^3y^2 - 30x^4y^2 + 5x^2y$$

Multiplying by a Binomial

In the same way that we used the distributive property to find the product of a monomial and a binomial, we will use it to find the product of two binomials.

$$\begin{aligned}(a+b)(c+d) &= (a+b)\cdot c + (a+b)\cdot d \\ &= ac + bc + ad + bd \\ &= ac + ad + bc + bd\end{aligned}$$

Here, we apply the distributive property multiple times to produce the final result. This same result is obtained in one step if we apply the distributive property to a and b separately as follows:

$$(a+b)(c+d) = ac + ad + bc + bd$$

This is often called the **FOIL method**. We add the products of the first terms of each binomial ac, the outer terms ad, the inner terms bc, and finally the last terms bd. This mnemonic device only works for products of binomials; hence, it is best to just remember that the distributive property applies.

How To

Given two binomials, use FOIL to simplify the expression.

1. Multiply the first terms of each binomial.
2. Multiply the outer terms of the binomials.
3. Multiply the inner terms of the binomials.

4. Multiply the last terms of each binomial.
5. Add the products.
6. Combine like terms and simplify.

Example 4

Multiply:

$$(2x+3)(5x-2)$$

Solution:

Distribute $2x$ and then distribute 3. Combine like terms.

$$\begin{aligned}(2x+3)(5x-2) &= 2x(5x-2)+3(5x-2)\\ &= 10x^2-4x+15x-6\\ &= 10x^2+11x-6\end{aligned}$$

Answer

$10x^2+11x-6$

Example 5

Multiply:

$$\left(\frac{1}{2}x-\frac{1}{4}\right)\left(\frac{1}{2}x+\frac{1}{4}\right)$$

Solution:

Distribute each term and combine like terms.

$$\begin{aligned}\left(\frac{1}{2}x-\frac{1}{4}\right)\left(\frac{1}{2}x+\frac{1}{4}\right) &= \frac{1}{2}x\frac{1}{2}x+\frac{1}{2}x\cdot\frac{1}{4}+\left(-\frac{1}{4}\right)\cdot\frac{1}{2}x+\left(-\frac{1}{4}\right)\cdot\frac{1}{4}\\ &= \frac{1}{4}x^2+\frac{1}{8}x-\frac{1}{8}x-\frac{1}{16}\\ &= \frac{1}{4}x^2-\frac{1}{16}\end{aligned}$$

Answer

$\frac{1}{4}x^2-\frac{1}{16}$

Example 6

Multiply:

$$(3y^2-1)(2y+1)$$

Solution:
Distribute and combine like terms.

$$\begin{aligned}(3y^2-1)(2y+1) &= 3y^2(2y+1)-1(2y+1)\\ &= 6y^3+3y^2-2y-1\end{aligned}$$

Answer

$6y^3+3y^2-2y-1$

After applying the distributive property, combine any like terms.

Example 7

Multiply:

$$(x^2 - 5)(3x^2 - 2x + 2)$$

Solution:

Multiply each term of the trinomial by x^2 and by -5, then combine like terms.

$$\begin{aligned}(x^2 - 5)(3x^2 - 2x + 2) &= x^2(3x^2 - 2x + 2) - 5(3x^2 - 2x + 2) \\ &= 3x^4 - 2x^3 + 2x^2 - 15x^2 + 10x - 10 \\ &= 3x^4 - 2x^3 - 13x^2 + 10x - 10\end{aligned}$$

Answer

$3x^4 - 2x^3 - 13x^2 + 10x - 10$

Example 8

Multiply:

$$(2x - 1)^3$$

Solution:

Perform one product at a time.

$$
\begin{aligned}
(2x-1)^3 &= (2x-1)\underbrace{(2x-1)(2x-1)}_{\text{Multiply First}} \\
&= (2x-1)\underbrace{4x^2-2x-2x+1}_{\text{Simplify}} \\
&= \underbrace{(2x-1)(4x^2-4x+1)}_{\text{Multiply Second}} \\
&= 8x^3-8x^2+2x-4x^2+4x-1 \\
&= 8x^3-12x^2+6x-1
\end{aligned}
$$

Answer

$8x^3-12x^2+6x-1$

At this point, it is worth pointing out a common mistake:

$$(2x-1)^3 \neq (2x)^3-1^3$$

The confusion comes from the product-to-a-power rule of exponents, where we apply the power to all factors. Since there are two terms within the parentheses, that rule does not apply. Care should be taken to understand what is different in the following two examples:

$$
\begin{aligned}
(xy)^2 &= x^2y^2 \quad \checkmark \\
(x+y)^2 &\neq x^2+y^2 \quad x
\end{aligned}
$$

Try It

Multiply: $(2x-3)(7x^2-5x+4)$

Show Solution

$14x^3-31x^2+23x-12$

Visualize

Visit the Geogebra site to practice Multiplying Polynomials. Link available on page 3.3 of the digital version of this textbook.

Product of Polynomials

When multiplying polynomials, we apply the distributive property many times. Multiply all of the terms of each polynomial and then combine like terms.

Example 9

Multiply:

$$(2x^2 + x - 3)(x^2 - 2x + 5)$$

Solution:

Multiply each term in the first trinomial by each term in the second trinomial, then combine like terms.

$$\begin{aligned}(2x^2 + x - 3)(x^2 - 2x + 5) &= 2x^2(x^2 - 2x + 5) + x(x^2 - 2x + 5) - 3(x^2 - 2x + 5) \\ &= (2x^4 - 4x^3 + 10x^2) + (x^3 - 2x^2 + 5x) + (-3x^2 + 6x - 15) \\ &= 2x^4 - 3x^3 + 5x^2 + 11x - 15\end{aligned}$$

Answer

$$2x^4 - 3x^3 + 5x^2 + 11x - 15$$

Notice that when multiplying a trinomial by a trinomial, we obtain nine terms before simplifying. In fact, when multiplying an n-term polynomial by an m-term polynomial, we will obtain $n \times m$ terms.

$$(2x^2 + x - 3)(x^2 - 2x + 5) = 2x^4 - 3x^3 + 5x^2 + 11x - 15$$

Because it is easy to make a small calculation error, it is a good practice to trace through the steps mentally to verify that the operations were performed correctly. Alternatively, we can check by evaluating any value for x in both expressions to verify that the results are the same. Here we choose $x = 2$:

$$\begin{aligned}(2x^2+x-3)(x^2-2x+5)\big|_{x=2} &= \big(2(2)^2+2-3\big)\big((2)^2-2(2)+5\big)\\ &= (8+2-3)(4-4+5)\\ &= 7\cdot 5\\ &= 35\end{aligned}$$

$$\begin{aligned}\big(2x^4-3x^3+5x^2+11x-15\big)\big|_{x=2} &= 2(2^4)-3(2^3)+5(2^2)+11(2)-15\\ &= 32-24+20+22-15\\ &= 35\end{aligned}$$

Because the results could coincidentally be the same, a check by evaluating does not necessarily prove that we have multiplied correctly. However, after verifying a few values, we can be fairly confident that the product is correct.

Visualize

Visit the Geogebra site to practice Operations on Polynomials. Link available on page 3.2 of the digital version of this textbook.

Try It

Multiply: $(x^2-2x-3)^2$

Show Solution

$x^4-4x^3-2x^2+12x+9$

Special Products

In this section, the goal is to recognize certain special products that occur often in our study of algebra. We will develop three formulas that will be very useful as we move along. The three should be memorized as knowing them will speed up your work later on. We begin by considering the following two calculations:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a^2+ab+ba+b^2 \\ &= a^2+2ab+b^2\end{aligned} \qquad \begin{aligned}(a-b)^2 &= (a-b)(a-b) \\ &= a^2-ab-ba+b^2 \\ &= a^2-2ab+b^2\end{aligned}$$

This leads us to two formulas that describe **perfect square trinomials**:

$$(a+b)^2 = a^2+2ab+b^2$$
$$(a-b)^2 = a^2-2ab+b^2$$

We can use these formulas to quickly square a binomial.

How To

Given a binomial, square it using the formula for perfect square trinomials.

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

Example 10

Multiply:

$$(3x+5)^2$$

Solution:

Here $a = 3x$ and $b = 5$. Apply the formula.

$$\begin{array}{rcccccc} (a+b) & = & a^2 & +2 & a & b & c \\ & & \downarrow & & \downarrow & \downarrow & \downarrow \\ (3x+5)^2 & = & (3x)^2 & +2\cdot & (3x) & (5) & (5^2) \\ & = & 9x^2 & + & 30x & + & 25 \end{array}$$

Answer

$9x^2 + 30x + 25$

This process should become routine enough to be performed mentally.

Example 11

Multiply:

$$(x-4)^2$$

Solution:

Here $a = x$ and $b = 4$. Apply the appropriate formula.

$$\begin{array}{rcccccc} (a-b) & = & a^2 & -2 & a & b & b^2 \\ & & \downarrow & & \downarrow & \downarrow & \downarrow \\ (x-4)^2 & = & (x)^2 & +2\cdot & (x) & (4) & (4)^2 \\ & = & x^2 & - & 8x & + & 16 \end{array}$$

Answer

$x^2 - 8x + 16$

Our third special product follows:

$$\begin{aligned} (a+b)(a-b) &= a^2 - ab + ba - b^2 \\ &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2 \end{aligned}$$

This product is called the **difference of squares**:

$$(a+b)(a-b) = a^2 - b^2$$

The binomials $(a+b)$ and $(a-b)$ are called **conjugate binomials**. Therefore, when conjugate binomials are multiplied, the middle term is eliminated, and the product is itself a binomial.

How To

Given a binomial multiplied by a binomial with the same terms but the opposite sign, find the difference of squares.

1. Square the first term of the binomials.
2. Square the last term of the binomials.
3. Subtract the square of the last term from the square of the first term.

Example 12

Multiply:

$$(7x+4)(7x-4)$$

Solution:

Use the difference of squares formula: $(a+b)(a-b)=a^2-b^2$.

$$49x^2-16$$

Answer

$49x^2-16$

Try It

Multiply: $(-5x+2)^2$

Show Solution

$25x^2-20x+4$

Key Takeaways

- To multiply a polynomial by a monomial, apply the distributive property and then simplify each of the resulting terms.
- To multiply polynomials, multiply each term in the first polynomial with each term in the second polynomial. Then combine like terms.
- The product of an n-term polynomial and an m-term polynomial results in an $m \times n$ term polynomial before like terms are combined.
- Check results by evaluating values in the original expression and in your answer to verify that the results are the same.
- Use the formulas for special products to quickly multiply binomials that occur often in algebra.

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3.4 Dividing Polynomials

Learning Objectives

By the end of this section, you should be able to:

- Divide by a monomial.
- Divide by a polynomial using the division algorithm.

Dividing by a Monomial

Recall the **quotient rule for exponents**: if x is nonzero and m and n are positive integers, then

$$\frac{x^m}{x^n} = x^{m-n}$$

In other words, when dividing two expressions with the same base, subtract the exponents. This rule applies when dividing a monomial by a monomial. In this section, we will assume that all variables in the denominator are nonzero.

Example 1

Divide:

$$\frac{28y^3}{7y}$$

Solution:
Divide the coefficients and subtract the exponents of the variable y.

$$\begin{aligned}\frac{28y^3}{7y} &= \frac{28}{7}y^{3-1} \\ &= 4y^2\end{aligned}$$

Answer

$4y^2$

Example 2

Divide:

$$\frac{24x^7y^5}{8x^3y^2}$$

Solution:
Divide the coefficients and apply the quotient rule by subtracting the exponents of the like bases.

$$\begin{aligned}\frac{24x^7y^5}{8x^3y^2} &= \frac{24}{8}x^{7-3}y^{5-2} \\ &= 3x^4y^3\end{aligned}$$

Answer

$3x^4y^3$

When dividing many terms by one term, we may treat the lone term as a common denominator and break up the fraction using the following property:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

Applying this property results in terms that can be treated as quotients of monomials.

Example 3

Divide:

$$\frac{-5x^4 + 25x^3 - 15x^2}{5x^2}$$

Solution:

Break up the fraction by dividing each term in the numerator by the monomial in the denominator, and then simplify each term.

$$\begin{aligned}\frac{-5x^4 + 25x^3 - 15x^2}{5x^2} &= \frac{-5x^4}{5x^2} + \frac{25x^3}{5x^2} - \frac{15x^2}{5x^2} \\ &= -x^2 + 5x - 3\end{aligned}$$

Answer

$-x^2 + 5x - 3$

Check your division by multiplying the answer, the **quotient**, by the monomial in the denominator, the **divisor**, to see if you obtain the original numerator, the **dividend**.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient}$$

or

$$\text{dividend} = \text{divisor} \cdot \text{quotient}$$

$$\begin{aligned}5x^2 \cdot (-x^2 + 5x - 3) &= 5x^2 \cdot (-x^2)5x^2 \cdot 5x - 5x^2 \cdot 3 \\ &= -5x^4 + 25x^3 - 15x^2 \quad \checkmark\end{aligned}$$

Example 4

Divide:

$$\frac{9a^4b - 7a^3b^2 + 3a^2b}{-3a^2b}$$

Solution:

$$\frac{9a^4b}{-3a^2b} + \frac{-7a^3b^2}{-3a^2b} + \frac{3a^2b}{-3a^2b} = -3a^2 + \frac{7}{3}ab - 1$$

Answer

$-3a^2 + \frac{7}{3}ab - 1$ The check is optional and is left to the reader.

Try It

$$(16x^5 - 8x^4 + 5x^3 + 2x^2) \div (2x^2)$$

Show Solution

$$8x^3 - 4x^2 + \frac{5}{2}x + 1$$

Dividing by a Polynomial

The same technique outlined for dividing by a monomial *does not* work for polynomials with two or more terms in the denominator. In this section, we will outline a process called **polynomial long division**, which is based on the division algorithm for real numbers. For the sake of clarity, we will assume that all expressions in the denominator are nonzero.

How To

Given a polynomial and a binomial, use long division to divide the polynomial by the binomial.

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom binomial from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Example 5

Divide:

$$\frac{x^3+3x^2-8x-4}{x-2}$$

Solution:

Here $x-2$ is the divisor and x^3+3x^2-8x-4 is the dividend.

Step 1: To determine the first term of the quotient, divide the leading term of the dividend by the leading term of the divisor.

$$x-2 \quad \overline{)\,x^3+3x^2-8x-4}^{\;?}$$

Begin by dividing the leading terms

$$x^3 \div x = x^2$$

Step 2: Multiply the first term of the quotient by the divisor, remembering to distribute, and line up like terms with the dividend.

$$\begin{array}{r} x^2 \\ x-2 \quad \overline{)x^3+3x^2-8x-4} \\ \underline{x^3-2x^2} \end{array}$$

Distribute and line up like terms

$x^2 \cdot (x-2) = x^3 - 2x^2$

Step 3: Subtract the resulting quantity from the dividend. Take care to subtract both terms.

$$\begin{array}{r} x^2 \\ x-2 \quad \overline{)x^3+3x^2-8x-4} \\ \underline{-(x^3-2x^2)} \\ 5x^2 \end{array}$$

Subtract.

$-(x^3-2x^2) = -x^3 + 2x^2$

Step 4: Bring down the remaining terms and repeat the process from step 1.

$$\begin{array}{r} x^2 \quad ? \\ x-2 \quad \overline{)x^3+3x^2-8x-4} \\ \underline{-(x^3-2x^2)} \\ 5x^2-8x-4 \end{array}$$

Bring down the remaining terms.

Notice that the leading term is eliminated and that the result has a degree that is one less than the dividend. The complete process is illustrated below:

$$\begin{array}{r} 9x^2-12x+16 \\ 3x+4 \quad \overline{)27x^3+0x^2+0x+64} \\ \underline{-(27x^3+36x^2)} \\ -36x^2+0x+64 \\ \underline{-(-36x^2-48x)} \\ 48x+64 \\ \underline{-(48x+64)} \\ 0 \end{array}$$

Polynomial long division ends when the degree of the **remainder** is less than the degree of the divisor. Here the remainder is 0.

Therefore, the binomial divides the polynomial evenly, and the answer is the quotient shown above the division line.

$$\frac{x^3+3x^2-8x-4}{x-2}=x^2+5x+2$$

To check the answer, multiply the divisor by the quotient to see if you obtain the dividend:

$$x^3+3x^2-8x-4=(x-2)(x^2+5x+2)$$

Answer

x^2+5x+2

Next, we demonstrate the case where there is a nonzero remainder.

$$\begin{array}{r} \text{quotient} \\ \text{divisor}\overline{)\,\text{dividend}} \\ \vdots \\ \hline \text{remainder} \end{array}$$

Just as with real numbers, the final answer adds the fraction where the remainder is the numerator and the divisor is the denominator to the quotient. In general, when dividing, we have

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

If we multiply both sides by the divisor, we obtain

$$\text{dividend} = \text{quotient} \times \text{divisor} + \text{remainder}$$

Example 6

Divide:

$$\frac{6x^2-5x+3}{2x-1}$$

Solution:

Since the denominator is a binomial, begin by setting up polynomial long division.

$$\begin{array}{r} ? \\ 2x-1\ \overline{)\,6x^2-5x+3} \end{array}$$

To start, determine what monomial times $2x-1$ results in a leading term $6x^2$. This is the quotient of

the given leading terms: $(6x^2) \div (2x) = 3x$. Multiply $3x$ times the divisor $2x - 1$ and line up the result with like terms of the dividend.

$$\begin{array}{r} \color{green}{3x} \\ 2x-1 \quad \overline{)6x^2-5x+3} \\ 6x^2-3x \end{array}$$

Subtract the result from the dividend and bring down the constant term $+3$.

$$\begin{array}{r} \color{green}{3x \quad ?} \\ 2x-1 \quad \overline{)6x^2-5x+3} \\ \underline{-(6x^2-3x)} \\ -2x+3 \end{array}$$

Subtracting eliminates the leading term and $-5x - (-3x) = -5x + 3x = -2x$. The quotient of $-2x$ and $2x$ is -1. Multiply $2x - 1$ by -1 and line up the result.

$$\begin{array}{r} \color{green}{3x-1} \\ 2x-1 \quad \overline{)6x^2-5x+3} \\ \underline{-(6x^2-3x)} \\ -2x+3 \\ -2x+1 \end{array}$$

Subtract again and notice that we are left with a remainder.

$$\begin{array}{r} \color{green}{3x-1} \\ 2x-1 \quad \overline{)6x^2-5x+3} \\ \underline{-(6x^2-3x)} \\ -2x+3 \\ \underline{-(-2x+1)} \\ 2 \end{array}$$

The constant term 2 has degree 0, and thus the division ends. We may write

$$\frac{6x^2-5x+3}{2x-1} = \color{green}{3x-1} + \frac{\color{blue}{2}}{2x-1}$$

Answer

$$3x - 1 + \frac{2}{2x-1}$$

To check that this result is correct, we multiply as follows:

$$\begin{aligned}\text{quotient} \times \text{divisor} + \text{ remainder} &= (3x-1)(2x-1)+2 \\ &= 6x^2-3x-2x+1+2 \\ &= 6x^2-5x+3 = \text{dividend}\checkmark\end{aligned}$$

Occasionally, some of the powers of the variables appear to be missing within a polynomial. This can lead to errors when lining up like terms. Therefore, when first learning how to divide polynomials using long division, fill in the missing terms with zero coefficients, called **placeholders**.

Example 7

Divide:

$$\frac{27x^3+64}{3x+4}$$

Solution:

Notice that the binomial in the numerator does not have terms with degree 2 or 1. The division is simplified if we rewrite the expression with placeholders:

$$27x^3+64 = 27x^3+0x^2+0x+64$$

Set up polynomial long division:

$$3x+4 \;\overline{)\,27x^3+0x^2+0x+64}$$

$$?$$

We begin with $27x^3 \div 3x = 9x^2$ and work the rest of the division algorithm.

$$\begin{array}{r} 9x^2-12x+16 \\ 3x+4 \;\overline{)\,27x^3+0x^2+0x+64} \\ \underline{-(27x^3+36x^2)} \\ -36x^2+0x+64 \\ \underline{-(-36x^2-48x)} \\ 48x+64 \\ \underline{-(48x+64)} \\ 0 \end{array}$$

Answer

$9x^2 - 12x + 16$

Example 8

Divide:

$$\frac{3x^4 - 2x^3 + 6x^2 + 23x - 7}{x^2 - 2x + 5}$$

Solution:

$$\begin{array}{r} ? \\ x^2 - 2x + 5 \quad \overline{)3x^4 - 2x^3 + 6x^2 + 23x - 7} \end{array}$$

Begin the process by dividing the leading terms to determine the leading term of the quotient $3x^4 \div x^2 = 3x^2$. Take care to distribute and line up the like terms. Continue the process until the remainder has a degree less than 2.

$$\begin{array}{r} 3x^2 - 4x - 1 \\ x^2 - 2x + 5 \quad \overline{)3x^4 - 2x^3 + 6x^2 + 23x - 7} \\ \underline{-(3x^4 - 6x^3 + 15x^2)} \\ 4x^3 - 9x^2 + 23x - 7 \\ \underline{-(4x^3 - 8x^2 + 20x)} \\ -x^2 + 3x - 7 \\ \underline{-(-x^2 + 2x - 5)} \\ x - 2 \end{array}$$

The remainder is $x - 2$. Write the answer with the remainder:

$$\frac{3x^4 - 2x^3 + 6x^2 + 23x - 7}{x^2 - 2x + 5} = 3x^2 + 4x - 1 + \frac{x - 2}{x^2 - 2x + 5}$$

Answer

$$3x^2 + 4x - 1 + \frac{x - 2}{x^2 - 2x + 5}$$

Polynomial long division takes time and practice to master. Work lots of problems and remember that you may check your answers by multiplying the quotient by the divisor (and adding the remainder if present) to obtain the dividend.

Try It

Divide: $\dfrac{20x^4 - 32x^3 + 7x^2 + 8x - 10}{5x - 3}$

Show Solution

$$4x^3 - 4x^2 - x + 1 - \frac{7}{5x - 3}$$

Key Takeaways

- When dividing by a monomial, divide all terms in the numerator by the monomial and then simplify each term. To simplify each term, divide the coefficients and apply the quotient rule for exponents.
- When dividing a polynomial by another polynomial, apply the division algorithm.
- To check the answer after dividing, multiply the divisor by the quotient and add the remainder (if necessary) to obtain the dividend.
- It is a good practice to include placeholders when performing polynomial long division.

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How To: "5.4 Dividing Polynomials" from College Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Used

How To box “Given a polynomial and a binomial, use long division to divide the polynomial by the binomial”; Reformatted.

3.5 Exercises

3.1 Polynomials

Classify the given polynomial as linear, quadratic, or cubic.

1. $2x + 1$
2. $x^2 + 7x + 2$
3. $2 - 3x^2 + x$
4. $4x$
5. $x^2 - x^3 + x + 1$
6. $5 - 10x^3$

Classify the given polynomial as a monomial, binomial, or trinomial and state the degree.

7. $x^3 - 1$
8. x^2y^2
9. $x - x^5 + 1$
10. $x^2 + 3x - 1$
11. $5ab^4$
12. $13x - 12$
13. $-5x^3 + 2x + 1$
14. $8x^2 - 9$
15. $4x^5 - 5x^3 + 6x$

16. $8x^4 - x^5 + 2x - 3$
17. $9x + 7$
18. $x^5 + x^4 + x^3 + x^2 - x + 1$
19. $6x - 1 + 5x^4 - 8$
20. $4x - 3x^2 + 3$
21. 7
22. x^2
23. $4x^2y - 3x^3y^3 + xy^3$
24. $a^3b^2 - 6ab$
25. a^3b^3
26. $x^2y - y^2x$
27. $xy - 3$
28. $a^5bc^2 + 3a^9 - 5a^4b^3c$
29. $-3x^{10}y^2z - xy^{12}z + 9x^{13} + 30$
30. $7x^0$

Write the following polynomials in standard form.

31. $1 - 6x + 7x^2$
32. $x - 9x^2 - 8$
33. $7 - x^3 + x^7 - x^2 + x - 5x^5$
34. $a^3 - a^9 + 6a^5 - a + 3 - a^4$
35. Fill in the following charts:

Table 3.5.1

X	X^2
-3	
-2	
0	
2	
3	

Table 3.5.2

X	X^3
-3	
-2	
0	
2	
3	

Evaluate.

36. $x^2 - 3x + 5$, where $x = -2$

37. $-12x + 13$, where $x = -13$

38. $-x^2 + 5x - 1$, where $x = -12$

39. $-2x^2 + 3x - 5$, where $x = 0$

40. $8x^5 - 27x^3 + 81x - 17$, where $x = 0$

41. $y^3 - 2y + 1$, where $y = -2$

42. $y^4 + 2y^2 - 32$, where $y = 2$

43. $a^3 + 2a^2 + a - 3$, where $a = -3$

44. $x^3 - x^2$, where $x = 5$

45. $34x^2 - 12x + 36$, where $x = -23$

46. $58x^2 - 14x + 12$, where $x = 4$

47. $x^2y + xy^2$, where $x = 2$ and $y = -3$

48. $2a^5b - ab^4 + a^2b^2$, where $a = -1$ and $b = -2$

49. $a^2 - b^2$, where $a = 5$ and $b = -6$

50. $a^2 - b^2$, where $a = 34$ and $b = -14$

51. $a^3 - b^3$, where $a = -2$ and $b = 3$

52. $a^3 + b^3$, where $a = 5$ and $b = -5$

For each problem, evaluate $b^2 - 4ac$, given the following values.

53. $a = -1, b = 2$, and $c = -1$

54. $a = 2, b = -2$, and $c = 12$

55. $a = 3, b = -5, c = 0$

56. $a = 1, b = 0$, and $c = -4$

57. $a = 14, b = -4$, and $c = 2$

58. $a = 1, b = 5$, and $c = 6$

The volume of a sphere in cubic units is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius. For each problem, calculate the volume of a sphere given the following radii.

59. $r = 3$ centimeters

60. $r = 1$ centimeter

61. $r = \frac{1}{2}$ feet

62. $r = \frac{3}{2}$ feet

63. $r = 0.15$ in

64. $r = 1.3$ inches

The height in feet of a projectile launched vertically from the ground with an initial velocity v_0 in feet per second is given by the formula $h = -16t^2 + v_0t$, where t represents time in seconds. For each problem, calculate the height of the projectile given the following initial velocity and times.

65. $v_0 = 64$ feet/second, at times $t = 0, 1, 2, 3, 4$ seconds

66. $v_0 = 80$ feet/second, at times $t = 0, 1, 2, 2.5, 3, 4, 5$ seconds

The stopping distance of a car, taking into account an average reaction time, can be estimated with the formula $d = 0.05v^2 + 1.5$, where d is in feet and v is the speed in miles per hour. For each problem, calculate the stopping distance of a car travelling at the given speeds.

67. 20 miles per hour

68. 40 miles per hour

69. 80 miles per hour

70. 100 miles per hour

Given the linear function $f(x) = \frac{2}{3}x + 6$, evaluate each of the following.

71. $f(-6)$

72. $f(-3)$

73. $f(0)$

74. $f(3)$

75. Find x when $f(x) = 10$

76. Find x when $f(x) = -4$

Show Solutions

1. Linear
2. Quadratic
3. Quadratic

4. Linear
5. Cubic
6. Cubic
7. Binomial; degree 3
8. Monomial; degree 4
9. Trinomial; degree 5
10. Trinomial; degree 2
11. Monomial; degree 5
12. Binomial; degree 1
13. Trinomial; degree 3
14. Binomial; degree 2
15. Trinomial; degree 5
16. Polynomial; degree 5
17. Binomial; degree 1
18. Polynomial; degree 5
19. Trinomial; degree 4
20. Trinomial; degree 2
21. Monomial; degree 0
22. Monomial; degree 2
23. Trinomial; degree 6

24. Binomial; degree 5

25. Monomial; degree 6

26. Binomial; degree 3

27. Binomial; degree 2

28. Trinomial; degree 9

29. Polynomial; degree 14

30. Monomial; degree 0

31. $7x^2 - 6x + 1$

32. $-9x^2 + x - 8$

33. $x^7 - 5x^5 - x^3 - x^2 + x + 7$

34. $-a^9 + 6a^5 - a^4 + a^3 - a + 3$

35. **Table 3.5.3.**

X	X^2
-3	9
-2	4
0	0
2	4
3	9

Table 3.5.4.

X	X^3
-3	-27
-2	-8
0	0
2	8
3	27

36. 15
37. 169
38. -205
39. -5
40. -17
41. -3
42. -8
43. -15
44. 100
45. $18,298$
46. 884
47. 6
48. 24
49. -11
50. 960
51. -35
52. 0
53. 0
54. -92
55. 25

56. 16

57. -96

58. 1

59. 36π cubic centimeters

60. $\frac{4}{3}\pi$ cubic centimeters

61. $\frac{\pi}{6}$ cubic feet

62. $\frac{9\pi}{2}$ cubic feet

63. 0.0045π cubic inches

64. $\frac{2197}{750}\pi$ cubic inches

65. $h = 0, 48, 64, 48, 0$ feet at $t = 0, 1, 2, 3, 4$ seconds

66. $h = 0, 64, 96, 100, 96, 64, 0$ feet at $t = 0, 1, 2, 2.5, 3, 4, 5$ seconds

67. 21.5 feet

68. 81.5 feet

69. 321.5 feet

70. 501.5 feet

71. 2

72. 4

73. 6

74. 8

75. $x = 6$

76. $x = -15$

3.2 Adding and Subtracting Polynomials

Add.

1. $(2x + 1) + (-x + 7)$
2. $(-6x + 5) + (3x - 1)$
3. $\left(\frac{2}{3}x + \frac{1}{2}\right) + \left(\frac{1}{3}x - 2\right)$
4. $\left(\frac{1}{3}x - \frac{3}{4}\right) + \left(\frac{5}{6}x + \frac{1}{8}\right)$
5. $(2x + 1) + (x - 3) + (5x - 2)$
6. $(2x - 8) + (-3x^2 + 7x - 5)$
7. $(x^2 - 3x + 7) + (3x^2 - 8x - 5)$
8. $(-5x^2 - 1 + x) + (-x + 7x^2 - 9)$
9. $\left(\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{6}\right) + \left(-\frac{3}{2}x^2 + \frac{2}{3}x - 1\right)$
10. $\left(-\frac{3}{5}x^2 + \frac{1}{4}x - 6\right) + \left(2x^2 - \frac{3}{8}x + \frac{5}{2}\right)$

11. $(x^2+5)+(3x^2-2x+1)+(x^2+x-3)$

12. $(a^3-a^2+a-8)+(a^3+a^2+6a-2)$

13. $(a^3-8)+(-3a^3+5a^2-2)$

14. $(4a^5+5a^3-a)+(3a^4-2a^2+7)$

15. $(2x^2+5x-12)+(7x-5)$

16. $(3x+5)+(x^2-x+1)+(x^3+2x^2-3x+6)$

17. $(6x^5-7x^3+x^2-15)+(x^4+2x^3-6x+12)$

18. $(1+7x-5x^3+4x^4)+(-3x^3+5-x^2+x)$

19. $(x^2y^2-7xy+7)+(4x^2y^2-3xy-8)$

20. $(x^2+xy-y^2)+(7x^2-5xy+2y^2)$

21. $(2x^2+3xy-7y^2)+(-5x^2-3xy+8y^2)$

22. $(a^2b^2-100)+(2a^2b^2-3ab+20)$

23. $(ab^2-3a^2b+ab-3)+(-2a^2b+ab^2-7ab-1)$

24. $(10a^2b-7ab+8ab^2)+(6a^2b-ab+5ab^2)$

25. Find the sum of $2x+8$ and $7x-1$

26. Find the sum of $13x-15$ and $16x+110$

27. Find the sum of $x^2-10x+8$ and $5x^2-2x-6$

28. Find the sum of $a^2-5a+10$ and $-9a^2+7a-11$

29. Find the sum of x^2y^2-xy+6 and x^2y^2+xy-7

30. Find the sum of $x^2-9xy+7y^2$ and $-3x^2-3xy+7y^2$

Subtract.

31. $(5x-3)-(2x-1)$

32. $(-4x+1)-(7x+10)$

33. $\left(\frac{1}{4}x-\frac{3}{4}\right)-\left(\frac{3}{4}x+\frac{1}{8}\right)$

34. $\left(-\frac{3}{5}x+\frac{3}{7}\right)-\left(\frac{2}{5}x-\frac{3}{2}\right)$

35. $(x^2+7x-5)-(4x^2-5x+1)$

36. $(-6x^2+3x-12)-(-6x^2+3x-12)$

37. $(-3x^3+4x-8)-(-x^2+4x+10)$

38. $\left(\frac{1}{2}x^2+\frac{1}{3}x-\frac{3}{4}\right)-\left(\frac{3}{2}x^2-\frac{1}{6}x+\frac{1}{2}\right)$

39. $\left(\frac{5}{9}x^2+\frac{1}{5}x-\frac{1}{3}\right)-\left(\frac{1}{3}x^2+\frac{3}{10}x+\frac{5}{9}\right)$

40. $(a^3-4a^2+3a-7)-(7a^3-2a^2-6a+9)$

41. $(3a^3+5a^2-2)-(a^3-a+8)$

42. $(5x^5+4x^3+x^2-6)-(4x^4-3x^3-x+3)$

43. $(3-5x-x^3+5x^4)-(-5x^3+2-x^2-7x)$

44. $(x^5-6x^3+9x)-(4x^4+2x^2-5)$

45. $(2x^2y^2-4xy+9)-(3x^2y^2-3xy-5)$

46. $(x^2+xy-y^2)-(x^2+xy-y^2)$

47. $(2x^2+3xy-7y^2)-(-5x^2-3xy+8y^2)$

48. $(ab^2-3a^2b+ab-3)-(-2a^2b+ab^2-7ab-1)$

49. $(10a^2b-7ab+8ab^2)-(6a^2b-ab+5ab^2)$

50. $(10a^2b^2+5ab-6)-(5a^2b^2+5ab-6)$

51. Subtract $3x + 1$ from $5x - 9$.

52. Subtract $x^2 - 5x + 10$ from $x^2 + 5x - 5$.

53. Find the difference of $3x - 7$ and $8x + 6$.

54. Find the difference of $2x^2 + 3x - 5$ and $x^2 - 9$.

55. The cost in dollars of producing customized coffee mugs with a company logo is given by the formula $C = 150 + 0.10x$, where x is the number of cups produced. The revenue from selling the cups in the company store is given by $R = 10x - 0.05x^2$, where x is the number of units sold.

    a. Find a formula for the profit. (profit = revenue – cost)

    b. Find the profit from producing and selling 100 mugs in the company store.

56. The cost in dollars of producing sweatshirts is given by the formula $C = 10q + 1200$, where C is the cost and q represents the quantity produced. The revenue generated by selling the sweatshirts for 37 each is given by $R = 37q$, where q represents the quantity sold. Determine the profit generated if 125 sweatshirts are produced and sold.

57. The outer radius of a washer is 3 times the radius of the hole.

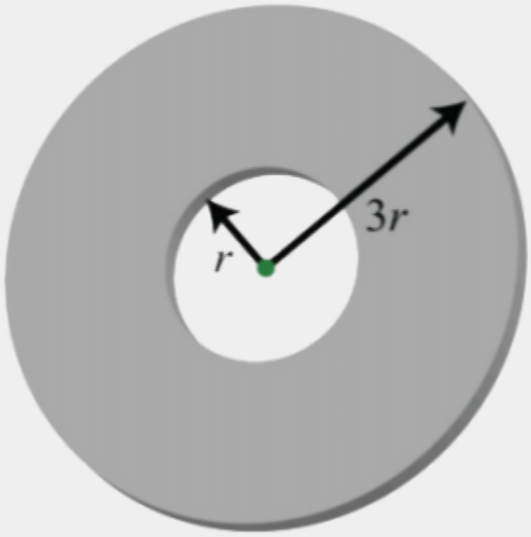

Figure 3.5.1.

    a. Derive a formula for the area of the face of the washer.

    b. What is the area of the washer if the hole has a diameter of 10 millimetres?

58. Derive a formula for the surface area of the following rectangular solid.

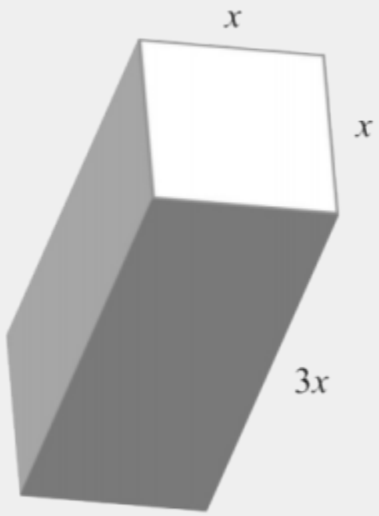

Figure 3.5.2.

Simplify.

59. $(2x+3)-(5x-8)+(x-7)$

60. $(3x-5)-(7x-11)-(5x+2)$

61. $(3x-2)-(4x-1)+(x+7)$

62. $(5x-3)-(2x+1)-(x-1)$

63. $(5x^2-3x+2)-(x^2+x-4)+(7x^2-2x-6)$

64. $(-2x^3+x^2-8)-(3x^2+x-6)-(2x-1)$

65. $(2x-7)-(x^2+3x-7)+(6x-1)$

66. $(6x^2-10x+13)+(4x^2-9)-(9-x^2)$

67. $(a^2-b^2)-(2a^2+3ab-4b^2)+(5ab-1)$

68. $(a^2-3ab+b^2)-(a^2+b^2)-(3ab-5)$

69. $\left(\frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{4}\right) - \left(\frac{3}{2}x - \frac{3}{4}\right) + \left(\frac{5}{4}x - \frac{1}{2}\right)$

70. $\left(\frac{9}{5}x^2 - \frac{1}{3}x + 2\right) - \left(\frac{3}{10}x^2 - \frac{4}{5}\right) - \left(x + \frac{5}{2}\right)$

Show Solutions

1. $x + 8$
2. $-3x + 4$
3. $x - \frac{3}{2}$
4. $\frac{7}{6}x - \frac{5}{8}$
5. $8x - 4$
6. $-3x^2 + 9x - 13$
7. $4x^2 - 11x + 2$
8. $2x^2 - 10$
9. $-x^2 + \frac{1}{3}x - \frac{5}{6}$
10. $\frac{7}{5}x^2 - \frac{1}{8}x - \frac{7}{2}$
11. $5x^2 - x + 3$
12. $2a^3 + 7a - 10$
13. $-2a^3 + 5a^2 - 10$

14. $4a^5 + 3a^4 + 5a^3 - 2a^2 - a + 7$
15. $2x^2 + 12x - 17$
16. $x^3 + 3x^2 - x + 12$
17. $6x^5 + x^4 - 5x^3 + x^2 - 6x - 3$
18. $4x^4 - 8x^3 - x^2 + 8x + 6$
19. $5x^2y^2 - 10xy - 1$
20. $8x^2 - 4xy + y^2$
21. $-3x^2 + y^2$
22. $3a^2b^2 - 3ab - 80$
23. $-5a^2b + 2ab^2 - 6ab - 4$
24. $16a^2b - 8ab + 13ab^2$
25. $9x + 7$
26. $29x + 95$
27. $6x^2 - 12x + 2$
28. $-8a^2 + 2a - 1$
29. $2x^2y^2 - 1$
30. $-2x^2 - 12xy + 14y^2$
31. $3x - 2$
32. $-11x - 9$

33. $-\frac{1}{2}x-\frac{7}{8}$

34. $-x+\frac{27}{14}$

35. $-3x^2+12x-6$

36. 0

37. $-3x^3+x^2-18$

38. $-x^2+\frac{1}{2}x-\frac{5}{4}$

39. $\frac{2}{9}x^2-\frac{1}{10}x-\frac{8}{9}$

40. $-6a^3-2a^2+9a-16$

41. $2a^3+5a^2+a-10$

42. $5x^5-4x^4+7x^3+x^2+x-9$

43. $5x^4+4x^3+x^2+2x+1$

44. $x^5-4x^4-6x^3-2x^2+9x+5$

45. $-x^2y^2-xy+14$

46. 0

47. $7x^2+6xy-15y^2$

48. $-a^2b+8ab-2$

49. $4a^2b+3ab^2-6ab$

50. $5a^2b^2$

51. $2x - 10$

52. $10x - 15$

53. $-5x - 13$

54. $x^2 + 3x + 4$

55. a. $P = -0.05x^2 + 9.9x - 150$; b. 340

56. $\$2,175$

57. a. $A = 8\pi r^2$; b. $200\pi \approx 628.32$ square millimeters

58. $14w^2$

59. $-2x + 4$

60. $-9x + 4$

61. 6

62. $2x - 3$

63. $11x^2 - 6x$

64. $-2x^3 - 2x^2 - 3x - 1$

65. $-x^2 + 5x - 1$

66. $11x^2 - 10x - 5$

67. $-a^2 + 2ab + 3b^2 - 1$

68. $-6ab + 5$

69. $\frac{1}{2}x^2 - x + \frac{1}{2}$

70. $\frac{3}{2}x^2 - \frac{4}{3}x + \frac{3}{10}$

3.3 Multiplying Polynomials

Multiply.

1. $5x(-3x^2y)$
2. $(-2x^3y^2)(-3xy^4)$
3. $\frac{1}{2}(4x - 3)$
4. $-\frac{3}{4}\left(\frac{2}{3}x - 6\right)$
5. $3x(5x - 2)$
6. $-4x(2x - 1)$
7. $x^2(3x + 2)$
8. $-6x^2(5x + 3)$
9. $2ab(4a - 2b)$
10. $5a^2b(a^2 - b^2)$
11. $6x^2y^3(-3x^3y + xy^2)$
12. $3ab^3(-5ab^3 + 6a^2b)$

13. $-\frac{1}{2}x^2y(4xy-10)$

14. $-3x^4y^2(3x^8y^3)$

15. $2x^2(-5x^3)(3x^4)$

16. $4ab(a^2b^3c)(a^4b^2c^4)$

17. $-2(5x^2-3x+4)$

18. $\frac{4}{5}(25x^2-50xy+5y^2)$

19. $3x(5x^2-2x+3)$

20. $-x(x^2+x-1)$

21. $x^2(3x^2-5x-7)$

22. $x^3(-4x^2-7x+9)$

23. $\frac{1}{4}x^4\left(8x^3-2x^2+\frac{1}{2}x-5\right)$

24. $-\frac{1}{3}x^3\left(\frac{3}{2}x^5-\frac{2}{3}x^3+\frac{9}{2}x-1\right)$

25. $a^2b(a^2-3ab+b^2)$

26. $6a^2bc^3(2a-3b+c^2)$

27. $\frac{2}{3}xy^2(9x^3y-27xy+3xy^3)$

28. $-3x^2y^2(12x^2-10xy-6y^2)$

29. Find the product of $3x$ and $2x^2-3x+5$.

30. Find the product of $-8y$ and $y^2-2y+12$.

31. Find the product of $-4x$ and $x^4-3x^3+2x^2-7x+8$.

32. Find the product of $3xy^2$ and $-2x^2y + 4xy - xy^2$.

33. $(3x - 2)(x + 4)$

34. $(x + 2)(x - 3)$

35. $(x - 1)(x + 1)$

36. $(3x - 1)(3x + 1)$

37. $(2x - 5)(x + 3)$

38. $(5x - 2)(3x + 4)$

39. $(-3x + 1)(x - 1)$

40. $(x + 5)(-x + 1)$

41. $\left(y - \frac{3}{2}\right)\left(y + \frac{3}{2}\right)$

42. $\left(\frac{1}{2}x + \frac{1}{3}\right)\left(\frac{2}{3}x - \frac{2}{3}\right)$

43. $\left(\frac{3}{4}x + \frac{1}{5}\right)\left(\frac{1}{4}x + \frac{2}{5}\right)$

44. $\left(\frac{1}{5}x + \frac{3}{10}\right)\left(\frac{3}{5}x - \frac{5}{2}\right)$

45. $(y^2 - 2)(y + 2)$

46. $(y^3 - 1)(y^2 + 2)$

47. $(a^2 - b^2)(a^2 + b^2)$

48. $(a^2 - 3b)^2$

49. $(x - 5)(2x^2 + 3x + 4)$

50. $(3x - 1)(x^2 - 4x + 7)$

51. $(2x-3)(4x^2+6x+9)$

52. $(5x+1)(25x^2-5x+1)$

53. $\left(x-\frac{1}{2}\right)(3x^2+4x-1)$

54. $\left(\frac{1}{3}x-\frac{1}{4}\right)(3x^2+9x-3)$

55. $(x+3)^3$

56. $(x-2)^3$

57. $(3x-1)^3$

58. $(2x+y)^3$

59. $(5x-2)(2x^3-4x^2+3x-2)$

60. $(x^2-2)(x^3-2x^2+x+1)$

61. $(x^2-x+1)(x^2+2x+1)$

62. $(3x^2-2x-1)(2x^2+3x-4)$

63. $(2x^2-3x+5)(x^2+5x-1)$

64. $(a+b+c)(a-b-c)$

65. $(a+2b-c)^2$

66. $(x+y+z)^2$

67. $(x-3)^4$

68. $(x+y)^4$

69. Find the volume of a rectangular solid with sides measuring x, $x+2$, and $x+4$ units.

70. Find the volume of a cube where each side measures $x-5$ units.

71. $(x+2)^2$

72. $(x-3)^2$

73. $(2x+5)^2$

74. $(3x-7)^2$

75. $(-x+2)^2$

76. $(-9x+1)^2$

77. $(a+6)^2$

78. $(2a-3b)^2$

79. $\left(\frac{2}{3}x+\frac{3}{4}\right)^2$

80. $\left(\frac{1}{2}x-\frac{3}{5}\right)^2$

81. $(x^2+2)^2$

82. $(x^2+y^2)^2$

83. $(x+4)(x-4)$

84. $(2x+1)(2x-1)$

85. $(5x+3)(5x-3)$

86. $\left(\frac{1}{5}x-\frac{1}{3}\right)\left(\frac{1}{5}x+\frac{1}{3}\right)$

87. $\left(\frac{2}{3}x+\frac{2}{5}\right)\left(\frac{2}{3}x-\frac{2}{5}\right)$

88. $(2x-3y)(2x+3y)$

89. $(4x-y)(4x+y)$

90. $(a^3 - b^3)(a^3 + b^3)$

91. A box is made by cutting out the corners and folding up the edges of a square piece of cardboard. A template for a cardboard box with a height of 2 inches is given. Find a formula for the volume if the initial piece of cardboard is a square with sides measuring x inches.

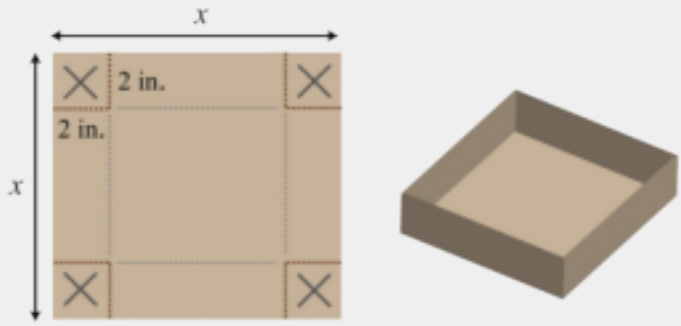

Figure 3.5.3.

92. A template for a cardboard box with a height of x inches is given. Find a formula for the volume if the initial piece of cardboard is a square with sides measuring 12 inches.

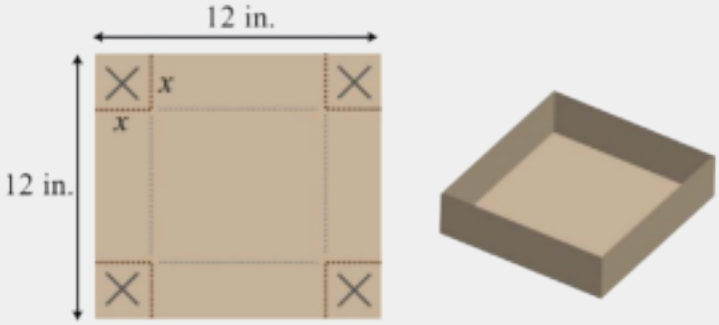

Figure 3.5.4.

Show Solutions

1. $-15x^3y$

2. $6x^4y^6$
3. $2x - \frac{3}{2}$
4. $-\frac{1}{2}x + \frac{9}{2}$
5. $15x^2 - 6x$
6. $-8x^2 + 4x$
7. $3x^3 + 2x^2$
8. $-30x^3 - 18x^2$
9. $8a^2b - 4ab^2$
10. $5a^4b - 5a^2b^3$
11. $-18x^5y^4 + 6x^3y^5$
12. $-15a^2b^6 + 18a^3b^4$
13. $-2x^3y^2 + 5x^2y$
14. $-9x^{12}y^5$
15. $-30x^9$
16. $4a^7b^6c^5$
17. $-10x^2 + 6x - 8$
18. $20x^2 - 40xy + 4y^2$
19. $15x^3 - 6x^2 + 9x$
20. $-x^3 - x^2 + x$

21. $3x^4 - 5x^3 - 7x^2$

22. $-4x^5 - 7x^4 + 9x^3$

23. $2x^7 - \frac{1}{2}x^6 + \frac{1}{8}x^5 - \frac{5}{4}x^4$

24. $-\frac{1}{2}x^8 + \frac{2}{9}x^6 - \frac{3}{2}x^4 + \frac{1}{3}x^3$

25. $a^4b - 3a^3b^2 + a^2b^3$

26. $12a^3bc^3 - 18a^2b^2c^3 + 6a^2bc^5$

27. $6x^4y^3 - 18x^2y^3 + 2x^2y^5$

28. $-36x^4y^2 + 30x^3y^3 + 18x^2y^4$

29. $6x^3 - 9x^2 + 15x$

30. $-8y^3 + 16y^2 - 96y$

31. $-4x^5 + 12x^4 - 8x^3 + 28x^2 - 32x$

32. $-6x^3y^3 + 12x^2y^3 - 3x^2y^4$

33. $3x^2 + 10x - 8$

34. $x^2 - x - 6$

35. $x^2 - 1$

36. $9x^2 - 1$

37. $2x^2 + x - 15$

38. $15x^2 + 14x - 8$

39. $-3x^2 + 4x - 1$

40. $-x^2 - 4x + 5$

41. $y^2 - \frac{9}{4}$

42. $\frac{1}{3}x^2 - \frac{1}{9}x - \frac{2}{9}$

43. $\frac{3}{16}x^2 + \frac{7}{20}x + \frac{2}{25}$

44. $\frac{3}{25}x^2 - \frac{8}{25}x - \frac{3}{4}$

45. $y^3 + 2y^2 - 2y - 4$

46. $y^5 + 2y^3 - y^2 - 2$

47. $a^4 - b^4$

48. $a^4 - 6a^2b + 9b^2$

49. $2x^3 - 7x^2 - 11x - 20$

50. $3x^3 - 13x^2 + 25x - 7$

51. $8x^3 - 27$

52. $125x^3 + 1$

53. $3x^3 + \frac{5}{2}x^2 - 3x + \frac{1}{2}$

54. $x^3 + \frac{9}{4}x^2 - \frac{13}{4}x + \frac{3}{4}$

55. $x^3 + 9x^2 + 27x + 27$

56. $x^3 - 6x^2 + 12x - 8$

57. $27x^3 - 27x^2 + 9x - 1$

58. $8x^3 + 12x^2y + 6xy^2 + y^3$

59. $10x^4 - 24x^3 + 23x^2 - 16x + 4$

60. $x^5 - 2x^4 - x^3 + 5x^2 - 2$

61. $x^4 + x^3 + x + 1$

62. $6x^4 + 5x^3 - 20x^2 + 5x + 4$

63. $2x^4 + 7x^3 - 12x^2 + 28x - 5$

64. $a^2 - b^2 - 2bc - c^2$

65. $a^2 + 4ab - 2ac + 4b^2 - 4bc + c^2$

66. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$

67. $x^4 - 12x^3 + 54x^2 - 108x + 81$

68. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

69. $x^3 + 6x^2 + 8x$

70. $x^3 - 15x^2 + 75x - 125$

71. $x^2 + 4x + 4$

72. $x^2 - 6x + 9$

73. $4x^2 + 20x + 25$

74. $9x^2 - 42x + 49$

75. $x^2 - 4x + 4$

76. $81x^2 - 18x + 1$

77. $a^2 + 12a + 36$

78. $4a^2 - 12ab + 9b^2$

79. $\frac{4}{9}x^2 + x + \frac{9}{16}$

80. $\frac{1}{4}x^2 - \frac{3}{5}x + \frac{9}{25}$

81. $x^4 + 4x^2 + 4$

82. $x^4 + 2x^2y^2 + y^4$

83. $x^2 - 16$

84. $4x^2 - 1$

85. $25x^2 - 9$

86. $\frac{1}{25}x^2 - \frac{1}{9}$

87. $\frac{4}{9}x^2 - \frac{4}{25}$

88. $4x^2 - 9y^2$

89. $16x^2 - y^2$

90. $a^6 - b^6$

91. $V = 2x^2 - 16x + 32extcubicinches$

92. $V = 4x^3 - 48x^2 + 144xextcubicinches$

3.4 Dividing Polynomials

Divide.

1. $\dfrac{81y^5}{9y^2}$
2. $\dfrac{36y^9}{9y^3}$
3. $\dfrac{52x^2y}{4xy}$
4. $\dfrac{24xy^5}{2xy^4}$
5. $\dfrac{25x^2y^5z^3}{5xyz}$
6. $\dfrac{-77x^4y^9z^2}{2x^3y^3z}$
7. $\dfrac{125a^3b^2c}{-10abc}$
8. $\dfrac{36a^2b^3c^5}{-6a^2b^2c^3}$
9. $\dfrac{9x^2+27x-3}{3}$
10. $\dfrac{10x^3-5x^2+40x-15}{5}$
11. $\dfrac{20x^3-10x^2+30x}{2x}$
12. $\dfrac{10x^4+8x^2-6x}{24x}$

13. $\dfrac{-6x^5 - 9x^3 + 3x}{-3x}$

14. $\dfrac{36a^{12} - 6a^9 + 12a^5}{-12a^5}$

15. $\dfrac{-12x^5 + 18x^3 - 6x^2}{-6x^2}$

16. $\dfrac{-49a^8 + 7a^5 - 21a^3}{7a^3}$

17. $\dfrac{9x^7 - 6x^4 + 12x^3 - x^2}{3x^2}$

18. $\dfrac{8x^9 + 16x^7 - 24x^4 + 8x^3}{-8x^3}$

19. $\dfrac{16a^7 - 32a^6 + 20a^5 - a^4}{4a^4}$

20. $\dfrac{5a^6 + 2a^5 + 6a^3 - 12a^2}{3a^2}$

21. $\dfrac{-4x^2y^3 + 16x^7y^8 - 8x^2y^5}{-4x^2y^3}$

22. $\dfrac{100a^{10}b^{30}c^5 - 50a^{20}b^5c^{40} + 20a^5b^{20}c^{10}}{10a^5b^5c^5}$

23. Find the quotient of $-36x^9y^7$ and $2x^8y^5$

24. Find the quotient of $144x^3y^{10}z^2$ and $-12x^3y^5z$

25. Find the quotient of $3a^4 - 18a^3 + 27a^2$ and $3a^2$

26. Find the quotient of $64a^2bc^3 - 16a^5bc^7$ and $4a^2bc^3$

27. $(2x^2 - 5x - 3) \div (x - 3)$

28. $(3x^2 + 5x - 2) \div (x + 2)$

29. $(6x^2 + 11x + 3) \div (3x + 1)$

30. $(8x^2 - 14x + 3) \div (2x - 3)$

31. $\dfrac{x^3 - x^2 - 2x - 12}{x - 3}$

32. $\dfrac{2x^3 + 11x^2 + 4x - 5}{x + 5}$

33. $\dfrac{2x^3 - x^2 - 4x + 3}{2x + 3}$

34. $\dfrac{-15x^3 - 14x^2 + 23x - 6}{5x - 2}$

35. $\dfrac{14x^4 - 9x^3 + 22x^2 + 4x - 1}{7x - 1}$

36. $\dfrac{8x^5 + 16x^4 - 8x^3 - 5x^2 - 21x + 10}{2x + 5}$

37. $\dfrac{x^2 + 8x + 17}{x + 5}$

38. $\dfrac{2x^2 - 5x + 5}{x - 2}$

39. $\dfrac{6x^2 - 13x + 9}{-2x + 1}$

40. $\dfrac{-12x^2 + x + 1}{3x + 2}$

41. $\dfrac{x^3 + 9x^2 + 19x + 1}{x + 4}$

42. $\dfrac{2x^3 - 13x^2 + 17x - 11}{x - 5}$

43. $\dfrac{9x^3 - 12x^2 + 16x - 15}{3x - 2}$

44. $\dfrac{3x^4 - 8x^3 + 5x^2 - 5x + 9}{x - 2}$

45. $(6x^5 - 13x^4 + 4x^3 - 3x^2 + 13x - 2) \div (3x + 1)$

46. $(8x^5 - 22x^4 + 19x^3 - 20x^2 + 23x - 3) \div (2x - 3)$

47. $\dfrac{5x^5 + 12x^4 + 12x^3 - 7x^2 - 19x + 3}{x^2 + 2x + 3}$

48. $\dfrac{6x^5 - 17x^4 + 5x^3 + 16x^2 - 7x - 3}{2x^2 - 3x - 1}$

49. $\dfrac{x^5 + 7x^4 - x^3 - 7x^2 - 49x + 9}{x^2 + 7x - 1}$

50. $\dfrac{5x^6 - 6x^4 - 4x^2 + x + 2}{5x^2 - 1}$

51. $\dfrac{x^3 - 27}{x - 3}$

52. $\dfrac{8x^3 + 125}{2x + 5}$

53. $(15x^5 - 9x^4 - 20x^3 + 12x^2 + 15x - 9) \div (5x - 3)$

54. $(2x^6 - 5x^5 - 4x^4 + 10x^3 + 6x^2 - 17x + 5) \div (2x - 5)$

55. $\dfrac{x^5 - 2x^3 + 3x - 1}{x - 1}$

56. $\dfrac{x^4 - 3x^2 + 5x - 1}{3x + 2}$

57. $\dfrac{a^2 - 4}{a + 2}$

58. $\dfrac{a^5 + 1}{a^5 + 1}$

59. $\dfrac{a^6 - 1}{a - 1}$

60. $\dfrac{x^5 - 1}{x - 1}$

61. $\dfrac{x^5 + x^4 + 6x^3 + 12x^2 - 4}{x^2 + x - 1}$

62. $\dfrac{50x^6 - 30x^5 - 5x^4 + 15x^3 - 5x + 1}{5x^2 - 3x + 2}$

63. $\dfrac{5x^5 - 15x^3 + 25x^2 - 5}{5x}$

64. $\dfrac{-36x^6 + 12x^4 - 6x^2}{6x^2}$

65. $\dfrac{150x^5y^2z^{15} - 10x^3y^6z^5 + 4x^3y^2z^4}{10x^3y^2z^5}$

66. $\dfrac{27m^6 + 9m^4 - 81m^2 + 1}{9m^2}$

67. Divide $3x^6 - 2x^5 + 27x^4 - 18x^3 - 6x^2 + 7x - 10$ by $3x - 2$

68. Divide $8x^6 + 4x^5 - 14x^4 - 5x^3 + x^2 - 2x - 3$ by $2x + 1$

Show Solutions

1. $9y^3$
2. $4y^6$
3. $13x$
4. $12y$

5. $5xy^4z^2$
6. $-\frac{77}{2}xy^6z$
7. $-\frac{25}{2}a^2b$
8. $-6bc^2$
9. $3x^2+9x-1$
10. $2x^3-x^2+8x-3$
11. $10x^2-5x+15$
12. $\frac{5}{12}x^3+\frac{1}{3}x-\frac{1}{4}$
13. $2x^4+3x^2-1$
14. $-3a^7+\frac{1}{2}a^4-1$
15. $2x^3-3x+1$
16. $-7a^5+a^2-3$
17. $3x^5-2x^2+4x-\frac{1}{3}$
18. $-x^6-2x^4+3x-1$
19. $4a^3-8a^2+5a-\frac{1}{4}$
20. $\frac{5}{3}a^4+\frac{2}{3}a^3+2a-4$
21. $-4x^5y^5+2y^2+1$

22. $10a^5b^{25} - 5a^{15}c^{35} + 2b^{15}c^5$

23. $-18xy^2$

24. $-12y^5z$

25. $a^2 - 6a + 9$

26. $16 - 4a^3c^4$

27. $2x + 1$

28. $3x - 1$

29. $2x + 3$

30. $4x - 1$

31. $x^2 + 2x + 4$

32. $2x^2 + x - 1$

33. $x^2 - 2x + 1$

34. $-3x^2 - 4x + 3$

35. $2x^3 - x^2 + 3x + 1$

36. $4x^4 - 2x^3 + x^2 - 5x + 2$

37. $x + 3 + \dfrac{2}{x + 5}$

38. $2x - 1 + \dfrac{3}{x - 2}$

39. $-3x + 5 + \dfrac{4}{-2x + 1}$

40. $-4x + 3 - \dfrac{5}{3x + 2}$

41. $x^2 + 5x - 1 + \dfrac{5}{x + 4}$

42. $2x^2 - 3x + 2 - \dfrac{1}{x - 5}$

43. $3x^2 - 2x + 4 - \dfrac{7}{3x - 2}$

44. $3x^3 - 2x^2 + x - 3 + \dfrac{3}{x - 2}$

45. $2x^4 - 5x^3 + 3x^2 - 2x + 5 - \dfrac{7}{3x + 1}$

46. $4x^4 - 5x^3 + 2x^2 - 7x + 1$

47. $5x^3 + 2x^2 - 7x + 1$

48. $3x^3 - 4x^2 - 2x + 3$

49. $x^3 - 7 + \dfrac{2}{x^2 + 7x - 1}$

50. $x^4 - x^2 - 1 + \dfrac{x + 1}{5x^2 - 1}$

51. $x^2 + 3x + 9$

52. $4x^2 - 10x + 25$

53. $3x^4 - 4x^2 + 3$

54. $x^5 - 2x^3 + 3x - 1$

55. $x^4 + x^3 - x^2 - x + 2 + \dfrac{1}{x - 1}$

56. $\frac{1}{3}x^3 - \frac{2}{9}x^2 - \frac{23}{27}x + \frac{181}{81} - \frac{443}{81(3x+2)}$

57. $a - 2$

58. 1

59. $a^5 + a^4 + a^3 + a^2 + a + 1$

60. $x^4 + x^3 + x^2 + x + 1$

61. $x^3 + 7x + 5 + \frac{2x+1}{x^2+x-1}$

62. $10x^4 - 5x^2 + 2 + \frac{x-3}{5x^2-3x+2}$

63. $x^4 - 3x^2 + 5x - \frac{1}{x}$

64. $-6x^4 + 2x^2 - 1$

65. $15x^2z^{10} - y^4 + \frac{2}{5z}$

66. $3m^4 + m^2 - 9 + \frac{1}{9m^2}$

67. $x^5 + 9x^3 - 2x + 1 - \frac{8}{3x-2}$

68. $4x^5 - 7x^3 + x^2 - 1 - \frac{2}{2x+1}$

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CHAPTER 4: SOLVING LINEAR EQUATIONS

Chapter Outline

4.1 Solving Linear Equations

- Linear Equations with One Variable and Their Solutions
- Solving Basic Linear Equations
- Isolating the Variable in Two Steps
- Multiplying by the Reciprocal
- Combining Like Terms and Simplifying
- General Guidelines for Solving Linear Equations

4.2 Applications of Linear Equations

- Key Words, Translation, and Strategy
- Problems Involving Relationships between Real Numbers
- Problems Involving Percents

4.3 Exercises

- 4.1 Solving Linear Equations
- 4.2 Applications of Linear Equations

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4.1 Solving Linear Equations

Learning Objectives

By the end of this section, you should be able to:

- Identify linear equations with one variable and verify their solutions.
- Use the properties of equality to solve basic linear equations.
- Use multiple steps to solve linear equations by isolating the variable.
- Solve linear equations where the coefficients are fractions or decimals.
- Solve general linear equations.

Linear Equations with One Variable and Their Solutions

Learning how to solve various algebraic equations is one of the main goals in algebra. This section introduces the basic techniques used for solving linear equations with one variable.

You will encounter linear equations in chemistry, biology and physics labs. Examples include the Ideal Gas Law $PV = nRT$ and $c_1v_1 = c_2v_2$. Being able to rearrange and solve these equations for the unknown variable will be helpful.

An equation is a statement indicating that two **algebraic expressions** are equal. A **linear equation with one variable**, x, is an equation that can be written in the general form $ax + b = 0$, where a and b are real numbers and $a \neq 0$. Here are some examples of linear equations, all of which are solved in this section:

$$x + 3 = -5 \qquad 2x - 5 = 15 \qquad \frac{5}{3}x + 2 = -8$$

A **solution** to a linear equation is any value that can replace the variable to produce a true statement (the left side is equal to the right side). The variable in the linear equation $2x + 3 = 13$ is x, and the solution is $x = 5$. To verify this, substitute the value 5 for x and check that you obtain a true statement.

$$\begin{aligned} 2x + 3 &= 13 \\ 2(5) + 3 &= 13 \\ 10 + 3 &= 13 \\ 13 &= 13 \quad \checkmark \end{aligned}$$

Alternatively, when an equation is equal to a constant, we can verify a solution by substituting the value for the variable and showing that the result is equal to that constant. In this sense, we say that solutions **satisfy the equation**.

How To

Given a linear equation in one variable, use algebra to solve it.

The following steps are used to manipulate an equation and isolate the unknown variable, so that the last line reads $x =$_____, if x is the unknown. There is no set order, as the steps used depend on what is given:

1. We may add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
2. Apply the distributive property as needed: $a(b + c) = ab + ac$.
3. Isolate the variable on one side of the equation.
4. When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

Example 1

Is $x = 3$ a solution to $-2x - 3 = -9$?

Solution:

$$-2x - 3 = -2(3) - 3 = -6 - 3 = -9 \quad \checkmark$$

Answer

Yes, it is a solution, because $x = 3$ satisfies the equation.

Example 2

Is $a = -\frac{1}{2}$ a solution to $-10a + 5 = 25$?

Solution:

$$-10a + 5 = -10\left(-\frac{1}{2}\right) + 5 = 5 + 5 = 10 \neq 25 \quad \times$$

Answer

No, it is not a solution, because $a = -\frac{1}{2}$ does not satisfy the equation.

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, then substitute the appropriate values. By making use of parentheses, we avoid some common errors when working with the order of operations.

Example 3

Is $y = -3$ a solution to $2y - 5 = -y - 14$?

Solution:

$$\begin{aligned} 2(\ ) - 5 &= -(\ ) - 14 && \text{Replace variables with parentheses.} \\ 2(-3) - 5 &= -(-3) - 14 && \text{Substitute the appropriate values.} \\ -6 - 5 &= 3 - 14 && \text{Simplify.} \\ -11 &= -11 \quad \checkmark && \text{True} \end{aligned}$$

Answer

Yes, it is a solution, because $y = -3$ produces a true statement.

Try It

Is $x = -3$ a solution to $-2x + 5 = -1$?

Show Solution

No

Solving Basic Linear Equations

We begin by defining **equivalent equations** as equations with the same solution set. Consider the following two linear equations and check to see if the solution is $x = 7$.

$$\begin{aligned} 3x - 5 &= 16 \\ 3(7) - 5 &= 16 \\ 21 - 5 &= 16 \\ 16 &= 16 \quad \checkmark \end{aligned} \qquad \begin{aligned} 3x &= 21 \\ 3(7) &= 21 \\ 21 &= 21 \quad \checkmark \end{aligned}$$

Here we can see that the two linear equations $3x - 5 = 16$ and $3x = 21$ are equivalent because they share the same solution set, namely, $\{7\}$. The goal is to develop a systematic process to find equivalent equations until the variable is isolated:

$$\left.\begin{aligned} 3x - 5 &= 16 \\ 3x &= 21 \\ x &= 7 \end{aligned}\right\} \text{ Equivalent equations}$$

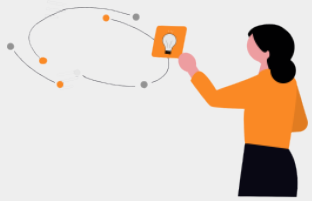
To do this, use the **properties of equality**. Given algebraic expressions A and B, where c is a real number, we have the following:

Addition Property of Equality: If $A = B$, then $A+c=B+c$

Subtraction Property of Equality: If $A = B$, then $A-c=B-c$

Multiplication Property of Equality: If $A = B$, then $cA = cB$

Division Property of Equality: If $A = B$, then $\frac{A}{c} = \frac{B}{c}$, $c \neq 0$



Note

Multiplying or dividing both sides of an equation by 0 is carefully avoided. Dividing by 0 is **undefined** and multiplying both sides by 0 results in the equation $0 = 0$.

To summarize, equality is retained, and you obtain an equivalent equation if you add, subtract, multiply, or divide both sides of an equation by any nonzero real number. The technique for solving linear equations involves applying these properties in order to isolate the variable on one side of the equation. If the linear equation has a constant term, then we add to or subtract it from both sides of the equation to obtain an equivalent equation where the variable term is isolated.

Science Application

A lab tech has a stock bottle of 12.0 M Hydrochloric Acid (HCl). They need to make 500 mL of a 1.0 M solution. How much of the stock acid should they use?

Use $c_1v_1 = c_2v_2$.

Three values are given, and one variable is unknown. Which variable is the unknown that you are solving for?

v_1

Solution

They should use 41.67 ml of 12 M HCl.

Let's try another one like this, since this equation is so critical to your lab performance.

Science Application

A student has 2.0 mL of a protein sample at a concentration of 89.4 mg/mL. They need to dilute it to 10 mg/mL. What should the final volume of the solution be?

Use $c_1v_1 = c_2v_2$.

You have three values and one unknown again. What is the unknown variable that you are solving for this time?

v_2

Solution

The final volume of the solution should be 19.68 ml.

Example 4

Solve:

$$x + 3 = -5$$

Solution:

To isolate the variable x on the left side, subtract 3 from both sides.

$$\begin{aligned} x + 3 - 3 &= -5 - 3 \qquad \text{Subtract 3 from both sides.} \\ x + 0 &= -8 \\ x &= -8 \end{aligned}$$

Answer

The solution is $x = -8$.

To check that this is true, substitute -8 into the original equation and simplify to see that it is satisfied: $x + 3 = -8 + 3 = -5$ ✓.

In the previous example, after subtracting 3 from both sides, you get $x + 0 = -8$. This is equivalent to $x = -8$. This step is often left out in the presentation of the solution.

Once the variable term of the equation (including the coefficient) is isolated, then apply the multiplication or division property of equality to obtain an equivalent equation with the variable isolated. In other words, our goal is to obtain an equivalent equation with x or $1x$ isolated on one side of the equal sign.

Example 5

Solve:

$$-5x = -35$$

Solution:

The coefficient of x is -5, so divide both sides by -5.

$$\frac{-5x}{-5} = \frac{-35}{-5} \qquad \text{Divide both sides by } -5.$$

$$1x = 7 \qquad \text{Here } \frac{-5}{-5} = 1 \text{ and } \frac{-35}{-5} = +7.$$

$$x = 7$$

Answer

The solution is $x = 7$. Perform the check mentally by substituting 7 for x in the original equation.

In the previous example, after dividing both sides by -5, x is left with a coefficient of 1, because $\frac{-5}{-5} = 1$.

In fact, when we say "isolate the variable," we mean to change the coefficient of the variable to 1, because $1x = 7$ is equivalent to $x = 7$. This step is often left out of the instructional examples, even though its omission is sometimes a source of confusion.

Another important property is the **symmetric property**: for any algebraic expressions A and B,

$$\text{If } A = B, \text{ then } B = A$$

The equation $2 = x$ is equivalent to $x = 2$. It does not matter on which side we choose to isolate the variable.

Example 6

Solve:

$$2 = 5 + x$$

Solution:
Isolate the variable x by subtracting 5 from both sides of the equation.

$$\begin{aligned} 2 &= 5 + x \\ 2-5 &= 5 + x-5 && \text{Subtract 5 from both sides.} \\ -3 &= x \\ x &= -3 && \text{(Symmetric property)} \end{aligned}$$

Answer

The solution is -3, and checking the solution shows that $2 = 5 - 3$.

Try It

Solve: $6 = x - 4$

Show Solution

$x = 10$

Isolating the Variable in Two Steps

A linear equation of the form $ax + b = c$ takes two steps to solve. First, use the appropriate equality property of addition or subtraction to isolate the variable term. Next, isolate the variable using the equality property of multiplication or division. Checking solutions in the following examples is left to the reader.

Example 7

Solve:

$$2x - 5 = 15$$

Solution:

$$\begin{aligned} 2x - 5 &= 15 \\ 2x - 5{\color{green}+5} &= 15{\color{green}+5} && \text{Add 5 to both sides.} \\ 2x &= 20 \\ \frac{2x}{{\color{green}2}} &= \frac{20}{{\color{green}2}} && \text{Divide both sides by 2.} \\ x &= 10 \end{aligned}$$

Answer

The solution is 10.

Example 8

Solve:

$$-3x - 2 = 9$$

Solution:

$$\begin{aligned} -3x - 2 &= 9 \\ -3x - 2{\color{green}+2} &= 9{\color{green}+2} && \text{Add 2 to both sides.} \\ -3x &= 11 \\ \frac{-3x}{{\color{green}-3}} &= \frac{11}{{\color{green}-3}} && \text{Divide both sides by } -3. \\ x &= -\frac{11}{3} \end{aligned}$$

Answer

The solution is $-\frac{11}{3}$.

Example 9

Solve:

$$6 - 5y = -14$$

Solution:

When no sign precedes the term, it is understood to be positive. In other words, think of this as $+6 - 5y = -14$. Begin by subtracting 6 from both sides of the equal sign.

$$\begin{aligned} 6 - 5y - 6 &= -14 - 6 && \text{Subtract 6 from both sides.} \\ -5y &= -20 \\ \frac{-5y}{-5} &= \frac{-20}{-5} && \text{Divide both sides by } -5. \\ y &= 4 \end{aligned}$$

Answer

The solution is 4.

Example 10

Solve:

$$3x + \frac{1}{2} = \frac{2}{3}$$

Solution:

$$
\begin{aligned}
3x + \frac{1}{2} &= \frac{2}{3} \\
3x + \frac{1}{2} - \frac{1}{2} &= \frac{2}{3} - \frac{1}{2} && \text{Subtract } \frac{1}{2} \text{ from both sides.} \\
3x &= \frac{4}{6} - \frac{3}{6} \\
3x &= \frac{1}{6} \\
\frac{3x}{3} &= \frac{\frac{1}{6}}{3} && \text{Divide both sides by 3.} \\
x &= \frac{1}{18}
\end{aligned}
$$

Answer

The solution is $\frac{1}{18}$.

Example 11

Solve:

$$3 - y = 1$$

Solution:

$$
\begin{aligned}
3 - y &= 1 \\
3 - y - 3 &= 1 - 3 && \text{Subtract 3 from both sides.} \\
-y &= -2 \\
\frac{-1y}{-1} &= \frac{-2}{-1} && \text{Divide both sides by } -1. \\
y &= 2
\end{aligned}
$$

Recall that $-y$ is equivalent to $-1y$. Alternatively, multiply both sides of $-y = -2$ by -1 and achieve the same result:

$$
\begin{aligned}
(-1)(-y) &= (-1)(-2) \\
y &= 2
\end{aligned}
$$

Answer

The solution is 2.

You may also need to use two equations sequentially to get the solution.

Science Application

A tech mixes Silver Nitrate (AgNO3) and Sodium Chloride (NaCl) to form a solid. If you want to produce 2.00 g of the solid AgCl, how many grams of NaCl do they need?

$$AgNO_3 + NaCl \rightarrow AgCl + NaNO_3$$

Molar masses: AgCl 143.32 g/mol, NaCl 58.44 g/mol

Use $n = \frac{m}{M}$ where n is the number of moles, m is the mass, and M is the molar mass.

First, determine the number of AgCl moles. Since AgCl and NaCl are in 1:1 molar ratio, you can determine the mass of NaCl directly from the moles, using the NaCl molar mass.

Solution

The tech should use 0.82 g NaCl.

Let's try one more.

Science Application

A tech reacts 10.00 g of Magnesium (Mg) with excess Hydrochloric Acid (HCl). The balanced equation is:

$$Mg + 2HCl \rightarrow MgCl_2 + H2$$

How many grams of Magnesium Chloride (MgCl) are formed?

Molar masses: Mg = 24.31 g/mol, MgCl2 = 95.21 g/mol

Solution

There are 39.17 g MgCl formed.

In summary, to retain equivalent equations, we must perform the same operation on both sides of the equation. First, apply the addition or subtraction property of equality to isolate the variable term and then apply the multiplication or division property of equality to isolate the variable on one side of the equation.

Try It

Solve: $-7x + 6 = 27$

Show Solution

$x = -3$

Multiplying by the Reciprocal

To solve an equation like $\frac{3}{4}x = 1$, we can isolate the variable by dividing both sides by the coefficient. For example

$$\begin{aligned} \frac{3}{4}x &= 1 \\ \frac{\frac{3}{4}x}{\frac{3}{4}} &= \frac{1}{\frac{3}{4}} \\ x &= 1 \cdot \frac{4}{3} \\ x &= \frac{4}{3} \end{aligned}$$

On the left side of the equal sign, the fraction cancels. On the right side, we have a complex fraction and multiply by the reciprocal of the coefficient. You can save a step by recognizing this and start by multiplying both sides of the equation by the reciprocal of the coefficient.

$$\begin{aligned} \frac{3}{4}x &= 1 \\ \frac{4}{3} \cdot \frac{3}{4}x &= \frac{4}{3} \cdot 1 \\ x &= \frac{4}{3} \end{aligned}$$

Recall that the product of reciprocals is 1, in this case $\frac{4}{3} \cdot \frac{3}{4} = 1$, leaving the variable isolated.

Example 12

Solve:

$$\frac{5}{3}x + 2 = -8$$

Solution:

Isolate the variable term using the addition property of equality and then multiply both sides of the equation by the reciprocal of the coefficient $\frac{5}{3}$.

$$\begin{aligned}
\frac{5}{3}x + 2 - 2 &= -8 - 2 && \text{Subtract 2 from both sides.} \\
\frac{5}{3}x &= -10 \\
\frac{3}{5}\cdot\frac{5}{3}x &= \frac{3}{5}\cdot(-10) && \text{Multiply both sides by } \frac{3}{5}. \\
x &= -6
\end{aligned}$$

Answer

The solution is -6.

Example 13

Solve:

$$-\frac{4}{5}x - 5 = 15$$

Solution:

$$\begin{aligned}
-\frac{4}{5}x - 5 &= 15 \\
-\frac{4}{5}x - 5 + 5 &= 15 + 5 \\
-\frac{4}{5}x &= 20 \\
\left(-\frac{5}{4}\right)\left(-\frac{4}{5}x\right) &= \left(-\frac{5}{4}\right)(20) \\
x &= -25
\end{aligned}$$

Answer

The solution is -25.

Try It

Solve: $\frac{2}{3}x - 9 = -4$

Show Solution

$$x = \frac{15}{2}$$

Combining Like Terms and Simplifying

Linear equations typically are not given in standard form, so solving them requires additional steps. These additional steps include simplifying expressions on each side of the equal sign using the order of operations.

Same-Side Like Terms

We will often encounter linear equations where the expressions on each side of the equal sign can be **simplified**. Typically, this involves combining **same-side like terms**. If this is the case, then it is best to simplify each side first before solving.

Example 14

Solve:

$$-4a + 2 - a = 3 - 2$$

Solution:
First, combine the like terms on each side of the equal sign.

$$
\begin{aligned}
-4a+2-a &= 3-2 && \text{Combine like terms on the same side.}\\
-5a+2 &= 1 \\
-5a+2-2 &= 1-2 && \text{Subtract 2 from both sides.}\\
-5a &= -1 \\
\frac{-5a}{-5} &= \frac{-1}{-5} && \text{Divide both sides by } -5.\\
a &= \frac{1}{5}
\end{aligned}
$$

Answer

The solution is $\frac{1}{5}$

Opposite-Side Like Terms

Given a linear equation in the form, we begin by **combining like terms** on opposite sides of the equal sign. To combine **opposite-side like terms**, use the addition or subtraction property of equality to effectively "move terms" from one side to the other so that they can be combined.

Example 15

Solve:

$$-2y-3=5y+11$$

Solution:

To "move" the term to the left side, subtract it on both sides.

$$
\begin{aligned}
-2y-3 &= 5y+11 \\
-2y-3-5y &= 5y+11-5y && \text{Subtract 5y from both sides}\\
-7y-3 &= 11
\end{aligned}
$$

From here, solve using the techniques developed previously.

Always check to see that the solution is correct by substituting the solution back into the original equation and simplifying to see if you obtain a true statement.

$$
\begin{aligned}
-2y - 3 &= 5y + 11 \\
-2(-2) - 3 &= 5y(-2) + 1 \\
4 - 3 &= -10 + 11 \\
1 &= 1 \quad \checkmark
\end{aligned}
$$

Answer

The solution is -2.

General Guidelines for Solving Linear Equations

When solving linear equations, the goal is to determine what value, if any, will produce a true statement when substituted in the original equation.

How to Solve Linear Equations

Do this by isolating the variable using the following steps:

Step 1: Simplify both sides of the equation using the order of operations and combine all same-side like terms.

Step 2: Use the appropriate properties of equality to combine opposite-side like terms with the variable term on one side of the equation and the constant term on the other.

Step 3: Divide or multiply as needed to isolate the variable.

Step 4: Check to see if the answer solves the original equation.

Example 16

Solve:

$$-\frac{1}{2}(10y - 2) + 3 = 14$$

Solution:
Simplify the linear expression on the left side before solving.

$$
\begin{aligned}
-\frac{1}{2}(10y-2)+3 &= 14 && \text{Distribute.}\\
-5y+1+3 &= 14 && \text{Combine same-side like terms.}\\
-5y+4 &= 14 && \text{Solve.}\\
-5y+4-4 &= 14-4\\
-5y &= 10\\
\frac{-5y}{-5} &= \frac{10}{-5}\\
y &= -2
\end{aligned}
$$

To check,

$$
\begin{aligned}
-\frac{1}{2}(10(-2)-2)+3 &= 14\\
-\frac{1}{2}(-20-2)+3 &= 14\\
-\frac{1}{2}(-22)+3 &= 14\\
11+3 &= 14\\
14 &= 14 \quad \checkmark
\end{aligned}
$$

Answer

The solution is -2.

Example 17

Solve:

$$5(3x+2)-2=-2(1-7x)$$

Solution:

First, simplify the expressions on both sides of the equal sign.

$$
\begin{aligned}
5(3x+2)-2 &= -2(1-7x) && \text{Distribute.}\\
15x+10-2 &= -2+14x && \text{Combine same-side like terms.}\\
15x+8 &= -2+14x &&\\
15x+8-14x &= -2+14x-14x && \text{Combine opposite-side like terms.}\\
x+8 &= -2 && \text{Solve.}\\
x+8-8 &= -2-8 &&\\
x &= -10 &&
\end{aligned}
$$

Answer

The solution is -10. The check is left as an exercise.

Try It

Solve: $6-3(4x-1)=4x-7$

Show Solution

$x=-1$

Key Takeaways

- Linear equations with one variable can be written in the form $ax+b=0$, where a and b are real numbers and $a \neq 0$.
- To "solve a linear equation" means to find a numerical value that can replace the variable and produce a true statement.
- The properties of equality provide tools for isolating the variable and solving equations.
- To solve a linear equation, first isolate the *variable term* by adding the opposite of the constant term to both sides of the equation. Then isolate the *variable* by dividing both sides of the equation by its coefficient.
- After isolating a variable term with a fraction coefficient, solve by multiplying both sides by the

reciprocal of the coefficient.

- Solving general linear equations involves isolating the variable with a coefficient on one side of the equal sign.
  The steps for solving linear equations are:
  - Simplify both sides of the equation and combine all same-side like terms.
  - Combine opposite-side like terms
    to obtain the variable term on one side of the equal sign and the constant term
    on the other.
  - Divide or multiply as needed to isolate the variable.
  - Check the answer.

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How To: "2.2 Linear Equations in One Variable" from College Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Use How To box "Given a linear equation in one variable, use algebra to solve it"; Reformatted.

4.2 Applications of Linear Equations

Learning Objectives

By the end of this section, you should be able to:

- Identify keywords and phrases, translate sentences to mathematical equations, and develop strategies to solve problems.
- Solve word problems involving relationships between numbers.
- Solve percent problems.
- Set up and solve uniform motion problems.

Key Words, Translation, and Strategy

Algebra simplifies the process of solving real-world problems. This is done by using letters to represent unknowns, restating problems in the form of equations, and offering systematic techniques for solving those equations. To solve problems using algebra, first translate the wording of the problem into mathematical statements that describe the relationships between the given information and the unknowns. Usually, this translation to mathematical statements is the difficult step in the process. The key to the translation is to carefully read the problem and identify certain key words and phrases.

Table 4.2.1.

Key Words	Translation
Sum, increased by, more than, plus, added to, total	$+$
Difference, decreased by, subtracted from, less, minus	$-$
Product, multiplied by, of, times, twice	$*$
Quotient, divided by, ratio, per	$/$
Is, total, result	$=$

Here are some examples of translated key phrases.

Table 4.2.2.

Key Phrases	Translation
The *sum* of a number and 7. Seven *more than* a number.	$x + 7$
The *difference of* a number and 7. Seven *less than* a number. Seven *subtracted from* a number.	$x - 7$
The *product of* 2 and a number. *Twice* a number.	$2x$
One-half *of a* number.	$\frac{1}{2}x$
The *quotient* of a number and 7.	$\frac{x}{7}$

When translating sentences into mathematical statements, be sure to read the sentence several times and identify the key words and phrases.

Example 1

Translate:

Four less than twice some number is 16.

Solution:

First, choose a variable for the unknown number and identify the key words and phrases. Let x represent the unknown indicated by "some number."

$$\overbrace{\text{twice some number}}^{2x} \quad \overbrace{\text{four less than}}^{-4} \quad \overbrace{\text{is}}^{=} \quad 16$$

Remember that subtraction is not commutative. For this reason, take care when setting up differences. In this example, $4 - 2x = 16$ is an incorrect translation.

Answer

$$2x - 4 = 16$$

It is important to first identify the variable—*let x represent...*—and state in words what the unknown quantity is. This step not only makes your work more readable but also forces you to think about what you are looking for. Usually, if you know what you are asked to find, then the task of finding it is achievable.

Example 2

Translate:

When 7 is subtracted from 3 times the sum of a number and 12, the result is 20.

Solution:

Let n represent the unknown number.

$$\overbrace{3}^{3\text{ times}}\cdot\overbrace{(n+12)}^{\text{the sum of a number and 12}}\overbrace{-7}^{7\text{ is subtracted from}}\overbrace{=}^{is}20$$

Answer

$$3(n+12)-7=20$$

To understand why parentheses are needed, study the structures of the following two sentences and their translations:

Table 4.2.3.

Sentence	Translation
"*3 times the sum* of a number and 12"	$3(n+12)$
"the sum of *3 times a number* and 12"	$3n+12$

The key is to focus on the phrase "*3 times the sum*." This prompts us to group the sum within parentheses and then multiply by 3. Once an application is translated into an algebraic equation, solve it using the techniques you have learned.

Guidelines for Setting Up and Solving Word Problems

- **Step 1**: Read the problem several times, identify the key words and phrases, and organize the given information.
- **Step 2**: Identify the variables by assigning a letter or expression to the unknown quantities.
- **Step 3**: Translate and set up an algebraic equation that models the problem.
- **Step 4**: Solve the resulting algebraic equation.
- **Step 5**: Finally, answer the question in sentence form and make sure it makes sense (check it).

For now, set up all of your equations using only one variable. Avoid two variables by looking for a relationship between the unknowns.

Problems Involving Relationships between Real Numbers

We classify applications involving relationships between real numbers broadly as number problems. These problems can sometimes be solved using some creative arithmetic, guessing, and checking. Solving in this manner is not a good practice and should be avoided. Begin by working through the basic steps outlined in the general guidelines for solving word problems.

Example 3

A larger integer is 2 less than 3 times a smaller integer. The sum of the two integers is 18. Find the integers.

Solution:

Step 1: Identify variables. Begin by assigning a variable to the smaller integer.

Let x represent the smaller integer.

Use the first sentence to identify the larger integer in terms of the variable x: "A larger integer is *2 less* than *3 times* a smaller."

Let $3x - 2$ represent the larger integer.

Step 2: Set up an equation. Add the expressions that represent the two integers, and set the resulting expression equal to 18 as indicated in the second sentence: "The *sum* of the two integers *is* 18."

$$x + (3x - 2) = 18$$

Step 3: Solve. Solve the equation to obtain the smaller integer x.

$$\begin{aligned} x + (3x - 2) &= 18 \\ x + 3x - 2 &= 18 \\ 4x - 2 &= 18 \\ 4x - 2{\color{green}+2} &= 18{\color{green}+2} \\ 4x &= 20 \\ \frac{4x}{{\color{green}4}} &= \frac{20}{{\color{green}4}} \\ x &= 5 \end{aligned}$$

Step 4: **Back substitute**. Use the expression $3x - 2$ to find the larger integer—this is called back-substituting

$$3x - 2 = 3({\color{green}5}) - 2 = 15 - 2 = 13$$

Step 5: Answer the question. The two integers are 5 and 13.

Step 6: Check. $5 + 13 = 18$. The answer makes sense.

Answer

5 and 13

Example 4

The difference between two integers is 2. The larger integer is 6 less than twice the smaller. Find the integers.

Solution:

Use the relationship between the two integers in the second sentence, "The larger integer is *6 less* than *twice the smaller*," to identify the unknowns in terms of one variable.

Let x represent the smaller integer.

Let $2x - 6$ represent the larger integer.

Since the difference is positive, subtract the smaller integer from the larger.

$$(2x - 6) - x = 2$$

Solve.

$$\begin{aligned} 2x - 6 - x &= 2 \\ x - 6 &= 2 \\ x - 6 + 6 &= 2 + 6 \\ x &= 8 \end{aligned}$$

Use $2x - 6$ to find the larger integer.

$$2x - 6 = 2(8) - 6 = 16 - 6 = 10$$

Answer

The two integers are 8 and 10. These integers clearly solve the problem.

It is worth mentioning again that you can often find solutions to simple problems by guessing and checking. This is so because the numbers are chosen to simplify the process of solving, so that the algebraic steps are not too tedious.

You learn how to set up algebraic equations with easier problems, so that you can use these ideas to solve more difficult problems later.

Example 5

The sum of two consecutive **even integers** is 46. Find the integers.

Solution:

The key phrase to focus on is "consecutive even integers."

Let x represent the first even integer.

Let $x + 2$ represent the next even integer.

Add the even integers and set them equal to 46.

$$x + (x + 2) = 46$$

Solve.

$$\begin{aligned} x + x + 2 &= 46 \\ 2x + 2 &= 46 \\ 2x + 2 - 2 &= 46 - 2 \\ 2x &= 44 \\ x &= 22 \end{aligned}$$

Use $x + 2$ to find the next even integer.

$$x + 2 = 22 + 2 = 24$$

Answer

The consecutive even integers are 22 and 24.

It should be clear that consecutive even integers are separated by two units. However, it may not be so clear that **odd integers** are as well.

Figure 4.2.1.

Example 6

The sum of two consecutive odd integers is 36. Find the integers.

Solution:

The key phrase to focus on is "consecutive odd integers."

Let x represent the first odd integer.

Let $x + 2$ represent the next odd integer.

Add the two odd integers and set the expression equal to 36.

$$x + (x + 2) = 36$$

Solve.

$$\begin{aligned} x + x + 2 &= 36 \\ 2x + 2 &= 36 \\ 2x + 2 - 2 &= 36 - 2 \\ 2x &= 34 \\ \frac{2x}{2} &= \frac{34}{2} \\ x &= 17 \end{aligned}$$

Use $x + 2$ to find the next odd integer.

$$x + 2 = 17 + 2 = 19$$

Answer

The consecutive odd integers are 17 and 19.

The algebraic setup for even and odd integer problems is the same. A common mistake is to use x and $x + 3$ when identifying the variables for consecutive odd integers. This is incorrect because adding 3 to an odd number yields an even number: for example, $5 + 3 = 8$. An incorrect setup is very likely to lead to a decimal answer, which may be an indication that the problem was set up incorrectly.

Example 7

The sum of three consecutive integers is 24. Find the integers.

Solution:
Consecutive integers are separated by one unit.

Let x represent the first integer.
Let $x+1$ represent the next integer.
Let $x+2$ represent the third integer.

Add the integers and set the sum equal to 24.

$$x+(x+1)+(x+2)=24$$

Solve.

$$\begin{aligned} x+x+1+x+2 &= 24 \\ 3x+3 &= 24 \\ 3x+3-3 &= 24-3 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

Back substitute to find the other two integers.

$$\begin{aligned} x+1 &= 7+1=8 \\ x+2 &= 7+2=9 \end{aligned}$$

Answer

The three consecutive integers are $7, 8$ and 9, where $7+8+9=24$.

Try It

The sum of three consecutive odd integers is 87. Find the integers.

Show Solution

The integers are 27, 29, and 31.

Problems Involving Percents

Whenever setting up an equation involving a percentage, we usually need to convert the percentage to a decimal or a fraction. If the question asks for a percentage, then do not forget to convert your answer to a percent at the end.

Uniform Motion Problems (Distance Problems)

Uniform motion refers to movement at a speed or rate that does not change. We can determine the distance travelled by multiplying the average rate by the time travelled at that rate with the formula $D = r \cdot t$. Applications involving uniform motion usually have a lot of data, so it helps to first organize the data in a chart and then set up an algebraic equation that models the problem.

Let's try a uniform motion problem from the chemistry lab.

Science Application

A glass tube is 100 cm long. Ammonia NH_3 is released from the left end. It travels at a rate of $r_1 = 30\ \text{cm/s}$. Hydrogen Chloride HCl is released from the right end at the exact same time. It travels at a rate of $r_2 = 20\ \text{cm/s}$.

How far from the Ammonia end (the left side) will the NH_4Cl first appear, making a white smoke?

Step 1: Set up the variables

Let t be the time (in seconds) until they meet.
Let d_1 be the distance the Ammonia travels: $d_1 = r_1 \cdot t = 30t$.
Let d_2 be the distance the HCl travels: $d_2 = r_2 \cdot t = 20t$.

Step 2: Create the equation

The total length of the tube is 100 cm. When they meet, the sum of their distances must equal the total length:

$$d_1 + d_2 = 100$$
$$30t + 20t = 100$$

Step 3: Solve for Time t

$$50t = 100$$
$$t = 2 \text{ seconds}$$

Step 4: Lastly, find the distance from the ammonia (left) end

The question asks for the distance from the left (ammonia) side, which we called d_1:

$$d_1 = 30 \text{ cm/s} \times 2 \text{ s}$$
$$d_1 = 60 \text{ cm}$$

The white smoke will appear 60 cm from the ammonia end.

Example 8

Two trains leave the station at the same time, travelling in opposite directions. One travels at 70 miles per hour and the other at 60 miles per hour. How long does it take for the distance between them to reach 390 miles?

Solution:

First, identify the unknown quantity and organize the data.

Let t represent the time it takes to separate 390 miles.

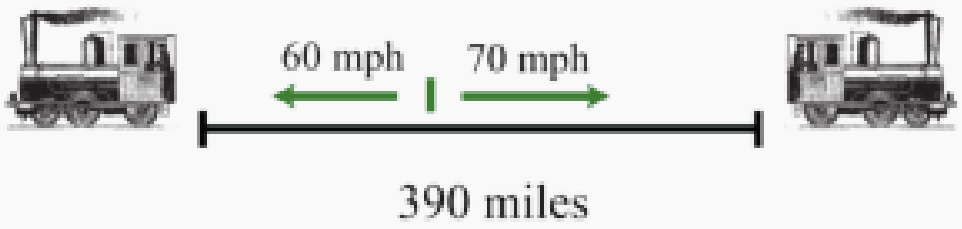

Figure 4.2.2.

The given information is filled in on the following chart. The time for each train is equal.

Table 4.2.4. Distance = Rate × Time

Train	Distance	Rate	Time
Train 1		70mph	t
Train 2		60mph	t
Total	390mi		

To avoid introducing two more variables, use the formula $D = r \cdot t$ to fill in the unknown distances travelled by each train.

Distance traveled by train 1: $D = r \cdot t = 70 \cdot t$

Distance traveled by train 2: $D = r \cdot t = 60 \cdot t$

We can now completely fill in the chart.

Table 4.2.5. Distance = Rate × Time

Train	Distance	Rate	Time
Train 1	$70t$	70mph	t
Train 2	$60t$	60mph	t
Total	390mi		

The algebraic setup is defined by the distance column. The problem asks for the time it takes for the total distance to reach 390 miles.

Train	Distance	Rate	Time
Train 1	$70t$	70mph	t
Train 2	$60t$	60mph	t
Total	390mi		

$$70t + 60t = 390$$

Figure 4.2.3.

Solve for t.

$$\begin{aligned} 70t + 60t &= 390 \\ 130t &= 390 \\ \frac{130t}{130} &= \frac{390}{130} \\ t &= 3 \end{aligned}$$

Answer

It takes 3 hours for the distance between the trains to reach 390 miles.

Example 9

A train travelling nonstop to its destination is able to make the trip at an average speed of 72 miles per hour. On the return trip, the train makes several stops and is only able to average 48 miles per hour. If the return trip takes 2 hours longer than the initial trip to the destination, then what is the travel time each way?)

Solution:

First, identify the unknown quantity and organize the data.

Let t represent the time it takes to arrive at the destination.

Let $t + 2$ represent the time it takes for the return trip.

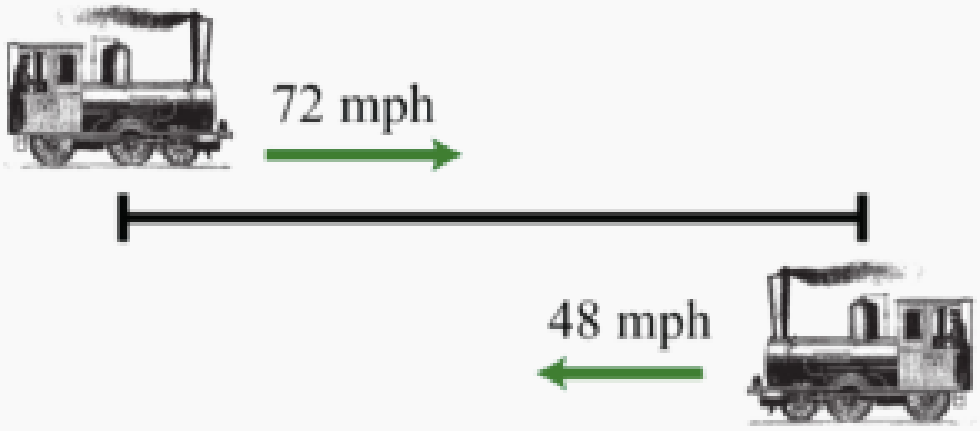

Figure 4.2.4.

The given information is filled in the following chart:

Table 4.2.6. Distance = Rate × Time

Trip	Distance	Rate	Time
To Destination		72mph	t
Return Trip		48mph	$t + 2$
Total			

Use the formula $D = r \cdot t$ to fill in the unknown distances.

Distance traveled on the destination: $D = r \cdot t = 72 \cdot t$

Distance traveled on the return trip: $D = r \cdot t = 48 \cdot (t + 2)$

Use these expressions to complete the chart.

Table 4.2.7. Distance = Rate × Time

Train	Distance	Rate	Time
To Destination	$72t$	72mph	t
Return	$48(t + 2)$	48mph	$t + 2$
Total			

The algebraic setup is again defined by the distance column. In this case, the distance to the destination and back is the same, and the equation is
$72t = 48(t + 2)$

Solve for t.

$$\begin{aligned} 72t &= 48(t + 2) \\ 72t &= 48t + 96 \\ 72t - 48t &= 48t + 96 - 48t \\ 24t &= 96 \\ \frac{24t}{24} &= \frac{96}{24} \\ t &= 4 \end{aligned}$$

The return trip takes $t + 2 = 4 + 2 = 6$ hours.

Answer

It takes 4 hours to arrive at the destination and 6 hours to return.

Try It

Mary departs for school on a bicycle at an average rate of 6 miles per hour. Her sister Kate, running late, leaves 15 minutes later and cycles at twice that speed. How long will it take Kate to catch up to Mary? Be careful! Pay attention to the units given in the problem.

Show Solution

It takes 15 minutes for Kate to catch up.

Key Takeaways

- Simplify the process of solving real-world problems by creating mathematical models that describe the relationship between unknowns. Use algebra to solve the resulting equations.
- Guessing and checking for solutions is a poor practice. This technique might sometimes produce correct answers, but it is unreliable, especially when problems become more complex.
- Read the problem several times and search for the keywords and phrases. Identify the unknowns and assign variables or expressions to the unknown quantities. Look for relationships that allow you to use only one variable. Set up a mathematical model for the situation and use algebra to solve the equation. Check to see if the solution makes sense and present the solution in sentence form.
- Do not avoid word problems: solving them can be fun and rewarding. With lots of practice, you will find that they really are not so bad after all. Modelling and solving applications is one of the major reasons to study algebra.
- Do not feel discouraged when the first attempt to solve a word problem does not work. This is part of the process. Try something different and learn from incorrect attempts.

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4.3 Exercises

4.1 Solving Linear Equations

Is the given value a solution to the linear equation?

1. $x - 6 = 20;\ x = 26$
2. $y + 7 = -6;\ y = -13$
3. $-x + 5 = 17;\ x = 12$
4. $-2y = 44;\ y = 11$
5. $4x = -24;\ x = -6$
6. $5x - 1 = 34;\ x = -7$
7. $-2a - 7 = -7;\ a = 0$
8. $-\frac{1}{3}x - 4 = -5;\ x = -3$
9. $-\frac{1}{2}x + \frac{2}{3} = -\frac{1}{4};\ x = \frac{11}{6}$
10. $-8x - 33 = 3x;\ x = 3$
11. $3y - 5 = -2y - 15;\ y = -2$
12. $3(2x + 1) = -4x - 3;\ x = -\frac{1}{2}$
13. $\frac{1}{2}y - \frac{1}{3} = \frac{1}{3}y + \frac{1}{6};\ y = 3$
14. $-\frac{4}{3}y + \frac{1}{9} = -\frac{2}{3}y - \frac{1}{9};\ y = \frac{1}{3}$

Solve.

15. $x + 3 = 13$
16. $y - 4 = 22$
17. $-6 + x = 12$
18. $9 + y = -4$
19. $x - \frac{1}{2} = \frac{1}{3}$
20. $x + \frac{2}{3} = -\frac{1}{5}$
21. $x + 2\frac{1}{2} = 3\frac{1}{3}$
22. $-37 + x = -37$
23. $4x = -44$
24. $-9x = 63$
25. $-y = 13$
26. $-x = -10$
27. $-9x = 0$
28. $-3a = -33$
29. $27 = 18y$
30. $14 = -7x$
31. $5.6a = -39.2$
32. $-1.2y = 3.72$
33. $\frac{1}{3}x = -\frac{1}{2}$

34. $-\frac{t}{12} = \frac{1}{4}$

35. $-\frac{7}{3}x = \frac{1}{2}$

36. $\frac{x}{5} = -3$

37. $\frac{4}{9}y = -\frac{2}{3}$

38. $-\frac{5}{8}y = -\frac{5}{2}$

39. $5x + 7 = 32$

40. $4x - 3 = 21$

41. $3a - 7 = 23$

42. $12y + 1 = 1$

43. $21x - 7 = 0$

44. $-3y + 2 = -13$

45. $-5x + 9 = 8$

46. $22x - 55 = -22$

47. $4.5x - 2.3 = 6.7$

48. $1.4 - 3.2x = 3$

49. $9.6 - 1.4y = -10.28$

50. $4.2y - 3.71 = 8.89$

51. $3 - 2y = -11$

52. $-4 - 7a = 24$

53. $-10 = 2x - 5$

54. $24 = 6 - 12y$

55. $\frac{5}{6}x - \frac{1}{2} = \frac{2}{3}$

56. $\frac{1}{2}x + \frac{1}{3} = \frac{2}{5}$

57. $4a - \frac{2}{3} = -\frac{1}{6}$

58. $\frac{3}{5}x - \frac{1}{2} = \frac{1}{10}$

59. $-\frac{4}{5}y + \frac{1}{3} = \frac{1}{15}$

60. $-\frac{9}{16}x + \frac{4}{3} = \frac{4}{3}$

61. $-x + 5 = 14$

62. $-y - 7 = -12$

63. $75 - a = 200$

64. $15 = 5 - x$

65. $-8 = 4 - 2x$

66. $33 - x = 33$

67. $18 = 6 - y$

68. $-12 = -2x + 3$

69. $-3 = 3.36 - 1.2a$

70. $0 = -3.1a + 32.55$

71. $\frac{1}{4} = -\frac{3}{8} + 10x$

72. $70 = 50 - \frac{1}{2}y$

Translate the following sentences into linear equations and then solve.

73. The sum of $2x$ and 5 is equal to 15.

74. The sum of $-3x$ and 7 is equal to 14.

75. The difference of $5x$ and 6 is equal to 4.

76. Twelve times x is equal to 36.

77. A number n divided by 8 is 5.

78. Six subtracted from two times a number x is 12.

79. Four added to three times a number n is 25.

80. Three-fourths of a number x is 9.

81. Negative two-thirds times a number x is equal to 20.

82. One-half of a number x plus 3 is equal to 10.

Find a linear equation of the form $ax + b = 0$ with the given solution, where a and b are integers. (Answers may vary.)

83. $x = 2$

84. $x = -3$

85. $x = -\frac{1}{2}$

86. $x = \frac{2}{3}$

Solve:

87. $4x - 7 = 7x + 5$

88. $-5x + 3 = -8x - 9$

89. $3x - 5 = 2x - 17$

90. $-2y - 52 = 3y + 13$

91. $-4x + 2 = 7x - 20$

92. $4x - 3 = 6x - 15$

93. $9x - 25 = 12x - 25$

94. $12y + 15 = -6y + 23$

95. $1.2x - 0.7 = 3x + 4.7$

96. $2.1x + 6.1 = -1.3x + 4.4$

97. $2.02x + 4.8 = 14.782 - 1.2x$

98. $-3.6x + 5.5 + 8.2x = 6.5 + 4.6x$

99. $\frac{1}{2}x - \frac{2}{3} = x + \frac{1}{5}$

100. $\frac{1}{3}x - \frac{1}{2} = -\frac{1}{4}x - \frac{1}{3}$

101. $-\frac{1}{10}y + \frac{2}{5} = \frac{1}{5}y + \frac{3}{10}$

102. $x - \frac{20}{3} = \frac{5}{2}x + \frac{5}{6}$

103. $\frac{2}{3}y + \frac{1}{2} = \frac{5}{8}y + \frac{37}{24}$

104. $\frac{1}{3} + \frac{4}{3}x = \frac{10}{7}x + \frac{1}{3} - \frac{2}{21}x$

105. $\frac{8}{9} - \frac{11}{18}x = \frac{7}{6} - 12x$

106. $\frac{1}{3} - 9x = \frac{4}{9} + \frac{1}{2}x$

107. $12x - 5 + 9x = 44$

108. $10 - 6x - 13 = 12$

109. $-2 + 4x + 9 = 7x + 8 - 2x$

110. $20x - 5 + 12x = 6 - x + 7$

111. $3a + 5 - a = 2a + 7$

112. $-7b + 3 = 2 - 5b + 1 - 2b$

113. $7x - 2 + 3x = 4 + 2x - 2$

114. $-3x + 8 - 4x + 2 = 10$

115. $6x + 2 - 3x = -2x - 13$

116. $3x - 0.75 + 0.21x = 1.24x + 7.13$

117. $-x - 2 + 4x = 5 + 3x - 7$

118. $-2y - 5 = 8y - 6 - 10y$

119. $\frac{1}{10}x - \frac{1}{3} = \frac{1}{30} - \frac{1}{15}x - \frac{7}{15}$

120. $\frac{5}{8} - \frac{4}{3}x + \frac{1}{3} = -\frac{3}{9}x - \frac{1}{4} + \frac{1}{3}x$

Show Solutions

1. Yes
2. Yes
3. No
4. No
5. Yes
6. No
7. Yes

8. No
9. Yes
10. No
11. Yes
12. No
13. Yes
14. Yes
15. 10
16. 26
17. 18
18. -13
19. $\frac{5}{6}$
20. $-\frac{13}{15}$
21. $\frac{5}{6}$
22. 0
23. -11
24. -7
25. -13
26. 10

27. 0

28. 11

29. $\frac{3}{2}$

30. -2

31. -7

32. -3.1

33. $-\frac{3}{2}$

34. -3

35. $-\frac{3}{14}$

36. -15

37. $-\frac{3}{2}$

38. 4

39. 5

40. 6

41. 10

42. 0

43. $\frac{1}{3}$

44. 5

45. $\frac{1}{5}$

46. $\frac{3}{2}$

47. 2

48. $-\frac{1}{2}$

49. 14.2

50. 3

51. 7

52. -4

53. $-\frac{5}{2}$

54. $-\frac{3}{2}$

55. $\frac{7}{5}$

56. $\frac{2}{15}$

57. $\frac{1}{8}$

58. 1

59. $\frac{1}{3}$

60. 0

61. -9

62. 5

63. -125

64. -10

65. 6

66. 0

67. -12

68. $\frac{15}{2}$

69. 5.3

70. 10.5

71. $\frac{1}{16}$

72. -40

73. $2x + 5 = 15;\ x = 5$

74. $-3x + 7 = 14;\ x = -\frac{7}{3}$

75. $5x - 6 = 4;\ x = 2$

76. $12x = 36;\ x = 3$

77. $\frac{n}{8} = 5;\ n = 40$

78. $2x - 6 = 12;\ x = 9$

79. $3n + 4 = 25;\ n = 7$

80. $\frac{3}{4}x = 9;\ x = 12$

81. $-\frac{2}{3}x = 20;\ x = -30$

82. $\frac{1}{2}x + 3 = 10;\ x = 14$

83. $x - 2 = 0$

84. $x + 3 = 0$

85. $2x + 1 = 0$

86. $3x - 2 = 0$

87. -4

88. -4

89. -12

90. -13

91. 2

92. 6

93. 0

94. $\frac{4}{9}$

95. -3

96. $-\frac{1}{2}$

97. 3.1

98. $\varnothing$

99. $-\dfrac{26}{15}$

100. $\dfrac{2}{7}$

101. $\dfrac{1}{3}$

102. -5

103. 25

104. R

105. $\dfrac{1}{41}$

106. $-\dfrac{2}{171}$

107. $\dfrac{7}{3}$

108. $-\dfrac{5}{2}$

109. -1

110. $\dfrac{9}{16}$

111. $\varnothing$

112. R

113. $\frac{1}{2}$

114. 0

115. -3

116. 4

117. R

118. $\varnothing$

119. $-\frac{3}{5}$

120. $\frac{29}{32}$

4.2 Applications of Linear Equations

Translate the following into algebraic equations.

1. The sum of a number and 6 is 37.
2. When 12 is subtracted from twice some number, the result is 6.
3. Fourteen less than 5 times a number is 1.
4. Twice some number is subtracted from 30, and the result is 50.
5. Five times the sum of 6 and some number is 20.
6. The sum of 5 times some number and 6 is 20.

7. When the sum of a number and 3 is subtracted from 10 the result is 5.

8. The sum of three times a number and five times that same number is 24.

9. Ten is subtracted from twice some number, and the result is the sum of the number and 2.

10. Six less than some number is ten times the sum of that number and 5.

Set up an algebraic equation and then solve.

11. A larger integer is 1 more than twice another integer. If the sum of the integers is 25, find the integers.

12. If a larger integer is 2 more than 4 times another integer and their difference is 32, find the integers.

13. One integer is 30 more than another integer. If the difference between the larger and twice the smaller is 8, find the integers.

14. The quotient of some number and 4 is 22. Find the number.

15. Eight times a number is decreased by three times the same number, giving a difference of 20. What is the number?

16. One integer is two units less than another. If their sum is -22, find the two integers.

17. The sum of two consecutive integers is 139. Find the integers.

18. The sum of three consecutive integers is 63. Find the integers.

19. The sum of three consecutive integers is 279. Find the integers.

20. The difference of twice the smaller of two consecutive integers and the larger is 39. Find the integers.

21. If the smaller of two consecutive integers is subtracted from two times the larger, then the result is 17. Find the integers.

22. The sum of two consecutive even integers is 46. Find the integers.

23. The sum of two consecutive even integers is 238. Find the integers.

24. The sum of three consecutive even integers is 96. Find the integers.

25. If the smaller of two consecutive even integers is subtracted from 3 times the larger, the result is 42

. Find the integers.

26. The sum of three consecutive even integers is 90. Find the integers.

27. The sum of two consecutive odd integers is 68. Find the integers.

28. The sum of two consecutive odd integers is 180. Find the integers.

29. The sum of three consecutive odd integers is 57. Find the integers.

30. If the smaller of two consecutive odd integers is subtracted from twice the larger, the result is 23. Find the integers.

31. Twice the sum of two consecutive odd integers is 32. Find the integers.

32. The difference between twice the larger of two consecutive odd integers and the smaller is 59. Find the integers.

Set up an algebraic equation and then solve.

33. If the perimeter of a square is 48 inches, then find the length of each side.

34. The length of a rectangle is 2 inches longer than its width. If the perimeter is 36 inches, find the length and width.

35. The length of a rectangle is 2 feet less than twice its width. If the perimeter is 26 feet, find the length and width.

36. The width of a rectangle is 2 centimetres less than one-half its length. If the perimeter is 56 centimetres, find the length and width.

37. The length of a rectangle is 3 feet less than twice its width. If the perimeter is 54 feet, find the dimensions of the rectangle.

38. If the length of a rectangle is twice as long as the width and its perimeter measures 72 inches, find the dimensions of the rectangle.

39. The perimeter of an equilateral triangle measures 63 centimetres. Find the length of each side.

40. An isosceles triangle whose base is one-half as long as the other two equal sides has a perimeter of 25 centimetres. Find the length of each side.

41. Each of the two equal legs of an isosceles triangle are twice the length of the base. If the perimeter is 105 centimetres, then how long is each leg?

42. A triangle has sides whose measures are consecutive even integers. If the perimeter is 42 inches, find the measure of each side.

43. A triangle has sides whose measures are consecutive odd integers. If the perimeter is 21 inches, find the measure of each side.

44. A triangle has sides whose measures are consecutive integers. If the perimeter is 102 inches, then find the measure of each side.

45. The circumference of a circle measures 50π units. Find the radius.

46. The circumference of a circle measures 10π units. Find the radius.

47. The circumference of a circle measures 100 centimetres. Determine the radius to the nearest tenth.

48. The circumference of a circle measures 20 centimetres. Find the diameter, rounded off to the nearest hundredth.

49. The diameter of a circle measures 5 inches. Determine the circumference to the nearest tenth.

50. The diameter of a circle is 13 feet. Calculate the exact value of the circumference.

Set up an algebraic equation, then solve.

51. Two cars leave a location traveling in opposite directions. If one car averages 55 miles per hour and the other averages 65 miles per hour, then how long will it take for them to separate a distance of 300 miles?

52. Two planes leave the airport at the same time, travelling in opposite directions. The average speeds for the planes are 450 miles per hour and 395 miles per hour. How long will it take the planes to be a distance of $1,478.75$ miles apart?

53. Bill and Ted are racing across the country. Bill leaves 1 hour earlier than Ted and travels at an average rate of 60 miles per hour. If Ted intends to catch up at a rate of 70 miles per hour, then how long will it take?

54. Two brothers leave from the same location, one in a car and the other on a bicycle, to meet up at their grandmother's house for dinner. If one brother averages 30 miles per hour in the car and the other averages 12 miles per hour on the bicycle, then it takes the brother on the bicycle 1 hour less than 3 times as long as the other in the car. How long does it take each of them to make the trip?

55. A commercial airline pilot flew at an average speed of 350 miles per hour before being informed that his destination airfield may be closed due to poor weather conditions. In an attempt to arrive before the storm, he increased his speed to 400 miles per hour and flew for another 3 hours. If the total distance flown was $2,950$ miles, then how long did the trip take?

56. Two brothers drove the $2,793$ miles from Los Angeles to New York. One of the brothers, driving during the day, was able to average 70 miles per hour, and the other, driving at night, was able to average 53 miles per hour. If the brother driving at night drove 3 hours less than the brother driving in the day, then how many hours did they each drive?

57. Joe and Ellen live 21 miles apart. Departing at the same time, they cycle toward each other. If Joe averages 8 miles per hour and Ellen averages 6 miles per hour, how long will it take them to meet?

58. If it takes 6 minutes to drive to the automobile repair shop at an average speed of 30 miles per hour, then how long will it take to walk back at an average rate of 4 miles per hour?

59. Jaime and Alex leave the same location and travel in opposite directions. Traffic conditions enabled Alex to average 14 miles per hour faster than Jaime. After $1\frac{1}{2}$ hours they are 159 miles apart. Find the speed at which each was able to travel.

60. Jane and Holly live 51 miles apart and leave at the same time, travelling toward each other to meet for lunch. Jane travelled on the freeway at twice the average speed as Holly. They were able to meet in a half hour. At what rate did each travel?

Show Solutions

1. $x + 6 = 37$
2. $2x - 12 = 6$
3. $5x - 14 = 1$
4. $30 - 2x = 50$
5. $5(x + 6) = 20$
6. $5x + 6 = 20$
7. $10 - (x + 3) = 5$
8. $3x + 5x = 24$
9. $2x - 10 = x + 2$

10. $x - 6 = 10(x + 5)$
11. $8, 17$
12. $10, 42$
13. $22, 52$
14. 88
15. 4
16. $-12, -10$
17. $69, 70$
18. $20, 21, 22$
19. $92, 93, 94$
20. $40, 41$
21. $15, 16$
22. $22, 24$
23. $118, 120$
24. $30, 32, 34$
25. $18, 20$
26. $28, 30, 32$
27. $33, 35$
28. $89, 91$
29. $17, 19, 21$

30. $19, 21$
31. $7, 9$
32. $55, 57$
33. 12 inches
34. Width: 8 inches; length: 10 inches
35. Width: 5 feet; length: 8 feet
36. Width: 8 centimeters; length: 20 centimeters
37. Width: 10 feet; length: 17 feet
38. Width: 12 inches; length: 24 inches
39. 21 centimeters
40. 5 centimeters, 10 centimeters, 10 centimeters
41. 42 centimeters
42. 12 inches, 14 inches, 16 inches
43. 5 inches, 7 inches, 9 inches
44. 33 inches, 34 inches, 35 inches
45. 25 units
46. 5 units
47. 15.9 centimeters
48. 6.37 centimeters
49. 15.7 inches

50. 13π feet

51. 2.5 hours

52. 1.75 hours

53. 6 hours

54. Car: 2 hours; bicycle: 5 hours

55. 8 hours

56. Day driver: 24 hours; night driver: 21 hours

57. $1\frac{1}{2}$ hours

58. 45 minutes

59. Jaime: 46 miles per hour; Alex: 60 miles per hour

60. Holly: 34 miles per hour; Jane: 68 miles per hour

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CHAPTER 5: GRAPHING LINEAR EQUATIONS

Chapter Outline

5.1 Rectangular Coordinate Systems

5.2 Graph Using the Slope and Y-Intercept

- Slope
- Slope-Intercept Form of a Line

5.3 Exercises

- 5.1 Rectangular Coordinate Systems
- 5.2 Graph Using the Slope and Y-Intercept

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5.1 Rectangular Coordinate Systems

Learning Objectives

By the end of this section, you should be able to:

- Plot points using the rectangular coordinate system.
- Calculate the distance between any two points in the rectangular coordinate plane.

We encounter graphs throughout science. Graphs are an efficient way to show your results. They also allow interpretation of trends. Graphs let you and others see your data. Graphing concentration and absorbance in a standard curve lets you calculate the concentration of an unknown sample based on absorbance. Graphing bacterial growth over time let's you calculate the generation time. To do these tasks, we need to know how to graph and what a linear equation means on a graph.

Rectangular Coordinate System

The horizontal number line is called the x-axis, and the vertical number line is called the y-axis. Each point on this plane (the flat surface of the graph) is associated with an ordered pair of real numbers (x, y). The first number is called the x-coordinate, and the second number is called the y-coordinate. The **intersection** of the two axes is known as the **origin**, which corresponds to the point $(0, 0)$.

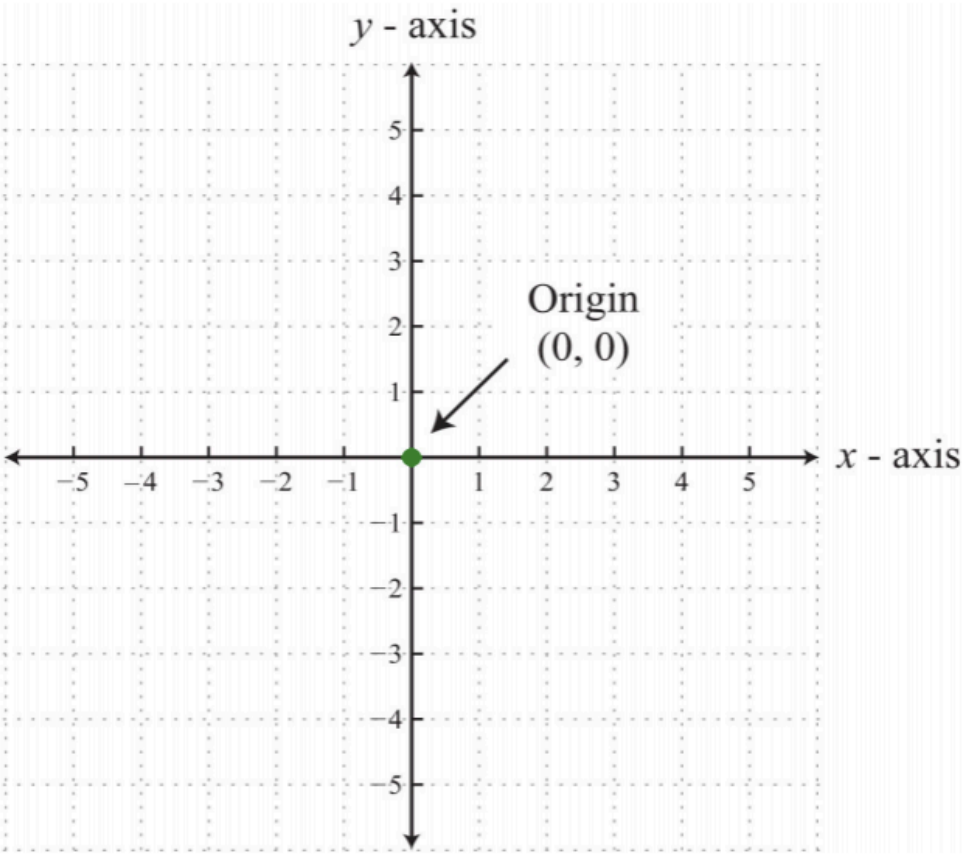

Figure 5.1.1.

An ordered pair (x, y) represents the position of a point relative to the origin. The x-coordinate represents a position to the right of the origin if it is positive and to the left of the origin if it is negative. The y-coordinate represents a position above the origin if it is positive and below the origin if it is negative. Using this system, every position (point) in the plane is uniquely identified. For example, the pair $(2, 3)$ denotes the position relative to the origin as shown:

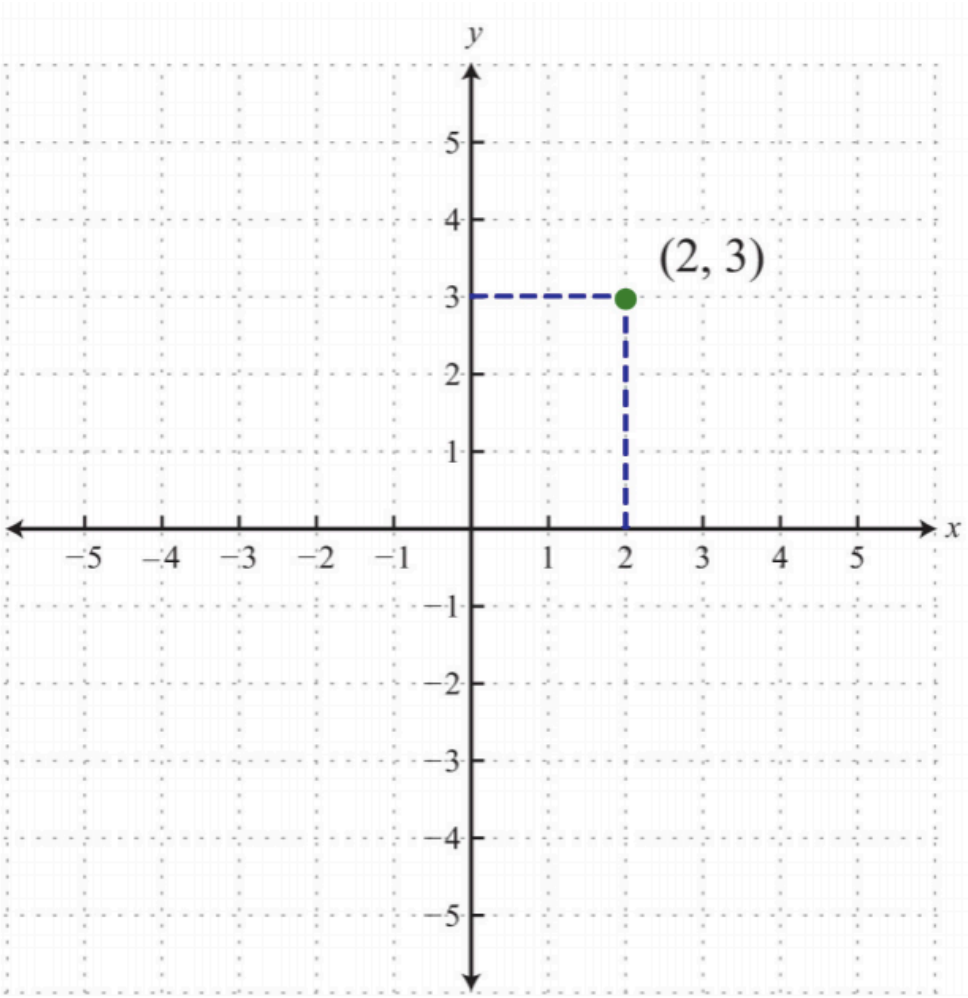

Figure 5.1.2.

This system is often called the **Cartesian coordinate system**, named after the French mathematician René Descartes (1596–1650).

The x– and y-axes break the plane into four regions called **quadrants**, named using roman numerals I, II, III, and IV, as pictured. In quadrant I, both coordinates are positive. In quadrant II, the x-coordinate is negative and the y-coordinate is positive. In quadrant III, both coordinates are negative. In quadrant IV, the x-coordinate is positive and the y-coordinate is negative.

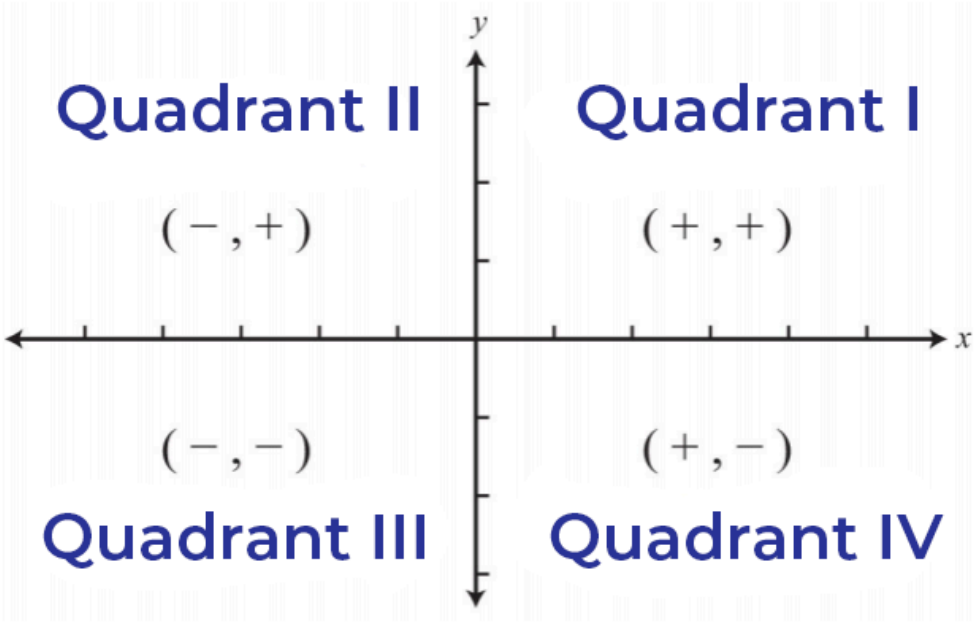

Figure 5.1.3.

How To

Given an equation, graph by plotting points.

1. Make a table with one column labelled x, a second column labelled with the equation, and a third column listing the resulting ordered pairs.
2. Enter x-values down the first column using positive and negative values. Selecting the x-values in numerical order will make the graphing simpler.
3. Select x-values that will yield y-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

Example 1

Plot the ordered pair $(-3, 5)$ and determine the quadrant in which it lies.

Solution:

The coordinates $x = -3$ and $y = 5$ indicate a point 3 units to the left of and 5 units above the origin.

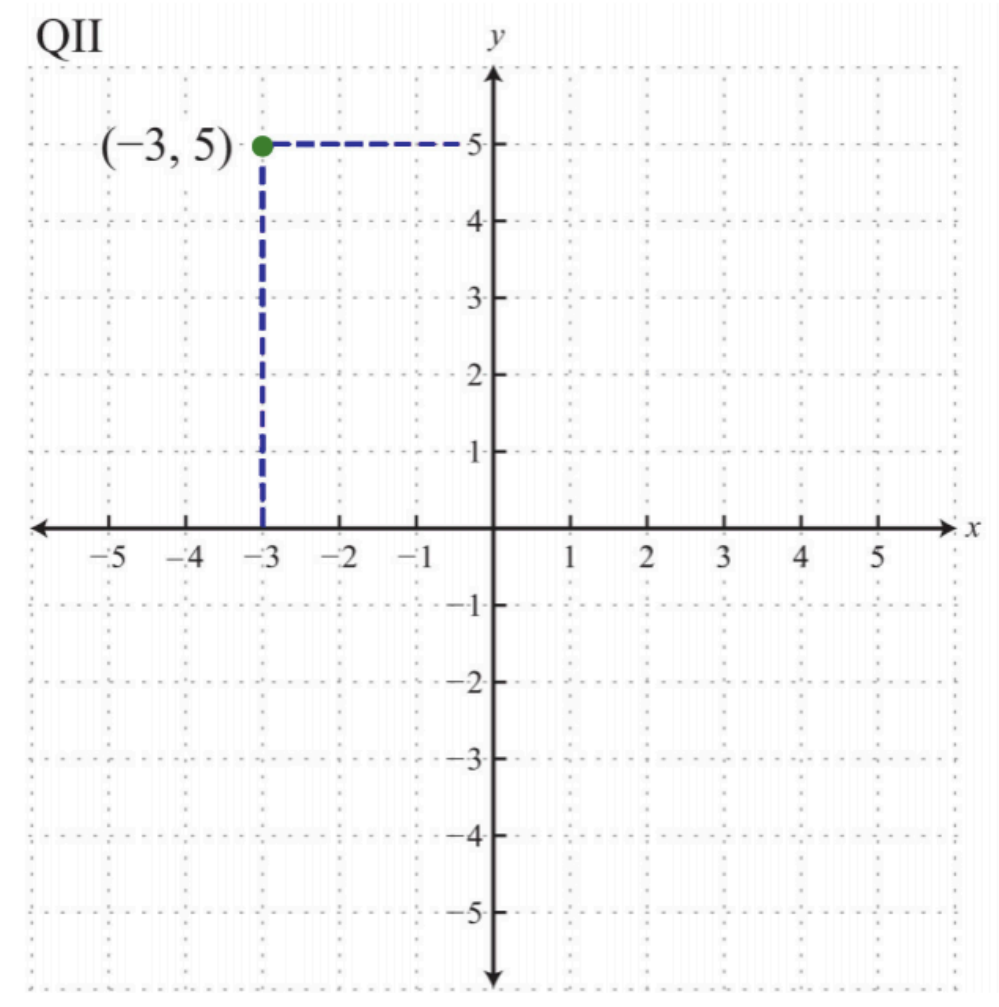

Figure 5.1.4

Answer

The point is plotted in quadrant II (QII) because the x-coordinate is negative and the y-coordinate is positive.

Ordered pairs with 0 as one of the coordinates do not lie in a quadrant; these points are on one axis or the other (or the point is the origin if both coordinates are 0). Also, the scale indicated on the x-axis may be different from the scale indicated on the y-axis. Choose a scale that is convenient for the given situation.

Example 2

Plot this set of ordered pairs:

$$\{(4,0),(-6,0),(0,3),(-2,6),(-4,-6)\}$$

Solution:

Each tick mark on the x-axis represents 2 units and each tick mark on the y-axis represents 3 units.

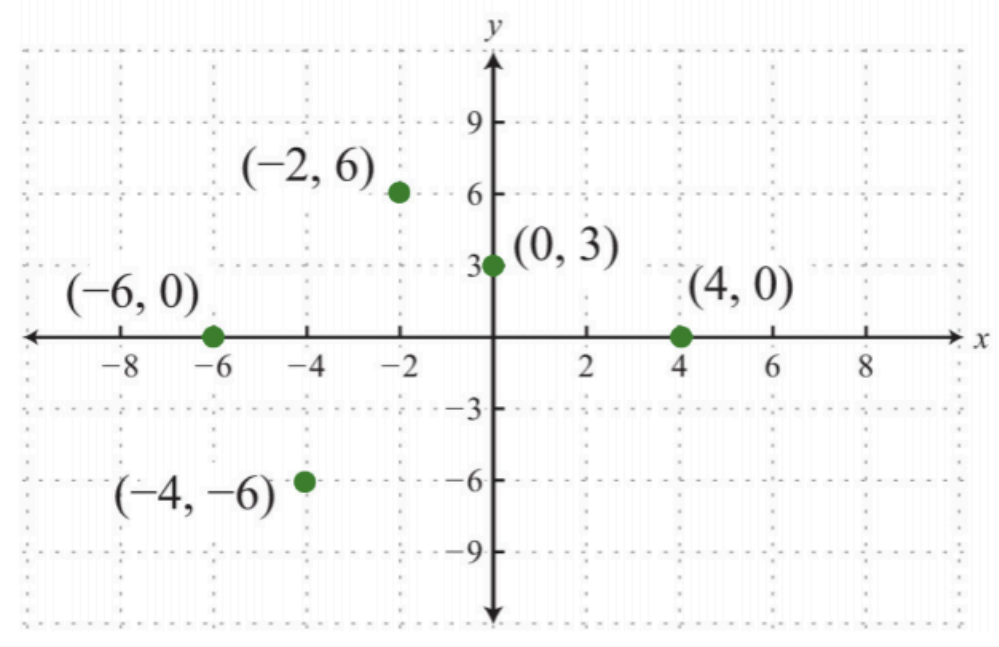

Figure 5.1.5

In this example, the points appear to be **collinear**, or to lie on the same line. The entire chapter focuses on finding and expressing points with this property.

Example 3

Plot this set of ordered pairs:

$$\{(-6,-5),(-3,-3),(0,-1),(3,1),(6,3)\}$$

Solution:

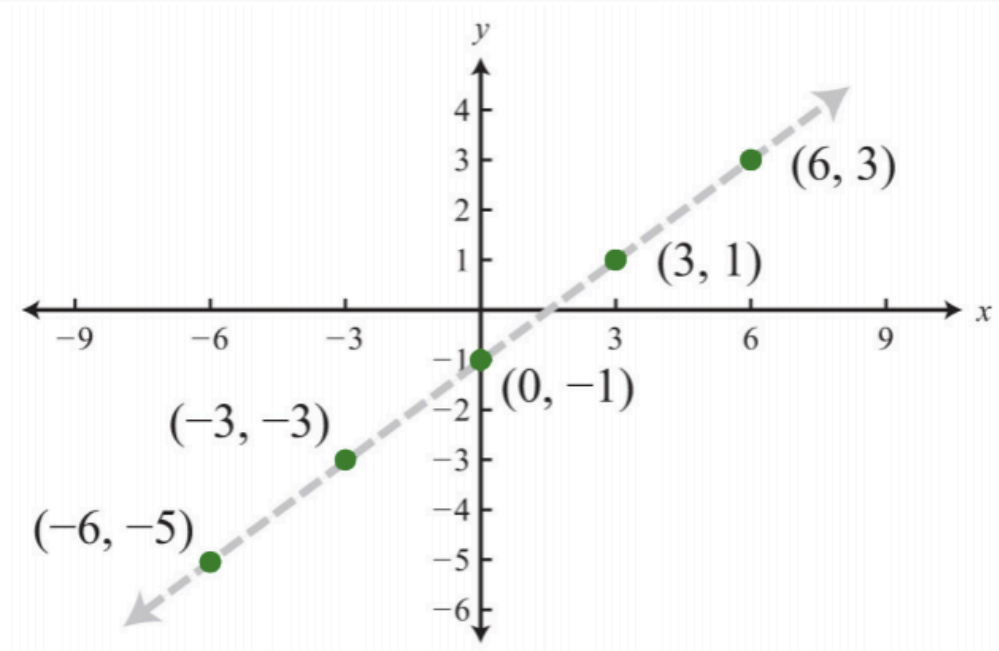

Figure 5.1.6

Try It

Plot the set of points $\{(5, 3), (-3, 2), (-2, -4), (4, -3)\}$ and indicate in which quadrant they lie.

Show Solution

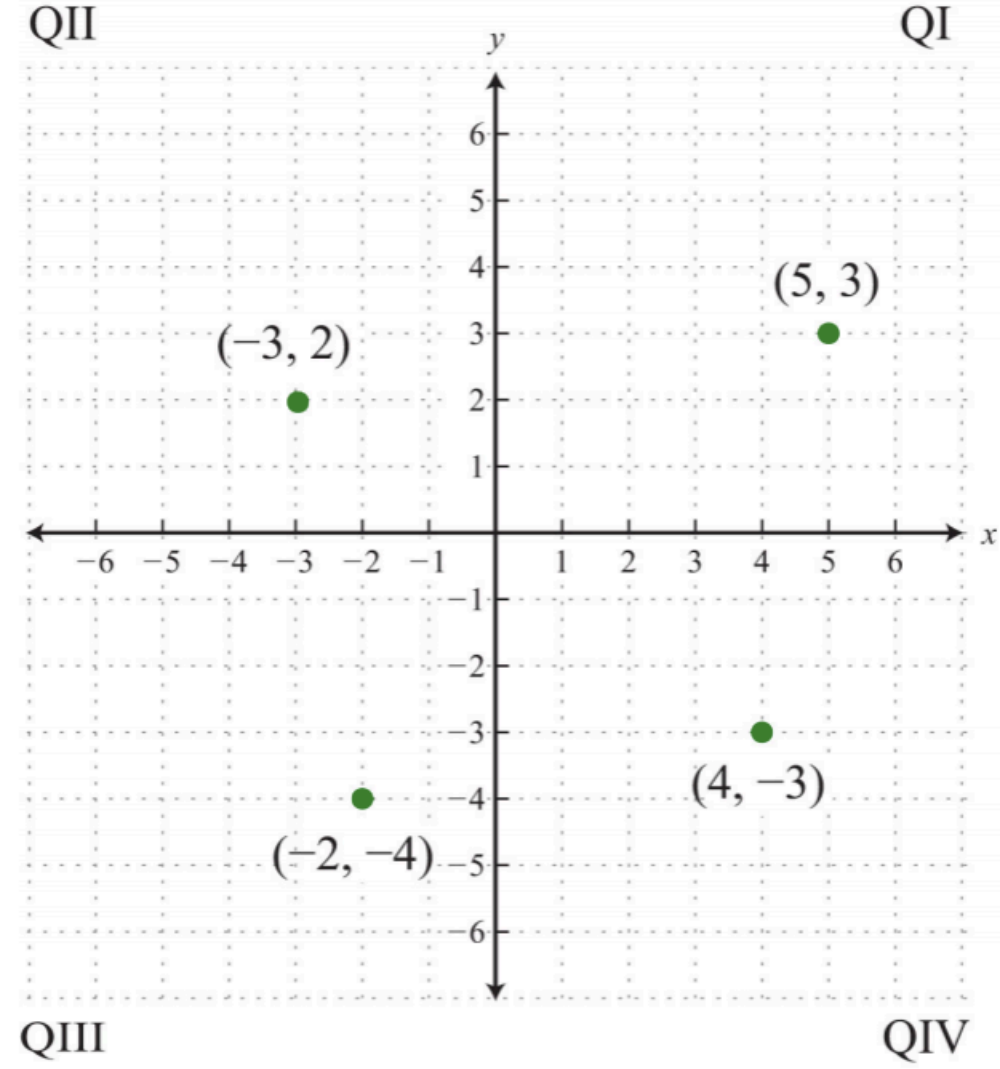

Figure 5.1.7

Graphs are used in everyday life to display data visually. A line graph consists of a set of related data values graphed on a coordinate plane and connected by line segments. Typically, the independent quantity, such as time, is displayed on the x-axis and the dependent quantity, such as distance travelled, on the y-axis.

Science Application

You are in the lab doing these experiments. You generate this data, now how do you graph it? Try this exercise.

Activity Description
For each experiment, indicate which variable is associated with which axis.

1. The Experiment: You add specific amounts of a base to an acid and measure how the acidity level changes after each addition.
   The Variables: pH, volume
2. The Experiment: You prepare a series of solutions with known strengths and measure absorbance in a spectrophotometer.
   The Variables: concentration, absorbance
3. The Experiment: You change the amount of reactant in a flask to determine how much faster the chemical reaction occurs at higher concentrations.
   The Variables: concentration, time
4. The Experiment: You grow a culture in an incubator and count the total number of bacteria every two hours to track how the population grows.
   The Variables: colonies/mL, time

---

1. x-axis: volume; y-axis: pH
2. x-axis: concentration; y-axis: absorbance
3. x-axis: concentration; y-axis: time
4. x-axis: colonies/mL; y-axis: time

Key Takeaways

- Use the rectangular coordinate system to uniquely identify points in a plane using ordered pairs (x, y). Ordered pairs indicate position relative to the origin.
  The x-coordinate indicates position to the left and right of the origin. The y-coordinate indicates position above or below the origin.
- The scales on the x-axis and y-axis may be different. Choose a scale for each axis that is appropriate for the given problem.
- Graphs are used to visualize real-world data. Typically, independent data is associated with the x -axis and dependent data is associated with the y-axis.

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How To: "2.1 The Rectangular Coordinate Systems and Graphs" from College Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Used How To box "Given an equation, graph by plotting points"; Reformatted.

5.2 Graph Using the Slope and Y-Intercept

Learning Objectives

By the end of this section, you should be able to:

- Identify and find the slope of a line.
- Graph a line using the slope and y-intercept.

Slope

The steepness of any incline can be measured as the **ratio** of the vertical change to the horizontal change. For example, a 5% incline can be written as $\frac{5}{100}$, which means that for every 100 metres forward, the height increases 5 metres.

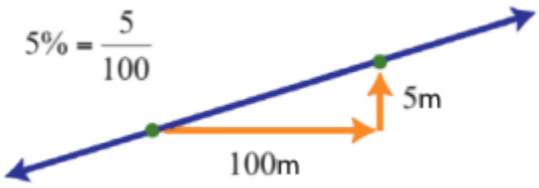

Figure 5.2.1.

In mathematics, we call the incline of a line the slope and use the letter m to denote it. The vertical change is called the **rise**, and the horizontal change is called the **run**.

$$\text{Slope} \qquad m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}$$

The rise and the run can be positive or negative. A positive rise corresponds to a vertical change up, and a negative rise corresponds to a vertical change down. A positive run denotes a horizontal change to the right, and a negative run corresponds to a horizontal change to the left. Given the graph, we can calculate the slope by determining the vertical and horizontal changes between any two points.

How To

Find the slope of a line from its graph using $m = \dfrac{\text{rise}}{\text{run}}$

1. Locate two points on the line whose coordinates are integers.
2. Starting with one point, sketch a right triangle, going from the first point to the second point.
3. Count the rise and the run on the legs of the triangle.
4. Take the ratio of rise to run to find the slope: $m = \dfrac{\text{rise}}{\text{run}}$

Example 1

Find the slope of the given line:

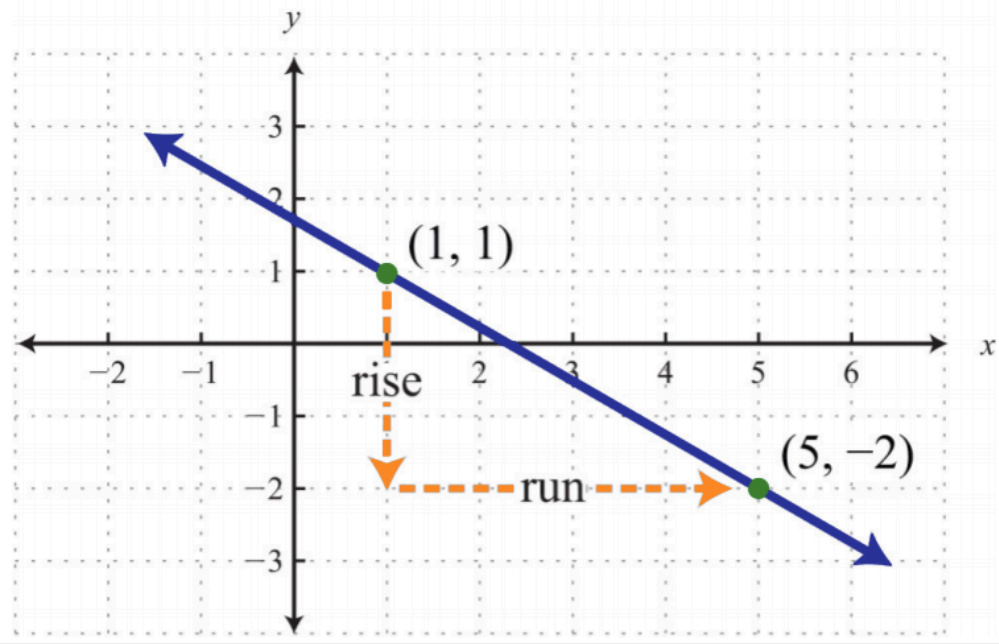

Figure 5.2.2.

Solution:

From the given points on the graph, count 3 units down and 4 units right.

$$m = \frac{rise}{run} = \frac{-3 \text{ units}}{4 \text{ units}} = -\frac{3}{4}$$

Answer

$$m = -\frac{3}{4}$$

Here we have a negative slope, which means that for every 4 units of movement to the right, the vertical change is 3 units downward. There are four geometric cases for the value of the slope.

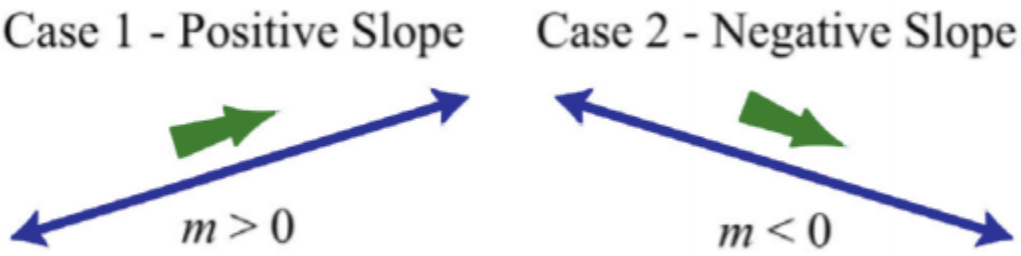

Figure 5.2.3

Reading the graph from left to right, we see that lines with an upward incline have positive slopes and lines with a downward incline have negative slopes.

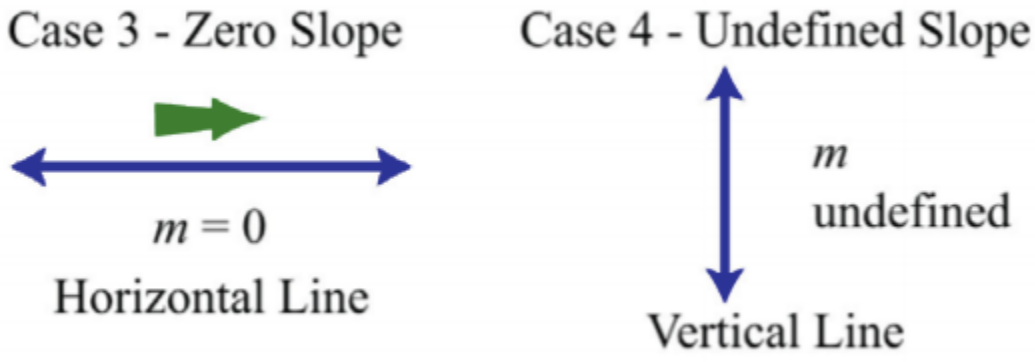

Figure 5.2.4.

If the line is horizontal, then the rise is 0:

$$m = \frac{\text{rise}}{\text{run}} = \frac{0}{\text{run}} = 0$$

The slope of a **horizontal line** is 0. If the line is vertical, then the run is 0:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{rise}}{0} \quad \text{Undefined}$$

The slope of a **vertical line** is undefined.

Try It

Find the slope of the given line:

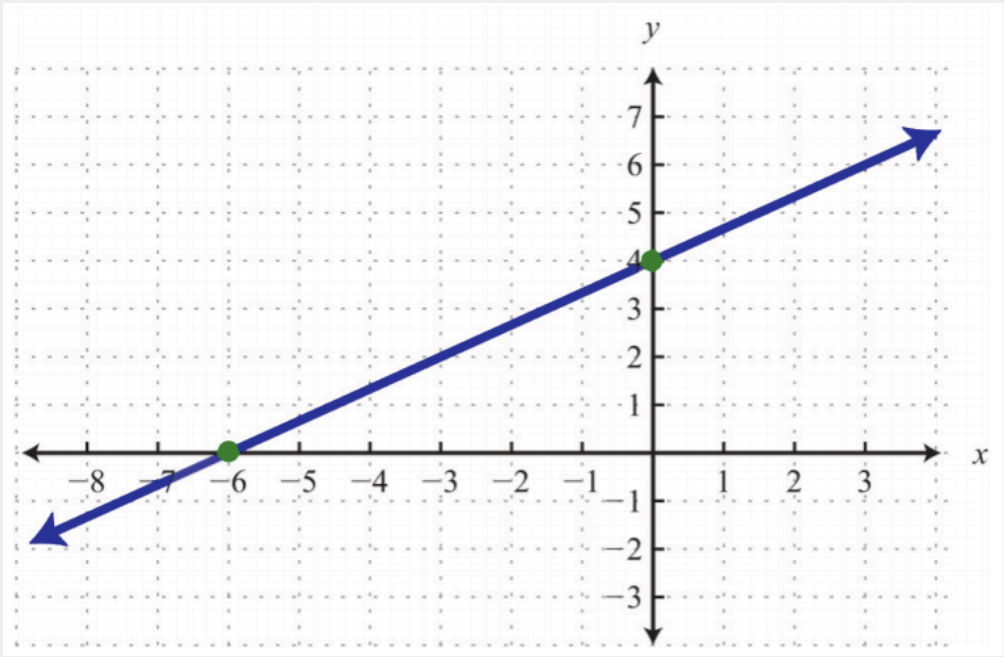

Figure 5.2.5.

Show Solution

$$m = \frac{2}{3}$$

Calculating the slope can be difficult if the graph does not have points with integer coordinates. Therefore, we next develop a formula that allows us to calculate the slope algebraically. Given any two points (x_1, y_1) and (x_2, y_2), we can obtain the rise and run by subtracting the corresponding coordinates.

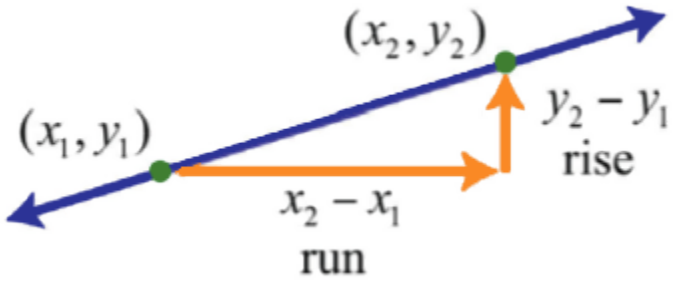

Figure 5.2.6.

This leads us to the **slope formula**. Given any two points (x_1, y_1) and (x_2, y_2), the slope is given by

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 2

Find the slope of the line passing through $(-3, -5)$ and $(2, 1)$

Solution:

Given $(-3, -5)$ and $(2, 1)$, calculate the difference of the y-values divided by the difference of the x-values. Since subtraction is not commutative, take care to be consistent when subtracting the coordinates.

$$\begin{array}{ll} (x_1, y_1) & (x_2, y_2) \\ (-3, -5) & (2, 1) \end{array}$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-5)}{2 - (-3)} \\ &= \frac{1 + 5}{2 + 3} \\ &= \frac{6}{5} \end{aligned}$$

Answer

$$m = \frac{6}{5}$$

We can graph the line described in the previous example and verify that the slope is $\frac{6}{5}$.

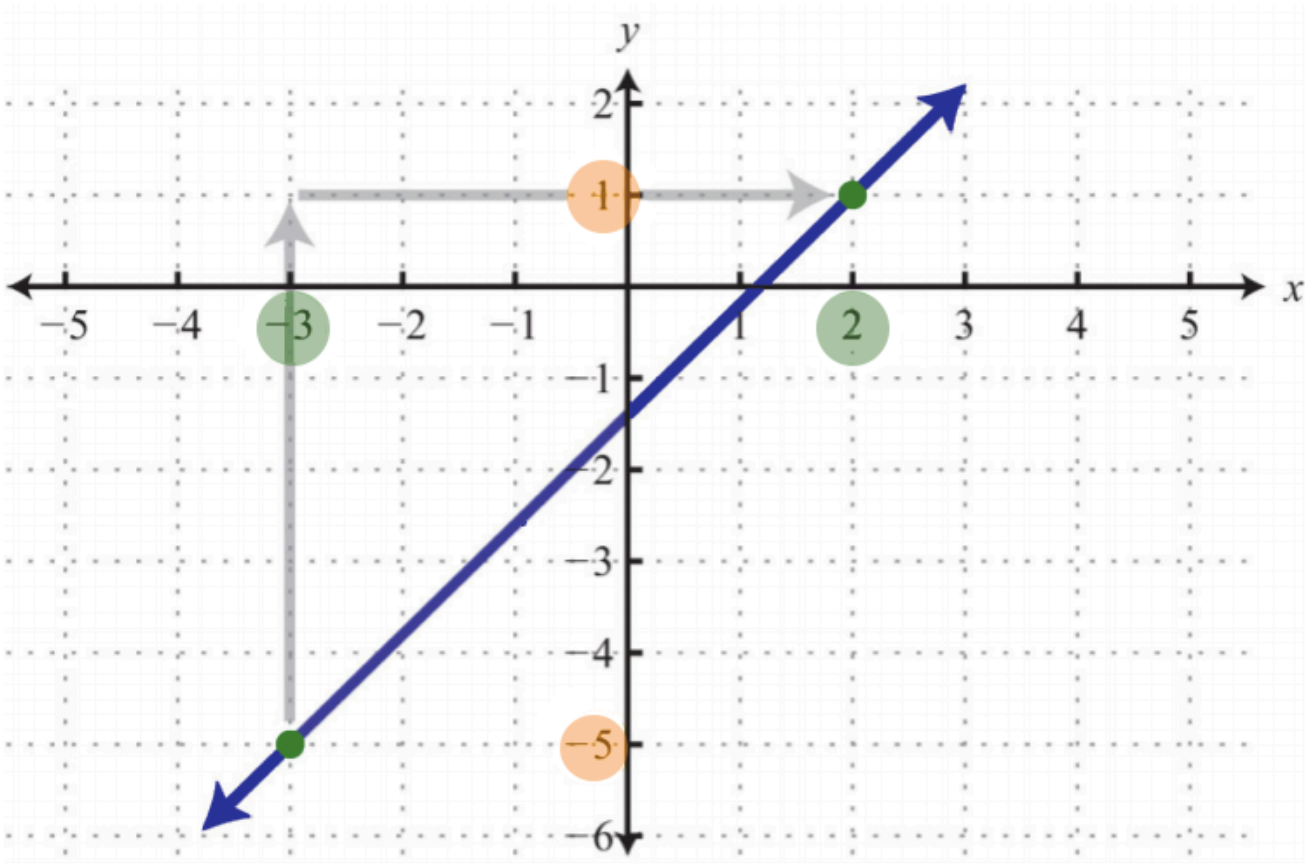

Figure 5.2.7.

Certainly, the graph is optional; the beauty of the slope formula is that we can obtain the slope, given two points, using only algebra.

Example 3

Find the slope of the line passing through $(-4, 3)$ and $(-1, -7)$.

Solution:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-4, 3) & (-1, -7) \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-7 - (3)}{-1 - (-4)}$$

$$= \frac{-7 - 3}{-1 + 4}$$

$$= \frac{-10}{3}$$

Answer

$m = -\frac{10}{3}$

When using the slope formula, take care to be consistent since order does matter. You must subtract the coordinates of the first point from the coordinates of the second point for both the numerator and the denominator in the same order.

Example 4

Find the slope of the line passing through $(7, -2)$ and $(-5, -2)$.

Solution:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (7, -2) & (-5, -2) \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-2 - (-2)}{-5 - (7)}$$

$$= \frac{-2 + 2}{-5 - 7}$$

$$= \frac{0}{-12} = 0$$

Answer

$m = 0$. As an exercise, plot the given two points and verify that they lie on a horizontal line.

Example 5

Find the slope of the line passing through $(-4, -3)$ and $(-4, 5)$.

Solution:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-4, -3) & (-4, 5) \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{5 - (-3)}{-4 - (-4)}$$

$$= \frac{5 + 3}{-4 + 4} = \frac{8}{0} \quad \text{Undefined}$$

Answer

The slope m is undefined. As an exercise, plot the given two points and verify that they lie on a vertical line.

Try It

Calculate the slope of the line passing through $(-2, 3)$ and $(5, -5)$.

Show Solution

$$m = -\frac{8}{7}$$

When considering the slope as a **rate** of change, it is important to include the correct units.

Example 6

A Corvette Coupe was purchased new in 1970 for about $\$5{,}200$ and depreciated in value over time until it was sold in 1985 for $\$1{,}300$. At this point, the car was beginning to be considered a classic and started to increase in value. In the year 2000, when the car was 30 years old, it sold at auction for $\$10{,}450$. The following line graph depicts the value of the car over time.

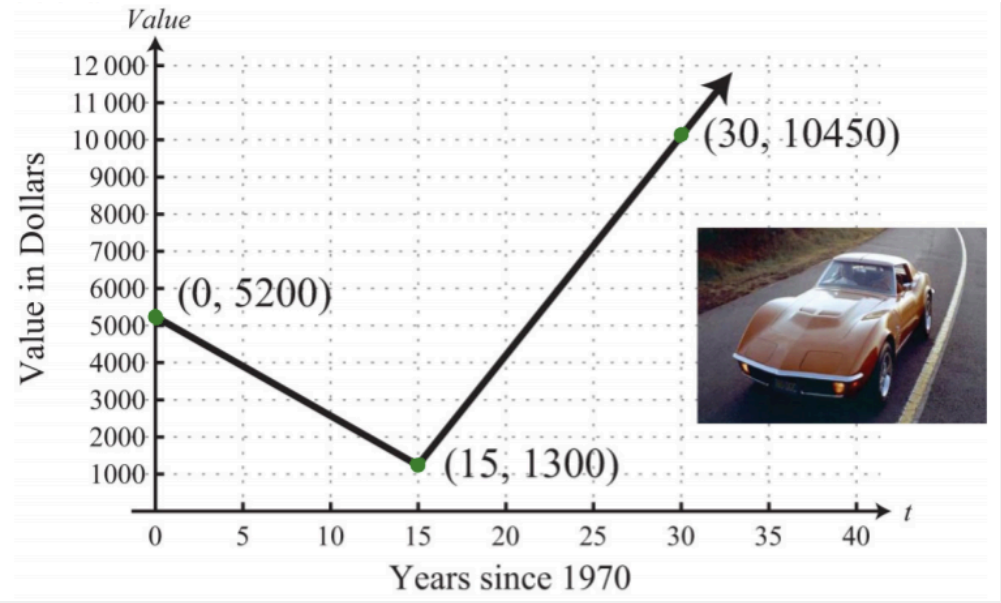

Figure 5.2.8.

a. Determine the rate at which the car depreciated in value from 1970 to 1985.
b. Determine the rate at which the car appreciated in value from 1985 to 2000.

Solution:

Notice that the value depends on the age of the car and that the slope measures the rate in dollars per year.

a. The slope of the line segment depicting the value for the first 15 years is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\$1{,}300 - \$5{,}200}{15 \text{ years} - 0 \text{ years}}$$

$$= \frac{-\$3{,}900}{15 \text{ years}}$$

$$= -\$260 \text{ per year}$$

b. The slope of the line segment depicting the value for the next 15 years is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\$10{,}450 - \$1{,}300}{30 \text{ years} - 15 \text{ years}}$$

$$= \frac{\$9{,}150}{15 \text{ years}}$$

$$= \$610 \text{ per year}$$

Answer

a. The value of the car depreciated $\$260$ per year from 1970 to 1985.
b. The value of the car appreciated $\$610$ per year from 1985 to 2000.

Slope-Intercept Form of a Line

To this point, we have learned how to graph lines by plotting points and by using the x-**intercept** and y-**intercepts**. In addition, we have seen that we need only two points to graph a line. In this section, we outline a process to easily determine two points using the y-intercept and the slope. The equation of any nonvertical line can be written in **slope-intercept form** $y = mx + b$. In this form, we can identify the slope, m, and the y-intercept, $(0, b)$.

How To

Graph a line given a point and the slope.

1. Plot the given point.
2. Use the slope formula $m = \dfrac{\text{rise}}{\text{run}}$ to identify the rise and the run.
3. Starting at the given point, count out the rise and run to mark the second point.
4. Connect the points with a line.

Example 7

Determine the slope and y-intercept:

$$y = -\frac{4}{5}x + 7$$

Solution:

In this form, the coefficient of x is the slope, and the constant is the y-value of the y-intercept. Therefore, by inspection, we have

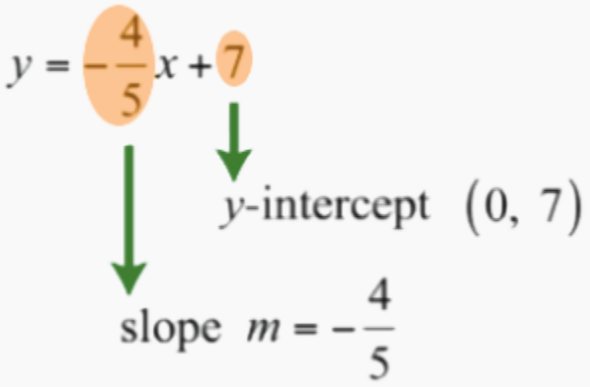

Figure 5.2.9.

Answer

The y-intercept is $(0, 7)$, and the slope is $m = -\frac{4}{5}$.

It is not always the case that the linear equation is given in slope-intercept form. When it is given in standard form, you have to first solve for y to obtain slope-intercept form.

Example 8

Express $3x + 5y = 30$ in slope-intercept form and then identify the slope and y-intercept.

Solution:
Begin by solving for y. To do this, apply the properties of equality to first isolate $5y$ and then divide both sides by 5.

$$\begin{aligned} 3x + 5y &= 30 \\ 3x + 5y - 3x &= 30 - 3x \\ 5y &= -3x + 30 \\ \frac{5y}{5} &= \frac{-3x + 30}{5} \\ y &= \frac{-3x}{5} + \frac{30}{5} \\ y &= -\frac{3}{5}x + 6 \end{aligned}$$

Answer

Slope-intercept form: $y = -\frac{3}{5}x + 6$; y-intercept: $(0, 6)$; slope: $m = -\frac{3}{5}$

Once the equation is in slope-intercept form, we immediately have one point to plot, the y-intercept. From the intercept, you can mark off the slope to plot another point on the line. From the previous example, we have

$$y = -\frac{3}{5}x + 6$$

y-intercept: $(0, 6)$

slope: $m = -\frac{3}{5} = \frac{-3}{5} = \frac{rise}{run}$

Starting from the point $(0, 6)$, use the slope to mark another point 3 units down and 5 units to the right.

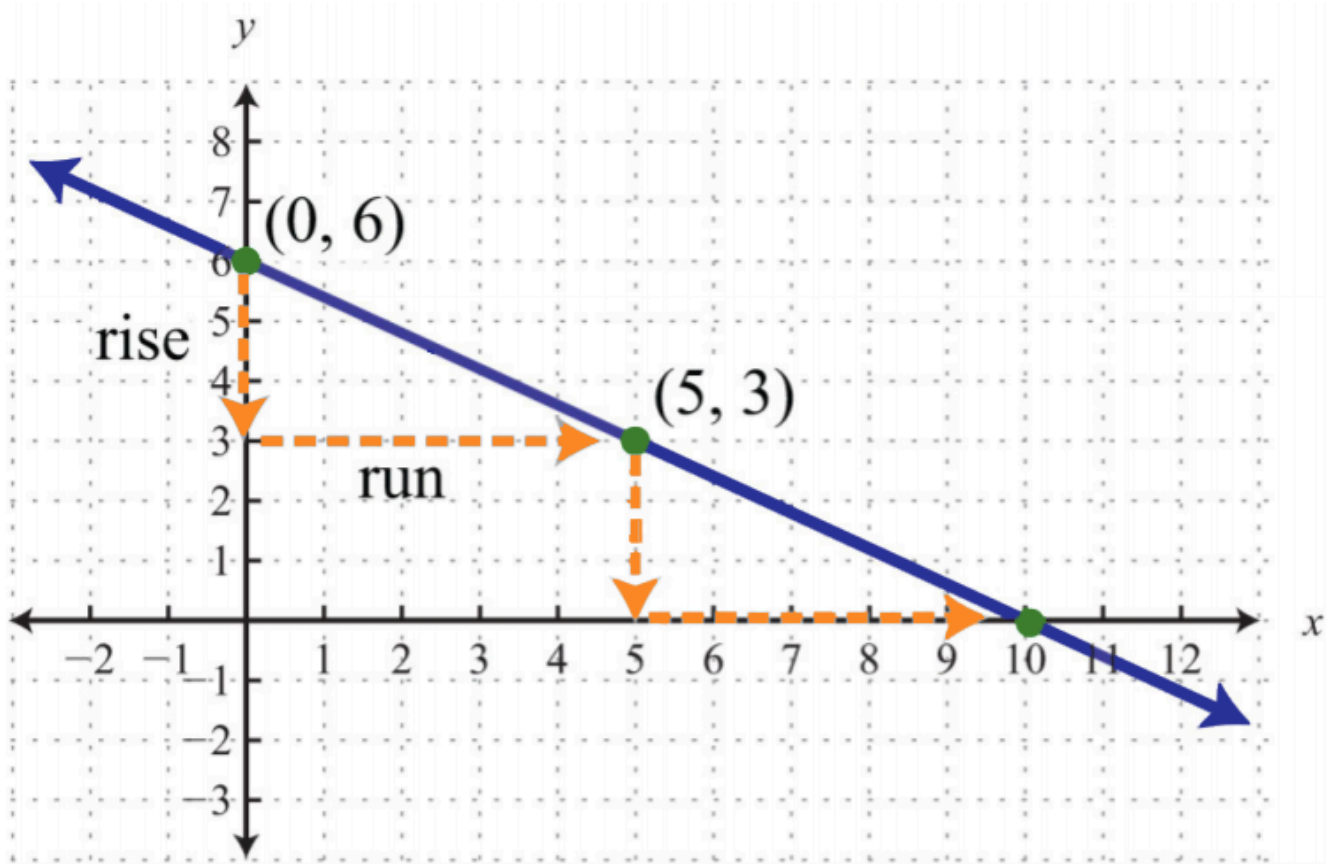

Figure 5.2.10.

It is not necessary to check that the second point, $(5, 3)$, solves the original linear equation. However, we do it here for the sake of completeness.

$$\begin{aligned} 3x + 5y &= 30 \\ 3(5) + 5(3) &= 30 \\ 15 + 15 &= 30 \\ 30 &= 30 \quad \checkmark \end{aligned}$$

Marking off the slope in this fashion produces as many ordered pair solutions as we desire. Notice that if we mark off the slope again, from the point $(5, 3)$, then we obtain the x-intercept, $(10, 0)$.

Science Application

A substance is heated from 20°C to 80°C. If the temperature increases at a constant rate of 10°C per minute, how long will it take to reach 80°C?

From this problem, can you see the graph? Which variable is dependent (y) and independent (x)? What is the y-intercept (b) and the slope?

Write the equation of the line, then substitute in 80°C and solve for the other variable.

Show Solution

Temperature is dependent on time, so temperature is on the y-axis and time is on the x-axis. The slope is the rate, $m = 10\frac{^\circ C}{min}$, $b = 20\,^\circ C$. The equation is $y = 10x + 20$. Sub $y = 80$ into the equation and solve for x. It will take six minutes to reach 80°C

Let's try two more, since making and interpreting standard curves is an important skill.

Science Applications

Standard Curve: You prepare a standard curve of DNA absorbance. The equation is $y = 0.0194x + 0.0228$. Concentration is in $\mu g/ml$. Identify which variable is on the x-axis and which is on the y-axis.

a. Your sample has an absorbance of 0.65. What is the DNA concentration?
b. What do you expect the absorbance to be if there is no DNA in the sample?
c. The linear range of the instrument is <1.0. At what expected DNA concentration do you need to make a dilution?

Solutions

a. $32\mu g$
b. $x = 0$, absorbance =0.0228
c. Set $y = 1$ and solve; $< 50.4\mu g$

Example 9

Graph:

$$-x + 2y = 4$$

Solution:
In this example, we outline the general steps for graphing a line using slope-intercept form.

Step 1: Solve for y to obtain slope-intercept form.

$$\begin{aligned} -x + 2y &= 4 \\ -x + 2y + x &= 4 + x \\ 2y &= x + 4 \\ \frac{2y}{2} &= \frac{x+4}{2} \\ y &= \frac{1x}{2} + \frac{4}{2} \\ y &= \frac{1}{2}x + 2 \end{aligned}$$

Step 2: Identify the y-intercept and slope.

$$y\text{-intercept: } (0, 2)$$

$$\text{slope: } m = \frac{1}{2} = \frac{rise}{run}$$

Step 3: Plot the y-intercept and use the slope to find another ordered pair solution. Starting from the y-intercept, mark off the slope and identify a second point. In this case, mark a point after a rise of 1 unit and a run of 2 units.

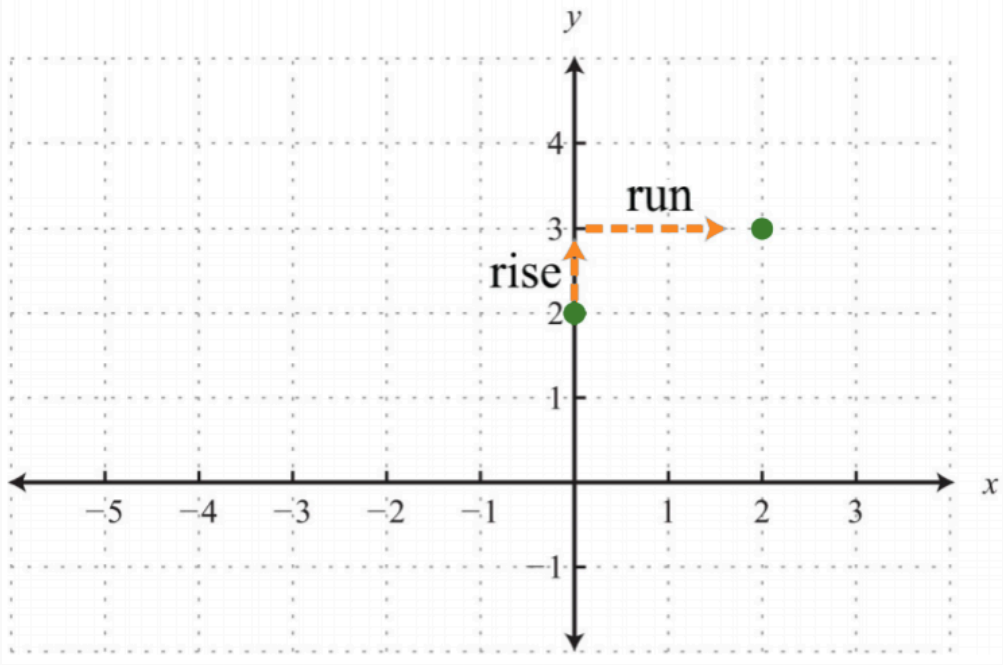

Figure 5.2.11.

Step 4: Draw the line through the two points with a straightedge.

Answer

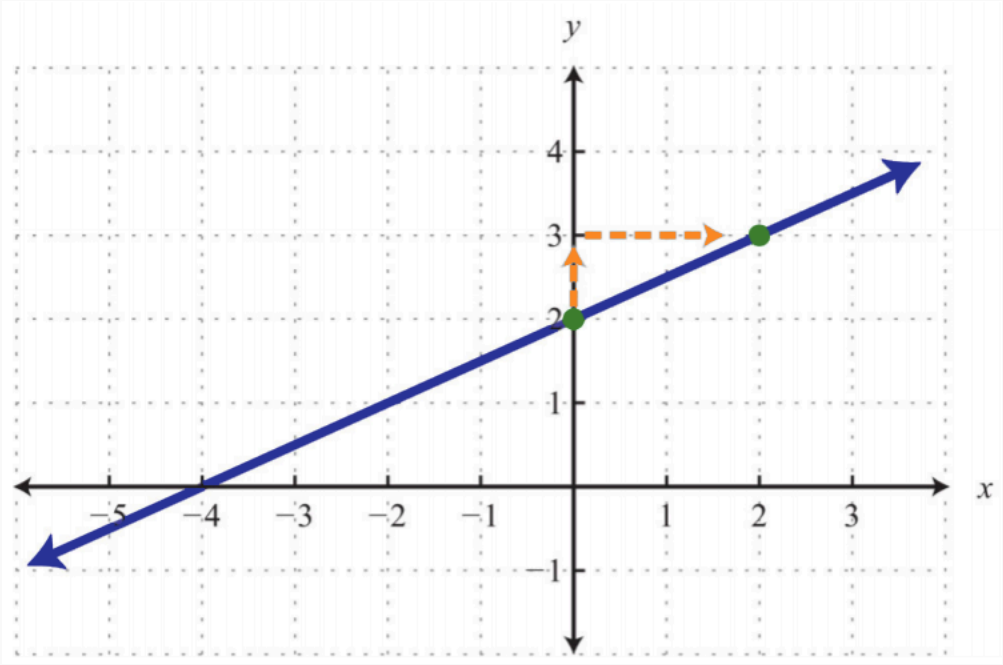

Figure 5.2.12.

In this example, we notice that we could get the x-intercept by marking off the slope in a different but equivalent manner. Consider the slope as the ratio of two negative numbers as follows:

$$m = \frac{1}{2} = \frac{-1}{-2} = \frac{rise}{run}$$

We could obtain another point on the line by marking off the equivalent slope down 1 unit and left 2 units. We do this twice to obtain the x-intercept, $(-4, 0)$.

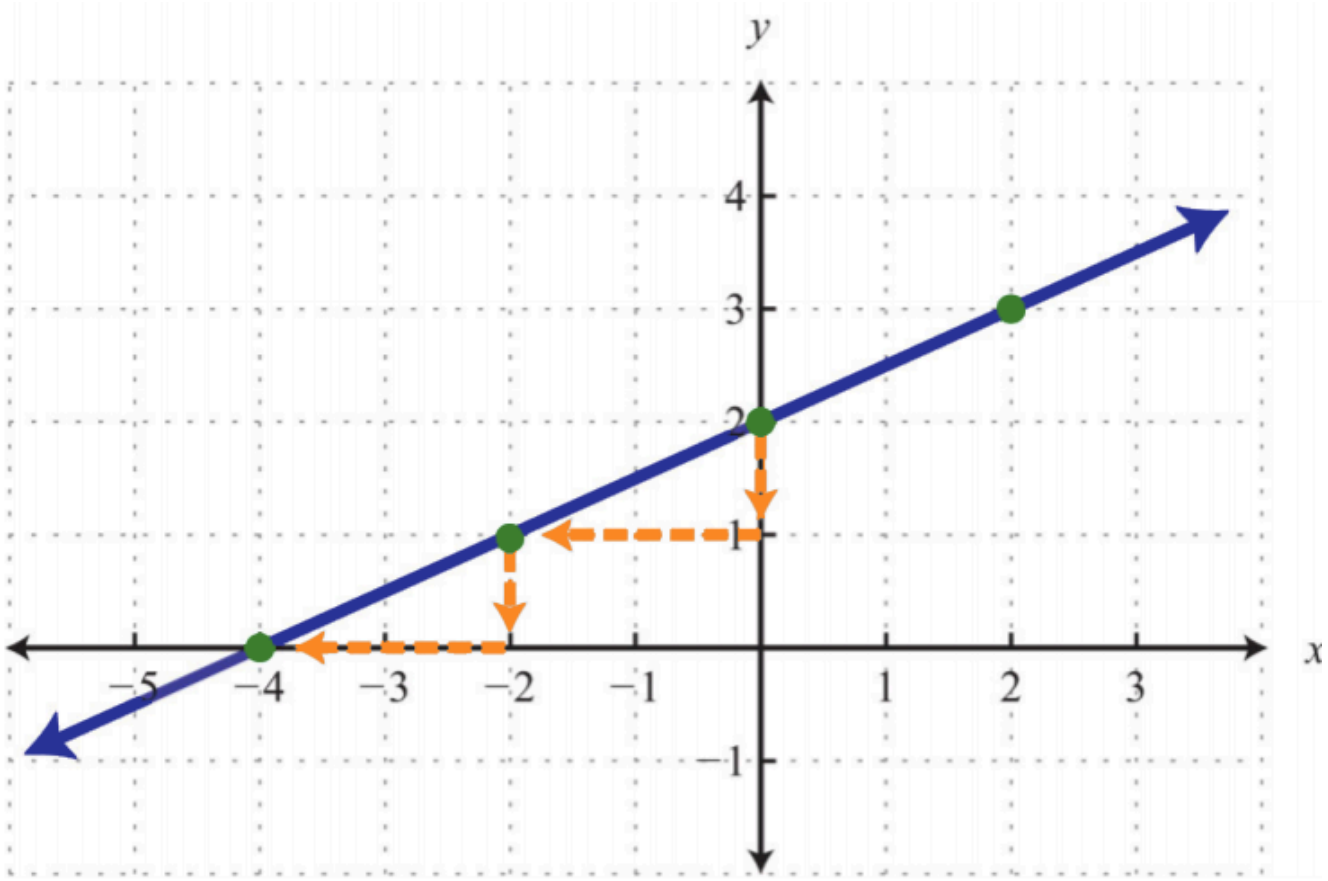

Figure 5.2.13.

Marking off the slope multiple times is not necessarily always going to give us the x-intercept, but when it does, we obtain a valuable point with little effort. In fact, it is a good practice to mark off the slope multiple times; doing so allows you to obtain more points on the line and produce a more accurate graph.

Example 10

Graph and find the x-intercept:

$$y = \frac{3}{4}x - 2$$

Solution:

The equation is given in slope-intercept form. Therefore, by inspection, we have the y-intercept and slope.

$$y\text{-intercept: } (0, -2)$$
$$\text{slope: } m = \frac{3}{4} = \frac{rise}{run}$$

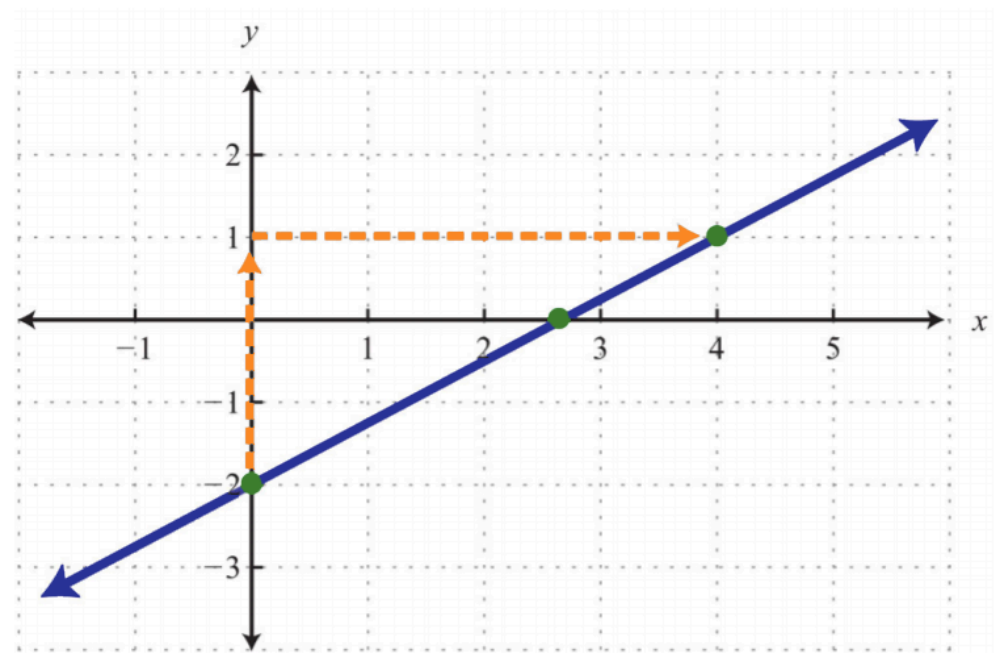

Figure 5.2.14.

We can see that the x-value of the x-intercept is a mixed number between 2 and 3. To algebraically find x-intercepts, recall that we must set $y = 0$ and solve for x.

$$0 = \frac{3}{4}x - 2$$

$$\frac{3}{4}x = -2$$

$$-\frac{4}{3}\cdot\left(-\frac{3}{4}\right) = -\frac{4}{3}\cdot(-2)$$

$$x = \frac{8}{3} \text{ or } 2\frac{2}{3}$$

Answer

The x-intercept is $\left(2\frac{2}{3}, 0\right)$.

Example 11

Graph:

$$x - y = 0$$

Solution:

Begin by solving for y.

$$\begin{aligned} x - y &= 0 \\ x - y - x &= 0 - x \\ -y &= -x \\ -1 \cdot (y) &= -1 \cdot (-x) \\ y &= x \end{aligned}$$

The equation $y = x$ can be written $y = 1x + 0$, and we have

$$y\text{-intercept: } (0, 0)$$

$$\text{slope: } m = 1 = \frac{1}{1} = \frac{rise}{run}$$

Answer

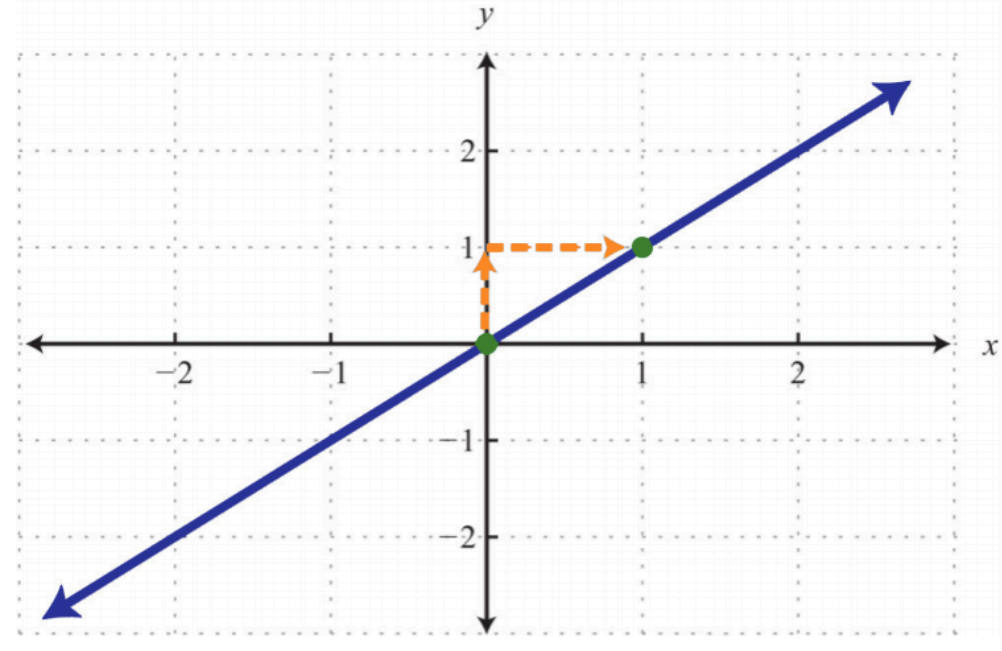

Figure 5.2.15.

Try It

Graph $-2x + 5y = 20$ and label the x-intercept.

Show Solution

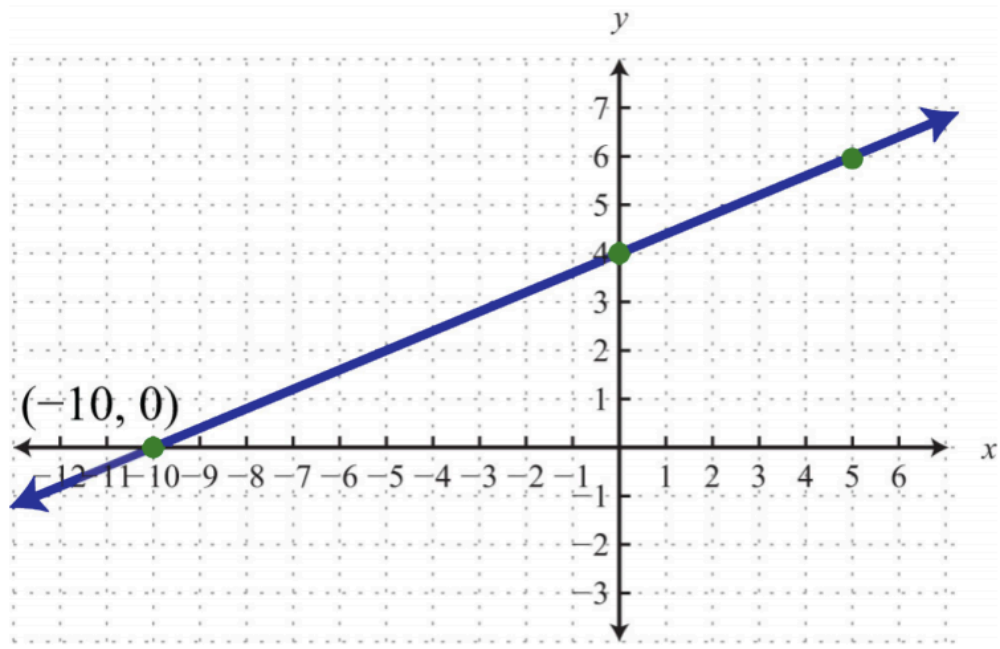

Figure 5.2.16.

Key Takeaways

- Slope measures the steepness of a line as *rise over run*. A positive rise denotes a vertical change up, and a negative rise denotes a vertical change down. A positive run denotes a horizontal change right, and a negative run denotes a horizontal change left.
- Horizontal lines have a slope of zero, and vertical lines have undefined slopes.
- Given any two points on a line, we can algebraically calculate the slope using the slope formula, $m = \dfrac{rise}{run} = \dfrac{y_2 - y_1}{x_2 - x_1}$.
- Any nonvertical line can be written in slope-intercept form, $y = mx + b$, from which we can determine, by inspection, the slope m and y-intercept $(0, b)$.
- If we know the y-intercept and slope of a line, then we can easily graph it. First, plot the y-intercept, and from this point use the slope as rise over run to mark another point on the line. Finally, draw a

line through these two points with a straightedge and add an arrow on either end to indicate that it extends indefinitely.
- We can obtain as many points on the line as we wish by marking off the slope multiple times.

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How To: "3.2 Slope of a Line" from Intermediate Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Used How To boxes "Find the slope of a line from its graph using $m = \frac{\text{rise}}{\text{run}}$" & "Graph a line given a point and the slope."; Reformatted.

5.3 Exercises

5.1 Rectangular Coordinate Systems

Graph the given set of ordered pairs.

1. $\{(-4,5),(-1,1),(-3,-2),(5,-1)\}$
2. $\{(-15,-10),(-5,10),(15,10),(5,-10)\}$
3. $\{(-2,5),(10,0),(2,-5),(6,-10)\}$
4. $\{(-8,3),(-4,6),(0,-6),(6,9)\}$
5. $\{(-10,5),(20,-10),(30,15),(50,0)\}$
6. $\left\{\left(-\frac{5}{3},-\frac{1}{2}\right),\left(-\frac{1}{3},\frac{1}{2}\right),\left(\frac{2}{3},-1\right),\left(\frac{5}{3},1\right)\right\}$
7. $\left\{\left(-\frac{5}{3},-\frac{4}{3}\right),\left(\frac{2}{5},\frac{4}{3}\right),(1,-\frac{2}{3}),(0,1)\right\}$
8. $\{(-3.5,0),(-1.5,2),(0,1.5),(2.5,-1.5)\}$
9. $\{(-0.8,0.2),(-0.2,-0.4),(0,-1),(0.6,-0.4)\}$
10. $\{(-1.2,-1.2),(-0.3,-0.3),(0,0),(0.6,0.6),(1.2,1.2)\}$

State the quadrant in which the given point lies.

11. $(-3,2)$
12. $(5,7)$
13. $(-12,-15)$
14. $(7,-8)$

15. $(-3.8, 4.6)$

16. $(17.3, 1.9)$

17. $\left(-\frac{1}{8}, -\frac{5}{8}\right)$

18. $\left(\frac{3}{4}, -\frac{1}{4}\right)$

19. $x > 0$ and $y < 0$

20. $x < 0$ and $y < 0$

21. $x < 0$ and $y > 0$

22. $x > 0$ and $y > 0$

The average price of a gallon of regular unleaded gasoline in US cities is given in the following line graph. Use the graph to answer the following questions.

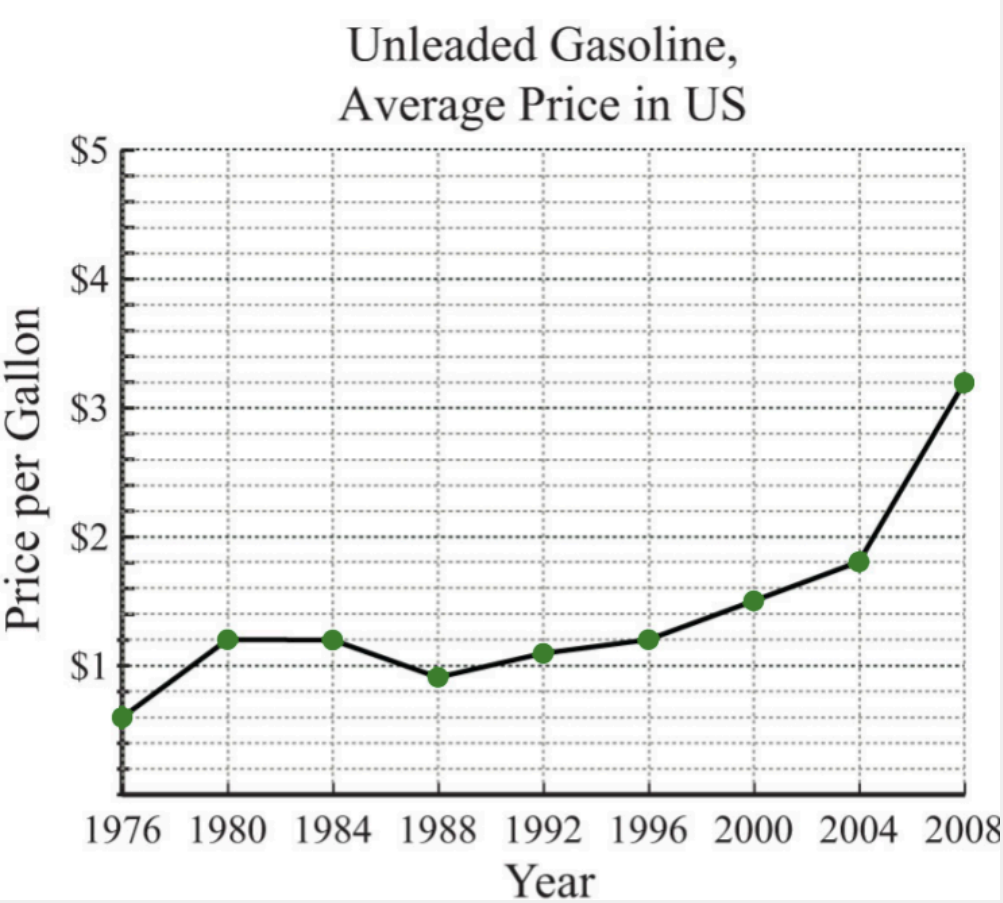


Figure 5.3.1. Unleaded Gasoline Average Price per Gallon in the US, 1976-2008. Data Source: Bureau of Labor Statistics.

23. What was the average price of a gallon of unleaded gasoline in 2004?

24. What was the average price of a gallon of unleaded gasoline in 1976?

25. In which years was the average price of a gallon of unleaded gasoline $1.20?

26. What is the price increase of a gallon of gasoline from 1980 to 2008?

27. What was the percentage increase in the price of a gallon of unleaded gasoline from 1976 to 1980?

28. What was the percentage increase in the price of a gallon of unleaded gasoline from 2000 to 2008?

The average price of all-purpose white flour in US cities from 1980 to 2008 is given in the following line graph. Use the graph to answer the questions that follow.

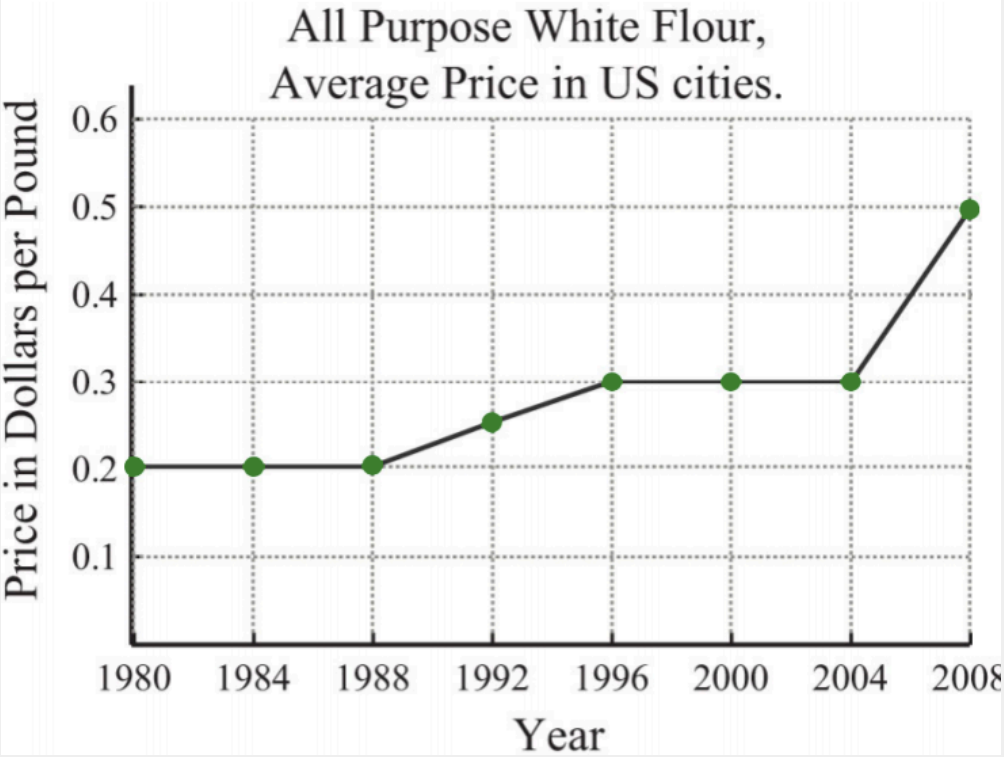


Figure 5.3.2. All Purpose While Flour, Average Price in US cities, Dollars per pound, 1980-2008. Data Source: Bureau of Labor Statistics.

29. What was the average price per pound of all-purpose white flour in 2000?
30. What was the average price per pound of all-purpose white flour in 2008?
31. In which year did the price of flour average $0.25 per pound?
32. In which years did the price of flour average $0.20 per pound?
33. What was the percentage increase in flour from the year 2000 to 2008?
34. What was the percentage increase in flour from the year 1992 to 2000?

Given the following data, create a line graph.

35. The percentage of total high school graduates who enrolled in college.

Table 5.5.1.

Year	Percentage
1969	36%
1979	40%
1989	47%
1999	42%

36. The average daily temperature is given in degrees Fahrenheit in May.

Table 5.5.2.

Date	Temperature (F)
1	60
5	72
10	75
15	67
20	60
25	55

Calculate the area of the shape formed by connecting the following set of vertices.

37. $\{(0,0),(0,3),(5,0),(5,3)\}$

38. $\{(-1,-1),(-1,1),(1,-1),(1,1)\}$

39. $\{(-2,-1),(-2,3),(5,3),(5,-1)\}$

40. $\{(-5,-4),(-5,5),(3,5),(3,-4)\}$

41. $\{(0,0),(4,0),(2,2)\}$

42. $\{(-2,-2),(2,-2),(0,2)\}$

43. $\{(0,0),(0,6),(3,4)\}$

44. $\{(-2,0),(5,0),(3,-3)\}$

Show Solutions

1. Plot the points $(-4, 5), (-1, 1), (-3, -2), (5, -1)$.
2. Plot the points $(-15, -10), (-5, 10), (15, 10), (5, -10)$.
3. Plot the points $(-2, 5), (10, 0), (2, -5), (6, -10)$.
4. Plot the points $(-8, 3), (-4, 6), (0, -6), (6, 9)$.
5. Plot the points $(-10, 5), (20, -10), (30, 15), (50, 0)$.
6. Plot the points $\left(-\frac{5}{3}, -\frac{1}{2}\right), \left(-\frac{1}{3}, \frac{1}{2}\right), \left(\frac{2}{3}, -1\right), \left(\frac{5}{3}, 1\right)$.
7. Plot the points $\left(-\frac{5}{3}, -\frac{4}{3}\right), \left(\frac{2}{5}, \frac{4}{3}\right), \left(1, -\frac{2}{3}\right), (0, 1)$.
8. Plot the points $(-3.5, 0), (-1.5, 2), (0, 1.5), (2.5, -1.5)$.
9. Plot the points $(-0.8, 0.2), (-0.2, -0.4), (0, -1), (0.6, -0.4)$.
10. Plot the points $(-1.2, -1.2), (-0.3, -0.3), (0, 0), (0.6, 0.6), (1.2, 1.2)$.
11. QII
12. QI
13. QIII
14. QIV
15. QII
16. QI
17. QIII
18. QIV

19. QIV
20. QIII
21. QII
22. QI
23. $1.80
24. $0.60
25. 1980, 1984, and 1996
26. $2.00
27. 100%
28. About 113%
29. $0.30
30. $0.50
31. 1992
32. 1980, 1984, and 1988
33. About 67%
34. 20%
35. Create a line graph through the points
$(1969, 36), (1979, 40), (1989, 47), (1999, 42)$.
36. Create a line graph through the points
$(1, 60), (5, 72), (10, 75), (15, 67), (20, 60), (25, 55)$.
37. 15 square units

38. 4 square units
39. 28 square units
40. 72 square units
41. 4 square units
42. 8 square units
43. 9 square units
44. 10.5 square units

5.2 Graph Using the Slope and Y-Intercept

Determine the slope and the y-intercept of the given graph.

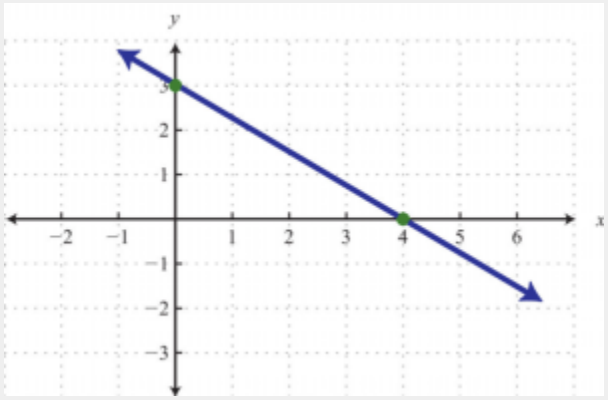
1. 

Figure 5.3.3.

2.

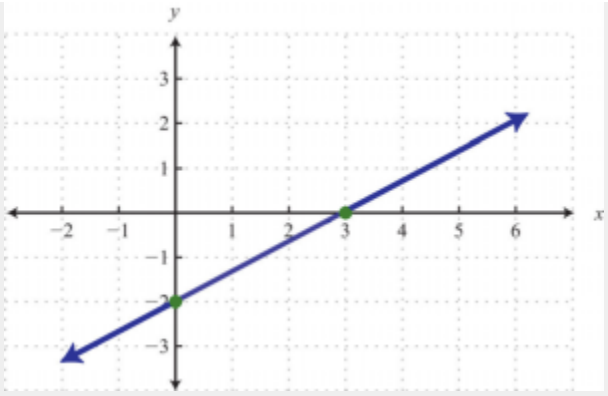


Figure 5.3.4.

3.

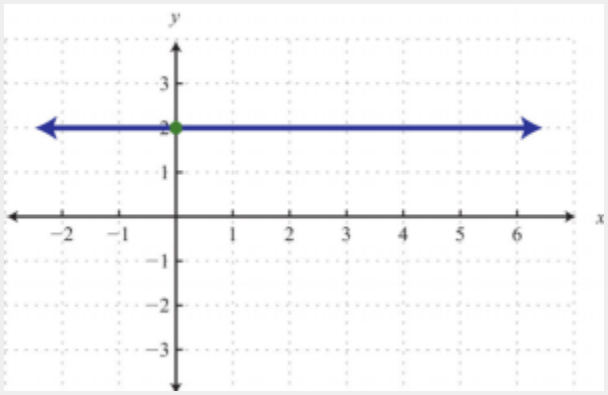


Figure 5.3.5.

4.

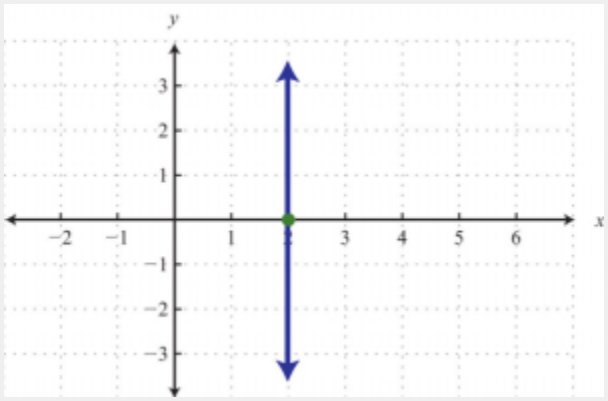


Figure 5.3.6.

5.

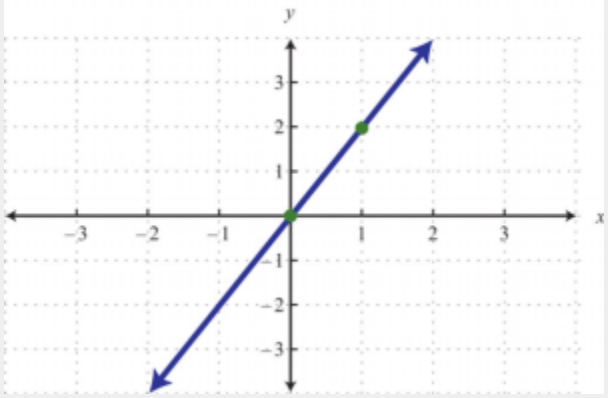


Figure 5.3.7.

6.

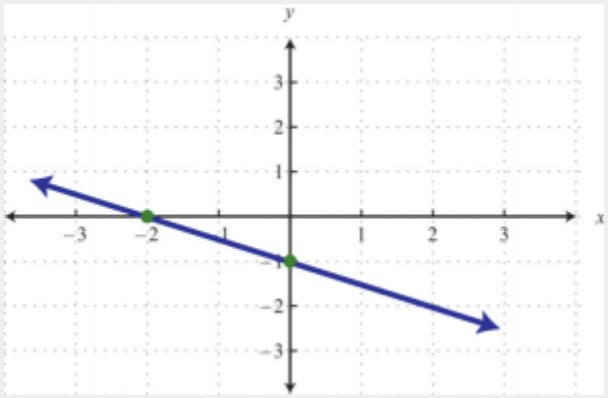


Figure 5.3.8.

Determine the slope, given two points.

7. $(3, 2)$ and $(5, 1)$
8. $(7, 8)$ and $(-3, 5)$
9. $(2, -3)$ and $(-3, 2)$
10. $(-3, 5)$ and $(7, -5)$
11. $(-1, -6)$ and $(3, 2)$
12. $(5, 3)$ and $(4, 12)$
13. $(-9, 3)$ and $(-6, -5)$

14. $(-22, 4)$ and $(-8, -12)$

15. $\left(\frac{1}{2}, -\frac{1}{3}\right)$ and $\left(-\frac{1}{2}, \frac{2}{3}\right)$

16. $\left(-\frac{3}{4}, \frac{3}{2}\right)$ and $\left(\frac{1}{4}, -\frac{1}{2}\right)$

17. $\left(-\frac{1}{3}, \frac{5}{8}\right)$ and $\left(\frac{1}{2}, -\frac{3}{4}\right)$

18. $\left(-\frac{3}{5}, -\frac{3}{2}\right)$ and $\left(1, \frac{4}{10}\right)$

19. $(3, -5)$ and $(5, -5)$

20. $(-3, 1)$ and $(-14, 1)$

21. $(-2, 3)$ and $(-2, -4)$

22. $(-4, -4)$ and $(5, 5)$

23. A roof drops 4 feet for every 12 feet forward. Determine the slope of the roof.

24. A road drops 300 feet for every 5,280 feet forward. Determine the slope of the road.

25. The following graph gives the US population of persons 65 years old and over. At what rate did this population increase from 2000 to 2008?

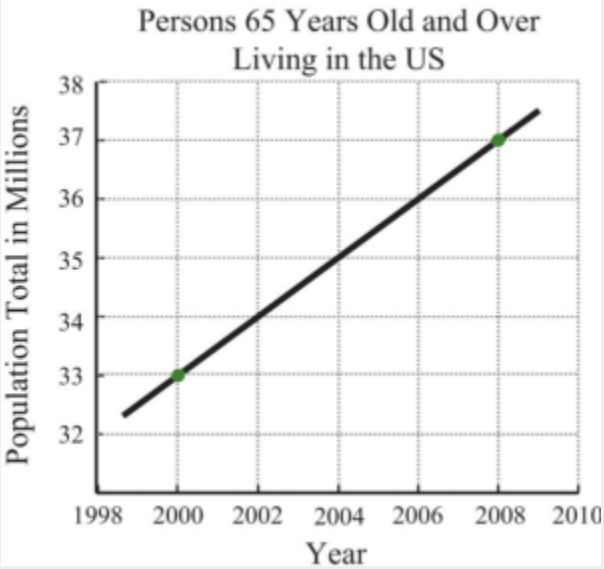


Figure 5.3.9.

26. The following graph shows total consumer credit outstanding in the United States. At what rate did consumer credit increase from 2002 to 2008?

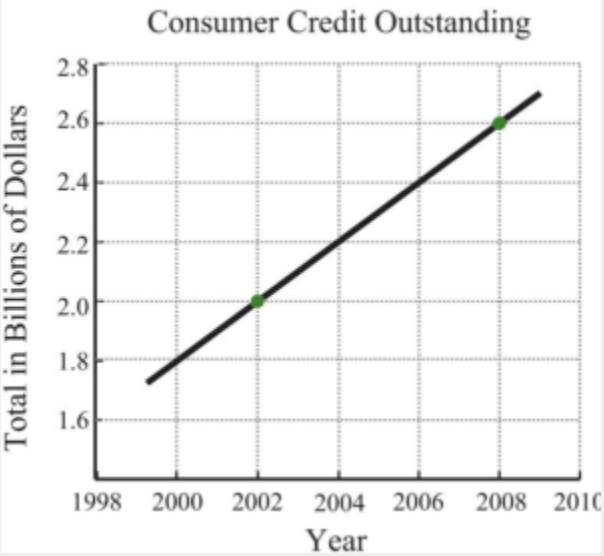


Figure 5.3.10.

27. A commercial van was purchased new for $20,000 and is expected to be worth $4,000 in 8 years. Determine the rate at which the van depreciated in value.

28. A commercial-grade copy machine was purchased new for $4,800 and will be considered worthless in 6 years. Determine the rate at which the copy machine depreciates in value.

29. Find y if the slope of the line passing through (-2,3) and (4,y) is 12.

30. Find y if the slope of the line passing through (5,y) and (6,−1) is 10.

31. Find y if the slope of the line passing through (5,y) and (−4,2 is 0.

32. Find x if the slope of the line passing through (−3,2) and (x,5) is undefined.

Express the given linear equation in slope-intercept form and identify the slope and y-intercept.

33. $6x - 5y = 30$

34. $-2x + 7y = 28$

35. $9x - y = 17$

36. $x - 3y = 18$

37. $2x - 3y = 0$

38. $-6x + 3y = 0$

39. $\frac{2}{3}x - \frac{5}{4}y = 10$

40. $-\frac{4}{3}x + \frac{1}{5}y = -5$

Graph the line given the slope and the y-intercept.

41. $m = \frac{1}{3}$ and $(0, -2)$

42. $m = -\frac{2}{3}$ and $(0, 4)$

43. $m = 3$ and $(0, 1)$

44. $m = -2$ and $(0, -1)$

45. $m = 0$ and $(0, 5)$

46. m undefined and $(0, 0)$

47. $m = 1$ and $(0, 0)$

48. $m = -1$ and $(0, 0)$

49. $m = -\frac{15}{3}$ and $(0, 20)$

50. $m = -10$ and $(0, -5)$

Graph using the slope and y-intercept.

51. $y = \frac{2}{3}x - 2$

52. $y = -\frac{1}{3}x + 1$

53. $y = -3x + 6$

54. $y = 3x + 1$

55. $y = \frac{3}{5}x$

56. $y = -\frac{3}{7}x$

57. $y = -8$

58. $y = 7$

59. $y = -x + 2$

60. $y = x + 1$

61. $y = \frac{1}{2}x + \frac{3}{2}$

62. $y = -\frac{3}{4}x + \frac{5}{2}$

63. $4x + y = 7$

64. $3x - y = 5$

65. $5x - 2y = 10$

66. $-2x + 3y = 18$

67. $x - y = 0$

68. $x + y = 0$

69. $\frac{1}{2}x - \frac{1}{3}y = 1$

70. $-\frac{2}{3}x + \frac{1}{2}y = 2$

71. $3x + 2y = 1$

72. $5x + 3y = 1$

73. On the same set of axes, graph the three lines, where $y = \frac{3}{2}x + b$ and $b = \{-2, 0, 2\}$.

74. On the same set of axes, graph the three lines, where $y = mx + 1$ and $m = \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$.

Show Solutions

1. y-intercept: $(0, 3)$; slope: $m = -\frac{3}{4}$
2. y-intercept: $(0, -2)$; slope: $m = \frac{2}{3}$
3. y-intercept: $(0, 2)$; slope: $m = 0$
4. Slope: undefined; no y-intercept
5. y-intercept: $(0, 0)$; slope: $m = 2$
6. y-intercept: $(0, -1)$; slope: $m = -\frac{1}{2}$

7. $-\frac{1}{2}$
8. $\frac{3}{10}$
9. -1
10. -1
11. 2
12. -9
13. $-\frac{8}{3}$
14. $-\frac{8}{7}$
15. -1
16. -2
17. $-\frac{33}{20}$
18. $\frac{19}{16}$
19. 0
20. 0
21. Undefined
22. 1
23. $-\frac{1}{3}$

24. $-\frac{5}{88}$

25. $\frac{1}{2}extmillionperyear$

26. 0.1 billion dollars per year

27. $\$2{,}000extperyear$

28. $\$800extperyear$

29. 75

30. -11

31. 2

32. -3

33. $y = \frac{6}{5}x - 6$; slope: $\frac{6}{5}$; y-intercept: $(0, -6)$

34. $y = \frac{2}{7}x + 4$; slope: $\frac{2}{7}$; y-intercept: $(0, 4)$

35. $y = 9x - 17$; slope: 9; y-intercept: $(0, -17)$

36. $y = \frac{1}{3}x - 6$; slope: $\frac{1}{3}$; y-intercept: $(0, -6)$

37. $y = \frac{2}{3}x$; slope: $\frac{2}{3}$; y-intercept: $(0, 0)$

38. $y = 2x$; slope: 2; y-intercept: $(0, 0)$

39. $y = \frac{8}{15}x - 8$; slope: $\frac{8}{15}$; y-intercept: $(0, -8)$

40. $y = \frac{20}{3}x - 25$; slope: $\frac{20}{3}$; y-intercept: $(0, -25)$

41. Graph $y = \frac{1}{3}x - 2$.

42. Graph $y = -\frac{2}{3}x + 4$.

43. Graph $y = 3x + 1$.

44. Graph $y = -2x - 1$.

45. Graph $y = 5$.

46. Graph $x = 0$.

47. Graph $y = x$.

48. Graph $y = -x$.

49. Graph $y = -5x + 20$.

50. Graph $y = -10x - 5$.

51. Graph $y = \frac{2}{3}x - 2$.

52. Graph $y = -\frac{1}{3}x + 1$.

53. Graph $y = -3x + 6$.

54. Graph $y = 3x + 1$.

55. Graph $y = \frac{3}{5}x$.

56. Graph $y = -\frac{3}{7}x$.

57. Graph $y = -8$.

58. Graph $y = 7$.

59. Graph $y = -x + 2$.

60. Graph $y = x + 1$.

61. Graph $y = \frac{1}{2}x + \frac{3}{2}$.

62. Graph $y = -\frac{3}{4}x + \frac{5}{2}$.

63. Graph $y = -4x + 7$.

64. Graph $y = 3x - 5$.

65. Graph $y = \frac{5}{2}x - 5$.

66. Graph $y = \frac{2}{3}x + 6$.

67. Graph $y = x$.

68. Graph $y = -x$.

69. Graph $y = \frac{3}{2}x - 3$.

70. Graph $y = \frac{4}{3}x + 4$.

71. Graph $y = -\frac{3}{2}x + \frac{1}{2}$.

72. Graph $y = -\frac{5}{3}x + \frac{1}{3}$.

73. Graph $y = \frac{3}{2}x - 2$, $y = \frac{3}{2}x$, and $y = \frac{3}{2}x + 2$.

74. Graph $y = -\frac{1}{2}x + 1$, $y = 1$, and $y = \frac{1}{2}x + 1$.

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CHAPTER 6: FACTORING

Chapter Outline

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6.1 Introduction to Factoring

- GCF of Natural Numbers
- GCF of Monomials
- Factoring out the GCF
- Factor by Grouping

6.2 Factoring Special Binomials

- Difference of Squares
- Sum and Difference of Cubes

6.3 Factoring Trinomials of the Form $x^2 + bx + c < /a >$

6.4 Factoring Trinomials of the Form ax^2+bx+c

- Factoring Trinomials of the Form $ax^2 + bx + c$
- Factoring Trinomials with Common Factors

6.5 General Guidelines for Factoring Polynomials

6.6 Factoring Fractions

- Rational Expressions, Evaluation, and Restrictions
- Simplifying Rational Expressions
- Opposite Binomial Factors or Factoring Out -1

6.7 Exercises

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6.1 Introduction to Factoring

Learning Objectives

By the end of this section, you should be able to:

- Determine the greatest common factor (GCF) of natural numbers.
- Determine the GCF of monomials.
- Factor out the GCF of a polynomial.
- Factor a four-term polynomial by grouping.

GCF of Natural Numbers

The process of writing a number or expression as a product is called factoring. If we write $60 = 5 \cdot 12$, we say that the product $5 \cdot 12$ is a factorization of 60 and that 5 and 12 are factors. Typically, there are many ways to factor a number. For example,

$$\begin{aligned} 60 &= 6 \cdot 10 \\ 60 &= 2 \cdot 30 \qquad \text{Factorizations of } 60 \\ 60 &= 4 \cdot 3 \cdot 5 \end{aligned}$$

Recall that a **prime number** is defined as a natural number with exactly two natural number factors, 1 and itself. The first ten prime numbers follow:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ \ldots$$

Any natural number greater than 1 can be uniquely written as a product of prime numbers. This product is called the **prime factorization**. The prime factorization of 60 can be determined by continuing to factor until only a product of prime numbers remains.

$$\begin{aligned} 60 &= 2 \cdot 30 \\ &= 2 \cdot 2 \cdot 15 \\ &= 2 \cdot 2 \cdot 3 \cdot 5 \end{aligned}$$

Since the prime factorization is unique, it does not matter how we choose to initially factor the number; the end result will be the same. The prime factorization of 60 follows:

$$60 = 2^2 \cdot 3 \cdot 5 \qquad \text{The prime factorization of } 60$$

Recall that the greatest common factor (GCF) of any two natural numbers is the product of all the common prime factors.

Example 1

Find the GCF of 60 and 140.

Solution:
First, determine the prime factorizations of both integers.

$$\begin{aligned} 140 &= 10 \cdot 14 \\ &= 2 \cdot 5 \cdot 2 \cdot 7 \\ &= 2^2 \cdot 5 \cdot 7 \end{aligned} \qquad \begin{aligned} 60 &= 6 \cdot 10 \\ &= 2 \cdot 3 \cdot 2 \cdot 5 \\ &= 2^2 \cdot 3 \cdot 5 \end{aligned}$$

The product of the common prime factors is $2^2 \cdot 5$; hence the GCF$(60, 140) = 2^2 \cdot 5 = 20$. To see that it is the greatest common factor, we can write the following:

$$\begin{aligned} 140 &= 20 \cdot 7 \\ 60 &= 20 \cdot 3 \end{aligned}$$

Answer

The greatest common factor of 60 and 140 is 20.

Example 2

Find the GCF of 504 and 1,080.

Solution:
First, determine the prime factorizations of both integers.

$$\begin{aligned} 504 &= 9 \cdot 56 \\ &= 3 \cdot 3 \cdot 7 \cdot 8 \\ &= 2^3 \cdot 3^2 \cdot 7 \end{aligned} \qquad \begin{aligned} 1080 &= 10 \cdot 108 \\ &= 2 \cdot 5 \cdot 9 \cdot 12 \\ &= 2^3 \cdot 3^3 \cdot 5 \end{aligned}$$

The product of the common prime factors is $2^3 \cdot 3^2$.

The GCF$(504, 1080) = 2^3 \cdot 3^2 = 72$. Note that we multiplied the common prime factors with the smallest exponent.

$$504 = 72 \cdot 7$$
$$1080 = 72 \cdot 15$$

The numbers 7 and 15 share no common natural number factor other than 1; we say that they are relatively prime.

Answer

The greatest common factor of 504 and $1,080$ is 72.

GCF of Monomials

We next consider factorizations of monomials. For example, $6x$ and x^4 are factors of $6x^5$ because $6x^5 = 6x \cdot x^4$. Typically, there are many ways to factor a monomial. Some factorizations of $6x^5$ follow:

$$6x^5 = 2x^3 \cdot 3x^2$$
$$6x^5 = 6x^4 \cdot x \qquad \text{Factorizations of } 6x^5$$
$$6x^5 = 2x \cdot 3x \cdot x^3$$

Given two or more monomials, it will be useful to find the greatest common monomial factor of each. For example, consider $6x^5y^3z$ and $8x^2y^3z^2$. The variable part of these two monomials looks very much like the prime factorization of natural numbers and, in fact, can be treated the same way.

How To

Find the greatest common factor (GCF) of two expressions.

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

Steps for finding the GCF of monomials are outlined in the following example.

Example 3

Find the GCF of $6x^5y^3z$ and $8x^2y^3z^2$.

Solution:

Step 1: Find the GCF of the coefficients.

$$6x^5y^3z \qquad \text{and} \qquad 8x^2y^3z^2$$

In this case, the GCF$(6, 8) = 2$.

Step 2: Determine the common variable factors with the smallest exponents.

$$6x^5y^3z \qquad \text{and} \qquad 8x^2y^3z^2$$

In this case, the common variables with the smallest exponents are x^2, y^3, and z^1.

Step 3: The GCF of the monomials is the product of the common variable factors and the GCF of the coefficients. Therefore, GCF

$$(6x^5y^3z, 8x^2y^3z^2) = 2 \cdot x^2 \cdot y^3 \cdot z$$

Answer

$2x^2y^3z$

It is worth pointing out that the GCF in the previous example divides both expressions evenly:

$$\frac{6x^5y^3z}{2x^2y^3z} = 3x^3 \qquad \text{and} \qquad \frac{8x^2y^3z^2}{2x^2y^3z} = 4z$$

Furthermore, we can write the following:

$$6x^5y^3z = 2x^2y^3z \cdot 3x^3 \qquad \text{and} \qquad 8x^2y^3z^2 = 2x^2y^3z \cdot 4z$$

The factors $3x^3$ and $4z$ share no common monomial factors other than 1; they are **relatively prime**.

Example 4

Determine the GCF of the following expressions:

$$30x^6y \text{ and } 18x^4y^2z$$

Solution:

The prime factorizations of the coefficients are

$$30 = 2 \cdot 3 \cdot 5$$
$$18 = 2 \cdot 3 \cdot 3$$

Thus the GCF$(30, 18) = 2 \cdot 3 = 6$. Next, consider the variable part:

$$30x^6y \qquad \text{and} \qquad 18x^4y^2z$$

The variable factors in common are x^4 and y. The factor z is not in common, and we have

$$\text{GCF} = 6 \cdot x^4 \cdot y$$

Answer

$6x^4y$

Example 5

Determine the GCF of the following three expressions:

$$12a^5b^2(a+b)^5, 60a^4b^3c(a+b)^3, \text{ and } 24a^2b^7c^3(a+b)^2$$

Solution:

First, determine the GCF of the coefficients.

$$12 = 2^2 \cdot 3$$
$$60 = 2^2 \cdot 3 \cdot 5$$
$$24 = 2^3 \cdot 3$$

The GCF $(12, 60, 24) = 2^2 \cdot 3 = 12$. Next, determine the common factors of the variable part:

$$12a^5b^2(a+b)^5 \qquad \text{and} \qquad 60a^4b^3c(a+b)^3 \qquad \text{and} \qquad 24a^2b^7c^3(a+b)^2$$

The variable factors in common are a^2, b^2, and $(a+b)^2$. Therefore,

$$\text{GCF} = 12 \cdot a^2 \cdot b^2 \cdot (a+b)^2$$

Answer

$12a^2b^2(a+b)^2$. Note that the variable c is not common to all three expressions and thus is not included in the GCF.

Try It

Determine the GCF of the following:

$$60x^4y^3(x+2y)^7, 45x^2y^5(x+2y)^4, \text{ and } 30x^7y^7(x+2y)^3.$$

Show Solution

$15x^2y^3(x+2y)^3$

Factoring out the GCF

We have seen that application of the distributive property is the key to multiplying polynomials. The process of factoring a polynomial involves using the distributive property in reverse to write each polynomial as a product of polynomial factors.

$$\begin{aligned} a(b+c) &= ab + ac && \text{Multiplying} \\ ab + ac &= a(b+c) && \text{Factoring} \end{aligned}$$

To demonstrate this idea, we multiply and factor side by side. Factoring utilizes the GCF of the terms.

Table 6.1.1.

Multiplying	Factoring
$3(5x+1) = 3\cdot 5x + 3\cdot 1$ $= 15x + 3$	$15x + 3 = 3\cdot 5x + 3\cdot 1$ $= 3(5x+1)$
$2x^2(3x^3+4) = 2x^2\cdot 3x^3 + 2x^2\cdot 4$ $= 6x^5 + 8x^2$	$6x^5 + 8x^2 = 2x^2\cdot 3x^3 + 2x^2\cdot 4$ $= 2x^2(3x^3+4)$

In the previous example, we see that the distributive property allows us to write the polynomial $6x^5 + 8x^2$ as a product of the two factors $2x^2$ and $(3x^3 + 4)$. Note that in this case, $2x^2$ is the GCF of the terms of the polynomial:

$$\text{GCF}\, 6x^5, 8x^2) = 2x^2$$

Factoring out the GCF involves rewriting a polynomial as a product where a factor is the GCF of all of its terms:

$$15x + 3 = 3(5x + 1) \qquad \text{Factoring out the GFC}$$
$$6x^5 + 8x^2 = 2x^2(3x^3 + 4)$$

How To

Factor the greatest common factor from a polynomial.

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the "reverse" Distributive Property to factor the expression.
4. Check by multiplying the factors.

The steps for factoring out the GCF of a polynomial are outlined in the following example.

Example 6

Factor out the GCF:

$$7x^4 + 21x^3 - 14x^2$$

Solution:

Step 1: Identify the GCF of all the terms. In this case, the GCF$(7, 21, 14) = 7$, and the common variable factor with the smallest exponent is x^2. The GCF of the polynomial is $7x^2$.

$$7x^4 + 21x^3 - 14x^2 = 7x^2(\quad ? \quad)$$

Step 2: Determine the terms of the missing factor by dividing each term of the original expression by the GCF. (*This step is usually performed mentally.*)

$$\frac{7x^4}{7x^2} = x^2 \qquad \frac{21x^3}{7x^2} = 3x \qquad \frac{-14x^2}{7x^2} = -2$$

Step 3: Apply the distributive property (in reverse) using the terms found in the previous step.

$$7x^4 + 21x^3 - 14x^2 = 7x^2(x^2 + 3x - 2)$$

Step 4: As a check, multiply using the distributive property to verify that the product equals the original expression. (*This step is optional and can be performed mentally.*)

$$\begin{aligned} 7x^2(x^2 + 3x - 2) &= 7x^2 \cdot x^2 + 7x^2 \cdot 3x - 7x^2 \cdot 2 \\ &= 7x^4 + 21x^3 - 14x^2 \quad \checkmark \end{aligned}$$

Answer

$7x^2(x^2 + 3x - 2)$

Example 7

Factor out the GCF:

$$48a - 16b + 4c$$

Solution:

There are no variable factors in common and the GCF$(48, 16, 4) = 4$

$$\begin{aligned} 48a - 16b + 4c &= 4(\quad ? \quad) \\ &= 4(12a - 4b + c) \end{aligned}$$

Answer

$4(12a - 4b + c)$

Example 8

Factor out the GCF:

$$25x^3 + 15x^2 + 5x$$

Solution:
The GCF$(25, 15, 5) = 5$ and the common variable factor with the smallest exponents is x^1. The GCF of all the terms is $5x$.

\begin{align*}25x^3+15x^2+5x&=5x(\qquad {\color{OliveGreen} ?} \qquad)\\&=5x(5x^2+3x+1) \end{align*}

Answer

$5x(5x^2 + 3x + 1)$

If the GCF is the same as one of the terms, then, after the GCF is factored out, a constant term 1 will remain. In the previous example, we can see that $\frac{5x}{5x} = 1$. The importance of remembering the constant term becomes clear when performing the check using the distributive property:

$$\begin{aligned} 5x(5x^2 + 3x + 1) &= 5x \cdot 5x^2 + 5x \cdot 3x + 5x \cdot 1 \\ &= 25x^3 + 15x^2 + 5x \quad \checkmark \end{aligned}$$

The constant term 1 allows us to obtain the same original expression after we distribute.

Example 9

Factor out the GCF:

$$15x^6y^4 + 10x^5y^3z^2 - 20xy^6z^3$$

Solution:
The GCF$(10, 15, 20) = 5$, and the common variables with the smallest exponent are x^1 and y^3. Therefore, the GCF of the terms is $5xy^3$. The first term does not have a variable factor of z and thus cannot be a part of the greatest common factor. If we divide each term by $5xy^3$, we obtain

$$\frac{15x^6y^4}{5xy^3} = 3x^5y \qquad \frac{10x^5y^3z^2}{5xy^3} = 2x^4z^2 \qquad \frac{-20xy^6z^3}{5xy^3} = -4y^3z^3$$

and can write

$$\begin{aligned} 15x^6y^4 + 10x^5y^3z^2 - 20xy^6z^3 &= 5xy^3(\quad ?\quad) \\ &= 5xy^3(3x^5y + 2x^4z^2 - 4y^3z^3) \end{aligned}$$

Answer

$5xy^3(3x^5y + 2x^4z^2 - 4y^3z^3)$

Example 10

Factor out the GCF:

$$24a^6b^2c^5 + 8a^7b^5c$$

Solution:

The GCF$(24, 8) = 8$, and the variable factors with the smallest exponents are a^6, b^2, and c. Therefore, the GCF of all the terms is $8a^6b^2c$.

$$\begin{aligned} 24a^6b^2c^5 + 8a^7b^5c &= 8a^6b^2c(\quad ?\quad) \\ &= 8a^6b^2c(3c^4 + ab^3) \end{aligned}$$

Answer

$8a^6b^2c(3c^4 + ab^3)$

Of course, not all polynomials with integer coefficients can be factored as a product of polynomials with integer coefficients other than 1 and itself. If this is the case, then we say that it is a **prime polynomial**.

Example 11

Factor:

$$3x - 5$$

Solution:
Prime: There are no polynomial factors other than 1 and itself.
Answer

Prime

Factor by Grouping

In this section, we outline a technique for factoring polynomials with four terms. First, review some preliminary examples where the terms have a common binomial factor.

Example 12

Factor:

$$5x(x-3) + 2(x-3)$$

Solution:
This expression is a binomial with terms $5x(x-3)$ and $2(x-3)$. In this case, $(x-3)$ is a common factor. Begin by factoring this common factor out:

$$5x(x-3) + 2(x-3) = (x-3)(\quad ? \quad)$$

To determine the terms of the remaining factor, divide each term by $(x-3)$:

$$\frac{5x(x-3)}{(x-3)} = 5x \qquad \frac{2(x-3)}{(x-3)} = 2$$

This step is typically performed mentally. We have

$$5x(x-3)+2(x-3) = (x-3)(\quad ?\quad)$$
$$= (x-3)(5x+2)$$

Answer

$(x-3)(5x+2)$

Recall that 1 is always a common factor. If the GCF is the same as a term, then the factor 1 remains after we factor out that GCF.

Example 13

Factor:

$$3x(4x+1)-(4x+1)$$

Solution:
Rewrite the second term $-(4x+1)$ as $-1(4x+1)$ and then factor out the common binomial factor $(4x+1)$.

$$3x(4x+1)-(4x+1) = 3x(4x+1)-1(4x+1)$$
$$= (4x+1)(\quad ?\quad)$$
$$= (4x+1)(3x-1)$$

Answer

$(4x+1)(3x-1)$

Remember that the goal for this section is to develop a technique that enables us to factor polynomials with four terms into a product of binomials. The intermediate step of this process looks like the previous two examples. For example, we wish to factor

$$5x^2-15x+2x-6$$

Begin by grouping the first two terms and the last two terms. Then factor out the GCF of each grouping:

$$\underbrace{5x^2-15x}_{group} \quad + \quad \underbrace{2x-6}_{group}$$
$$= 5x(x-3)+2(x-3)$$

In this form, it is a binomial with a common binomial factor, $(x-3)$.

The steps that follow outline a technique for factoring four-term polynomials called **factor by grouping**.

$$= (x-3)(\quad ?\quad)$$
$$= (x-3)(5x+2)$$

How To

Factor by grouping.

Step 1. Group terms with common factors.
Step 2. Factor out the common factor in each group.
Step 3. Factor the common factor from the expression.
Step 4. Check by multiplying the factors.

Example 14

Factor:

$$2x^3+4x^2+3x+6$$

Solution:
Group terms in such a way as to obtain a binomial with common factors.

Step 1: Group the first two and last two terms and then factor out the GCF of each.

$$\underbrace{2x^3+4x^2}_{group} \quad + \quad \underbrace{3x+6}_{group}$$

The GCF of the first two terms is $2x^2$, and the GCF of the second two terms is 3.

$$2x^3+4x^2+3x+6 = \underbrace{2x^3+4x^2}_{group} \quad + \quad \underbrace{3x+6}_{group}$$
$$= 2x^2(\quad ?\quad)+3(\quad ?\quad)$$
$$= 2x^2(x+2)+3(x+2)$$

Step 2: At this point, the polynomial is a binomial. Factor out any factors common to both terms. Here $(x+2)$ is a common factor.

$$= 2x^2(x+2) + 3(x+2)$$
$$= (x+2)(\quad ? \quad)$$
$$= (x+2)(2x^2+3)$$

Step 3: Optional check: multiply to verify that we obtain the original expression.

$$(x+2)(2x^2+3) = 2x^3 + 3x + 4x^2 + 6$$
$$= 2x^3 + 4x^2 + 3x + 6 \quad \checkmark$$

Answer

$(x+2)(2x^2+3)$

Example 15

Factor:

$$2a^3 - 3a^2 + 2a - 3$$

Solution:

The GCF of the first two terms is a^2 and the GCF of the second two terms is 1.

$$2a^3 - 3a^2 + 2a - 3 = \underbrace{2a^3 - 3a^2}_{group} + \underbrace{2a - 3}_{group}$$
$$= a^2(\quad ? \quad) + 1(\quad ? \quad)$$
$$= a^2(2a-3) + 1(2a-3)$$
$$= (2a-3)(\quad ? \quad)$$
$$= (2a-3)(a^2+1)$$

Answer

$(2a-3)(a^2+1)$. The check is left to the reader.

Example 16

Factor:

$$6x^4 - 24x^3 - 5x + 20$$

Solution:

The GCF for the first group is $6x^3$. We have to choose 5 or -5 to factor out of the second group.

$$\underbrace{6x^4 - 24x^3}_{group} \quad - \quad \underbrace{5x + 20}_{group}$$

$$= 6x^3(x - 4) + 5(-x + 4) \qquad x$$
$$= 6x^3(x - 4) - 5(x - 4) \qquad \checkmark$$

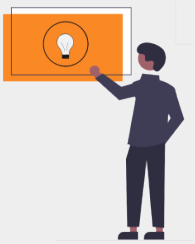
Factoring out a $+5$ does not result in a common binomial factor. If we choose to factor out -5, then we obtain a common binomial factor and can proceed. Note that when factoring out a negative number, we change the signs of the factored terms.

$$6x^4 - 24x^3 - 5x + 20 = \underbrace{6x^4 - 24x^3}_{group} \quad - \quad \underbrace{5x + 20}_{group}$$

$$= 6x^3(\quad ?\quad) - 5(\quad ?\quad)$$
$$= 6x^3(x - 4) - 5(x - 4)$$
$$= (x - 4)(\quad ?\quad)$$
$$= (x - 4)(6x^3 - 5)$$

Answer

$(x - 4)(6x^3 - 5)$. The check is left to the reader.



Note

The sign of the leading coefficient in the second grouping usually indicates whether or not to factor out a negative factor. If that coefficient is positive, factor out a positive factor. If it is negative, factor out a negative factor.

When all the terms of a polynomial have a GCF other than 1, it is a best practice to factor that out before factoring by grouping.

Example 17

Factor:

$$3y^4 + 9y^2 - 6y^3 - 18y$$

Solution:

Here, we notice that the greatest common factor of all the terms is $3y$. Begin by factoring out the GCF and then factor the result by grouping.

$$\begin{aligned} & 3y^4 + 9y^2 - 6y^3 - 18y \\ &= 3y[y^3 + 3y - 2y^2 - 6] && \text{Factor out the GCF.} \\ &= 3y[y(y^2 + 3) - 2(y^2 + 3)] && \text{Factor by grouping.} \\ &= 3y[y(y^2 + 3)(y - 2)] \\ &= 3y(y^2 + 3)(y - 2) \end{aligned}$$

Answer

$3y(y^2 + 3)(y - 2)$

Sometimes we must first rearrange the terms in order to obtain a common factor.

Example 18

Factor:

$$ab - 2a^2b + a^3 - 2b^3$$

Solution:

Simply factoring the GCF out of the first group and last group does not yield a common binomial factor.

$$\underbrace{ab - 2a^2b}_{group} + \underbrace{a^3 - 2b^3}_{group}$$

$$= ab(1 - 2a) + 1(a^3 - 2b^3)$$

We must rearrange the terms, searching for a grouping that produces a common factor. In this example, we have a workable grouping if we switch the terms a^3 and ab.

$$ab - 2a^2b + a^3 - 2b^3 = \underbrace{a^3 - 2a^2b}_{group} + \underbrace{ab - 2b^3}_{group}$$

$$= a^2(a - 2b) + b(a - 2b)$$
$$= (a - 2b)(a^2 + b)$$

Answer

$(a - 2b)(a^2 + b)$

Not all factorable four-term polynomials can be factored with this technique. For example,

$$3x^3 + 5x^2 - x + 2$$

This four-term polynomial cannot be grouped in any way to produce a common binomial factor. Despite this, the polynomial is not prime and can be written as a product of polynomials. It can be factored as follows:

$$3x^3 + 5x^2 - x + 2 = (x + 2)(3x^2 - x + 1)$$

Factoring such polynomials is something that we will learn to do as we move further along in our study of algebra. For now, we will limit our attempt to factor four-term polynomials to using the factor-by-grouping technique.

Try It

Factor: $x^3 - x^2y - xy + y^2$

Show Solution

$(x - y)(x^2 - y)$

Key Takeaways

- To find the greatest common factor (GCF) of any collection of natural numbers, first find the prime factorization of each. The GCF is the product of all the common prime factors.
- The GCF of two or more monomials is the product of the GCF of the coefficients and the common variable factors with the smallest power.
- If the terms of a polynomial have a greatest common factor, then factor out that GCF using the distributive property. Divide each term of the polynomial by the GCF to determine the terms of the remaining factor.
- Some four-term polynomials can be factored by grouping the first two terms and the last two terms. Factor out the GCF of each group and then factor out the common binomial factor.
- When factoring by grouping, you sometimes have to rearrange the terms to find a common binomial factor. After factoring out the GCF, the remaining binomial factors must be the same for the technique to work.
- Not all polynomials can be factored as the product of polynomials with integer coefficients. In this case, we call it a prime polynomial.

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where otherwise noted. Modifications: Used How To boxes “Find the greatest common factor (GCF) of two expressions”, “Factor the greatest common factor from a polynomial”, & “Factor by grouping”; Reformatted.

6.2 Factoring Special Binomials

Learning Objectives

By the end of this section, you should be able to:

- Factor binomials that are differences of squares.
- Factor binomials that are sums and differences of cubes.

Difference of Squares

A binomial is a polynomial with two terms. We begin with our first special binomial called the *difference of squares*:

$$a^2 - b^2 = (a + b)(a - b)$$

To verify the above formula, multiply:

$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ba - b^2 \\ &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

We use this formula to factor certain special binomials.

How To

Factor differences of squares.

$$a^2 - b^2$$

1. Does the binomial fit the pattern?
   - Is this a difference? ____–____
   - Are the first and last terms perfect squares?

2. Write them as squares. $(a)^2 - (b^2)$

3. Write the product of conjugates. $(a - b)(a + b)$

4. Check by multiplying.

Visualize

Visit the Geogebra site to practice Difference of Two Squares. Link available on page 6.2 of the digital version of this textbook.

Example 1

Factor:

$$x^2 - 16$$

Solution:

Step 1: Identify the binomial as the difference of squares and determine the square factors of each term.

$$\begin{array}{ccc} \text{Difference} & \text{of} & \text{Squares} \\ \downarrow & & \downarrow \quad \downarrow \\ x^2 - 16 & & x^2 - 16 \end{array}$$

Here we can write

$$x^2 - 16 = (x)^2 - (4)^2$$

The terms are squares of x and 4. Hence $a = x$ and $b = 4$.

Step 2: Substitute into the difference of squares formula.

$$\begin{array}{rcccccccc} a^2 - b^2 & = (& a & + & b)(& a & - & b) \\ & & \downarrow & & \downarrow & \downarrow & & \downarrow \\ x^2 - 16 & = (& x & + & 4)(& x & - & 4) \end{array}$$

Step 3: Multiply to check. This step is optional.

$$(x+4)(x-4) = x^2 - 16 \quad \checkmark$$

Answer

$(x+4)(x-4)$

It is worth taking some extra time at this point to review all of the squares of integers from 1 to 12.

$$\begin{aligned} 1^2 &= 1 & 7^2 &= 49 \\ 2^2 &= 4 & 8^2 &= 64 \\ 3^2 &= 9 & 9^2 &= 81 \\ 4^2 &= 16 & 10^2 &= 100 \\ 5^2 &= 25 & 11^2 &= 121 \\ 6^2 &= 36 & 12^2 &= 144 \end{aligned}$$

Recognizing these perfect square integers helps speed the factoring process.

How To

Factor perfect square trinomials.

1. Does the trinomial fit the pattern?

   $a^2 + 2ab + b^2$ $\qquad$ $a^2 - 2ab + b^2$

   - Is the first term a perfect square?
     - Write it as a square.

   $(a)^2$ $\qquad$ $(a)^2$

   - Is the last term a perfect square?
     - Write it as a square.

   $(a)^2 \quad (b)^2$ $\qquad$ $(a)^2 \quad (b)^2$

   - Check the middle term.
     - Is it $2ab$?

   $(a)^2 \quad (b)^2$
 $2ab$ $\qquad$ $(a)^2 \quad (b)^2$
 $2ab$

2. Write the square of the binomial.

   $(a+b)^2$ $\qquad$ $(a-b)^2$

3. Check by multiplying.

Example 2

Factor:

$$9x^2 - 121$$

Solution:
The subtraction indicates that this is a difference. Furthermore, we recognize that the terms are squares.

$$9x^2 - 121 = (3x)^2 - (11)^2$$

In this case, $a = 3x$ and $b = 11$. Substitute into the formula for the difference of squares.

$$a^2 - b^2 = (a + b)(a - b)$$
$$9x^2 - 121 = (3x + 11)(3x - 11)$$

Answer

$$(3x + 11)(3x - 11)$$

It may be the case that the terms of the binomial have a common factor. If so, it will be difficult to identify the perfect squares until we first factor out the GCF.

Example 3

Factor:

$$12y^2 - 75$$

Solution:
The terms are not perfect squares. However, notice that they do have a common factor. First, factor out the GCF, 3.

$$12y^2 - 75 = 3(4y^2 - 25)$$

The resulting binomial factor is a difference of squares with $a = 2y$ and $b = 5$.

$$\begin{aligned} 12y^2 - 75 &= 3(4y^2 - 25) \\ &= 3(2y + 5)(2y - 5) \end{aligned}$$

Answer

$3(2y + 5)(2y - 5)$

Example 4

Factor:

$$49x^2 - 100y^2$$

Solution:
Here we have a binomial with two variables and recognize that it is a difference of squares.

$$49x^2 - 100y^2 = (7x)^2 - (10y)^2$$

Therefore, $a = 7x$ and $b = 10y$. Substitute into the formula for the difference of squares.

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b) \\ 49x^2 - 100y^2 &= (7x + 10y)(7x - 10y) \end{aligned}$$

Answer

$(7x + 10y)(7x - 10y)$

Try It

Factor: $36x^2 - 1$

Show Solution

$(6x + 1)(6x - 1)$

Given any real number b, a polynomial of the form $x^2 + b^2$ is prime.

Furthermore, the sum of squares $a^2 + b^2$ does not have a general factored equivalent. Care should be taken not to confuse this with a perfect square trinomial:

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

Therefore, note that:

$$(a + b)^2 \neq a^2 + b^2$$

When the degree of the special binomial is greater than two, we may need to apply the difference of squares formula multiple times. A polynomial is completely factored when none of the factors can be factored any further.

Example 5

Factor completely:

$$x^4 - 16$$

Solution:
First, identify what is being squared:

$$x^4 - 16 = (\quad)^2 - (\quad)^2$$

To do this, recall the **power rule for exponents**, $(x^m)^n = x^{mn}$. When exponents are raised to a power, multiply them. With this in mind, determine that $(x^2)^2 = x^4$ and write

$$x^4 - 16 = (x^2)^2 - (4)^2$$

Therefore, $a = x^2$ and $b = 4$. Substitute into the formula for the difference of squares.

$$x^4 - 16 = (x^2 + 4)(x^2 - 4)$$

At this point, notice that the factor $(x^2 - 4)$ is itself a difference of two squares and thus can be further factored using $a = x$ and $b = 2$. The factor $(x^2 + 4)$ is a sum of squares, which cannot be factored using real numbers.

$$\begin{aligned} x^4 - 16 &= (x^2 + 4)(x^2 - 4) \\ &= (x^2 + 4)(x + 2)(x - 2) \end{aligned}$$

Answer

$(x^2 + 4)(x + 2)(x - 2)$

Try It

Factor completely: $81x^4 - 1$

Show Solution

$$(9x^2+1)(3x+1)(3x-1)$$

Sum and Difference of Cubes

Two other special binomials of interest are the sum and **difference of cubes**:

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

We can verify these formulas by multiplying:

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a^3-a^2b+ab^2+a^2b-ab^2+b^3\\ &= a^3+b^3 \quad \checkmark\end{aligned}$$

$$\begin{aligned}(a-b)(a^2+ab+b^2) &= a^3+a^2b+ab^2-a^2b-ab^2-b^3\\ &= a^3-b^3 \quad \checkmark\end{aligned}$$

The process for factoring the sum and difference of cubes is very similar to that for the difference of squares. We first identify a and b and then substitute them into the appropriate formula. The separate formulas for sum and difference of cubes allow us to always choose a and b to be positive.

How To

Factor the sum or difference of cubes.

1. Does the binomial fit the sum or difference of cubes pattern?
   - Is it a sum or difference?
   - Are the first and last terms perfect cubes?
2. Write them as cubes.
3. Use either the sum or difference of cubes pattern.
4. Simplify inside the parentheses.
5. Check by multiplying the factors.

This visual shows you the relationship between the factors and the product:

Visualize

Visit the Geogebra site to practice Sum & Difference of 2 Cubes. Link available on page 6.2 of the digital version of this textbook.

Example 6

Factor:

$$x^3 + 8$$

Solution:

The plus sign and the fact that the terms are cubes indicate to us that this is a sum of cubes.

$$\begin{array}{ccc} \text{Sum} & \text{of} & \text{Cubes} \\ \downarrow & & \downarrow \quad \downarrow \\ x^3+8 & & x^3+8 \end{array}$$

Next, identify what is being cubed.

$$x^3 + 8 = (x)^3 + (2)^3$$

In this case, $a = x$ and $b = 2$. Substitute into the sum of cubes formula.

$$\begin{array}{rcccccccccc} a^3+b^3 & = (& a & + & b)(& a^2 & - & ab & + & b^2) \\ & & \downarrow & & \downarrow & \downarrow & & \downarrow & & \downarrow \\ x^3+8 & = (& x & + & 2)(& x^2 & - & x\cdot 2 & + & 2^2) \\ & = (& x & + & 2)(& x^2 & - & 2x & + & 4) \end{array}$$

The resulting trinomial is prime, and the factoring is complete. We can check this factorization by multiplying.

$$\begin{aligned}(x+2)(x^2-2x+4) &= x^3-2x^2+4x+2x^2-4x+8 \\ &= x^3+8 \quad \checkmark\end{aligned}$$

Answer

$$(x+2)(x^2-2x+4)$$

It is helpful to review the perfect cubes of integers from 1 to 12. This will aid you in identifying sums and differences of cubes.

$$\begin{aligned} 1^3 &= 1 & 7^3 &= 343 \\ 2^3 &= 8 & 8^3 &= 512 \\ 3^3 &= 27 & 9^3 &= 729 \\ 4^3 &= 64 & 10^3 &= 1000 \\ 5^3 &= 125 & 11^3 &= 1331 \\ 6^3 &= 216 & 12^3 &= 1728 \end{aligned}$$

Example 7

Factor:

$$y^3 - 125$$

Solution:

In this case, we have a difference of cubes.

$$\begin{array}{ccc} \text{Difference} & \text{of} & \text{Cubes} \\ \downarrow & & \downarrow \quad \downarrow \\ y^3 - 125 & & y^3 - 125 \end{array}$$

We can write

$$y^3 - 125 = (y)^3 - (5)^3$$

Substitute $a = y$ and $b = 5$ into the formula for difference of cubes.

$$\begin{array}{rcccccccc} a^3 - b^3 & = (& a & - & b)(& a^2 & + & ab & + & b^2) \\ & & \downarrow & & \downarrow & \downarrow & & \downarrow & & \downarrow \\ y^3 - 125 & = (& y & - & 5)(& y^2 & + & y \cdot 5 & + & 5^2) \\ & = (& y & - & 5)(& y^2 & + & 5y & + & 25) \end{array}$$

Answer

$$(y - 5)(y^2 + 5y + 25)$$

Always look for common factors when factoring. If the terms of the binomial have a GCF other than 1, then factor that out first.

Example 8

Factor:

$$54x^4 + 128x$$

Solution:

Begin by factoring out the GCF $2x$.

$$54x^4 + 128x = 2x(27x^3 + 64)$$

The resulting binomial factor is a sum of cubes, where $a = 3x$ and $b = 4$.

$$\begin{aligned} 54x^4 + 128x &= 2x(27x^3 + 64) \\ &= 2x(3x + 4)\left((3x)^2 - (3x)(4) + 4^2\right) \\ &= 2x(3x + 4)(9x^2 - 12x + 16) \end{aligned}$$

Answer

$2x(3x + 4)(9x^2 - 12x + 16)$

Example 9

Factor:

$$x^3y^3 - 1$$

Solution:

This binomial is a difference of cubes with two variables. Identify what is being cubed.

$$x^3y^3 - 1 = (xy)^3 - (1)^3$$

Here $a = xy$ and $b = 1$. Substitute into the appropriate formula and simplify.

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$\begin{aligned} x^3y^3 - 1 &= (xy-1)\left((xy)^2 + xy \cdot 1 + 1^2\right) \\ &= (xy-1)(x^2y^2 + xy + 1) \end{aligned}$$

Answer

$(xy-1)(x^2y^2 + xy + 1)$

Try It

Factor: $8x^3 + 343$

Show Solution

$(2x+7)(4x^2 - 14x + 49)$

When factoring, always look for resulting factors to factor further.

Example 10

Factor completely:

$$x^6 - 64$$

Solution:
When confronted with a binomial that is both a difference of squares and cubes, as this is, make it a rule to factor using difference of squares first.

$$x^6 - 64 = (x^3)^2 - (8)^2$$

Therefore, $a = x^3$ and $b = 8$. Substitute into the difference of squares formula.

$$x^6 - 64 = (x^3 + 8)(x^3 - 8)$$

The resulting two binomial factors are a sum and a difference of cubes. Each can be factored further.

$$x^3+8=(x+2)(x^2-2x+4)$$
$$x^3-8=(x-2)(x^2+2x+4)$$

Therefore, we have

$$\begin{aligned} x^6-64 &= (x^3+8)\cdot(x^3-8) \\ &= \overbrace{(x+2)(x^2-2x+4)}\cdot\overbrace{(x-2)(x^2+2x+4)} \end{aligned}$$

The trinomial factors are prime, and the expression is completely factored.

Answer

$(x+2)(x^2-2x+4)(x-2)(x^2+2x+4)$

As an exercise, factor the previous example as a difference of cubes first and then compare the results. Why do you think we make it a rule to factor using the difference of squares first?

Try It

Factor: x^6-y^6

Show Solution

$(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

Key Takeaways

- When factoring special binomials, the first step is to identify them as a sum or a difference. Once we identify the binomial, we then determine the values of a and b and substitute them into the appropriate formula.
- The formulas for all of the special binomials should be memorized. In addition, to help facilitate the identification of special binomials, memorize the squares and cubes of integers up to at least 12.
- If a binomial is both a difference of squares and cubes, then first factor it as a difference of squares.

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How To: "6.3 Factor Special Products" from Intermediate Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Used How To boxes "Factor differences of squares", "Factor perfect square trinomials", & "Factor the sum or difference of cubes"; Reformatted.

6.3 Factoring Trinomials of the Form x^2+bx+c

Learning Objectives

By the end of this section, you should be able to:

- Factor trinomials of the form $x^2 + bx + c$
- Factor trinomials using the AC method.

Factoring Trinomials of the Form $x^2 + bx + c$

Some trinomials of the form $x^2 + bx + c$ can be factored as a product of binomials. For example,

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

We can verify this factorization by multiplying:

$$\begin{aligned}(x + 2)(x + 5) &= x^2 + 5x + 2x + 10 \\ &= x^2 + 7x + 10 \quad \checkmark\end{aligned}$$

Factoring trinomials requires that we work the distributive process in reverse. Notice that the product of the first terms of each binomial is equal to the first term of the trinomial.

$$x \cdot x = x^2$$

The middle term of the trinomial, $7x$, is the sum of the products of the outer and inner terms of the binomial:

$$5x + 2x = 7x$$

And the product of the last terms of each binomial is equal to the last term of the trinomial.

$$2 \cdot 5 = 10$$

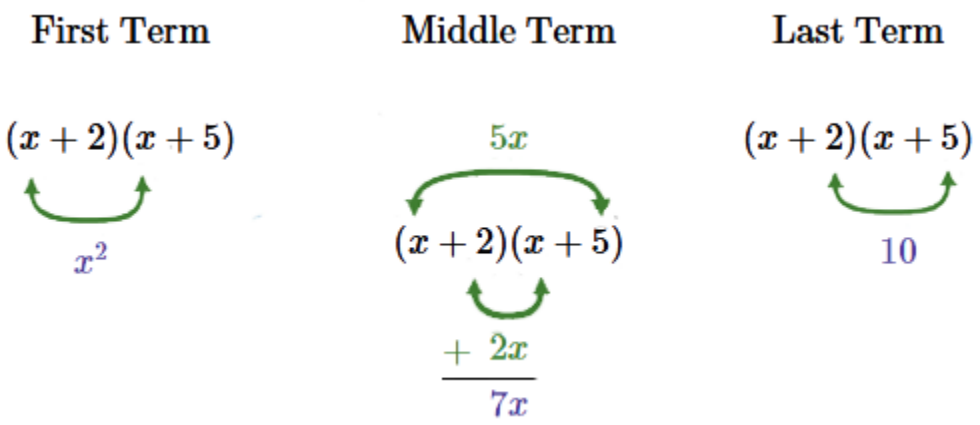

Figure 6.3.1

If a trinomial of this type factors, then these relationships will be true:

$$\begin{aligned} x^2 + bx + c &= (x+p)(x+q) \\ &= x^2 + px + qx + pq \\ &= x^2 + (p+q)x + pq \end{aligned}$$

This gives us

$$b = p + q \quad \text{and} \quad c = pq$$

In short, if the leading coefficient of a factorable trinomial is one, then the factors of the last term must add up to the coefficient of the middle term. This observation is the key to factoring trinomials using the technique known as trial and error (or **guess and check**). The steps are outlined in the following example.

How To

Given a trinomial in the form $x^2 + bx + x$, *factor it.*

1. List factors of c.
2. Find p and q, a pair of factors of c with a sum of b.
3. Write the factored expression $(x+p)(x+q)$.

Example 1

Factor:

$$x^2 + 7x + 12$$

Solution:
Note that the polynomial to be factored has three terms; it is a trinomial with a leading coefficient of 1. Use trial and error to factor as follows:

Step 1: Write two sets of blank parentheses. If a trinomial of this form factors, then it will factor into two linear binomial factors.

$$x^2 + 7x + 12 = (\quad)(\quad)$$

Step 2: Write the factors of the first term in the first space of each set of parentheses. In this case, factor $x^2 = x \cdot x$.

$$x^2 + 7x + 12 = (x \quad)(x \quad)$$

Step 3: Determine the factors of the last term whose sum equals the coefficient of the middle term. To do this, list all of the factorizations of 12 and search for factors whose sum equals the coefficient of the middle term, 7.

$$\begin{aligned} 12 &= 1 \cdot 12 && \rightarrow 1 + 12 = 13 \\ &= 2 \cdot 6 && \rightarrow 2 + 6 = 8 \\ &= 3 \cdot 4 && \rightarrow 3 + 4 = 7 \end{aligned}$$

Choose $12 = 3 \cdot 4$ because $3 + 4 = 7$.

Step 4: Write in the last term of each binomial using the factors determined in the previous step.

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

Step 5: Check by multiplying the two binomials.

$$\begin{aligned} (x + 3)(x + 4) &= x^2 + 4x + 3x + 12 \\ &= x^2 + 7x + 12 \quad \checkmark \end{aligned}$$

Answer

$(x + 3)(x + 4)$

Since multiplication is commutative, the order of the factors does not matter.

$$\begin{aligned} x^2 + 7x + 12 &= (x + 3)(x + 4) \\ &= (x + 4)(x + 3) \end{aligned}$$

If the last term of the trinomial is positive, then either both of the constant factors must be negative or both must be positive. Therefore, when looking at the list of factorizations of the last term, we are searching for sums that are equal to the coefficient of the middle term.

Example 2

Factor:

$$x^2 - 9x + 20$$

Solution:

First, factor $x^2 = x \cdot x$.

$$x^2 - 9x + 20 = (x \quad ?)(x \quad ?)$$

Next, determine which factors of 20 add up to -9:

$$\begin{aligned} 20 &= 1 \cdot 20 \\ &= (-2) \cdot (-10) \\ &= 4 \cdot 5 \end{aligned}$$

In this case, choose -4 and -5 because $(-4)(-5) = +20$ and $-4 + (-5) = -9$.

$$x^2 - 9x + 20 = (x - 4)(x - 5)$$

Check.

$$\begin{aligned} (x - 4)(x - 5) &= x^2 - 5x - 4x + 20 \\ &= x^2 - 9x + 20 \quad \checkmark \end{aligned}$$

Answer

$$(x - 4)(x - 5)$$

If the last term of the trinomial is negative, then one of its factors must be negative. In this case, search the list of factorizations of the last term for differences that equal the coefficient of the middle term.

Example 3

Factor:

$$x^2 - 4x - 12$$

Solution:

Begin by factoring the first term $x^2 = x \cdot x$.

$$x^2 - 4x - 12 = (x \quad ?)(x \quad ?)$$

The factors of 12 are listed below. In this example, we are looking for factors whose sum is -4:

$$\begin{aligned} 12 &= 1 \cdot 12 && \to 1 + (-12) = -11 \\ &= 2 \cdot 6 && \to 2 + (-6) = -4 \\ &= 3 \cdot 4 && \to 3 + (-4) = -1 \end{aligned}$$

Choose the factors 2 and -6 because $2 + (-6) = -4$.

$$x^2 - 4x - 12 = (x + 2)(x - 6)$$

Multiply to check.

$$\begin{aligned} (x + 2)(x - 6) &= x^2 - 6x + 2x - 12 \\ &= x^2 - 4x - 12 \quad \checkmark \end{aligned}$$

Answer

$$(x + 2)(x - 6)$$

Often, our first guess will not produce a correct factorization. This process may require repeated trials. For this reason, the check is very important and is not optional.

Example 4

Factor:

$$x^2 + 5x - 6$$

Solution:

The first term of this trinomial x^2 factors as $x \cdot x$

$$x^2 + 5x - 6 = (x \quad ?)(x \quad ?)$$

Consider the factors of 6:

$$\begin{aligned} 6 &= 1 \cdot 6 \\ &= 2 \cdot 3 \end{aligned}$$

Suppose we choose the factors 2 and 3 because $2 + 3 = 5$, the coefficient of the middle term. Then we have the following incorrect factorization:

$$x^2 + 5x - 6 \overset{?}{=} (x + 2)(x + 3)$$

When we multiply to check, we find the error.

$$\begin{aligned} (x + 2)(x + 3) &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6 \end{aligned}$$

In this case, the middle term is correct, but the last term is not. Since the last term in the original expression is negative, we need to choose factors that are opposite in sign. Therefore, we must try again. This time we choose the factors -1 and 6 because $-1 + 6 = 5$.

$$x^2 + 5x - 6 = (x - 1)(x + 6)$$

Now the check shows that this factorization is correct.

$$\begin{aligned} (x - 1)(x + 6) &= x^2 + 6x - x - 6 \\ &= x^2 + 5x - 6 \quad \checkmark \end{aligned}$$

Answer

$$(x - 1)(x + 6)$$

If we choose the factors wisely, then we can reduce much of the guesswork in this process. However, if a guess is not correct, do not get discouraged; just try a different set of factors.

Example 5

Factor:

$$x^2 + 3x + 20$$

Solution:

$$\begin{aligned} x^2 + 3x + 20 &= (x \quad ?)(x \quad ?) \\ 20 &= 1 \cdot 20 \\ &= 2 \cdot 10 \\ &= 4 \cdot 5 \end{aligned}$$

Here there are no factors of 20 whose sum is 3. Therefore, the original trinomial cannot be factored as a product of two binomials. This trinomial is prime.

Answer

Prime

Try It

Factor: $x^2 - 13x - 30$

Show Solution

$(x + 2)(x - 15)$

The techniques described can also be used to factor trinomials with more than one variable.

Example 6

Factor:

$$x^2 - 14xy - 72y^2$$

Solution

The first term x^2 factors as $x \cdot x$.

$$x^2 - 14xy - 72y^2 = (x \quad ?)(x \quad ?)$$

Next, look for factors of 72 whose sum is -14.

$$\begin{aligned} 72 &= 1 \cdot 72 \\ &= 2 \cdot 36 \\ &= 3 \cdot 24 \\ &= 4 \cdot 18 \rightarrow 4 + (-18) = -14 \\ &= 6 \cdot 12 \\ &= 8 \cdot 9 \end{aligned}$$

Therefore, factor $72y^2$ as $4y \cdot (-18y)$ and try the following factorization:

$$x^2 - 14xy - 72y^2 = (x + 4y)(x - 18y)$$

Multiply to check.

$$\begin{aligned} (x + 4y)(x - 18y) &= x^2 - 18xy + 4xy - 72y^2 \\ &= x^2 - 14xy - 72y^2 \quad \checkmark \end{aligned}$$

Visually, we have the following:

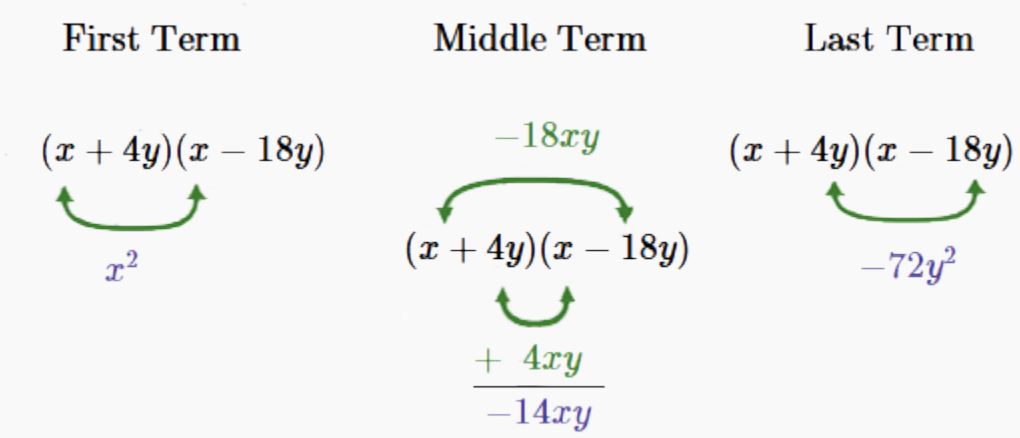

Figure 6.3.2.

Answer

$(x+4y)(x-18y)$

Try It

Factor: $x^2y^2+9xy-10$

Show Solution

$(xy-1)(xy+10)$

At this point, it is recommended that you stop and factor as many trinomials of the form x^2+bx+c as time allows before moving on to the next section. Factoring trinomials is one of the more important skills that we learn in this course and should be mastered. Practice with the *Chapter Exercises*.

Key Takeaways

- Factor a trinomial by systematically guessing what factors give two binomials whose product is the original trinomial.
- If a trinomial of the form $x^2 + bx + c$ factors into the product of two binomials, then the coefficient of the middle term is the sum of the factors of the last term.
- Not all trinomials can be factored as the product of binomials with integer coefficients. In this case, we call it a prime trinomial.
- Factoring is one of the more important skills required in algebra. For this reason, you should practice working as many problems as it takes to become proficient.

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How To: "1.5 Factoring Polynomials" from College Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Used How To box "Given a trinomial in the form $x^2 + bx + x$, factor it"; Reformatted.

6.4 Factoring Trinomials of the Form ax^2+bx+c

Learning Objectives

By the end of this section, you should be able to:

- Factor trinomials of the form $ax^2 + bx + c$.
- Factor trinomials with a **common factor**.

Factoring Trinomials of the Form $ax^2 + bx + c$

Factoring trinomials of the form $ax^2 + bx + c$ can be challenging because the middle term is affected by the factors of both a and c. To illustrate this, consider the following factored trinomial:

$$10x^2 + 17x + 3 = (2x + 3)(5x + 1)$$

We can multiply to verify that this is the correct factorization.

$$\begin{aligned}(2x + 3)(5x + 1) &= 10x^2 + 2x + 15x + 3 \\ &= 10x^2 + 17x + 3 \quad \checkmark\end{aligned}$$

As we have seen before, the product of the first terms of each binomial is equal to the first term of the trinomial. The middle term of the trinomial is the sum of the products of the outer and inner terms of the binomials. The product of the last terms of each binomial is equal to the last term of the trinomial. Visually, we have the following:

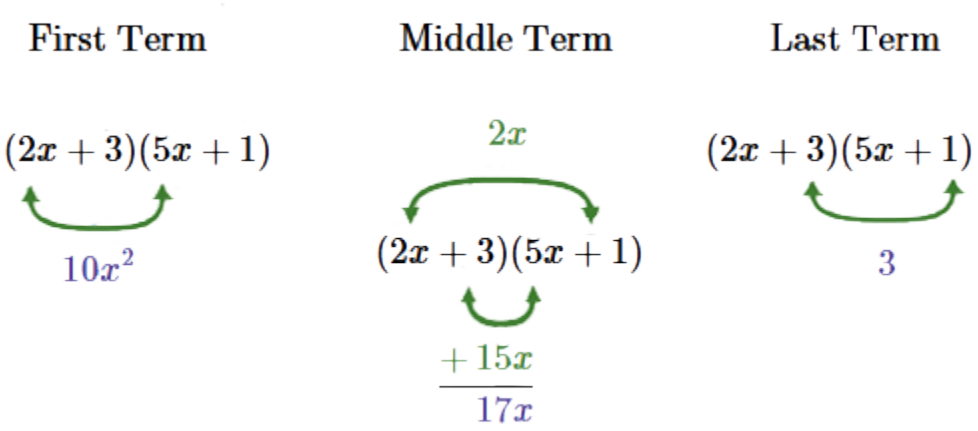

Figure 6.4.1.

In general,

$$\begin{aligned} ax^2+bx+c &= (px+m)(qx+n) \\ &= pqx^2+pnx+qmx+mn \\ &= pqx^2+(pn+qm)x+mn \end{aligned}$$

This gives us,

$$a=pq \quad \text{and} \quad b=pn+qm, \quad \text{where} \quad c=mn$$

In short, when the leading coefficient of a trinomial is something other than 1, there will be more to consider when determining the factors using the trial and error method. The key lies in the understanding of how the middle term is obtained. Multiply $(2x+5)(3x+7)$ and carefully follow the formation of the middle term.

$$(2x+5)(3x+7) = 2x\cdot 3x + \underbrace{2x\cdot 7+5\cdot 3x}_{\text{middle term}} + 5\cdot 7$$

$$\begin{aligned} &= 6x^2+14x+15x+35 \\ &= 6x^2+29x+35 \end{aligned}$$

If we think of the FOIL method for multiplying binomials, then the middle term results from the sum of the inner product and the outer product. In this case, $14x+15x=29x$, as illustrated below:

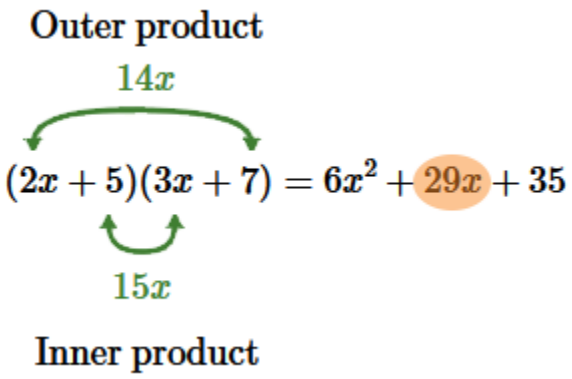

Figure 6.4.2.

For this reason, we need to look for products of the factors of the first and last terms whose sum is equal to the coefficient of the middle term. For example, to factor $6x^2 + 29x + 35$, look at the factors of 6 and 35.

$$\begin{aligned} 6 &= 1 \cdot 6 & 35 &= 1 \cdot 35 \\ &= 2 \cdot 3 & &= 5 \cdot 7 \end{aligned}$$

The combination that produces the coefficient of the middle term is $2 \cdot 7 + 3 \cdot 5 = 14 + 15 = 29$. Make sure that the outer terms have coefficients 2 and 7, and that the inner terms have coefficients 5 and 3. Use this information to factor the trinomial:

$$\begin{aligned} 6x^2 + 29x + 35 &= (2x \quad ?)(3x \quad ?) \\ &= (2x + 5)(3x + 7) \end{aligned}$$

How To

Given a trinomial in the form $ax^2 + bx + c$ factor by grouping.

1. List factors of ac.
2. Find p and q, a pair of factors of ac with a sum of b.
3. Rewrite the original expression as $ax^2 + px + qx + c$.
4. Pull out the GCF of $ax^2 + px$.
5. Pull out the GCF of $qx + c$.
6. Factor out the GCF of the expression.

Example 1

Factor:

$$3x^2 + 7x + 2$$

Solution:
Since the leading coefficient and the last term are both prime, there is only one way to factor each.

$$3 = 1 \cdot 3 \quad \text{and} \quad 2 = 1 \cdot 2$$

Begin by writing the factors of the first term, $3x^2$, as follows:

$$3x^2 + 7x + 2 = (x \quad ?)(3x \quad ?)$$

The middle term and last term are both positive; therefore, the factors of 2 are chosen as positive numbers. In this case, the only choice is in which grouping to place these factors.

$$(x + 1)(3x + 2) \quad \text{or} \quad (x + 2)(3x + 1)$$

Determine which grouping is correct by multiplying each expression.

$$\begin{aligned}(x + 1)(3x + 2) &= 3x^2 + 2x + 3x + 2 \\ &= 3x^2 + 5x + 2 \quad \times \\ (x + 2)(3x + 1) &= 3x^2 + x + 6x + 2 \\ &= 3x^2 + 7x + 2 \quad \checkmark\end{aligned}$$

Notice that these products differ only in their middle terms. Also, notice that the middle term is the sum of the inner and outer product, as illustrated below:

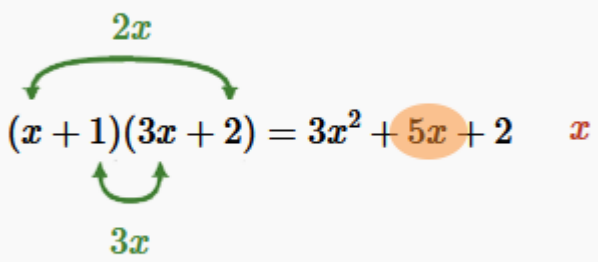

Figure 6.4.3.

x

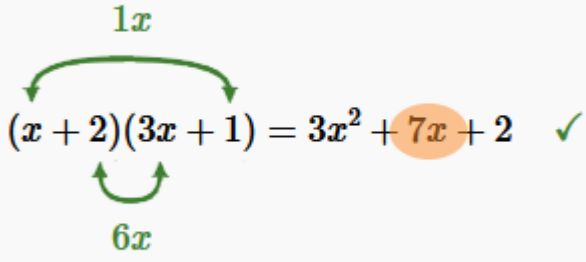

Figure 6.4.4.

✓

Answer

$(x+2)(3x+1)$

Example 2

Factor:

$$12x^2 + 38x + 20$$

Solution:
First, consider the factors of the first and last terms.

$$\begin{aligned} 12 &= 1 \cdot 12 \\ &= 2 \cdot 6 \\ &= 3 \cdot 4 \end{aligned} \qquad \begin{aligned} 20 &= 1 \cdot 20 \\ &= 2 \cdot 10 \\ &= 4 \cdot 5 \end{aligned}$$

We search for products of factors whose sum equals the coefficient of the middle term, 38. For brevity, the thought process is illustrated starting with the factors 2 and 6. Factoring begins at this point with the first term.

$$12x^2 + 38x + 20 = (2x \quad ?)(6x \quad ?)$$

We search for factors of 20 that, along with the factors of 12, produce a middle term of $38x$.

Table 6.4.1.

Factors of 20	Possible	Factorization
$1 \cdot 20$	$(2x+1)(6x+20)$	middle term $\Rightarrow 46x$
$1 \cdot 20$	$(2x+20)(6x+1)$	middle term $\Rightarrow 122x$
$2 \cdot 10$	$(2x+2)(6x+10)$	middle term $\Rightarrow 32x$
$2 \cdot 10$	$(2x+10)(6x+2)$	middle term $\Rightarrow 64x$
$4 \cdot 5$	$(2x+4)(6x+5)$	middle term $\Rightarrow 34x$
$4 \cdot 5$	$(2x+5)(6x+4)$	middle term $\Rightarrow$ $38x$ ✓

Here, the last combination produces a middle term of $38x$.

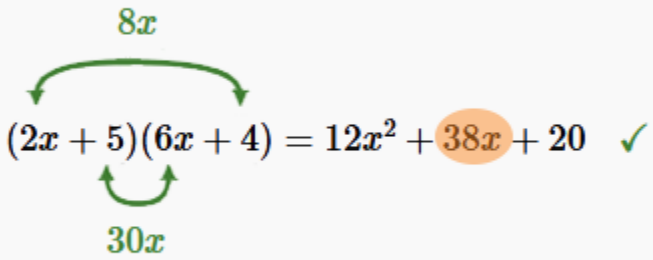

Figure 6.4.5.

Answer

$(2x+5)(6x+4)$

Example 3

Factor:

$$10x^2 - 23x + 6$$

Solution:
First, consider the factors of the first and last terms.

$$\begin{aligned} 10 &= 1 \cdot 10 \\ &= 2 \cdot 5 \end{aligned} \qquad \begin{aligned} 6 &= 1 \cdot 6 \\ &= 2 \cdot 3 \end{aligned}$$

We are searching for products of factors whose sum equals the coefficient of the middle term, -23. Factoring begins at this point with two sets of blank parentheses:

$$10x^2 - 23x + 6 = (\quad)(\quad)$$

Since the last term is positive and the middle term is negative, we know that both factors of the last term must be negative. Here we list all possible combinations with the factors of $10x^2 = 2x \cdot 5x$.

$$10x^2 - 23x + 6 = (2x \quad ?)(5x \quad ?)$$

$$\begin{aligned} &(2x-1)(5x-6) && \text{middle term} \Rightarrow -17x \\ &(2x-6)(5x-1) && \text{middle term} \Rightarrow -32x \\ &(2x-2)(5x-3) && \text{middle term} \Rightarrow -16x \\ &(2x-3)(5x-2) && \text{middle term} \Rightarrow -19x \end{aligned}$$

There is no combination that produces a middle term of $-23x$. We then move on to the factors of $10x^2 = 10x \cdot x$ and list all possible combinations:

$$10x^2 - 23x + 6 = (10x \quad ?)(x \quad ?)$$

$$\begin{aligned} &(10x-1)(x-6) && \text{middle term} \Rightarrow -61x \\ &(10x-6)(x-1) && \text{middle term} \Rightarrow -162x \\ &(10x-2)(x-3) && \text{middle term} \Rightarrow -32x \\ &(10x-3)(x-2) && \text{middle term} \Rightarrow -23x \quad \checkmark \end{aligned}$$

And we can write

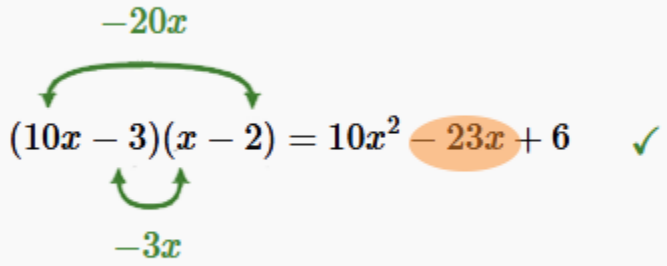

Figure 6.4.6.

Answer

$(10x-3)(x-2)$. The complete check is left to the reader.

We can reduce much of the guesswork involved in factoring trinomials if we consider all of the factors of the first and last terms and their products.

Example 4

Factor:

$$5x^2+38x-16$$

Solution:
We begin with the factors of 5 and 16.

$$\begin{aligned} 5 &= 1\cdot 5 \\ 16 &= 1\cdot 16 \\ &= 2\cdot 8 \\ &= 4\cdot 4 \end{aligned}$$

Since the leading coefficient is prime, we can begin with the following:

$$5x^2+38x-16=(x\quad ?)(5x\quad ?)$$

We look for products of the factors of 5 and 16 that could possibly add to 38.

Table 6.4.2.

Factors of 16	Possible	Products
$1 \cdot 16$	$1 \cdot 1$ and $5 \cdot 16$	products $\Rightarrow 1$ and 80
$1 \cdot 16$	$1 \cdot 16$ and $5 \cdot 1$	products $\Rightarrow 16$ and 5
$2 \cdot 8$	$1 \cdot 2$ and $5 \cdot 8$	products $\Rightarrow 2$ and 40 ✓
$2 \cdot 8$	$1 \cdot 8$ and $5 \cdot 2$	products $\Rightarrow 8$ and 10
$4 \cdot 5$	$1 \cdot 4$ and $5 \cdot 4$	products $\Rightarrow 4$ and 20

Since the last term is negative, we must look for factors with opposite signs. Here we can see that the products 2 and 40 add up to 38 if they have opposite signs:

$$1 \cdot (-2) + 5 \cdot 8 = -2 + 40 = 38$$

Therefore, use -2 and 8 as the factors of 16, making sure that the inner and outer products are $-2x$ and $40x$:

$$(x+8)(5x-2) = 5x^2 + \underbrace{38x}_{\substack{-2x \\ 40x}} - 16 \quad \checkmark$$

Figure 6.4.7.

Answer

$(x+8)(5x-2)$. The complete check is left to the reader.

After lots of practice, the process described in the previous example can be performed mentally.

Try It

Factor: $12x^2 - 31x - 30$

Show Solution

$$(3x - 10)(4x + 3)$$

When given trinomials with multiple variables, the process is similar.

Example 5

Factor:

$$9x^2 + 30xy + 25y^2$$

Solution:
Search for factors of the first and last terms such that the sum of the inner and outer products equals the middle term.

$$\begin{aligned} 9x^2 &= 1x \cdot 9x \\ &= 3x \cdot 3x \end{aligned} \qquad \begin{aligned} 25y^2 &= 1y \cdot 25y \\ &= 5y \cdot 5y \end{aligned}$$

Add the following products to obtain the middle term: $3x \cdot 5y + 3x \cdot 5y = 30xy$.

$$\begin{aligned} 9x^2 + 30xy + 25y^2 &= (3x \quad)(3x \quad) \\ &= (3x + 5y)(3x + 5y) \\ &= (3x + 5y)^2 \end{aligned}$$

In this example, we have a perfect square trinomial. Check.

$$\begin{aligned} (3x + 5y)^2 &= 9x^2 + 2 \cdot 3x \cdot 5y + 25y^2 \\ &= 9x^2 + 30xy + 25y^2 \quad \checkmark \end{aligned}$$

Answer

$(3x + 5y)^2$

Try It

Factor: $16x^2 - 24xy + 9y^2$

Show Solution

$(4x-3y)^2$

Factoring Trinomials with Common Factors

It is a good practice to first factor out the GCF, if there is one. Doing this produces a trinomial factor with smaller coefficients. As we have seen, trinomials with smaller coefficients require much less effort to factor. This commonly overlooked step is worth identifying early.

Example 6

Factor:

$$12x^2-27x+6$$

Solution:
Begin by factoring out the GCF.

$$12x^2-27x+6=3(4x^2-9x+2)$$

After factoring out 3, the coefficients of the resulting trinomial are smaller and have fewer factors.

$$\begin{aligned} 4&=1\cdot 4 & 2&=1\cdot 2\\ &=2\cdot 2 \end{aligned}$$

After some thought, we can see that the combination that gives the coefficient of the middle term is $4(-2)+1(-1)=-8-1=-9$.

$$\begin{aligned} 3(4x^2-9x+2)&=3(4x\quad ?)(x\quad ?)\\ &=3(4x-1)(x-2) \end{aligned}$$

Check.

$$\begin{aligned} 3(4x-1)(x-2) &= 3(4x^2-8x-x+2) \\ &= 3(4x^2-9x+2) \\ &= 12x^2-27x+6 \quad \checkmark \end{aligned}$$

The factor 3 is part of the factored form of the original expression; be sure to include it in the answer.

Answer

$3(4x-1)(x-2)$

It is a good practice to consistently work with trinomials where the leading coefficient is positive. Factor out -1 if the leading coefficient is negative.

Example 7

Factor:

$$-x^2+2x+15$$

Solution:

In this example, the leading coefficient is -1. Before beginning the factoring process, factor out the -1:

$$-x^2+2x+15 = -1(x^2-2x-15)$$

At this point, factor the remaining trinomial as usual, remembering to write the -1 as a factor in your final answer. Because $3+(-5)=-2$, use 3 and -5 as the factors of 15.

$$\begin{aligned} -x^2+2x+15 &= -1(x^2-2x-15) \\ &= -1(x+3)(x-5) \end{aligned}$$

Answer

$-1(x+3)(x-5)$. The check is left to the reader.

Example 8

Factor:

$$-60a^2 - 5a + 30$$

Solution:

The GCF of all the terms is 5. However, in this case, factor out -5 because this produces a trinomial factor where the leading coefficient is positive.

$$-60a^2 - 5a + 30 = -5(12a^2 + a - 6)$$

Focus on the factors of 12 and 6 that combine to give the middle coefficient, 1.

$$\begin{aligned} 12 &= 1 \cdot 12 \\ &= 2 \cdot 6 \\ &= 3 \cdot 4 \end{aligned} \qquad \begin{aligned} 6 &= 1 \cdot 6 \\ &= 2 \cdot 3 \end{aligned}$$

After much thought, we find that $3 \cdot 3 - 4 \cdot 2 = 9 - 8 = 1$. Factor the remaining trinomial.

$$\begin{aligned} -60a^2 - 5a + 30 &= -5(12a^2 + a - 6) \\ &= -5(4a + 3)(3a - 2) \end{aligned}$$

Answer

$-5(4a + 3)(3a - 2)$. The check is left to the reader.

Try It

Factor: $24 + 2x - x^2$

Show Solution

$$-1(x-6)(x+4)$$

Key Takeaways

- If a trinomial of the form ax^2+bx+c factors into the product of two binomials, then the coefficient of the middle term will be the sum of certain products of factors of the first and last terms.
- If the trinomial has a greatest common factor, then it is a best practice to first factor out the GCF before attempting to factor it into a product of binomials.
- If the leading coefficient of a trinomial is negative, then it is a best practice to factor that negative factor out before attempting to factor the trinomial.
- Factoring trinomials of the form ax^2+bx+c takes lots of practice and patience. It is extremely important to take the time to become proficient by working lots of exercises.

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6.5 General Guidelines for Factoring Polynomials

Learning Objectives

By the end of this section, you should be able to:

- Develop a general strategy for factoring polynomials.

General Factoring Strategy

We have learned various techniques for factoring polynomials with up to four terms. The challenge is to identify the type of polynomial and then decide which method to apply. The following outlines a general guideline for factoring polynomials:

1. Check for common factors. If the terms have common factors, then factor out the greatest common factor (GCF) and look at the resulting polynomial factors to factor further.
2. Determine the number of terms in the polynomial.
   a. Factor four-term polynomials by grouping.
   b. Factor trinomials (three terms) using "trial and error" or the AC method.
   c. Factor binomials (two terms) using the following special products:

Table 6.5.1.

Difference of Squares:	$a^2 - b^2 = (a + b)(a - b)$
Sum of Squares:	$a^2 + b^2$ (no general formula)
Difference of Cubes:	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
Sum of Cubes:	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

3. Look for factors that can be factored further.
4. Check by multiplying.

How To

Use a general strategy for factoring polynomials.

1. Is there a greatest common factor?
    - Factor it out.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
    - If it is a binomial:
    - Is it a sum?
        - Of squares? Sums of squares do not factor.
        - Of cubes? Use the sum of cubes pattern.
    - Is it a difference?
        - Of squares? Factor as the product of conjugates.
        - Of cubes? Use the difference of cubes pattern.
    - If it is a trinomial:
        - Is it of the form $x^2 + bx + c$? Undo FOIL.
        - Is it of the form $ax^2 + bx + c$?
        - If a and c are squares, check if it fits the trinomial square pattern.
        - Use the trial and error or "ac" method.
    - If it has more than three terms:
        - Use the grouping method.
3. Check.
   Is it factored completely?
   Do the factors multiply back to the original polynomial?

Note

- If a binomial is both a difference of squares and a difference of cubes, then first factor it as difference of squares and then as a sum and difference of cubes to obtain a more complete factorization.
- Not all polynomials with integer coefficients factor. When this is the case, we say that the polynomial is prime.

If an expression has a GCF, then factor this out first. Doing so is often overlooked and typically results in factors that are easier to work with. Also, look for the resulting factors to factor further; many factoring problems require more than one step. A polynomial is completely factored when none of the factors can be factored further.

Example 1

Factor:

$$6x^4 - 3x^3 - 24x^2 + 12x$$

Solution:

This four-term polynomial has a GCF of $3x$. Factor this out first.

$$6x^4 - 3x^3 - 24x^2 + 12x = 3x(2x^3 - x^2 - 8x + 4)$$

Now factor the resulting four-term polynomial by grouping.

$$6x^4 - 3x^3 - 24x^2 = 3x\underbrace{(2x^3 - x^2}_{\text{group}}\underbrace{-8x + 4)}_{\text{group}}$$

$$\begin{aligned} &= 3x(x^2(2x - 1) - 4(2x - 1)) \\ &= 3x((2x - 1)(x^2 - 4)) \\ &= 3x(2x - a)(x^2 - 4) \end{aligned}$$

The factor $(x^2 - 4)$ is a difference of squares and can be factored further.

$$\begin{aligned} &= 3x(2x - 1)(x^2 - 4) \\ &= 3x(2x - 1)(x + 2)(x - 2) \end{aligned}$$

Answer

$3x(2x - 1)(x + 2)(x - 2)$

Example 2

Factor:

$$18x^3y - 60x^2y + 50xy$$

Solution:

This trinomial has a GCF of $2xy$. Factor this out first.

$$18x^3y - 60x^2y + 50xy = 2xy(9x^2 - 30x + 25)$$

The trinomial factor can be factored further using the trial-and-error method. Use the factors $9 = 3 \cdot 3$ and $25 = (-5) \cdot (-5)$. These combine to generate the correct coefficient for the middle term: $3(-5) + 3(-5) = -15 - 15 = -30$.

$$\begin{aligned} 18x^3y - 60x^2y + 50xy &= 2xy(9x^2 - 30x + 25) \\ &= 2xy(3x - 5)(3x - 5) \\ &= 2xy(3x - 5)^2 \end{aligned}$$

Check.

$$\begin{aligned} 2xy(3x-5)^2 &= 2xy(9x^2 - 2\cdot 3x\cdot 5 + 5^2) \\ &= 2xy(9x^2 - 30x + 25) \\ &= 18x^3y - 60x^2y + 50xy \quad \checkmark \end{aligned}$$

Answer

$2xy(3x-5)^2$

Example 3

Factor:

$$5a^3b^4 + 10a^2b^3 - 75ab^2$$

Solution:

This trinomial has a GCF of $5ab^2$. Factor this out first.

$$5a^3b^4 + 10a^2b^3 - 75ab^2 = 5ab^2(a^2b^2 + 2ab - 15)$$

The resulting trinomial factor can be factored as follows:

$$\begin{aligned} 5a^3b^4 + 10a^2b^3 - 75ab^2 &= 5ab^2(a^2b^2 + 2ab - 15) \\ &= 5ab^2(ab+5)(ab-3) \end{aligned}$$

Answer

$5ab^2(ab+5)(ab-3)$

Try It

Factor: $3x^3y - 12x^2y^2 + 12xy^3$

Show Solution

$3xy(x-2y)^2$

Example 4

Factor:

$$16y^4-1$$

Solution:
This binomial does not have a GCF. Therefore, begin factoring by identifying it as a difference of squares.

$$16y^4-1=(4y^2)^2-(1)^2$$

Here $a=4y^2$ and $b=1$. Substitute into the formula for difference of squares.

$$16y^4-1=(4y^2+1)(4y^2-1)$$

The factor $(4y^2+1)$ is a sum of squares and is prime. However, $(4y^2-1)$ is a difference of squares and can be factored further.

$$\begin{aligned}16y^4-1&=(4y^2+1)(4y^2-1)\\&=(4y^2+1)(2y+1)(2y-1)\end{aligned}$$

Answer

$(4y^2+1)(2y+1)(2y-1)$

Example 5

Factor:

$$x^6 - 64y^6$$

Solution:
This binomial is a difference of squares and a difference of cubes. When this is the case, first factor it as a difference of squares.

$$x^6 - 64y^6 = (x^3)^2 - (8y^3)^2$$

We can write

$$x^6 - 64y^6 = (x^3 + 8y^3)(x^3 - 8y^3)$$

Each factor can be further factored either as a sum or difference of cubes, respectively.

$$x^3 + 8y^3 = (x)^3 + (2y)^3 = (x + 2y)(x^2 - 2xy + 4y^2)$$
$$x^3 - 8y^3 = (x)^3 - (2y)^3 = (x - 2y)(x^2 + 2xy + 4y^2)$$

Therefore,

$$\begin{aligned} x^6 - 64y^6 &= (x^3 + 8y^3)(x^3 - 8y^3) \\ &= (x + 2y)(x^2 - 2xy + 4y^2)(x - 2y)(x^2 + 2xy + 4y^2) \end{aligned}$$

Answer

$$(x + 2y)(x^2 - 2xy + 4y^2)(x - 2y)(x^2 + 2xy + 4y^2)$$

Example 6

First, identify this expression as a difference of squares.:

$$x^2 - (2x - 1)^2 = (x)^2 - (2x - 1)^2$$

Solution:

Here, use $a = x$ and $b = 2x - 1$ in the formula for a difference of squares.

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b) \\ x^2 - (2x - 1)^2 &= [(x + (2x - 1))][(x - (2x - 1))] \\ &= (x + 2x - 1)(x - 2x + 1) \\ &= (3x - 1)(-x + 1) \end{aligned}$$

Answer

$(3x - 1)(-x + 1)$

Try It

Factor: $x^4 + 2x^3 + 27x + 54$

Show Solution

$(x + 2)(x + 3)(x^2 - 3x + 9)$

Key Takeaways

- Use the polynomial type to determine the method used to factor it.
- It is a best practice to look for and factor out the greatest common factor (GCF) first. This will facilitate further factoring and simplify the process. Be sure to include the GCF as a factor in the final answer.
- Look for resulting factors to factor further. It is often the case that factoring requires more than one step.
- If a binomial can be considered as both a difference of squares and a difference of cubes, then factor it first as a difference of squares. This results in a more complete factorization.

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6.6 Factoring Fractions

Learning Objectives

By the end of this section, you should be able to:

- Determine the restrictions to the domain of a rational expression.
- Simplify rational expressions.
- Simplify expressions with opposite binomial factors.
- Simplify and evaluate rational functions.

Rational Expressions, Evaluation, and Restrictions

A rational number, or fraction $\frac{a}{b}$, is a real number defined as a quotient of two integers a and b, where $b \neq 0$. Similarly, we define a rational expression, or algebraic fraction $\frac{P}{Q}$, as the quotient of two polynomials P and Q, where $Q \neq 0$. Some examples of rational expressions follow:

$$\frac{(x+3)}{(x-5)} \qquad \frac{x+7}{2x^2+x-6} \qquad \frac{x^4+x^3-2x^2-x}{x^2-1}$$

The example $\frac{x+3}{x-5}$ consists of linear expressions in both the numerator and denominator. Because the denominator contains a variable, this expression is not defined for all values of x.

Example 1

Evaluate:

$$\frac{x+3}{x-5} \text{ for the set of } x\text{-values } \{-3, 4, 5\}$$

Solution:

Substitute the values in for x.

Table 6.6.1.

$x=-3$	$x=4$	$x=5$
$\frac{x+3}{x-5}=\frac{(-3)+3}{(-3)-5}$ $=\frac{0}{-8}$ $=0$	$\frac{x+3}{x-5}=\frac{(4)+3}{(4)-5}$ $=\frac{7}{-1}$ $=-7$	$\frac{x+3}{x-5}=\frac{(5)+3}{(5)-5}$ $=\frac{8}{0}$ $=$ Undefined

Answer

When $x=-3$, the value of the rational expression is 0; when $x=4$, the value of the rational expression is -7; and when $x=5$, the value of the rational expression is undefined.

This example illustrates that variables are restricted to values that do not make the denominator equal to 0. The domain of a rational expression is the set of real numbers for which it is defined, and restrictions are the real numbers for which the expression is not defined. We often express the domain of a rational expression in terms of its restrictions.

Example 2

Find the domain of the following:

$$\frac{x+7}{2x^2+x-6}$$

Solution:
In this example, the numerator $x+7$ is a linear expression, and the denominator $2x^2+x-6$ is a quadratic expression. If we factor the denominator, then we will obtain an equivalent expression.

$$\frac{x+7}{2x^2+x-6}=\frac{x+7}{(2x-3)(x+2)}$$

Because rational expressions are undefined when the denominator is 0, we wish to find the values for x that make it 0. To do this, apply the zero product property. Set each factor in the denominator equal to 0 and solve.

$$(2x-3)(x+2)=0$$

$$\begin{aligned} 2x-3=0 \quad &\text{or} \quad x+2=0 \\ 2x=3 \quad &\quad\quad\;\; x=-2 \\ x=\frac{3}{2} \quad &\end{aligned}$$

We conclude that the original expression is defined for any real number except $\frac{3}{2}$ and -2. These two values are the restrictions to the domain. It is important to note that -7 is not a restriction to the domain because the expression is defined as 0 when the numerator is 0.

$$\begin{aligned} \frac{x+7}{2x^2+x-6} &= \frac{(-7)+7}{2(-7)^2+(-7)-6} \\ &= \frac{0}{98-7-6} \\ &= \frac{0}{85} \\ &= 0 \end{aligned}$$

Answer

The domain consists of any real number x, where $x \neq \frac{3}{2}$ and $x \neq -2$.

Remember to state restrictions (determined by the denominator). Ignore the numerator when finding those restrictions.

Example 3

Determine the domain:

$$\frac{x^4+x^3-2x^2-x}{x^2-1}$$

Solution:

To find the restrictions to the domain, set the denominator equal to 0 and solve:

$$\begin{aligned} x^2-1&=0 \\ (x+1)(x-1)&=0 \\ x+1=0 \quad &\text{or} \quad x-1=0 \\ x=-1 \quad &\quad\quad\;\; x=1 \end{aligned}$$

These two values cause the denominator to be 0. Hence, they are restricted from the domain.

Answer

The domain consists of any real number x, where $x \neq \pm 1$.

Example 4

Determine the domain:

$$\frac{x^2 - 25}{4}$$

Solution:

There is no variable in the denominator and thus no restriction to the domain.

Answer

The domain consists of all real numbers, R.

Simplifying Rational Expressions

When simplifying fractions, look for common factors that cancel. For example,

$$\frac{12}{60} = \frac{1 \cdot \cancel{12}}{5 \cdot \cancel{12}} = \frac{1}{5}$$

We say that the fraction $\frac{12}{60}$ is equivalent to $\frac{1}{5}$. Fractions are in simplest form if the numerator and denominator share no common factor other than 1. Similarly, when working with rational expressions, look for factors to cancel. For example,

$$\frac{x+4}{(x-3)(x+4)} = \frac{1 \cdot \cancel{(x+4)}}{(x-3)\cancel{(x+4)}} = \frac{1}{x-3}$$

The resulting rational expression is equivalent if it shares the same domain. Therefore, we must make note of the restrictions and write

$$\frac{x+4}{(x-3)(x+4)} = \frac{1}{x-3}, \text{ where } x \neq 3 \text{ and } x \neq -4$$

In words, $\dfrac{x+4}{(x-3)(x+4)}$ is equivalent to $\dfrac{1}{x-3}$, if $x \neq 3$ and $x \neq -4$. We can verify this by choosing a few values with which to evaluate both expressions to see if the results are the same. Here we choose $x = 7$ and evaluate as follows:

$$\begin{aligned}
\frac{x+4}{(x-3)(x+4)} &= \frac{1}{x-3} \\
\frac{(7)+4}{(7-3)(7+4)} &= \frac{1}{(7)-3} \\
\frac{11}{(4)(11)} &= \frac{1}{4} \\
\frac{1}{4} &= \frac{1}{4} \quad \checkmark
\end{aligned}$$

It is important to state the restrictions before simplifying rational expressions because the simplified expression may be defined for restrictions of the original. In this case, the expressions are not equivalent. Here -4 is defined for the simplified equivalent but not for the original, as illustrated below:

$$\begin{aligned}
\frac{x+4}{(x-3)(x+4)} &= \frac{1}{x-3} \\
\frac{(-4)+4}{(-4-3)(-4+4)} &= \frac{1}{(-4)-3} \\
\frac{0}{(-7)(0)} &= \frac{1}{-7} \\
\frac{0}{0} &= -\frac{1}{7} \quad \text{x}
\end{aligned}$$

Example 5

Simplify and state the restriction:

$$\frac{25x^2}{15x^3}$$

Solution:

In this example, the expression is undefined when x is 0.

$$\frac{25x^2}{15x^3} = \frac{25(0)^2}{15(0)^3} = \frac{0}{0} \text{ Undefined}$$

Therefore, the domain consists of all real numbers x, where $x \neq 0$. With this understanding, we can cancel common factors.

$$\begin{aligned} \frac{25x^2}{15x^3} &= \frac{5 \cdot \cancel{5x^2}}{3x \cdot \cancel{5x^2}} \\ &= \frac{5}{3x} \end{aligned}$$

Answer

$\frac{5}{3x}$, where $x \neq 0$

Example 6

State the restrictions and simplify:

$$\frac{3x(x-5)}{(2x+1)(x-5)}$$

Solution:

To determine the restrictions, set the denominator equal to 0 and solve.

$$(2x+1)(x-5)=0$$

$$\begin{array}{rcl} 2x+1=0 & \text{or} & x-5=0 \\ 2x=-1 & & x=5 \\ x=-\dfrac{1}{2} & & \end{array}$$

The domain consists of all real numbers except $-\dfrac{1}{2}$ and 5. Next, we find an equivalent expression by cancelling common factors.

$$\begin{aligned} \frac{3x(x-5)}{(2x+1)(x-5)} &= \frac{3x\,\cancel{(x-5)}}{(2x+1)\,\cancel{(x-5)}} \\ &= \frac{3x}{2x+1} \end{aligned}$$

Answer

$\dfrac{3x}{2x+1}$, where $x \neq -\dfrac{1}{2}$ and $x \neq 5$

Typically, rational expressions are not given in factored form. If this is the case, factor first and then cancel. The steps are outlined in the following example.

Example 7

State the restrictions and simplify:

$$\frac{3x+6}{x^2+x-2}$$

Solution:

Step 1: Completely factor the numerator and denominator.

$$\frac{3x+6}{x^2+x-2}=\frac{3(x+2)}{(x-1)(x+2)}$$

Step 2: Determine the restrictions to the domain. To do this, set the denominator equal to 0 and solve.

$$(x-1)(x+2)=0$$

$$\begin{array}{lll} x-1=0 & \text{or} & x+2=0 \\ x=1 & & x=-2 \end{array}$$

The domain consists of all real numbers except -2 and 1.

Step 3: Cancel common factors, if any.

$$\begin{aligned} \frac{3x+6}{x^2+x-2} &= \frac{3\cancel{(x+2)}}{(x-1)\cancel{(x+2)}} \\ &= \frac{3}{x-1} \end{aligned}$$

Answer

$\frac{3}{x-1}$, where $x \neq 1$ and $x \neq -2$

Example 8

State the restrictions and simplify.

Solution:
First, factor the numerator and denominator.

Any value of x that results in a value of 0 in the denominator is a restriction. By inspection, we determine that the domain consists of all real numbers except 4 and 3. Next, cancel common factors.

$$\begin{aligned} &= \frac{\cancel{(x-3)}(x+10)}{(x-4)\cancel{(x-3)}} \\ &= \frac{x+10}{x-4} \end{aligned}$$

Answer

$\frac{x+10}{x-4}$, where $x \neq 3$ and $x \neq 4$

It is important to remember that we can only cancel factors of a product. A common mistake is to cancel terms. Do not do this!

$\frac{\cancel{x^2} + 7x - 30}{\cancel{x^2} - 7x + 12}$ Incorrect

$\frac{\cancel{x} + 10}{\cancel{x} - 4}$ Incorrect

$\frac{2\,\cancel{x-1}}{\cancel{x-1}}$ Incorrect

Try It

State the restrictions and simplify: $\frac{x^2 - 16}{5x^2 - 20x}$

Show Solution

$\frac{x+4}{5x}$, where $x \neq 0$ and $x \neq 4$

In some examples, we will make a broad assumption that the denominator is nonzero. When we make that assumption, we do not need to determine the restrictions.

Example 9

Simplify. *(Assume all denominators are nonzero.)*

$$\frac{xy + y^2 - 3x - 3y}{x^2 - y^2}$$

Solution:
Factor the numerator by grouping. Factor the denominator using the formula for a difference of squares.

Next, cancel common factors.

$$= \frac{\cancel{(x+y)}\,(y-3)}{\cancel{(x+y)}\,(x-y)}$$
$$= \frac{y-3}{x-y}$$

Answer

$$\frac{y-3}{x-y}$$

Opposite Binomial Factors or Factoring Out -1

Recall that the opposite of the real number a is $-a$. Similarly, we can define the opposite of a polynomial P to be $-P$. We first consider the opposite of the binomial $a-b$:

$$-(a-b) = -a+b = b-a$$

This leads us to the opposite binomial property:

$$-(a-b) = (b-a)$$

This is equivalent to factoring out a -1.

$$(b-a) = -1(a-b)$$

If $a \neq b$, then we can divide both sides by $(a-b)$ and obtain the following:

$$\frac{b-a}{a-b} = -1$$

Example 10

State the restrictions and simplify:

$$\frac{3-x}{x-3}$$

Solution:

By inspection, we can see that the denominator is 0 if $x = 3$. Therefore, 3 is the restriction to the domain. Apply the opposite binomial property to the numerator and then cancel.

$$\begin{aligned}\frac{3-x}{x-3} &= \frac{-1(x-3)}{x-3} \\ &= \frac{-1 \cdot \cancel{(x-3)}}{\cancel{(x-3)}} \\ &= -1\end{aligned}$$

Answer

-1, where $x \neq 3$

Since addition is commutative, we have

$$(a+b) = (b+a)$$

or

$$\frac{b+a}{a+b} = 1$$

Take care not to confuse this with the opposite binomial property. Also, it is important to recall that

$$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}$$

In other words, show a negative fraction by placing the negative sign in the numerator, in front of the fraction bar, or in the denominator. Generally, negative denominators are avoided.

Example 11

Simplify and state the restrictions:

$$\frac{4-x^2}{x^2+3x-10}$$

Solution:

Begin by factoring the numerator and denominator.

$$\begin{aligned}\frac{4-x^2}{x^2+3x-10} &= \frac{(2+x)(2-x)}{(x-2)(x+5)} \\ &= \frac{(2+x)\cdot(-1)\,\cancel{(x-2)}}{\cancel{(x-2)}\,(x+5)} \\ &= \frac{(2+x)\cdot(-1)}{x+5} \\ &= -\frac{x+2}{x+5}\end{aligned}$$

The restrictions are $x \neq 2$ and $x \neq 5$.

Apply the opposite binomial property, then cancel.

Answer

$-\frac{x+2}{x+5}$, where $x \neq 2$ and $x \neq -5$

Try It

Simplify and state the restrictions: $\frac{2x^2-7x-15}{25-x^2}$

Show Solution

$\frac{-2x+3}{x+5}$, where $x \neq \pm 5$

Key Takeaways

- Rational expressions usually are not defined for all real numbers. The real numbers that give a value of 0 in the denominator are not part of the domain. These values are called restrictions.
- Simplifying rational expressions is similar to simplifying fractions. First, factor the numerator and denominator and then cancel the common factors. Rational expressions are simplified if there are no common factors other than 1 in the numerator and the denominator.
- Simplified rational expressions are equivalent for values in the domain of the original expression. Be sure to state the restrictions if the denominators are not assumed to be nonzero.
- Use the opposite binomial property to cancel binomial factors that involve subtraction. Use $-(a-b)=b-a$ to replace factors that will then cancel. Do not confuse this with factors that involve addition, such as $(a+b)=(b+a)$.

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6.7 Exercises

6.1 Introduction to Factoring

Give the prime factorization of each number and determine the GCF.

1. $18, 24$
2. $45, 75$
3. $72, 60$
4. $168, 175$
5. $144, 245$
6. $15, 50, 60$
7. $14, 63, 70$
8. $12, 48, 125$
9. $60, 72, 900$
10. $252, 336, 360$

Determine the GCF of all the terms.

11. $15x, 30$
12. $14x, 21$
13. $45x^4, 8x^3$
14. $36x^5, 35y^2$
15. $6x, 27x, 36x$

16. $12x^3, 4x^2, 6x$

17. $12x^2y, 60xy^3$

18. $7ab^2, 2a^2b, 3a^3b^3$

19. $6a^2b^2, 18a^3b^2, 9ab^2$

20. $15x(x+2), 9(x+2)$

21. $20x(2x-1), 16(2x-1)$

22. $20x^3(x+y)^5, 10x^5(x+y)^2$

Given the GCF, determine the missing factor.

23. $25x^2 + 10x = 5x(?)$

24. $12y^5 + 7y^2 = y^2(?)$

25. $22x^4 - 121x^2 + 11x = 11x(?)$

26. $30y^3 - 45y^2 - 3y = 3y(?)$

27. $36a^5b^7 - 60a^6b^5 = 12a^5b^5(?)$

28. $24x^2y + 48xy^2 - 12xy = 12xy(?)$

Factor out the GCF.

29. $4x - 8$

30. $27x - 9$

31. $3x - 18$

32. $5x - 10$

33. $25x - 16$

34. $72x - 35$

35. $15x^2 + 30x$

36. $14a^2 - 7a$

37. $30a^5 - 10a^2$

38. $8x^4 - 16x^3$

39. $5x^6 + x^3$

40. $3x^7 - 9x^5$

41. $18a^2 + 30a - 6$

42. $24a^2 - 36a - 12$

43. $27x^3 - 6x^2 + 3x$

44. $8x^3 - 12x^2 + 2x$

45. $9x^4 + 18x^3 - 3x^2$

46. $12y^4 - 16y^3 + 20y^2$

47. $7x^5 - 21x^3 - 14x^2 + 28x$

48. $36y^{10} + 12y^8 - 18y^4 - 6y^3$

49. $12x^5y^2 - 8x^3y$

50. $125a^8b^4c^3 - 25a^2b^3c^3$

51. $6x^4y^3 - 4x^3y^2 + 8x^2y$

52. $15x^4y^2 - 30x^3y^3 + 15x^2y^4$

53. $81x^7y^6z^2 - 18x^2y^8z^4 + 9x^2y^5z^2$

54. $4x^5y^4z^9 + 26x^5y^3z^4 - 14x^6y^8z^5$

55. $2x(x - 3) + 5(x - 3)$

56. $3x(2x + 1) - 4(2x + 1)$

57. $5x(5x + 2) - (5x + 2)$

58. $2x(3x+4)+(3x+4)$

59. $x^2(4x-7)-5(4x-7)$

60. $(x+6)-3x^2(x+6)$

61. $(a+b)^2-3a(a+b)^2$

62. $(ab+2)^3+3ab(ab+2)^3$

63. $7x(x+7)^5+14x^2(x+7)^5$

64. $36x^5(3x+2)^4-12x^3(3x+2)^4$

Are the following factored correctly? Check by multiplying.

65. $4x^2-16x=4x(x-4)$

66. $3a^3-3a=3a(a^2)$

67. $3x^3-5x^6=x^3(3-x^2)$

68. $5x^3-10x^4+15x^5=5x^3(1-2x+3x^2)$

69. $x^3-x^2+x=x(x^2-x)$

70. $12x^4y^3-16x^5y^2+8x^6y^7=4x^4y^2(3y-4x+2x^2y^5)$

Use polynomial long division to show that the given factor divides the polynomial evenly.

71. Show that $(x-1)$ is a factor of $(2x^3-5x^2+4x-1)$.

72. Show that $(x+3)$ is a factor of $(3x^3+7x^2-4x+6)$.

73. Show that $(3x-2)$ is a factor of $(3x^3+4x^2-7x+2)$.

74. Show that $(2x+1)$ is a factor of $(2x^3-5x^2+x+2)$.

75. The height in feet of an object tossed into the air is given by the function $h(t)=16^2+32t$, where t is the time in seconds after it is tossed. Write the function in factored form.

76. The height in feet of an object dropped from a 16-foot ladder is given by the function $h(t)=16^2+16$, where t is the time in seconds after it is tossed. Write the function in factored

form.

77. The surface area of a cylinder is given by the formula $SA = 2\pi r^2 + 2\pi rh$, where r represents the radius of the base and h is the height of the cylinder. Express this formula in factored form.

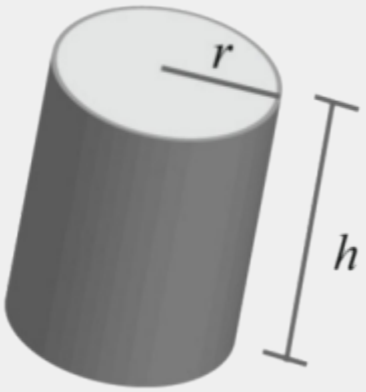

Figure 6.7.1.

78. The surface area of a cone is given by the formula $SA = \pi r^2 + \pi rs$, where $r$represents the radius of the base and s represents the slant height. Express this formula in factored form.

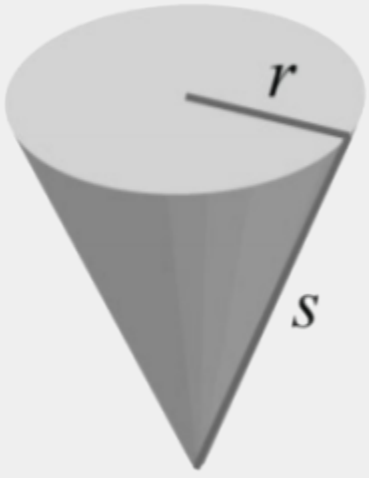

Figure 6.7.2.

Factor by grouping.

79. $x^2 - 10x + 2x - 20$

80. $x^2 - 6x - 3x + 18$

81. $x^3 + 2x^2 + 2x + 4$

82. $x^3 - 3x^2 + 5x - 15$

83. $x^3 + 7x^2 - 2x - 14$

84. $2x^3 + 2x^2 - x - 1$

85. $x^3 - 5x^2 + 4x - 20$

86. $6x^3 - 3x^2 + 2x - 1$

87. $9x^3 - 6x^2 - 3x + 2$

88. $2x^4 - x^3 - 6x + 3$

89. $x^5 + x^3 + 2x^2 + 2$

90. $6x^5 - 4x^3 - 9x^2 + 6$

91. $3a^3b + 3ab^2 + 2a^2 + 2b$

92. $2a^3 + 2ab^3 - 3a^2b - 3b^4$

93. $2a^3 - a^2b^2 - 2ab + b^3$

94. $a^4 - 3a^3b^2 + ab^2 - 3b^4$

95. $3a^2b^4 - 6b^3 - a^2b + 2$

96. $3x^3 + 2y^3 + x^3y^3 + 6$

97. $-3x^3 - 5y^3 + x^3y^3 + 15$

98. $2x^3y^3 + 2 - y^3 - 4x^3$

99. $3x^2 - y^3 + xy^2 - 3xy$

100. $2x^2 + y^3 - 2xy - xy^2$

Factor out the GCF first and then factor by grouping.

101. $5x^2 - 35x - 15x + 105$

102. $12x^2 - 30x - 12x + 30$

103. $2x^3 + 6x^2 - 10x - 30$

104. $6x^3 - 3x^2 - 42x + 21$

105. $4x^4 + 4x^3 - 12x^2 - 12x$

106. $-9x^4 + 6x^3 - 45x^2 + 30x$

107. $-12x^5 + 4x^4 + 6x^3 - 2x^2$

108. $24x^5 - 36x^4 + 8x^3 - 12x^2$

109. $24a^3b^2 - 60a^3b + 40ab^2 - 100ab$

110. $a^4b^2 - 2a^3b^3 + a^2b^3 - 2ab^4$

Show Solutions

1. $18 = 2 \cdot 3^2, 24 = 2^3 \cdot 3, \text{GCF} = 6$
2. $45 = 3^2 \cdot 5, 75 = 3 \cdot 5^2, \text{GCF} = 15$
3. $72 = 2^3 \cdot 3^2, 60 = 2^2 \cdot 3 \cdot 5, \text{GCF} = 12$
4. $168 = 2^3 \cdot 3 \cdot 7, 175 = 5^2 \cdot 7, \text{GCF} = 7$
5. $144 = 2^4 \cdot 3^2, 245 = 5 \cdot 7^2, \text{GCF} = 1$
6. $15 = 3 \cdot 5, 50 = 2 \cdot 5^2, 60 = 2^2 \cdot 3 \cdot 5, \text{GCF} = 5$

7. $14 = 2 \cdot 7, 63 = 3^2 \cdot 7, 70 = 2 \cdot 5 \cdot 7, \text{GCF} = 7$
8. $12 = 2^2 \cdot 3, 48 = 2^4 \cdot 3, 125 = 5^3, \text{GCF} = 1$
9. $60 = 2^2 \cdot 3 \cdot 5, 72 = 2^3 \cdot 3^2, 900 = 2^2 \cdot 3^2 \cdot 5^2, \text{GCF} = 12$
10. $252 = 2^2 \cdot 3^2 \cdot 7, 336 = 2^4 \cdot 3 \cdot 7, 360 = 2^3 \cdot 3^2 \cdot 5, \text{GCF} = 12$
11. 15
12. 7
13. x^3
14. 1
15. $3x$
16. $2x$
17. $12xy$
18. ab
19. $3ab^2$
20. $3(x + 2)$
21. $4(2x - 1)$
22. $10x^3(x + y)^2$
23. $5x + 2$
24. $12y^3 + 7$
25. $2x^3 - 11x + 1$
26. $10y^2 - 15y - 1$

27. $3b^2 - 5a$
28. $2x + 4y - 1$
29. $4(x - 2)$
30. $9(3x - 1)$
31. $3(x - 6)$
32. $5(x - 2)$
33. Prime
34. Prime
35. $15x(x + 2)$
36. $7a(2a - 1)$
37. $10a^2(3a^3 - 1)$
38. $8x^3(x - 2)$
39. $x^3(5x^3 + 1)$
40. $3x^5(x^2 - 3)$
41. $6(3a^2 + 5a - 1)$
42. $12(a^2 - 3a - 1)$
43. $3x(9x^2 - 2x + 1)$
44. $2x(4x^2 - 6x + 1)$
45. $3x^2(3x^2 + 6x - 1)$

46. $4y^2(3y^2 - 4y + 5)$

47. $7x(x^4 - 3x^2 - 2x + 4)$

48. $6y^3(6y^7 + 2y^5 - 3y - 1)$

49. $4x^3y(3x^2y - 2)$

50. $25a^2b^3c^3(5a^6b - 1)$

51. $2x^2y(3x^2y^2 - 2xy + 4)$

52. $15x^2y^2(x^2 - 2xy + y^2)$

53. $9x^2y^5z^2(9x^5y - 2y^3z^2 + 1)$

54. $2x^5y^3z^4(2yz^5 + 13 - 7xy^5z)$

55. $(x - 3)(2x + 5)$

56. $(2x + 1)(3x - 4)$

57. $(5x + 2)(5x - 1)$

58. $(3x + 4)(2x + 1)$

59. $(4x - 7)(x^2 - 5)$

60. $(x + 6)(1 - 3x^2)$

61. $(a + b)^2(1 - 3a)$

62. $(ab + 2)^3(1 + 3ab)$

63. $7x(x + 7)^5(1 + 2x)$

64. $12x^3(3x + 2)^4(3x^2 - 1)$

65. Yes

66. No

67. No

68. Yes

69. No

70. Yes

71. $(2x^3 - 5x^2 + 4x - 1) \div (x - 1) = 2x^2 - 3x + 1$

72. $(3x^3 + 7x^2 - 4x + 6) \div (x + 3) = 3x^2 - 2x + 2$

73. $(3x^3 + 4x^2 - 7x + 2) \div (3x - 2) = x^2 + 2x - 1$

74. $(2x^3 - 5x^2 + x + 2) \div (2x + 1) = x^2 - 3x + 2$

75. $h(t) = -16t(t - 2)$

76. $h(t) = -16(t - 1)(t + 1)$

77. $SA = 2\pi r(r + h)$

78. $SA = \pi r(r + s)$

79. $(x - 10)(x + 2)$

80. $(x - 6)(x - 3)$

81. $(x + 2)(x^2 + 2)$

82. $(x - 3)(x^2 + 5)$

83. $(x + 7)(x^2 - 2)$

84. $(x + 1)(2x^2 - 1)$

85. $(x-5)(x^2+4)$

86. $(2x-1)(3x^2+1)$

87. $(3x-2)(3x^2-1)$

88. $(2x-1)(x^3+3)$

89. $(x^2+1)(x^3+2)$

90. $(3x^2-2)(2x^3-3)$

91. $(a^2+b)(3ab+2)$

92. $(a^2+b^3)(2a-3b)$

93. $(a^2-b)(2a-b^2)$

94. $(a^3+b^2)(a-3b^2)$

95. $(3b^3-1)(a^2b-2)$

96. $(x^3+2)(y^3+3)$

97. $(x^3-5)(y^3-3)$

98. $(2x^3-1)(y^3-2)$

99. $(x-y)(3x+y^2)$

100. $(x-y)(2x-y^2)$

101. $5(x-7)(x-3)$

102. $6(2x-5)(x-1)$

103. $2(x+3)(x^2-5)$

104. $3(2x-1)(x^2-7)$

105. $4x(x+1)(x^2-3)$

106. $-3x(3x-2)(x^2+5)$

107. $-2x^2(3x-1)(2x^2-1)$

108. $4x^2(2x-3)(3x-1)$

109. $4ab(3a^2+5)(2b-5)$

110. $ab(a-b)(a^2b-2b^2)$

6.2 Factoring Special Binomials

Factor completely.

1. x^2-9

2. x^2-100

3. y^2-36

4. y^2-144

5. x^2+4

6. x^2-5

7. m^2-81

8. m^2-64

9. $16x^2 - 9$
10. $25x^2 - 4$
11. $144x^2 - 1$
12. $9x^2 - 121$
13. $4y^2 - 81$
14. $100y^2 - 49$
15. $9 - 4x^2$
16. $100 - x^2$
17. $1 - y^2$
18. $25 - 9y^2$
19. $-3x^2 + 75$
20. $-16x^2 + 25$
21. $2x^2 - 72$
22. $20x^3 - 45x$
23. $-48x + 27x^3$
24. $36x^2 - 100$
25. $x^2 - y^2$
26. $25x^2 - 9y^2$
27. $a^2 - 4b^2$
28. $a^2b^2 - 36$
29. $4x^2y^2 - 1$
30. $x^2y^2 - 25$

31. $2a^3 - 8ab^2$

32. $3a^3b^4 - 75ab^2$

33. $-100xy^3 + 4x^3y$

34. $-18x^3y^3 + 32xy$

35. $(x+1)^2 - y^2$

36. $x^2 - (y-2)^2$

37. $(x-3)^2 - (y+3)^2$

38. $(x^2+2)^2 - (x-1)^2$

39. $(x^2-1)^2 - (2x+3)^2$

40. $x^4 - 1$

41. $x^4 - y^4$

42. $16x^4 - 81$

43. $a^4b^4 - 16$

44. $a^4 - 16b^4$

45. $x^8 - 1$

46. $25x^8 - 1$

47. $a^8 - b^2$

48. $a^4 - 9$

49. $x^8 - y^8$

50. $81x^8 - 1$

51. The height of a projectile dropped from a 64-foot tower is given by the function $h(t) = -16t^2 + 64$, where t represents the time in seconds after it is dropped. Rewrite this function in factored form. (Hint: Factor out -16 first.)

52. The height of a projectile dropped from a 36-foot tower is given by the function $h(t) = -16t^2 + 36$, where t represents the time in seconds after it is dropped. Rewrite this function in factored form.

53. $x^3 - 1$

54. $x^3 + 1$

55. $y^3 - 27$

56. $y^3 - 8$

57. $8y^3 + 1$

58. $27y^3 - 1$

59. $64a^3 - 125$

60. $8a^3 + 27$

61. $a^3 + 216$

62. $a^3 - 125$

63. $x^3 - 1000$

64. $343m^3 - 1$

65. $512n^3 + 1$

66. $8x^3 + 343$

67. $40y^3 - 135$

68. $27y^3 + 729$

69. $27y^3 - 64$

70. $x^3 + 3$

71. $5x^3 + 1$

72. $1 - y^3$

73. $27 - 1000y^3$

74. $343 + 125a^3$

75. $x^3 - y^3$

76. $x^3 + y^3$

77. $x^3y^3 + 125$

78. $8x^3y^3 - 27$

79. $27a^3 - 8b^3$

80. $16x^3 - 250y^3$

81. $128x^3 + 2y^3$

82. $54x^3 - 2y^3$

83. $3a^4b - 24ab^4$

84. $a^3b^3c^3 - 1$

85. $(x + 1)^3 - 8$

86. $8x^3 - (x - 5)^3$

87. $(x - 2)^3 + (x + 2)^3$

88. $(a + 3)^3 + (b - 3)^3$

89. $x^6 - 1$

90. $x^6 + 1$

91. $64a^6 - 1$

92. $a^6 - 81b^6$

93. $x^6 - y^6$

94. $x^6 + y^6$

Show Solutions

1. $(x+3)(x-3)$
2. $(x+10)(x-10)$
3. $(y+6)(y-6)$
4. $(y+12)(y-12)$
5. Prime
6. Prime
7. $(m+9)(m-9)$
8. $(m+8)(m-8)$
9. $(4x+3)(4x-3)$
10. $(5x+2)(5x-2)$
11. $(12x+1)(12x-1)$
12. $(3x+11)(3x-11)$
13. $(2y+9)(2y-9)$
14. $(10y+7)(10y-7)$
15. $(3+2x)(3-2x)$
16. $(10+x)(10-x)$
17. $(1+y)(1-y)$

18. $(5+3y)(5-3y)$
19. $-3(x+5)(x-5)$
20. $(5+4x)(5-4x)$
21. $2(x+6)(x-6)$
22. $5x(2x+3)(2x-3)$
23. $3x(3x+4)(3x-4)$
24. $4(3x+5)(3x-5)$
25. $(x+y)(x-y)$
26. $(5x+3y)(5x-3y)$
27. $(a+2b)(a-2b)$
28. $(ab+6)(ab-6)$
29. $(2xy+1)(2xy-1)$
30. $(xy+5)(xy-5)$
31. $2a(a+2b)(a-2b)$
32. $3ab^2(ab+5)(ab-5)$
33. $4xy(x+5y)(x-5y)$
34. $-2xy(3xy+4)(3xy-4)$
35. $(x+1+y)(x+1-y)$
36. $(x+y-2)(x-y+2)$
37. $(x-y-6)(x+y)$

38. $(x^2+x+1)(x^2-x+3)$

39. $(x^2+2x+2)(x^2-2x-4)$

40. $(x^2+1)(x+1)(x-1)$

41. $(x^2+y^2)(x+y)(x-y)$

42. $(4x^2+9)(2x+3)(2x-3)$

43. $(a^2b^2+4)(ab+2)(ab-2)$

44. $(a^2+2b^2)(a+2b)(a-2b)$

45. $(x^4+1)(x^2+1)(x+1)(x-1)$

46. $(5x^4+1)(5x^4-1)$

47. $(a^4+b)(a^4-b)$

48. $(a^2+3)(a+\sqrt{3})(a-\sqrt{3})$

49. $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$

50. $(9x^4+1)(3x^2+1)(3x^2-1)$

51. $h(t)=-16(t+2)(t-2)$

52. $h(t)=-(4t+6)(4t-6)$

53. $(x-1)(x^2+x+1)$

54. $(x+1)(x^2-x+1)$

55. $(y-3)(y^2+3y+9)$

56. $(y-2)(y^2+2y+4)$

57. $(2y+1)(4y^2-2y+1)$

58. $(3y-1)(9y^2+3y+1)$

59. $(4a-5)(16a^2+20a+25)$

60. $(2a+3)(4a^2-6a+9)$

61. $(a+6)(a^2-6a+36)$

62. $(a-5)(a^2+5a+25)$

63. $(x-10)(x^2+10x+100)$

64. $(7m-1)(49m^2+7m+1)$

65. $(8n+1)(64n^2-8n+1)$

66. $(2x+7)(4x^2-14x+49)$

67. $5(2y-3)(4y^2+6y+9)$

68. $27(y+3)(y^2-3y+9)$

69. $(3y-4)(9y^2+12y+16)$

70. Prime

71. Prime

72. $(1-y)(1+y+y^2)$

73. $(3-10y)(9+30y+100y^2)$

74. $(7+5a)(49-35a+25a^2)$

75. $(x-y)(x^2+xy+y^2)$

76. $(x+y)(x^2-xy+y^2)$

77. $(xy+5)(x^2y^2-5xy+25)$

78. $(2xy-3)(4x^2y^2+6xy+9)$

79. $(3a-2b)(9a^2+6ab+4b^2)$

80. $2(2x-5y)(4x^2+10xy+25y^2)$

81. $2(4x+y)(16x^2-4xy+y^2)$

82. $2(3x-y)(9x^2+3xy+y^2)$

83. $3ab(a-2b)(a^2+2ab+4b^2)$

84. $(abc-1)(a^2b^2c^2+abc+1)$

85. $(x-1)(x^2+4x+7)$

86. $(x+5)(7x^2-7x+25)$

87. $2x(x^2+12)$

88. $(a+b)(a^2-ab+b^2+9a-9b+27)$

89. $(x+1)(x^2+x+1)(x-1)(x^2+x+1)$

90. $(x^2+1)(x^4-x^2+1)$

91. $(2a+1)(4a^2-2a+1)(2a-1)(4a^2+2a+1)$

92. $(a^3+9b^3)(a^3-9b^3)$

93. $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

94. $(x^2+y^2)(x^4-x^2y^2+y^4)$

6.3 Factoring Trinomials of the Form $x^2 + bx + c$

Are the following factored correctly? Check by multiplying.

1. $x^2 + 5x - 6 = (x + 2)(x + 3)$
2. $x^2 + 6x + 16 = (x + 8)(x - 2)$
3. $y^2 + 2y - 8 = (y + 4)(y - 2)$
4. $y^2 - 10y + 21 = (y - 3)(y - 7)$
5. $a^2 - 10a + 25 = (a - 5)^2$
6. $a^2 + 6a + 9 = (a - 3)^2$
7. $x^2 + 10x - 25 = (x + 5)(x - 5)$
8. $x^2 + 5x + 14 = (x - 2)(x + 7)$
9. $y^2 + 50y - 600 = (y + 60)(y - 10)$
10. $y^2 - 3y + 2 = (y - 2)(y - 1)$

Factor.

11. $x^2 + 6x + 8$
12. $x^2 + 4x + 3$
13. $x^2 + 3x + 2$
14. $x^2 + x - 2$
15. $x^2 + 3x - 10$
16. $x^2 - 2x - 35$
17. $x^2 - 13x + 12$

18. $x^2 - 15x + 36$

19. $x^2 - 12x + 36$

20. $x^2 + 18x + 81$

21. $x^2 - 2x + 1$

22. $x^2 - 18x + 81$

23. $x^2 + 5x + 5$

24. $x^2 - 4x + 6$

25. $x^2 - 20x + 91$

26. $x^2 + 20x + 91$

27. $x^2 - 2x - 48$

28. $x^2 + 16x + 48$

29. $x^2 + 22x + 48$

30. $x^2 + 22x - 48$

31. $y^2 + 7y + 12$

32. $y^2 + 8y - 20$

33. $y^2 - 16y + 60$

34. $y^2 - 31y - 32$

35. $a^2 - 11a - 42$

36. $a^2 - 14a - 51$

37. $a^2 + 26a + 25$

38. $a^2 - 22a + 120$

39. $a^2 + 4a - 1$

40. $a^2 - 6a + 2$

41. $y^2 - 14y + 40$

42. $y^2 - 4y - 96$

43. $x^2 - 2xy + y^2$

44. $x^2 + 2xy + y^2$

45. $x^2 - 16xy + 15y^2$

46. $x^2 - 4xy - 32y^2$

47. $x^2 + 2xy - 15y^2$

48. $x^2 - 12xy + 32y^2$

49. $x^2y^2 - 6xy + 9$

50. $x^2y^2 + 25xy - 26$

51. $a^2 + 4ab + 4b^2$

52. $a^2 - 19ab - 20b^2$

53. $a^2 - ab - 12b^2$

54. $a^2 - 10ab - 56b^2$

55. $a^2b^2 - 2ab - 15$

56. $a^2b^2 - 10ab + 24$

57. The area of a square is given by the function $A(x) = x^2 - 14x + 49$, where x is measured in meters. Rewrite this function in factored form.

58. The area of a square is given by the function $A(x) = x^2 + 16x + 64$, where x is measured in meters. Rewrite this function in factored form.

Factor using the AC method.

59. $x^2 + 5x - 14$

60. $x^2 + 2x - 48$

61. $x^2 - 9x + 8$

62. $x^2 - 14x + 24$

63. $x^2 - x - 72$

64. $x^2 - x - 90$

65. $y^2 - 8y + 16$

66. $y^2 + 16y + 64$

67. $x^2 + 4x + 12$

68. $x^2 + 5x - 8$

69. $x^2 + 3xy - 18y^2$

70. $x^2 - 15xy + 50y^2$

Show Solutions

1. No
2. No
3. Yes
4. Yes
5. Yes
6. No
7. No

8. No
9. Yes
10. Yes
11. $(x+2)(x+4)$
12. $(x+1)(x+3)$
13. $(x+1)(x+2)$
14. $(x+2)(x-1)$
15. $(x+5)(x-2)$
16. $(x-7)(x+5)$
17. $(x-1)(x-12)$
18. $(x-3)(x-12)$
19. $(x-6)^2$
20. $(x+9)^2$
21. $(x-1)^2$
22. $(x-9)^2$
23. Prime
24. Prime
25. $(x-7)(x-13)$
26. $(x+7)(x+13)$
27. $(x-8)(x+6)$

28. $(x+4)(x+12)$
29. $(x+4)(x+18)$
30. $(x+24)(x-2)$
31. $(y+3)(y+4)$
32. $(y+10)(y-2)$
33. $(y-6)(y-10)$
34. $(y-32)(y+1)$
35. $(a-14)(a+3)$
36. $(a-17)(a+3)$
37. $(a+1)(a+25)$
38. $(a-10)(a-12)$
39. Prime
40. Prime
41. $(y-10)(y-4)$
42. $(y-12)(y+8)$
43. $(x-y)^2$
44. $(x+y)^2$
45. $(x-15y)(x-y)$
46. $(x-8y)(x+4y)$
47. $(x+5y)(x-3y)$

48. $(x-8y)(x-4y)$
49. $(xy-3)^2$
50. $(xy+26)(xy-1)$
51. $(a+2b)^2$
52. $(a-20b)(a+b)$
53. $(a-4b)(a+3b)$
54. $(a-14b)(a+4b)$
55. $(ab+3)(ab-5)$
56. $(ab-6)(ab-4)$
57. $A(x)=(x-7)^2$
58. $A(x)=(x+8)^2$
59. $(x+7)(x-2)$
60. $(x+8)(x-6)$
61. $(x-8)(x-1)$
62. $(x-12)(x-2)$
63. $(x-9)(x+8)$
64. $(x-10)(x+9)$
65. $(y-4)^2$
66. $(y+8)^2$
67. Prime

68. Prime

69. $(x+6y)(x-3y)$

70. $(x-10y)(x-5y)$

6.4 Factoring Trinomials of the Form ax^2+bx+c

Factor.

1. $3x^2-14x-5$
2. $5x^2+7x+2$
3. $2x^2+5x-3$
4. $2x^2+13x-7$
5. $2x^2+9x-5$
6. $7x^2+20x-3$
7. $7x^2-46x-21$
8. $3x^2+x-2$
9. $5x^2+34x-7$
10. $5x^2-28x-12$
11. $9x^2-12x+4$
12. $4x^2-20x+25$

13. $49x^2 + 14x + 1$

14. $25x^2 - 10x + 1$

15. $2x^2 + 7x + 16$

16. $6x^2 - 19x - 10$

17. $27x^2 + 66x - 16$

18. $12x^2 - 88x - 15$

19. $12y^2 - 8y + 1$

20. $16y^2 - 66y - 27$

21. $9x^2 - 12xy + 4y^2$

22. $25x^2 + 40x + 16$

23. $15x^2 - 26xy + 8y^2$

24. $12a^2 - 4ab - 5b^2$

25. $4x^2y^2 + 16xy - 9$

26. $20x^2y^2 + 4xy - 7$

27. The area of a rectangle is given by $A(x) = 3x^2 - 10x + 3$, where x is measured in meters. Rewrite this function in factored form.

28. The area of a rectangle is given by $A(x) = 10x^2 - 59x - 6$, where x is measured in meters. Rewrite this function in factored form.

29. $6x^2 - 20x - 16$

30. $45x^2 + 27x - 18$

31. $20x^2 - 20x + 5$

32. $3x^2 + 39x - 90$

33. $16x^2 + 26x - 10$

34. $54x^2 - 15x + 6$

35. $45x^2 - 45x - 20$

36. $90x^2 + 300x + 250$

37. $40x^2 - 36xy + 8y^2$

38. $24a^2b^2 + 18ab - 81$

39. $6x^2y^2 + 46xy + 28$

40. $2x^5 + 44x^4 + 144x^3$

41. $5x^3 - 65x^2 + 60x$

42. $15a^4b^2 - 25a^3b - 10a^2$

43. $6a^4b + 2a^3b^2 - 4a^2b^3$

44. $20a^3b^2 - 60a^2b^3 + 45ab^4$

Factor out -1 and then factor further.

45. $-x^2 - 4x + 21$

46. $-x^2 + x + 12$

47. $-x^2 + 15x - 56$

48. $-x^2 + x + 72$

49. $-y^2 + 10y - 25$

50. $-y^2 - 16y - 64$

51. $36 - 9a - a^2$

52. $72 - 6a - a^2$

53. $32 + 4x - x^2$

54. $200 + 10x - x^2$

Factor out a negative common factor first, and then factor further if possible.

55. $-8x^2 + 6x + 9$

56. $-4x^2 + 28x - 49$

57. $-18x^2 - 6x + 4$

58. $2 + 4x - 30x^2$

59. $15 + 39x - 18x^2$

60. $90 + 45x - 10x^2$

61. $-2x^2 + 26x + 28$

62. $-18x^3 - 51x^2 + 9x$

63. $-3x^2y^2 + 18xy^2 - 24y^2$

64. $-16a^4 + 16a^3b - 4a^2b^2$

65. The height is $h(t) = -16t^2 + 64t + 80$. Rewrite in factored form.

66. The height is $h(t) = -16t^2 + 64t + 192$. Rewrite in factored form.

Factor using the AC method.

67. $2x^2 + 5x - 7$

68. $3x^2 + 7x - 10$

69. $4x^2 - 25x + 6$

70. $16x^2 - 38x - 5$

71. $6x^2 + 23x - 18$

72. $8x^2 + 10x - 25$

73. $4x^2 + 28x + 40$

74. $-6x^2 - 3x + 30$

75. $12x^2 - 56xy + 60y^2$

76. $20x^2 + 80xy + 35y^2$

Show Solutions

1. $(x - 5)(3x + 1)$
2. $(5x + 2)(x + 1)$
3. $(x + 3)(2x - 1)$
4. $(2x - 1)(x + 7)$
5. $(x + 5)(2x - 1)$
6. $(7x - 1)(x + 3)$
7. $(x - 7)(7x + 3)$
8. $(3x - 2)(x + 1)$
9. $(5x - 1)(x + 7)$
10. $(5x + 2)(x - 6)$
11. $(3x - 2)^2$
12. $(2x - 5)^2$
13. $(7x + 1)^2$
14. $(5x - 1)^2$
15. Prime
16. $(3x - 5)(2x + 2)$

17. $(3x+8)(9x-2)$
18. $(6x+1)(2x-15)$
19. $(6y-1)(2y-1)$
20. $(8y+3)(2y-9)$
21. $(3x-2y)^2$
22. $(5x+4)^2$
23. $(3x-4y)(5x-2y)$
24. $(2a+b)(6a-5b)$
25. $(2xy-1)(2xy+9)$
26. $(2xy-1)(10xy+7)$
27. $A(x)=(3x-1)(x-3)$
28. $A(x)=(10x+1)(x-6)$
29. $2(x-4)(3x+2)$
30. $9(5x-2)(x+1)$
31. $5(2x-1)^2$
32. $3(x+15)(x-2)$
33. $2(8x-5)(x+1)$
34. $3(18x^2-5x+2)$
35. $5(3x-4)(3x+1)$
36. $10(3x+5)^2$

37. $4(5x-2y)(2x-y)$
38. $3(2ab+9)(4ab-3)$
39. $2(xy+7)(3xy+2)$
40. $2x^3(x+4)(x+18)$
41. $5x(x-12)(x-1)$
42. $5a^2(3a^2b^2-5ab-2)$
43. $2a^2b(3a-2b)(a+b)$
44. $5ab^2(4a^2-12ab+9b^2)$
45. $-1(x-3)(x+7)$
46. $-1(x-4)(x+3)$
47. $-1(x-7)(x-8)$
48. $-1(x-9)(x+8)$
49. $-1(y-5)^2$
50. $-1(y+8)^2$
51. $-1(a-3)(a+12)$
52. $-1(a-6)(a+12)$
53. $-1(x-8)(x+4)$
54. $-1(x-20)(x+10)$
55. $-(2x-3)(4x+3)$

56. $-(2x-7)^2$

57. $-2(3x-1)(3x+2)$

58. $-2(3x-1)(5x+1)$

59. $-3(2x-5)(3x+1)$

60. $-5(2x-9)(x+2)$

61. $-2(x-14)(x+1)$

62. $-3x(2x-1)(3x+3)$

63. $-3y^2(x-4)(x-2)$

64. $-4a^2(2a-b)^2$

65. $h(t)=-16(t+1)(t-5)$

66. $h(t)=-16(t-6)(t+2)$

67. $(x-1)(2x+7)$

68. $(x-1)(3x+10)$

69. $(x-6)(4x-1)$

70. $(2x-5)(8x+1)$

71. $(2x+9)(3x-2)$

72. $(4x-5)(2x+5)$

73. $4(x+2)(x+5)$

74. $-3(2x+5)(x-2)$

75. $4(x-3y)(3x-5y)$

76. $5(2x+y)(2x+7y)$

6.5 General Guidelines for Factoring Polynomials

Factor completely.

1. $2x^3y^2 - 12x^4y^3$
2. $18x^3y^3 - 6x^4y^3$
3. $5x^2 + 20x - 25$
4. $4x^2 + 10x - 6$
5. $24x^3 - 9x^2 - 9x$
6. $30x^3 - 65x^2 + 10x$
7. $6x^3 + 27x^2 - 9x$
8. $21x^3 + 49x^2 - 28x$
9. $6x^3 - 9x^2 - 15x + 90$
10. $6x^4 + 24x^3 - 2x^2 - 8x$
11. $x^4 - 6x^3 + 8x - 48$
12. $x^4 - 6x^3 + 27x - 135$
13. $4x^3 - 4x^2 - 9x + 9$
14. $50x^3 + 25x^2 - 50x - 16$

15. $2x^3 + 250$

16. $3x^3 - 81x^2$

17. $2x^3 - 162x^2$

18. $4x^4 - 96$

19. $x^4 + 16$

20. $x^3 + 9$

21. $72 - 2x^2$

22. $6x^4 - 25x^2$

23. $7x^3 - 14x$

24. $36x^2 - 12x + 1$

25. $25x^2 + 10x + 1$

26. $250x^3 + 200x^4 + 40x^5$

27. $-7x^2 + 19x + 6$

28. $-8x^4 + 40x^3 - 50x^2$

29. $a^4 - 16$

30. $10a^4 - 81b^4$

31. $y^2 + y^4 - y - 1$

32. $4y^3 + 2y^4 - 4y^2 - 2y$

33. $3x^5 - 192x^3$

34. $4x^7 + 4x$

35. $4x^2 - 19xy + 12y^2$

36. $16x^2 - 64xy - 27y^2$

37. $6x^5 - 8x^4 - 5x^3 + 8x^2$

38. $4n^4 - 4n^2y^2 - y^2 + y^3$

39. $15a^2 - 4ab - 4b^2$

40. $6a^2 - 25ab + 4b^2$

41. $6x^2 + 5xy + 6y^2$

42. $9x^2 + 5xy - 12y^2$

43. $(3x - 1)^2 - 64$

44. $(x - 5)^2 - (x - 2)^2$

45. $(x + 1)^2 + 8$

46. $(x - 4)^2 - 27$

47. $(2x - 1)^2 - (2x - 1) - 12$

48. $(x - 4)^2 + 5(x - 4) + 6$

49. $a^3b - 10a^2b^2 + 25ab^3$

50. $2a^3b^2 - 12a^2b + 18a$

51. $15a^2b^2 - 57ab - 12$

52. $-60x^3 + 4x^2 + 24x$

53. $-24x^4 + 78x^3 - 54x$

54. $9y^5 - 15y^4 + 4y^3$

55. $36 - 15a - 6x^2$

56. $60x^4y^2 + 8x^2y^2 - 8x^3y^2$

57. $x^4 - 1$

58. $10x^4 - 64$

59. $x^8 - 1$

60. $81x^8 - 1$

61. $x^{16} - 1$

62. $x^{12} - 1$

63. $64x^6 - 216x^4 - 2x^3 + 8x$

64. $4n^4 - 4n^2y^2 - y^2 + y^3$

65. $32y^5 + 82y^3 - 18y - 18$

66. $3a^3 + a^2b - 12ab - 4b^2$

67. $18m^2 - 21mn - 8n^2$

68. $6m^2n^2 + 10mn - 15$

69. The volume of a certain rectangular solid is given by the function $V(x) = x^3 - 2x^2 - 3x$. Write the function in its factored form.

70. The volume of a certain right circular cylinder is given by the function $V(x) = 4\pi x^3 - 4\pi x^2 + \pi r$. Write the function in its factored form.

Show Solutions

1. $2x^3y^2(1 - 6xy)$
2. $6x^3y^3(3 - x)$
3. $5(x - 1)(x + 5)$
4. $2(2x - 1)(x + 3)$

5. $3x(2x-3)(4x+1)$
6. $5x(6x^2-13x+2)$
7. $3x(2x^2+9x-3)$
8. $7x(3x^2+7x-4)$
9. $3(2x^3-3x^2-5x+30)$
10. $2x(x+4)(3x^2-1)$
11. $(x-6)(x+2)(x^2-2x+4)$
12. $(x-5)(x^3-x^2-5x+27)$
13. $(x-1)(2x-3)(2x+3)$
14. Prime
15. $2(x+5)(x^2-5x+25)$
16. $3x^2(x-27)$
17. $2x^2(x-9)(x+9)$
18. $4(x^2+2\sqrt{6})(x^2-2\sqrt{6})$
19. Prime
20. Prime
21. $2(6+x)(6-x)$
22. $x^2(6x^2-25)$
23. $7x(x^2-2)$

24. $(6x-1)^2$

25. $(5x+1)^2$

26. $10x^3(5+2x)^2$

27. $-(x-3)(7x+2)$

28. $-2x^2(2x-5)^2$

29. $(a^2+4)(a+2)(a-2)$

30. Prime

31. $(y^2+1)(y-1)(y+1)$

32. $2y(y+1)(y-1)(y+2)$

33. $3x^3(x-8)(x+8)$

34. $4x(x^6+1)$

35. $(x-4y)(4x-3y)$

36. $(8x+3y)(2x-9y)$

37. $x^2(6x^3-8x^2-5x+8)$

38. $(4n^2-y)(n+y)(n-y)$

39. $(3a-2b)(5a+2b)$

40. $(3a-4b)(2a-b)$

41. Prime

42. $(3x+4y)(3x-3y)$

43. $(3x-9)(3x+7)$

44. $3(2x-7)$

45. Prime

46. Prime

47. $(2x-5)(2x+3)$

48. $(x-2)(x-1)$

49. $ab(a-5b)^2$

50. $2a(ab-3)^2$

51. $3(5ab+1)(ab-4)$

52. $-4x(15x^2-x-6)$

53. $-6x(4x^3-13x^2+9)$

54. $y^3(3y-4)(3y-1)$

55. $-3(2x^2+5a-12)$

56. $4x^2y^2(15x^2-2x+2)$

57. $(x^2+1)(x+1)(x-1)$

58. $2(5x^4-32)$

59. $(x^4+1)(x^2+1)(x+1)(x-1)$

60. $(9x^4+1)(3x^2+1)(3x^2-1)$

61. $(x^8+1)(x^4+1)(x^2+1)(x+1)(x-1)$

62. $(x^6+1)(x^3+1)(x^3-1)$

63. $2x(x+2)(x-2)(3x-1)(9x^2+3x+1)$

64. $(4n^2-y)(n+y)(n-y)$

65. $2(y+1)(4y-3)(4y+3)$

66. $(3a+b)(a^2-4b)$

67. $(3m+n)(6m-8n)$

68. Prime

69. $V(x)=x(x-3)(x+1)$

70. $V(x)=\pi x(2x-1)^2$

6.6 Factoring Fractions

Evaluate for the given set of *x*-values.

1. $5x;\ \{-1,0,1\}$

2. $\dfrac{4x^3}{x^2};\ \{-1,0,1\}$

3. $\dfrac{1}{x+9};\ \{-10,-9,0\}$

4. $\dfrac{x+6}{x-5};\ \{-6,0,5\}$

5. $\dfrac{3x(x-2)}{2x-1};\ \left\{0, \dfrac{1}{2}, 2\right\}$

6. $\dfrac{9x^2-1}{x-7};\ \left\{0, \dfrac{1}{3}, 7\right\}$

7. $5x^2-9;\ \{-3, 0, 3\}$

8. $\dfrac{x^2-2}{5x^2-3x-10};\ \{-5, -4, 5\}$

9. Fill in the following chart:

x	$\dfrac{1}{x}$
-3	
-2	
0	
2	
3	

10. Fill in the following chart:

x	$\dfrac{1}{x-2}$
-1	
0	
1	
2	
3	

11. Fill in the following chart:

x	$\dfrac{1}{x+2}$
-4	
-3	
-2	
-1	
0	

12. Fill in the following chart:

x	$\frac{1}{x} - 2$
-2	
-1	
0	
1	
2	

An object's weight depends on its height above the surface of Earth. If an object weighs 120 pounds on the surface of Earth, then its weight in pounds, *W*, *x* miles above the surface, is approximated by the formula

$$W = \frac{120 \cdot 4000^2}{(4000 + x)^2}$$

For each problem below, approximate the weight of a 120-pound object at the given height above the surface of Earth. (1 mile = 5,280 feet)

13. 100 miles

14. 1,000 miles

15. 44,350 feet

16. 90,000 feet

The price-to-earnings ratio (P/E) is a metric used to compare the valuations of similar publicly traded companies. The P/E ratio is calculated using the stock price and the earnings per share (EPS) over the previous 12-month period as follows:

$$\text{P/E} = \frac{\text{price per share}}{\text{earnings per share}}$$

If each share of a company's stock is priced at $22.40, then calculate the P/E ratio given the following values for the earnings per share.

17. $1.40

18. $1.21

19. What happens to the P/E ratio when earnings decrease?

20. What happens to the P/E ratio when earnings increase?

State the restrictions to the domain.

21. $\dfrac{1}{3x}$

22. $\dfrac{3x^2}{7x^5}$

23. $\dfrac{3x(x+1)}{x+4}$

24. $\dfrac{2x^2(x-3)}{x-1}$

25. $\dfrac{1}{5x-1}$

26. $\dfrac{x-2}{3x-2}$

27. $\dfrac{x-9}{5x(x-2)}$

28. $\dfrac{1}{(x-3)(x+6)}$

29. $\dfrac{x}{1-x^2}$

30. $\dfrac{x^2-9}{x^2-36}$

31. $\dfrac{1}{2x(x+3)(2x-1)}$

32. $\dfrac{x-3}{(3x-1)(2x+3)}$

33. $\dfrac{4x(2x+1)}{12x^2+x-1}$

34. $\dfrac{x-5}{3x^2-15x}$

State the restrictions and then simplify.

35. $\dfrac{5x^2}{20x^3}$

36. $\dfrac{12x^6}{60x}$

37. $\dfrac{3x^2(x-2)}{9x(x-2)}$

38. $\dfrac{20(x-3)(x-5)}{6(x-3)(x+1)}$

39. $\dfrac{6x^2(x-8)}{36x(x+9)(x-8)}$

40. $\dfrac{16x^2-1}{(4x+1)^2}$

41. $\dfrac{9x^2-6x+1}{(3x-1)^2}$

42. $\dfrac{x-7}{x^2-49}$

43. $\dfrac{x^2-64}{x^2+8x}$

44. $\dfrac{x+10}{x^2-100}$

45. $\dfrac{2x^3-12x^2}{5x^2-30x}$

46. $\dfrac{30x^5+60x^4}{2x^3-8x}$

47. $\dfrac{2x-12}{x^2+x-6}$

48. $\dfrac{x^2-x-6}{3x^2-8x-3}$

49. $\dfrac{6x^2 - 25x + 25}{3x^2 + 16x - 35}$

50. $\dfrac{3x^2 + 4x - 1}{5x^2 - 9}$

51. $\dfrac{x^2 - 10x + 21}{x^2 - 4x - 21}$

52. $\dfrac{x^3 - 1}{x^2 - 1}$

53. $\dfrac{x^3 + 8}{x^2 - 4}$

54. $\dfrac{x^4 - 1}{6x^2 - 4}$

55. $\dfrac{x - 9}{9 - x}$

56. $\dfrac{3x - 2}{2 - 3x}$

57. $\dfrac{x + 6}{6 + x}$

58. $\dfrac{3x + 1}{1 + 3x}$

59. $\dfrac{(2x - 5)(x - 7)}{(7 - x)(2x - 1)}$

60. $\dfrac{(3x + 2)(x + 5)}{(x - 5)(2 + 3x)}$

61. $\dfrac{x^2 - 4}{(2 - x)^2}$

62. $\dfrac{16 - 9x^2}{(3x + 4)^2}$

63. $\dfrac{4x^2(10-x)}{3x^3-300x}$

64. $-\dfrac{2x+1}{4x^3-49x}$

65. $\dfrac{2x^2-7x-4}{1-4x^2}$

66. $\dfrac{9x^2-4}{4x-6x^2}$

67. $\dfrac{x^2-5x-14}{7-15x+2x^2}$

68. $\dfrac{2x^3+x^2-2x-1}{1+x-2x^2}$

69. $\dfrac{x^3+2x-3x^2-6}{2+x^2}$

70. $\dfrac{27+x^3}{x^2+6x+9}$

71. $\dfrac{64-x^3}{x^2-8x+16}$

72. $\dfrac{x^2+4}{4-x^2}$

Simplify. (Assume all denominators are nonzero.)

73. $-\dfrac{15x^3y^2}{5xy^2(x+y)}$

74. $\dfrac{14x^7y^2(x-2y)^4}{7x^8y(x-2y)^2}$

75. $\dfrac{y+x}{x^2-y^2}$

76. $\dfrac{y-x}{x^2-y^2}$

77. $\dfrac{x^2-y^2}{(x-y)^2}$

78. $\dfrac{a^2-ab-6b^2}{a^2-6ab+9b^2}$

79. $\dfrac{2a^2-11a+12}{-32+2a^2}$

80. $\dfrac{a^2b-3a^2}{3a^2-3ab}$

81. $\dfrac{xy^2-x+y^3-y}{x-xy^2}$

82. $\dfrac{x^3-xy^2-x^2y+y}{3x^2-2xy+y^2}$

83. $\dfrac{x^3-27}{x^2+3x+9}$

84. $\dfrac{x^2-x+1}{x^3+1}$

Show Solutions

1. $-5, 0, 5$
2. Undefined, undefined, 4
3. -1, undefined, $\dfrac{1}{9}$

4. $0, -\frac{6}{5}$, undefined

5. 0, undefined, 0

6. $-\frac{1}{7}, 0$, undefined

7. $36, -9, 36$

8. $\frac{23}{130}, \frac{14}{82}, \frac{23}{100}$

9.

x	$\frac{1}{x}$
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
0	undefined
2	$\frac{1}{2}$
3	$\frac{1}{3}$

10.

x	$\frac{1}{x-2}$
-1	$-\frac{1}{3}$
0	$-\frac{1}{2}$
1	-1
2	undefined
3	1

11.

x	$\dfrac{1}{x+2}$
-4	$-\frac{1}{2}$
-3	-1
-2	undefined
-1	1
0	$\frac{1}{2}$

12.

x	$\dfrac{1}{x}-2$
-2	$-\frac{5}{2}$
-1	-3
0	undefined
1	-1
2	$-\frac{3}{2}$

13. About 114 pounds
14. About 77 pounds
15. About 119.5 pounds
16. About 118.98 pounds
17. 16
18. About 18.5
19. The P/E ratio increases.
20. The P/E ratio decreases.
21. $x \neq 0$
22. $x \neq 0$

23. $x \neq -4$

24. $x \neq 1$

25. $x \neq \frac{1}{5}$

26. $x \neq \frac{2}{3}$

27. $x \neq 0, 2$

28. $x \neq 3, -6$

29. $x \neq \pm 1$

30. $x \neq \pm 6$

31. $x \neq 0, -3, \frac{1}{2}$

32. $x \neq \frac{1}{3}, -\frac{3}{2}$

33. $x \neq \frac{1}{4}, -\frac{1}{3}$

34. $x \neq 0, 5$

35. $\frac{1}{4x};\ x \neq 0$

36. $\frac{x^5}{5};\ x \neq 0$

37. $\frac{x}{3};\ x \neq 0, 2$

38. $\frac{10(x-5)}{3(x+1)};\ x \neq 3, -1$

39. $\dfrac{x}{6(x+9)};\ x \neq 0, -9, 8$

40. $\dfrac{4x-1}{4x+1};\ x \neq -\dfrac{1}{4}$

41. $1;\ x \neq \dfrac{1}{3}$

42. $\dfrac{1}{x+7};\ x \neq \pm 7$

43. $\dfrac{x-8}{x};\ x \neq 0, -8$

44. $\dfrac{1}{x-10};\ x \neq \pm 10$

45. $\dfrac{2x}{5};\ x \neq 0, 6$

46. $\dfrac{15x^2}{x-2};\ x \neq 0, \pm 2$

47. $\dfrac{2}{x+3};\ x \neq -3, 2$

48. $\dfrac{x+2}{3x+1};\ x \neq 3, -\dfrac{1}{3}$

49. $\dfrac{2x-5}{x+7};\ x \neq -7, \dfrac{5}{3}$

50. Prime; $x \neq \pm \dfrac{3}{\sqrt{5}}$

51. $\dfrac{x-3}{x+3};\ x \neq -3, 7$

52. $\dfrac{x^2+x+1}{x+1};\ x \neq \pm 1$

53. $\dfrac{x^2-2x+4}{x-2};\ x \neq \pm 2$

54. $\dfrac{(x^2+1)(x^2-1)}{2(3x^2-2)};\ x \neq \pm\sqrt{\dfrac{2}{3}}$

55. $-1;\ x \neq 9$

56. $-1;\ x \neq \dfrac{2}{3}$

57. $1;\ x \neq -6$

58. $1;\ x \neq -\dfrac{1}{3}$

59. $-\dfrac{2x-5}{2x-1};\ x \neq 7, \dfrac{1}{2}$

60. $\dfrac{x+5}{x-5};\ x \neq 5, -\dfrac{2}{3}$

61. $\dfrac{x+2}{x-2};\ x \neq 2$

62. $\dfrac{4-3x}{3x+4};\ x \neq -\dfrac{4}{3}$

63. $-\dfrac{4x}{3(x+10)};\ x \neq 0, \pm 10$

64. $-\dfrac{1}{x(2x-7)};\ x \neq 0, \pm\dfrac{7}{2}$

65. $\dfrac{x-4}{1-2x};\ x \neq \pm\dfrac{1}{2}$

66. $-\dfrac{3x+2}{2x};\ x \neq 0, \dfrac{2}{3}$

67. $\dfrac{x+2}{2x-1};\ x \neq 7, \dfrac{1}{2}$

68. $-\dfrac{(x+1)^2}{2x-1};\ x \neq 1, -\dfrac{1}{2}$

69. $x-3;\ x \neq \sqrt{-2}$

70. $\dfrac{x^2-3x+9}{x+3};\ x \neq -3$

71. $\dfrac{x^2+4x+16}{4-x};\ x \neq 4$

72. $-\dfrac{x^2+4}{(x+2)(x-2)};\ x \neq \pm 2$

73. $-\dfrac{3x^2}{x+y}$

74. $\dfrac{2y(x-2y)^2}{x}$

75. $\dfrac{1}{x-y}$

76. $-\dfrac{1}{x+y}$

77. $\dfrac{x+y}{x-y}$

78. $\dfrac{a+2b}{a-3b}$

79. $\dfrac{2a-3}{2(a+4)}$

80. $-\dfrac{a}{3}$

81. $-\dfrac{x+y}{x}$

82. $\dfrac{x-y}{3x^2-2xy+y^2}$

83. $x-3$

84. $\dfrac{1}{x+1}$

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CHAPTER 7: RATIONAL EXPRESSIONS

Chapter Outline

7.1 Multiplying and Dividing Rational Expressions

- Multiplying Rational Expressions
- Dividing Rational Expressions

7.2 Adding and Subtracting Rational Expressions

- Adding and Subtracting with Common Denominators
- Adding and Subtracting with Unlike Denominators

7.3 Exercises

- 7.1 Multiplying and Dividing Rational Expressions
- 7.2 Adding and Subtracting Rational Expressions

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7.1 Multiplying and Dividing Rational Expressions

Learning Objectives

By the end of this section, you should be able to:

- Multiply rational expressions.
- Divide rational expressions.

Multiplying Rational Expressions

When multiplying fractions, we can multiply the numerators and denominators together and then reduce, as illustrated:

$$\frac{3}{5}\cdot\frac{5}{9}=\frac{3\cdot 5}{5\cdot 9}=\frac{\overset{1}{\cancel{3}}\cdot\overset{1}{\cancel{5}}}{\underset{1}{\cancel{5}}\cdot\underset{3}{\cancel{9}}}=\frac{1}{3}$$

Multiplying rational expressions is performed in a similar manner. For example,

$$\frac{y}{x}\cdot\frac{x}{y^2}=\frac{y\cdot x}{x\cdot y^2}=\frac{\overset{1}{\cancel{y}}\cdot\overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}}\cdot\underset{3}{\cancel{y^2}}}=\frac{1}{y}$$

In general, given polynomials P, Q, R, and S, where $Q\neq 0$ and $S\neq 0$, we have

$$\frac{P}{Q}\cdot\frac{R}{S}=\frac{PR}{QS}$$

In this section, assume that all variable expressions in the denominator are nonzero unless otherwise stated.

How To

Given two rational expressions, multiply them.

1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

Example 1

Multiply:

$$\frac{12x^2}{5y^3} \cdot \frac{20y^4}{6x^3}$$

Solution:

Multiply numerators and denominators and then cancel common factors.

$$\frac{12x^2}{5y^3} \cdot \frac{20y^4}{6x^3} = \frac{240x^2y^4}{30x^3y^3} \qquad \text{Multiply.}$$

$$= \frac{\cancel{240}^{8}\ \cancel{x^2}^{1}\ \cancel{y^4}^{y}}{\cancel{30}_{1}\ \cancel{x^3}_{x}\ \cancel{y^3}_{1}} \qquad \text{Cancel.}$$

$$= \frac{8y}{x}$$

Answer

$$\frac{8y}{x}$$

Example 2

Multiply:

$$\frac{x-3}{x+5} \cdot \frac{x+5}{x+7}$$

Solution:

Leave the product in factored form and cancel the common factors.

$$\frac{x-3}{x+5} \cdot \frac{x+5}{x+7} = \frac{(x-3) \cdot \cancel{(x+5)}}{\cancel{(x+5)} \cdot (x+7)}$$

$$= \frac{x-3}{x+7}$$

Answer

$$\frac{x-3}{x+7}$$

Example 3

Multiply:

$$\frac{15x^2y^3}{(2x-1)} \cdot \frac{x(2x-1)}{3x^2y(x+3)}$$

Solution:

Leave the polynomials in the numerator and denominator factored so that we can cancel the factors. In other words, do not apply the distributive property.

$$\frac{15x^2y^3}{(2x-1)} \cdot \frac{x(2x-1)}{3x^2y(x+3)} = \frac{15x^3y^3(2x-1)}{3x^2y(2x-1)(x+3)}$$ Multiply.

$$= \frac{\cancel{15}^{5}\ \cancel{x^3}^{x}\ \cancel{y^2}^{y}\ \cancel{(2x-1)}^{1}}{\cancel{3}\ \cancel{x^2}\ \cancel{y}\ \cancel{(2x-1)}\ (x+3)}$$ Cancel.

$$= \frac{5xy^2}{x+3}$$

Answer

$$\frac{5xy^2}{x+3}$$

Typically, rational expressions will not be given in factored form. In this case, first factor all numerators and denominators completely. Next, multiply and cancel any common factors, if there are any.

Example 4

Multiply:

$$\frac{x+5}{x-5} \cdot \frac{x-5}{x^2-25}$$

Solution:

Factor the denominator $x^2 - 25$ as a difference of squares. Then multiply and cancel.

$$\frac{x+5}{(x-5)} \cdot \frac{x-5}{x^2-25} = \frac{x+5}{x-5} \cdot \frac{x-5}{(x+5)(x-5)}$$ Factor.

$$= \frac{\cancel{(x+5)}^{1}\ \cancel{x-5}^{1}}{\cancel{(x-5)}\ \cancel{(x+5)}\ (x-5)}$$ Cancel.

$$= \frac{1}{x-5}$$

Keep in mind that 1 is always a factor; so when the entire numerator cancels out, make sure to write the factor 1.

Answer

$$\frac{1}{x-5}$$

Example 5

Multiply:

$$\frac{x^2+3x+2}{x^2-5x+6} \cdot \frac{x^2-7x+12}{x^2+8x+7}$$

Solution:

$$\begin{aligned}\frac{x^2+3x+2}{x^2-5x+6} \cdot \frac{x^2-7x+12}{x^2+8x+7} &= \frac{(x+1)(x+2)}{(x-2)(x-3)} \cdot \frac{(x-3)(x-4)}{(x+1)(x+7)} \\ &= \frac{(x+1)(x+2)(x-3)(x-4)}{(x-2)(x-3)(x+1)(x+7)} \\ &= \frac{(x+2)(x-4)}{(x-2)(x+7)}\end{aligned}$$

It is a best practice to leave the final answer in factored form.

Answer

$$\frac{(x+2)(x-4)}{(x-2)(x+7)}$$

Example 6

Multiply:

$$\frac{-2x^2+x+3}{x^2+2x-8}\cdot\frac{3x-6}{x^2+x}$$

Solution:

The trinomial $-2x^2+x+3$ in the numerator has a negative leading coefficient. Recall that it is a best practice to first factor out a -1 and then factor the resulting trinomial.

$$\begin{aligned}\frac{-2x^2+x+3}{x^2+2x-8}\cdot\frac{3x-6}{x^2+x} &= \frac{-1(2x^2-x-3)}{x^2+2x-8}\cdot\frac{3(x-2)}{x(x+1)}\\ &= \frac{-1(2x-3)(x+1)}{(x-2)(x+4)}\cdot\frac{3(x-2)}{x(x+1)}\\ &= \frac{-1(2x-3)\cdot 3}{(x+4)\cdot x}\\ &= -\frac{3(2x-3)}{x(x+4)}\end{aligned}$$

Answer

$$-\frac{3(2x-3)}{x(x+4)}$$

Example 7

Multiply:

$$\frac{7-x}{x^2+3x} \cdot \frac{x^2+10x+21}{x^2-49}$$

Solution:

We replace $7-x$ with $-1(x-7)$ so that we can cancel this factor.

$$\begin{aligned}\frac{7-x}{x^2+3x} \cdot \frac{x^2+10x+21}{x^2-49} &= \frac{-1(x-7)}{x(x+3)} \cdot \frac{(x+3)(x+7)}{(x+7)(x-7)} \\ &= \frac{-1\ \cancel{(x-7)}\ \cancel{(x+3)}\ \cancel{(x+7)}}{x\ \cancel{(x+3)}\ \cancel{(x+7)}\ \cancel{(x-7)}} \\ &= \frac{-1}{x} \\ &= -\frac{1}{x}\end{aligned}$$

Answer

$$-\frac{1}{x}$$

Try It

Multiply: $\frac{x^2-64}{8-x} \cdot \frac{x+x^2}{x^2+9x+8}$

Show Solution

$-x$

Dividing Rational Expressions

To divide two fractions, we multiply by the reciprocal of the divisor, as illustrated:

$$\frac{5}{8} \div \frac{1}{2} = \frac{5}{8} \cdot \frac{2}{1} = \frac{5 \cdot \overset{1}{\cancel{2}}}{\underset{4 \cdot 1}{\cancel{8}}} = \frac{5}{4}$$

Dividing rational expressions is performed in a similar manner. For example,

$$\frac{x}{y^2} \div \frac{1}{y} = \frac{x}{y^2} \cdot \frac{y}{1} = \frac{x \cdot \overset{1}{\cancel{y}}}{\underset{y \cdot 1}{\cancel{y^2}}} = \frac{x}{y}$$

In general, given polynomials P, Q, R, and S, where $Q \neq 0$, $R \neq 0$, and $S \neq 0$, we have

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}$$

How To

Given two rational expressions, divide them.

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.
4. Multiply the denominators.
5. Simplify.

Example 8

Divide:

$$\frac{8x^5y}{25z^6} \div \frac{20xy^4}{15z^3}$$

Solution:

First, multiply by the reciprocal of the divisor and then cancel.

$$\frac{8x^5y}{25z^6} \div \frac{20xy^4}{15z^3} = \frac{8x^5y}{25z^6} \cdot \frac{15z^3}{20xy^4}$$

Multiply by the reciprocal of the divisor.

$$= \frac{120x^5yz^3}{500xy^4z^6}$$

$$= \frac{\cancel{120}^{6}\ \cancel{x^5}^{x^4}\ \cancel{y}\ \cancel{z^3}}{\cancel{500}_{25}\ \cancel{x}\ \cancel{y^4}_{y^3}\ \cancel{z^6}_{z^3}}$$

Cancel.

$$= \frac{6x^4}{25y^3z^3}$$

Answer

$$\frac{6x^4}{25y^3z^3}$$

Example 9

Divide:

$$\frac{x+2}{x^2-4} \div \frac{x+3}{x-2}$$

Solution:

After multiplying by the reciprocal of the divisor, factor and cancel.

$$\begin{aligned}
\frac{x+2}{x^2-4} \div \frac{x+3}{x-2} &= \frac{x+2}{x^2-4} \cdot \frac{x-2}{x+3} && \text{Multiply by the reciprocal of the divisor.} \\
&= \frac{(x+2)}{(x+2)(x-2)} \cdot \frac{(x-2)}{(x+3)} && \text{Factor.} \\
&= \frac{\cancel{(x+2)}\ \cancel{x-2}}{\cancel{(x+2)}\ \cancel{(x-2)}\ (x+3)} && \text{Cancel.} \\
&= \frac{1}{x+3}
\end{aligned}$$

Answer

$$\frac{1}{x+3}$$

Example 10

Divide:

$$\frac{x^2-6x-16}{x^2+4x-21} \div \frac{x^2+9x+14}{x^2-8x+15}$$

Solution:

Begin by multiplying by the reciprocal of the divisor. After doing so, factor and cancel.

$$\begin{aligned}
\frac{x^2-6x-16}{x^2+4x-21} \div \frac{x^2+9x+14}{x^2-8x+15} &= \frac{x^2-6x-16}{x^2+4x-21} \cdot \frac{x^2-8x+15}{x^2+9x+14} \\
&= \frac{(x+2)(x-8)}{(x-3)(x+7)} \cdot \frac{(x-3)(x-5)}{(x+2)(x+7)} \\
&= \frac{(x-8)(x-5)}{(x+7)^2}
\end{aligned}$$

Answer

$$\frac{(x-8)(x-5)}{(x+7)^2}$$

Example 11

Divide:

$$\frac{9-4x^2}{x+2} \div (2x-3)$$

Solution:

Just as we do with fractions, think of the divisor $(2x-3)$ as an algebraic fraction over 1.

$$\begin{aligned}\frac{9-4x^2}{x+2} \div (2x-3) &= \frac{9-4x^2}{x+2} \div \frac{2x-3}{1} \\ &= \frac{(3+2x)(3-2x)}{x+2} \cdot \frac{1}{2x-3} \\ &= \frac{(3+2x)\cdot(-1)(2x-3)}{(x+2)(2x-3)} \\ &= -\frac{3+2x}{x+2} \text{ or } -\frac{2x+3}{x+2}\end{aligned}$$

Answer

$$-\frac{2x+3}{x+2}$$

Try It

Divide: $\frac{4x^2+7x-2}{25x^2} \div \frac{1-4x}{100x^4}$

Show Solution

$-4x^3 - 8x^2$

Key Takeaways

- After multiplying rational expressions, factor both the numerator and denominator and then cancel common factors. Make note of the restrictions to the domain. The values that give a value of 0 in the denominator are the restrictions.
- To divide rational expressions, multiply by the reciprocal of the divisor.
- The restrictions to the domain of a product consist of the restrictions to the domain of each factor.
- The restrictions to the domain of a quotient consist of the restrictions to the domain of each rational expression, as well as the restrictions on the reciprocal of the divisor.

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7.2 Adding and Subtracting Rational Expressions

Learning Objectives

By the end of this section, you should be able to:

- Add and subtract rational expressions with common denominators.
- Add and subtract rational expressions with unlike denominators.

Adding and Subtracting with Common Denominators

Adding and subtracting rational expressions is similar to adding and subtracting fractions. Recall that if the denominators are the same, we can add or subtract the numerators and write the result over the common denominator.

$$\begin{aligned}\frac{3}{13}+\frac{7}{13} &= \frac{3+7}{13}\\ &= \frac{10}{13}\end{aligned}$$

When working with rational expressions, the common denominator will be a polynomial. In general, given polynomials P, Q, and R, where $Q \neq 0$, we have the following:

$$\frac{P}{Q} \pm \frac{R}{Q} = \frac{P \pm R}{Q}$$

In this section, assume that all variable factors in the denominator are nonzero.

Example 1

Add:

$$\frac{3}{y} + \frac{7}{y}$$

Solution:
Add the numerators 3 and 7, and write the result over the common denominator, y.

$$\begin{aligned} \frac{3}{y} + \frac{7}{y} &= \frac{3+7}{y} \\ &= \frac{10}{y} \end{aligned}$$

Answer

$$\frac{10}{y}$$

Example 2

Subtract:

$$\frac{x-5}{2x-1} - \frac{1}{2x-1}$$

Solution: Subtract the numerators $x - 5$ and 1, and write the result over the common denominator, $2x - 1$.

$$\frac{x-5}{2x-1}-\frac{1}{2x-1}=\frac{x-5-1}{2x-1}$$

Simplify the numerator.

$$=\frac{x-6}{2x-1}$$

Answer

$$\frac{x-6}{2x-1}$$

Example 3

Subtract:

$$\frac{2x+7}{(x+5)(x-3)}-\frac{x+10}{(x+5)(x-3)}$$

Solution:

Use parentheses to remind us to subtract the entire numerator of the second rational expression.

$$\frac{2x+7}{(x+5)(x-3)}-\frac{x+10}{(x+5)(x-3)}=\frac{(2x+7)-(x+10)}{(x+5)(x-3)}$$

Simplify the numerator.

$$=\frac{2x+7-x-10}{(x+5)(x-3)}$$

Leave the denominator factored.

$$=\frac{\cancel{x-3}^{\,1}}{(x+5)\,\cancel{(x-3)}}$$

Cancel common factors.

$$=\frac{1}{x+5}$$

Answer

$$\frac{1}{x+5}$$

Example 4

Simplify:

$$\frac{2x^2+10x+3}{x^2-36}-\frac{x^2+6x+5}{x^2-36}+\frac{x-4}{x^2-36}$$

Solution:

Subtract and add the numerators, then write the result over the common denominator x^2-36.

$$\begin{aligned}\frac{2x^2+10x+3}{x^2-36}-\frac{x^2+6x+5}{x^2-36}+\frac{x-4}{x^2-36} &= \frac{(2x^2+10x+3)-(x^2+6x+5)+(x-4)}{x^2-36}\\ &= \frac{x^2+5x-6}{x^2-36}\\ &= \frac{(x-1)(x+6)}{(x-6)(x+6)}\\ &= \frac{x-1}{x-6}\end{aligned}$$

Answer

$$\frac{x-1}{x-6}$$

Try It

Subtract: $\dfrac{x^2+1}{2x^2-7x-4}-\dfrac{x^2-2x}{2x^2-7x-4}$

Show Solution

$$\frac{1}{x-4}$$

Adding and Subtracting with Unlike Denominators

To add rational expressions with unlike denominators, first find equivalent expressions with common denominators. Do this just as you have with fractions. If the denominators of fractions are relatively prime, then the least common denominator (LCD) is their product. For example,

$$\frac{1}{3}+\frac{1}{5} \quad \Rightarrow \quad \text{LCD} = 3 \cdot 5 = 15$$

Multiply each fraction by the appropriate form of 1 to obtain equivalent fractions with a common denominator.

$$\begin{aligned} \frac{1}{3}+\frac{1}{5} &= \frac{1 \cdot 5}{3 \cdot 5}+\frac{1 \cdot 3}{5 \cdot 3} \\ &= \frac{5}{15}+\frac{3}{15} \\ &= \frac{5+3}{15} \\ &= \frac{8}{15} \end{aligned}$$

Equivalent fractions with a common denominator.

The process of adding and subtracting rational expressions is similar. In general, given polynomials P, Q, R, and S, where $Q \neq 0$ and $S \neq 0$, we have the following:

$$\frac{P}{Q} \pm \frac{R}{S} = \frac{PS \pm QR}{QS}$$

In this section, assume that all variable factors in the denominator are nonzero.

How To

Given two rational expressions, add or subtract them.

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

Example 5

Add:

$$\frac{1}{x}+\frac{1}{y}$$

Solution:

Here the $LCD = xy$. Multiply the first term by $\frac{y}{y}$ and the second by $\frac{x}{x}$.

$$\begin{aligned}\frac{1}{x}+\frac{1}{y} &= \frac{1}{x}\cdot\frac{y}{y}+\frac{1}{y}\cdot\frac{x}{x} \\ &= \frac{y}{xy}+\frac{x}{xy} \qquad \text{Equivalent terms with a common denominator.} \\ &= \frac{y+x}{xy} \qquad \text{Add the numerators and place the result over the common denominator, } xy.\end{aligned}$$

Answer

$$\frac{y+x}{xy}$$

Example 6

Subtract:

$$\frac{1}{y} - \frac{1}{y-3}$$

Solution:

Since the $LCD = y(y-3)$, multiply the first term by $\frac{y-3}{y-3}$ and the second term by $\frac{y}{y}$.

$$\begin{aligned}\frac{1}{y} - \frac{1}{y-3} &= \frac{1}{y} \cdot \frac{y-3}{y-3} - \frac{1}{y-3} \cdot \frac{y}{y} \\ &= \frac{y-3}{y(y-3)} - \frac{y}{y(y-3)} \\ &= \frac{y-3-y}{y(y-3)} \\ &= \frac{-3}{y(y-3)}\end{aligned}$$

Answer

$$\frac{-3}{y(y-3)}$$

It is not always the case that the LCD is the product of the given denominators. Typically, the denominators are not relatively prime; thus, determining the LCD requires some thought. Begin by factoring all denominators. The LCD is the product of all factors with the highest power. For example, given

$$\frac{1}{x^3(x+2)(x-3)} \quad \text{and} \quad \frac{1}{x(x+2)^2}$$

there are three base factors in the denominator: x, $(x+2)$, and $(x-3)$. The highest powers of these factors are x^3, $(x+2)^2$, and $(x-3)^1$. Therefore,

$$\text{LCD} = x^3(x+2)^2(x-3)$$

Example 7

Subtract:

$$\frac{x}{x^2+4x+3}-\frac{3}{x^2-4x-5}$$

Solution:

Step 1: Factor all denominators to determine the LCD.

$$\frac{x}{x^2+4x+3}-\frac{3}{x^2-4x-5}=\frac{x}{(x+1)(x+3)}-\frac{3}{(x+1)(x-5)}$$

The LCD is $(x+1)(x+3)(x-5)$.

Step 2: Multiply by the appropriate factors to obtain equivalent terms with a common denominator.

$$\begin{aligned}&=\frac{x}{(x+1)(x+3)}\cdot\frac{x-5}{x-5}-\frac{3}{(x+1)(x-5)}\cdot\frac{x+3}{x+3}\\&=\frac{x(x-5)}{(x+1)(x+3)(x-5)}-\frac{3(x+3)}{(x+1)(x+3)(x-5)}\end{aligned}$$

Step 3: Subtract the numerators and place the result over the common denominator.

$$\frac{x(x-5)-3(x+3)}{(x+1)(x+3)(x-5)}$$

Step 4: Simplify the resulting rational expression.

$$\begin{aligned}\frac{x(x-5)-3(x+3)}{(x+1)(x+3)(x-5)} &= \frac{x^2-5x-3x-9}{(x+1)(x+3)(x-5)} \\ &= \frac{x^2-8x-9}{(x+1)(x+3)(x-5)} \\ &= \frac{(x+1)(x-9)}{(x+1)(x+3)(x-5)} \\ &= \frac{x-9}{(x+3)(x-5)}\end{aligned}$$

Answer

Answer: $\dfrac{x-9}{(x+3)(x-5)}$

Example 8

Subtract:

$$\frac{x^2-9x+18}{x^2-13x+36}-\frac{x}{x-4}$$

Solution:

$$\begin{aligned}\frac{x^2-9x+18}{x^2-13x+36}-\frac{x}{x-4} &= \frac{x^2-9x+18}{(x-4)(x-9)}-\frac{x}{x-4}\cdot\frac{x-9}{x-9} \\ &= \frac{x^2-9x+18-x(x-9)}{(x-4)(x-9)} \\ &= \frac{x^2-9x+18-x^2+9x}{(x-4)(x-9)} \\ &= \frac{18}{(x-4)(x-9)}\end{aligned}$$

Answer

$$\frac{18}{(x-4)(x-9)}$$

Example 9

Subtract:

$$\frac{1}{x^2-4}-\frac{1}{2-x}$$

Solution:

Factor denominators and apply the **opposite binomial property** to rewrite $2-x=-(x-2)$.

$$\begin{aligned}\frac{1}{x^2-4}-\frac{1}{2-x}&=\frac{1}{(x+2)(x-2)}-\frac{1}{-(x-2)}\\&=\frac{1}{(x+2)(x-2)}+\frac{1}{x-2}\end{aligned}$$

The LCD is $(x+2)(x-2)$. Multiply the second term by $\frac{x+2}{x+2}$.

$$\begin{aligned}&=\frac{1}{(x+2)(x-2)}+\frac{1}{x-2}\cdot\frac{x+2}{x+2}\\&=\frac{1}{(x+2)(x-2)}+\frac{x+2}{(x-2)(x+2)}\\&=\frac{1+x+2}{(x+2)(x-2)}\\&=\frac{x+3}{(x+2)(x-2)}\end{aligned}$$

Answer

$$\frac{x+3}{(x+2)(x-2)}$$

Example 10

Simplify:

$$\frac{y-1}{y+1}-\frac{y+1}{y-1}+\frac{y^2-5}{y^2-1}$$

Solution: Factor $y^2-1=(y+1)(y-1)$ and use LCD $(y+1)(y-1)$.

$$\begin{aligned}\frac{y-1}{y+1}-\frac{y+1}{y-1}+\frac{y^2-5}{y^2-1} &= \frac{(y-1)(y-1)}{(y+1)(y-1)}-\frac{(y+1)(y+1)}{(y-1)(y+1)}+\frac{y^2-5}{(y+1)(y-1)}\\ &= \frac{(y^2-2y+1)-(y^2+2y+1)+(y^2-5)}{(y+1)(y-1)}\\ &= \frac{y^2-4y-5}{(y+1)(y-1)}\\ &= \frac{(y+1)(y-5)}{(y+1)(y-1)}\\ &= \frac{y-5}{y-1}\end{aligned}$$

Answer

$$\frac{y-5}{y-1}$$

Try It

Simplify: $-\frac{2x^2-1+x}{1+x}-\frac{5}{1-x}$

Show Solution

$$\frac{-2x^2+3x+4}{x-1}$$

Rational expressions are sometimes expressed using **negative exponents**. In this case, apply the rules for negative exponents before simplifying the expression.

Example 11

Simplify:

$$y^{-2}+(y-1)^{-1}$$

Solution:
Rewrite negative exponents as rational expressions, then add using a common denominator.

$$\begin{aligned}
&y^{-2}+(y-1)^{-1} \\
&=\frac{1}{y^2}+\frac{1}{y-1} && \text{Replace negative exponents.} \\
&=\frac{1}{y^2}\cdot\frac{y-1}{y-1}+\frac{1}{y-1}\cdot\frac{y^2}{y^2} && \text{Multiply by factors to obtain expressions with a common denominator.} \\
&=\frac{y-1}{y^2(y-1)}+\frac{y^2}{y^2(y-1)} \\
&=\frac{(y-1)+y^2}{y^2(y-1)} && \text{Add and simplify.} \\
&=\frac{y^2+y-1}{y^2(y-1)} && \text{The trinomial does not factor.}
\end{aligned}$$

Answer

$$\frac{y^2+y-1}{y^2(y-1)}$$

Key Takeaways

- When adding or subtracting rational expressions with a common denominator, add or subtract the expressions in the numerator and write the result over the common denominator.
- To find equivalent rational expressions with a common denominator, first factor all denominators and determine the least common multiple. Then multiply the numerator and denominator of each term by the appropriate factor to obtain a common denominator. Finally, add or subtract the expressions in the numerator and write the result over the common denominator.

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7.3 Exercises

7.1 Multiplying and Dividing Rational Expressions

Multiply. (Assume all denominators are nonzero.)

1. $\frac{2x}{3} \cdot \frac{9}{4x^2}$
2. $-\frac{5x}{3y} \cdot \frac{y^2}{25x}$
3. $\frac{5x^2}{2y} \cdot \frac{4y^2}{15x^3}$
4. $\frac{16a^4}{7b^2} \cdot \frac{49b^3}{2a^3}$
5. $\frac{x-6}{12x^3} \cdot \frac{24x^2}{x-6}$
6. $\frac{x+10}{2x-1} \cdot \frac{x-2}{x+10}$
7. $\frac{(y-1)^2}{y+1} \cdot \frac{1}{y-1}$
8. $\frac{y^2-9}{y+3} \cdot \frac{2y-3}{y-3}$
9. $\frac{2a-5}{a-5} \cdot \frac{2a+5}{4a^2-25}$
10. $\frac{2a^2-9a+4}{a^2-16} \cdot (a^2+4a)$

11. $\dfrac{2x^2+3x-2}{(2x-1)^2}\cdot\dfrac{2x}{x+2}$

12. $\dfrac{9x^2+19x+2}{4-x^2}\cdot\dfrac{x^2-4x+4}{9x^2-8x-1}$

13. $\dfrac{x^2+8x+16}{16-x^2}\cdot\dfrac{x^2-3x-4}{x^2+5x+4}$

14. $\dfrac{x^2-x-2}{x^2+8x+7}\cdot\dfrac{x^2+2x-15}{x^2-5x+6}$

15. $\dfrac{x+1}{x-3}\cdot\dfrac{3-x}{x+5}$

16. $\dfrac{2x-1}{x-1}\cdot\dfrac{x+6}{1-2x}$

17. $\dfrac{9+x}{3x+1}\cdot\dfrac{3}{x+9}$

18. $\dfrac{1}{2+5x}\cdot\dfrac{5x+2}{5x}$

19. $\dfrac{100-y^2}{y-10}\cdot\dfrac{25y^2}{y+10}$

20. $\dfrac{3y^3}{6y-5}\cdot\dfrac{36y^2-25}{5+6y}$

21. $\dfrac{3a^2+14a-5}{a^2+1}\cdot\dfrac{3a+1}{1-9a^2}$

22. $\dfrac{4a^2-16a}{4a-1}\cdot\dfrac{1-16a^2}{4a^2-15a-4}$

23. $\dfrac{x+9}{-x^2+14x-45}\cdot\left(x^2-81\right)$

24. $\dfrac{1}{2+5x}\cdot\left(25x^2+20x+4\right)$

25. $\dfrac{x^2+x-6}{3x^2+15x+18} \cdot \dfrac{2x^2-8}{x^2-4x+4}$

26. $\dfrac{5x^2-4x-1}{5x^2-6x+1} \cdot \dfrac{25x^2-10x+1}{3-75x^2}$

Divide. (Assume all denominators are nonzero.)

27. $\dfrac{5x}{8} \div \dfrac{15x^2}{4}$

28. $\dfrac{3}{8y} \div \dfrac{15}{2y^2}$

29. $\dfrac{\dfrac{5x^9}{3y^3}}{\dfrac{25x^{10}}{9y^5}}$

30. $\dfrac{\dfrac{12x^4y^2}{21z^5}}{\dfrac{6x^3y^2}{7z^3}}$

31. $\dfrac{(x-4)^2}{30x^4} \div \dfrac{x-4}{15x}$

32. $\dfrac{5y^4}{10(3y-5)^2} \div \dfrac{10y^5}{2(3y-5)^3}$

33. $\dfrac{x^2-9}{5x} \div (x-3)$

34. $\dfrac{y^2-64}{8y} \div (8+y)$

35. $\dfrac{(a-8)^2}{2a^2+10a} \div \dfrac{a-8}{a}$

36. $\dfrac{2}{4a^2b^3(a-2b)} \div 12ab(a-2b)^5$

37. $\dfrac{x^2+7x+10}{x^2+4x+4} \div \dfrac{1}{x^2-4}$

38. $\dfrac{2x^2-x-1}{2x^2-3x+1} \div \dfrac{1}{4x^2-1}$

39. $\dfrac{y+1}{y^2-3y} \div \dfrac{y^2-1}{y^2-6y+9}$

40. $\dfrac{9-a^2}{a^2-8a+15} \div \dfrac{2a^2-10a}{a^2-10a+25}$

41. $\dfrac{a^2-3a-18}{2a^2-11a-6} \div \dfrac{a^2+a-6}{2a^2-a-1}$

42. $\dfrac{y^2-7y+10}{y^2+5y-14} \div \dfrac{2y^2-9y-5}{y^2+14y+49}$

43. $\dfrac{6y^2+y-1}{4y^2+4y+1} \div \dfrac{3y^2+2y-1}{2y^2-7y-4}$

44. $\dfrac{x^2-7x-18}{x^2+8x+12} \div \dfrac{x^2-81}{x^2+12x+36}$

45. $\dfrac{4a^2-b^2}{b+2a} \div (b-2a)^2$

46. $\dfrac{x^2-y^2}{y+x} \div (y-x)^2$

47. $\dfrac{5y^2(y-3)}{4x^3} \div \dfrac{25y(3-y)}{2x^2}$

48. $\dfrac{15x^3}{3(y+7)} \div \dfrac{25x^6}{9(7+y)^2}$

49. $\dfrac{3x+4}{x-8} \div \dfrac{7x}{8-x}$

50. $\dfrac{3x-2}{2x+1} \div \dfrac{2-3x}{3x}$

51. $\dfrac{(7x-1)^2}{4x+1} \div \dfrac{28x^2-11x+1}{1-4x}$

52. $\dfrac{4x}{(x+2)^2} \div \dfrac{2-x}{x^2-4}$

53. $\dfrac{a^2-b^2}{a} \div (b-a)^2$

54. $\dfrac{(a-2b)^2}{2b} \div (2b^2+ab-a^2)$

55. $\dfrac{x^2-6x+9}{x^2+7x+12} \div \dfrac{9-x^2}{x^2+8x+16}$

56. $\dfrac{2x^2-9x-5}{25-x^2} \div \dfrac{1-4x+4x^2}{-2x^2-9x+5}$

57. $\dfrac{3x^2-16x+5}{100-4x^2} \div \dfrac{9x^2-6x+1}{3x^2+14x-5}$

58. $\dfrac{10x^2-25x-15}{x^2-6x+9} \div \dfrac{9-x^2}{x^2+6x+9}$

Recall that multiplication and division are to be performed in the order they appear from left to right. Simplify the following.

59. $\dfrac{1}{x^2} \cdot \dfrac{x-1}{x+3} \div \dfrac{x-1}{x^3}$

60. $\dfrac{x-7}{x+9} \cdot \dfrac{1}{x^3} \div \dfrac{x-7}{x}$

61. $\dfrac{x+1}{x-2} \div \dfrac{x}{x-5} \cdot \dfrac{x^2}{x+1}$

62. $\dfrac{x+4}{2x+5} \div \dfrac{x-3}{2x+5} \cdot \dfrac{x+4}{x-3}$

63. $\dfrac{2x-1}{x+1} \div \dfrac{x-4}{x^2+1} \cdot \dfrac{x-4}{2x-1}$

64. $\dfrac{4x^2-1}{3x+2} \div \dfrac{2x-1}{x+5} \cdot \dfrac{3x+2}{2x+1}$

Show Solutions

1. $\dfrac{3}{2x}$
2. $-\dfrac{y}{15}$
3. $\dfrac{2y}{3x}$
4. $56ab$
5. $\dfrac{2}{x}$
6. $\dfrac{x-2}{2x-1}$
7. $\dfrac{y-1}{y+1}$
8. $2y-3$
9. $\dfrac{1}{a-5}$
10. $a(2a-1)$
11. $\dfrac{2x}{2x-1}$
12. $-\dfrac{x-2}{x-1}$

13. -1

14. $\dfrac{(x+2)(x-1)(x+5)}{(x+1)(x+7)(x-2)}$

15. $-\dfrac{x+1}{x+5}$

16. $-\dfrac{x+6}{x-1}$

17. $\dfrac{3}{3x+1}$

18. $\dfrac{1}{5x}$

19. $-25y^2$

20. $3y^3$

21. $-\dfrac{a+5}{a^2+1}$

22. $-4a$

23. $-\dfrac{(x+9)^2}{x-5}$

24. $5x+2$

25. $\dfrac{2}{3}$

26. $-\dfrac{1}{3}$

27. $\dfrac{1}{6x}$

28. $\dfrac{y}{20}$

29. $\dfrac{3y^2}{5x}$

30. $\dfrac{2x}{3z^2}$

31. $\dfrac{x-4}{2x^3}$

32. $\dfrac{3y-5}{10y}$

33. $\dfrac{x+3}{5x}$

34. $\dfrac{y-8}{8y}$

35. $\dfrac{a-8}{2(a+5)}$

36. $\dfrac{1}{24a^3b^4(a-2b)^6}$

37. $(x+5)(x-2)$

38. $(2x+1)^2$

39. $\dfrac{y-3}{y(y-1)}$

40. $-\dfrac{a+3}{2a}$

41. $\dfrac{a-1}{a-2}$

42. $\dfrac{y-4}{y+1}$

43. $\dfrac{y-4}{y+1}$

44. $\dfrac{x+6}{x+9}$

45. $\dfrac{1}{2a-b}$

46. $\dfrac{1}{x-y}$

47. $-\dfrac{y}{10x}$

48. $\dfrac{9(y+7)}{5x^3}$

49. $-\dfrac{3x+4}{7x}$

50. $-\dfrac{3x}{2x+1}$

51. $-\dfrac{7x-1}{4x+1}$

52. $-\dfrac{4x}{x+2}$

53. $\dfrac{a+b}{a(a-b)}$

54. $-\dfrac{a-2b}{2b(a+b)}$

55. $-\dfrac{(x-3)(x+4)}{(x+3)^2}$

56. $\dfrac{2x+1}{2x-1}$

57. $-\dfrac{1}{4}$

58. $-\dfrac{5(2x+1)(x+3)}{(x-3)^2}$

59. $\dfrac{x}{x+3}$

60. $\dfrac{1}{x^2(x+9)}$

61. $\dfrac{x(x-5)}{x-2}$

62. $\dfrac{(x+4)^2}{(x-3)^2}$

63. $\dfrac{x^2+1}{x+1}$

64. $x+5$

7.2 Adding and Subtracting Rational Expressions

Simplify. (Assume all denominators are nonzero).

1. $\frac{3}{x} + \frac{7}{x}$
2. $\frac{9}{x} - \frac{10}{x}$
3. $\frac{x}{y} - \frac{3}{y}$
4. $\frac{4}{x-3} + \frac{6}{x-3}$
5. $\frac{7}{2x-1} - \frac{x}{2x-1}$
6. $\frac{8}{3x-8} - \frac{3x}{3x-8}$
7. $\frac{2}{x-9} + \frac{x-11}{x-9}$
8. $\frac{y+2}{2y+3} - \frac{y+3}{2y+3}$
9. $\frac{2x-3}{4x-1} - \frac{x-4}{4x-1}$
10. $\frac{2x}{x-1} - \frac{3x+4}{x-1} + \frac{x-2}{x-1}$
11. $\frac{1}{3y} - \frac{2y-9}{3y} - \frac{13-5y}{3y}$
12. $\frac{-3y+2}{5y-10} + \frac{y+7}{5y-10} - \frac{3y+4}{5y-10}$

13. $\dfrac{x}{(x+1)(x-3)} - \dfrac{3}{(x+1)(x-3)}$

14. $\dfrac{3x+5}{(2x-1)(x-6)} - \dfrac{x+6}{(2x-1)(x-6)}$

15. $\dfrac{x}{x^2-36} + \dfrac{6}{x^2-36}$

16. $\dfrac{x}{x^2-81} - \dfrac{9}{x^2-81}$

17. $\dfrac{x^2+2}{x^2+3x-28} + \dfrac{x-2}{2x^2+3x-28}$

18. $\dfrac{x^2}{x^2-x-3} - \dfrac{3-x^2}{x^2-x-3}$

19. $\dfrac{1}{2} + \dfrac{1}{3x}$

20. $\dfrac{1}{5x^2} - \dfrac{1}{x}$

21. $\dfrac{1}{12y^2} + \dfrac{3}{10y^3}$

22. $\dfrac{1}{x} - \dfrac{1}{2y}$

23. $\dfrac{1}{y} - 2$

24. $\dfrac{3}{y+2} - 4$

25. $\dfrac{2}{x+4} + 2$

26. $\dfrac{2}{y} - \dfrac{1}{y^2}$

27. $\dfrac{3}{x+1}+\dfrac{1}{x}$

28. $\dfrac{1}{x-1}-\dfrac{2}{x}$

29. $\dfrac{1}{x-3}+\dfrac{1}{x+5}$

30. $\dfrac{1}{x+2}-\dfrac{1}{x-3}$

31. $\dfrac{x}{x+1}-\dfrac{2}{x-2}$

32. $\dfrac{2x-3}{x+5}-\dfrac{x}{x-3}$

33. $\dfrac{y+1}{y-1}+\dfrac{y-1}{y+1}$

34. $\dfrac{3y-1}{3y}-\dfrac{y+4}{y-2}$

35. $\dfrac{2x-5}{2x+5}-\dfrac{2x+5}{2x-5}$

36. $\dfrac{2}{2x-1}-\dfrac{2x+1}{1-2x}$

37. $\dfrac{3x+4}{x-8}-\dfrac{2}{8-x}$

38. $\dfrac{1}{y-1}+\dfrac{1}{1-y}$

39. $\dfrac{2x^2}{x^2-9}+\dfrac{x+15}{9-x^2}$

40. $\dfrac{x}{x+3}+\dfrac{1}{x-3}-\dfrac{1}{5}-\dfrac{x}{(x+3)(x-3)}$

41. $\dfrac{2x}{3x-1} - \dfrac{1}{3x+1} + \dfrac{2(x-1)}{(3x-1)(3x+1)}$

42. $\dfrac{4x}{2x+1} - \dfrac{x}{x-5} + \dfrac{16x-3}{(2x+1)(x-5)}$

43. $\dfrac{x}{3x} + \dfrac{2}{x-2} + \dfrac{4}{3x(x-2)}$

44. $-\dfrac{2x}{x+6} - \dfrac{3x}{6-x} - \dfrac{18(x-2)}{(x+6)(x-6)}$

45. $\dfrac{x}{x+5} - \dfrac{1}{x-7} - \dfrac{25-7x}{(x+5)(x-7)}$

46. $\dfrac{x}{x^2} - \dfrac{2}{x-3} + \dfrac{2}{x-3}$

47. $\dfrac{1}{x+5} - \dfrac{x^2}{x^2-25}$

48. $\dfrac{5x-2}{x^2-4} - \dfrac{2}{x-2}$

49. $\dfrac{1}{x+1} - \dfrac{6x-3}{x^2-7x-8}$

50. $\dfrac{3x}{9x^2-16} - \dfrac{1}{3x+4}$

51. $\dfrac{2x}{x^2-1} + \dfrac{1}{x^2+x}$

52. $\dfrac{x(4x-1)}{2x^2} + \dfrac{7}{x-4} - \dfrac{x}{4+x}$

53. $\dfrac{3x^2}{3x^2+5x-2} - \dfrac{2x}{3x-1}$

54. $\dfrac{2x}{x-4} - \dfrac{11x+4}{x^2-2x-8}$

55. $\dfrac{x}{2x+1}+\dfrac{6x-24}{2x^2-7x-4}$

56. $\dfrac{1}{x^2-x-6}+\dfrac{1}{x^2-3x-10}$

57. $\dfrac{x}{x^2+4x+3}-\dfrac{3}{x^2-4x-5}$

58. $\dfrac{y+1}{2y^2+5y-3}-\dfrac{y}{4y^2-1}$

59. $\dfrac{y-1}{y^2-25}-\dfrac{2}{y^2-10y+25}$

60. $\dfrac{3x^2+2}{4x^2-2x-8}-\dfrac{1}{2x-4}$

61. $\dfrac{4x^2+2}{8x^2-6x-7}-\dfrac{2}{8x-7}$

62. $\dfrac{a}{4-a+a^2}-\dfrac{9a+18}{a^2-13a+36}$

63. $\dfrac{3a-12}{a^2-8a+16}-\dfrac{a+2}{4-a}$

64. $\dfrac{a^2-14}{2a^2-7a-4}-\dfrac{5}{1+2a}$

65. $\dfrac{1}{x+3}-\dfrac{x}{x^2-6x+9}+\dfrac{3}{x^2-9}$

66. $\dfrac{3x}{x+7}-\dfrac{2x}{x-2}+\dfrac{23x-10}{x^2+5x-14}$

67. $\dfrac{x+3}{x-1}+\dfrac{x-1}{x+2}-\dfrac{x(x+11)}{x^2+x-2}$

68. $-\dfrac{2x}{3x+1}-\dfrac{4}{x-2}+\dfrac{4(x+5)}{3x^2-5x-2}$

69. $\dfrac{x-1}{4x-1} - \dfrac{x+3}{2x+3} - \dfrac{3(x+5)}{8x^2+10x-3}$

70. $\dfrac{3x}{2x-3} - \dfrac{2}{2x+3} - \dfrac{6x^2-5x-9}{4x^2-9}$

71. $\dfrac{1}{y+1} + \dfrac{1}{y} + \dfrac{2}{y^2-1}$

72. $\dfrac{1}{y} - \dfrac{1}{y+1} + \dfrac{1}{y-1}$

73. $5^{-2} + 2^{-1}$

74. $6^{-1} + 4^{-2}$

75. $x^{-1} + y^{-1}$

76. $x^{-2} - y^{-1}$

77. $(2x-1)^{-1} - x^{-2}$

78. $(x-4)^{-1} - (x+1)^{-1}$

79. $3x^2(x-1)^{-1} - 2x$

80. $2(y-1)^{-2} - (y-1)^{-1}$

Show Solutions

1. $\dfrac{10}{x}$

2. $-\dfrac{1}{x}$

3. $\dfrac{x-3}{y}$
4. $\dfrac{10}{x-3}$
5. $\dfrac{7-x}{2x-1}$
6. -1
7. 1
8. $-\dfrac{1}{2y+3}$
9. $\dfrac{x+1}{4x-1}$
10. $-\dfrac{6}{x-1}$
11. $\dfrac{y-1}{y}$
12. $-\dfrac{y-1}{y-2}$
13. $\dfrac{1}{x+1}$
14. $\dfrac{1}{x-6}$
15. $\dfrac{1}{x-6}$
16. $\dfrac{1}{x+9}$

17. $\dfrac{x(2x^2+8x+9)}{(x-4)(x+7)(2x+7)}$

18. $\dfrac{2x^2-3}{x^2-x-3}$

19. $\dfrac{3x+2}{6x}$

20. $\dfrac{1-5x}{5x^2}$

21. $\dfrac{5y+18}{60y^3}$

22. $\dfrac{2y-x}{2xy}$

23. $\dfrac{1-2y}{y}$

24. $-\dfrac{4y+5}{y+2}$

25. $\dfrac{2(x+5)}{x+4}$

26. $\dfrac{2y-1}{y^2}$

27. $\dfrac{4x+1}{x(x+1)}$

28. $\dfrac{2-x}{x(x-1)}$

29. $\dfrac{2(x+1)}{(x-3)(x+5)}$

30. $-\dfrac{5}{(x+2)(x-3)}$

31. $\dfrac{x^2-4x-2}{(x-2)(x+1)}$

32. $\dfrac{x^2-14x+9}{(x+5)(x-3)}$

33. $\dfrac{2(y^2+1)}{(y+1)(y-1)}$

34. $\dfrac{2-19y}{3y(y-2)}$

35. $\dfrac{40x}{(2x+5)(2x-5)}$

36. $\dfrac{2x+3}{2x-1}$

37. $\dfrac{3(x+2)}{x-8}$

38. 0

39. $\dfrac{2x+5}{x+3}$

40. $\dfrac{4x^2-15x+24}{5(x+3)(x-3)}$

41. $\dfrac{2x+1}{3x+1}$

42. $\dfrac{x-3}{x-5}$

43. $\dfrac{x^2+4x+4}{3x(x-2)}$

44. $\dfrac{x+6}{x-6}$

45. $\dfrac{x-6}{x-7}$

46. $\dfrac{1}{x}$

47. $\dfrac{x-5-x^2}{(x+5)(x-5)}$

48. $\dfrac{3}{x+2}$

49. $\dfrac{5}{x-8}$

50. $\dfrac{4}{(3x-4)(3x+4)}$

51. $\dfrac{2x-1}{x(x-1)}$

52. $\dfrac{2x^3+21x^2-8x+16}{2x(x-4)(x+4)}$

53. $\dfrac{x(x-4)}{(x+2)(3x-1)}$

54. $\dfrac{2x+1}{x+2}$

55. $\dfrac{x+6}{2x+1}$

56. $\dfrac{2(x-4)}{(x+2)(x-3)(x-5)}$

57. $\dfrac{x-9}{(x-5)(x+3)}$

58. $\dfrac{y^2+1}{(2y-1)(y+3)(2y+1)}$

59. $\dfrac{y^2-8y-5}{(y+5)(y-5)^2}$

60. $\dfrac{x(3x^2-8x+3)}{2(x-2)(2x^2-x-4)}$

61. $\dfrac{4x}{x+1}$

62. $-\dfrac{2(4a^3+11a^2-9a+36)}{(a^2-a+4)(a-9)(a-4)}$

63. $\dfrac{a+5}{a-4}$

64. $\dfrac{(a-2)(a-3)}{(2a+1)(a-4)}$

65. $\dfrac{6x}{(x+3)(x-3)^2}$

66. $\dfrac{x+5}{x+7}$

67. $\dfrac{x-7}{x+2}$

68. $-\dfrac{2(x+4)}{3x+1}$

69. $\dfrac{x-5}{4x-1}$

70. $\dfrac{5}{2x-3}$

71. $\dfrac{2y-1}{y(y-1)}$

72. $\dfrac{y^2+2y-1}{y(y+1)(y-1)}$

73. $\dfrac{27}{50}$

74. $\dfrac{11}{48}$

75. $\dfrac{x+y}{xy}$

76. $\dfrac{y-x^2}{x^2y}$

77. $\dfrac{(x-1)^2}{x^2(2x-1)}$

78. $\dfrac{5}{(x-4)(x+1)}$

79. $\dfrac{x(x+2)}{x-1}$

80. $\dfrac{3-y}{(y-1)^2}$

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CHAPTER 8: APPLICATIONS OF FRACTIONS

Chapter Outline

8.1 Complex Rational Expressions

- Definitions
- Method 1: Simplify Using Division
- Method 2: Simplify Using the LCD

8.2 Equations with Fractions

- Solving Rational Equations
- Literal Equations

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- 8.1 Complex Rational Expressions
- 8.2 Equations with Fractions
- 8.3 Word Problems with Fractions

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8.1 Complex Rational Expressions

Learning Objectives

By the end of this section, you should be able to:

- Simplify complex rational expressions by multiplying the numerator by the reciprocal of the divisor.
- Simplify complex rational expressions by multiplying the numerator and denominator by the least common denominator (LCD).

Definitions

A complex fraction is a fraction where the numerator or denominator consists of one or more fractions. For example,

$$\frac{\frac{3}{4}}{\frac{1}{2}}$$

Simplifying such a fraction requires us to find an equivalent fraction with an integer numerator and denominator. One way to do this is to divide. Recall that dividing fractions involves multiplying by the reciprocal of the divisor.

$$\frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{\cancel{4}^{2}} \cdot \frac{\cancel{2}^{1}}{1} = \frac{3}{2}$$

Method 1: Using Division

An alternative method for simplifying this complex fraction involves multiplying both the numerator and denominator by the LCD of all the given fractions. In this case, the LCD = 4.

$$\frac{\frac{3}{4} \cdot 4}{\frac{1}{2} \cdot 4} = \frac{3}{2}$$

Method 2: Using the LCD.

A **complex rational expression** is defined as a rational expression that contains one or more rational expressions in the numerator or denominator or both. For example,

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}$$

We simplify a complex rational expression by finding an equivalent fraction where the numerator and denominator are polynomials. As illustrated above, there are two methods for simplifying complex rational expressions, and we will outline the steps for both methods. For the sake of clarity, assume that variable expressions used as denominators are nonzero.

Method 1: Simplify Using Division

We begin our discussion on simplifying complex rational expressions using division. Before we can multiply by the reciprocal of the divisor, we must simplify the numerator and denominator separately. The goal is to first obtain single algebraic fractions in the numerator and the denominator. The steps for simplifying a complex algebraic fraction are illustrated in the following example.

How To

Given a rational expression, simplify it.

1. Factor the numerator and denominator.
2. Cancel any common factors.

Example 1

Simplify:

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}$$

Solution:

Step 1: Simplify the numerator and denominator. The goal is to obtain a single algebraic fraction

divided by another single algebraic fraction. In this example, find equivalent terms with a common denominator in both the numerator and denominator before adding and subtracting.

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}} = \frac{\frac{1}{2}\cdot\frac{x}{x}+\frac{1}{x}\cdot\frac{2}{2}}{\frac{1}{4}\cdot\frac{x^2}{x^2}-\frac{1}{x^2}\cdot\frac{4}{4}}$$

$$= \frac{\frac{x}{2x}+\frac{2}{2x}}{\frac{x^2}{4x^2}-\frac{4}{4x^2}}$$ Equivalent fractions with common denominators.

$$= \frac{\frac{x+2}{2x}}{\frac{x^2-4}{4x^2}}$$ Add the fractions in the numerator and denominator.

At this point, we have a single algebraic fraction divided by a single algebraic fraction.

Step 2: Multiply the numerator by the reciprocal of the divisor.

$$\frac{\frac{x+2}{2x}}{\frac{x^2-4}{4x^2}} = \frac{x+2}{2x}\cdot\frac{4x^2}{x^2-4}$$

Step 3: Factor all numerators and denominators completely.

$$= \frac{x+2}{2x}\cdot\frac{4x^2}{(x+2)(x-2)}$$

Step 4: Cancel all common factors.

$$= \frac{4x^2\cdot(x+2)}{2x(x+2)(x-2)}$$

$$= \frac{\overset{2x}{\cancel{4x^2}}\cdot\cancel{(x+2)}}{\cancel{2x}\,\cancel{(x+2)}\,(x-2)}$$

$$= \frac{2x}{x-2}$$

Answer

$$\frac{2x}{x-2}$$

Example 2

Simplify:

$$\frac{\frac{1}{x}-\frac{1}{x-2}}{\frac{4}{x^2-2x}}$$

Solution:

$$\frac{\frac{1}{x}-\frac{1}{x-2}}{\frac{4}{x^2-2x}}=\frac{\frac{1}{x}\cdot\frac{x-2}{x-2}-\frac{1}{x-2}\cdot\frac{x}{x}}{\frac{4}{x^2-2x}}$$

$$=\frac{\frac{x-2}{x(x-2)}-\frac{x}{x(x-2)}}{\frac{4}{x^2-2x}}$$

$$=\frac{\frac{x-2-x}{x(x-2)}}{\frac{4}{x^2-2x}}$$

$$=\frac{\frac{-2}{x(x-2)}}{\frac{4}{x^2-2x}}$$

$$=\frac{-2}{x(x-2)}\cdot\frac{x^2-2x}{4}$$

$$=\frac{-2}{x(x-2)}\cdot\frac{x(x-2)}{4}$$

$$=-\frac{1}{2}$$

Answer

$-\frac{1}{2}$

Example 3

Simplify:

$$\frac{1-\frac{4}{x}-\frac{21}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}}$$

Solution:
The LCD of the rational expressions in both the numerator and denominator is x^2. Multiply by the appropriate factors to obtain equivalent terms with this as the denominator and then subtract.

$$\frac{1-\frac{4}{x}-\frac{21}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}} = \frac{\frac{1}{1}\cdot\frac{x^2}{x^2}-\frac{4}{x}\cdot\frac{x}{x}-\frac{21}{x^2}}{\frac{1}{1}\cdot\frac{x^2}{x^2}-\frac{2}{x}\cdot\frac{x}{x}-\frac{15}{x^2}}$$

$$= \frac{\frac{x^2}{x^2}-\frac{4x}{x^2}-\frac{21}{x^2}}{\frac{x^2}{x^2}-\frac{2x}{x^2}-\frac{15}{x^2}}$$

$$= \frac{\frac{x^2-4x-21}{x^2}}{\frac{x^2-2x-15}{x^2}}$$

$$= \frac{x^2-4x-21}{x^2}\cdot\frac{x^2}{x^2-2x-15} = \frac{(x+3)(x-7)}{(x+3)(x-5)} = \frac{x-7}{x-5}$$

Answer

$$\frac{x-7}{x-5}$$

Example 4

Simplify:

$$\frac{1-\dfrac{1}{x^2}}{\dfrac{1}{x}-1}$$

Solution:

$$\frac{1-\dfrac{1}{x^2}}{\dfrac{1}{x}-1} = \frac{\dfrac{1}{1}\cdot\dfrac{x^2}{x^2}-\dfrac{1}{x^2}}{\dfrac{1}{x}-\dfrac{1}{1}\cdot\dfrac{x}{x}}$$

$$= \frac{\dfrac{x^2-1}{x^2}}{\dfrac{1-x}{x}}$$

$$= \frac{x^2-1}{x^2}\cdot\frac{x}{1-x}$$

$$= \frac{(x+1)\cancel{(x-1)}}{\cancel{x^2}\,x}\cdot\frac{\cancel{x}}{-1\cdot\cancel{(x-1)}}$$

$$= \frac{x+1}{-1\cdot x}$$

$$= -\frac{x+1}{x}$$

Answer

$$-\frac{x+1}{x}$$

Try It

Simplify: $\dfrac{\frac{1}{81}-\frac{1}{x^2}}{\frac{1}{9}+\frac{1}{x}}$

Show Solution

$$\frac{x-9}{9x}$$

Science Application

Often, equations that we use in the lab contain complex fractions. If you can properly track the units in these fractions, your answer will have the correct units. Let's see:

When calculating percent yield in chemistry, you use the formula:

$$\%\text{yield} = \frac{\text{actual mass}}{(\frac{\text{mass used}}{\text{molar mass}})(\text{molar mass})} \times 100\%$$

What are the units of percent yield? We can determine them by using a complex fraction and inserting just the units into the equation, and cancelling:

$$\%\text{yield} = \frac{g}{(\frac{g}{\frac{g}{mol}})(\frac{g}{mol})} \times 100\%$$

That's a lot of fractions! Invert and multiply in the denominator to cancel out units:

$$\%\text{yield} = \frac{g}{g\frac{\cancel{mol}}{\cancel{g}}\frac{\cancel{g}}{\cancel{mol}}} \times 100\%$$

This leaves you with just:

$$\%\text{yield} = \frac{\cancel{g}}{\cancel{g}} \times 100\%$$

So the units are:

$$\%\text{yield} = \%$$

Method 2: Simplify Using the LCD

An alternative method for simplifying complex rational expressions involves clearing the fractions by multiplying the expression by a special form of 1. In this method, multiply the numerator and denominator by the least common denominator (LCD) of all given fractions.

Example 5

Simplify:

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}$$

Solution:

Step 1: Determine the LCD of all the fractions in the numerator and denominator. In this case, the denominators of the given fractions are $2, x, 4$, and x^2. Therefore, the LCD is $4x^2$.

Step 2: Multiply the numerator and denominator by the LCD. This step should clear the fractions in both the numerator and denominator.

$$\begin{aligned}
\frac{\frac{1}{2}+\frac{1}{x^2}}{\frac{1}{4}-\frac{1}{x^2}} &= \frac{\left(\frac{1}{2}+\frac{1}{x}\right)\cdot 4x^2}{\left(\frac{1}{4}-\frac{1}{x^2}\right)\cdot 4x^2} && \text{Distribute.}\\
&= \frac{\frac{1}{2}\cdot 4x^2+\frac{1}{x}\cdot 4x^2}{\frac{1}{4}\cdot 4x^2-\frac{1}{x^2}\cdot 4x^2} && \text{Cancel.}\\
&= \frac{2x^2+4x}{x^2-4}
\end{aligned}$$

This leaves us with a single algebraic fraction with a polynomial in the numerator and in the denominator.

Step 3: Factor the numerator and denominator completely.

$$\begin{aligned}
&= \frac{2x^2+4x}{x^2-4}\\
&= \frac{2x(x+2)}{(x+2)(x-2)}
\end{aligned}$$

Step 4: Cancel all common factors.

$$\begin{aligned}
&= \frac{2x\,\cancel{(x+2)}}{\cancel{(x+2)}\,(x-2)}\\
&= \frac{2x}{x-2}
\end{aligned}$$

Answer

$$\frac{2x}{x-2}$$

Note

This was the same problem that we began this section with, and the results here are the same. It is worth taking the time to compare the steps involved using both methods on the same problem.

Example 6

Simplify:

$$\frac{1-\frac{2}{x}-\frac{15}{x^2}}{3-\frac{14}{x}-\frac{5}{x^2}}$$

Solution:
Considering all of the denominators, we find that the LCD is x^2. Therefore, multiply the numerator and denominator by x^2:

$$\frac{1-\frac{2}{x}-\frac{15}{x^2}}{3-\frac{14}{x}-\frac{5}{x^2}} = \frac{x^2-2x-15}{3x^2-14x-5}$$

At this point, we have a rational expression that can be simplified by factoring and then cancelling the common factors.

$$= \frac{(x+3)\cancel{(x-5)}}{(3x+1)\cancel{(x-5)}} \qquad \text{Cancel.}$$

$$= \frac{x+3}{3x+1}$$

Answer

$$\frac{x+3}{3x+1}$$

Multiplying the numerator and denominator by the same nonzero factor is equivalent to multiplying by 1 and does not change the problem. Because $\frac{x^2}{x^2} = 1$, we can multiply the numerator and denominator by x^2 in the previous example and obtain an equivalent expression.

Example 7

Simplify:

$$\frac{\frac{1}{x+1} + \frac{3}{x-3}}{\frac{2}{x-3} - \frac{1}{x+1}}$$

Solution:
The LCM of all the denominators is $(x+1)(x-3)$. Begin by multiplying the numerator and denominator by these factors.

$$\frac{\frac{1}{x+1}+\frac{3}{x-3}}{\frac{2}{x-3}-\frac{1}{x+1}}=\frac{\left(\frac{1}{x+1}+\frac{3}{x-3}\right)\cdot(x+1)(x-3)}{\left(\frac{2}{x-3}-\frac{1}{x+1}\right)\cdot(x+1)(x-3)}$$ Distribute.

$$=\frac{\frac{1\cancel{(x+1)}(x-3)}{\cancel{x+1}}+\frac{3(x+1)\cancel{(x-3)}}{\cancel{x-3}}}{\frac{2(x+1)\cancel{(x-3)}}{\cancel{x-3}}-\frac{1\cancel{(x+1)}(x-3)}{\cancel{x+1}}}$$ Cancel.

$$=\frac{(x-3)+3(x+1)}{2(x+1)-1(x-3)}$$ Simplify.

$$=\frac{x-3+3x+3}{2x+2-x+3}$$

$$=\frac{4x}{x+5}$$

Answer

$$\frac{4x}{x+5}$$

Try It

Simplify: $\dfrac{\frac{1}{y}-\frac{1}{4}}{\frac{1}{16}-\frac{1}{y^2}}$

Show Solution

$$-\frac{4y}{y+4}$$

Key Takeaways

- Complex rational expressions can be simplified into equivalent expressions with a polynomial numerator and polynomial denominator.
- One method of simplifying a complex rational expression requires us to first write the numerator and denominator as a single algebraic fraction. Then multiply the numerator by the reciprocal of the divisor and simplify the result.
- Another method for simplifying a complex rational expression requires that we multiply it by a special form of 1. Multiply the numerator and denominator by the LCM of all the denominators as a means to clear the fractions. After doing this, simplify the remaining rational expression.
- An algebraic fraction is reduced to lowest terms if the numerator and denominator are polynomials that share no common factors other than 1.

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8.2 Equations with Fractions

Learning Objectives

By the end of this section, you should be able to:

- Solve rational equations.
- Solve **literal equations**, or formulas, involving rational expressions.

Solving Rational Equations

A rational equation is an equation containing at least one rational expression. Rational expressions typically contain a variable in the denominator. For this reason, we will take care to ensure that the denominator is not 0 by making note of restrictions and checking our solutions.

Solve rational equations by clearing the fractions by multiplying both sides of the equation by the least common denominator (LCD).

How To

Given a rational equation, solve it.

1. Factor all denominators in the equation.
2. Find and exclude values that set each denominator equal to zero.
3. Find the LCD.
4. Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.
5. Solve the remaining equation.
6. Make sure to check solutions back in the original equations to avoid a solution producing zero in a denominator.

Example 1

Solve:

$$\frac{5}{x} - \frac{1}{3} = \frac{1}{x}$$

Solution:
We first make a note that $x \neq 0$ and then multiply both sides by the LCD, $3x$:

$$\begin{aligned} 3x\left(\frac{5}{x} - \frac{1}{3}\right) &= 3x\left(\frac{1}{x}\right) && \text{Multiply both sides by the LCD.} \\ 3x \cdot \frac{5}{x} - 3x \cdot \frac{1}{3} &= 3x \cdot \frac{1}{x} && \text{Distribute.} \\ 15 - x &= 3 && \text{Simplify, and then solve.} \\ -x &= -12 \\ x &= 12 \end{aligned}$$

Check your answer by substituting 12 for x to see if you obtain a true statement.

$$\begin{aligned} \frac{5}{x} - \frac{1}{3} &= \frac{1}{x} \\ \frac{5}{12} - \frac{1}{3} &= \frac{1}{12} \\ \frac{5}{12} - \frac{4}{12} &= \frac{1}{12} \\ \frac{1}{12} &= \frac{1}{12} \quad \checkmark \end{aligned}$$

Answer

The solution is 12.

After multiplying both sides of the previous example by the LCD, we were left with a linear equation to solve. This is not always the case; sometimes we will be left with a quadratic equation, which you will learn in the next chapter.

Example 2

Solve:

$$2 - \frac{1}{x(x+1)} = \frac{3}{x+1}$$

Solution:
In this example, there are two restrictions, $x \neq 0$ and $x \neq -1$. Begin by multiplying both sides by the LCD, $x(x+1)$.

$$x(x+1)\left(2 - \frac{1}{x(x+1)}\right) = x(x+1) \cdot \frac{3}{x+1}$$

$$x(x+1) \cdot 2 - \frac{x(x+1) \cdot 1}{x(x+1)} = \frac{x(x+1) \cdot 3}{x+1}$$

$$2x(x+1) - 1 = 3x$$

$$2x^2 + 2x - 1 = 3x$$

After distributing and dividing out the common factors, a quadratic equation remains. To solve it, rewrite it in standard form, factor, and then set each factor equal to 0.

$$\begin{aligned} 2x + 1 &= 0 \quad \text{or} \quad x - 1 = 0 \\ 2x &= -1 \qquad\quad x = 1 \\ x &= -\tfrac{1}{2} \end{aligned}$$

Check to see if these values solve the original equation.

$$2 - \frac{1}{x(x+1)} = \frac{3}{x+1}$$

$$\begin{aligned} \text{Check } x &= -\tfrac{1}{2} \\ 2-\frac{1}{(-\frac{1}{2})((-\frac{1}{2})+1)} &= \frac{3}{(-\frac{1}{2})+1} \\ 2-\frac{1}{(-\frac{1}{4})} &= \frac{3}{\frac{1}{2}} \\ 2+1\cdot\tfrac{4}{1} &= 3\cdot\tfrac{2}{1} \\ 2+4 &= 6 \\ 6 &= 6 \quad \checkmark \end{aligned} \qquad \begin{aligned} \text{Check } x &= 1 \\ 2-\frac{1}{1(1+1)} &= \frac{3}{1+1} \\ \tfrac{4}{2}-\tfrac{1}{2} &= \tfrac{3}{2} \\ \tfrac{3}{2} &= \tfrac{3}{2} \quad \checkmark \end{aligned}$$

Answer

The solutions are $-\dfrac{1}{2}$ and 1.

Up to this point, all of the possible solutions have solved the original equation. However, this may not always be the case. Multiplying both sides of an equation by variable factors may lead to **extraneous solutions**, which are solutions that do not solve the original equation. This is why you need to check all solutions in the original equation to see if they work. A complete list of steps for solving a rational equation is outlined in the following example.

Example 3

Solve:

$$\frac{x}{x+2}+\frac{2}{x^2+5x+6}=\frac{5}{x+3}$$

Solution:

Step 1: Factor all denominators and determine the LCD.

$$\frac{x}{x+2}+\frac{2}{x^2+5x+6}=\frac{5}{x+3}$$

$$\frac{x}{(x+2)}+\frac{2}{(x+2)(x+3)}=\frac{5}{(x+3)}$$

The LCD is $(x+2)(x+3)$.

Step 2: Identify the restrictions. In this case, they are $x \neq -2$ and $x \neq -3$.

Step 3: Multiply both sides of the equation by the LCD. Distribute carefully and then simplify.

$$(x+2)(x+3)\cdot\left(\frac{x}{(x+2)}+\frac{2}{(x+2)(x+3)}\right)=(x+2)(x+3)\cdot\frac{5}{(x+3)}$$

$$\frac{x\cdot\cancel{(x+2)}\,(x+3)}{\cancel{(x+2)}}+\frac{2\cdot\cancel{(x+2)}\,\cancel{(x+3)}}{\cancel{(x+2)}\,\cancel{(x+3)}}=\frac{5\cdot(x+2)\,\cancel{(x+3)}}{\cancel{(x+3)}}$$

$$x(x+3)+2=5(x+2)$$

$$x^2+3x+2=5x+10$$

Step 4: Solve the resulting equation. Here, the result is a quadratic equation. Rewrite it in standard form, factor, and then set each factor equal to 0.

$$\begin{aligned}x^2+3x+2&=5x+10\\x^2-2x-8&=0\\(x+2)(x-4)&=0\end{aligned}$$

$$\begin{aligned}x+2&=0\\x&=-2\end{aligned}\qquad\text{or}\qquad\begin{aligned}x-4&=0\\x&=4\end{aligned}$$

Step 5: Check for extraneous solutions. Always substitute into the original equation or the factored equivalent. In this case, choose the factored equivalent to check:

$$\frac{x}{(x+2)}+\frac{2}{(x+2)(x+3)}=\frac{5}{(x+3)}$$

Check $x = -2$		Check $x = 4$	
$\frac{-2}{(-2+2)} + \frac{2}{(-2+2)(-2+3)} = \frac{5}{(-2+3)}$		$\frac{4}{(4+2)} + \frac{2}{(4+2)(4+3)} = \frac{5}{(4+3)}$	
$-\frac{2}{0} + \frac{2}{0(1)} = \frac{5}{1}$	x	$\frac{4}{6} + \frac{2}{6 \cdot 7} = \frac{5}{7}$	
Undefined terms!		$\frac{2}{3} + \frac{1}{21} = \frac{5}{7}$	
		$\frac{14}{21} + \frac{1}{21} = \frac{5}{7}$	
		$\frac{15}{21} = \frac{5}{7}$	
		$\frac{5}{7} = \frac{5}{7}$	✓

Here -2 is an extraneous solution and is not included in the solution set. It is important to note that -2 is a restriction.

Answer

The solution is 4.

If this process produces a solution that happens to be a restriction, then disregard it as an extraneous solution.

Science Application

The decomposition of ammonium nitrate NH_4NO_3 is monitored over 15 seconds, and it is found that the rate of formation of N_2O is $0.02\frac{mol}{L \cdot s}$

a) Calculate the total amount of NH_4NO_3 decomposed and the total amount of H_2O formed in moles?

We first need the balanced chemical equation for the decomposition of ammonium nitrate NH_4NO_3:

$$NH_4NO_3(s) \rightarrow N_2O(g) + 2H_2O(g)$$

From this balanced equation, the molar ratios are:

1 mole of NH_4NO_3 produces 1 mole of N_2O.

1 mole of NH_4NO_3 produces 2 moles of H_2O.

Find the total moles of N_2O produced:

The rate of formation of N_2O is given as 0.02 $\text{mol}/(\text{L} \cdot \text{s})$. Assuming a volume of 1 L (which is standard when volume isn't specified), we find the total moles by multiplying the rate by the total time: $\text{Moles of } N_2O = \text{Rate} \times \text{Time}$

$$\text{Moles of } N_2O = 0.02 \frac{\text{mol}}{\text{s}} \times 15 \text{ s} = 0.30 \text{ moles}$$

Find the moles of NH_4NO_3 decomposed:

According to the balanced equation, the ratio of NH_4NO_3 to N_2O is 1:1.

$$\text{Moles of } NH_4NO_3 \text{ decomposed} = 0.30 \text{ moles}$$

Find the moles of H_2O formed:

The ratio of N_2O to H_2O is $1:2$. This means twice as much water is formed as N_2O.

$$\text{Moles of } H_2O \text{ formed} = 0.30 \text{ moles} \times 2 = 0.60 \text{ moles}$$

b) If you started with 1 mole of NH_4NO_3, what proportion is left after 15 seconds?

Calculate the moles remaining: If we started with 1.00 mole and 0.30 moles decomposed, we subtract to find what is left:

$$\text{Moles left} = \text{Initial moles} - \text{Decomposed moles}$$
$$\text{Moles left} = 1.00 - 0.30 = 0.70 \text{ moles}$$

Calculate the proportion: The proportion is the amount remaining divided by the starting amount.

$$\text{Proportion left} = \frac{0.70 \text{ moles}}{1.00 \text{ mole}} = 0.70 \text{ (or } 70\%)$$

Try It

Solve: $\dfrac{x}{x-5} + \dfrac{3}{x+2} = \dfrac{7x}{x^2-3x-10}$

Show Solution

-3

Sometimes all potential solutions are extraneous, in which case we say that there is no solution to the original equation. In the next two examples, we demonstrate two ways in which a rational equation can have no solutions.

Example 4

Solve:

$$\frac{3x}{x^2-4}-\frac{2}{x+2}=\frac{1}{x+2}$$

Solution:
To identify the LCD, first factor the denominators.

$$\frac{3x}{x^2-4}-\frac{2}{x+2}=\frac{1}{x+2}$$
$$\frac{3x}{(x+2)(x-2)}-\frac{2}{(x+2)}=\frac{1}{(x+2)}$$

Multiply both sides by the least common denominator (LCD), $(x+2)(x-2)$, distributing carefully.

$$\frac{3x\cdot(x+2)(x-2)}{(x+2)(x-2)}-\frac{2\cdot(x+2)(x-2)}{(x+2)}=\frac{1\cdot(x+2)(x-2)}{(x+2)}$$
$$3x-2(x-2)=1(x-2)$$
$$3x-2x+4=x-2$$
$$x+4=x-2$$
$$4=-2 \quad \text{False}$$

The equation is a **contradiction** and thus has no solution.
Answer

No solution, $\emptyset$

Example 5

Solve:

$$\frac{x}{x-4}-\frac{4}{x+5}=\frac{36}{x^2+x-20}$$

Solution:
First, factor the denominators.

$$\frac{x}{(x-4)}-\frac{4}{(x+5)}=\frac{36}{(x-4)(x+5)}$$

Take note that the restrictions are $x \neq 4$ and $x \neq -5$. To clear the fractions, multiply by the LCD, $(x-4)(x+5)$.

$$\frac{x\cdot(x-4)(x+5)}{(x-4)}-\frac{4\cdot(x-4)(x+5)}{(x+5)}=\frac{36\cdot(x-4)(x+5)}{(x-4)(x+5)}$$

$$\begin{aligned} x(x+5)-4(x-4)&=36\\ x^2+5x-4x+16&=36\\ x^2+x-20&=0\\ (x-4)(x+5)&=0 \end{aligned}$$

$$\begin{aligned} x-4&=0 \quad \text{or} \quad x+5=0\\ x&=4 \qquad\qquad x=-5 \end{aligned}$$

Both of these values are restrictions of the original equation; hence, both are extraneous.

Answer

No solution, $\emptyset$

Try It

Solve: $\dfrac{1}{x+1}+\dfrac{x}{x-3}=\dfrac{4x}{x^2-2x-3}$.

Show Solution

$\emptyset$

It is important to point out that this technique for clearing algebraic fractions only works for equations. Do not try to clear algebraic fractions when simplifying expressions. As a reminder, we have

Expression	Equation
$\frac{1}{x}+\frac{x}{2x+1}$	$\frac{1}{x}+\frac{x}{2x+1}=0$

Expressions are to be simplified, and equations are to be solved. If we multiply the expression by the LCD, $x(2x+1)$, we obtain another expression that is not equivalent.

Incorrect	Correct
$\frac{1}{x}+\frac{x}{2x+1}$	$\frac{1}{x}+\frac{x}{2x+1}=0$
$\neq x(2x+1)\cdot\left(\frac{1}{x}+\frac{x}{2x+1}\right)$	$x(2x+1)\cdot\left(\frac{1}{x}+\frac{x}{2x+1}\right)=x(2x+1)\cdot 0$
$=2x+1+x^2$ x	$2x+1+x^2=0$
	$x^2+2x+1=0$ ✓

Literal Equations

Literal equations, or formulas, are often rational equations. Hence, the techniques described in this section can be used to solve for particular variables. Assume that all variable expressions in the denominator are nonzero.

Example 6

Solve for x:

$$z = \frac{x - 5}{y}$$

Solution:
The goal is to isolate x. Assuming that y is nonzero, multiply both sides by y and then add 5 to both sides.

$$\begin{aligned} y \cdot z &= y \cdot \frac{x - 5}{y} \\ yz &= x - 5 \\ yz + 5 &= x - 5 + 5 \\ yz + 5 &= x \end{aligned}$$

Answer

$x = yz + 5$

Example 7

Solve for c:

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$$

Solution:
In this example, the goal is to isolate c. We begin by multiplying both sides by the LCD, abc, distributing carefully.

$$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$$
$$abc \cdot \frac{1}{c} = abc \cdot \frac{1}{a} + abc \cdot \frac{1}{b}$$
$$ab = bc + ac$$

On the right side of the equation, factor out c.

$$ab = c(b + a)$$

Next, divide both sides of the equation by the quantity $(b + a)$.

$$\frac{ab}{(b+a)} = \frac{c(b+a)}{(b+a)}$$
$$\frac{ab}{b+a} = c$$

Answer

$$c = \frac{ab}{b + a}$$

Try It

Solve for y: $x = \dfrac{y + 1}{y - 1}$

Show Solution

$$y = \frac{1 + x}{x - 1}$$

Key Takeaways

- Begin solving rational equations by multiplying both sides by the LCD. The resulting equivalent equation can be solved using the techniques learned up to this point.
- Multiplying both sides of a rational equation by a variable expression introduces the possibility of extraneous solutions. Therefore, we must check the solutions against the set of restrictions. If a solution is a restriction, then it is not part of the domain and is extraneous.
- When multiplying both sides of an equation by an expression, distribute carefully and multiply each term by that expression.
- If all of the resulting solutions are extraneous, then the original equation has no solutions.

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8.3 Word Problems with Fractions

Learning Objectives

By the end of this section, you should be able to:

- Solve applications involving relationships between real numbers.
- Solve applications involving uniform motion (distance problems).
- Solve work-rate applications.

Number Problems

Recall that the reciprocal of a nonzero number n is $\frac{1}{n}$. For example, the reciprocal of 5 is $\frac{1}{5}$ and $5 \cdot \frac{1}{5} = 1$.

In this section, the applications will often involve the keyword “reciprocal.” When this is the case, we will see that the algebraic setup results in a rational equation.

Example 1

A positive integer is 4 less than another. The sum of the reciprocals of the two positive integers is $\frac{10}{21}$. **Find the two integers.**

Solution:
Begin by assigning variables to the unknowns.

- Let n represent the larger positive integer.
- Let $n - 4$ represent the smaller positive integer.

Next, use the reciprocals $\frac{1}{n}$ and $\frac{1}{n-4}$ to translate the sentences into an algebraic equation.

$$\frac{1}{n}+\frac{1}{n-4}=\frac{10}{21}$$

We can solve this rational equation by multiplying both sides by the least common denominator (LCD).

In this case, the LCD is $21n(n-4)$.

$$\begin{aligned}\frac{1}{n}+\frac{1}{n-4}&=\frac{10}{21}\\ 21n(n-4)\left(\frac{1}{n}+\frac{1}{n-4}\right)&=21n(n-4)\left(\frac{10}{21}\right)\\ 21(n-4)+21n&=10n(n-4)\end{aligned}$$

Solve the resulting quadratic equation.

$$\begin{aligned}21(n-4)+21n&=10n(n-4)\\ 42n-84&=10n^2-40n\\ 0&=10n^2-82n+84\\ 0&=2\left(5n^2-41n+42\right)\\ 0&=2(5n-6)(n-7)\end{aligned}$$

$$\begin{aligned}5n-6=0 \quad &\text{or} \quad n-7=0\\ n=\frac{6}{5} \quad &\text{or} \quad n=7\end{aligned}$$

The question calls for integers, and the only integer solution is $n=7$. Hence disregard $\frac{6}{5}$.

Use the expression $n-4$ to find the smaller integer.

$$n-4=7-4=3$$

Answer

The two positive integers are 3 and 7. The check is left to the reader.

Example 2

A positive integer is 4 less than another. If the reciprocal of the smaller integer is subtracted from twice the reciprocal of the larger, then the result is $\frac{1}{30}$. **Find the two integers.**

Solution:

- Let n represent the larger positive integer.
- Let $n-4$ represent the smaller positive integer.

Set up an algebraic equation.

$$\frac{2}{n}-\frac{1}{n-4}=\frac{1}{30}$$

Solve this rational equation by multiplying both sides by the LCD. The LCD is $30n(n-4)$.

$$\begin{aligned}
\frac{2}{n}-\frac{1}{n-4}&=\frac{1}{30}\\
30n(n-4)\left(\frac{2}{n}-\frac{1}{n-4}\right)&=30n(n-4)\left(\frac{1}{30}\right)\\
60(n-4)-30n&=n(n-4)\\
60n-240-30n&=n^2-4n\\
30n-240&=n^2-4n\\
0&=n^2-34n+240\\
0&=(n-10)(n-24)
\end{aligned}$$

$$n=10 \quad \text{or} \quad n=24$$

Here we have two viable possibilities for the larger integer. For this reason, we have two solutions.

If $n=10$, then $n-4=10-4=6$.

If $n=24$, then $n-4=24-4=20$.

As a check, perform the operations indicated in the problem.

Check 6 and 10	Check 20 and 24
$2\left(\frac{1}{10}\right)-\frac{1}{6}=\frac{1}{5}-\frac{1}{6}$	$2\left(\frac{1}{24}\right)-\frac{1}{20}=\frac{1}{12}-\frac{1}{20}$
$=\frac{6}{30}-\frac{5}{30}$	$=\frac{5}{60}-\frac{3}{60}$
$=\frac{1}{30}$ ✓	$=\frac{1}{30}$ ✓

Answer

Two sets of positive integers solve this problem: $(6, 10)$ and $(20, 24)$.

Try It

The difference between the reciprocals of two consecutive positive odd integers is $\frac{2}{15}$. **Find the integers.**

Show Solution

The integers are 3 and 5.

Science Applications

In the first step of glycolysis, glucose reacts with ATP to form glucose-6-phosphate and ADP. The reaction follows second-order kinetics with respect to both glucose and ATP. The rate constant (k) for the reaction is $(0.02/M \cdot s)$. Initially, the concentration of glucose $([A] = 0.1M)$. Calculate the concentration of glucose $([A]0)$ after 500 seconds (t).

$$\frac{1}{[A]} = \frac{1}{[A_o]} + kt$$

Substitute values into the equation:

$$\frac{1}{[A]} = \frac{1}{0.1} + (0.02 \times 500)$$

Solve the right side by following the order of operations (first divide/multiply, then add/subtract):

$$\frac{1}{[A]} = 10 + 10$$

$$\frac{1}{[A]} = 20$$

Now, take the reciprocal to find [A]. You are really multiplying both sides by [A] and dividing both sides by 20:

$$[A] = 0.05 \text{ M}$$

During the first step of glycolysis, glucose is being used up to create glucose-6-phosphate. Because this is a second-order reaction, the concentration doesn't drop in a straight line; it slows down as there is less glucose available. After 500 seconds, exactly half of your starting glucose (0.1 M) has been converted, leaving you with 0.05 M.

Uniform Motion Problems

Uniform motion problems, also referred to as distance problems, involve the formula

$$D = rt$$

where the distance, D, is given as the product of the average rate, r, and the time, t, travelled at that rate. If we divide both sides by the average rate, r, then we obtain the formula

$$t = \frac{D}{r}$$

For this reason, when the unknown quantity is time, the algebraic setup for distance problems often results in a rational equation. Similarly, when the unknown quantity is the rate, the setup also may result in a rational equation.

We begin any uniform motion problem by first organizing our data with a chart. Use this information to set up an algebraic equation that models the application.

Example 3

Mary spent the first 120 miles of her road trip in traffic. When the traffic cleared, she was able to drive twice as fast for the remaining 300 miles. If the total trip took 9 hours, then **how fast was she moving in traffic?**

Solution:
First, identify the unknown quantity and organize the data.

- Let x represent Mary's average speed (miles per hour) in traffic.
- Let $2x$ represent her average speed after the traffic cleared.

Table 8.3.1. Distance = Rate × Time

Traffic	Distance	Rate	Time
In traffic	120 mi	x	
Clear traffic	300 mi	$2x$	
Total			9 hours

To avoid introducing two more variables for the time column, use the formula $t = \dfrac{D}{r}$. Here, the time for each leg of the trip is calculated as follows:

$$\text{Time spent in traffic:} \quad t = \frac{120}{x}$$

$$\text{Time clear of traffic:} \quad t = \frac{300}{2x}$$

Use these expressions to complete the chart.

Traffic	Distance	Rate	Time
In traffic	120 mi	x	$\frac{120}{x}$
Clear traffic	300 mi	$2x$	$\frac{300}{2x}$
Total			9 hours

$$\frac{120}{x}+\frac{300}{2x}=9$$

Figure 8.3.1.

The algebraic setup is defined by the time column. Add the times for each leg of the trip to obtain a total of 9 hours:

$$\frac{120}{x}+\frac{300}{2x}=9$$

We begin solving this equation by first multiplying both sides by the LCD, $2x$.

$$\begin{aligned}\frac{120}{x}+\frac{300}{2x}&=9\\ 2x\left(\frac{120}{x}+\frac{300}{2x}\right)&=2x\cdot 9\\ 240+300&=18x\\ 540&=18x\\ 30&=x\end{aligned}$$

Answer

Mary averaged 30 miles per hour in traffic.

Example 4

A passenger train can travel, on average, 20 miles per hour faster than a freight train. If the passenger train covers 390 miles in the same time it takes the freight train to cover 270 miles, then **how fast is each train?**

Solution:

- Let x represent the average speed of the freight train.
- Let $x + 20$ represent the speed of the passenger train.

Next, organize the given data in a chart.

Table 8.3.2. Distance = Rate × Time

Train	Distance	Rate	Time
Passenger Train	390 mi	$x + 20$	
Freight Train	270 mi	x	
Total			

Use the formula $t = \dfrac{D}{r}$ to fill in the time column for each train.

$$\text{Passenger: } t = \frac{390}{x + 20} \qquad \text{Freight: } t = \frac{270}{x}$$

Table 8.3.3. Distance = Rate × Time

Train	Distance	Rate	Time
Passenger Train	390 mi	$x + 20$	$\dfrac{390}{x + 20}$
Freight Train	270 mi	x	$\dfrac{270}{x}$
Total			

Because the trains travel the same amount of time, finish the algebraic setup by equating the expressions that represent the times:

$$\frac{390}{x+20} = \frac{270}{x}$$

Solve this equation by first multiplying both sides by the LCD, $x(x+20)$.

$$\begin{aligned} x(x+20)\left(\frac{390}{x+20}\right) &= x(x+20)\left(\frac{270}{x}\right) \\ 390x &= 270(x+20) \\ 390x &= 270x + 5400 \\ 120x &= 5400 \\ x &= 45 \end{aligned}$$

Use $x+20$ to find the speed of the passenger train.

$$x + 20 = 45 + 20 = 65$$

Answer

The speed of the passenger train is 65 miles per hour, and the speed of the freight train is 45 miles per hour.

Example 5

Brett lives on the river 8 miles upstream from town. When the current is 2 miles per hour, he can row his boat downstream to town for supplies and back in 3 hours. **What is his average rowing speed in still water?**

Solution:
Let x represent Brett's average rowing speed in still water.

- Rowing downstream, the current increases his speed, and his rate is $x + 2$ miles per hour.
- Rowing upstream, the current decreases his speed, and his rate is $x - 2$ miles per hour.

Begin by organizing the data in the following chart:

Table 8.3.4. Distance = Rate × Time

Trip	Distance	Rate	Time
Trip downstream	8 mi	$x+2$	
Trip upstream	8 mi	$x-2$	
Total			3 hours

Use the formula $t = \dfrac{D}{r}$ to fill in the time column for each leg of the trip.

$$\text{Downstream: } t = \frac{8}{x+2} \qquad \text{Upstream: } t = \frac{8}{x-2}$$

Table 8.3.5. Distance = Rate × Time

Trip	Distance	Rate	Time
Trip downstream	8 mi	$x+2$	$\dfrac{8}{x+2}$
Trip upstream	8 mi	$x-2$	$\dfrac{8}{x-2}$
Total			3 hours

The algebraic setup is defined by the time column. Add the times for each leg of the trip to obtain a total of 3 hours:

$$\frac{8}{x+2} + \frac{8}{x-2} = 3$$

Solve this equation by first multiplying both sides by the LCD, $(x+2)(x-2)$.

$$\begin{aligned}(x+2)(x-2)\left(\frac{8}{x+2} + \frac{8}{x-2}\right) &= 3(x+2)(x-2)\\ 8(x-2) + 8(x+2) &= 3(x^2-4)\\ 16x &= 3x^2 - 12\\ 0 &= 3x^2 - 16x - 12\end{aligned}$$

Next, solve the resulting quadratic equation.

$$3x^2 - 16x - 12 = 0$$
$$(3x + 2)(x - 6) = 0$$
$$3x + 2 = 0 \text{ or } x - 6 = 0$$
$$x = -\frac{2}{3} \text{ or } x = 6$$

Use only the positive solution, $x = 6$ miles per hour.

Answer

His rowing speed is 6 miles per hour.

Try It

Dwayne drove 18 miles to the airport to pick up his father and then returned home. On the return trip, he was able to drive an average of 15 miles per hour faster than he did on the trip there. If the total driving time was 1 hour, then **what was his average speed driving to the airport?**

Show Solution

His average speed driving to the airport was 30 miles per hour.

Science Application

The dilution formula is: $[C_1V_1 = C_2V_2]$. A biologist has a buffer with concentration $(C_1 = \frac{3}{4}, \text{X})$ and wants to dilute it to $(C_2 = \frac{1}{2}, \text{X})$. If the final volume is 200 mL, what was the initial volume (V_1) of the solution that should be used?

Substitute values into the equation:

$$\left(\frac{3}{4}\right) \cdot V_1 = \left(\frac{1}{2}\right) \cdot 200$$

Solve for V_1

Calculate the right side of the equation: $\frac{1}{2} \times 200 = 100$

To get V_1 by itself, we multiply both sides by the reciprocal of $\frac{3}{4}$, which is $\frac{4}{3}V_1 = 100 \times \frac{4}{3}$

$$V_1 = \frac{400}{3}$$

Now use a calculator to get the decimal value:

The initial volume should be approximately 133.33 mL.

Work-Rate Problems

The rate at which a task can be performed is called a **work rate**.

Example 6

For example, if a painter can paint a room in 8 hours, then the task is to paint the room, and we can write

$$\frac{1 \text{ task}}{8 \text{ hours}}$$

In other words, the painter can complete $\frac{1}{8}$ of the task per hour.

a. **Calculate the fraction of the task that one painter can accomplish in 2, 4, and 8 hours.**
b. **If an apprentice would take 10 hours to paint the room, how long would it take for the apprentice and the master painter to paint the room?**

Solution:
a. If he works for less than 8 hours, then he will perform a fraction of the task. For example,

$$\text{work rate} \times \text{time} = \text{work completed}$$

$$\frac{1}{8} \times 2 \text{ hrs} = \frac{1}{4} \qquad \text{One-quarter of the room painted}$$

$$\frac{1}{8} \times 4 \text{ hrs} = \frac{1}{2} \qquad \text{One-half of the room painted}$$

$$\frac{1}{8} \times 8 \text{ hrs} = 1 \qquad \text{One whole room painted}$$

Obtain the amount of the task completed by multiplying the work rate by the amount of time the painter works.

Typically, work-rate problems involve people working together to complete tasks. When this is the case, we can organize the data in a chart, just as we have done with distance problems.

b. Suppose an apprentice painter can paint the same room by himself in 10 hours. Then we say that he can complete $\frac{1}{10}$ of the task per hour.

Let t represent the time it takes both painters, working together, to paint the room.

Table 8.3.6. Amount of Task Completed = Work Rate × Time

Person	Amount of Task Completed	Work Rate	Time
Painter		$\frac{1}{8}$	t
Apprentice		$\frac{1}{10}$	t
Total	1		

To complete the chart, multiply the work rate by the time for each person.

The portion of the room each can paint adds to a total of 1 task completed.

This is represented by the equation obtained from the first column of the chart:

Person	Amount of Task Completed	Work Rate	Time
Painter	$\frac{1}{8}t$	$\frac{1}{8}$	t
Apprentice	$\frac{1}{10}t$	$\frac{1}{10}$	t
Total	1		

$$\frac{1}{8}t+\frac{1}{10}t=1$$

Figure 8.3.2.

This setup results in a rational equation that can be solved for t by multiplying both sides by the LCD, 40.

$$\begin{aligned}\frac{1}{8}t+\frac{1}{10}t&=1\\ 40\left(\frac{1}{8}t+\frac{1}{10}t\right)&=40\\ 5t+4t&=40\\ 9t&=40\\ t&=\frac{40}{9}\\ &=4\frac{4}{9}\end{aligned}$$

Therefore, the two painters, working together, complete the task in $4\frac{4}{9}$ hours.

In general, we have the following work-rate formula:

$$\frac{1}{t_1}t+\frac{1}{t_2}t=1$$

Here $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates, and t is the time it takes to complete one task working together.

If we factor out the time, t, and then divide both sides by t, we obtain an equivalent work-rate formula:

$$\begin{aligned} \frac{1}{t_1}t + \frac{1}{t_2}t &= 1 \\ t\left(\frac{1}{t_1} + \frac{1}{t_2}\right) &= 1 \\ \frac{1}{t_1} + \frac{1}{t_2} &= \frac{1}{t} \end{aligned}$$

In summary, we have the following equivalent work-rate formulas:

$$\frac{t}{t_1} + \frac{t}{t_2} = 1 \qquad \text{or} \qquad \frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{t}$$

In general, we have the following work-rate formula:

$$\frac{1}{t_1}t + \frac{1}{t_2}t = 1$$

Here $\frac{1}{t_1 t}$ and $\frac{1}{t_2 t}$ are the individual work rates, and t is the time it takes to complete one task working together. If we factor out the time, t, and then divide both sides by t, we obtain an equivalent work-rate formula:

$$\begin{aligned} \frac{1}{t_1}t + \frac{1}{t_2}t &= 1 \\ t\left(\frac{1}{t_1}t + \frac{1}{t_2}t\right) &= 1 \\ \frac{1}{t_1}t + \frac{1}{t_2}t &= \frac{1}{t} \end{aligned}$$

In summary, we have the following equivalent work-rate formulas:

Work rate formulas

$\frac{1}{t_1}t \quad$ or $\quad \frac{t}{t_1} + \frac{t}{t_2} = 1 \quad$ or $\quad \frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{t}$

Example 7

Working alone, Billy's dad can complete the yard work in 3 hours. If Billy helps his dad, then the yard work takes 2 hours. **How long would it take Billy, working alone, to complete the yard work?**

Solution:

Billy's dad has an individual work rate of $\frac{1}{3}$ task per hour. If we let x represent the time it takes Billy working alone to complete the yard work, then Billy's individual work rate is $\frac{1}{x}$.

Table 8.3.7. Amount of Task Completed = Work Rate × Time

Person	Amount of Task Completed	Work Rate	Time
Billy's Dad		$\frac{1}{3}$	2
Billy		$\frac{1}{x}$	2
Total	1		

Working together, they can complete the task in 2 hours. Multiply the individual work rates by 2 hours to fill in the chart.

Person	Amount of Task Completed	Work Rate	Time
Billy's Dad	$\frac{1}{3} \cdot 2$	$\frac{1}{3}$	2
Billy	$\frac{1}{x} \cdot 2$	$\frac{1}{x}$	2
Total	1		

$$\frac{1}{3} \cdot 2 + \frac{1}{x} \cdot 2 = 1$$

Figure 8.3.3.

The amount of the task each completes will total 1 completed task. To solve for x, multiply both sides by the LCD, $3x$.

$$\begin{aligned} \frac{1}{3} \cdot 2 + \frac{1}{x} \cdot 2 &= 1 \\ 3x\left(\frac{2}{3} + \frac{2}{x}\right) &= 3x \\ 2x + 6 &= 3x \\ 6 &= x \end{aligned}$$

Answer

It takes Billy 6 hours to complete the yard work alone.

Of course, the unit of time for the work rate need not always be in hours.

Example 8

Working together, two construction crews can build a shed in 5 days. Working separately, the less experienced crew takes twice as long to build a shed than the more experienced crew. Working separately, **how long does it take each crew to build a shed?**

Solution:

- Let x represent the time it takes the more experienced crew to build a shed.
- Let $2x$ represent the time it takes the less experienced crew to build a shed.

Working together, the job is completed in 5 days. This gives the following setup:

Table 8.3.8. Amount of Task Completed = Work Rate × Time

Crew	Amount of Task	Work Rate	Time
Experienced Crew	$\frac{1}{x} \cdot 5$	$\frac{1}{x}$	5 days
Inexperienced Crew	$\frac{1}{2x} \cdot 5$	$\frac{1}{2x}$	5 days
Total	1		

The first column in the chart gives an algebraic equation that models the problem:

$$\frac{1}{x} \cdot 5 + \frac{1}{2x} \cdot 5 = 1 \quad \Rightarrow \quad \frac{5}{x} + \frac{5}{2x} = 1$$

Solve the equation by multiplying both sides by $2x$.

$$\begin{aligned} 2x\left(\frac{5}{x} + \frac{5}{2x}\right) &= 2x \\ 10 + 5 &= 2x \\ 15 &= 2x \\ x &= \frac{15}{2} = 7\frac{1}{2} \end{aligned}$$

To determine the time it takes the less experienced crew, use $2x$:

$$2x = 2\left(\frac{15}{2}\right) = 15$$

Answer

Working separately, the experienced crew takes $7\frac{1}{2}$ days to build a shed, and the less experienced crew takes 15 days.

Try It

Joe's garden hose fills the pool in 12 hours. His neighbour has a thinner hose that fills the pool in 15 hours. **How long will it take to fill the pool using both hoses?**

Show Solution

It will take both hoses $6\frac{2}{3}$ hours to fill the pool.

Key Takeaways

- In this section, all of the steps outlined for solving general word problems apply. Look for the keyword "reciprocal," which indicates that you should write the quantity in the denominator of a fraction with numerator 1.
- When solving distance problems where the time element is unknown, use the equivalent form of the uniform motion formula, $t = \frac{D}{r}$, to avoid introducing more variables.
- When solving work-rate problems, multiply the individual work rate by the time to obtain the portion of the task completed.

The sum of the portions results in the total amount of work completed.

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8.4 Exercises

8.1 Complex Rational Expressions

Simplify. (Assume all denominators are nonzero.)

1. $\dfrac{\frac{1}{2}}{\frac{5}{4}}$
2. $\dfrac{\frac{7}{8}}{\frac{5}{4}}$
3. $\dfrac{\frac{10}{3}}{\frac{20}{9}}$
4. $-\dfrac{\frac{4}{21}}{\frac{8}{7}}$
5. $\dfrac{\frac{2}{3}}{\frac{5}{6}}$
6. $\dfrac{\frac{7}{4}}{\frac{14}{3}}$

7. $\dfrac{1-\dfrac{3}{2}}{\dfrac{5}{4}-\dfrac{1}{3}}$

8. $\dfrac{\dfrac{1}{2}-5}{\dfrac{1}{2}+\dfrac{1}{3}}$

9. $\dfrac{1+\dfrac{3}{2}}{1-\dfrac{1}{4}}$

10. $\dfrac{2-\dfrac{1}{2}}{1+\dfrac{3}{4}}$

11. $\dfrac{\dfrac{5x^2}{x+1}}{\dfrac{25x}{x+1}}$

12. $\dfrac{\dfrac{7+x}{7x}}{\dfrac{x+7}{14x^2}}$

13. $\dfrac{\dfrac{3y}{x}}{\dfrac{y^2}{x-1}}$

14. $\dfrac{\dfrac{5a^2}{b-1}}{\dfrac{15a^3}{(b-1)^2}}$

15. $\dfrac{1+\dfrac{1}{x}}{2-\dfrac{1}{x}}$

16. $\dfrac{\dfrac{2}{x}+1}{3-\dfrac{1}{x}}$

17. $\dfrac{\dfrac{2}{3y}-4}{6-\dfrac{1}{y}}$

18. $\dfrac{\dfrac{5}{y-12}}{\dfrac{10-y}{y^2}}$

19. $\dfrac{\dfrac{1}{5}-\dfrac{1}{x}}{\dfrac{1}{25}-\dfrac{1}{x^2}}$

20. $\dfrac{\dfrac{1}{x}+\dfrac{1}{5}}{\dfrac{1}{25}-\dfrac{1}{x^2}}$

21. $\dfrac{\dfrac{1}{x}-\dfrac{1}{3}}{\dfrac{1}{9}-\dfrac{1}{x^2}}$

22. $\dfrac{\dfrac{1}{4}+\dfrac{1}{x}}{\dfrac{1}{x^2}-\dfrac{1}{16}}$

23. $\dfrac{16-\dfrac{1}{x^2}}{\dfrac{1}{x}-4}$

24. $\dfrac{2-\dfrac{1}{y}}{1-\dfrac{1}{4y^2}}$

25. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{y^2}-\dfrac{1}{x^2}}$

26. $\dfrac{\dfrac{1}{2x}-\dfrac{4}{3}}{\dfrac{1}{4x^2}-\dfrac{16}{9}}$

27. $\dfrac{\dfrac{2}{25}-\dfrac{1}{2x^2}}{\dfrac{1}{5}-\dfrac{1}{2x}}$

28. $\dfrac{\dfrac{4}{25}-\dfrac{1}{4x^2}}{\dfrac{1}{5}+\dfrac{1}{4x}}$

29. $\dfrac{\dfrac{1}{y}-\dfrac{1}{x}}{4-\dfrac{2}{xy}}$

30. $\dfrac{\dfrac{1}{ab}+2}{\dfrac{1}{a}+\dfrac{1}{b}}$

31. $$\frac{\frac{1}{y}+\frac{1}{x}}{xy}$$

32. $$\frac{\frac{3}{x}}{\frac{1}{3}-\frac{1}{x}}$$

33. $$\frac{1-\frac{4}{x}-\frac{21}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}}$$

34. $$\frac{1-\frac{3}{x}-\frac{4}{x^2}}{1-\frac{16}{x^2}}$$

35. $$\frac{3-\frac{1}{2x}-\frac{1}{2x^2}}{2-\frac{2}{x}+\frac{1}{2x^2}}$$

36. $$\frac{\frac{1}{2}-\frac{5}{x}+\frac{12}{x^2}}{\frac{1}{2}-\frac{6}{x}+\frac{18}{x^2}}$$

37. $$\frac{\frac{1}{x}-\frac{4}{3x^2}}{3-\frac{8}{x}+\frac{16}{3x^2}}$$

38. $$\frac{1+\frac{3}{10x}-\frac{1}{10x^2}}{\frac{3}{5}-\frac{1}{10x}-\frac{1}{5x^2}}$$

39. $\dfrac{\dfrac{x-1}{1+\dfrac{4}{x}}-\dfrac{5}{x^2}}{}$

40. $2-\dfrac{5}{2x}-\dfrac{3x}{24x+3}$

41. $\dfrac{\dfrac{1}{x-3}+\dfrac{2}{x}}{\dfrac{1}{x}-\dfrac{3}{x-3}}$

42. $\dfrac{\dfrac{14}{x}-5+\dfrac{1}{x^2}}{\dfrac{1}{x^2}+\dfrac{13}{x}-10}$

43. $\dfrac{\dfrac{1}{x+5}+\dfrac{4}{x-2}}{\dfrac{2}{x-2}-\dfrac{1}{x+5}}$

44. $\dfrac{\dfrac{3}{x-1}-\dfrac{2}{x+3}}{\dfrac{2}{x+3}+\dfrac{1}{x-3}}$

45. $\dfrac{\dfrac{x}{x+1}-\dfrac{2}{x+3}}{\dfrac{x}{3x+4}+\dfrac{1}{x+1}}$

46. $\dfrac{\dfrac{x}{x-9}+\dfrac{2}{x+1}}{\dfrac{x}{7x-9}-\dfrac{1}{x+1}}$

47. $\dfrac{\dfrac{x}{3x+2}-\dfrac{1}{x+2}}{\dfrac{x}{x+2}-\dfrac{2}{x+2}}$

48. $$\frac{\frac{x}{x-4}+\frac{1}{x+2}}{\frac{x}{3x+4}+\frac{1}{x+2}}$$

49. $$\frac{\frac{a}{3-8b^3}}{\frac{7}{a}-\frac{2}{b}}$$

50. $$\frac{\frac{27a}{3+b^3}}{\frac{\frac{a}{b}}{3a+b}}$$

51. $$\frac{\frac{1}{b^3}+\frac{1}{a^3}}{\frac{1}{b}+\frac{1}{a}}$$

52. $$\frac{\frac{1}{b^3}-\frac{1}{a^3}}{\frac{1}{a}-\frac{1}{b}}$$

53. $$\frac{\frac{x^2+y^2}{\frac{x}{y}+2}}{\frac{x^2-y^2}{\frac{2}{xy}}}$$

54. $$\frac{x}{\frac{y+4+4yx}{xy+3+2yx}}$$

55. $$1+\frac{1}{1+\frac{1}{2}}$$

56. $2 - \dfrac{1}{1 + \dfrac{1}{3}}$

57. $\dfrac{1}{1 + \dfrac{1}{1 + x}}$

58. $\dfrac{x + \dfrac{1}{x}}{1 - \dfrac{1}{x} + 1}$

59. $\dfrac{1 - \dfrac{1}{x}}{x - \dfrac{1}{x}}$

60. $\dfrac{\dfrac{1}{x} - x}{x - \dfrac{1}{x^2}}$

Show Solutions

1. $\dfrac{2}{5}$
2. $\dfrac{7}{10}$
3. $\dfrac{3}{2}$
4. $-\dfrac{1}{6}$

5. $\frac{4}{5}$
6. $\frac{3}{8}$
7. $-\frac{6}{11}$
8. $-\frac{27}{5}$
9. $\frac{10}{3}$
10. $\frac{6}{7}$
11. $\frac{x}{5}$
12. $2x$
13. $\frac{3(x-1)}{xy}$
14. $\frac{b-1}{3a}$
15. $\frac{x+1}{2x-1}$
16. $\frac{x+2}{3x-1}$
17. $-\frac{2}{3}$
18. $-\frac{5y^2}{(y-12)(y-10)}$

19. $\dfrac{5x}{x+5}$

20. $\dfrac{5x}{x-5}$

21. $-\dfrac{3x}{x+3}$

22. $\dfrac{4x}{4-x}$

23. $-\dfrac{4x+1}{x}$

24. $\dfrac{4y}{2y+1}$

25. $\dfrac{xy}{x-y}$

26. $\dfrac{6x}{8x+3}$

27. $\dfrac{2x+5}{5x}$

28. $\dfrac{4x-5}{5x}$

29. $\dfrac{x-y}{4xy-2}$

30. $\dfrac{2ab+1}{a+b}$

31. $\dfrac{x+y}{x^2y^2}$

32. $\dfrac{9}{x-3}$

33. $\dfrac{x-7}{x-5}$

34. $\dfrac{x+1}{x+4}$

35. $\dfrac{3x+1}{2x-1}$

36. $\dfrac{x-4}{x-6}$

37. $\dfrac{1}{3x-4}$

38. $\dfrac{5x-1}{3x-2}$

39. $\dfrac{x(x-1)}{x+4}-\dfrac{5}{x^2}$

40. $2-\dfrac{5}{2x}-\dfrac{x}{8x+1}$

41. $\dfrac{3x-6}{-2x-3}$

42. $\dfrac{-5x^2+14x+1}{-10x^2+13x+1}$

43. $\dfrac{5x+18}{x+12}$

44. $\dfrac{(x-1)(3x+4)}{(x+2)(x+3)}$

45. $\dfrac{x+1}{3x+2}$

46. $\dfrac{(x+1)(8x-9)}{(x-9)(6x+1)}$

47. $\dfrac{x+1}{3x+2}$

48. $\dfrac{(x+6)(3x+4)}{(x-4)(x+8)}$

49. $\dfrac{a(7b-2a)}{2b(3-8b^3)}$

50. $\dfrac{27b(3a+b)}{b^3+3}$

51. $\dfrac{a^2-ab+b^2}{a^2b^2}$

52. $\dfrac{a^2+ab+b^2}{a^2b^2}$

53. $\dfrac{2y(x^2+y^2)}{x(x+2y)(x^2-y^2)}$

54. $\dfrac{x(xy+3+2yx)}{y+4+4yx}$

55. $\dfrac{5}{3}$

56. $\dfrac{5}{4}$

57. $\dfrac{x+1}{x+2}$

58. $\dfrac{x^2+1}{2x-1}$

59. $\dfrac{1}{x+1}$

60. $-\dfrac{x(x+1)}{x^3-1}$

8.2 Equations with Fractions

Solve.

1. $\dfrac{1}{2}+\dfrac{1}{x}=\dfrac{1}{8}$

2. $\dfrac{1}{3}-\dfrac{1}{x}=\dfrac{2}{9}$

3. $\dfrac{1}{3x}-\dfrac{2}{3}=\dfrac{1}{x}$

4. $\dfrac{2}{5x}-\dfrac{1}{x}=\dfrac{3}{10}$

5. $\dfrac{1}{2x+1}=5$

6. $\dfrac{3}{3x-1}+4=5$

7. $\dfrac{2x-3}{x+5}=\dfrac{2}{x+5}$

8. $\dfrac{5x}{2x-1} = \dfrac{x-1}{2x-1}$

9. $\dfrac{5}{x-7} = \dfrac{6}{x-9}$

10. $\dfrac{5}{x+5} = \dfrac{3}{x+1}$

11. $\dfrac{x}{6} - \dfrac{6}{x} = 0$

12. $\dfrac{5x+x}{5} = -2$

13. $\dfrac{x}{x+12} = \dfrac{2}{x}$

14. $\dfrac{2x}{x+5} = \dfrac{1}{6-x}$

15. $\dfrac{1}{x} + \dfrac{x}{2x+1} = 0$

16. $\dfrac{9x}{3x-1} - \dfrac{4}{x} = 0$

17. $1 - \dfrac{2}{x} = \dfrac{48}{x^2}$

18. $2 - \dfrac{9}{x} = \dfrac{5}{x^2}$

19. $1 + \dfrac{12}{x} = \dfrac{12}{x-2}$

20. $1 - \dfrac{3x-5}{x(3x-4)} = -\dfrac{1}{x}$

21. $\dfrac{x}{2} = \dfrac{14}{x+3}$

22. $\dfrac{3x}{2} = \dfrac{x+1}{3-x}$

23. $6 = \dfrac{-3x + 3}{x - 1}$

24. $\dfrac{1}{2x - 2} = 2 + \dfrac{6(4 - x)}{x - 2}$

25. $2 + \dfrac{2x}{x - 3} = \dfrac{3(x - 1)}{x - 3}$

26. $\dfrac{x}{x - 1} + \dfrac{1}{6x - 1} = \dfrac{x(x - 1)}{6x - 1}$

27. $\dfrac{12}{x^2 - 81} = \dfrac{1}{x + 9} - \dfrac{2}{x - 9}$

28. $\dfrac{14}{x^2 - 49} = \dfrac{2}{x - 7} - \dfrac{3}{x + 7}$

29. $\dfrac{6x}{x + 3} + \dfrac{4}{x - 3} = \dfrac{3x}{x^2 - 9}$

30. $\dfrac{3x}{x + 2} - \dfrac{17}{x - 2} = -\dfrac{48}{x^2 - 4}$

31. $x^{-1} + 3 = 0$

32. $4^{-y} - 1 = 0$

33. $y^{-2} - 4 = 0$

34. $9x^{-2} - 1 = 0$

35. $3(x - 1)^{-1} + 5 = 0$

36. $5 - 2(3x + 1)^{-1} = 0$

37. $3 + 2x^{-3} = 2x^{-3}$

38. $\dfrac{1}{x} = \dfrac{1}{x + 1}$

39. $\dfrac{x}{x + 1} = \dfrac{x + 1}{x}$

40. $\dfrac{3x-1}{3x} = \dfrac{x}{x+3}$

41. $\dfrac{4x-7}{x-5} = \dfrac{3x-2}{x-5}$

42. $\dfrac{x}{x^2-9} = \dfrac{1}{x-3}$

43. $\dfrac{3x+4}{x-8} - \dfrac{2}{8-x} = 1$

44. $\dfrac{1}{x} = \dfrac{6}{x(x+3)}$

45. $\dfrac{3}{x} = \dfrac{1}{x+1} + \dfrac{13}{x(x+1)}$

46. $\dfrac{x}{x-1} - \dfrac{3}{4x-1} = \dfrac{9x}{(4x-1)(x-1)}$

47. $\dfrac{1}{x-4} + \dfrac{x}{x-2} = 2x^2 - 6x + 8$

48. $\dfrac{x}{x-5} + \dfrac{x-1}{x^2-11x+30} = \dfrac{5}{x-6}$

49. $\dfrac{x}{x+1} - \dfrac{6}{5x^2+4x-1} = -\dfrac{5}{5x-1}$

50. $\dfrac{-8x^2-4}{x-12} + \dfrac{2(x+2)}{x^2+4x-60} = \dfrac{1}{x+2}$

51. $\dfrac{x}{x+2} - \dfrac{20}{x^2-6-x} = \dfrac{-4}{x-3}$

52. $\dfrac{x+7}{x-1} + \dfrac{x-1}{x+1} = \dfrac{4}{x^2-1}$

53. $\dfrac{x-1}{x-3} + \dfrac{x-3}{x-1} = -\dfrac{x+5}{x-3}$

54. $\dfrac{x-2}{x-5}-\dfrac{x-5}{x-2}=\dfrac{8-x}{x-5}$

55. $\dfrac{x+7}{x-2}-\dfrac{81}{x^2+5x-14}=\dfrac{9}{x+7}$

56. $\dfrac{x}{x-6}+1=\dfrac{5x+30}{36-x^2}$

57. $\dfrac{2x}{x+1}-\dfrac{4}{4x-3}=-\dfrac{7}{4x^2+x-3}$

58. $\dfrac{x-5}{x-10}+\dfrac{5}{x-5}=-\dfrac{5x}{x^2-15x+50}$

59. $\dfrac{5}{x^2+5x+4}+\dfrac{x+1}{x^2+3x-4}=\dfrac{5}{x^2-1}$

60. $\dfrac{1}{x^2-2x-63}+\dfrac{x-9x^2+10}{x+21}=\dfrac{1}{x^2-6x-27}$

61. $\dfrac{4}{x^2-4}+\dfrac{2(x-2)}{x^2-4x-12}=\dfrac{x+2}{x^2-8x+12}$

62. $\dfrac{x+2}{x^2-5x+4}+\dfrac{x+2}{x^2+x-2}=\dfrac{x-1}{x^2-2x-8}$

63. $\dfrac{6x}{x-1}-\dfrac{11x+1}{2x^2-x-1}=\dfrac{6x}{2x+1}$

64. $\dfrac{8x}{2x-3}+\dfrac{4x^2}{x^2-7x+6}=\dfrac{1}{x-2}$

Solve for the indicated variable.

65. Solve for r: $t=\dfrac{D}{r}$

66. Solve for b: $h=\dfrac{2A}{b}$

67. Solve for P: $t=\dfrac{I}{Pr}$

68. Solve for π: $r = \dfrac{C}{2\pi}$

69. Solve for c: $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$

70. Solve for y: $m = \dfrac{y - y_1}{x - x_1}$

71. Solve for w: $P = 2(l + w)$

72. Solve for t: $A = P(1 + rt)$

73. Solve for m: $s = \dfrac{1}{n + m}$

74. Solve for S: $h = S2\pi r - r$

75. Solve for x: $y = \dfrac{x}{x + 2}$

76. Solve for x: $y = 2x + 15x$

77. Solve for R: $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

78. Solve for S_1: $\dfrac{1}{f} = \dfrac{1}{S_1} + \dfrac{1}{S_2}$

Show Solutions

1. $-\dfrac{8}{3}$
2. 9
3. -1

4. -2
5. $-\frac{2}{5}$
6. $\frac{4}{3}$
7. $\frac{5}{2}$
8. -1
9. -3
10. -11
11. $-6, 6$
12. $-\frac{5}{3}$
13. $-4, 6$
14. $\frac{7 \pm \sqrt{73}}{4}$
15. -1
16. $\frac{4}{3}, \frac{1}{12}$
17. $-6, 8$
18. $\frac{5 \pm \sqrt{65}}{4}$
19. $-4, 6$

20. $\frac{4}{3}$

21. $-7, 4$

22. $\frac{9 \pm \sqrt{105}}{6}$

23. $\emptyset$

24. $\frac{31}{16}$

25. 5

26. $\emptyset$

27. $\emptyset$

28. $\frac{13}{5}$

29. -39

30. $\frac{5}{3}$

31. $-\frac{1}{3}$

32. 0

33. $-\frac{1}{2}, \frac{1}{2}$

34. $-3, 3$

35. $\frac{2}{5}$

36. $-\frac{1}{5}$

37. $\emptyset$

38. $\emptyset$

39. $\emptyset$

40. $-\frac{1}{2}$

41. 5

42. $\emptyset$

43. -7

44. 3

45. 5

46. $\frac{11}{4}$

47. -1

48. 7

49. $\emptyset$

50. 12

51. -4

52. -1

53. $\emptyset$

54. -7

55. $\frac{5}{3}$

56. -6

57. $\emptyset$

58. 15

59. $\frac{1}{2}$

60. $-6, 4$

61. $-\frac{1}{3}$

62. 4

63. 10

64. $\frac{3}{2}$

65. $r = \frac{D}{t}$

66. $b = \frac{2A}{h}$

67. $P = \frac{I}{tr}$

68. $\pi = \frac{C}{2r}$

69. $c = \frac{ab}{b - a}$

70. $y = m(x - x_1) + y_1$

71. $w = \frac{P}{2} - l$

72. $t = \frac{A - P}{Pr}$

73. $m = \frac{1}{s} - n$

74. $S = \frac{h + r}{2\pi r}$

75. $x = -\frac{2y}{y - 1}$

76. $x = \frac{y}{17}$

77. $R = \frac{R_1 R_2}{R_1 + R_2}$

78. $S_1 = \frac{fS_2}{S_2 - f}$

8.3 Word Problems with Fractions

Use algebra to solve the following applications.

1. A positive integer is twice another. The sum of the reciprocals of the two positive integers is $\frac{3}{10}$. Find the two integers.

2. A positive integer is twice another. The sum of the reciprocals of the two positive integers is $\frac{3}{12}$.

Find the two integers.

3. A positive integer is twice another. The difference of the reciprocals of the two positive integers is $\frac{1}{8}$. Find the two integers.

4. A positive integer is twice another. The difference of the reciprocals of the two positive integers is $\frac{1}{18}$. Find the two integers.

5. A positive integer is 2 less than another. If the sum of the reciprocal of the smaller and twice the reciprocal of the larger is $\frac{5}{12}$, then find the two integers.

6. A positive integer is 2 more than another. If the sum of the reciprocal of the smaller and twice the reciprocal of the larger is $\frac{17}{35}$, then find the two integers.

7. The sum of the reciprocals of two consecutive positive even integers is $\frac{11}{60}$. Find the two even integers.

8. The sum of the reciprocals of two consecutive positive odd integers is $\frac{16}{63}$. Find the integers.

9. The difference of the reciprocals of two consecutive positive even integers is $\frac{1}{24}$. Find the two even integers.

10. The difference of the reciprocals of two consecutive positive odd integers is $\frac{2}{99}$. Find the integers.

11. If 3 times the reciprocal of the larger of two consecutive integers is subtracted from 2 times the reciprocal of the smaller, then the result is $\frac{1}{2}$. Find the two integers.

12. If 3 times the reciprocal of the smaller of two consecutive integers is subtracted from 7 times the reciprocal of the larger, then the result is $\frac{1}{2}$. Find the two integers.

13. A positive integer is 5 less than another. If the reciprocal of the smaller integer is subtracted from 3 times the reciprocal of the larger, then the result is $\frac{1}{12}$. Find the two integers.

14. A positive integer is 6 less than another. If the reciprocal of the smaller integer is subtracted from

10 times the reciprocal of the larger, then the result is $\frac{3}{7}$. Find the two integers.

Use algebra to solve the following applications.

15. James can jog twice as fast as he can walk. He jogged the first 9 miles to his grandmother's house, then walked the remaining 1.5 miles. If the total trip took 2 hours, what was his average jogging speed?

16. On a business trip, an executive travelled 720 miles by jet aircraft and then 80 miles by helicopter. If the jet averaged 3 times the speed of the helicopter and the total trip took 4 hours, what was the average speed of the jet?

17. Sally drove 23 miles in traffic, then 99 miles after the traffic cleared at an average of 20 mph faster. If the total trip took 2 hours, what was her average speed in traffic?

18. Harry travelled 15 miles on the bus and then 72 miles on a train. If the train was 18 mph faster than the bus and the total trip took 2 hours, what was the average speed of the train?

19. A bus averages 6 mph faster than a trolley. If the bus travels 90 miles in the same time it takes the trolley to travel 75 miles, what is the speed of each?

20. A passenger car averages 16 mph faster than the bus. If the bus travels 56 miles in the same time it takes the passenger car to travel 84 miles, what is the speed of each?

21. A light aircraft travels 2 mph less than twice as fast as a passenger car. If the passenger car can travel 231 miles in the same time it takes the aircraft to travel 455 miles, what is the average speed of each?

22. Mary can run 1 mph more than twice as fast as Bill can walk. If Bill can walk 3 miles in the same time it takes Mary to run 7.2 miles, what is Bill's average walking speed?

23. An airplane traveling with a 20-mph tailwind covers 270 miles. On the return trip against the wind, it covers 190 miles in the same amount of time. What is the speed of the airplane in still air?

24. A jet airliner travelling with a 30-mph tailwind covers 525 miles in the same time it can travel 495 miles after the tailwind eases to 10 mph. What is the speed of the airliner in still air?

25. A boat averages 16 mph in still water. With the current, the boat can travel 95 miles in the same time it takes to travel 65 miles against it. What is the speed of the current?

26. A river tour boat averages 7 mph in still water. If the total 24-mile tour downriver and 24 miles back takes 7 hours, how fast is the river current?

27. If the river current flows at an average 3 mph, then a tour boat makes the 9-mile tour downstream

and back upstream in 4 hours. What is the average speed of the boat in still water?

28. Jane rowed her canoe against a 1-mph current upstream 12 miles and then returned 12 miles back downstream. If the total trip took 5 hours, at what speed can Jane row in still water?

29. Jose drove 15 miles to pick up his sister and then returned home. On the return trip, he averaged 15 mph faster than on the trip there. If the total trip took 1 hour, what was Jose's average speed on the return trip?

30. Barry drove 24 miles to town and back in 1 hour. On the return trip, he averaged 14 mph faster than on the trip to town. What was his average speed on the trip to town?

31. Jerry paddled his kayak upstream against a 1-mph current for 12 miles. The return trip downstream took 1 hour less time. How fast can Jerry paddle in still water?

32. It takes a light aircraft 1 hour more to fly 360 miles against a 30-mph headwind than it does with it. What is the speed of the aircraft in calm air?

33. James can paint the office by himself in 7 hours. Manny paints the office in 10 hours. How long will it take them to paint the office working together?

34. Barry can lay a brick driveway by himself in 12 hours. Robert does the same job in 10 hours. How long will it take them to lay the brick driveway working together?

35. Jerry can detail a car by himself in 50 minutes. Sally does the same job in 1 hour. How long will it take them to detail a car working together?

36. Jose can build a small shed by himself in 26 hours. Alex builds the same small shed in 2 days. How long would it take them to build the shed working together?

37. Allison can complete a sales route by herself in 6 hours. Working with an associate, she completes the route in 4 hours. How long would it take her associate to complete the route by herself?

38. James can prepare and paint a house by himself in 5 days. Working with his brother, Bryan, they can do it in 3 days. How long would it take Bryan to prepare and paint the house by himself?

39. Joe can assemble a computer by himself in 1 hour. Working with an assistant, he can assemble a computer in 40 minutes. How long would it take his assistant to assemble a computer working alone?

40. The teacher's assistant can grade class homework assignments by herself in 1 hour. If the teacher helps, then the grading can be completed in 20 minutes. How long would it take the teacher to grade the papers, working alone?

41. A larger pipe fills a water tank twice as fast as a smaller pipe. When both pipes are used, they fill the

tank in 5 hours. If the larger pipe is left off, how long would it take the smaller pipe to fill the tank?

42. A newer printer can print twice as fast as an older printer. If both printers working together can print a batch of flyers in 45 minutes, how long would it take the newer printer to print the batch working alone?

43. Working alone, Henry takes 9 hours longer than Mary to clean the carpets in the entire office. Working together, they clean the carpets in 6 hours. How long would it take Mary to clean the office carpets if Henry were not there to help?

44. Working alone, Monique takes 4 hours longer than Audrey to record the inventory of the entire shop. Working together, they take inventory in 1.5 hours. How long would it take Audrey to record the inventory working alone?

45. Jerry can lay a tile floor in 3 hours less time than Jake. If they work together, the floor takes 2 hours. How long would it take Jerry to lay the floor by himself?

46. Jeremy can build a model airplane in 5 hours, less time than his brother. Working together, they need 6 hours to build the plane. How long would it take Jeremy to build the model airplane working alone?

47. Harry can paint a shed by himself in 6 hours. Jeremy can paint the same shed by himself in 8 hours. How long will it take them to paint two sheds working together?

48. Joe assembles a computer by himself in 1 hour. Working with an assistant, he can assemble 10 computers in 6 hours. How long would it take his assistant to assemble 1 computer working alone?

49. Jerry can lay a tile floor in 3 hours, and his assistant can do the same job in 4 hours. If Jerry starts the job and his assistant joins him 1 hour later, then how long will it take to lay the floor?

50. Working alone, Monique takes 6 hours to record the inventory of the entire shop, while it takes Audrey only 4 hours to do the same job. How long will it take them working together if Monique leaves 2 hours early?

Show Solutions

1. $(5, 10)$
2. $(6, 12)$

3. $(4, 8)$
4. $(9, 18)$
5. $(6, 8)$
6. $(5, 7)$
7. $(10, 12)$
8. $(7, 9)$
9. $(6, 8)$
10. $(9, 11)$
11. $(1, 2)$ or $(-4, -3)$
12. $(5, 6)$
13. $(4, 9)$ or $(15, 20)$
14. $(4, 10)$
15. 6 miles per hour
16. 600 miles per hour
17. 46 miles per hour
18. 54 miles per hour
19. Trolley: 30 mph; bus: 36 mph
20. Bus: 32 mph; car: 48 mph
21. Passenger car: 66 mph; aircraft: 130 mph
22. 5 mph

23. 115 miles per hour
24. 690 miles per hour
25. 3 miles per hour
26. 1 mile per hour
27. 6 miles per hour
28. 5 miles per hour
29. 40 miles per hour
30. 34 miles per hour
31. 5 miles per hour
32. 210 miles per hour
33. $4\frac{2}{17}$ hours
34. $5\frac{5}{11}$ hours
35. $27\frac{3}{11}$ minutes
36. $16\frac{8}{37}$ hours
37. 12 hours
38. $7\frac{1}{2}$ days
39. 2 hours
40. 30 minutes

41. 15 hours
42. $67\frac{1}{2}$ minutes
43. 9 hours
44. 30 minutes
45. 3 hours
46. 3 hours
47. $6\frac{6}{7}$ hours
48. 3 hours
49. $2\frac{1}{7}$ hours
50. $3\frac{1}{5}$ hours

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CHAPTER 9: QUADRATIC EQUATIONS

Chapter Outline

9.1 Solving Equations with Factoring

- Solving Quadratic Equations by Factoring
- Finding Equations with Given Solutions

9.2 Completing the Square

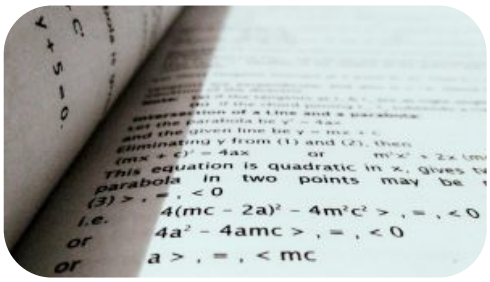
9.3 The Quadratic Formula

9.4 Solving Quadratic Equations

- Discriminant
- Applications Involving Quadratic Equations

9.5 Exercises

- 9.1 Solving Equations with Factoring
- 9.2 Completing the Square
- 9.3 The Quadratic Formula
- 9.4 Solving Quadratic Equations



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9.1 Solving Equations with Factoring

Learning Objectives

By the end of this section, you should be able to:

- Verify solutions to quadratic equations.
- Solve quadratic equations by factoring.
- Determine a quadratic equation with given solutions.

Solving Quadratic Equations by Factoring

Learning how to solve equations is one of our main goals in algebra. Up to this point, we have solved linear equations, which are of degree 1 (x). In this section, we will learn a technique that can be used to solve certain equations of degree 2 (x^2). A quadratic equation is any equation that can be written in the standard form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$. The following are some examples of quadratic equations, all of which will be solved in this section:

$$x^2 + x - 6 = 0$$

$$4x^2 - 9 = 0$$

$$2x^2 + 10x + 20 = -3x + 5$$

A solution of a quadratic equation in standard form is called a **root**. Quadratic equations can have two real solutions, one real solution, or no real solution. The quadratic equation $x^2 + x - 6 = 0$ has two solutions, namely,

$$x = -3 \text{ and } x = 2$$

This is just like factoring a trinomial, only now, you can solve for x in each of the factors.

How To

Given a quadratic equation with the leading coefficient of 1, factor it.

1. Find two numbers whose product equals c and whose sum equals b.
2. Use those numbers to write two factors of the form $(x + k)$ or $(x - k)$, where k is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are 1 and -2, the factors are $(x + 1)(x - 2)$.
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

Example 1

Verify that:

$$x = -3 \text{ and } x = 2 \text{ are solutions to } x^2 + x - 6 = 0$$

Solution:
To verify solutions, substitute the values for x and then simplify to see if a true statement results.

$$\begin{aligned} \text{Check: } x = -3 \qquad & \qquad \text{Check: } x = 2 \\ x^2 + x - 6 = 0 \qquad & \qquad x^2 + x - 6 = 0 \\ (-3)^2 + (-3) - 6 = 0 \qquad & \qquad (2)^2 + (2) - 6 = 0 \\ 9 - 3 - 6 = 0 \qquad & \qquad 4 + 2 - 6 = 0 \\ \checkmark \quad 0 = 0 \qquad & \qquad \checkmark \quad 0 = 0 \end{aligned}$$

Answer

Both values produce true statements. Therefore, they are both solutions to the equation.

Our goal is to develop algebraic techniques for finding solutions to quadratic equations. The first technique requires the **zero-product property**:

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0$$

In other words, if any product is equal to zero, then one or both of the variable factors must be equal to zero.

Example 2

Solve:

$$(x-8)(x+7)=0$$

Solution:
This equation consists of a product of two quantities equal to zero; therefore, the zero-product property applies. One or both of the quantities must be zero.

$$\begin{aligned} (x-8)&=0 \\ x-8+8&=0+8 \\ x&=8 \end{aligned} \quad \text{or} \quad \begin{aligned} (x+7)&=0 \\ x+7-7&=0-7 \\ x&=-7 \end{aligned}$$

To verify that these are solutions, substitute them for the variable x.

$$\begin{aligned} \text{Check: } x&=8 \\ (x-8)(x+7)&=0 \\ (8-8)(8+7)&=0 \\ (0)(15)&=0 \\ \checkmark \quad 0&=0 \end{aligned} \qquad \begin{aligned} \text{Check: } x&=-7 \\ (x-8)(x+7)&=0 \\ (-7+8)(-7+7)&=0 \\ (1)(0)&=0 \\ \checkmark \quad 0&=0 \end{aligned}$$

Notice that each solution produces a factor that is equal to zero.

Answer

The solutions are 8 and -7.

The quadratic equation may not be given in its factored form.

Example 3

Solve:

$$x^2+3x-10=0$$

Solution:
The goal is to produce a product that is equal to zero. We can do that by factoring the trinomial on the left side of the equation.

$$\begin{aligned} x^2+3x-10&=0 \\ (x\quad ?)(x\quad ?)&=0 \qquad -10=5(-2) \text{ and } 3=5+(-2) \\ (x+5)(x-2)&=0 \end{aligned}$$

Next, apply the zero-product property and set each factor equal to zero.

$$x+5=0 \quad \text{or} \quad x-2=0$$

This leaves us with two linear equations, each of which can be solved for x.

$$\begin{aligned} x+5-5&=0-5 & \qquad x-2+2&=0+2 \\ x&=-5 & \qquad x&=2 \end{aligned}$$

Check the solutions by substituting into the original equation to verify that we obtain true statements.

$$\begin{aligned} \text{Check: } x=-5 \qquad & \qquad \text{Check: } x=2 \\ x^2+3x-10&=0 & \qquad x^2+3x-10&=0 \\ (-5)^2=3(-5)-10&=0 & \qquad (2)^2+3(2)-10&=0 \\ 25-15-10&=0 & \qquad 4+6-10&=0 \\ \checkmark \quad 0&=0 & \qquad \checkmark \quad 0&=0 \end{aligned}$$

Answer

The solutions are -5 and 2.

Using the zero-product property after factoring a quadratic equation in standard form is the key to this technique. However, the quadratic equation may not be given in standard form, and so there may be some preliminary steps before factoring.

The steps required to solve by factoring are outlined in the following example.

Example 4

Solve:

$$2x^2+10x+20=-3x+5$$

Solution:

Step 1: Express the quadratic equation in standard form. For the zero-product property to apply, the quadratic expression must be equal to zero. Use the addition and subtraction properties of equality to combine opposite-side like terms and obtain zero on one side of the equation. In this example, add $3x$ to and subtract 5 from both sides.

$$\begin{aligned} 2x^2+10x+20+3x &= -3x+5+3x \\ 2x^2+13x+20 &= 5 \\ 2x^2+13x+20-5 &= 5-5 \\ 2x^2+13x+15 &= 0 \end{aligned}$$

Step 2: Factor the quadratic expression.

$$(2x+3)(x+5)=0$$

Step 3: Apply the zero-product property and set each variable factor equal to zero.

$$2x+3=0 \quad \text{or} \quad x+5=0$$

Step 4: Solve the resulting linear equations.

$$\begin{aligned} 2x+3&=0 \\ 2x&=-3 \\ \frac{2x}{2}&=\frac{-3}{2} \\ x&=-\frac{3}{2} \end{aligned} \qquad \text{or} \qquad \begin{aligned} x+5&=0 \\ x&=-5 \end{aligned}$$

Answer

The solutions are -5 and $-\frac{3}{2}$. The check is optional.

Example 5

Solve:

$$9x^2+1=6x$$

Solution:
Write this in standard form by subtracting $6x$ from both sides.

$$9x^2 + 1 - 6x = 6x - 6x$$
$$9x^2 - 6x + 1 = 0$$

Once the equation is in standard form, equal to zero, factor.

$$(3x - 1)(3x - 1) = 0$$

This is a perfect square trinomial. Hence, setting each factor equal to zero results in a repeated solution.

$$\begin{aligned} 3x - 1 &= 0 \\ 3x &= 1 \\ x &= \frac{1}{3} \end{aligned} \quad \text{or} \quad \begin{aligned} 3x - 1 &= 0 \\ 3x &= 1 \\ x &= \frac{1}{3} \end{aligned}$$

A repeated solution is called a **double root** and does not have to be written twice.

Answer

The solution is $\frac{1}{3}$.

Try It

Solve: $x^2 - 3x = 28$

Show Solution

$x = -4$ or $x = 7$

Not all quadratic equations in standard form are trinomials. We often encounter binomials.

How To

Given a quadratic equation with an x^2 term but no x term, use the square root property to solve it.

1. Isolate the x^2 term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a $\pm$ sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the $\pm$ sign.

Try It

Try this exercise to gauge your learning:

Activity Description
Match the options to the question.

Rewrite in standard form, then factor. Set each factor to zero, then solve for x.

Questions:

a. $3x = 4x^2, x =$ *Hint: Rearrange so all variables are on one side, then factor.*
b. $x^2 - 4x - 32 = 0, x =$
c. $x(x-5) = 2(x+5) + 8, x =$ *Hint: Expand first.*
d. $2x^2 + 7x - 4 = 0, x =$

Options:

$$\frac{1}{2}, -4 \qquad 8, -4 \qquad 9, -2 \qquad 0, \frac{3}{4}$$

Answers:

a. $0, \frac{3}{4}$
b. $8, -4$
c. $9, -2$
d. $\frac{1}{2}, -4$

Example 6

Solve:

$$x^2 - 9 = 0$$

Solution:
This quadratic equation is given in standard form, where the binomial on the left side is a difference of squares. Factor as follows:

$$\begin{aligned} x^2 - 9 &= 0 \\ (x+3)(x-3) &= 0 \end{aligned}$$

Next, set each factor equal to zero and solve.

$$\begin{aligned} x + 3 &= 0 \\ x &= -3 \end{aligned} \quad \text{or} \quad \begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

Answer

The solutions are 3 and -3, which can also be written as ± 3.

Example 7

Solve:

$$5x^2 = 15x$$

Solution:
By inspection, we see that $x = 0$ is a solution to this quadratic equation. Since dividing by zero is undefined, we want to avoid dividing both sides of this equation by x. In general, we wish to avoid dividing both sides of any equation by a variable or an expression containing a variable. We will discuss this in more detail later.

The first step is to rewrite this equation in standard form with zero on one side.

$$\begin{aligned} 5x^2 &= 15x \\ 5x^2-15x &= 15x-15x \quad \text{Subtract } 15x \text{ from both sides.} \\ 5x^2-15x &= 0 \end{aligned}$$

Next, factor the expression. Notice that the binomial on the left has a GCF of $5x$.

$$\begin{aligned} 5x^2-15x &= 0 \\ 5x(x-3) &= 0 \end{aligned}$$

Set each factor to equal to zero.

$$\begin{aligned} 5x &= 0 \qquad \text{or} \qquad x-3=0 \\ \frac{5x}{5} &= \frac{0}{5} \qquad\qquad x=3 \\ x &= 0 \end{aligned}$$

Answer

The solutions are 0 and 3.

Example 8

Solve:

$$(2x+1)(x+5)=11$$

Solution:
This quadratic equation appears to be factored; hence, it might be tempting to set each factor equal to 11. However, this would lead to incorrect results. We must rewrite the equation in standard form, equal to zero, so that we can apply the zero-product property.

$$\begin{aligned} (2x+1)(x+5) &= 11 \\ 2x^2+10x+x+5 &= 11 \\ 2x^2+11x+5-11 &= 11-11 \\ 2x^2+11x-6 &= 0 \end{aligned}$$

Once it is in standard form, we can factor and then set each factor equal to zero.

$$(2x-1)(x+6)=0$$

$$\begin{aligned} 2x-1&=0 \quad \text{or} \quad x+6=0 \\ 2x&=1 \qquad\qquad x=-6 \\ x&=\frac{1}{2} \end{aligned}$$

Answer

The solutions are $\frac{1}{2}$ and -6.

Example 9

Solve:

$$15x^2-25x+10=0$$

Solution:
We begin by **factoring out the GCF** of 5. Then factor the resulting trinomial.

$$\begin{aligned} 15x^2-25x+10&=0 \\ 5(3x^2-5x+2)&=0 \\ 5(3x-2)(x-1)&=0 \end{aligned}$$

Next, we set each variable factor equal to zero and solve for x.

$$\begin{aligned} 3x-2&=0 \quad \text{or} \quad x-1=0 \\ 3x&=2 \qquad\qquad x=1 \\ x&=\frac{2}{3} \end{aligned}$$

Notice that the factor 5 is not a variable factor and thus did not contribute to the solution set.

Answer

The solutions are $\frac{2}{3}$ and 1.

Example 10

Solve:

$$52x^2 + 76x - 13 = 0$$

Solution:
Clear the fractions by multiplying both sides of the equation by the LCD, which is equal to 6.

$$\frac{5}{2}x^2 + \frac{7}{6}x - \frac{1}{3} = 0$$
$$6 \cdot \left(\frac{5}{2}x^2 + \frac{7}{6}x - \frac{1}{3}\right) = 6 \cdot (0)$$
$$15x^2 + 7x - 2 = 0$$

At this point, we have an equivalent equation with integer coefficients and can factor as usual. Begin with the factors of 15 and 2.

$$15 = 1 \cdot 15 \qquad 2 = 1 \cdot 2$$
$$= 3 \cdot 5$$

The coefficient of the middle term is $7 = 3(-1) + 5(2)$. Factor as follows:

$$15x^2 + 7x - 2 = 0$$
$$(3x + 2)(5x - 1) = 0$$

Set each factor equal to zero and solve.

$$3x + 2 = 0 \quad \text{or} \quad 5x - 1 = 0$$
$$3x = -2 \qquad\qquad 5x = 1$$
$$x = -\frac{2}{3} \qquad\qquad x = \frac{1}{5}$$

Answer

The solutions are $-\frac{2}{3}$ and $\frac{1}{5}$.

Try It

Solve: $4x^2 - 9 = 0$

Show Solution

$-\frac{3}{2}$ and $\frac{3}{2}$

Finding Equations with Given Solutions

The zero-product property states,

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0$$

And, in fact, the converse is true as well:

$$\text{If } a = 0 \text{ or } b = 0, \text{ then } ab = 0$$

When this is the case, we can write the following:

$$a \cdot b = 0, \text{ if and only if } a = 0 \text{ or } b = 0$$

We use this property to find equations, given the solutions. To do this, the steps for solving by factoring are performed in reverse.

Example 11

Find a quadratic equation with solutions -7 and 2.

Solution:
Given the solutions, we can determine two linear factors.

$$\begin{aligned} x &= -7 \quad \text{or} \quad x = 2 \\ x + 7 &= 0 \qquad\quad x - 2 = 0 \end{aligned}$$

The product of these linear factors is equal to zero when $x = -7$ or $x = 2$:

$$(x + 7)(x - 2) = 0$$

Multiply the binomials and present the equation in standard form.

$$\begin{aligned} x^2 - 2x + 7x - 14 &= 0 \\ x^2 + 5x - 14 &= 0 \end{aligned}$$

Answer

$x^2 + 5x - 14 = 0$

We may check our equation by substituting the given answers to see if we obtain a true statement. Also, the equation found above is not unique, and so the check becomes essential when our equation looks different from someone else's. This is left as an exercise.

Example 12

Find a quadratic equation with integer coefficients, given solutions $\frac{1}{2}$ and $-\frac{3}{4}$.

Solution:
To avoid fractional coefficients, we first clear the fractions by multiplying both sides by the denominator.

$$\begin{aligned} x &= \frac{1}{2} \\ 2 \cdot x &= 2 \cdot \frac{1}{2} \\ 2x &= 1 \\ 2x - 1 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= -\frac{3}{4} \\ 4 \cdot x &= -\frac{3}{4} \cdot 4 \\ 4x &= -3 \\ 4x + 3 &= 0 \end{aligned}$$

Apply the zero-product property and multiply.

$$\begin{aligned} (2x - 1)(4x + 3) &= 0 \\ 8x^2 + 6x - 4x - 3 &= 0 \\ 8x^2 + 2x - 3 &= 0 \end{aligned}$$

Answer

$8x^2 + 2x - 3 = 0$

Try It

Find a quadratic equation with integer coefficients, given solutions -1 and $\frac{2}{3}$.

Show Solution

$$3x^2 + x - 2 = 0$$

Key Takeaways

- A polynomial can have at most a number of solutions equal to its degree. Therefore, quadratic equations can have up to two real solutions.
- To solve a quadratic equation, first write it in standard form. Once the quadratic expression is equal to zero, factor it and then set each variable factor equal to zero. The solutions to the resulting linear equations are the solutions to the quadratic equation.
- Not all quadratic equations can be solved by factoring. We will learn how to solve quadratic equations that do not factor later in the course.
- To find a quadratic equation with given solutions, perform the process of solving by factoring in reverse.

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9.2 Completing the Square

Learning Objectives

By the end of this section, you should be able to:

- Solve quadratic equations by completing the square.

Completing the Square

In this section, we will devise a method for rewriting any quadratic equation of the form

$$ax^2 + bx + c = 0$$

in the form

$$(x - p)^2 = q$$

This process is called **completing the square**. As we have seen, quadratic equations in this form can easily be solved by extracting roots. We begin by examining perfect square trinomials:

$$\begin{array}{rcccc} (x+3)^2 = x^2 & + & 6x & + & 9 \\ & & \downarrow & & \uparrow \\ & & \left(\frac{6}{2}\right)^2 = (3)^2 & = & 9 \end{array}$$

The last term, 9, is the square of one-half of the coefficient of x. In general, this is true for any perfect square trinomial of the form $x^2 + bx + c$.

$$\begin{aligned} \left(x + \frac{b}{2}\right)^2 &= x^2 + 2 \cdot \frac{b}{2}x + \left(\frac{b}{2}\right)^2 \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2 \end{aligned}$$

In other words, any trinomial of the form $x^2 + bx + c$ will be a perfect square trinomial if

$$c = \left(\frac{b}{2}\right)^2$$

Note:

It is important to point out that the leading coefficient must be equal to 1 for this to be true.

Example 1

Complete the square:

$$x^2 + 8x + \ ? \ = (x + \ ?)^2$$

Solution:
In this example, the coefficient of the middle term $b = 8$, so find the value that completes the square as follows:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{8}{2}\right)^2 = (4)^2 = 16$$

The value that completes the square is 16.

$$\begin{aligned} x^2 + 8x + 16 &= (x + 4)(x + 4) \\ &= (x + 4)^2 \end{aligned}$$

Answer

$$x^2 + 8x + 16 = (x + 4)^2$$

Example 2

Complete the square:

$$x^2 + 3x + \ ? \ = (x + \ ?)^2$$

Solution:

Here, $b = 3$, so find the value that will complete the square as follows:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

The value $\frac{9}{4}$ completes the square:

$$\begin{aligned} x^2 + 3x + \frac{9}{4} &= \left(x + \frac{3}{2}\right)\left(x + \frac{3}{2}\right) \\ &= \left(x + \frac{3}{2}\right)^2 \end{aligned}$$

Answer

$$x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$$

We can use this technique to solve quadratic equations. The idea is to take any quadratic equation in standard form and complete the square so that we can solve it by extracting roots. The following are general steps for solving a quadratic equation with a leading coefficient of 1 in standard form by completing the square.

Example 3

Solve by completing the square:

$$x^2 + 14x + 46 = 0$$

Solution:

Step 1: Add or subtract the constant term to obtain the equation in the form $x^2 + bx = c$. In this example, subtract 46 to move it to the right side of the equation.

$$\begin{aligned} x^2 + 14x + 46 &= 0 \\ x^2 + 14x &= -46 \end{aligned}$$

Step 2: Use $\left(\frac{b}{2}\right)^2$ to determine the value that completes the square. Here $b = 14$:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{14}{2}\right)^2 = (7)^2 = 49$$

Step 3: Add $\left(\frac{b}{2}\right)^2$ to both sides of the equation and complete the square.

$$\begin{aligned} x^2 + 14x + 49 &= -46 + 49 \\ (x + 7)^2 &= 3 \end{aligned}$$

Step 4: Solve by extracting roots.

$$\begin{aligned} x + 7 &= \pm\sqrt{3} \\ x &= -7 \pm \sqrt{3} \end{aligned}$$

Answer

The solutions are $-7 - \sqrt{3}$ or $-7 + \sqrt{3}$. The check is optional.

Example 4

Solve by completing the square:

$$x^2 - 18x + 72 = 0$$

Solution:
Begin by subtracting 72 from both sides.

$$\begin{aligned} x^2 - 18x + 72 &= 0 \\ x^2 - 18x &= -72 \end{aligned}$$

Next, find the value that completes the square using $b = -18$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-18}{2}\right)^2 = (-9)^2 = 81$$

To complete the square, add 81 to both sides, complete the square, and then solve by **extracting the roots**.

$$\begin{aligned} x^2 - 18x + 81 &= -72 + 81 \\ (x-9)^2 &= 9 \\ x - 9 &= \pm\sqrt{9} \\ x - 9 &= \pm 3 \\ x &= 9 \pm 3 \end{aligned}$$

At this point, separate the "plus or minus" into two equations and solve each.

$$\begin{aligned} x &= 9 - 3 \quad \text{or} \quad x = 9 + 3 \\ x &= 6 \qquad\qquad\quad x = 12 \end{aligned}$$

Answer

The solutions are 6 and 12.

Note that in the previous example, the solutions are integers. If this is the case, then the original equation will factor.

$$(x-6)(x-12) = 0$$

Practice completing the square with the *Chapter Exercises*.

Visualize

Visit the Geogebra site to practice Completing the Square: Choose Your Own Expression. Link available on page 9.2 of the digital version of this textbook.

If it factors, we can solve it by factoring. However, not all quadratic equations will factor.

Example 5

Solve by completing the square:

$$x^2 + 10x + 1 = 0$$

Solution:
Begin by subtracting 1 from both sides of the equation.

$$\begin{aligned} x^2 + 10x + 1 &= 0 \\ x^2 + 10x &= -1 \end{aligned}$$

Here, $b = 10$, and we determine the value that completes the square as follows:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = (5)^2 = 25$$

To complete the square, add 25 to both sides of the equation.

$$\begin{aligned} x^2 + 10x &= -1 \\ x^2 + 10x + 25 &= -1 + 25 \\ (x+5)^2 &= 24 \end{aligned}$$

Factor and then solve by extracting roots.

$$\begin{aligned} x + 5 &= \pm\sqrt{24} = \pm 2\sqrt{6} \\ x &= -5 \pm 2\sqrt{6} \end{aligned}$$

Answer

The solutions are $-5 - 2\sqrt{6}$ and $-5 + 2\sqrt{6}$

Sometimes, quadratic equations do not have real solutions.

Example 6

Solve by completing the square:

$$x^2 - 2x + 3 = 0$$

Solution:
Begin by subtracting 3 from both sides of the equation.

$$\begin{aligned} x^2 - 2x + 3 &= 0 \\ x^2 - 2x &= -3 \end{aligned}$$

Here $b = -2$, and we have

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$$

Therefore,

$$\begin{aligned} x^2 - 2x+1 &= -3+1 \\ (x-1)^2 &= -2 \end{aligned}$$

At this point, we see that extracting the root leads to the square root of a negative number.

Answer

No real solution

Try It

Solve by completing the square: $x^2 - 2x - 27 = 0$

Show Solution

$x = 1 \pm 2\sqrt{7}$

The coefficient of x is not always divisible by 2.

Example 7

Solve by completing the square:

$$x^2 + 3x - 2 = 0$$

Solution:
Begin by adding 2 to both sides.

$$x^2 + 3x - 2 = 0$$
$$x^2 + 3x = 2$$

Use $b = 3$ to find the value that completes the square:

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

To complete the square, add $\frac{9}{4}$ to both sides of the equation.

$$x^2 + 3x = 2$$
$$x^2 + 3x + \tfrac{9}{4} = 2 + \tfrac{9}{4}$$
$$\left(x + \tfrac{3}{2}\right)^2 = \tfrac{17}{4}$$

Solve by extracting roots.

$$x + \frac{3}{2} = \pm\sqrt{\frac{17}{4}} = \pm\frac{\sqrt{17}}{2}$$
$$x = \frac{-3 \pm \sqrt{17}}{2}$$

Answer

The solutions are $\dfrac{-3 \pm \sqrt{17}}{2}$

So far, all of the examples have had a leading coefficient of 1. The formula $\left(\dfrac{b}{2}\right)^2$ determines the value that completes the square only if the leading coefficient is 1. If this is not the case, then simply divide both sides by the leading coefficient.

Example 8

Solve by completing the square:

$$2x^2 + 5x - 1 = 0$$

Solution:
Notice that the leading coefficient is 2. Therefore, divide both sides by 2 before beginning the steps required to solve by completing the square.

$$\frac{2x^2+5x-1}{2} = \frac{0}{2}$$
$$x^2 + \frac{5}{2}x - \frac{1}{2} = 0$$

Begin by adding $\dfrac{1}{2}$ to both sides of the equation.

$$x^2 + \frac{5}{2}x - \frac{1}{2} = 0$$
$$x^2 + \frac{5}{2}x = \frac{1}{2}$$

Here $b = \dfrac{5}{2}$, and we can find the value that completes the square as follows:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{\frac{5}{2}}{2}\right)^2 = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$$

To complete the square, add $\dfrac{25}{16}$ to both sides of the equation.

$$x^2+\frac{5}{2}x=\frac{1}{2}$$
$$x^2+\frac{5}{2}x+\frac{25}{16}=\frac{1}{2}+\frac{25}{16}$$
$$\left(x+\frac{5}{4}\right)^2=\frac{33}{16}$$

Next, solve by extracting roots.

$$x+\frac{5}{4}=\pm\sqrt{\frac{33}{16}}=\pm\frac{\sqrt{33}}{4}$$
$$x=\frac{-5\pm\sqrt{33}}{4}$$

Answer

The solutions are $\frac{-5\pm\sqrt{33}}{4}$

Try It

Solve: $2x^2-2x-3=0$

Show Solution

$$\frac{1\pm\sqrt{7}}{2}$$

Key Takeaways

- Solve any quadratic equation by completing the square.
- You can apply the square root property to solve an equation if you can first convert the equation to the form $(x-p)^2=q$.
- To complete the square, first make sure the equation is in the form $x^2+bx=c$. Then add the value $\left(\frac{b}{2}\right)^2$ to both sides and factor.
- The process for completing the square always works, but it may lead to some tedious calculations with fractions. This is the case when the middle term, b, is not divisible by 2.

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9.3 The Quadratic Formula

Learning Objectives

By the end of this section, you should be able to:

- Solve quadratic equations with real solutions using the quadratic formula.

The Quadratic Formula

The quadratic formula solves any quadratic equation in standard form. Given $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, then the solutions can be calculated using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Consider the quadratic equation $2x^2 - 7x + 3 = 0$. It can be solved by factoring as follows:

$$(2x - 1)(x - 3) = 0$$

$$\begin{aligned} 2x - 1 = 0 \quad &\text{or} \quad x - 3 = 0 \\ 2x = 1 \quad &\quad\quad x = 3 \\ x = \frac{1}{2} \quad &\end{aligned}$$

The solutions are $\frac{1}{2}$ and 3. The following example shows that we can obtain the same results using the quadratic formula.

How To

Given a quadratic equation, solve it using the quadratic formula

1. Make sure the equation is in standard form: $ax^2 + bx + c = 0$.

2. Make note of the values of the coefficients and constant term, a, b, and c.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

Visualize

Visit the Geogebra site to practice The Quadratic Formula: Choose Your Own Equation. Link available on page 9.3 of the digital version of this textbook.

Example 1

Solve using the quadratic formula:

$$2x^2 - 7x + 3 = 0$$

Solution:

Begin by identifying a, b, and c as the coefficients of each term.

$$a = 2 \qquad b = -7 \qquad c = 3$$

Substitute these values into the quadratic formula and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(3)}}{2(2)} \\ &= \frac{7 \pm \sqrt{49 - 24}}{4} \\ &= \frac{7 \pm \sqrt{25}}{4} \\ &= \frac{7 \pm 5}{4} \end{aligned}$$

Separate the “plus or minus” into two equations and simplify each individually.

$$x = \frac{7-5}{4} \quad \text{or} \quad x = \frac{7+5}{4}$$
$$x = \frac{2}{4} \qquad\qquad x = \frac{12}{3}$$
$$x = \frac{1}{2} \qquad\qquad x = 3$$

Answer

The solutions are $\frac{1}{2}$ and 3.

Of course, if the quadratic factors, then it is a best practice to solve it by factoring. However, not all quadratic polynomials factor; nevertheless, the quadratic formula provides us with a means to solve such equations.

Example 2

Solve using the quadratic formula:

$$5x^2 + 2x - 1 = 0$$

Solution:
Begin by identifying a, b, and c.

$$a = 5 \qquad b = 2 \qquad c = -1$$

Substitute these values into the quadratic formula.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-2 \pm \sqrt{2^2 - 4(5)(-1)}}{2(5)} \\
&= \frac{-2 \pm \sqrt{4 + 20}}{10} \\
&= \frac{-2 \pm \sqrt{24}}{10} \\
&= \frac{-2 \pm 2\sqrt{6}}{10} \\
&= \frac{-1 \pm \sqrt{6}}{5}
\end{aligned}
$$

Answer

The solutions are $\frac{-1 \pm \sqrt{6}}{5}$

Science Application

When a chemical reaction reaches equilibrium, we use a constant K_c to describe the ratio of products to reactants. A technician places 0.1 M of nitrous acid (HNO_2) into a sealed container. It decomposes into H^+ and NO_2^-. The equilibrium equation for this specific setup is:

$$4.5x10^{-4} = \frac{x^2}{0.1 - x}$$

Solve for x (the amount of reactant that was transformed into product before equilibrium was reached). You need to re-arrange the equation to isolate x, then re-arrange into standard form for a quadratic ($ax^2 + bx + c$). Then input the a, b and c values into the quadratic formula.

Answer:

$x = 6.5x10^{-3}M$

Try It

Try using the quadratic formula in this activity:

Activity Description
Drag the numbers into the correct boxes.

1. Use the quadratic formula to solve. Keep the answers exact.

Questions:

a. $2x^2 - 5x + 1 = 0$
$x =$
b. $12x^2 = 5 + 11x$
$x =$

Options:

$$-\frac{5 \pm \sqrt{17}}{4} \qquad \frac{5 \pm \sqrt{17}}{4} \qquad \frac{5}{4}, -\frac{1}{3} \qquad -\frac{11}{12}, \frac{1}{3}$$

Answers:

a. $x = \dfrac{5 \pm \sqrt{17}}{4}$
b. $\dfrac{5}{4}, -\dfrac{1}{3}$

---

2. Use the quadratic formula to solve. Keep the answers exact.

Questions:

a. $4.5x^2 - 12.8x - 5.2 = 0, x \approx$
b. $1.85(x^2 + 4.2x) = 5.1x(2.2x - 1.5), x \approx$
c. $5.4 \times 10^5 x^2 + 8.2 \times 10^5 x + 1.1 \times 10^5 = 0, x \approx$
d. $-0.006x^2 + 0.015x + 0.009 = 0, x \approx$

Options:

$1.64, 0$ $-7.41, -8.99$ $3.19, -0.632$ 1.64 $3.00, -0.500$ $2.36, 0.489$ $-3.00, 0.500$ $-0.147, -1.37$

Answers:

a. $3.19, -0.362$
b. $-7.41, -8.99$
c. $3.00, -0.500$
d. $-3.00, 0.500$

Often, terms are missing. When this is the case, use 0 as the coefficient.

Example 3

Solve using the quadratic formula:

$$x^2 - 18 = 0$$

Solution:
Think of this equation with the following coefficients:

$$1x^2 + 0x - 18 = 0$$

Here

$$a = 1 \qquad b = 0 \qquad c = -18$$

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-0 \pm \sqrt{0^2 - 4(1)(-18)}}{2(1)} \\ &= \frac{\pm\sqrt{72}}{2} \\ &= \frac{\pm 6\sqrt{2}}{2} \\ &= \pm 3\sqrt{2} \end{aligned}$$

Answer

The solutions are $\pm 3\sqrt{2}$

Since the coefficient of x was 0, we could have solved the equation by extracting the roots. As an exercise, solve the previous example using this method and verify that the results are the same.

Example 4

Solve using the quadratic formula:

$$9x^2 - 12x + 4 = 0$$

Solution:
In this case,

$$a = 9 \qquad b = -12 \qquad c = 4$$

Substitute these values into the quadratic formula and then simplify.

$$\begin{aligned} x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(9)(4)}}{2(9)} \\ &= \frac{12 \pm \sqrt{144 - 144}}{18} \\ &= \frac{12 \pm 0}{18} \\ &= \frac{2}{3} \end{aligned}$$

In this example, notice that the radicand of the square root is 0. This results in only one solution to this quadratic equation. Normally, we expect two solutions. When we find only one solution, the solution is called a double root. If we solve this equation by factoring, then the solution appears twice.

$$(3x - 2)(3x - 2) = 0$$

$$\begin{aligned} 3x - 2 &= 0 \quad \text{or} \quad 3x - 2 = 0 \\ 3x &= 2 \qquad\qquad 3x = 2 \\ x &= \frac{2}{3} \qquad\qquad x = \frac{2}{3} \end{aligned}$$

Answer

$\frac{2}{3}$, double root

Example 5

Solve using the quadratic formula:

$$x^2 + x + 1 = 0$$

Solution:
In this case,

$$a = 1 \qquad b = 1 \qquad c = 1$$

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1 - 4}}{2} \\ &= \frac{-1 \pm \sqrt{-3}}{2} \end{aligned}$$

The solution involves the square root of a negative number; hence, the solutions are not real. This quadratic equation has two non-real solutions and will be discussed in further detail as we continue in our study of algebra. For now, simply state that the equation does not have real solutions.

Answer

No real solutions

Try It

Solve: $x^2 - 2x - 2 = 0$

Show Solution

$1 \pm \sqrt{3}$

It is important to place the quadratic equation in standard form before using the quadratic formula.

Science Application

An environmental scientist is studying how sediment travels down a stream after a heavy rain. A small sediment particle is caught in a steady current. Initially, the particle is moving at a slow speed of $0.2m/s$. As it enters a steeper section of the stream, it begins to accelerate constantly at a rate of $0.15m/s^2$ due to gravity and the increased water pressure.

The student needs to know how long it takes for the particle to travel exactly 5.0 meters down the stream. **How many seconds (t) does it take for the particle to travel 5.0 meters?**

v: Final Velocity. This is how fast the object is moving at the very end of the time we are measuring.
v_0: Initial Velocity. This is the starting speed of the object (at 0 seconds).
a: Acceleration. This is how much the speed increases every second.
t: Time. This is how long the object has been moving or accelerating.

We will use the first kinematic equation:

$$\delta x = v_0 t + \frac{1}{2} a t^2$$

List the known values:

$v_0 = 0.2m/s$ (Starting speed)
$a = 0.15m/s^2$ (Rate of speeding up)
x = displacement distance (the total distance the particle travelled), 5 m
t = what you are solving

Once you put those numbers into the equation and rearrange into standard form (with t is the variable), you will find the coefficients for the quadratic formula:

$a = 0.075$
$b = 0.2$
$c = -5$

Put these numbers into the quadratic formula

$$t = \frac{-0.2 \pm \sqrt{(0.2)^2 - 4(0.075)(-5.0)}}{2(0.075)}$$
$$t = \frac{-0.2 \pm \sqrt{0.04 + 1.5}}{0.15}$$
$$t = \frac{-0.2 \pm \sqrt{1.54}}{0.15}$$

The two possible values for t are:

t is approx 6.94 seconds
t is approx -9.61 seconds

Since time can't be negative, the particle travelled for about 6.9 seconds.

Example 6

Solve using the quadratic formula:

$$(2x + 1)(2x - 1) = 24x + 8$$

Solution:
Begin by using the distributive property to expand the left side and combining like terms to obtain an equation in standard form, equal to 0.

$$(2x + 1)(2x - 1) = 24x + 8$$
$$4x^2 - 1 = 24x + 8$$
$$4x^2 - 24x - 9 = 0$$

Once the equation is in standard form, identify a, b, and c. Here

$$a = 4 \qquad b = -24 \qquad c = -9$$

Substitute these values into the quadratic formula and then simplify.

$$
\begin{aligned}
x &= \frac{-(-24) \pm \sqrt{(-24)^2 - 4(4)(-9)}}{2(4)} \\
&= \frac{24 \pm \sqrt{576 + 144}}{8} \\
&= \frac{24 \pm \sqrt{720}}{8} \\
&= \frac{24 \pm \sqrt{144 \cdot 5}}{8} \\
&= \frac{24 \pm 12\sqrt{5}}{8} \\
&= \frac{3(2 \pm \sqrt{5})}{2} \\
&= \frac{6 \pm 3\sqrt{5}}{2}
\end{aligned}
$$

Answer

The solutions are $\frac{6 \pm 3\sqrt{5}}{2}$

Try It

Solve: $3x(x - 2) = 1$

Show Solution

$$\frac{3 \pm 2\sqrt{3}}{3}$$

Key Takeaways

- Use the quadratic formula to solve any quadratic equation in standard form.
- To solve any quadratic equation, first rewrite in standard form, $ax^2 + bx + c = 0$, substitute the appropriate coefficients into the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and then simplify.

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9.4 Solving Quadratic Equations

Learning Objectives

By the end of this section, you should be able to:

- Use the discriminant to determine the number and type of solutions to any quadratic equation.
- Develop a general strategy for solving quadratic equations.
- Solve applications involving quadratic equations.

Discriminant

If given a quadratic equation in standard form, $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, then the solutions can be calculated using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions are rational, irrational, or not real (having a negative square root). We can determine the type and number of solutions by studying the discriminant, the expression inside the radical, $b^2 - 4ac$. If the value of this expression is negative, then the equation has no real solutions. If the **discriminant** is positive, then we have two real solutions. And if the discriminant is 0, then we have one real solution.

Example 1

Determine the type and number of solutions:

$$x^2 - 10x + 30 = 0$$

Solution:
We begin by identifying a, b, and c. Here

$$a = 1 \qquad b = -10 \qquad c = 30$$

Substitute these values into the discriminant and simplify.

$$\begin{aligned} b^2 - 4ac &= (-10)^2 - 4(1)(30) \\ &= 100 - 120 \\ &= -20 \end{aligned}$$

Since the discriminant is negative, we conclude that there are no real solutions.

Answer

No real solution

If we use the quadratic formula in the previous example, we find that a negative radicand stops the process of simplification and shows that there is no real solution.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{-20}}{2(1)} \\ &= \frac{10 \pm \sqrt{-20}}{2} \end{aligned}$$

No real solution.

Note: We will study quadratic equations with no real solutions as we progress in our study of algebra.

Example 2

Determine the type and number of solutions:

$$7x^2 - 10x + 1 = 0$$

Solution:

Here

$$a = 7 \qquad b = -10 \qquad c = 1$$

Substitute these values into the discriminant:

$$\begin{aligned} b^2 - 4ac &= (-10)^2 - 4(7)(1) \\ &= 100 - 28 \\ &= 72 \end{aligned}$$

Since the discriminant is positive, we can conclude that there are two real solutions.

Answer

Two real solutions

If we use the quadratic formula in the previous example, we find that a positive radicand in the quadratic formula leads to two real solutions.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(7)(1)}}{2(7)} \qquad \text{Positive discriminant} \\ &= \frac{10 \pm \sqrt{72}}{14} \\ &= \frac{10 \pm 6\sqrt{2}}{14} \\ &= \frac{5 \pm 3\sqrt{2}}{7} \qquad \text{Two real solutions} \end{aligned}$$

The two real solutions are $\frac{5 - 3\sqrt{2}}{7}$ and $\frac{5 + 3\sqrt{2}}{7}$. Note that these solutions are irrational; we can approximate the values on a calculator.

Example 3

Determine the type and number of solutions:

$$2x^2 - 7x - 4 = 0$$

Solution:
In this example,

$$a = 2 \qquad b = -7 \qquad c = -4$$

Substitute these values into the discriminant and simplify.

$$\begin{aligned} b^2 - 4ac &= (-7)^2 - 4(2)(-4) \\ &= 49 + 32 \\ &= 81 \end{aligned}$$

Since the discriminant is positive, we conclude that there are two real solutions. Furthermore, since the discriminant is a perfect square, we obtain two rational solutions.

Answer

Two real solutions

We could solve the previous quadratic equation using the quadratic formula as follows:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{81}}{2(2)} \\ &= \frac{7 \pm 9}{4} \end{aligned}$$

$$\begin{array}{lll} x = \frac{7-9}{4} & \text{or} & x = \frac{7+9}{4} \\ x = \frac{-2}{4} & & x = \frac{16}{4} \\ x = -\frac{1}{2} & & x = 4 \end{array}$$

Note

If the discriminant is a perfect square, then we could have factored the original equation.

Factoring the original equation we get:

$$(2x+1)(x-4)=0$$

$$\begin{aligned} 2x+1=0 \quad &\text{or} \quad x-4=0 \\ 2x=-1 \quad &\qquad\quad x=4 \\ x=-\frac{1}{2} \quad &\end{aligned}$$

Given the special condition where the discriminant is 0, we obtain only one solution, a double root.

Example 4

Determine the type and number of solutions:

$$9x^2-6x+1=0$$

Solution:
Here $a=9, b=-6$, and $c=1$, and we have

$$\begin{aligned} b^2-4ac &= (-6)^2-4(9)(1) \\ &= 36-36 \\ &= 0 \end{aligned}$$

Since the discriminant is 0, we conclude that there is only one real solution, a double root.

Answer

One real solution

Since 0 is a perfect square, we can solve the equation above by factoring.

$$(3x-1)(3x-1)=0$$

$$\begin{aligned} 3x-1=0 \quad &\text{or} \quad 3x-1=0 \\ 3x=1 \quad &\qquad\quad 3x=1 \\ x=\frac{1}{3} \quad &\qquad\quad x=\frac{1}{3} \end{aligned}$$

Here $\frac{1}{3}$ is a solution that occurs twice; it is a double root.

In summary, if given any quadratic equation in standard form, $ax^2+bx+c=0$, where a, b, and c are real numbers, and $a \neq 0$, then we have the following:

Table 9.4.1: Discriminants

Positive discriminant:	$b^2 - 4ac > 0$	Two real solutions
Zero discriminant:	$b^2 - 4ac = 0$	One real solution
Negative discriminant:	$b^2 - 4ac < 0$	No real solution

As we will see, knowing the number and type of solutions ahead of time helps us determine which method is best for solving a quadratic equation.

Try It

Determine the number and type of solutions: $3x^2 - 5x + 4 = 0$

Show Solution

No real solution

General Guidelines for Solving Quadratic Equations

Use the coefficients of a quadratic equation to help decide which method is most appropriate for solving it. While the quadratic formula always works, it is sometimes not the most efficient method. Given any quadratic equation in standard form, $ax^2 + bx + c = 0$, general guidelines for determining the method for solving it follow:

1. If $c = 0$, then factor out the GCF and solve by factoring.
2. If $b = 0$, then solve by extracting the roots.
3. If a, b, and c are all nonzero, then determine the value for the discriminant, $b^2 - 4ac$:
    a. If the discriminant is a perfect square, then solve by factoring.
    b. If the discriminant is not a perfect square, then solve using the quadratic formula.
    c. If the discriminant is positive, we obtain two real solutions.
    d. If the discriminant is negative, then there is no real solution.

Example 5

Solve:

$$15x^2 - 5x = 0$$

Solution:
In this case, $c = 0$, and we can solve by factoring out the GCF.

$$5x(3x - 1) = 0$$

$$\begin{aligned} 5x &= 0 \quad \text{or} \quad 3x - 1 = 0 \\ x &= 0 \qquad\quad 3x = 1 \\ & \qquad\quad\; x = \frac{1}{3} \end{aligned}$$

Answer

The solutions are 0 and $\frac{1}{3}$

Example 6

Solve:

$$3x^2 - 5 = 0$$

Solution:
In this case, $b = 0$, and we can solve by extracting the roots.

$$
\begin{aligned}
3x^2 - 5 &= 0 \\
3x^2 &= 5 \\
x^2 &= \frac{5}{3} && \text{Apply the square root property.} \\
x &= \pm\sqrt{\frac{5}{3}} \\
&= \pm\frac{\sqrt{5}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\
&= \pm\frac{\sqrt{15}}{3}
\end{aligned}
$$

Answer

The solutions are $\pm\frac{\sqrt{15}}{3}$

Example 7

Solve:

$$9x^2 - 6x - 7 = 0$$

Solution:
Begin by identifying a, b, and c as the coefficients of each term. Here

$$a = 9 \qquad b = -6 \qquad c = -7$$

Substitute these values into the discriminant and then simplify.

$$
\begin{aligned}
b^2 - 4ac &= (-6)^2 - 4(9)(-7) \\
&= 36 + 252 \\
&= 288
\end{aligned}
$$

Since the discriminant is positive and not a perfect square, use the quadratic formula and expect two real solutions.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{288}}{2(9)} \\ &= \frac{6 \pm \sqrt{144 \cdot 2}}{18} \\ &= \frac{6 \pm 12\sqrt{2}}{18} \\ &= \frac{1 \pm 2\sqrt{2}}{3} \end{aligned}$$

Answer

The solutions are $\dfrac{1 \pm 2\sqrt{2}}{3}$

Example 8

Solve:

$$4x(x-2) = -7$$

Solution:
Begin by rewriting the quadratic equation in standard form.

$$\begin{aligned} 4x(x-2) &= -7 \\ 4x^2 - 8x &= -7 \\ 4x^2 - 8x + 7 &= 0 \end{aligned}$$

Here

$$a = 4 \qquad b = -8 \qquad c = 7$$

Substitute these values into the discriminant and then simplify.

$$\begin{aligned} b^2 - 4ac &= (-8)^2 - 4(4)(7) \\ &= 64 - 112 \\ &= -48 \end{aligned}$$

Since the discriminant is negative, the solutions are not real numbers.

Answer

No real solution

Example 9

Solve:

$$(3x+5)(3x+7)=6x+10$$

Solution:

Begin by rewriting the quadratic equation in standard form.

$$\begin{aligned}(3x+5)(3x+7)&=6x+10\\9x^2+21x+15x+35&=6x+10\\9x^2+36x+35&=6x+10\\9x^2+30x+25&=0\end{aligned}$$

Substitute $a=9, b=30$, and $c=25$ into the discriminant.

$$\begin{aligned}b^2-4ac&=(30)^2-4(9)(25)\\&=900-900\\&=0\end{aligned}$$

Since the discriminant is 0, solve by factoring and expect one real solution, a double root.

$$(3x+5)(3x+5)=0$$

$$\begin{aligned}3x+5&=0 \quad \text{or} \quad 3x+5=0\\3x&=-5 \qquad\qquad 3x=-5\\x&=-\frac{5}{3} \qquad\qquad x=-\frac{5}{3}\end{aligned}$$

Answer

The solution is $-\frac{5}{3}$

Try It

Solve: $5x^2 + 2x - 7 = 2x - 3$

Show Solution

$$\pm\frac{2\sqrt{5}}{5}$$

Applications Involving Quadratic Equations

In this section, the algebraic setups usually consist of a quadratic equation where the solutions may not be integers.

Example 10

The height of a triangle is 2 inches less than twice the length of its base. If the total area of the triangle is 11 square inches, then **find the lengths of the base and height.** Round answers to the nearest hundredth.

Solution:

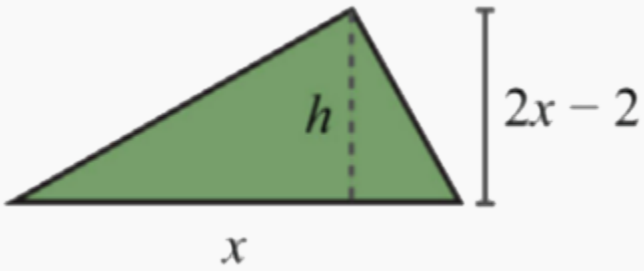

Figure 9.4.1.

Let x represent the length of the base of the triangle.

$$\overbrace{2x}^{\text{twice the base}} \quad \overbrace{-2}^{\text{2 inches less than}}$$

Let $2x - 2$ represent the height of the triangle.

Use the formula $A = \frac{1}{2}bh$ and the fact that the area is 11 square inches to set up an algebraic equation.

$$A = \frac{1}{2}b \cdot h$$
$$11 = \frac{1}{2}x(2x - 2)$$

To rewrite this quadratic equation in standard form, first distribute $\frac{1}{2}x$.

$$11 = \frac{1}{2}x(2x - 2)$$
$$11 = x^2 - x$$
$$0 = x^2 - x - 11$$

Use the coefficients, $a = 1, b = -1$, and $c = -11$, to determine the type of solutions.

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(1)(-11) \\ &= 1 + 44 \\ &= 45 \end{aligned}$$

Since the discriminant is positive, expect two real solutions.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{45}}{2(1)} \\ &= \frac{1 \pm 3\sqrt{5}}{2} \end{aligned}$$

In this problem, disregard the negative solution and consider only the positive solution.

$$x = \frac{1 + 3\sqrt{5}}{2}$$

Back substitute to find the height.

$$\begin{aligned} h &= 2x - 2 \\ &= 2\left(\frac{1 + 3\sqrt{5}}{2}\right) - 2 \\ &= 1 + 3\sqrt{5} - 2 \\ &= -1 + 3\sqrt{5} \end{aligned}$$

Answer

The base measures $\frac{-1 + 3\sqrt{5}}{2} \approx 3.85$ inches and the height is $-1 + 3\sqrt{5} \approx 5.71$ inches.

Example 11

The sum of the squares of two consecutive positive integers is 481. **Find the integers.**

Solution:
Let n represent the first positive integer.

Let $n + 1$ represent the next positive integer.

The algebraic setup follows:

$$\overbrace{n^2 + (n+1)^2}^{\text{The sum of the squares of ...}} \quad \overbrace{= 481}^{\text{... is 481}}$$

Rewrite the quadratic equation in standard form.

$$\begin{aligned} n^2+(n+1)^2 &= 481 \\ 2n^2+2n+1 &= 481 \\ 2n^2+2n-480 &= 0 \\ 2(n^2+n-240) &= 0 \\ n^2+n-240 &= 0 \end{aligned}$$

When the coefficients are large, sometimes it is less work to use the quadratic formula instead of trying to factor it. In this case, $a=1, b=1$, and $c=-240$. Substitute into the quadratic formula and then simplify.

$$\begin{aligned} n &= \frac{-1\pm\sqrt{1^2-4(1)(-240)}}{2} \\ &= \frac{-1\pm\sqrt{961}}{2} \\ &= \frac{-1\pm 31}{2} \end{aligned}$$

$$\begin{array}{ll} n=\frac{-1-31}{2} & n=\frac{-1+31}{2} \\ n=\frac{-32}{2} & n=\frac{30}{2} \\ n=-16 & n=15 \end{array}$$

Since the problem calls for positive integers, disregard the negative solution and choose $n=15$.

$$\begin{aligned} n+1 &= 15+1 \\ &= 16 \end{aligned}$$

Answer

The positive integers are 15 and 16.

Key Takeaways

- Determine the number and type of solutions to any quadratic equation in standard form using the discriminant, b^2-4ac. If the discriminant is negative, then the solutions are not real. If the discriminant is positive, then the solutions are real. If the discriminant is 0, then there is only one solution, a double root.
- Choose the appropriate method for solving a quadratic equation based on the value of its discriminant. While the quadratic formula will solve any quadratic equation, it may not be the most efficient

method.

- When solving applications, use the key words and phrases to set up an algebraic equation that models the problem. In this section, the setup typically involves a quadratic equation.

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9.5 Exercises

9.1 Solving Equations with Factoring

Determine whether the given set of values are solutions to the quadratic equation.

1. $\{-3, 5\};\ x^2 - 2x - 15 = 0$
2. $\{7, -1\};\ x^2 - 6x - 7 = 0$
3. $\{-\frac{1}{2}, \frac{1}{2}\};\ x^2 - 14 = 0$
4. $\{-\frac{3}{4}, \frac{3}{4}\};\ x^2 - \frac{9}{16} = 0$
5. $\{-3, 2\};\ x^2 - x - 6 = 0$
6. $\{-5, 1\};\ x^2 - 4x - 5 = 0$

Solve.

7. $(x - 3)(x + 2) = 0$
8. $(x + 5)(x + 1) = 0$
9. $(2x - 1)(x - 4) = 0$
10. $(3x + 1)(3x - 1) = 0$
11. $(x - 2)^2 = 0$
12. $(5x + 3)^2 = 0$
13. $7x(x - 5) = 0$
14. $-2x(2x - 3) = 0$

15. $(x-12)(x+34)=0$

16. $(x+58)(x-38)=0$

17. $(14x+12)(16x-23)=0$

18. $(15x-3)^2=0$

19. $-5(x+1)(x-2)=0$

20. $12(x-7)(x-6)=0$

21. $(x+5)(x-1)=0$

22. $(x+5)(x+1)=0$

23. $-2(3x-2)(2x+5)=0$

24. $5(7x-8)^2=0$

25. $x^2-x-6=0$

26. $x^2+3x-10=0$

27. $y^2-10y+24=0$

28. $y^2+6y-27=0$

29. $x^2-14x+40=0$

30. $x^2+14x+49=0$

31. $x^2-10x+25=0$

32. $3x^2+2x-1=0$

33. $5x^2-9x-2=0$

34. $7y^2+20y-3=0$

35. $9x^2-42x+49=0$

36. $25x^2+30x+9=0$

37. $2y^2 + y - 3 = 0$

38. $7x^2 - 11x - 6 = 0$

39. $2x^2 = -15x + 8$

40. $8x - 5 = 3x^2$

41. $x^2 - 36 = 0$

42. $x^2 - 100 = 0$

43. $4x^2 - 81 = 0$

44. $49x^2 - 4 = 0$

45. $x^2 = 4$

46. $9y^2 = 1$

47. $16y^2 = 25$

48. $36x^2 = 25$

49. $4x^2 - 36 = 0$

50. $2x^2 - 18 = 0$

51. $10x^2 + 20x = 0$

52. $-3x^2 + 6x = 0$

53. $25x^2 = 50x$

54. $x^2 = 0$

55. $(x + 1)^2 - 25 = 0$

56. $(x - 2)^2 - 36 = 0$

57. $5x(x - 4) = -4 + x$

58. $(x - 1)(x - 10) = 22$

59. $(x-3)(x-5)=24$

60. $-2x(x-9)=x+21$

61. $(x+1)(6x+1)=2x$

62. $(x-2)(x+12)=15x$

63. $(x+1)(x+2)=2(x+16)$

64. $(x-9)(2x+3)=2(x-9)$

Clear the fractions by first multiplying both sides by the LCD and then solve.

65. $115x^2+13x+25=0$

66. $114x^2-12x+37=0$

67. $32x^2-23=0$

68. $52x^2-110=0$

69. $314x^2-212=0$

70. $13x^2-15x=0$

71. $132x^2-12x+2=0$

72. $13x^2+56x-12=0$

73. The sides of a square measure $x+3$ units. If the area is 25 square units, then find x.

74. The height of a triangle is 2 units more than its base. If the area is 40 square units, then find the length of the base.

75. The sides of a right triangle have measures that are consecutive integers. Find the length of the hypotenuse.

76. The profit in dollars generated by producing and selling x custom lamps is given by $P(x)=-10x^2+800x-12000$. How many lamps must be sold and produced to break even?

Assuming dry road conditions and average reaction times, the safe stopping distance, d in feet of an average car is given using the formula $d=120v^2+v$, where v represents the speed of the car in miles per hour. For each problem below, given the stopping distance, determine the safe speed.

77. 15 feet

78. 40 feet

79. 75 feet

80. 120 feet

Find a quadratic equation with integer coefficients, given the following solutions.

81. $-3, 1$

82. $-5, 3$

83. $-10, -3$

84. $-7, -4$

85. $-1, 0$

86. $0, \frac{3}{5}$

87. $-2, 2$

88. $-\frac{1}{2}, \frac{1}{2}$

89. $-4, \frac{1}{3}$

90. $\frac{2}{3}, \frac{2}{5}$

91. $-\frac{1}{5}, -\frac{2}{3}$

92. $-\frac{3}{2}, \frac{3}{4}$

93. 3, double root

94. -5, double root

Solve.

95. $7x(x+5)(x-9)=0$

96. $(x-1)(x-2)(x-3)=0$

97. $-2x(x-10)(x-1)=0$

98. $8x(x-4)^2=0$

99. $4(x+3)(x-2)(x+1)=0$

100. $-2(3x+1)(3x-1)(x-1)(x+1)=0$

101. $x^3-x^2-2x=0$

102. $2x^3+5x^2-3x=0$

103. $5x^3-15x^2+10x=0$

104. $-2x^3+2x^2+12x=0$

105. $3x^3-27x=0$

106. $-2x^3+8x=0$

107. $x^3+x^2-x-1=0$

108. $x^3+2x^2-16x-32=0$

109. $8x^3-4x^2-18x+9=0$

110. $12x^3=27x$

Show Solutions

1. Yes
2. Yes

3. No
4. Yes
5. Yes
6. No
7. $-2,\ 3$
8. $-5,\ -1$
9. $\frac{1}{2},\ 4$
10. $-\frac{1}{3},\ \frac{1}{3}$
11. 2
12. $-\frac{3}{5}$
13. $0,\ 5$
14. $0,\ \frac{3}{2}$
15. $-34,\ 12$
16. $-58,\ 38$
17. $-\frac{6}{7},\ \frac{23}{16}$
18. $\frac{1}{5}$
19. $-1,\ 2$

20. $6,\ 7$

21. $-5,\ 1$

22. $-5,\ -1$

23. $-\frac{5}{2},\ \frac{2}{3}$

24. $\frac{8}{7}$

25. $-2,\ 3$

26. $-5,\ 2$

27. $4,\ 6$

28. $-9,\ 3$

29. $4,\ 10$

30. -7

31. 5

32. $-\frac{1}{3},\ 1$

33. $-\frac{1}{5},\ 2$

34. $-3,\ \frac{1}{7}$

35. $\frac{7}{3}$

36. $-\frac{3}{5}$

37. $-\frac{3}{2}, 1$

38. $-\frac{6}{7}, 1$

39. $-8, \frac{1}{2}$

40. $1, \frac{5}{3}$

41. $-6, 6$

42. $-10, 10$

43. $-\frac{9}{2}, \frac{9}{2}$

44. $-\frac{2}{7}, \frac{2}{7}$

45. $-2, 2$

46. $-\frac{1}{3}, \frac{1}{3}$

47. $-\frac{5}{4}, \frac{5}{4}$

48. $-\frac{5}{6}, \frac{5}{6}$

49. $-3, 3$

50. $-3, 3$

51. $-2, 0$

52. $0, 2$

53. $0,\ 2$

54. 0

55. $-6,\ 4$

56. $-4,\ 8$

57. $\frac{1}{5},\ 4$

58. $-1,\ 12$

59. $-3,\ 5$

60. $-\frac{1}{2},\ 9$

61. $-\frac{1}{2},\ -\frac{1}{3}$

62. $-3,\ 8$

63. $-6,\ 5$

64. $\frac{3}{2},\ 9$

65. No real solutions

66. No real solutions

67. $\pm\sqrt{\frac{23}{32}}$

68. $\pm\sqrt{\frac{55}{26}}$

69. $\pm\sqrt{\frac{106}{157}}$

70. $0,\ \frac{15}{13}$

71. No real solutions

72. $\frac{-28 \pm 2\sqrt{235}}{13}$

73. 2

74. 8 units

75. 5 units

76. 20 or 60 lamps

77. 10 mph

78. 20 mph

79. 30 mph

80. 40 mph

81. $x^2 + 2x - 3 = 0$

82. $x^2 + 2x - 15 = 0$

83. $x^2 + 13x + 30 = 0$

84. $x^2 + 11x + 28 = 0$

85. $x^2 + x = 0$

86. $5x^2 - 3x = 0$

87. $x^2 - 4 = 0$

88. $4x^2 - 1 = 0$

89. $3x^2 + 11x - 4 = 0$

90. $15x^2 - 16x + 4 = 0$

91. $15x^2 + 13x + 2 = 0$

92. $8x^2 + 6x - 9 = 0$

93. $x^2 - 6x + 9 = 0$

94. $x^2 + 10x + 25 = 0$

95. $-5,\ 0,\ 9$

96. $1,\ 2,\ 3$

97. $0,\ 1,\ 10$

98. $0,\ 4$

99. $-3,\ -1,\ 2$

100. $-1,\ -\frac{1}{3},\ \frac{1}{3},\ 1$

101. $-1,\ 0,\ 2$

102. $-3,\ 0,\ \frac{1}{2}$

103. $0,\ 1,\ 2$

104. $-2,\ 0,\ 3$

105. $-3,\ 0,\ 3$

106. $-2,\ 0,\ 2$

107. $-1,\ 1$

108. $-4,\ -2,\ 2$

109. $-\frac{3}{2},\ \frac{1}{2},\ \frac{3}{2}$

110. $-\frac{3}{2},\ 0,\ \frac{3}{2}$

9.2 Completing the Square

Complete the square.

1. $x^2+6x+\ ?\ =(x+\ ?)^2$
2. $x^2+8x+\ ?\ =(x+\ ?)^2$
3. $x^2-2x+\ ?\ =(x-\ ?)^2$
4. $x^2-4x+\ ?\ =(x-\ ?)^2$
5. $x^2+7x+\ ?\ =(x+\ ?)^2$
6. $x^2+3x+\ ?\ =(x+\ ?)^2$
7. $x^2+23x+\ ?\ =(x+\ ?)^2$
8. $x^2+45x+\ ?\ =(x+\ ?)^2$
9. $x^2+34x+\ ?\ =(x+\ ?)^2$

10. $x^2 + 53x + \ ? \ = (x + \ ?)^2$

Solve by factoring and then solve by completing the square. Check answers.

11. $x^2 + 2x - 8 = 0$
12. $x^2 - 8x + 15 = 0$
13. $y^2 + 2y - 24 = 0$
14. $y^2 - 12y + 11 = 0$
15. $t^2 + 3t - 28 = 0$
16. $t^2 - 7t + 10 = 0$
17. $2x^2 + 3x - 2 = 0$
18. $3x^2 - x - 2 = 0$
19. $2y^2 - y - 1 = 0$
20. $2y^2 + 7y - 4 = 0$

Show Solutions

1. $x^2 + 6x + 9 = (x + 3)^2$
2. $x^2 + 8x + 16 = (x + 4)^2$
3. $x^2 - 2x + 1 = (x - 1)^2$
4. $x^2 - 4x + 4 = (x - 2)^2$
5. $x^2 + 7x + \frac{49}{4} = (x + \frac{7}{2})^2$

6. $x^2 + 3x + \frac{9}{4} = (x + \frac{3}{2})^2$
7. $x^2 + 23x + \frac{529}{4} = (x + \frac{23}{2})^2$
8. $x^2 + 45x + \frac{2025}{4} = (x + \frac{45}{2})^2$
9. $x^2 + 34x + 289 = (x + 17)^2$
10. $x^2 + 53x + \frac{2809}{4} = (x + \frac{53}{2})^2$
11. $-4,\ 2$
12. $3,\ 5$
13. $-6,\ 4$
14. $1,\ 11$
15. $-7,\ 4$
16. $2,\ 5$
17. $-2,\ \frac{1}{2}$
18. $-\frac{2}{3},\ 1$
19. $-\frac{1}{2},\ 1$
20. $-4,\ \frac{1}{2}$

9.3 The Quadratic Formula

Identify the coefficients a, b, and c used in the quadratic formula. Do not solve.

1. $x^2 - x + 5 = 0$
2. $x^2 - 3x - 1 = 0$
3. $3x^2 - 10 = 0$
4. $-y^2 + 5 = 0$
5. $5t^2 - 7t = 0$
6. $-y^2 + y = 0$
7. $-x^2 + x = -6$
8. $-2x^2 - x = -15$
9. $(3x + 1)(2x + 5) = 19x + 4$
10. $(4x + 1)(2x + 1) = 16x + 4$

Solve by factoring and then solve using the quadratic formula. Check answers.

11. $x^2 - 10x + 24 = 0$
12. $x^2 - 3x - 18 = 0$
13. $t^2 + 6t + 5 = 0$
14. $t^2 + 9t + 14 = 0$
15. $2x^2 - 7x - 4 = 0$
16. $3x^2 - x - 2 = 0$
17. $-2x^2 - x + 3 = 0$

18. $-6x^2 + x + 1 = 0$

19. $y^2 - 2y + 1 = 0$

20. $y^2 - 1 = 0$

Use the quadratic formula to solve the following.

21. $x^2 - 6x + 4 = 0$

22. $x^2 - 4x + 1 = 0$

23. $x^2 + 2x - 5 = 0$

24. $x^2 + 4x - 6 = 0$

25. $t^2 - 4t - 1 = 0$

26. $t^2 - 8t - 2 = 0$

27. $-y^2 + y + 1 = 0$

28. $-y^2 - 3y + 2 = 0$

29. $-x^2 + 16x - 62 = 0$

30. $-x^2 + 14x - 46 = 0$

31. $2t^2 - 4t - 3 = 0$

32. $4t^2 - 8t - 1 = 0$

33. $-4y^2 + 12y - 9 = 0$

34. $-25x^2 + 10x - 1 = 0$

35. $3x^2 + 6x + 2 = 0$

36. $5x^2 + 10x + 2 = 0$

37. $9t^2 + 6t - 11 = 0$

38. $8t^2 + 8t + 1 = 0$

39. $x^2 - 2 = 0$

40. $x^2 - 18 = 0$

41. $9x^2 - 3 = 0$

42. $2x^2 - 5 = 0$

43. $y^2 + 9 = 0$

44. $y^2 + 1 = 0$

45. $2x^2 = 0$

46. $x^2 = 0$

47. $-2y^2 + 5y = 0$

48. $-3y^2 + 7y = 0$

49. $t^2 - t = 0$

50. $t^2 + 2t = 0$

51. $x^2 - 0.6x - 0.27 = 0$

52. $x^2 - 1.6x - 0.8 = 0$

53. $y^2 - 1.4y - 0.15 = 0$

54. $y^2 - 3.6y + 2.03 = 0$

55. $12t^2 + 5t + 32 = 0$

56. $-t^2 + 3t - 34 = 0$

57. $3y^2 + 12y - 13 = 0$

58. $-2y^2 + 13y + 12 = 0$

59. $2x^2 - 10x + 3 = 4$

60. $3x^2 + 6x + 1 = 8$

61. $-2y^2 = 3(y-1)$

62. $3y^2 = 5(2y-1)$

63. $(t+1)^2 = 2t+7$

64. $(2t-1)^2 = 73-4t$

65. $(x+5)(x-1) = 2x+1$

66. $(x+7)(x-2) = 3(x+1)$

67. $x(x+5) = 3(x-1)$

68. $x(x+4) = -7$

69. $(5x+3)(5x-3) - 10(x-1) = 0$

70. $(3x+4)(3x-1) - 33x = -20$

71. $27y(y+1) + 2(3y-2) = 0$

72. $8(4y^2+3) - 3(28y-1) = 0$

73. $(x+2)^2 - 2(x+7) = 4(x+1)$

74. $(x+3)^2 - 10(x+5) = -2(x+1)$

Show Solutions

1. $a = 1,\ b = -1,\ c = 5$
2. $a = 1,\ b = -3,\ c = -1$
3. $a = 3,\ b = 0,\ c = -10$
4. $a = -1,\ b = 0,\ c = 5$

5. $a = 5,\ b = -7,\ c = 0$
6. $a = -1,\ b = 1,\ c = 0$
7. $a = -1,\ b = 1,\ c = 6$
8. $a = -2,\ b = -1,\ c = 15$
9. $a = 6,\ b = -2,\ c = 1$
10. $a = 8,\ b = -10,\ c = -3$
11. $4,\ 6$
12. $-3,\ 6$
13. $-5,\ -1$
14. $-7,\ -2$
15. $-\frac{1}{2},\ 4$
16. $-\frac{2}{3},\ 1$
17. $-\frac{3}{2},\ 1$
18. $-\frac{1}{3},\ \frac{1}{2}$
19. 1, double root
20. $-1,\ 1$
21. $3 \pm \sqrt{5}$
22. $2 \pm \sqrt{3}$

23. $-1 \pm \sqrt{6}$

24. $-2 \pm \sqrt{10}$

25. $2 \pm \sqrt{5}$

26. $4 \pm 3\sqrt{2}$

27. $\dfrac{1 \pm \sqrt{5}}{2}$

28. $\dfrac{-3 \pm \sqrt{17}}{2}$

29. $8 \pm \sqrt{2}$

30. $7 \pm \sqrt{3}$

31. $1 \pm \dfrac{\sqrt{10}}{2}$

32. $1 \pm \dfrac{\sqrt{5}}{2}$

33. $\dfrac{3}{2}$, double root

34. $\dfrac{1}{5}$, double root

35. $\dfrac{-3 \pm \sqrt{3}}{3}$

36. $\dfrac{-5 \pm \sqrt{15}}{5}$

37. $\dfrac{-1 \pm 2\sqrt{3}}{3}$

38. $\dfrac{-2 \pm \sqrt{2}}{4}$

39. $\pm\sqrt{2}$

40. $\pm 3\sqrt{2}$

41. $\pm\dfrac{\sqrt{3}}{3}$

42. $\pm\sqrt{\dfrac{5}{2}}$

43. No real solutions

44. No real solutions

45. 0, double root

46. 0, double root

47. $0,\ \dfrac{5}{2}$

48. $0,\ \dfrac{7}{3}$

49. $0,\ 1$

50. $-2,\ 0$

51. $-0.3,\ 0.9$

52. $-0.4,\ 2$

53. $-0.1,\ 1.5$

54. $0.7,\ 2.9$

55. No real solutions

56. No real solutions

57. $\frac{-6 \pm 5\sqrt{3}}{3}$

58. $\frac{13 \pm \sqrt{265}}{4}$

59. $\frac{5 \pm 3\sqrt{3}}{2}$

60. $\frac{-3 \pm \sqrt{30}}{3}$

61. $\frac{-3 \pm \sqrt{33}}{4}$

62. $\frac{5 \pm \sqrt{10}}{3}$

63. $\pm\sqrt{6}$

64. $\pm 3\sqrt{2}$

65. $-1 \pm \sqrt{7}$

66. $-1 \pm 3\sqrt{2}$

67. No real solutions

68. No real solutions

69. $\frac{1}{5}$, double root

70. $\frac{4}{3}$, double root

71. $-\frac{4}{3}, \frac{1}{9}$

72. $\frac{3}{8}, \frac{9}{4}$

73. $1 \pm \sqrt{15}$

74. $-5, 7$

9.4 Solving Quadratic Equations

Calculate the discriminant and use it to determine the number and type of solutions. Do not solve.

1. $x^2 + 2x + 3 = 0$
2. $x^2 - 2x - 3 = 0$
3. $3x^2 - x - 2 = 0$
4. $3x^2 - x + 2 = 0$
5. $9y^2 + 2 = 0$
6. $9y^2 - 2 = 0$
7. $5x^2 + x = 0$
8. $5x^2 - x = 0$
9. $12x^2 - 2x + 52 = 0$
10. $12x^2 - x - 12 = 0$

11. $-x^2 - 2x + 4 = 0$

12. $-x^2 - 4x + 2 = 0$

13. $4t^2 - 20t + 25 = 0$

14. $9t^2 - 6t + 1 = 0$

Choose the appropriate method to solve the following.

15. $x^2 - 2x - 3 = 0$

16. $x^2 + 2x + 3 = 0$

17. $3x^2 - x - 2 = 0$

18. $3x^2 - x + 2 = 0$

19. $9y^2 + 2 = 0$

20. $9y^2 - 2 = 0$

21. $5x^2 + x = 0$

22. $5x^2 - x = 0$

23. $12x^2 - 2x + 52 = 0$

24. $12x^2 - x - 12 = 0$

25. $-x^2 - 2x + 4 = 0$

26. $-x^2 - 4x + 2 = 0$

27. $4t^2 - 20t + 25 = 0$

28. $9t^2 - 6t + 1 = 0$

29. $y^2 - 4y - 1 = 0$

30. $y^2 - 6y - 3 = 0$

31. $25x^2 + 1 = 0$

32. $36x^2 + 4 = 0$

33. $5t^2 - 4 = 0$

34. $2t^2 - 9 = 0$

35. $12x^2 - 94x + 1 = 0$

36. $3x^2 + 12x - 16 = 0$

37. $36y^2 = 2y$

38. $50y^2 = -10y$

39. $x(x - 6) = -29$

40. $x(x - 4) = -16$

41. $4y(y + 1) = 5$

42. $2y(y + 2) = 3$

43. $-3x^2 = 2x + 1$

44. $3x^2 + 4x = -2$

45. $6(x + 1)^2 = 11x + 7$

46. $2(x + 2)^2 = 7x + 11$

47. $9t^2 = 4(3t - 1)$

48. $5t(5t - 6) = -9$

49. $(x + 1)(x + 7) = 3$

50. $(x - 5)(x + 7) = 14$

51. A positive real number is 2 less than another. When 4 times the larger is added to the square of the smaller, the result is 49. Find the numbers.

52. A positive real number is 1 more than another. When twice the smaller is subtracted from the square of the larger, the result is 4. Find the numbers.

53. A positive real number is 6 less than another. If the sum of the squares of the two numbers is 38, then find the numbers.

54. A positive real number is 1 more than twice another. If 4 times the smaller number is subtracted from the square of the larger, then the result is 21. Find the numbers.

55. The area of a rectangle is 60 square inches. If the length is 3 times the width, then find the dimensions of the rectangle.

56. The area of a rectangle is 6 square feet. If the length is 2 feet more than the width, then find the dimensions of the rectangle.

57. The area of a rectangle is 27 square meters. If the length is 6 meters less than 3 times the width, then find the dimensions of the rectangle.

58. The area of a triangle is 48 square inches. If the base is 2 times the height, then find the length of the base.

59. The area of a triangle is 14 square feet. If the base is 4 feet more than 2 times the height, then find the length of the base and the height.

60. The area of a triangle is 8 square meters. If the base is 4 meters less than the height, then find the length of the base and the height.

61. The perimeter of a rectangle is 54 centimetres, and the area is 180 square centimetres. Find the dimensions of the rectangle.

62. The perimeter of a rectangle is 50 inches, and the area is 126 square inches. Find the dimensions of the rectangle.

63. George maintains a successful 6-meter-by-8-meter garden. Next season, he plans on doubling the planting area by increasing the width and height by an equal amount. By how much must he increase the length and width?

64. A uniform brick border is to be constructed around a 6-foot-by-8-foot garden. If the total area of the garden, including the border, is to be 100 square feet, then find the width of the brick border.

65. If the sides of a square measure $\sqrt{106}$ units, then find the length of the diagonal.

66. If the diagonal of a square measures $\sqrt{310}$ units, then find the length of each side.

67. The diagonal of a rectangle measures $\sqrt{63}$ inches. If the width is 4 inches less than the length, then find the dimensions of the rectangle.

68. The diagonal of a rectangle measures $\sqrt{23}$ inches. If the width is 2 inches less than the length, then find the dimensions of the rectangle.

69. The top of a 20-foot ladder, leaning against a building, reaches a height of 18 feet. How far is the base of the ladder from the wall? Round off to the nearest hundredth.

70. To safely use a ladder, the base should be placed about 1/4 of the ladder's length away from the wall. If a 20-foot ladder is to be safely used, then how high against a building will the top of the ladder reach? Round off to the nearest hundredth.

71. The diagonal of a television monitor measures 32 inches. If the monitor has a 3:2 aspect ratio, then determine its length and width. Round off to the nearest hundredth.

72. The diagonal of a television monitor measures 52 inches. If the monitor has a 16:9 aspect ratio, then determine its length and width. Round off to the nearest hundredth.

73. The profit in dollars of running an assembly line that produces custom uniforms each day is given by the function $P(t) = -40t^2 + 960t - 4,000$, where t represents the number of hours the line is in operation.

    a. Calculate the profit on running the assembly line for 10 hours a day.

    b. Calculate the number of hours the assembly line should run in order to break even. Round off to the nearest tenth of an hour.

74. The profit in dollars generated by producing and selling x custom lamps is given by the function $P(x) = -10x^2 + 800x - 12,000$.

    a. Calculate the profit on the production and sale of 35 lamps.

    b. Calculate the number of lamps that must be sold to make a profit of $3,000.

75. If $1,200 is invested in an account earning an annual interest rate r, then the amount A that is in the account at the end of 2 years is given by the formula $A = 1,200(1 + r)^2$. If at the end of 2 years the amount in the account is $1,335.63, then what was the interest rate?

76. A manufacturing company has determined that the daily revenue, R, in thousands of dollars, depends on the number, n, of palettes of product sold according to the formula $R = 12n - 0.6n^2$. Determine the number of palettes that must be sold in order to maintain revenues at $60,000 per day.

77. The height of a projectile launched upward at a speed of 32 feet/second from a height of 128 feet is given by the function $h(t) = -16t^2 + 32t + 128$.

    a. What is the height of the projectile at 1/2 second?

b. At what time after launch will the projectile reach a height of 128 feet?

78. The height of a projectile launched upward at a speed of 16 feet/second from a height of 192 feet is given by the function $h(t) = -16t^2 + 16t + 192$.

    a. What is the height of the projectile at 3/2 seconds?

    b. At what time will the projectile reach 128 feet?

79. The height of an object dropped from the top of a 144-foot building is given by $h(t) = -16t^2 + 144$. How long will it take to reach a point halfway to the ground?

80. The height of a projectile shot straight up into the air at 80 feet/second from the ground is given by $h(t) = -16t^2 + 80t$. At what time will the projectile reach 95 feet?

Show Solutions

1. -8, no real solution
2. 16, two real solutions
3. 25, two real solutions
4. -23, no real solution
5. -72, no real solution
6. 72, two real solutions
7. 1, two real solutions
8. 1, two real solutions
9. -2492, no real solution
10. 577, two real solutions
11. 20, two real solutions

12. 24, two real solutions
13. 0, one real solution
14. 0, one real solution
15. $-1,\ 3$
16. No real solution
17. $-\frac{2}{3},\ 1$
18. No real solution
19. No real solution
20. $\pm\frac{\sqrt{2}}{3}$
21. $-\frac{1}{5},\ 0$
22. $0,\ \frac{1}{5}$
23. No real solution
24. $\frac{1\pm\sqrt{577}}{24}$
25. $-1\pm\sqrt{5}$
26. $-2\pm\sqrt{6}$
27. $\frac{5}{2}$
28. $\frac{1}{3}$

29. $2 \pm \sqrt{5}$

30. $3 \pm 2\sqrt{3}$

31. No real solution

32. No real solution

33. $\pm \dfrac{2\sqrt{5}}{5}$

34. $\pm \dfrac{3\sqrt{2}}{2}$

35. $\dfrac{47 \pm \sqrt{2197}}{12}$

36. $\dfrac{-6 \pm 2\sqrt{21}}{3}$

37. $0,\ \dfrac{1}{18}$

38. $-\dfrac{1}{5},\ 0$

39. No real solution

40. No real solution

41. $\dfrac{-1 \pm \sqrt{6}}{2}$

42. $\dfrac{-2 \pm \sqrt{10}}{2}$

43. No real solution

44. No real solution

45. $-\frac{1}{2}, \frac{1}{3}$

46. $-\frac{3}{2}, 1$

47. $\frac{2}{3}$

48. $\frac{3}{5}$

49. $-4 \pm 2\sqrt{3}$

50. $-1 \pm 5\sqrt{2}$

51. $3\sqrt{5} - 2$ and $3\sqrt{5}$

52. $\sqrt{3}$ and $\sqrt{3} + 1$

53. $\sqrt{10} - 3$ and $\sqrt{10} + 3$

54. $\sqrt{5}$ and $2\sqrt{5} + 1$

55. Length: $6\sqrt{5}$; width: $2\sqrt{5}$

56. Length: $1 + \sqrt{7}$; width: $\sqrt{7} - 1$

57. Length: $3\sqrt{10} - 3$; width: $1 + \sqrt{10}$

58. Base: $8\sqrt{3}$; height: $4\sqrt{3}$

59. Base: $2 + 2\sqrt{15}$; height: $\sqrt{15} - 1$

60. Base: $2\sqrt{5} - 2$; height: $2 + 2\sqrt{5}$

61. 12 inches by 15 inches

62. 9 inches by 14 inches

63. $\sqrt{97} - 7$ feet, or about 2.85 feet

64. $\dfrac{\sqrt{101} - 7}{2}$ feet, or about 1.52 feet

65. $2\sqrt{53}$ inches

66. $\sqrt{155}$ centimeters

67. Length: $\dfrac{4 + \sqrt{110}}{2}$; width: $\dfrac{\sqrt{110} - 4}{2}$

68. Length: $\dfrac{2 + \sqrt{42}}{2}$; width: $\dfrac{\sqrt{42} - 2}{2}$

69. $2\sqrt{19}$ feet, or about 8.72 feet

70. $5\sqrt{15}$ feet, or about 19.36 feet

71. Length: $\dfrac{96}{\sqrt{13}}$ inches; width: $\dfrac{64}{\sqrt{13}}$ inches

72. Length: $\dfrac{832}{\sqrt{337}}$ inches; width: $\dfrac{468}{\sqrt{337}}$ inches

73. $P(10) = 1600$. The break-even points are $12 \pm 2\sqrt{11}$, or about 5.4 and 18.6 uniforms.

74. $P(35) = 3750$. The company makes $\$3000$ profit when 30 or 50 lamps are sold.

75. 5.5%

76. 10 ads

77. 140 feet; 0 seconds and 2 seconds

78. 180 feet; $\dfrac{1 + \sqrt{17}}{2}$ seconds, or about 2.56 seconds

79. $\frac{3}{\sqrt{2}}$ seconds, or about 2.12 seconds

80. $\frac{10-\sqrt{5}}{4}$ seconds and $\frac{10+\sqrt{5}}{4}$ seconds

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CHAPTER 10: RADICALS

Chapter Outline

10.1 Rational Exponents

- Definition of Rational Exponents
- Operations Using the Rules of Exponents
- Radical Expressions with Different Indices

10.2 Simplifying Radical Expressions

- Simplifying Radical Expressions
- Formulas Involving Radicals

10.3 Adding and Subtracting Radicals

10.4 Exercises

- 10.1 Rational Exponents
- 10.2 Simplifying Radical Expressions
- 10.3 Adding and Subtracting Radicals

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10.1 Rational Exponents

Learning Objectives

By the end of this section, you should be able to:

- Write expressions with rational exponents in radical form.
- Write radical expressions with rational exponents.
- Perform operations and simplify expressions with rational exponents.
- Perform operations on radicals with different indices.

Definition of Rational Exponents

So far, exponents have been limited to integers. In this section, we will define what **rational (or fractional) exponents** mean and how to work with them. All of the rules for exponents developed up to this point apply. In particular, recall the product rule for exponents. Given any rational numbers m and n, then

$$x^m \cdot x^n = x^{m+n}$$

For example, if we have an exponent of $\frac{1}{2}$, then the product rule for exponents implies the following:

$$5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$$

Here $5^{\frac{1}{2}}$ is one of two equal factors of 5; hence, it is a square root of 5, and we can write

$$5^{1/2} = \sqrt{5}$$

Furthermore, we can see that $2^{\frac{1}{3}}$ is one of three equal factors of 2.

$$2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 2^{\frac{3}{3}} = 2^1 = 2$$

Therefore, $2^{\frac{1}{3}}$ is the cube root of 2, and we can write

$$2^{1/3} = \sqrt[3]{2}$$

This is true in general, given any nonzero real number a,

$$a^{1/n} = \sqrt[n]{a}$$

In other words, the denominator of a fractional exponent determines the index of an n^{th} root.

How To

Given an expression with a rational exponent, write the expression as a radical.

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Example 1

Rewrite as a radical.

a. $7^{1/2}$
b. $7^{1/3}$

Answer

a. $7^{1/2} = \sqrt{7}$
b. $7^{1/3} = \sqrt[3]{7}$

Example 2

Rewrite as a radical and then simplify.

a. $81^{1/2}$
b. $81^{1/4}$

Solution:

a. $81^{1/2} = \sqrt{81} = 9$
b. $81^{1/4} = \sqrt[4]{81} = \sqrt[4]{3^4} = 3$

Answer

a. 9
b. 3

Example 3

Rewrite as a radical and then simplify.

a. $(125x^3)^{1/3}$
b. $(-32y^{10})^{1/5}$

Solution:

a.
$$\begin{aligned}(125x^3)^{1/3} &= \sqrt[3]{125x^3} \\ &= \sqrt[3]{5^3x^3} \\ &= 5x\end{aligned}$$

b.
$$\begin{aligned}(-32y^{10})^{1/5} &= \sqrt[5]{-32y^{10}} \\ &= \sqrt[5]{(-2)^5(y^2)^5} \\ &= -2y^2\end{aligned}$$

Answer

a. $5x$
b. $-2y^2$

Next, consider fractional exponents where the numerator is an integer other than 1. For example, consider the following:

$$5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} = 5^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = 5^{\frac{6}{3}} = 5^2$$

This shows that $5^{2/3}$ is one of three equal factors of 5^2. In other words, $5^{2/3}$ is the cube root of 5^2 and we can write:

$$5^{2/3} = \sqrt[3]{5^2}$$

In general, given any real number a,

$$a^{m/n} = \sqrt[n]{a^m}$$

An expression with a rational exponent is equivalent to a radical where the denominator is the index and the numerator is the exponent. Any radical expression can be written with a rational exponent, which we call exponential form.

Radical form $\sqrt[5]{x^2}$ $=$ Exponential form $x^{2/5}$

Example 4

Rewrite as a radical.

a. $7^{2/5}$
b. $2^{3/4}$

Solution:

a. $7^{2/5} = \sqrt[5]{7^2} = \sqrt[5]{49}$
b. $2^{3/4} = \sqrt[4]{2^3} = \sqrt[4]{8}$

Answer

a. $7^{2/5} = \sqrt[5]{7^2} = \sqrt[5]{49}$
b. $2^{3/4} = \sqrt[4]{2^3} = \sqrt[4]{8}$

Example 5

Rewrite as a radical and then simplify.

a. $8^{2/3}$
b. $(32)^{3/5}$

Solution:

a.

$$\begin{aligned} 8^{2/3} &= \sqrt[3]{8^2} \\ &= \sqrt[3]{64} \\ &= \sqrt[3]{4^3} \\ &= 4 \end{aligned}$$

b. We can often avoid very large integers by working with their prime factorization.

$$\begin{aligned} (32)^{3/5} &= \sqrt[5]{(32)^3} && \text{Replace } 32 \text{ with } 2^5. \\ &= \sqrt[5]{(2^5)^3} && \text{Apply the power rule for exponents.} \\ &= \sqrt[5]{2^{15}} && 15 \div 5 = 3, \text{ so } 2^{15} = (2^3)^5. \\ &= \sqrt[5]{(2^3)^5} && \text{Simplify.} \\ &= 2^3 \\ &= 8 \end{aligned}$$

Answer

a. 4
b. 8

Try It

Activity Description
Fill in the blanks:

$$(\sqrt[4]{5})^3 = 5^{\frac{\square}{\square}}$$

$$(\sqrt[\square]{2xy})^{\square} = 2xy^{\frac{2}{5}}$$

$$(2x\sqrt[\square]{y})^{\square} = 2xy^{\frac{2}{3}}$$

$$(\sqrt{x})^7 = x^{\frac{\square}{\square}}$$

Options:

$$2 \quad 3 \quad 4 \quad 5 \quad 7$$

Answers:

$$(\sqrt[4]{5})^3 = 5^{\frac{3}{4}}$$

$$(\sqrt[5]{2xy})^2 = 2xy^{\frac{2}{5}}$$

$$(2x\sqrt[3]{y})^2 = 2xy^{\frac{2}{3}}$$

$$(\sqrt{x})^7 = x^{\frac{7}{2}}$$

Given a radical expression, we will be asked to find the equivalent in exponential form. Assume all variables are positive.

Example 6

Rewrite using rational exponents:

$$\sqrt[3]{x^2}$$

Solution:
Here, the index is 3, and the power is 2. We can write

$$\sqrt[3]{x^2} = x^{2/3}$$

Answer

$x^{2/3}$

Example 7

Rewrite using rational exponents:

$$\sqrt[6]{y^3}$$

Solution:
Here, the index is 6 and the power is 3. We can write

$$\begin{aligned}\sqrt[6]{y^3} &= y^{3/6} \\ &= y^{1/2}\end{aligned}$$

Answer

$y^{1/2}$

It is important to note that the following are equivalent.

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

In other words, it does not matter if we apply the power first or the root first. For example, we can apply the power before the root:

$$27^{2/3} = \sqrt[3]{27^2} = \sqrt[3]{(3^3)^2} = \sqrt[3]{3^6} = 3^2 = 9$$

Or we can apply the nth root before the power:

$$27^{2/3} = (\sqrt[3]{27})^2 = (\sqrt[3]{3^3})^2 = 3^2 = 9$$

The results are the same.

Example 8

Rewrite as a radical and then simplify:

$$(-8)^{2/3}$$

Solution:
Here, the index is 3, and the power is 2. We can write

$$(-8)^{2/3} = (\sqrt[3]{-8})^2 = (-2)^2 = 4$$

Answer

4

Try It

Rewrite as a radical and then simplify: $25^{3/2}$

Show Solution

Answer: 125

Some calculators have a caret button ^. If so, we can calculate approximations for radicals using it and rational exponents. For example, to calculate $\sqrt{2} = 2^{1/2} = 2 \wedge (1/2) \approx 1.414$, we would type

$$2 \ \wedge \ (1 \div 2) =$$

To calculate $\sqrt[3]{2^2} = 2^{2/3} = 2 \wedge (2/3) \approx 1.587$, we would type

$$2 \ \wedge \ (2 \div 3) =$$

Operations Using the Rules of Exponents

In this section, we review all of the rules of exponents, which extend to include rational exponents. If given any rational numbers m and n, then we have

Table 10.1.1

Product rule:	$x^m \cdot x^n = x^{m+n}$
Quotient rule:	$\dfrac{x^m}{x^n} = x^{m-n}, \ x \neq 0$
Power rule:	$(x^m)^n = x^{m \cdot n}$
Power rule for a product:	$(xy)^n = x^n y^n$
Power rule for a quotient:	$\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}, \ y \neq 0$
Negative exponents:	$x^{-n} = \dfrac{1}{x^n}$
Zero exponent:	$x^0 = 1, \ x \neq 0$

These rules allow us to perform operations with rational exponents.

Example 9

Simplify:

$$2^{\frac{2}{3}} \cdot 2^{\frac{1}{6}}$$

Solution:

$$\begin{aligned} 2^{\frac{2}{3}} \cdot 2^{\frac{1}{6}} &= 2^{\frac{2}{3}+\frac{1}{6}} && \text{Apply the product rule } x^m \cdot x^n = x^{m+n} \\ &= 2^{\frac{4}{6}+\frac{1}{6}} && \text{Use a common denominator and add.} \\ &= 2^{\frac{5}{6}} \end{aligned}$$

Answer

$2^{\frac{5}{6}}$

Example 10

Simplify:

$$\frac{x^{1/2}}{x^{1/3}}$$

Solution:

$$\begin{aligned} \frac{x^{1/2}}{x^{1/3}} &= x^{\frac{1}{2}-\frac{1}{3}} && \text{Apply the quotient rule } \frac{x^m}{x^n} = x^{m-n} \\ &= x^{\frac{3}{6}-\frac{2}{6}} && \text{Use a common denominator and subtract.} \\ &= x^{\frac{1}{6}} \end{aligned}$$

Answer

$x^{\frac{1}{6}}$

Example 11

Simplify:

$$\left(y^{3/4}\right)^{2/3}$$

Solution:

$$\begin{aligned}\left(y^{3/4}\right)^{2/3} &= y^{\frac{3}{4}\cdot\frac{2}{3}} && \text{Apply the power rule } (x^m)^n = x^{m\cdot n}\\ &= y^{\frac{6}{12}} && \text{Multiply and reduce.}\\ &= y^{\frac{1}{2}}\end{aligned}$$

Answer

$y^{\frac{1}{2}}$

Example 12

Simplify:

$$\left(16a^4b^8\right)^{3/4}$$

Solution:

$$\begin{aligned}\left(16a^4b^8\right)^{3/4} &= \left(2^4a^4b^8\right)^{3/4} && \text{Rewrite 16 as } 2^4\\ &= (2^4)^{3/4}(a^4)^{3/4}(b^8)^{3/4} && \text{Apply the power rule for a product.}\\ &= 2^{4\cdot\frac{3}{4}}a^{4\cdot\frac{3}{4}}b^{8\cdot\frac{3}{4}} && \text{Apply the power rule to each factor.}\\ &= 2^3a^3b^6 && \text{Simplify.}\\ &= 8a^3b^6\end{aligned}$$

Answer

$8a^3b^6$

Try It

Simplify: $(8a^{3/4}b^3)^{2/3}a^{1/3}$

Show Solution

Answer: $4a^{5/6}b^2$

Radical Expressions with Different Indices

To apply the product or **quotient rule for radicals**, the indices of the radicals involved must be the same. If the indices are different, then first rewrite the radicals in exponential form and then apply the rules for exponents.

Example 13

Multiply:

$$\sqrt{2}\cdot\sqrt[3]{2}$$

Solution:
In this example, the index of each radical factor is different. Hence, the product rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents. Then apply the product rule for exponents.

$$\begin{aligned}\sqrt{2}\cdot\sqrt[3]{2} &= 2^{\frac{1}{2}}\cdot 2^{\frac{1}{3}} && \text{Equivalents using rational exponents.}\\ &= 2^{\frac{1}{2}+\frac{1}{3}} && \text{Apply the product rule for exponents.}\\ &= 2^{\frac{3+2}{6}}\\ &= 2^{\frac{5}{6}}\\ &= \sqrt[6]{2^5}\end{aligned}$$

Answer

$\sqrt[6]{2^5}$

Example 14

Divide:

$$\frac{\sqrt[3]{4}}{\sqrt[5]{2}}$$

Solution:
In this example, the index of the radical in the numerator is different from the index of the radical in the denominator. Hence, the quotient rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents, and then apply the quotient rule for exponents.

$$\begin{aligned}\frac{\sqrt[3]{4}}{\sqrt[5]{2}} &= \frac{\sqrt[3]{2^2}}{\sqrt[5]{2}}\\ &= \frac{2^{\frac{2}{3}}}{2^{\frac{1}{5}}} && \text{Equivalents using rational exponents.}\\ &= 2^{\frac{2}{3}-\frac{1}{5}} && \text{Apply the quotient rule for exponents.}\\ &= 2^{\frac{10-3}{15}}\\ &= 2^{\frac{7}{15}}\\ &= \sqrt[15]{2^7}\end{aligned}$$

Answer

$\sqrt[15]{2^7}$

Example 15

Simplify:

$$\sqrt{\sqrt[3]{4}}$$

Solution:
Here, the radicand of the square root is a cube root. After rewriting this expression using rational exponents, we will see that the power rule for exponents applies.

$$\begin{aligned}\sqrt{\sqrt[3]{4}} &= \sqrt{\sqrt[3]{2^2}} \\ &= \left(2^{2/3}\right)^{1/2} && \text{Equivalents using rational exponents.} \\ &= 2^{\frac{2}{3}\cdot\frac{1}{2}} && \text{Apply the power rule for exponents.} \\ &= 2^{\frac{1}{3}} \\ &= \sqrt[3]{2}\end{aligned}$$

Answer

$\sqrt[3]{2}$

Key Takeaways

- When converting fractional exponents to radicals, use the numerator as the power and the denominator as the index of the radical.
- All the rules of exponents apply to expressions with rational exponents.

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10.2 Simplifying Radical Expressions

Learning Objectives

By the end of this section, you should be able to:

- Simplify radical expressions using the product and quotient rule for radicals.
- Use formulas involving radicals.

Simplifying Radical Expressions

An algebraic expression that contains radicals is called a **radical expression**. We use the product and quotient rules to simplify them.

Example 1

Simplify:

$$\sqrt[3]{8y^3}$$

Solution:
Use the fact that $\sqrt[n]{a^n} = a$ when n is odd.

$$\begin{aligned} \sqrt[3]{8y^3} &= \sqrt[3]{2^3 \cdot y^3} && \text{Apply the product rule for radicals.} \\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{y^3} && \text{Simplify.} \\ &= 2y \end{aligned}$$

Answer

$2y$

Example 2

Simplify:

$$\sqrt{9x^2}$$

Solution:
The square root has index 2; use the fact that $\sqrt[n]{a^n} = a$ when n is even.

$$\begin{aligned} \sqrt{9x^2} &= \sqrt{3^2x^2} && \text{Apply the product rule for radicals.} \\ &= \sqrt{3^2} \cdot \sqrt{x^2} && \text{Simplify.} \\ &= 3|x| \end{aligned}$$

Since x is a variable, it may represent a negative number. Thus, we need to ensure that the result is positive by including the absolute value operator.

Answer

$3|x|$

Note

In this section, we will assume that all variables are positive. This allows us to focus on calculating n^{th} roots without the technicalities associated with the principal n^{th} root problem. For this reason, we will use the following property for the rest of the section:

$$\sqrt[n]{a^n} = a, \text{ if } a \geq 0$$

When simplifying radical expressions, look for factors with powers that match the index.

How To

Given the product of multiple radical expressions, use the product rule to combine them into one radical expression.

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

How To

Given a radical expression, use the quotient rule to simplify it.

1. Write the radical expression as the quotient of two radical expressions.
2. Simplify the numerator and denominator.

Example 3

Simplify:

$$\sqrt{18x^3y^4}$$

Solution:
Begin by determining the square factors of 18, x^3, and y^4.

$$\begin{aligned} 18 &= 2 \cdot 3^2 \\ x^3 &= x^2 \cdot x \qquad \text{Square factors.} \\ y^4 &= (y^2)^2 \end{aligned}$$

Make these substitutions and then apply the **product rule for radicals** and simplify.

$$\begin{aligned} \sqrt{18x^3y^4} &= \sqrt{2 \cdot 3^2 \cdot x^2 \cdot x \cdot (y^2)^2} && \text{Apply the product rule for radicals.} \\ &= \sqrt{3^2} \cdot \sqrt{x^2} \cdot \sqrt{(y^2)^2} \cdot \sqrt{2x} && \text{Simplify.} \\ &= 3 \cdot x \cdot y^2 \cdot \sqrt{2x} \\ &= 3xy^2\sqrt{2x} \end{aligned}$$

Answer

$3xy^2\sqrt{2x}$

Example 4

Simplify:

$$\sqrt{\frac{4a^5}{b^6}}$$

Solution:
Begin by determining the square factors of 4, a^5, and b^6.

$$\begin{aligned} 4 &= 2^2 \\ a^5 &= a^2 \cdot a^2 \cdot a = (a^2)^2 \cdot a \\ b^6 &= b^3 \cdot b^3 = (b^3)^2 \end{aligned}$$

Square factors.

Make these substitutions and then apply the product and quotient rules for radicals and simplify.

$$\begin{aligned} \sqrt{\frac{4a^5}{b^6}} &= \sqrt{\frac{2^2(a^2)^2 \cdot a}{(b^3)^2}} \\ &= \frac{\sqrt{2^2} \cdot \sqrt{(a^2)^2} \cdot \sqrt{a}}{\sqrt{(b^3)^2}} \\ &= \frac{2a^2\sqrt{a}}{b^3} \end{aligned}$$

Apply the product and quotient rules for radicals.

Simplify.

Answer

$$\frac{2a^2\sqrt{a}}{b^3}$$

Example 5

Simplify:

$$\sqrt[3]{80x^5y^7}$$

Solution:
Begin by determining the cubic factors of 80, x^5, and y^7.

$$\begin{aligned} 80 &= 2^4 \cdot 5 = 2^3 \cdot 2 \cdot 5 \\ x^5 &= x^3 \cdot x^2 \\ y^7 &= y^6 \cdot y = (y^2)^3 \cdot y \end{aligned}$$

Cubic factors.

Make these substitutions and then apply the product rule for radicals and simplify.

$$\begin{aligned}\sqrt[3]{80x^5y^7} &= \sqrt[3]{2^3 \cdot 2 \cdot 5 \cdot x^3 \cdot x^2 \cdot (y^2)^3 \cdot y} && \text{Apply the product and quotient rules for radicals.}\\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{(y^2)^3} \cdot \sqrt[3]{10x^2y} && \text{Simplify.}\\ &= 2 \cdot x \cdot y^2 \cdot \sqrt[3]{10x^2y}\\ &= 2xy^2\sqrt[3]{10x^2y}\end{aligned}$$

Answer

$2xy^2\sqrt[3]{10x^2y}$

Example 6

Simplify:

$$\sqrt[3]{\frac{9x^6}{y^3z^9}}$$

Solution:
The coefficient $9 = 3^2$ and thus does not have any perfect cube factors. It will be left as the only remaining radicand because all of the other factors are cubes, as illustrated below:

$$\begin{aligned}x^6 &= (x^2)^3\\ y^3 &= (y)^3 \qquad \text{Cubic factors}\\ z^9 &= (z^3)^3\end{aligned}$$

Replace the variables with these equivalents, apply the product and quotient rules for radicals, and then simplify.

$$\begin{aligned}\sqrt[3]{\frac{9x^6}{y^3z^9}} &= \sqrt[3]{\frac{3^2\cdot(x^2)^3}{y^3\cdot(z^3)^3}}\\ &= \frac{\sqrt[3]{3^2}\cdot\sqrt[3]{(x^2)^3}}{\sqrt[3]{y^3}\cdot\sqrt[3]{(z^3)^3}}\\ &= \frac{\sqrt[3]{3^2}\cdot x^2}{y\cdot z^3}\\ &= \frac{\sqrt[3]{9}\cdot x^2}{y\cdot z^3}\end{aligned}$$

Answer

$$\frac{\sqrt[3]{9}\cdot x^2}{y\cdot z^3}$$

Example 7

Simplify:

$$\sqrt[4]{81a^4b^5}$$

Solution:
Determine all factors that can be written as perfect powers of 4. Here it is important to see that $b^5 = b^4\cdot b$. Hence, the factor b will be left inside the radical.

$$\begin{aligned}\sqrt[4]{81a^4b^5} &= \sqrt[4]{3^4\cdot a^4\cdot b^4\cdot b}\\ &= \sqrt[4]{3^4}\cdot\sqrt[4]{a^4}\cdot\sqrt[4]{b^4}\cdot\sqrt[4]{b}\\ &= 3ab\sqrt[4]{b}\end{aligned}$$

Answer

$$3ab\sqrt[4]{b}$$

Example 8

Simplify:

$$\sqrt[5]{-32x^3y^6z^5}$$

Solution:
Notice that the variable factor x cannot be written as a power of 5 and thus will be left inside the radical. In addition, $y^6 = y^5 \cdot y$; the factor y will be left inside the radical as well.

Answer

$$-2yz\sqrt[5]{x^3y}$$

Try It

Simplify: $\sqrt{192x^6y^7z^{12}}$

(Assume all variables are positive.)

Show Solution

$$8x^3y^3z^6\sqrt{3y}$$

Note

To easily simplify an n^{th} root, we can divide the powers by the index.

$$\sqrt{a^6} = a^3 \text{ (since } a^{6\div 2} = a^3)$$
$$\sqrt[3]{b^6} = b^2 \text{ (since } b^{6\div 3} = b^2)$$
$$\sqrt[6]{c^6} = c \text{ (since } c^{6\div 6} = c^1)$$

If the index does not divide evenly into the power, then we can use the quotient and remainder to simplify. For example,

$$\sqrt{a^5} = a^2\sqrt{a} \text{ (since } a^{5\div 2} = a^2_{r\ 1})$$
$$\sqrt[3]{b^5} = b\sqrt[3]{b^2} \text{ (since } b^{5\div 3} = b^1_{r\ 2})$$
$$\sqrt[5]{c^{14}} = c^2\sqrt[5]{c^4} \text{ (since } c^{14\div 5} = c^2_{r\ 4})$$

The quotient is the exponent of the factor outside of the radical, and the remainder is the exponent of the factor left inside the radical.

Try It

Activity Description
Fill in the blanks:

$$\sqrt{x^4y^6z^2} = x^{\square}y^{\square}z^{\square}$$

$$\sqrt{12x^5} = \sqrt{4x^4}\sqrt{3x} = \square x^{\square}\sqrt{3x}$$

$$\sqrt{75p^9} = \sqrt{25p^8}\sqrt{3p} = \square p^{\square}\sqrt{3p}$$

$$\sqrt{48x^4y^5} = \sqrt{16x^4y^4}\sqrt{3y} = \square x^{\square}y^{\square}\sqrt{3y}$$

Options:

1 2 3 4 5

Answers:

$$\begin{aligned}
\sqrt{x^4y^6z^2} &= x^2y^3z^1 \\
\sqrt{12x^5} &= \sqrt{4x^4}\sqrt{3x} &&= 2x^2\sqrt{3x} \\
\sqrt{75p^9} &= \sqrt{25p^8}\sqrt{3p} &&= 5p^4\sqrt{3p} \\
\sqrt{48x^4y^5} &= \sqrt{16x^4y^4}\sqrt{3y} &&= 4x^2y^2\sqrt{3y}
\end{aligned}$$

Formulas Involving Radicals

We next review the **distance formula**. Given two points (x_1, y_1) and (x_2, y_2).

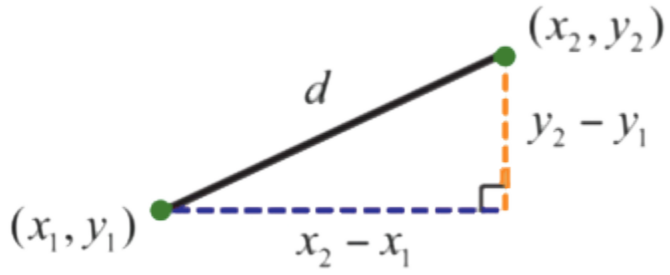

Figure 10.2.1

The distance d between them is given by the following formula:

Distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Recall that this formula was derived from the **Pythagorean theorem**.

Example 9

Calculate the distance between $(-4, 7)$ and $(2, 1)$.

Solution:
Use the distance formula with the following points.

$$\begin{array}{ll} (x_1, y_1) & (x_2, y_2) \\ (-4, 7) & (2, 1) \end{array}$$

It is a good practice to include the formula in its general form before substituting values for the variables; this improves readability and reduces the probability of making errors.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - (-4))^2 + (1 - 7)^2} \\ &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{72} \\ &= \sqrt{36 \cdot 2} \\ &= 6\sqrt{2} \end{aligned}$$

Answer

$6\sqrt{2}$ units

Example 10

The period T of a pendulum in seconds is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L represents the length of the pendulum in feet. If the length of a pendulum measures 6 feet, then **calculate the period, rounded off to the nearest tenth of a second.**

Solution:
Substitute 6 for L and then simplify.

$$\begin{aligned} T &= 2\pi\sqrt{\frac{L}{32}} \\ &= 2\pi\sqrt{\frac{6}{32}} && \text{Reduce.} \\ &= 2\pi\sqrt{\frac{3}{16}} && \text{Apply the product and quotient rules for radicals.} \\ &= 2\pi\frac{\sqrt{3}}{\sqrt{16}} && \text{Simplify.} \\ &= \frac{2\pi\sqrt{3}}{4} && \text{Use a calculator.} \\ &\approx 2.7 \end{aligned}$$

Answer

The period is approximately 2.7 seconds.

Key Takeaways

- In beginning algebra, we typically assume that all variable expressions within the radical are positive. This allows us to focus on simplifying radicals without the technical issues associated with the principal nth root.
- To simplify radical expressions, look for factors of the radicand with powers that match the index. If found, they can be simplified by applying the product and quotient rules for radicals, as well as the property $\sqrt[n]{a^n} = a$, where a is positive.

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10.3 Adding and Subtracting Radicals

Learning Objectives

By the end of this section, you should be able to:

- Add and subtract like radicals.
- Simplify radical expressions involving like radicals.

Adding and Subtracting Radical Expressions

Adding and subtracting radical expressions is similar to adding and subtracting like terms. Radicals are considered to be like radicals (or similar radicals) when they share the same index and radicand. For example, the terms $3\sqrt{5}$ and $4\sqrt{5}$ contain like radicals and can be added using the distributive property as follows:

$$\begin{aligned} 3\sqrt{5}+4\sqrt{5} &= (3+4)\sqrt{5} \\ &= 7\sqrt{5} \end{aligned}$$

When adding terms with like radicals, add only the coefficients; the radical part remains the same.

How To

Given a radical expression requiring addition or subtraction of square roots, simplify.

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

Example 1

Add:

$$3\sqrt{2}+2\sqrt{2}$$

Solution: The terms contain like radicals; therefore, add the coefficients.

$$3\sqrt{2}+2\sqrt{2}=5\sqrt{2}$$

Answer

$5\sqrt{2}$

Subtraction is performed in a similar manner.

Example 2

Subtract:

$$2\sqrt{7}-3\sqrt{7}$$

Solution:

$$\begin{aligned}2\sqrt{7}-3\sqrt{7}&=(2-3)\sqrt{7}\\&=-1\sqrt{7}\\&=-\sqrt{7}\end{aligned}$$

Answer

$-\sqrt{7}$

If the radicand and the index are not exactly the same, then the radicals are not similar, and we cannot combine them.

Example 3

Simplify:

$$10\sqrt{5}+6\sqrt{2}-9\sqrt{5}-7\sqrt{2}$$

Solution:

Combine like radicals ($\sqrt{5}$ with $\sqrt{5}$, and $\sqrt{2}$ with $\sqrt{2}$).

Answer

$$\sqrt{5}-\sqrt{2}$$

Caution

It is important to point out that $\sqrt{5}-\sqrt{2}\neq\sqrt{5-2}$. We can verify this by calculating the value of each side with a calculator.

$$\sqrt{5}-\sqrt{2}\approx 0.82$$
$$\sqrt{5-2}=\sqrt{3}\approx 1.73$$

In general, note that $\sqrt[n]{a}\pm\sqrt[n]{b}\neq\sqrt[n]{a\pm b}$.

Example 4

Simplify:

$$3\sqrt[3]{6}+2\sqrt{6}-\sqrt[3]{6}-3\sqrt{6}$$

Solution:

$$\begin{aligned}3\sqrt[3]{6}+2\sqrt{6}-\sqrt[3]{6}-3\sqrt{6}&=(3\sqrt[3]{6}-\sqrt[3]{6})+(2\sqrt{6}-3\sqrt{6})\\&=2\sqrt[3]{6}-\sqrt{6}\end{aligned}$$

We cannot simplify any further because $\sqrt[3]{6}$ and $\sqrt{6}$ are not like radicals; the indices are not the same.

Answer

$2\sqrt[3]{6} - \sqrt{6}$

Often, we will have to simplify before we can identify the like radicals within the terms.

Example 5

Subtract:

$$\sqrt{12} - \sqrt{48}$$

Solution:

At first glance, the radicals do not appear to be similar. However, after simplifying completely, we will see that we can combine them.

$$\begin{aligned} \sqrt{12} - \sqrt{48} &= \sqrt{4 \cdot 3} - \sqrt{16 \cdot 3} \\ &= 2\sqrt{3} - 4\sqrt{3} \\ &= -2\sqrt{3} \end{aligned}$$

Answer

$-2\sqrt{3}$

Example 6

Simplify:

$$\sqrt{20} + \sqrt{27} - 3\sqrt{5} - 2\sqrt{12}$$

Solution:

$$\begin{aligned}\sqrt{20}+\sqrt{27}-3\sqrt{5}-2\sqrt{12} &= \sqrt{4\cdot 5}+\sqrt{9\cdot 3}-3\sqrt{5}-2\sqrt{4\cdot 3}\\ &= 2\sqrt{5}+3\sqrt{3}-3\sqrt{5}-4\sqrt{3}\\ &= -\sqrt{5}-\sqrt{3}\end{aligned}$$

Answer

$-\sqrt{5}-\sqrt{3}$

Try It

Subtract: $2\sqrt{50}-6\sqrt{8}$

Show Solution

$-2\sqrt{2}$

Next, we work with radical expressions involving variables. In this section, assume all radicands containing variable expressions are not negative.

Example 7

Simplify:

$$-6\sqrt[3]{2x} - \sqrt[3]{3x} + 7\sqrt[3]{2x}$$

Solution:

$$\begin{aligned} -6\sqrt[3]{2x} - \sqrt[3]{3x} + 7\sqrt[3]{2x} &= (-6\sqrt[3]{2x} + 7\sqrt[3]{2x}) - \sqrt[3]{3x} \\ &= \sqrt[3]{2x} - \sqrt[3]{3x} \end{aligned}$$

We cannot combine any further because the remaining radical expressions do not share the same radicand; they are not like radicals. Note that $\sqrt[3]{2x} - \sqrt[3]{3x} \neq \sqrt[3]{2x - 3x}$.

Answer

$\sqrt[3]{2x} - \sqrt[3]{3x}$

Try It

Activity Description
Fill in the blanks:

$$\begin{aligned} \sqrt[5]{27n} - \sqrt[3]{12n} &= 5(\square)\sqrt{3n} - 3(\square)\sqrt{3n} &&= \square\sqrt{3n} \\ x^2\sqrt{9x^3y} - \sqrt{25xy^3} + x^2 &= \sqrt{xy}(\square x^3 - \square y + x^2) \\ \sqrt{16b^2} - 3\sqrt{50b^3} &= \square b - 15b^{\square}\sqrt{2b} \end{aligned}$$

Options:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

Answers:

$$\begin{aligned}
\sqrt[5]{27n} - \sqrt[3]{12n} &= 5(3)\sqrt{3n} - 3(2)\sqrt{3n} &= 9\sqrt{3n} \\
x^2\sqrt{9x^3y} - \sqrt{25xy^3} + x^2 &= \sqrt{xy}(3x^3 - 5y + x^2) \\
\sqrt{16b^2} - 3\sqrt{50b^3} &= 4b - 15b^1\sqrt{2b}
\end{aligned}$$

We will often find the need to subtract a radical expression with multiple terms. If this is the case, remember to apply the distributive property before combining like terms.

Example 8

Simplify:

$$(9\sqrt{x} - 2\sqrt{y}) - (10\sqrt{x} + 7\sqrt{y})$$

Solution:

$$\begin{aligned}
(9\sqrt{x} - 2\sqrt{y}) - (10\sqrt{x} + 7\sqrt{y}) &= 9\sqrt{x} - 2\sqrt{y} - 10\sqrt{x} - 7\sqrt{y} \\
&= (9\sqrt{x} - 10\sqrt{x}) + (-2\sqrt{y} - 7\sqrt{y}) \\
&= -\sqrt{x} - 9\sqrt{y}
\end{aligned}$$

Answer

$-\sqrt{x} - 9\sqrt{y}$

Until we simplify, it is often unclear which terms involving radicals are similar.

Example 9

Simplify:

$$5\sqrt[3]{2y} - (\sqrt[3]{54y} - \sqrt[3]{16})$$

Solution:

$$\begin{aligned} 5\sqrt[3]{2y}-(\sqrt[3]{54y}-\sqrt[3]{16}) &= 5\sqrt[3]{2y}-\sqrt[3]{54y}+\sqrt[3]{16} \\ &= 5\sqrt[3]{2y}-\sqrt[3]{3^3\cdot 2y}+\sqrt[3]{2^3\cdot 2} \\ &= 5\sqrt[3]{2y}-3\sqrt[3]{2y}+2\sqrt[3]{2} \\ &= 2\sqrt[3]{2y}+2\sqrt[3]{2} \end{aligned}$$

Answer

$2\sqrt[3]{2y}+2\sqrt[3]{2}$

Example 10

Simplify:

$$2a\sqrt{125a^2b}-a^2\sqrt{80b}+4\sqrt{20a^4b}$$

Solution:

$$\begin{aligned} 2a\sqrt{125a^2b}-a^2\sqrt{80b}+4\sqrt{20a^4b} &= 2a\sqrt{25\cdot 5\cdot a^2\cdot b}-a^2\sqrt{16\cdot 5\cdot b}+4\sqrt{4\cdot 5\cdot (a^2)^2\cdot b} \\ &= 2a\cdot 5a\sqrt{5b}-a^2\cdot 4\sqrt{5b}+4\cdot 2a^2\sqrt{5b} \\ &= 10a^2\sqrt{5b}-4a^2\sqrt{5b}+8a^2\sqrt{5b} \\ &= 14a^2\sqrt{5b} \end{aligned}$$

Answer

$14a^2\sqrt{5b}$

Try It

Simplify: $\sqrt{45x^3} - (\sqrt{20x^3} - \sqrt{80x})$

Show Solution

Answer: $x\sqrt{5x} + 4\sqrt{5x}$

Note

Take careful note of the differences between products and sums within a radical.

Table 10.3.1

Products	Sums
$\sqrt{x^2y^2} = xy$	$\sqrt{x^2 + y^2} \neq x + y$
$\sqrt[3]{x^3y^3} = xy$	$\sqrt[3]{x^3 + y^3} \neq x + y$

The property $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ says that we can simplify radicals when the operation in the radicand is multiplication. There is no corresponding property for addition.

Key Takeaways

- Add and subtract terms that contain like radicals just as you do like terms. If the index and radicand are exactly the same, then the radicals are similar and can be combined. This involves adding or

subtracting only the coefficients; the radical part remains the same.
- Simplify each radical completely before combining like terms.

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How To: "1.3 Radicals and Rational Exponents" from College Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Used How To box "Given a radical expression requiring addition or subtraction of square roots, simplify"; Reformatted.

10.4 Exercises

10.1 Rational Exponents

Express using rational exponents.

1. $\sqrt{6}$
2. $\sqrt{10}$
3. $\sqrt[3]{11}$
4. $\sqrt[4]{2}$
5. $\sqrt[3]{5^2}$
6. $\sqrt[4]{2^3}$
7. $\sqrt[5]{x}$
8. $\sqrt[6]{x}$
9. $\sqrt[6]{x^7}$
10. $\sqrt[5]{x^4}$

Express in radical form.

11. $2^{1/2}$
12. $5^{1/3}$
13. $7^{2/3}$
14. $2^{3/5}$
15. $x^{3/4}$

16. $x^{5/6}$

17. $x^{-1/2}$

18. $x^{-3/4}$

19. $\left(\frac{1}{x}\right)^{-1/3}$

20. $\left(\frac{1}{x}\right)^{-3/5}$

Write as a radical and then simplify.

21. $25^{1/2}$

22. $36^{1/2}$

23. $121^{1/2}$

24. $144^{1/2}$

25. $\left(\frac{1}{4}\right)^{1/2}$

26. $\left(\frac{4}{9}\right)^{1/2}$

27. $(4)^{-1/2}$

28. $(9)^{-1/2}$

29. $\left(\frac{1}{4}\right)^{-1/2}$

30. $\left(\frac{1}{16}\right)^{-1/2}$

31. $8^{1/3}$

32. $125^{1/3}$

33. $\left(\frac{1}{27}\right)^{1/3}$

34. $\left(\frac{8}{125}\right)^{1/3}$

35. $(-27)^{1/3}$

36. $(-64)^{1/3}$

37. $16^{1/4}$

38. $625^{1/4}$

39. $81^{-1/4}$

40. $16^{-1/4}$

41. $100{,}000^{1/5}$

42. $(-32)^{1/5}$

43. $\left(\frac{1}{32}\right)^{1/5}$

44. $\left(\frac{1}{243}\right)^{1/5}$

45. $9^{3/2}$

46. $4^{3/2}$

47. $8^{5/3}$

48. $27^{2/3}$

49. $16^{3/2}$

50. $32^{2/5}$

51. $\left(\frac{1}{16}\right)^{3/4}$

52. $\left(\frac{1}{81}\right)^{3/4}$

53. $(-27)^{2/3}$

54. $(-27)^{4/3}$

55. $(-32)^{3/5}$

56. $(-32)^{4/5}$

Use a calculator to approximate an answer rounded to the nearest hundredth.

57. $2^{3/4}$

58. $3^{2/3}$

59. $5^{1/5}$

60. $7^{1/7}$

61. $(-9)^{3/2}$

62. $-9^{3/2}$

63. Explain why $(-4)^{(3/2)}$ gives an error on a calculator and $-4^{(3/2)}$ gives an answer of -8.

64. Marcy received a text message from Mark asking her how old she was. In response, Marcy texted back "$125^{(2/3)}$ years old." Help Mark determine how old Marcy is.

Perform the operations and simplify. Leave answers in exponential form.

65. $2^{2/3} \cdot 2^{4/3}$

66. $3^{3/2} \cdot 3^{1/2}$

67. $5^{1/2} \cdot 5^{1/3}$

68. $2^{1/6} \cdot 2^{3/4}$

69. $y^{1/4} \cdot y^{2/5}$

70. $x^{1/2} \cdot x^{1/4}$

71. $\dfrac{5^{7/3}}{5^{1/3}}$

72. $\dfrac{2^{9/2}}{2^{1/2}}$

73. $\dfrac{2a^{2/3}}{a^{1/6}}$

74. $\dfrac{3b^{1/2}}{b^{1/3}}$

75. $\left(8^{1/2}\right)^{2/3}$

76. $\left(3^{6}\right)^{2/3}$

77. $\left(x^{2/3}\right)^{1/2}$

78. $\left(y^{3/4}\right)^{4/5}$

79. $\left(\dfrac{4x^2}{y^4}\right)^{1/2}$

80. $\left(\dfrac{9x^6}{y^2}\right)^{1/2}$

81. $\left(\dfrac{2x^{1/3}}{y^{2/3}}\right)^{3}$

82. $\left(\dfrac{8x^{3/2}}{y^{1/2}}\right)^{2}$

83. $\left(\dfrac{a^{3/4}}{a^{1/2}}\right)^{4/3}$

84. $\left(\dfrac{b^{4/5}}{b^{1/10}}\right)^{10/3}$

85. $\left(\dfrac{4x^{2/3}}{y^4}\right)^{1/2}$

86. $\left(\dfrac{27x^{3/4}}{y^9}\right)^{1/3}$

87. $y^{1/2} \cdot \dfrac{y^{2/3}}{y^{1/6}}$

88. $x^{2/5} \cdot \dfrac{x^{1/2}}{x^{1/10}}$

89. $\dfrac{xy}{x^{1/2}y^{1/3}}$

90. $\dfrac{x^{5/4}y}{xy^{2/5}}$

91. $\dfrac{49a^{5/7}b^{3/2}}{7a^{3/7}b^{1/4}}$

92. $\dfrac{16a^{5/6}b^{5/4}}{8a^{1/2}b^{2/3}}$

93. $\left(\dfrac{9x^{2/3}}{y^6}\right)^{3/2} x^{1/2}y$

94. $\left(\dfrac{125x^3}{y^{3/5}}\right)^{2/3} xy^{1/3}$

95. $\dfrac{\left(27a^{1/4}b^{3/2}\right)^{2/3}}{a^{1/6}b^{1/2}}$

96. $\dfrac{\left(25a^{2/3}b^{4/3}\right)^{3/2}}{a^{1/6}b^{1/3}}$

Perform the operations.

97. $\sqrt[3]{9} \cdot \sqrt[5]{3}$

98. $\sqrt{5} \cdot \sqrt[5]{25}$

99. $\sqrt{x} \cdot \sqrt[3]{x}$

100. $\sqrt{y} \cdot \sqrt[4]{y}$

101. $\sqrt[3]{x^2} \cdot \sqrt[4]{x}$

102. $\sqrt[5]{x^3} \cdot \sqrt[3]{x}$

103. $\dfrac{\sqrt[3]{100}}{\sqrt{10}}$

104. $\dfrac{\sqrt[5]{16}}{\sqrt[3]{4}}$

105. $\dfrac{\sqrt[3]{a^2}}{\sqrt{a}}$

106. $\dfrac{\sqrt[5]{b^4}}{\sqrt[3]{b}}$

107. $\dfrac{\sqrt[3]{x^2}}{\sqrt[5]{x^3}}$

108. $\dfrac{\sqrt[4]{x^3}}{\sqrt[3]{x^2}}$

109. $\sqrt[5]{\sqrt{16}}$

110. $\sqrt[3]{\sqrt{9}}$

111. $\sqrt[3]{\sqrt[5]{2}}$

112. $\sqrt[3]{\sqrt[5]{5}}$

113. $\sqrt[3]{\sqrt{7}}$

114. $\sqrt[3]{\sqrt{3}}$

Show Solutions

1. $6^{1/2}$
2. $10^{1/2}$
3. $11^{1/3}$
4. $2^{1/4}$
5. $5^{2/3}$
6. $2^{3/4}$
7. $x^{1/5}$
8. $x^{1/6}$
9. $x^{7/6}$
10. $x^{4/5}$
11. $\sqrt{2}$
12. $\sqrt[3]{5}$
13. $\sqrt[3]{7^2}$

14. $\sqrt[5]{2^3}$
15. $\sqrt[4]{x^3}$
16. $\sqrt[6]{x^5}$
17. $\sqrt{\frac{1}{x}}$
18. $\sqrt[4]{\frac{1}{x^3}}$
19. $\sqrt[3]{x}$
20. $\sqrt[5]{x^3}$
21. 5
22. 6
23. 11
24. 12
25. $\frac{1}{2}$
26. $\frac{2}{3}$
27. $\frac{1}{2}$
28. $\frac{1}{3}$
29. 2
30. 4

31. 2

32. 5

33. $\frac{1}{3}$

34. $\frac{2}{5}$

35. -3

36. -4

37. 2

38. 5

39. $\frac{1}{3}$

40. $\frac{1}{2}$

41. 10

42. -2

43. $\frac{1}{2}$

44. $\frac{1}{3}$

45. 27

46. 8

47. 32

48. 9

49. 64

50. 4

51. $\frac{1}{8}$

52. $\frac{1}{27}$

53. 9

54. 81

55. -8

56. 16

57. Approximately 1.68

58. Approximately 2.08

59. Approximately 1.38

60. Approximately 1.32

61. Not a real number

62. -27

63. In the first expression, the square root of a negative number creates an error condition on the calculator. The square root of a negative number is not real. In the second expression, because of the order of operations, the negative sign is applied to the answer after 4 is raised to the $3/2$ power.

64. 25 years old

65. 4

66. 9

67. $5^{5/6}$

68. $2^{11/12}$

69. $y^{13/20}$

70. $x^{3/4}$

71. 25

72. 16

73. $2a^{1/2}$

74. $3b^{1/6}$

75. 2

76. 81

77. $x^{1/3}$

78. $y^{3/5}$

79. $\dfrac{2x}{y^2}$

80. $\dfrac{3x^3}{y}$

81. $\dfrac{8x}{y^2}$

82. $\dfrac{64x^3}{y}$

83. $a^{1/3}$

84. $b^{7/3}$

85. $\dfrac{2x^{1/3}}{y^2}$

86. $\dfrac{3x^{1/4}}{y^3}$

87. y

88. $x^{4/5}$

89. $x^{1/2}y^{2/3}$

90. $x^{1/4}y^{3/5}$

91. $7a^{2/7}b^{5/4}$

92. $2a^{1/3}b^{7/12}$

93. $\dfrac{27x^{3/2}}{y^8}$

94. $\dfrac{25x^3}{y^{1/15}}$

95. $9b^{1/2}$

96. $125a^{5/6}b^{5/3}$

97. $\sqrt[15]{3^{13}}$

98. $\sqrt[10]{5^9}$

99. $x^{5/6}$

100. $y^{3/4}$

101. $\sqrt[12]{x^{11}}$

102. $\sqrt[15]{x^{14}}$

103. $\sqrt[6]{10}$

104. $\sqrt[15]{4}$

105. $\sqrt[6]{a}$

106. $\sqrt[15]{b^7}$

107. $\sqrt[15]{x}$

108. $\sqrt[12]{x}$

109. $\sqrt[5]{4}$

110. $\sqrt[3]{3}$

111. $\sqrt[15]{2}$

112. $\sqrt[15]{5}$

113. $\sqrt[6]{7}$

114. $\sqrt[6]{3}$

10.2 Simplifying Radical Expressions

Simplify. (Assume all variables represent positive numbers.)

1. $\sqrt{36a^2}$

2. $\sqrt{121b^2}$
3. $\sqrt{x^2y^2}$
4. $\sqrt{25x^2y^2z^2}$
5. $\sqrt{180x^3}$
6. $\sqrt{150y^3}$
7. $\sqrt{49a^3b^2}$
8. $\sqrt{4a^4b^3c}$
9. $\sqrt{45x^5y^3}$
10. $\sqrt{50x^6y^4}$
11. $\sqrt{64r^2s^6t^5}$
12. $\sqrt{144r^8s^6t^2}$
13. $\sqrt{(x+1)^2}$
14. $\sqrt{(2x+3)^2}$
15. $\sqrt{4(3x-1)^2}$
16. $\sqrt{9(2x+3)^2}$
17. $\sqrt{9x^3 \cdot 5y^2}$
18. $\sqrt{4x^5 \cdot 9y^4}$
19. $\sqrt{36m^7n^4}$
20. $\sqrt{147m^9n^6}$

21. $\sqrt{2r^2s^5 \cdot 25t^4}$

22. $\sqrt{36r^5s^2t^6}$

23. $\sqrt[3]{27a^3}$

24. $\sqrt[3]{125b^3}$

25. $\sqrt[3]{250x^4y^3}$

26. $\sqrt[3]{162a^3b^5}$

27. $\sqrt[3]{64x^3y^6z^9}$

28. $\sqrt[3]{216x^{12}y^3}$

29. $\sqrt[3]{8x^3y^4}$

30. $\sqrt[3]{27x^5y^3}$

31. $\sqrt[3]{a^4b^5c^6}$

32. $\sqrt[3]{a^7b^5c^3}$

33. $\sqrt[3]{8x^4 \cdot 27y^3}$

34. $\sqrt[3]{x^5 \cdot 125y^6}$

35. $\sqrt[3]{360r^5s^{12}t^{13}}$

36. $\sqrt[3]{540r^3s^2t^9}$

37. $\sqrt[4]{81x^4}$

38. $\sqrt[4]{x^4y^4}$

39. $\sqrt[4]{16x^4y^8}$

40. $\sqrt[4]{81x^{12}y^4}$

41. $\sqrt[4]{a^4b^5c^6}$

42. $\sqrt[4]{54a^6c^8}$

43. $\sqrt[4]{128x^6}$

44. $\sqrt[4]{243y^7}$

45. $\sqrt[5]{32m^{10}n^5}$

46. $\sqrt[5]{37m^9n^{10}}$

47. $-3\sqrt{4x^2}$

48. $7\sqrt{9y^2}$

49. $-5\sqrt{4x^3y}$

50. $-3\sqrt{x^3y^{18}}$

51. $12ab\sqrt{a^5b^3}$

52. $6a^2b^9\sqrt{a^7b^2}$

53. $2x\sqrt[3]{8x^6}$

54. $-5x^2\sqrt[3]{27x^3}$

55. $2ab\sqrt[3]{-8a^4b^5}$

56. $5a^2b\sqrt[3]{-27a^3b^3}$

Rewrite the following as a radical expression with a coefficient of 1.

57. $5\sqrt{2x}$

58. $2\sqrt{3y}$

59. $2x\sqrt{3}$

60. $3y\sqrt{2}$

61. $ab\sqrt{10a}$

62. $2ab\sqrt{2a}$

63. $m^2n\sqrt{mn}$

64. $2m^2n^3\sqrt{3n}$

65. $5\sqrt[3]{2x}$

66. $3\sqrt[3]{5y}$

67. $2x\sqrt[3]{3}$

68. $3y\sqrt[3]{2}$

Assume that the variable could represent any real number and then simplify.

69. $\sqrt{4x^2}$

70. $\sqrt{25y^2}$

71. $\sqrt[3]{8y^3}$

72. $\sqrt[3]{125a^3}$

73. $\sqrt[4]{64x^4}$

74. $\sqrt[4]{81y^4}$

75. $\sqrt{36a^4}$

76. $\sqrt{100a^8}$

77. $\sqrt{4a^6}$

78. $\sqrt{a^{10}}$

79. $\sqrt{18a^4b^5}$

80. $\sqrt{48a^5b^3}$

81. $\sqrt[6]{128x^6y^8}$

82. $\sqrt[6]{a^6b^7c^8}$

The y-intercepts for any graph will have the form $(0, y)$, where y is a real number. Therefore, to find y-intercepts, set $x = 0$ and solve for y. Find the y-intercepts for the following.

83. $y = \sqrt{x + 4} - 1$

84. $y = \sqrt{x + 1} - 3$

85. $y = \sqrt[3]{x - 1} + 2$

86. $y = \sqrt[3]{x + 1} - 3$

Factor the radicand and then simplify. (Assume that all expressions are positive.)

87. $\sqrt{x^2 - 6x + 9}$

88. $\sqrt{x^2 - 10x + 25}$

89. $\sqrt{4x^2 + 12x + 9}$

90. $\sqrt{9x^2 + 6x + 1}$

91. The speed of a vehicle before the brakes were applied can be estimated by the length of the skid marks left on the road. On dry pavement, the speed v in miles per hour can be estimated by the formula $v = \sqrt{5d}$, where d represents the length of the skid marks in feet. Estimate the speed of a vehicle before applying the brakes on dry pavement if the skid marks measure 36 feet.

92. The radius r of a sphere can be calculated using the formula $r = \sqrt[3]{\frac{3V}{4\pi}}$, where V represents the sphere's volume. What is the radius of a sphere if the volume is 36π cubic centimetres?

The period T of a pendulum in seconds is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L represents the length in feet. Calculate the period, given the following lengths. Give the exact value and the approximate value rounded off to the nearest tenth of a second.

93. 8 feet

94. 32 feet

95. $\frac{1}{2}$ foot

96. $\frac{1}{8}$ foot

The time t in seconds that an object is in free fall is given by the formula

$$s = 16t^2$$

where s represents the distance it has fallen in feet. Calculate the time it takes an object to fall, given the following distances. Give the exact value and the approximate value rounded off to the nearest tenth of a second.

97. 48 feet

98. 80 feet

99. 192 feet

100. 288 feet

Given the function, calculate the following.

101. $f(x) = \sqrt{x-1}$, find $f(1)$, $f(2)$, and $f(5)$

102. $f(x) = \sqrt{x+5}$, find $f(-5)$, $f(-1)$, and $f(20)$

103. $f(x) = \sqrt{x} + 3$, find $f(0)$, $f(1)$, and $f(16)$

104. $f(x) = \sqrt{x} - 5$, find $f(0)$, $f(1)$, and $f(25)$

105. $g(x) = \sqrt[3]{x}$, find $g(-1)$, $g(0)$, and $g(1)$

106. $g(x) = \sqrt[3]{x+7}$, find $g(-15)$, $g(-7)$, and $g(20)$

107. $g(x) = \sqrt[3]{x} - 2$, find $g(-1)$, $g(0)$, and $g(8)$

108. $g(x) = \sqrt[3]{x-1} + 2$, find $g(0)$, $g(2)$, and $g(9)$

For each function, fill in the table.

109. $f(x) = \sqrt{x+1}$

x	$f(x)$
-1	
0	
3	
8	
15	

110. $f(x) = \sqrt{x-2}$

x	$f(x)$
2	
3	
6	
11	
18	

111. $f(x) = \sqrt[3]{x} + 1$

x	$f(x)$
-8	
-1	
0	
1	
8	

112. $f(x) = \sqrt[3]{x+2}$

x	$f(x)$
-10	
-3	
-2	
-1	
6	

Show Solutions

1. $6a$
2. $11b$
3. xy
4. $5xyz$
5. $6x\sqrt{5x}$
6. $5y\sqrt{6y}$
7. $7ab\sqrt{a}$
8. $2a^2b\sqrt{bc}$
9. $3x^2y\sqrt{5xy}$
10. $5x^3y^2\sqrt{2}$
11. $8rs^3t^2\sqrt{t}$
12. $12r^4s^3t$
13. $x+1$
14. $2x+3$
15. $2(3x-1)$
16. $3(2x+3)$
17. $3xy\sqrt{5x}$
18. $6x^2y^2\sqrt{x}$

19. $6m^3n^2\sqrt{m}$
20. $7m^4n^3\sqrt{3m}$
21. $5rs^2t^2\sqrt{2s}$
22. $6r^2st^3\sqrt{r}$
23. $3a$
24. $5b$
25. $5xy\sqrt[3]{2x}$
26. $3ab\sqrt[3]{6b^2}$
27. $4xy^2z^3$
28. $6x^4y$
29. $2xy\sqrt[3]{y}$
30. $3xy\sqrt[3]{x^2}$
31. $abc^2\sqrt[3]{ab^2}$
32. $a^2bc\sqrt[3]{ab^2}$
33. $6xy\sqrt[3]{x}$
34. $5xy^2\sqrt[3]{x^2}$
35. $2rs^4t^4\sqrt[3]{45r^2t}$
36. $3rt^3\sqrt[3]{20s^2}$
37. $3x$

38. xy

39. $2xy^2$

40. $3x^3y$

41. $abc\sqrt[4]{bc^2}$

42. $ac^2\sqrt[4]{54a^2}$

43. $2x\sqrt[4]{8x^2}$

44. $3y\sqrt[4]{3y^3}$

45. $2m^2n$

46. $mn^2\sqrt[5]{37m^4}$

47. $-6x$

48. $21y$

49. $-10x\sqrt{xy}$

50. $-3xy^9\sqrt{x}$

51. $12a^3b^2\sqrt{ab}$

52. $6a^5b^{10}\sqrt{a}$

53. $4x^3$

54. $-15x^3$

55. $-4a^2b^2\sqrt[3]{ab^2}$

56. $-15a^3b^2$

57. $\sqrt{50x}$

58. $\sqrt{12y}$

59. $\sqrt{12x^2}$

60. $\sqrt{18y^2}$

61. $\sqrt{10a^3b^2}$

62. $\sqrt{8a^3b^2}$

63. $\sqrt{m^5n^3}$

64. $\sqrt{12m^4n^7}$

65. $\sqrt[3]{250x}$

66. $\sqrt[3]{135y}$

67. $\sqrt[3]{24x^3}$

68. $\sqrt[3]{54y^3}$

69. $2|x|$

70. $5|y|$

71. $2y$

72. $5a$

73. $2|x|$

74. $3|y|$

75. $6a^2$

76. $10a^4$

77. $2|a^3|$

78. $|a^5|$

79. $3a^2b^2\sqrt{2b}$

80. $4a^2b\sqrt{3ab}$

81. $2|xy|\sqrt[6]{2y^2}$

82. $|abc|\sqrt[6]{bc^2}$

83. $(0, 1)$

84. $(0, -2)$

85. $(0, 1)$

86. $(0, -2)$

87. $x - 3$

88. $x - 5$

89. $2x + 3$

90. $3x + 1$

91. 30 miles per hour

92. 3 centimetres

93. $\pi \approx 3.1$ seconds

94. $2\pi \approx 6.3$ seconds

95. $\frac{\pi}{4} \approx 0.8$ seconds

96. $\frac{\pi}{8} \approx 0.4$ seconds

97. $\sqrt{3} \approx 1.7$ seconds

98. $\sqrt{5} \approx 2.2$ seconds

99. $2\sqrt{3} \approx 3.5$ seconds

100. $3\sqrt{2} \approx 4.2$ seconds

101. $f(1) = 0$, $f(2) = 1$, and $f(5) = 2$

102. $f(-5) = 0$, $f(-1) = 2$, and $f(20) = 5$

103. $f(0) = 3$, $f(1) = 4$, and $f(16) = 7$

104. $f(0) = -5$, $f(1) = -4$, and $f(25) = 0$

105. $g(-1) = -1$, $g(0) = 0$, and $g(1) = 1$

106. $g(-15) = -2$, $g(-7) = 0$, and $g(20) = 3$

107. $g(-1) = -3$, $g(0) = -2$, and $g(8) = 0$

108. $g(0) = 1$, $g(2) = 3$, and $g(9) = 4$

109.

x	$f(x)$
-1	0
0	1
3	2
8	3
15	4

110.

x	$f(x)$
2	0
3	1
6	2
11	3
18	4

111.

x	$f(x)$
-8	-1
-1	0
0	1
1	2
8	3

112.

x	$f(x)$
-10	-2
-3	-1
-2	0
-1	1
6	2

10.3 Adding and Subtracting Radicals

Simplify.

1. $9\sqrt{3}+5\sqrt{3}$
2. $12\sqrt{6}+3\sqrt{6}$
3. $4\sqrt{5}-7\sqrt{5}$
4. $3\sqrt{10}-8\sqrt{10}$
5. $\sqrt{6}-4\sqrt{6}+2\sqrt{6}$
6. $5\sqrt{10}-15\sqrt{10}-2\sqrt{10}$
7. $13\sqrt{7}-6\sqrt{2}-5\sqrt{7}+5\sqrt{2}$
8. $10\sqrt{13}-12\sqrt{15}+5\sqrt{13}-18\sqrt{15}$
9. $6\sqrt{5}-(4\sqrt{3}-3\sqrt{5})$
10. $-12\sqrt{2}-(6\sqrt{6}+\sqrt{2})$
11. $(2\sqrt{5}-3\sqrt{10})-(\sqrt{10}+3\sqrt{5})$
12. $(-8\sqrt{3}+6\sqrt{15})-(\sqrt{3}-\sqrt{15})$
13. $4\sqrt[3]{6}-3\sqrt[3]{5}+6\sqrt[3]{6}$
14. $\sqrt[3]{10}+5\sqrt[3]{10}-4\sqrt[3]{10}$
15. $(7\sqrt[3]{9}-4\sqrt[3]{3})-(\sqrt[3]{9}-3\sqrt[3]{3})$
16. $(-8\sqrt[3]{5}+\sqrt[3]{25})-(2\sqrt[3]{5}+6\sqrt[3]{25})$

Simplify. (Assume all radicands containing variable expressions are positive.)

17. $9\sqrt{x}+7\sqrt{x}$
18. $-8\sqrt{y}+4\sqrt{y}$
19. $7\sqrt{xy}-3\sqrt{xy}+\sqrt{xy}$
20. $10\sqrt{y^2x}-12\sqrt{y^2x}-2\sqrt{y^2x}$

21. $2\sqrt{ab}-5\sqrt{a}+6\sqrt{ab}-10\sqrt{a}$

22. $-3\sqrt{xy}+6\sqrt{y}-4\sqrt{xy}-7\sqrt{y}$

23. $5\sqrt{xy}-(3\sqrt{xy}-7\sqrt{xy})$

24. $-8\sqrt{ab}-(2\sqrt{ab}-4\sqrt{ab})$

25. $(3\sqrt{2x}-\sqrt{3x})-(\sqrt{2x}-7\sqrt{3x})$

26. $(\sqrt{y}-4\sqrt{2y})-(\sqrt{y}-5\sqrt{2y})$

27. $5\sqrt[3]{x}-12\sqrt[3]{x}$

28. $-2\sqrt[3]{y}-3\sqrt[3]{y}$

29. $a\sqrt[5]{3b}+4a\sqrt[5]{3b}-a\sqrt[5]{3b}$

30. $-8\sqrt[4]{ab}+3\sqrt[4]{ab}-2\sqrt[4]{ab}$

31. $6\sqrt{2a}-4\sqrt[3]{2a}+7\sqrt{2a}-\sqrt[3]{2a}$

32. $4\sqrt[5]{3a}+\sqrt[3]{3a}-9\sqrt[5]{3a}+\sqrt[3]{3a}$

33. $(\sqrt[4]{4xy}-\sqrt[3]{xy})-(2\sqrt[4]{4xy}-\sqrt[3]{xy})$

34. $(5\sqrt[6]{6y}-\sqrt{5y})-(2\sqrt[6]{6y}+\sqrt{3y})$

Simplify.

35. $\sqrt{75}-\sqrt{12}$

36. $\sqrt{24}-\sqrt{54}$

37. $\sqrt{32}+\sqrt{27}-\sqrt{8}$

38. $\sqrt{20}+\sqrt{48}-\sqrt{45}$

39. $\sqrt{28}-\sqrt{27}+\sqrt{63}-\sqrt{12}$

40. $\sqrt{90}+\sqrt{24}-\sqrt{40}-\sqrt{54}$

41. $\sqrt{45}-\sqrt{80}+\sqrt{245}-\sqrt{5}$

42. $\sqrt{108}+\sqrt{48}-\sqrt{75}-\sqrt{3}$

43. $4\sqrt{2}-(\sqrt{27}-\sqrt{72})$

44. $-\sqrt{35}-(\sqrt{20}-\sqrt{50})$

45. $\sqrt[3]{16}-\sqrt[3]{54}$

46. $\sqrt[3]{81}-\sqrt[3]{24}$

47. $\sqrt[3]{135}+\sqrt[3]{40}-\sqrt[3]{5}$

48. $\sqrt[3]{108}-\sqrt[3]{32}-\sqrt[3]{4}$

49. $2\sqrt{27}-2\sqrt{12}$

50. $3\sqrt{50}-4\sqrt{32}$

51. $3\sqrt{243}-2\sqrt{18}-\sqrt{48}$

52. $6\sqrt{216}-2\sqrt{24}-2\sqrt{96}$

53. $2\sqrt{18}-3\sqrt{75}-2\sqrt{98}+4\sqrt{48}$

54. $\sqrt{245}-\sqrt{12}+\sqrt{220}-\sqrt{108}$

55. $(2\sqrt{363}-3\sqrt{96})-(7\sqrt{12}-2\sqrt{54})$

56. $(2\sqrt{288}+3\sqrt{360})-(2\sqrt{72}-7\sqrt{40})$

57. $3\sqrt[3]{54}+5\sqrt[3]{250}-4\sqrt[3]{16}$

58. $4\sqrt[3]{162}-2\sqrt[3]{384}-3\sqrt[3]{750}$

Simplify. (Assume all radicands containing variable expressions are positive.)

59. $\sqrt{81b}+\sqrt{4b}$

60. $\sqrt{100a}+\sqrt{a}$

61. $\sqrt{9a^2b} - \sqrt{36a^2b}$

62. $\sqrt{50a^2} - \sqrt{18a^2}$

63. $\sqrt{49x} - \sqrt{9y} + \sqrt{x} - \sqrt{4y}$

64. $\sqrt{9x} + \sqrt{64y} - \sqrt{25x} - \sqrt{y}$

65. $7\sqrt{8x} - (3\sqrt{16y} - 2\sqrt{18x})$

66. $2\sqrt{64y} - (3\sqrt{32y} - \sqrt{81y})$

67. $2\sqrt{9m^2n} - 5\sqrt{m^2n} + \sqrt{m^2n}$

68. $4\sqrt{18n^2m} - 2\sqrt{n^8m} + \sqrt{n^2m}$

69. $\sqrt{4x^2y} - \sqrt{9xy^2} - \sqrt{16x^2y} + \sqrt{xy^2}$

70. $\sqrt{32x^2y^2} + \sqrt{12x^2y} - \sqrt{18x^2y^2} - \sqrt{27x^2y}$

71. $(\sqrt{9x^2y} - \sqrt{16y}) - (\sqrt{49x^2y} - 4\sqrt{y})$

72. $(\sqrt{72x^2y^2} - \sqrt{18x^2y}) - (\sqrt{50x^2y^2} + \sqrt{x^2y})$

73. $\sqrt{12m^4n} - m\sqrt{75m^2n} + 2\sqrt{27m^4n}$

74. $5n\sqrt{27mn^2} + 2\sqrt{12mn^4} - n\sqrt{3mn^2}$

75. $2\sqrt{27a^3b} - a\sqrt{48ab} - a\sqrt{144a^3b}$

76. $\sqrt{298a^4b} - 2a\sqrt{162a^2b} + a\sqrt{200b}$

77. $\sqrt[3]{125a} - \sqrt[3]{27a}$

78. $\sqrt[3]{1000a^2} - \sqrt[3]{64a^2}$

79. $2x\sqrt[3]{54x} - 2\sqrt[3]{16x^4} + 5\sqrt[3]{2x^4}$

80. $\sqrt[3]{54x^3} - \sqrt[3]{250x^6} + 2x\sqrt[3]{2}$

81. $\sqrt[4]{16y^2} + \sqrt[4]{81y^2}$

82. $\sqrt[5]{32y^4} - \sqrt[5]{y^4}$

83. $\sqrt[4]{32a^3} - \sqrt[4]{162a^3} + 5\sqrt[4]{2a^3}$

84. $\sqrt[4]{80a^4b} + \sqrt[4]{5a^4b} - a\sqrt[4]{5b}$

85. $\sqrt[3]{27x^3} + \sqrt[3]{8x} - \sqrt[3]{125x^3}$

86. $\sqrt[3]{24x} - \sqrt[3]{128x} - \sqrt[3]{81x}$

87. $\sqrt[3]{27x^4y} - \sqrt[3]{8xy^3} + x\sqrt[3]{64xy} - y\sqrt[3]{x}$

88. $\sqrt[3]{125xy^3} + \sqrt[3]{8x^3y} - \sqrt[3]{216xy^3} + 10x\sqrt[3]{y}$

89. $(\sqrt[3]{162x^4y} - \sqrt[3]{250x^4y^2}) - (\sqrt[3]{2x^4y^2} - \sqrt[3]{384x^4y})$

90. $(\sqrt[5]{32x^2y^6} - \sqrt[5]{243x^6y^2}) - (\sqrt[5]{x^2y^6} - x\sqrt[5]{xy^2})$

Show Solutions

1. $14\sqrt{3}$
2. $15\sqrt{6}$
3. $-3\sqrt{5}$
4. $-5\sqrt{10}$
5. $-\sqrt{6}$
6. $-12\sqrt{10}$

7. $8\sqrt{7}-\sqrt{2}$
8. $15\sqrt{13}-30\sqrt{15}$
9. $9\sqrt{5}-4\sqrt{3}$
10. $-13\sqrt{2}-6\sqrt{6}$
11. $-\sqrt{5}-4\sqrt{10}$
12. $-9\sqrt{3}+7\sqrt{15}$
13. $10\sqrt[3]{6}-3\sqrt[3]{5}$
14. $2\sqrt[3]{10}$
15. $6\sqrt[3]{9}-\sqrt[3]{3}$
16. $-10\sqrt[3]{5}-5\sqrt[3]{25}$
17. $16\sqrt{x}$
18. $-4\sqrt{y}$
19. $5\sqrt{xy}$
20. $-4y\sqrt{x}$
21. $8\sqrt{ab}-15\sqrt{a}$
22. $-7\sqrt{xy}-\sqrt{y}$
23. $9\sqrt{xy}$
24. $-6\sqrt{ab}$
25. $2\sqrt{2x}+6\sqrt{3x}$

26. $\sqrt{2y}$

27. $-7\sqrt[3]{x}$

28. $-5\sqrt[3]{y}$

29. $4a\sqrt[5]{3b}$

30. $-7\sqrt[4]{ab}$

31. $13\sqrt{2a}-5\sqrt[3]{2a}$

32. $-5\sqrt[5]{3a}+2\sqrt[3]{3a}$

33. $-\sqrt[4]{4xy}$

34. $3\sqrt[6]{6y}-\sqrt{5y}-\sqrt{3y}$

35. $3\sqrt{3}$

36. $-\sqrt{6}$

37. $2\sqrt{2}+3\sqrt{3}$

38. $4\sqrt{3}-\sqrt{5}$

39. $5\sqrt{7}-5\sqrt{3}$

40. $\sqrt{10}-\sqrt{6}$

41. $5\sqrt{5}$

42. $4\sqrt{3}$

43. $10\sqrt{2}-3\sqrt{3}$

44. $5\sqrt{2}-2\sqrt{5}-\sqrt{35}$

45. $-\sqrt[3]{2}$

46. $\sqrt[3]{3}$

47. $4\sqrt[3]{5}$

48. 0

49. $2\sqrt{3}$

50. $-\sqrt{2}$

51. $23\sqrt{3}-6\sqrt{2}$

52. $24\sqrt{6}$

53. $-8\sqrt{2}+\sqrt{3}$

54. $7\sqrt{5}+2\sqrt{55}-8\sqrt{3}$

55. $8\sqrt{3}-6\sqrt{6}$

56. $12\sqrt{2}+32\sqrt{10}$

57. $26\sqrt[3]{2}$

58. $-11\sqrt[3]{6}$

59. $11\sqrt{b}$

60. $11\sqrt{a}$

61. $-3a\sqrt{b}$

62. $2a\sqrt{2}$

63. $8\sqrt{x}-5\sqrt{y}$

64. $-2\sqrt{x}+7\sqrt{y}$

65. $20\sqrt{2x}-12\sqrt{y}$

66. $25\sqrt{y}-12\sqrt{2y}$

67. $2m\sqrt{n}$

68. $12n\sqrt{2m}+(n-2n^4)\sqrt{m}$

69. $-2x\sqrt{y}-2y\sqrt{x}$

70. $xy\sqrt{2}-x\sqrt{3y}$

71. $-4x\sqrt{y}$

72. $xy\sqrt{2}-3x\sqrt{2y}-x\sqrt{y}$

73. $3m^2\sqrt{3n}$

74. $18n^2\sqrt{3m}$

75. $2a\sqrt{3ab}-12a^2\sqrt{ab}$

76. $a^2\sqrt{298b}-18a^2\sqrt{2b}+10a\sqrt{2b}$

77. $2\sqrt[3]{a}$

78. $6\sqrt[3]{a^2}$

79. $7x\sqrt[3]{2x}$

80. $5x\sqrt[3]{2}-5x^2\sqrt[3]{2}$

81. $5\sqrt{y}$

82. $\sqrt[5]{y^4}$

83. $4\sqrt[4]{2a^3}$

84. $2a\sqrt[4]{5b}$

85. $-2x+2\sqrt[3]{x}$

86. $-\sqrt[3]{3x}-4\sqrt[3]{2x}$

87. $7x\sqrt[3]{xy}-3y\sqrt[3]{x}$

88. $12x\sqrt[3]{y}-y\sqrt[3]{x}$

89. $7x\sqrt[3]{6xy}-6x\sqrt[3]{2xy^2}$

90. $y\sqrt[5]{x^2y}-2x\sqrt[5]{xy^2}$

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CHAPTER 11: RADICALS CONTINUED

Chapter Outline

11.1 Multiplying and Dividing Radicals

- Multiplying Radical Expressions
- Dividing Radical Expressions (Rationalizing the Denominator)

11.2 Radical Equations

11.3 Exercises

- 11.1 Multiplying and Dividing Radicals
- 11.2 Radical Equations

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11.1 Multiplying and Dividing Radicals

Learning Objectives

By the end of this section, you should be able to:

- Multiply radical expressions.
- Divide radical expressions.
- Rationalize the denominator.

Multiplying Radical Expressions

When multiplying radical expressions with the same index, we use the product rule for radicals. If a and b represent positive real numbers,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

Example 1

Multiply:

$$\sqrt{2} \cdot \sqrt{6}$$

Solution:
This problem is a product of two square roots. Apply the product rule for radicals and then simplify.

$$\begin{aligned} \sqrt{2} \cdot \sqrt{6} &= \sqrt{2 \cdot 6} && \text{Multiply the radicands.} \\ &= \sqrt{12} && \text{Simplify.} \\ &= \sqrt{2^2 \cdot 3} \\ &= 2\sqrt{3} \end{aligned}$$

Answer

$2\sqrt{3}$

Example 2

Multiply:

$$\sqrt[3]{9} \cdot \sqrt[3]{6}$$

Solution:
This problem is a product of cube roots. Apply the product rule for radicals and then simplify.

$$\begin{aligned} \sqrt[3]{9} \cdot \sqrt[3]{6} &= \sqrt[3]{9 \cdot 6} && \text{Multiply the radicands.} \\ &= \sqrt[3]{54} && \text{Simplify.} \\ &= \sqrt[3]{3^3 \cdot 2} \\ &= 3\sqrt[3]{2} \end{aligned}$$

Answer

$3\sqrt[3]{2}$

Often, there will be coefficients in front of the radicals.

Example 3

Multiply:

$$2\sqrt{3} \cdot 5\sqrt{2}$$

Solution:
Using the product rule for radicals and the fact that multiplication is commutative, we can multiply the coefficients and the radicands as follows.

$$\begin{aligned} 2\sqrt{3}\cdot 5\sqrt{2} &= 2\cdot 5\cdot \sqrt{3}\cdot \sqrt{2} && \text{Multiplication is commutative.} \\ &= 10\cdot \sqrt{6} && \text{Multiply the coefficients and the radicands.} \\ &= 10\sqrt{6} \end{aligned}$$

Typically, the first step involving the application of the **commutative property** is not shown.

Answer

$10\sqrt{6}$

Example 4

Multiply:

$$-2\sqrt[3]{5x}\cdot 3\sqrt[3]{25x^2}$$

Solution:

Multiply coefficients and radicands, then simplify the cube root.

$$\begin{aligned} -2\sqrt[3]{5x}\cdot 3\sqrt[3]{25x^2} &= -6\sqrt[3]{(5x)(25x^2)} \\ &= -6\sqrt[3]{125x^3} \\ &= -6(5x) \\ &= -30x \end{aligned}$$

Answer

$-30x$

Use the **distributive property** when multiplying radical expressions with more than one term.

Example 5

Multiply:

$$4\sqrt{3}\,(2\sqrt{2}-\sqrt{6})$$

Solution:
Apply the distributive property and simplify each product.

$$\begin{aligned} 4\sqrt{3}(2\sqrt{2}-\sqrt{6}) &= 4\sqrt{3}\cdot 2\sqrt{2}-4\sqrt{3}\cdot\sqrt{6} \\ &= 8\sqrt{6}-4\sqrt{18} \\ &= 8\sqrt{6}-12\sqrt{2} \end{aligned}$$

Answer

$8\sqrt{6}-12\sqrt{2}$

The process for multiplying radical expressions with multiple terms is the same process used when multiplying polynomials. Apply the distributive property, simplify each radical, and then combine like terms.

Try It

Activity Description
Match the options to the correct equation.
Perform the operation and simplify

1. $\sqrt{5x}\cdot\sqrt{10x}=$
2. $2\sqrt{ab}\cdot\sqrt{a}=$
3. $\sqrt{\dfrac{20y^6}{5y^2}}=$
4. $\sqrt{2b}(\sqrt{8b}-3\sqrt{50b^3})=$

Options:

$$\sqrt{50x^2} \qquad \sqrt{2a^2vb} \qquad 4b-30b^2 \qquad 2a\sqrt{b} \qquad 5x\sqrt{2} \qquad \sqrt{50x}$$
$$\sqrt{16b^2}-3\sqrt{100b^4} \qquad 2y^2 \qquad \sqrt{4y^4}$$

1. $5x\sqrt{2}$
2. $2a\sqrt{b}$
3. $2y^2$
4. $4b - 30b^2$

Example 6

Multiply:

$$(\sqrt{5}+2)(\sqrt{5}-4)$$

Solution:
Begin by applying the distributive property.

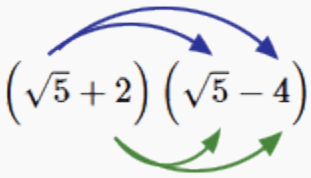

Figure 11.1.1

$$\begin{aligned}(\sqrt{5}+2)(\sqrt{5}-4) &= \sqrt{5}\cdot\sqrt{5} - 4\sqrt{5} + 2\sqrt{5} - 8 && \text{Simplify.}\\ &= 5 - 4\sqrt{5} + 2\sqrt{5} - 8 \\ &= 5 - 8 - 4\sqrt{5} + 2\sqrt{5} && \text{Combine like terms.}\\ &= -3 - 2\sqrt{5}\end{aligned}$$

Answer

$-3 - 2\sqrt{5}$

Example 7

Multiply:

$$(3\sqrt{x}-\sqrt{y})^2$$

Solution:

$$\begin{aligned}(3\sqrt{x}-\sqrt{y})^2 &= (3\sqrt{x}-\sqrt{y})(3\sqrt{x}-\sqrt{y}) \\ &= 9x-3\sqrt{xy}-3\sqrt{xy}+y \qquad \text{Simplify.} \\ &= 9x-6\sqrt{xy}+y\end{aligned}$$

Answer

$9x-6\sqrt{xy}+y$

Try It

Multiply: $(2\sqrt{3}+5\sqrt{2})(\sqrt{3}-2\sqrt{6})$

Show Solution

Answer: $6-12\sqrt{2}+5\sqrt{6}-20\sqrt{3}$

The expressions $(a+b)$ and $(a-b)$ are called **conjugates**. When multiplying conjugates, the sum of the products of the inner and outer terms results in 0.

Example 8

Multiply:

$$(\sqrt{2}+\sqrt{5})(\sqrt{2}-\sqrt{5})$$

Solution:

Apply the distributive property and then combine like terms.

$$\begin{aligned}(\sqrt{2}+\sqrt{5})(\sqrt{2}-\sqrt{5}) &= 2-\sqrt{10}+\sqrt{10}-5\\ &= -3\end{aligned}$$

Answer

-3

It is important to note that when multiplying conjugate radical expressions, we obtain a rational expression. This is true in general and is often used in our study of algebra.

$$\begin{aligned}(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) &= \sqrt{a^2}-\sqrt{ab}+\sqrt{ab}-\sqrt{b^2}\\ &= a-b\end{aligned}$$

$$\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$$

Try It

Activity Description

Match the correct answer to the question.

Perform the operation, then simplify.

1. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) =$
2. $(3+\sqrt{2})(3-\sqrt{2}) =$
3. $(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3}) =$
4. $(\sqrt{5x}+2)(\sqrt{5x}-2) =$
5. What special product were these?

Options:

2 $5x-4$ perfect square difference of squares cube of bionmial

16 $a-b$ $9+\sqrt{2}$ $a+b$ 1 7

Answers:

1. $a-b$
2. 7
3. 2
4. $5x-4$
5. difference of squares

Dividing Radical Expressions (Rationalizing the Denominator)

To divide radical expressions with the same index, we use the quotient rule for radicals. If a and b represent nonnegative numbers, where $b \neq 0$, then we have

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example 9

Divide:

$$\frac{\sqrt{80}}{\sqrt{10}}$$

Solution:

In this case, we can see that 10 and 80 have common factors. If we apply the quotient rule for radicals and write it as a single square root, we will be able to reduce the fractional radicand.

$$\begin{aligned}\frac{\sqrt{80}}{\sqrt{10}} &= \sqrt{\frac{80}{10}} && \text{Apply the quotient rule for radicals and reduce the radicand.}\\ &= \sqrt{8} && \text{Simplify.}\\ &= \sqrt{4\cdot 2}\\ &= 2\sqrt{2}\end{aligned}$$

Answer

$2\sqrt{2}$

Example 10

Divide:

$$\frac{\sqrt{16x^5y^4}}{\sqrt{2xy}}$$

Solution:

$$\begin{aligned}\frac{\sqrt{16x^5y^4}}{\sqrt{2xy}} &= \sqrt{\frac{16x^5y^4}{2xy}} && \text{Apply the quotient rule for radicals and cancel.}\\ &= \sqrt{8x^4y^3} && \text{Simplify.}\\ &= \sqrt{4\cdot 2\cdot (x^2)^2\cdot y^2\cdot y}\\ &= 2x^2y\sqrt{2y}\end{aligned}$$

Answer

$2x^2y\sqrt{2y}$

Example 11

Divide:

$$\frac{\sqrt[3]{54a^3b^5}}{\sqrt[3]{16a^2b^2}}$$

Solution:

$$\begin{aligned}
\frac{\sqrt[3]{54a^3b^5}}{\sqrt[3]{16a^2b^2}} &= \sqrt[3]{\frac{54\cdot a^3b^5}{16a^2b^2}} && \text{Apply the quotient rule for radicals and then cancel.}\\
&= \sqrt[3]{\frac{27ab^3}{8}} && \text{Replace 27 and 8 withtheir prime factorizations.}\\
&= \sqrt[3]{\frac{3^3ab^3}{2^3}} && \text{Simplify.}\\
&= \frac{\sqrt[3]{3^3ab^3}}{\sqrt[3]{2^3}}\\
&= \frac{3b\sqrt[3]{a}}{2}
\end{aligned}$$

Answer

$$\frac{3b\sqrt[3]{a}}{2}$$

When the divisor of a radical expression contains a radical, it is a common practice to find an equivalent expression where the denominator is a rational number. Finding such an equivalent expression is called **rationalizing the denominator**.

Radical Expression	Rational Denominator
$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{3}}{3}$

To do this, multiply the fraction by a special form of 1 so that the radicand in the denominator can be written with a power that matches the index. After doing this, simplify and eliminate the radical in the denominator. For example,

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3^2}} = \frac{\sqrt{3}}{3}$$

Remember, to obtain an equivalent expression, you must multiply the numerator and denominator by the exact same nonzero factor.

How To

Given an expression with a single square root radical term in the denominator, rationalize the denominator.

1. Multiply the numerator and denominator by the radical in the denominator.
2. Simplify.

Example 12

Rationalize the denominator:

$$\frac{\sqrt{3}}{\sqrt{2}}$$

Solution:

The goal is to find an equivalent expression without a radical in the denominator. In this example, multiply by 1 in the form $\frac{\sqrt{2}}{\sqrt{2}}$

$$\begin{aligned} \frac{\sqrt{3}}{\sqrt{2}} &= \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} && \text{Multiply by } \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{6}}{\sqrt{2^2}} && \text{Simplify.} \\ &= \frac{\sqrt{6}}{2} && \text{Rational Denominator.} \end{aligned}$$

Answer

$$\frac{\sqrt{6}}{2}$$

Example 13

Rationalize the denominator:

$$\frac{1}{2\sqrt{3x}}$$

Solution:

The radicand in the denominator determines the factors that you need to use to rationalize it. In this example, multiply by 1 in the form $\frac{1}{2\sqrt{3x}}$

$$\begin{aligned}
\frac{1}{2\sqrt{3x}} &= \frac{1}{2\sqrt{3x}} \cdot \frac{\sqrt{3x}}{\sqrt{3x}} && \text{Multiply by } \frac{\sqrt{3x}}{\sqrt{3x}} \\
&= \frac{\sqrt{3x}}{2\sqrt{3^2x^2}} && \text{Simplify.} \\
&= \frac{\sqrt{3x}}{2 \cdot 3x} \\
&= \frac{\sqrt{3x}}{6x}
\end{aligned}$$

Answer

$$\frac{\sqrt{3x}}{6x}$$

Typically, we will find the need to reduce or cancel after rationalizing the denominator.

Example 14

Rationalize the denominator:

$$\frac{5\sqrt{2}}{\sqrt{5ab}}$$

Solution:

In this example, we will multiply by 1 in the form $\frac{\sqrt{5ab}}{\sqrt{5ab}}$

$$\begin{aligned}
\frac{5\sqrt{2}}{\sqrt{5ab}} &= \frac{5\sqrt{2}}{\sqrt{5ab}} \cdot \frac{\sqrt{5ab}}{\sqrt{5ab}} \\
&= \frac{5\sqrt{10ab}}{25\sqrt{25a^2b^2}} && \text{Simplify.} \\
&= \frac{5\sqrt{10ab}}{5ab} && \text{Cancel.} \\
&= \frac{\sqrt{10ab}}{ab}
\end{aligned}$$

Notice that a and b do not cancel in this example. Do not cancel factors inside a radical with those that are outside.

Answer

$$\frac{\sqrt{10ab}}{ab}$$

Try It

Rationalize the denominator: $\sqrt{\frac{4a}{3b}}$

Show Solution

$$\frac{2\sqrt{3ab}}{3b}$$

Up to this point, we have seen that multiplying a numerator and a denominator by a square root with the exact same radicand results in a rational denominator. In general, this is true only when the denominator contains a square root. However, this is not the case for a cube root. For example,

$$\frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}}{\sqrt[3]{x^2}}$$

Note that multiplying by the same factor in the denominator does not rationalize it. In this case, if we multiply by 1 in the form of $\frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}}$, then we can write the radicand in the denominator as a power of 3. Simplifying the result then yields a rationalized denominator. For example,

$$\frac{1}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2}}{x}$$

To rid the denominator of radical expressions, we "ratoinalize the denominator". Begin by factoring the radicand of the denominator (everything under the roots sign). Look at the index (the little number beside the root sign) to tell you how many of each factor you need to remove it from the root sign (e.g. if you have $\sqrt[3]{2}$, then you need 2^3 to remove it from the root sign. You would need to multiply numerator and denominator by $\sqrt[3]{2^2}$ to remove the radical in the denominator).

Try It

Activity Description
Select the value below to remove the radical from each denominator.

1. $\frac{1}{\sqrt{3}} \cdot \square = \frac{\sqrt{3}}{3}$
2. $\frac{7}{3\sqrt{2}} \cdot \square = \frac{7\sqrt{2}}{6}$

3. $\dfrac{1}{\sqrt[3]{2x}} \cdot \square = \dfrac{\sqrt[3]{4x^2}}{2x}$
4. $\dfrac{\sqrt{2}}{\sqrt{5}-\sqrt{2}} \cdot \square = \dfrac{\sqrt{10}+2}{3}$

Options:

$\frac{\sqrt{5+2}}{\sqrt{5+2}}$ $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ $\frac{\sqrt[3]{4x^2}}{\sqrt[3]{4x^2}}$ $\frac{\sqrt[3]{2x}}{\sqrt[3]{2x}}$ $\frac{3\sqrt{2}}{3\sqrt{2}}$ $\frac{\sqrt{3}}{\sqrt{3}}$

Answers

1. $\dfrac{\sqrt{3}}{\sqrt{3}}$
2. $\dfrac{\sqrt[3]{2x}}{\sqrt[3]{2x}}$
3. $\dfrac{\sqrt[3]{2x}}{\sqrt[3]{2x}}$
4. $\dfrac{\sqrt{5+2}}{\sqrt{5+2}}$

Example 15

Rationalize the denominator:

$$\frac{1}{\sqrt[3]{25}}$$

Solution:

The radical in the denominator is equivalent to $\sqrt[3]{5^2}$. To rationalize the denominator, it should be $\sqrt[3]{5^3}$. To obtain this, we need one more factor of 5. Therefore, multiply by 1 in the form of $\dfrac{\sqrt[3]{5}}{\sqrt[3]{5}}$

$$\frac{1}{\sqrt[3]{25}} = \frac{1}{\sqrt[3]{5^2}} \cdot \frac{\sqrt[3]{5}}{\sqrt[3]{5}}$$ Multiply by the cube root of factors that result in powers of 3

$$= \frac{\sqrt[3]{5}}{\sqrt[3]{5^3}}$$ Simplify.

$$= \frac{\sqrt[3]{5}}{5}$$

Answer

$$\frac{\sqrt[3]{5}}{5}$$

Example 16

Rationalize the denominator:

$$\sqrt[3]{\frac{27a}{2b^2}}$$

Solution:

In this example, we will multiply by 1 in the form $\sqrt[3]{\frac{2^2b}{2^2b}}$

$$\sqrt[3]{\frac{27a}{2b^2}} = \frac{\sqrt[3]{3^3a}}{\sqrt[3]{2b^2}}$$ Apply the quotient rule for radicals

$$= \frac{3\sqrt[3]{a}}{\sqrt[3]{2b^2}} \cdot \frac{\sqrt[3]{2^2b}}{\sqrt[3]{2^2b}}$$ Multiply by the cube root of factors that result in powers of 3.

$$= \frac{3\sqrt[3]{2^2ab}}{\sqrt[3]{2^3b^3}}$$ Simplify.

$$= \frac{3\sqrt[3]{4ab}}{2b}$$

Answer

$$\frac{3\sqrt[3]{4ab}}{2b}$$

Example 17

Rationalize the denominator:

$$\frac{1}{\sqrt[5]{4x^3}}$$

Solution:

In this example, we will multiply by 1 in the form $\frac{\sqrt[5]{2^3x^2}}{\sqrt[5]{2^3x^2}}$

$$\frac{1}{\sqrt[5]{4x^3}} = \frac{1}{\sqrt[5]{2^2x^3}} \cdot \frac{\sqrt[5]{2^3x^2}}{\sqrt[5]{2^3x^2}}$$

Multiply by the fifth root of factors that results in powers of 5.

$$= \frac{\sqrt[5]{2^2x^5}}{\sqrt[5]{2^5x^5}}$$

Simplify.

$$= \frac{\sqrt[5]{8x^2}}{2x}$$

Answer

$$\frac{\sqrt[5]{8x^2}}{2x}$$

When two terms involving square roots appear in the denominator, we can rationalize it using a very special technique. This technique involves multiplying the numerator and the denominator of the fraction by the conjugate of the denominator. Recall that multiplying a radical expression by its conjugate produces a rational number.

How To

Given an expression with a radical term and a constant in the denominator, rationalize the denominator.

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

Example 18

Rationalize the denominator:

$$\frac{1}{\sqrt{3}-\sqrt{2}}$$

Solution:

In this example, the conjugate of the denominator is $\sqrt{3}+\sqrt{2}$. Therefore, multiply by 1 in the form $\frac{(\sqrt{3}+\sqrt{2})}{(\sqrt{3}+\sqrt{2})}$

$$\frac{1}{\sqrt{3}-\sqrt{2}} = \frac{1}{(\sqrt{3}-\sqrt{2})} \cdot \frac{(\sqrt{3}+\sqrt{2})}{(\sqrt{3}+\sqrt{2})}$$

Multiply numerator and denominator by the conjugate of the denominator

$$= \frac{\sqrt{3}+\sqrt{2}}{\sqrt{9}+\sqrt{6}-\sqrt{6}-\sqrt{4}}$$

$$= \frac{\sqrt{3}+\sqrt{2}}{3-2}$$

$$= \frac{\sqrt{3}+\sqrt{2}}{1}$$

$$= \sqrt{3}+\sqrt{2}$$

Simplify.

Answer

$\sqrt{3}+\sqrt{2}$

Notice that the terms involving the square root in the denominator are eliminated by multiplying by the conjugate. We can use the property $\dfrac{(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})}=a-b$ to expedite the process of multiplying the expressions in the denominator.

Example 19

Rationalize the denominator:

$$\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}+\sqrt{6}}$$

Solution:

Multiply by 1 in the form $\dfrac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}$

$$\begin{aligned}
\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}+\sqrt{6}} &= \frac{(\sqrt{2}-\sqrt{6})}{(\sqrt{2}+\sqrt{6})}\cdot\frac{(\sqrt{2}-\sqrt{6})}{(\sqrt{2}-\sqrt{6})} && \text{Multiply by the conjugate of the denominator.}\\
&= \frac{\sqrt{4}-\sqrt{12}-\sqrt{12}+\sqrt{36}}{2-6} && \text{Simplify.}\\
&= \frac{2-2\sqrt{12}+6}{-4}\\
&= \frac{8-2\sqrt{4\cdot 3}}{-4}\\
&= \frac{8-4\sqrt{3}}{-4}\\
&= \frac{8}{-4}-\frac{4\sqrt{3}}{-4}\\
&= -2+\sqrt{3}
\end{aligned}$$

Answer

$-2+\sqrt{3}$

Example 20

Rationalize the denominator:

$$\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Solution:

In this example, we will multiply by 1 in the form $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

$$\begin{aligned}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} &= \frac{(\sqrt{x}-\sqrt{y})}{(\sqrt{x}+\sqrt{y})}\cdot\frac{(\sqrt{x}-\sqrt{y})}{(\sqrt{x}-\sqrt{y})} && \text{Multiply by the conjugate of the denominator.}\\ &= \frac{\sqrt{x^2}-\sqrt{xy}-\sqrt{xy}+\sqrt{y^2}}{x-y} && \text{Simplify.}\\ &= \frac{x-2\sqrt{xy}+y}{x-y}\end{aligned}$$

Answer

$$\frac{x-2\sqrt{xy}+y}{x-y}$$

Try It

Rationalize the denominator: $\frac{3\sqrt{5}+5}{2\sqrt{5}-3}$

Show Solution

$$\frac{45+19\sqrt{5}}{11}$$

Key Takeaways

- To multiply two single-term radical expressions, multiply the coefficients and multiply the radicands. If possible, simplify the result.
- Apply the distributive property when multiplying radical expressions with multiple terms. Then simplify and combine all like radicals.
- Multiplying a two-term radical expression involving square roots by its conjugate results in a rational expression.
- It is common practice to write radical expressions without radicals in the denominator. The process of finding such an equivalent expression is called rationalizing the denominator.
- If an expression has one term in the denominator involving a radical, then rationalize it by multiplying the numerator and denominator by the n^{th} root of the factors of the radicand so that their powers equal the index.
- If a radical expression has two terms in the denominator involving square roots, then rationalize it by multiplying the numerator and denominator by its conjugate.

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11.2 Radical Equations

Learning Objectives

By the end of this section, you should be able to:

- Solve equations involving square roots.
- Solve equations involving cube roots.

Radical Equations

A radical equation is any equation that contains one or more radicals with a variable in the radicand. The following are some examples of radical equations, all of which will be solved in this section:

$$\sqrt{x-1}=5$$
$$\sqrt{2x-5}+4=x$$
$$\sqrt[3]{x^2+4}-2=0$$

We begin with the **squaring property of equality**; given real numbers a and b, we have the following:

$$\text{If } a=b, \text{ then } a^2=b^2$$

In other words, equality is retained if we square both sides of an equation.

$$\begin{aligned} -3=-3 \quad \Rightarrow \quad (-3)^2 &= (-3)^2 \\ 9 &= 9 \ \checkmark \end{aligned}$$

The converse, on the other hand, is not necessarily true:

$$3^2 = (-3)^2 \quad \Rightarrow \quad 3 \neq -3 \quad \text{x}$$

This is important because we will use this property to solve radical equations. Consider a very simple radical equation that can be solved by inspection:

$$\sqrt{x}=3$$

Here we can see that $x=9$ is a solution. To solve this equation algebraically, make use of the squaring property of equality and the fact that $(\sqrt{a})^2=\sqrt{a^2}=a$ when a is positive.

Eliminate the square root by squaring both sides of the equation as follows:

$$(\sqrt{x})^2=(3)^2$$
$$x = 9$$

As a check, we can see that $\sqrt{9} = 3$ as expected. Because the converse of the squaring property of equality is not necessarily true, solutions to the squared equation may not be solutions to the original. Hence, squaring both sides of an equation introduces the possibility of extraneous solutions, or solutions that do not solve the original equation. For this reason, we must check the answers that result from squaring both sides of an equation.

How To

Given a radical equation, solve it.

1. Isolate the radical expression on one side of the equal sign.
2. Put all remaining terms on the other side.
3. If the radical is a square root, then square both sides of the equation. If it is a cube root, then raise both sides of the equation to the third power. In other words, for an nth root radical, raise both sides to the nth power. Doing so eliminates the radical symbol.
4. Solve the remaining equation.
5. If a radical term still remains, repeat steps 1–2.
6. Confirm solutions by substituting them into the original equation.

Example 1

Solve:

$$\sqrt{x-1} = 5$$

Solution:
We can eliminate the square root by applying the squaring property of equality.

$$\begin{aligned} (\sqrt{x-1})^2 &= 5^2 && \text{Square both sides.} \\ x-1 &= 25 && \text{Simplify.} \\ x &= 26 && \text{Solve.} \end{aligned}$$

Next, we must check.

$$\sqrt{26-1} = 5$$
$$\sqrt{25} = 5$$
$$5 = 5 \quad \checkmark$$

Answer

The solution is 26.

Example 2

Solve:

$$\sqrt{5-4x} = x$$

Solution:
Begin by squaring both sides of the equation.

$$(\sqrt{5-4x})^2 = x^2$$
$$5-4x = x^2$$

You are left with a quadratic equation that can be solved by factoring.

$$0 = x^2 + 4x - 5$$
$$0 = (x+5)(x-1)$$
$$x+5=0 \quad \text{or} \quad x-1=0$$
$$x=-5 \qquad\qquad x=1$$

Since you squared both sides, you must check your solutions.

$$\sqrt{5-4x} = x \qquad \sqrt{5-4x} = x$$
$$\sqrt{5-4(-5)} = -5 \qquad \sqrt{5-4(1)} = 1$$
$$\sqrt{25} = -5 \qquad \sqrt{1} = 1$$
$$\text{x}\ \ 5 = -5 \qquad \checkmark\ \ 1 = 1$$

After checking, you can see that $x = -5$ was extraneous; it did not solve the original radical equation. Disregard that answer. This leaves $x = 1$ as the only solution.

Answer

The solution is $x = 1$.

In the previous two examples, notice that the radical is isolated on one side of the equation. Typically, this is not the case. The steps for solving radical equations involving square roots are outlined in the following example.

Visualize

Visit the Geogebra site to practice Solving Radical Equations with Quadratics. Link available on page 11.2 of the digital version of this textbook.

Science Application

An oxygen molecule (O_2) in the upper atmosphere has an RMS speed (v_{rms}) of $450\ \text{m/s}$. Given the molar mass $M = 0.032\ \text{kg/mol}$ and the gas constant $R = 8.314$, find the temperature (T) in Kelvin.

Use the equation: $v_{rms} = \sqrt{\dfrac{3RMT}{M}}$

Sub into the equation:

$$450 = \sqrt{\frac{3 \times 8.314 \times T}{0.032}}$$

Square both sides to get rid of the radical:

$$450^2 = \frac{3 \times 8.314 \times T}{0.032}$$

Rearrange to clear the fraction:

$$450^2 \times 0.032 = 3 \times 8.314 \times T$$

Solve for T:

$$T = \frac{450^2 \times 0.032}{3 \times 8.314}$$

Use a calculator to get a decimal, because this is science.

The temperature is approximately $259.78\ \text{K}$.

Example 3

Solve:

$$\sqrt{2x-5}+4=x$$

Solution:

Step 1: Isolate the square root. Begin by subtracting 4 from both sides of the equation.

$$\sqrt{2x-5}=x-4$$

Step 2: Square both sides. Squaring both sides eliminates the square root.

$$\begin{aligned}(\sqrt{2x-5})^2&=(x-4)^2\\2x-5&=x^2-8x+16\end{aligned}$$

Step 3: Solve the resulting equation. Here you are left with a quadratic equation that can be solved by factoring.

$$\begin{aligned}0&=x^2-10x+21\\0&=(x-3)(x-7)\end{aligned}$$

$$\begin{aligned}x-3=0 \quad &\text{or} \quad x-7=0\\x=3 \quad &\quad\quad x=7\end{aligned}$$

Step 4: Check the solutions in the original equation. Squaring both sides introduces the possibility of extraneous solutions; hence, the check is required.

$$\begin{aligned}\text{Check } x=3 \qquad & \qquad \text{Check } x=7\\\sqrt{2x-5}+4=x \qquad & \qquad \sqrt{2x-5}+4=x\\\sqrt{2(3)-5}+4=3 \qquad & \qquad \sqrt{2(7)-5}+4=7\\\sqrt{1}+4=3 \qquad & \qquad \sqrt{9}+4=7\\\text{x}\ \ 5=3 \qquad & \qquad \checkmark\ \ 7=7\end{aligned}$$

After checking, we can see that $x=3$ is an extraneous root; it does not solve the original radical equation.
This leaves $x=7$ as the only solution.

Answer

The solution is $x=7$.

Example 4

Solve:

$$3\sqrt{x+1}-2x=0$$

Solution:
Begin by isolating the term with the radical.

$$\begin{aligned} 3\sqrt{x+1}-2x&=0 && \text{Add } 2x \text{ to both sides}\\ 3\sqrt{x+1}&=2x \end{aligned}$$

Despite the fact that the term on the left side has a coefficient, it is still considered isolated. Recall that terms are separated by addition or subtraction operations.

$$\begin{aligned} 3\sqrt{x+1}&=2x\\ (3\sqrt{x+1})^2&=(2x)^2 && \text{Square both sides.}\\ 9(x+1)&=4x^2 \end{aligned}$$

Solve the resulting quadratic equation.

$$\begin{aligned} 4x^2-9x-9&=0\\ 0&=(4x+3)(x-3)\\ 4x+3=0 \quad &\text{or} \quad x-3=0\\ x=-\frac{3}{4} \quad &\quad x=3 \end{aligned}$$

Since we squared both sides, we must check our solutions

Check $x = -\frac{3}{4}$	Check $x = 3$
$3\sqrt{x+1} - 2x = 0$	$3\sqrt{x+1} - 2x = 0$
$3\sqrt{(-\frac{3}{4})+1} - 2(-\frac{3}{4}) = 0$	$3\sqrt{(3)+1} - 2(3) = 0$
$3\sqrt{\frac{1}{4}} + \frac{3}{2} = 0$	$3\sqrt{4} - 6 = 0$
$3 \cdot \frac{1}{2} + \frac{3}{2} = 0$	$3 \cdot 2 - 6 = 0$
$\frac{3}{2} + \frac{3}{2} = 0$	$6 - 6 = 0$
$\frac{6}{2} = 0$	$\checkmark 0 = 0$
$\text{x}\ 3 = 0$	

After checking, we can see that $x = -\dfrac{3}{4}$ was extraneous.

Answer

The solution is 3.

Sometimes, both of the possible solutions are extraneous.

Example 5

Solve:

$$\sqrt{4 - 11x} - x + 2 = 0$$

Solution:

Begin by isolating the radical.

$$\begin{aligned} \sqrt{4 - 11x} &= x - 2 \\ (\sqrt{4 - 11x})^2 &= (x - 2)^2 \\ 4 - 11x &= x^2 - 4x + 4 \\ 0 &= x^2 + 7x \\ x(x + 7) &= 0 \end{aligned}$$

$$x = 0 \quad \text{or} \quad x = -7$$

Since we squared both sides, we must check our solutions.

Check $x=-7$	Check $x=0$
$\sqrt{4-11x}-x+2=0$	$\sqrt{4-11x}-x+2=0$
$\sqrt{4-11(-7)}-(-7)+2=0$	$\sqrt{4-11(0)}-(0)+2=0$
$\sqrt{4+77}+7+2=0$	$\sqrt{4}+2=0$
$\sqrt{81}+9=0$	$2+2=0$
$9+9=0$	x $4=0$
x $18=0$	

Since both possible solutions are extraneous, the equation has no solution.

Answer

No solution, $\varnothing$.

The squaring property of equality extends to any positive integer power n. Given real numbers a and b, we have the following:

$$\text{If } a=b, \text{ then } a^n=b^n$$

This is often referred to as the **power property of equality**. Use this property, along with the fact that $\left(\sqrt[n]{a}\right)^n=\sqrt[n]{a^n}=a$, when a is positive, to solve radical equations with indices greater than 2.

Science Application

A mass is hanging on a spring with a spring constant $k=50\ \text{N/m}$. The system bounces with a period (time) of $T=2\ \text{seconds}$. What is the mass m? Use the formula $T=2\pi\sqrt{\dfrac{m}{k}}$

Sub values into the equation:

$$2=2\pi\sqrt{\frac{m}{50}}$$

Re-arrange to isolate the radical:

$$\frac{2}{2\pi}=\sqrt{\frac{m}{50}} \implies \frac{1}{\pi}=\sqrt{\frac{m}{50}}$$

Square both sides to remove the root:

$$\left(\frac{1}{\pi}\right)^2=\frac{m}{50}$$

Multiply by 50 to find m:

$$m = 50 \times \left(\frac{1}{\pi}\right)^2$$

We can use a calculator to get a decimal, since this is science!

The mass is approximately 5.07 kg.

Example 6

Solve:

$$\sqrt[3]{x^2+4} - 2 = 0$$

Solution:

Isolate the radical and then cube both sides of the equation.

$$\begin{aligned}
\sqrt[3]{x^2+4} - 2 &= 0 && \text{Isolate the radical.}\\
\sqrt[3]{x^2+4} &= 2 \\
\left(\sqrt[3]{x^2+4}\right)^3 &= 2^3 && \text{Cube both sides.}\\
x^2+4 &= 8 && \text{Solve.}\\
x^2-4 &= 0 \\
(x+2)(x-2) &= 0
\end{aligned}$$

$$\begin{aligned} x+2 &= 0 \\ x &= -2 \end{aligned} \quad \text{or} \quad \begin{aligned} x-2 &= 0 \\ x &= 2 \end{aligned}$$

Check.

$$\begin{aligned}
\text{Check } x = -2 \qquad & \qquad \text{Check } x = 2\\
\sqrt[3]{x^2+4} - 2 &= 0 & \sqrt[3]{x^2+4} - 2 &= 0\\
\sqrt[3]{(-2)^2+4} - 2 &= 0 & \sqrt[3]{(2)^2+4} - 2 &= 0\\
\sqrt[3]{8} - 2 &= 0 & \sqrt[3]{8} - 2 &= 0\\
\checkmark 0 &= 0 & \checkmark 0 &= 0
\end{aligned}$$

Answer

The solutions are -2 and 2.

Try It

Solve: $\sqrt{2x-1}+2=x$

Show Solution

$x=5$ ($x=1$ is extraneous)

It may be the case that the equation has two radical expressions.

Example 7

Solve:

$$\sqrt{3x-4}=\sqrt{2x+9}$$

Solution:
Both radicals are considered isolated on separate sides of the equation.

$$\begin{aligned}(\sqrt{3x-4})^2&=(\sqrt{2x+9})^2\\ 3x-4&=2x+9 \qquad \text{Solve.}\\ x&=13\end{aligned}$$

Check $x=13$.

$$\begin{aligned}\sqrt{3x-4}&=\sqrt{2x+9}\\ \sqrt{3(13)-4}&=\sqrt{2(13)+9}\\ \sqrt{35}&=\sqrt{35} \quad \checkmark\end{aligned}$$

Answer

The solution is 13.

Example 8

Solve:

$$\sqrt[3]{x^2+x-14}=\sqrt[3]{x+50}$$

Solution:
Eliminate the radicals by cubing both sides.

$$\begin{aligned} x^2+x-14&=x+50 \qquad \text{Solve.}\\ x^2-64&=0\\ (x+8)(x-8)&=0\\ x+8=0 \quad &\text{or} \quad x-8=0\\ x=-8 \quad &\qquad\quad x=8 \end{aligned}$$

Check.

$$\begin{aligned} \text{Check } x=-8 \qquad & \qquad \text{Check } x=8\\ \sqrt[3]{x^2+x-14}=\sqrt[3]{x+50} \qquad & \qquad \sqrt[3]{x^2+x-14}=\sqrt[3]{x+50}\\ \sqrt[3]{64-8-14}=\sqrt[3]{42} \qquad & \qquad \sqrt[3]{64+8-14}=\sqrt[3]{58}\\ \checkmark\sqrt[3]{42}=\sqrt[3]{42} \qquad & \qquad \checkmark\sqrt[3]{58}=\sqrt[3]{58} \end{aligned}$$

Answer

The solutions are -8 and 8.

We will learn how to solve some of the more advanced radical equations in the next course, Intermediate Algebra.

Try It

Solve: $\sqrt{3x+1} = \sqrt{2x-3}$

Show Solution

No solution for x

Key Takeaways

- Solve equations involving square roots by first isolating the radical and then squaring both sides. Squaring a square root eliminates the radical, leaving us with an equation that can be solved using the techniques learned earlier in our study of algebra. However, squaring both sides of an equation introduces the possibility of extraneous solutions, so check your answers in the original equation.
- Solve equations involving cube roots by first isolating the radical and then cubing both sides. This eliminates the radical and results in an equation that may be solved with techniques you have already mastered.

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11.3 Exercises

11.1 Multiplying and Dividing Radicals

Multiply. (Assume all variables are nonnegative.)

1. $\sqrt{3} \cdot \sqrt{5}$
2. $\sqrt{7} \cdot \sqrt{3}$
3. $\sqrt{2} \cdot \sqrt{6}$
4. $\sqrt{5} \cdot \sqrt{15}$
5. $\sqrt{7} \cdot \sqrt{7}$
6. $\sqrt{12} \cdot \sqrt{12}$
7. $2\sqrt{5} \cdot 7\sqrt{10}$
8. $3\sqrt{15} \cdot 2\sqrt{6}$
9. $(2\sqrt{5})^2$
10. $(6\sqrt{2})^2$
11. $\sqrt{2x} \cdot \sqrt{2x}$
12. $\sqrt{5y} \cdot \sqrt{5y}$
13. $\sqrt{3a} \cdot \sqrt{12}$
14. $\sqrt{3a} \cdot \sqrt{2a}$
15. $4\sqrt{2x} \cdot 3\sqrt{6x}$

16. $5\sqrt{10y} \cdot 2\sqrt{2y}$

17. $\sqrt[3]{5} \cdot \sqrt[3]{25}$

18. $\sqrt[3]{4} \cdot \sqrt[3]{2}$

19. $\sqrt[3]{4} \cdot \sqrt[3]{10}$

20. $\sqrt[3]{18} \cdot \sqrt[3]{6}$

21. $(5\sqrt[3]{9})(2\sqrt[3]{6})$

22. $(2\sqrt[3]{4})(3\sqrt[3]{4})$

23. $(2\sqrt[3]{2})^3$

24. $(3\sqrt[3]{4})^3$

25. $\sqrt[3]{3a^2} \cdot \sqrt[3]{9a}$

26. $\sqrt[3]{7b} \cdot \sqrt[3]{49b^2}$

27. $\sqrt[3]{6x^2} \cdot \sqrt[3]{4x^2}$

28. $\sqrt[3]{12y} \cdot \sqrt[3]{9y^2}$

29. $\sqrt[3]{20x^2y} \cdot \sqrt[3]{10x^2y^2}$

30. $\sqrt[3]{63xy} \cdot \sqrt[3]{12x^4y^2}$

31. $\sqrt{5}(3 - \sqrt{5})$

32. $\sqrt{2}(\sqrt{3} - \sqrt{2})$

33. $3\sqrt{7}(2\sqrt{7} - \sqrt{3})$

34. $2\sqrt{5}(6 - 3\sqrt{10})$

35. $\sqrt{6}(\sqrt{3} - \sqrt{2})$

36. $\sqrt{15}(\sqrt{5}+\sqrt{3})$

37. $\sqrt{x}(\sqrt{x}+\sqrt{xy})$

38. $\sqrt{y}(\sqrt{xy}+\sqrt{y})$

39. $\sqrt{2ab}(\sqrt{14a}-2\sqrt{10b})$

40. $\sqrt{6ab}\left(5\sqrt{2a}-\sqrt{3b}\right)$

41. $\left(\sqrt{2}-\sqrt{5}\right)\left(\sqrt{3}+\sqrt{7}\right)$

42. $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{5}-\sqrt{7}\right)$

43. $\left(2\sqrt{3}-4\right)\left(3\sqrt{6}+1\right)$

44. $\left(5-2\sqrt{6}\right)\left(7-2\sqrt{3}\right)$

45. $\left(\sqrt{5}-\sqrt{3}\right)^2$

46. $\left(\sqrt{7}-\sqrt{2}\right)^2$

47. $\left(2\sqrt{3}+\sqrt{2}\right)\left(2\sqrt{3}-\sqrt{2}\right)$

48. $\left(\sqrt{2}+3\sqrt{7}\right)\left(\sqrt{2}-3\sqrt{7}\right)$

49. $\left(\sqrt{a}-\sqrt{2b}\right)^2$

50. $\left(\sqrt{ab}+1\right)^2$

51. What are the perimeter and area of a rectangle with a length of $5\sqrt{3}$ centimetres and a width of $3\sqrt{2}$ centimetres?

52. What are the perimeter and area of a rectangle with a length of $2\sqrt{6}$ centimetres and a width of $\sqrt{3}$ centimetres?

53. If the base of a triangle measures $6\sqrt{2}$ meters and the height measures $3\sqrt{2}$ meters, then what is the area?

54. If the base of a triangle measures $6\sqrt{3}$ meters and the height measures $3\sqrt{6}$ meters, then what is the area?

Divide.

55. $\dfrac{\sqrt{75}}{\sqrt{3}}$

56. $\dfrac{\sqrt{360}}{\sqrt{10}}$

57. $\dfrac{\sqrt{72}}{\sqrt{75}}$

58. $\dfrac{\sqrt{90}}{\sqrt{98}}$

59. $\dfrac{\sqrt{90x^6}}{\sqrt{2x}}$

60. $\dfrac{\sqrt{96y^3}}{\sqrt{3y}}$

61. $\dfrac{\sqrt{162x^7y^5}}{\sqrt{2xy}}$

62. $\dfrac{\sqrt{363x^4y^9}}{\sqrt{3xy}}$

63. $\dfrac{\sqrt[3]{16a^5b^2}}{\sqrt[3]{2a^2b^2}}$

64. $\dfrac{\sqrt[3]{192a^2b^7}}{\sqrt[3]{2a^2b^2}}$

Rationalize the denominator.

65. $\dfrac{1}{\sqrt{5}}$

66. $\dfrac{1}{\sqrt{6}}$

67. $\dfrac{\sqrt{2}}{\sqrt{3}}$

68. $\dfrac{\sqrt{3}}{\sqrt{7}}$

69. $\dfrac{5}{2\sqrt{10}}$

70. $\dfrac{3}{5\sqrt{6}}$

71. $\dfrac{\sqrt{3}-\sqrt{5}}{\sqrt{3}}$

72. $\dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{2}}$

73. $\dfrac{1}{\sqrt{7x}}$

74. $\dfrac{1}{\sqrt{3y}}$

75. $\dfrac{a}{5\sqrt{ab}}$

76. $\dfrac{3b^2}{2\sqrt{3ab}}$

77. $\dfrac{2}{\sqrt[3]{36}}$

78. $\dfrac{14}{\sqrt[3]{7}}$

79. $\dfrac{1}{\sqrt[3]{4x}}$

80. $\dfrac{1}{\sqrt[3]{3y^2}}$

81. $\dfrac{9x \cdot \sqrt[3]{2}}{\sqrt[3]{9xy^2}}$

82. $\dfrac{5y^2 \cdot \sqrt[3]{x}}{\sqrt[3]{5x^2y}}$

83. $\dfrac{3a}{2\sqrt[3]{3a^2b^2}}$

84. $\dfrac{25n}{3\sqrt[3]{25m^2n}}$

85. $\sqrt[5]{\dfrac{3}{27x^2y}}$

86. $\sqrt[5]{\dfrac{2}{16xy^2}}$

87. $\dfrac{ab}{\sqrt[5]{9a^2b}}$

88. $\dfrac{abc}{\sqrt[5]{ab^2c^3}}$

89. $\dfrac{3}{\sqrt{10}-3}$

90. $\dfrac{2}{\sqrt{6}-2}$

91. $\dfrac{1}{\sqrt{5}+\sqrt{3}}$

92. $\dfrac{1}{\sqrt{7}-\sqrt{2}}$

93. $\dfrac{\sqrt{3}}{\sqrt{3}+\sqrt{6}}$

94. $\dfrac{\sqrt{5}}{\sqrt{5}+\sqrt{15}}$

95. $\dfrac{10}{5-3\sqrt{5}}$

96. $\dfrac{-2\sqrt{2}}{4-3\sqrt{2}}$

97. $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}}$

98. $\dfrac{\sqrt{10}-\sqrt{2}}{\sqrt{10}+\sqrt{2}}$

99. $\dfrac{2\sqrt{3}-3\sqrt{2}}{4\sqrt{3}+\sqrt{2}}$

100. $\dfrac{6\sqrt{5}+2}{2\sqrt{5}-\sqrt{2}}$

101. $\dfrac{x+\sqrt{y}}{x-\sqrt{y}}$

102. $\dfrac{x-\sqrt{y}}{x+\sqrt{y}}$

103. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

104. $\dfrac{\sqrt{ab}+\sqrt{2}}{\sqrt{ab}-\sqrt{2}}$

105. $\dfrac{\sqrt{x}}{5-2\sqrt{x}}$

106. $\dfrac{1}{\sqrt{x}-y}$

Show Solutions

1. $\sqrt{15}$
2. $\sqrt{21}$
3. $2\sqrt{3}$
4. $5\sqrt{3}$
5. 7
6. 12
7. $70\sqrt{2}$
8. $18\sqrt{10}$
9. 20
10. 72
11. $2x$
12. $5y$
13. $6\sqrt{a}$

14. $a\sqrt{6}$
15. $24x\sqrt{3}$
16. $20y\sqrt{5}$
17. 5
18. 2
19. $2\sqrt[3]{5}$
20. $3\sqrt[3]{4}$
21. $30\sqrt[3]{2}$
22. $12\sqrt[3]{2}$
23. 16
24. 108
25. $3a$
26. $7b$
27. $2x\sqrt[3]{3x}$
28. $3y\sqrt[3]{4}$
29. $2xy\sqrt[3]{25x}$
30. $3xy\sqrt[3]{28x^2}$
31. $3\sqrt{5}-5$
32. $\sqrt{6}-2$

33. $42 - 3\sqrt{21}$

34. $12\sqrt{5} - 30\sqrt{2}$

35. $3\sqrt{2} - 2\sqrt{3}$

36. $5\sqrt{3} + 3\sqrt{5}$

37. $x + x\sqrt{y}$

38. $y\sqrt{x} + y$

39. $2a\sqrt{7b} - 4b\sqrt{5a}$

40. $10a\sqrt{3b} - 3b\sqrt{2a}$

41. $\sqrt{6} + \sqrt{14} - \sqrt{15} - \sqrt{35}$

42. $\sqrt{15} - \sqrt{21} + \sqrt{10} - \sqrt{14}$

43. $18\sqrt{2} + 2\sqrt{3} - 12\sqrt{6} - 4$

44. $35 - 10\sqrt{3} - 14\sqrt{6} + 12\sqrt{2}$

45. $8 - 2\sqrt{15}$

46. $9 - 2\sqrt{14}$

47. 10

48. -61

49. $a - 2\sqrt{2ab} + 2b$

50. $ab + 2\sqrt{ab} + 1$

51. Perimeter: $10\sqrt{3} + 6\sqrt{2}$ centimeters; area: $15\sqrt{6}$ square centimeters

52. Perimeter: $4\sqrt{6}+2\sqrt{3}$ centimeters; area: $6\sqrt{2}$ square centimeters

53. 18 square meters

54. $27\sqrt{2}$ square meters

55. 5

56. 6

57. $\frac{2\sqrt{6}}{5}$

58. $\frac{3\sqrt{5}}{7}$

59. $3x^2\sqrt{5x}$

60. $4y\sqrt{2}$

61. $9x^3y^2$

62. $11xy^4\sqrt{x}$

63. $2a$

64. $2b\sqrt[3]{12b^2}$

65. $\frac{\sqrt{5}}{5}$

66. $\frac{\sqrt{6}}{6}$

67. $\frac{\sqrt{6}}{3}$

68. $\dfrac{\sqrt{21}}{7}$

69. $\dfrac{\sqrt{10}}{4}$

70. $\dfrac{\sqrt{6}}{10}$

71. $\dfrac{3-\sqrt{15}}{3}$

72. $\sqrt{3}-1$

73. $\dfrac{\sqrt{7x}}{7x}$

74. $\dfrac{\sqrt{3y}}{3y}$

75. $\dfrac{\sqrt{ab}}{5b}$

76. $\dfrac{b\sqrt{3ab}}{2a}$

77. $\dfrac{\sqrt[3]{6}}{3}$

78. $2\sqrt[3]{49}$

79. $\dfrac{\sqrt[3]{2x^2}}{2x}$

80. $\dfrac{\sqrt[3]{9y}}{3y}$

81. $\dfrac{3\sqrt[3]{6x^2y}}{y}$

82. $\dfrac{y\sqrt[3]{25x^2y^2}}{x}$

83. $\dfrac{\sqrt[3]{9ab}}{2b}$

84. $\dfrac{5\sqrt[3]{5mn^2}}{3m}$

85. $\dfrac{\sqrt[5]{27x^3y^4}}{3xy}$

86. $\dfrac{\sqrt[5]{4x^4y^3}}{2xy}$

87. $\dfrac{\sqrt[5]{27a^3b^4}}{3}$

88. $\sqrt[5]{a^4b^3c^2}$

89. $3\sqrt{10}+9$

90. $\sqrt{6}+2$

91. $\dfrac{\sqrt{5}-\sqrt{3}}{2}$

92. $\dfrac{\sqrt{7}+\sqrt{2}}{5}$

93. $-1+\sqrt{2}$

94. $\dfrac{\sqrt{3}-1}{2}$

95. $\dfrac{-5-3\sqrt{5}}{2}$

96. $6+4\sqrt{2}$

97. $-4-\sqrt{15}$

98. $\dfrac{3-\sqrt{5}}{2}$

99. $\dfrac{15-7\sqrt{6}}{23}$

100. $\dfrac{30+3\sqrt{10}+2\sqrt{5}+\sqrt{2}}{9}$

101. $\dfrac{x^2+2x\sqrt{y}+y}{x^2-y}$

102. $\dfrac{x^2-2x\sqrt{y}+y}{x^2-y}$

103. $\dfrac{a-2\sqrt{ab}+b}{a-b}$

104. $\dfrac{ab+2\sqrt{2ab}+2}{ab-2}$

105. $\dfrac{5\sqrt{x}+2x}{25-4x}$

106. $\dfrac{\sqrt{x}+y}{x-y^2}$

11.2 Radical Equations

Solve.

1. $\sqrt{x} = 2$
2. $\sqrt{x} = 7$
3. $\sqrt{x} + 7 = 8$
4. $\sqrt{x} + 4 = 9$
5. $\sqrt{x} + 6 = 3$
6. $\sqrt{x} + 2 = 1$
7. $5\sqrt{x} - 1 = 0$
8. $3\sqrt{x} - 2 = 0$
9. $\sqrt{x - 3} = 3$
10. $\sqrt{x + 5} = 6$
11. $\sqrt{3x + 1} = 2$
12. $\sqrt{5x - 4} = 4$
13. $\sqrt{7x + 4} + 6 = 11$
14. $\sqrt{3x - 5} + 9 = 14$
15. $\sqrt{2x - 1} - 3 = 0$
16. $\sqrt{3x + 1} - 2 = 0$
17. $\sqrt[3]{x} = 2$

18. $\sqrt[3]{x} = 5$

19. $\sqrt[3]{2x+9} = 3$

20. $\sqrt[3]{4x-11} = 1$

21. $\sqrt[3]{5x+7} + 3 = 1$

22. $\sqrt[3]{3x-6} + 5 = 2$

23. $2\sqrt[3]{x+2} - 1 = 0$

24. $2\sqrt[3]{2x-3} - 1 = 0$

25. $\sqrt{8x+11} = \sqrt{3x+1}$

26. $2\sqrt{3x-4} = \sqrt{2(3x+1)}$

27. $\sqrt{2(x+10)} = \sqrt{7x-15}$

28. $\sqrt{5(x-4)} = \sqrt{x+4}$

29. $\sqrt[3]{5x-2} = \sqrt[3]{4x}$

30. $\sqrt[3]{9(x-1)} = \sqrt[3]{3(x+7)}$

31. $\sqrt[3]{3x+1} = \sqrt[3]{2(x-1)}$

32. $\sqrt[3]{9x} = \sqrt[3]{3(x-6)}$

33. $\sqrt{4x+21} = x$

34. $\sqrt{8x+9} = x$

35. $\sqrt{4(2x-3)} = x$

36. $\sqrt{3(4x-9)} = x$

37. $2\sqrt{x-1}=x$

38. $3\sqrt{2x-9}=x$

39. $\sqrt{9x+9}=x+1$

40. $\sqrt{3x+10}=x+4$

41. $\sqrt{x-1}=x-3$

42. $\sqrt{2x-5}=x-4$

43. $\sqrt{16-3x}=x-6$

44. $\sqrt{7-3x}=x-3$

45. $3\sqrt{2x+10}=x+9$

46. $2\sqrt{2x+5}=x+4$

47. $3\sqrt{x-1}-1=x$

48. $2\sqrt{2x+2}-1=x$

49. $\sqrt{10x+41}-5=x$

50. $\sqrt{6(x+3)}-3=x$

51. $\sqrt{8x^2-4x+1}=2x$

52. $\sqrt{18x^2-6x+1}=3x$

53. $5\sqrt{x+2}=x+8$

54. $4\sqrt{2(x+1)}=x+7$

55. $\sqrt{x^2-25}=x$

56. $\sqrt{x^2+9}=x$

57. $3+\sqrt{6x-11}=x$

58. $2+\sqrt{9x-8}=x$

59. $\sqrt{4x+25}-x=7$

60. $\sqrt{8x+73}-x=10$

61. $2\sqrt{4x+3}-3=2x$

62. $2\sqrt{6x+3}-3=3x$

63. $2x-4=\sqrt{14-10x}$

64. $3x-6=3\sqrt{3-24x}$

65. $\sqrt[3]{x^2-24}=1$

66. $\sqrt[3]{x^2-54}=3$

67. $\sqrt[3]{x^2+6x}+1=4$

68. $\sqrt[3]{x^2+2x}+5=7$

69. $\sqrt[3]{25x^2-10x-7}=-2$

70. $\sqrt[3]{9x^2-12x-23}=-3$

71. $\sqrt{2x^2-15x+25}=\sqrt{(x+5)(x-5)}$

72. $\sqrt{x^2-4x+4}=\sqrt{x(5-x)}$

73. $\sqrt[3]{2(x^2+3x-20)}=\sqrt[3]{(x+3)^2}$

74. $\sqrt[3]{3x^2+3x+40}=\sqrt[3]{(x-5)^2}$

75. $x^{1/2}-10=0$

76. $x^{1/2}-6=0$

77. $x^{1/3}+2=0$

78. $x^{1/3} + 4 = 0$

79. $(x - 1)^{1/2} - 3 = 0$

80. $(x + 2)^{1/2} - 6 = 0$

81. $(2x - 1)^{1/3} + 3 = 0$

82. $(3x - 1)^{1/3} - 2 = 0$

83. $(4x + 15)^{1/2} - 2x = 0$

84. $(3x + 2)^{1/2} - 3x = 0$

85. $(2x + 12)^{1/2} - x = 6$

86. $(4x + 36)^{1/2} - x = 9$

87. $2(5x + 26)^{1/2} = x + 10$

88. $3(x - 1)^{1/2} = x + 1$

89. The square root of 1 less than twice a number is equal to 2 less than the number. Find the number.

90. The square root of 4 less than twice a number is equal to 6 less than the number. Find the number.

91. The square root of twice a number is equal to one-half of that number. Find the number.

92. The square root of twice a number is equal to one-third of that number. Find the number.

93. The distance, d, measured in miles, that a person can see an object is given by the formula $d = \sqrt{\frac{3h}{2}}$ where h represents the person's height above sea level, measured in feet. How high must a person be to see an object 5 miles away?

94. The current, I, measured in amperes, is given by the formula $I = \sqrt{\frac{P}{R}}$ where P is the power usage, measured in watts, and R is the resistance, measured in ohms. If a light bulb requires $1/2$ ampere of current and uses 60 watts of power, then what is the resistance of the bulb?

The period, T, of a pendulum in seconds is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L represents the length in feet. For each problem below, calculate the length of a pendulum, given the period. Give the exact value and the approximate value rounded off to the nearest tenth of a foot.

95. 1 second

96. 2 seconds

97. $\frac{1}{2}$ second

98. $\frac{1}{3}$ second

The time, t, in seconds, an object is in free fall is given by the formula

$$s = 16t^2$$

where s represents the distance in feet the object has fallen. For each problem below, calculate the distance an object falls, given the amount of time.

99. 1 second

100. 2 seconds

101. $\frac{1}{2}$ second

102. $\frac{1}{4}$ second

The x-intercepts for any graph have the form $(x, 0)$, where x is a real number. Therefore, to find x -intercepts, set $y = 0$ and solve for x. Find the x-intercepts for each of the following.

103. $y = \sqrt{x - 3} - 1$

104. $y = \sqrt{x + 2} - 3$

105. $y = \sqrt[3]{x - 1} + 2$

106. $y = \sqrt[3]{x + 1} - 3$

Show Solutions

1. 4
2. 49
3. 1
4. 25
5. $\varnothing$
6. $\varnothing$
7. $\frac{1}{25}$
8. $\frac{4}{9}$
9. 12
10. 31
11. 1
12. 4
13. 3
14. 10
15. 5
16. 1
17. 8

18. 125
19. 9
20. 3
21. -3
22. -7
23. $-\frac{15}{8}$
24. $\frac{25}{16}$
25. $\varnothing$
26. 3
27. 7
28. 6
29. 2
30. 5
31. -3
32. -3
33. 7
34. 9
35. $2,\ 6$
36. $3,\ 9$

37. 2

38. 9

39. $-1,\ 8$

40. $-3,\ -2$

41. 5

42. 7

43. $\varnothing$

44. $\varnothing$

45. $-3,\ 3$

46. $-2,\ 2$

47. $2,\ 5$

48. $-1,\ 7$

49. $4,\ -4$

50. $-3,\ 3$

51. $\frac{1}{2}$

52. $\frac{1}{3}$

53. $2,\ 7$

54. $1,\ 17$

55. $\varnothing$

56. $\varnothing$

57. 10

58. 12

59. $-6,\ -4$

60. $-9,\ -3$

61. $-\frac{1}{2},\ \frac{3}{2}$

62. $-\frac{1}{3},\ 1$

63. $\varnothing$

64. $\varnothing$

65. $-5,\ 5$

66. $-9,\ 9$

67. $-9,\ 3$

68. $-4,\ 2$

69. $\frac{1}{5}$

70. $\frac{2}{3}$

71. $5,\ 10$

72. $\frac{1}{2},\ 4$

73. $-7,\ 7$

74. $-5, \ -\frac{3}{2}$

75. 100

76. 36

77. -8

78. -64

79. 10

80. 34

81. -13

82. 3

83. $\frac{5}{2}$

84. $\frac{2}{3}$

85. $-6, \ -4$

86. $-9, \ -5$

87. $-2, \ 2$

88. $2, \ 5$

89. 5

90. 10

91. $8, \ 0$

92. $18, \ 0$

93. $16\frac{2}{3}$ feet

94. 240 ohms

95. $\frac{8}{\pi^2} \approx 0.8$ foot

96. $\frac{32}{\pi^2} \approx 3.2$ feet

97. $\frac{2}{\pi^2} \approx 0.2$ foot

98. $\frac{8}{9\pi^2} \approx 0.1$ foot

99. 16 feet

100. 64 feet

101. 4 feet

102. 1 foot

103. $(4, 0)$

104. $(7, 0)$

105. $(-7, 0)$

106. $(26, 0)$

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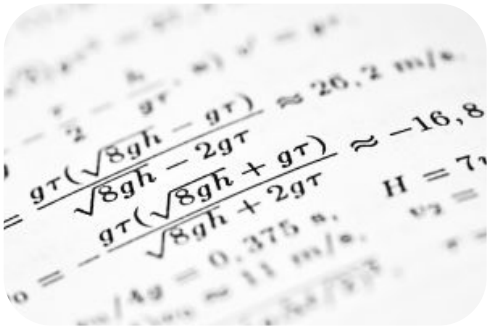
CHAPTER 12: EQUATIONS IN QUADRATIC FORM

Chapter Outline

12.1 Equations in Quadratic Form

- Quadratic Form
- Solving Rational Equations Resulting in a Quadratic

12.2 Exercises



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12.1 Equations in Quadratic Form

Learning Objectives

By the end of this section, you should be able to:

- Solve equations in quadratic form

Solving Equations in Quadratic Form

Equations in quadratic form are equations with three terms. The first term has a power other than 2. The middle term has an exponent that is one-half the exponent of the leading term. The third term is a constant. We can solve equations in this form as if they were quadratic. A few examples of these equations are:

$$x^4 - 5x^2 + 4 = 0$$
$$x^6 + 7x^3 - 8 = 0$$
$$x^{\frac{2}{3}} + 4x^{\frac{1}{3}} + 2 = 0$$

In each one, doubling the exponent of the middle term equals the exponent on the leading term. We can solve these equations by substituting a variable for the middle term.

Quadratic Form

If the exponent on the middle term is one-half of the exponent on the leading term, we have an equation in quadratic form, which we can solve as if it were a quadratic. We substitute a variable for the middle term to solve equations in quadratic form.

How To

Given an equation in quadratic form, solve it.

1. Identify the exponent on the leading term and determine whether it is double the exponent on the middle term.
2. If it is, substitute a variable, such as u, for the variable portion of the middle term.

3. Rewrite the equation so that it takes on the standard form of a quadratic.
4. Solve using one of the usual methods for solving a quadratic.
5. Replace the substitution variable with the original term.
6. Solve the remaining equation.

Example 1

Solve this fourth-degree equation:

$$3x4 - 2x^2 - 1 = 0$$

Solution:

This equation fits the main criteria, that the power on the leading term is double the power on the middle term. Next, we will make a substitution for the variable term in the middle. Let $u = x^2$. Rewrite the equation in u.

$$3u^2 - 2u - 1 = 0$$

Now solve the quadratic.

$$\begin{aligned} 3u^2 - 2u - 1 &= 0 \\ (3u + 1)(u - 1) &= 0 \end{aligned}$$

Solve each factor and replace the original term for u.

$$\begin{aligned} 3u + 1 &= 0 \\ 3u &= -1 \\ u &= -\frac{1}{3} \\ x^2 &= -\frac{1}{3} \\ x &= \pm i\sqrt{\frac{1}{3}} \end{aligned}$$

$$\begin{aligned} u - 1 &= 0 \\ u &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

The solutions are $\pm i\sqrt{\frac{1}{3}}$ and ± 1.

Answer

$\pm i\sqrt{\frac{1}{3}}$ and ± 1

Try It

Solve using substitution: $x^4 - 8x^2 - 9 = 0$.

Show Solution

$x = \pm 3$ and $x = \pm i$

Example 2

Solve the equation in quadratic form:

$$(x+2)^2 = 11(x+2) - 12 = 0$$

Solution:

This equation contains a binomial in place of the single variable. The tendency is to expand what is presented. However, recognizing that it fits the criteria for being in quadratic form makes all the difference in the solving process. First, make a substitution, letting $u = x + 2$. Then rewrite the equation in u.

$$u^2 + 11u - 12 = 0$$
$$(u+12)(u-1) = 0$$

Solve using the zero-factor property and then replace u with the original expression.

$$u + 12 = 0$$
$$u = -12$$
$$x + 2 = 12$$
$$x = -14$$

The second factor results in

$$u - 1 = 0$$
$$u = 1$$
$$x + 2 = 1$$
$$x = -1$$

We have two solutions: $-14 \text{ and } -1$

Answer

$-14 \text{ and } -1$

Try It

Solve: $(x-5)^2 - 4(x-5) - 21 = 0$

Show Solution

$x = 12 \text{ and } x = 2$

Solving Rational Equations Resulting in a Quadratic

Earlier, we solved rational equations. Sometimes, solving a rational equation results in a quadratic. When this happens, we continue the solution by simplifying the quadratic equation using one of the methods we have seen. It may turn out that there is no solution.

Example 3

Solve the following rational equation:

$$\frac{-4x}{x-1}+\frac{4}{x+1}=\frac{-8}{x^2-1}$$

Solution:

We want all denominators in factored form to find the LCD. Two of the denominators cannot be factored further. However, $x^2-1=(x+1)(x-1)$. Then, the LCD is $(x+1)(x-1)$.

Next, we multiply the whole equation by the LCD.

$$\begin{aligned}(x+1)(x-1)[\frac{-4x}{x-1}+\frac{4}{x+1}] &= [\frac{-8}{(x+1)(x-1)}](x+1)(x-1)\\ (-4x(x+1)+4(x-1) &= -8\\ (-4x^2-4x+4x-4 &= -8\\ (-4x^2+4 &= 0\\ (-4(x^2-1) &= 0\\ (-4(x+1)(x-1) &= 0\\ x &= -1\\ x &= 1\end{aligned}$$

In this case, either solution produces a zero in the denominator in the original equation. Thus, there is no solution.

Answer

No Solution

Try It

Solve: $\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$

Show Solution

$-1, 0$ is not a solution

Key Takeaways

- Equations in quadratic form are easy to spot, as the exponent on the first term is double the exponent on the second term, and the third term is a constant. We may also see a binomial in place of the single variable. We use substitution to solve.
- Solving a rational equation may also lead to a quadratic equation or an equation in quadratic form.

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12.2 Exercises

12.1 Equations in Quadratic Form

For the following exercises, solve the equation by identifying the quadratic form. Use a substitute variable and find all real solutions by factoring.

1. $x^4 - 10x^2 + 9 = 0$
2. $4(t-1)^2 - 9(t-1) = -2$
3. $(x^2-1)^2 + (x^2-1) - 12 = 0$
4. $(x+1)^2 - 8(x+1) - 9 = 0$
5. $(x-3)^2 - 4 = 0$

Show Solutions

1. $x = 1, -1, 3, -3$
2. $t = \frac{5}{4}, 3$
3. $x = 2, -2$
4. $x = 8, -2$
5. $x = 1, 5$

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CHAPTER 13: LOGARITHMS

Chapter Outline

13.1 Properties of Logarithms

- Properties of Logarithms
- Inverse Properties of Logarithms
- Product Property of Logarithms
- Quotient Property of Logarithms
- Power Property of Logarithms
- Summary of Properties
- Change-of-Base Formula

13.2 Exercises

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13.1 Properties of Logarithms

Learning Objectives

By the end of this section, you should be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

Properties of Logarithms

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0 = 1$, we can convert this to logarithmic form and get $\log_a 1 = 0$. Also, since $a^1 = a$, we get $\log_a a = 1$.

$$\log_a 1 = 0 \quad \log_a a = 1$$

In the next example, we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

Example 1

Evaluate using the properties of logarithms:

a. $\log_8 1$
b. $\log_6 6$

Solution:

a.

$$\log_8 1$$

Use the property, $\log_a 1 = 0$.

$$0 \quad \log_8 1 = 0$$

b.

$$\log_6 6$$

Use the property, $\log_a a = 1$.

$$1 \quad \log_6 6 = 1$$

Answer

a. $0 \quad \log_8 1 = 0$
b. $1 \quad \log_6 6 = 1$

Try It

Evaluate using the properties of logarithms:

a. $\log_{13} 1$
b. $\log_9 9$

Show Solution

a. 0
b. 1

Try It

Evaluate using the properties of logarithms:

a. $\log_5 1$

b. $\log_7 7$

Show Solution

a. 0
b. 1

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $a^{\log_a x} = x$ converts to the logarithmic equation $\log_a x = \log_a x$, which is a true statement for positive values for x only.

The logarithmic equation $\log_a a^x = x$ converts to the exponential equation $a^x = a^x$, which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power. These two properties show the composition of functions. Both ended up with the **identity** function, which shows again that the exponential and logarithmic functions are inverse functions.

Inverse Properties of Logarithms

For $a > 0, x > 0$ and $a \neq 1$,

$$a^{\log_a x} = x \quad \log_a a^x = x$$

In the next example, apply the inverse properties of logarithms.

Example 2

Evaluate using the properties of logarithms:

a. $4^{\log_4 9}$
b. $\log_3 3^5$

Solution:

a.

$$4^{\log_4 9}$$

Use the property, $a^{\log_a x} = x$.

$$9 \quad 4^{\log_4 9} = 9$$

b.

$$\log_3 3^5$$

Use the property, $\log_a(a^x) = x$.

$$5 \quad \log_3 3^5 = 5$$

Answer

a. $9 \quad 4^{\log_4 9} = 9$
b. $5 \quad \log_3 3^5 = 5$

Try It

Evaluate using the properties of logarithms:

a. $5^{\log_5 15}$
b. $\log_7 7^4$

Show Solution

a. 15
b. 4

Try It

Evaluate using the properties of logarithms:

a. $2^{\log_2 8}$
b. $\log_2 2^{15}$

Show Solution

a. 8
b. 15

There are three more properties of logarithms that will be useful in our work. We know exponential functions and logarithmic functions are very interrelated. Our definition of a logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $a^m \cdot a^n = a^{m+n}$, we see that to multiply the same base, we add the exponents. The **Product Property** of Logarithms, $\log_a(M \cdot N) = \log_a M + \log_a N$ tells us to take the log of a product, we add the log of the factors.

Try It

Apply what you've just learned here:

Activity Description
Match the options to the questions.

For the equation:

$$\log_5 25 = x$$

1. The argument is:
2. The base is:
3. The answer is:
4. The question you are asking is to what _____ do we raise the _____ to get 25?

Options:

$$\text{base} \qquad 5 \qquad \text{exponent} \qquad 25 \qquad x$$

5. log_{2}1=_____ because _____.
6. log_{3}3=_____ because _____.

Options:

$$2 \qquad 1 \qquad 3^1 = 3 \qquad 3 \qquad 2^0 = 1 \qquad 0$$

7. $2^x = 8$ is the same as _____.
8. $e^t = 7$ is the same as _____.
9. $3^k = 81$ is the same as _____.

Options:

$$\log_e 7 = t \qquad \log_2 8 = x \qquad \log_3 81 = k \qquad \log_2 16 = x \qquad \log_3 27 = k$$

1. $\log_5 25 = x$ is the same as _____.
2. $\log 1000 = x$ is the same as _____.

Options:

$$5^x = 125 \qquad 5^x = 25 \qquad 10^x = 1000 \qquad 1000^x = 10$$

Answers:

1. 25
2. 5
3. x
4. exponent, base
5. $0, 2^0 = 1$
6. $1, 3^1 = 3$
7. $\log_2 8 = x$
8. $\log_e 7 = t$
9. $\log_3 81 = k$
10. $5^x = 25$
11. $10^x = 1000$

Product Property of Logarithms

If $M > 0, N > 0, \text{a} > 0$ and $\text{a} \neq 1,$ then

$$\log_a(M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

How To

Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Example 3

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

a. $\log_3 7x$
b. $\log_4 64xy$

Solution:

a.

$$\log_3 7x$$

Use the Product Property, $\log_a(M \cdot N) = \log_a M + \log_a N$.

$$\log_3 7 + \log_3 x$$

$$\log_3 7x = \log_3 7 + \log_3 x$$

b.

$$\log_4 64xy$$

Use the Product Property, $\log_a(M \cdot N) = \log_a M + \log_a N$.

$$\log_4 64 + \log_4 x + \log_4 y$$

Simplify be evaluating, $\log_4 64$.

$$3 + \log_4 x + \log_4 y$$
$$\log_4 64xy = 3 + \log_4 x + \log_4 y$$

Answer

a. $\log_3 7 + \log_3 x$
b. $3 + \log_4 x + \log_4 y$

Try It

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

a. $\log_3 3x$
b. $\log_2 8xy$

Show Solution

a. $1 + \log_3 x$
b. $3 + \log_2 x + \log_2 y$

Try It

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

a. $\log_9 9x$
b. $\log_3 27xy$

Show Solution

a. $1 + \log_9 x$
b. $3 + \log_3 x + \log_3 y$

Similarly, in the **Quotient Property** of Exponents, $\frac{a^m}{a^n} = a^{m-n}$, we see that to divide by the same base, we subtract the exponents. The Quotient Property of Logarithms, $\log_a \frac{M}{N} = \log_a M - \log_a N$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Quotient Property of Logarithms

If $M > 0, N > 0, \text{a} > 0$ and $\text{a} \neq 1,$ then

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_a M = \log_a N \neq \log_a (M - N)$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

How To

Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1. Express the argument in lowest terms by factoring the numerator and denominator and cancelling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Example 4

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_5 \frac{5}{7}$

b. $\log \frac{x}{100}$

Solution:

a.

$$\log_5 \frac{5}{7}$$

Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.

$$\log_5 5 - \log_5 7$$

Simplify.

$$1 - \log_5 7$$

$$\log_5 \frac{5}{7} = 1 - \log_5 7$$

b.

$$\log \frac{x}{100}$$

Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.

$$\log x - \log 100$$

Simplify.

$$\log \frac{x}{100} = \log x - 2$$

Answer

a. $1 - \log_5 7$
b. $\log x - 2$

Try It

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_4 \frac{3}{4}$
b. $\log \frac{x}{1000}$

Show Solution

a. $\log_4 3 - 1$
b. $\log x - 3$

Try It

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_2 \frac{5}{4}$

b. $\log \frac{10}{y}$

Show Solution

a. $\log_2 5 - 2$

b. $1 - \log y$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$. We see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_a M^p = p \log_a M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Power Property of Logarithms

If $M > 0, \mathrm{a} > 0, \mathrm{a} \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

How To

Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Example 5

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_5 4^3$
b. $\log x^{10}$

Solution:

a.

$$\log_5 4^3$$

Use the Power Property, $\log_a M^p = p \log_a M$.

$$3 \log_5 4$$

$$log_5 4^3 = 3 \log_5 4$$

b.

$$\log x^{10}$$

Use the Power Property, $\log_a M^p = p \log_a M$.

$$10 \log x$$

$$\log x^{10} = 10 \log x$$

Answer

a. $3 \log_5 4$
b. $10 \log x$

Try It

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_7 5^4$
b. $\log x^{100}$

Show Solution

a. $4 \log_7 5$
b. $100 \cdot \log x$

Try It

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_2 3^7$
b. $\log x^{20}$

Show Solution

a. $7 \log_2 3$
b. $20 \cdot \log x$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Try It

Try these questions to gauge your understanding:

Activity Description
Match the options to the correct box.

1. $\log_2 16 =$
2. $\log_2(\frac{1}{8}) =$
3. $\log_e(e^5) =$

Options:

$$0 \quad -3 \quad 3 \quad 2 \quad 4 \quad 5$$

Simplify as much as possible. Express as a sum or difference of logs.

4. $\log_3 \sqrt{6} = \square$
5. $\log 3000 = \square, \square, \square$
6. $\log_2(\frac{4}{7}) = \square, \square, \square$

Options:

$$\log 3 \quad 3 \quad \log_3 \sqrt{2} + \log_3 \sqrt{3} \quad \frac{1}{2}\log_3 2 + \frac{1}{2} \quad + \quad - \quad 2 \quad \log_2 7$$

Expand these as much as possible.

7. $\log_7 \dfrac{11}{9} =$
8. $8 \log_7 \sqrt[4]{3^3} =$
9. $\log_2 \dfrac{\sqrt{x}}{a^5} =$
10. $\log_2 \dfrac{x^{-3}}{\sqrt{y}} =$

Options:

$$\tfrac{1}{2}\log_3 x + 5\log_3 a \qquad \tfrac{1}{2}\log_3 x - 5\log_3 a \qquad -3\log_2 x - \tfrac{1}{2}\log_2 y$$
$$\log_7 11 - 2\log_7 3 \qquad 8\log_7 3^{\frac{3}{4}} \qquad 6\log_7 3$$

Condense into a single log.

11. $\log_2 4 + \log_2 R =$
12. $-\log_b \sqrt{x} + \log_b x^2 =$
13. $2\log_e 2 + 3\log_e \pi - \log_e 3 =$

Options:

$$\log_3 \frac{2\pi^3}{3} \qquad 2\log_2 4R \qquad \log_3 \frac{4\pi^3}{3} \qquad \frac{3}{2}\log x \qquad 2\log_2 2R^2$$

Change the base, then use your calculator to find a value if possible.

14. $\log_2 x^4 = \square/\square$
15. $\log_5 \sqrt{11} = \square/\square = 0.75$
16. $\log_7 \dfrac{8}{3} = \square/\square = 0.5$

Options:

$$\log_2 \qquad \log \frac{8}{3} \qquad \log \sqrt{11} \qquad \log 7 \qquad \log 5 \qquad \log x^4$$

Answers:

1. 4
2. -3
3. 5
4. $\dfrac{1}{2}\log_3 2 + \dfrac{1}{2}$

5. $3, +, log 3$
6. $2, -, log_2 7$
7. $\log_7 11 - 2\log_7 3$
8. $6\log_7 3$
9. $\frac{1}{2}\log_3 x + 5\log_3 a$
10. $-3\log_2 x - \frac{1}{2}\log_2 y$
11. $2\log_2 4R$
12. $\frac{3}{2}\log x$
13. $\log_3 \frac{4\pi^3}{3}$
14. $\log x^4, \log 2$
15. $\log \sqrt{11}, \log 5$
16. $\log \frac{8}{3}, \log 7$

Summary of Properties

If $M > 0, \text{a} > 0, \text{a} \neq 1$ and p is any real number then,

Table 13.1.1 Properties of Logarithms

Property	Base a	Base e
	$\log_a 1 = 0$ $\log_a a = 1$	$\ln 1 = 0$ $\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a(M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p\log_a M$	$\ln M^p = p\ln M$

Now that we have the properties, we can use them to “expand” a logarithmic expression. This means to write the logarithm as a sum or difference, and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

Example 6

Use the Properties of Logarithms to expand the logarithm:

$$\log_4\left(2x^3y^2\right)$$

Simplify, if possible.

Solution:

Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$.

Use the Power Property, $\log_a M^p = p\log_a M$, on the last two terms. Simplify.

Answer

$$\log_4\left(2x^3y^2\right) = \frac{1}{2} + 3\log_4(x) + 2\log_4(y)$$

Try It

Use the Properties of Logarithms to expand the logarithm: $\log_2\left(5x^4y^2\right)$. Simplify, if possible.

Show Solution

$$\log_2 5 + 4\log_2 x + 2\log_2 y$$

Try It

Use the Properties of Logarithms to expand the logarithm: $\log_3\left(7x^5y^3\right)$. Simplify, if possible.

Show Solution

$$\log_3 7 + 5\log_3 x + 3\log_3 y$$

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

Example 7

Use the Properties of Logarithms to expand the logarithm:

$$\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$$

Simplify, if possible.

Solution:

$$\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$$

Rewrite the radical with a rational exponent.

$$\log_2 \left(\frac{x^3}{3y^2z}\right)^{\frac{1}{4}}$$

Use the Power Property, $\log_a M^p = p\log_a M$.

$$\frac{1}{4}\log_2\left(\frac{x^3}{3y^2z}\right)$$

Use the Quotient Property, $\log_a M\,N = \log_a M - \log_a N$.

$$\frac{1}{4}\left(\log_2\left(x^3\right) - \log_2\left(3y^2z\right)\right)$$

Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$, in the second term.

$$\frac{1}{4}\left(\log_2\left(x^3\right) - \left(\log_2 3 + \log_2 y^2 + \log_2 z\right)\right)$$

Use the Power Property, $\log_a M^p = p\log_a M$, inside the parentheses.

$$\frac{1}{4}\left(3\log_2 x - \left(\log_2 3 + 2\log_2 y + \log_2 z\right)\right)$$

Simplify by distributing.

$$\frac{1}{4}\left(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z\right)$$

$$log_2\sqrt[4]{\frac{x^3}{3y^2z}} = \frac{1}{4}\left(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z\right)$$

Answer

$$\log_2\sqrt[4]{\frac{x^3}{3y^2z}} = \frac{1}{4}\left(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z\right)$$

Try It

Use the Properties of Logarithms to expand the logarithm $\log_4\sqrt[5]{\frac{x^4}{2y^3z^2}}$. Simplify, if possible.

Show Solution

$$\frac{1}{5}\left(4\log_4 x - \frac{1}{2} - 3\log_4 y - 2\log_4 z\right)$$

Try It

Use the Properties of Logarithms to expand the logarithm $\log_3 \sqrt[3]{\frac{x^2}{5yz}}$. Simplify, if possible.

Show Solution

$$\frac{1}{3}(2\log_3 x - \log_3 5 - \log_3 y - \log_3 z)$$

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

How To

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.

2. Next, apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Example 8

Use the Properties of Logarithms to condense the logarithm:

$$\log_4 3 + \log_4 x - \log_4 y$$

Simplify, if possible.

Solution:

The log expressions all have the same base, 4.

The first two terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.

Since the logs are subtracted, we use the Quotient Property, $\log_a M - \log_a N = \log_a \frac{M}{N}$.

Answer

$$\log_4 \frac{3x}{y}$$

Try It

Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

Show Solution

$$\log_2 \frac{5x}{y}$$

Try It

Use the Properties of Logarithms to condense the logarithm $\log_3 6 - \log_3 x - \log_3 y$. Simplify, if possible.

Show Solution

$$\log_3 \frac{6}{xy}$$

Example 9

Use the Properties of Logarithms to condense the logarithm:

$$2\log_3 x + 4\log_3(x+1)$$

Simplify, if possible.

Solution:

The log expressions have the same base, 3.

$$2\log_3 x + 4\log_3(x+1)$$

Use the Power Property, $\log_a M^p = p\log_a M$.

$$\log_3 x^2 + \log_3 (x+1)^4$$

The terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a (M \cdot N)$.

$$\log_3 x^2 (x+1)^4$$

$$(2\log_3 x + 4\log_3 (x+1) = \log_3 x^2 (x+1)^4$$

Answer

$\log_3 x^2 (x+1)^4$

Try It

Use the Properties of Logarithms to condense the logarithm: $3\log_2 x + 2\log_2 (x-1)$. Simplify, if possible.

Show Solution

$\log_2 x^3 (x-1)^2$

Try It

Use the Properties of Logarithms to condense the logarithm $2\log x + 2\log (x+1)$. Simplify, if possible.

Show Solution

$\log x^2 (x+1)^2$

Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the Change-of-Base Formula. We will show how this is derived.

Suppose we want to evaluate $\log_a M$	$\log_a M$
Let $y = \log_a M$.	$y = \log_a M$
Rewrite the expression in exponential form.	$a^y = M$
Take the $\log_b$ of each side.	$\log_b a^y = \log_b M$
Use the Power Property.	$y \log_b a = \log_b M$
Solve for y.	$y = \dfrac{\log_b M}{\log_b a}$
Substitute $y = \log_a M$.	$\log_a M = \dfrac{\log_b M}{\log_b a}$

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b > 0, b \neq 1$. Because our calculators have keys for logarithms base 10 and base e, we will rewrite the Change-of-Base Formula with the new base as 10 or e.

Change-of-Base Formula

For any logarithmic bases a, b and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \quad \log_a M = \frac{\log M}{\log a} \quad \log_a M = \frac{\ln M}{\ln a}$$

new base b | new base 10 | new base e

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value, and so we use the approximately equal symbol $(\approx)$.

How To

Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1. Determine the new base n, remembering that the common log, $log(x)$, has base 10, and the natural log, $\ln(x)$, has base e.
2. Rewrite the log as a quotient using the change-of-base formula
3. The numerator of the quotient will be a logarithm with base n and argument M.
4. The denominator of the quotient will be a logarithm with base n and argument b.

Example 10

Rounding to three decimal places, approximate:

$$\log_4 35$$

Solution:

Begin

$$\log_4 35$$

Use the Change-of-Base Formula.

$$\log_4 M = \frac{\log_b M}{\log_b a}$$

Identify a and M. Choose 10 for b.

$$\log_4 35 = \frac{\log 35}{\log 4}$$

Enter the expression $\frac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places.

$$\log_4 35 \approx 2.565$$

Answer

$\log_4 35 \approx 2.565$

Try It

Rounding to three decimal places, approximate: $\log_3 42$.

Show Solution

3.402

Try It

Rounding to three decimal places, approximate: $\log_5 46$.

Show Solution

2.379

Access these online resources for additional instruction and practice in using the properties of logarithms.

- Using Properties of Logarithms to Expand Logs
- Using Properties of Logarithms to Condense Logs
- Change of Base

Key Takeaways

- $\log_a 1 = 0 \quad \log_a a = 1$
- **Inverse Properties of Logarithms**
  - For $a > 0, x > 0$ and $a \neq 1 a^{\log_a x} = x \quad \log_a a^x = x$

- **Product Property of Logarithms**
  - If $M > 0, N > 0, a > 0$ and $a \neq 1$, then, $\log_a(M \cdot N) = \log_a M + \log_a N$. The logarithm of a product is the sum of the logarithms.
- **Quotient Property of Logarithms**
  - If $M > 0, N > 0, \mathrm{a} > 0$ and $a \neq 1$, then $\log_a \frac{M}{N} = \log_a M - \log_a N$. The logarithm of a quotient is the difference of the logarithms.
- **Power Property of Logarithms**
  - If $M > 0, a > 0, a \neq 1$ and p is any real number, then $\log_a M^p = p \log_a M$. The log of a number raised to a power is the product of the power times the log of the number.
- **Properties of Logarithms Summary**

  If $M > 0, a > 0, a \neq 1$ and p is any real number then,

Table 13.1.3 Properties of Logarithms (Repeated)

Property	Base a $\log_a 1 = 0$ $\log_a a = 1$	Base e $\ln 1 = 0$ $\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a(M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

- **Change-of-Base Formula**

  For any logarithmic bases a and b, and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \quad \log_a M = \frac{\log M}{\log a} \quad \log_a M = \frac{\ln M}{\ln a}$$

$$\text{new base } b \qquad \text{new base } 10 \qquad \text{new base } e$$

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How To: "6.5 Logarithmic Properties" from College Algebra 2e by OpenStax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted. Modifications: Used How To boxes "Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms", "Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms", "Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm", "Given a sum, difference, or product of logarithms with the same

base, write an equivalent expression as a single logarithm", & "Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$"; Reformatted.

13.2 Exercises

13.1 Properties of Logarithms

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

1. $\log_b(7x \cdot 2y)$
2. $\ln(3ab \cdot 5c)$
3. $\log_b(\frac{13}{17})$
4. $\log_4(\frac{\frac{x}{y}}{w})$
5. $\ln(\frac{1}{4^k})$
6. $\log_2(y^x)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

7. $\log(\frac{x^{15}y^{13}}{z^{19}})$
8. $\ln(\frac{a^{-2}}{b^{-4}c^5})$
9. $\log(\sqrt{x^3y^{-4}})$
10. $\ln(y\sqrt{\frac{y}{1-y}})$
11. $\log(x^2y^3\sqrt[3]{x^2y^5})$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

12. $\log(2x^4) + \log(3x^5)$

13. $\ln(6x^9) - \ln(3x^2)$

14. $2\log(x) + 3\log(x+1)$

15. $\log(x) - \frac{1}{2}log(y) + 3\log(z)$

16. $4\log_7(c) + \frac{\log_7(a)}{3} + \frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

17. $\log_7(15)$ to base e

18. $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6) = a$ and $\log_5(11) = b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

19. $\log_{11}(5)$

20. $\log_6(55)$

21. $\log_{11}(\frac{6}{11})$

For the following exercises, use properties of logarithms to evaluate without using a calculator.

22. $\log_3(\frac{1}{9}) - 3\log_3(3)$

23. $6\log_8(2) + (\frac{\log_8(64)}{3\log_8(4)})$

24. $2\log_9(3) - 4\log_9(3) + \log_9(\frac{1}{729})$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

25. $\log_3(22)$

26. $\log_8(65)$

27. $\log_6(5.38)$

28. $\log_4(\frac{15}{2})$

29. $\log_{\frac{1}{2}}(4.7)$

Solve.

30. Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2$. Show the steps for solving.

31. Use the quotient rule for logarithms to find all x values such that $\log_6(x+2)-\log_6(x-3)=1$. Show the steps for solving.

32. Can the power property of logarithms be derived from the power property of exponents using the equation $b^x=m$? If not, explain why. If so, show the derivation.

33. Prove that $\log_b(n)=\dfrac{1}{log_n(b)}$ for any positive integers $b>1$ and $n>1$

34. Does $\log_{81}(2401)=\log_3(7)$? Verify the claim algebraically.

Show Solutions

1. $\log_b(2)+\log_b(7)+\log_b(x)+\log_b(y)$

3. $\log_b(13)-\log_b(17)$

5. $-k\ln(4)$

7. $15\log(x)+13\log(y)-19\log(z)$

9. $\frac{3}{2}\log(x) - 2\log(y)$

11. $\frac{8}{3}\log(x) + \frac{14}{3}\log(y)$

13. $\ln(2x^7)$

15. $\log(\frac{xz^3}{\sqrt{y}})$

17. $\log_7(15) = \frac{\ln(15)}{\ln(7)}$

19. $\log_{11}(5) = \frac{\log_5(5)}{\log_5(11)} = \frac{1}{b}$

21. $\log_{11}(\frac{6}{11} = \frac{\log_5(\frac{6}{11})}{\log_5(11)} = \frac{\log_5(6) - \log_5(11)}{\log_5(11)} = \frac{a-b}{b} = \frac{a}{b} - 1$

23. 3

25. 2.81359

27. 0.93913

29. -2.23266

31. $x = 4$ By the quotient rule: $\log_6(x+2) - \log_6(x-3) = \log_6 \frac{x+2}{x-3} = 1$

Rewriting as an exponential equation and solving for x:

$$
\begin{aligned}
6^1 &= \frac{x+2}{x-3} && \text{(After multiplying both sides by } x-3\text{)} \\
6(x-3) &= x+2 && \text{(Carry out distributive multiplication on left)} \\
6x - 18 &= x+2 && \text{(Add 18 to both sides, subtract } x \text{ from both sides)} \\
5x &= 20 \\
x &= 4
\end{aligned}
$$

Checking, we find that $\log_6(4+2) - \log_6(4-3) = \log_6(6) - \log_6(1)$

33. Let b and n be positive integers greater than 1. Then, by the change-of-base formula,

$$\log_b(n) = \frac{\log_n(n)}{\log_n(b)} = \frac{1}{\log_n(b)}$$

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CHAPTER 14: EXPONENTIAL AND LOG EQUATIONS

Chapter Outline

14.1 Exponential and Log Equations

- Using Like Bases to Solve Exponential Equations

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- Rewriting Equations So All Powers Have the Same Base
- Solving Exponential Equations Using Logarithms
- Equations Containing e
- Extraneous Solutions
- Using the Definition of a Logarithm to Solve Logarithmic Equations
- Using the One-to-One Property of Logarithms to Solve Logarithmic Equations
- Solving Applied Problems Using Exponential and Logarithmic Equations

14.2 Exercises

14.1 Exponential and Log Equations

Learning Objectives

By the end of this section, you should be able to:

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Uncontrolled population growth, as in the wild rabbits in Australia, can be modelled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T $b > 0, b \neq 1, b^S = b^T$ if and only if $S = T$.

In other words, when an exponential equation has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.

For example, consider the equation $3^{4x-7} = \dfrac{3^{2x}}{3}$. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x:

$$3^{4x-7} = \frac{3^{2x}}{3}$$

$$c^{4x-7} = \frac{3^{2x}}{3^1} \qquad \text{Rewrite 3 as } 3^1$$

$$3^{4x-7} = 3^{2x-1} \qquad \text{Use the division property of exponents}$$

$$4x - 7 = 2x - 1 \qquad \text{Apply the one-to-one property of exponents}$$

$$2x = 6 \qquad \text{Subtract } 2x \text{ and add 7 to both sides}$$

$$x = 3 \qquad \text{Divide by 2}$$

How To

Given an exponential equation with the form $b^S = b^T$, where S and T are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example 1

Solve:

$$2^{x-1} = 2^{2x-4}$$

Solution:

$$2^{x-1} = 2^{2x-4} \qquad \text{The common base is 2}$$

$$x - 1 = 2x - 4 \qquad \text{By the one-to-one property the exponents must be equal}$$

$$x = 3 \qquad \text{Solve for } x$$

Answer

$x = 3$

Try It

Solve: $5^{2x} = 5^{3x+2}$

Show Solution

$x = -2$

Rewriting Equations So All Powers Have the Same Base

Sometimes the common base for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x:

$$
\begin{aligned}
256 &= 4^{x-5} && \\
2^8 &= (2^2)^{x-5} && \text{Rewrite each side as a power with base 2} \\
2^8 &= 2^{2x-10} && \text{Use the rules of exponenets} \\
8 &= 2x - 10 && \text{Apply the one-to-one property of exponents} \\
18 &= 2x && \text{Add 10 to both sides} \\
x &= 9 && \text{Divide by 2}
\end{aligned}
$$

How To

Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side of the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S = T$, for the unknown.

Example 2

Solve:

$$8^{x+2} = 16^{x+1}$$

Solution:

$$\begin{aligned}
8^{x+2} &= 16^{x+1} && \\
(2^3)^{x+2} &= (2^4)^{x+1} && \text{Write 8 and 16 as powers of 2} \\
2^{3x+6} &= 2^{4x+4} && \text{To take a power of a power, multiply exponents} \\
3x+6 &= 4x+4 && \text{Use the one-to-one property to set the exponents equal} \\
x &= 2 && \text{Solve for } x
\end{aligned}$$

Answer

$x = 2$

Example 3

Solve

$$2^{5x} = \sqrt{2}$$

Solution:

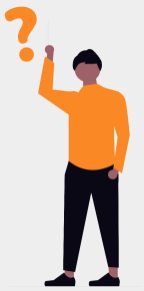
$$\begin{aligned}
2^{5x} &= 2^{\frac{1}{2}} && \text{Write the square root of 2 as a power of 2} \\
5x &= \frac{1}{2} && \text{Use the one-to-one property} \\
x &= \frac{1}{10} && \text{Solve for } x
\end{aligned}$$

Answer

$x = \frac{1}{10}$

Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?

No, every equation does not have a solution. The range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.



Example 4

Solve

$$3^{x+1} = -2$$

Solution:

This equation has no solution. There is no real value of x

The following diagram shows that the two graphs do not cross, so the left side is never equal to the right side. Thus, the equation has no solution.

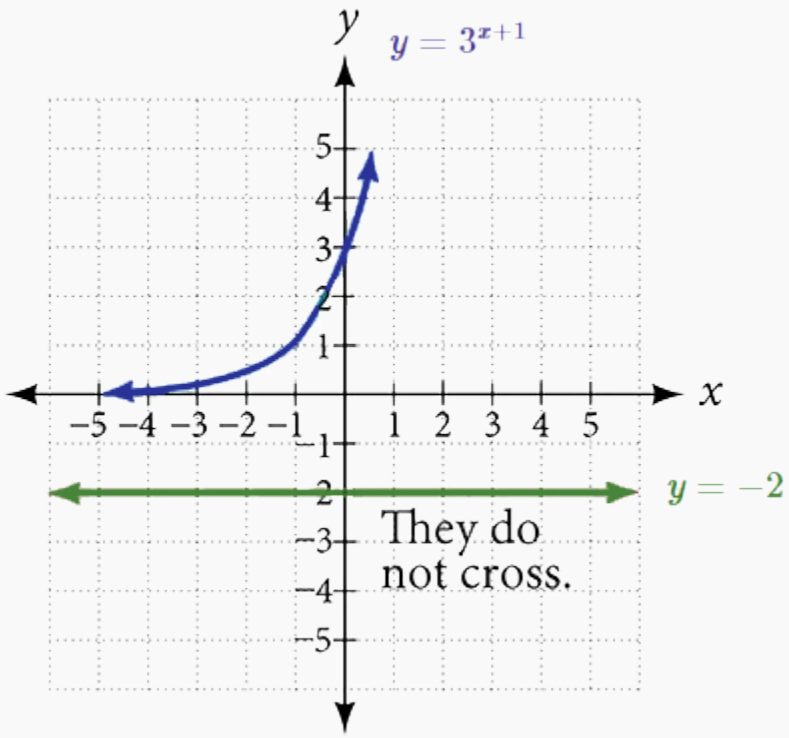


Figure 14.1.1

Answer

No Solution.

Try It

Solve 1: $5^{2x} = 25^{3x+2}$

Solve 2: $5^x = \sqrt{5}$

Solve 3: $2^x = -100$

Show Solution

1. $x = -1$

2. $x = \frac{1}{2}$
3. No Solution.

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

How To

Given an exponential equation in which a common base cannot be found, solve for the unknown.

1. Apply the logarithm to both sides of the equation.
2. If one of the terms in the equation has base 10, use the common logarithm.
3. If none of the terms in the equation has base 10, use the natural logarithm.
4. Use the rules of logarithms to solve for the unknown.

Example 5

Solve

$$5^{x+2} = 4^x$$

Solution:
There is no easy way to get the powers to have the same base.

$$
\begin{aligned}
5^{x+2} &= 4 && \\
\ln(5^{x+2}) &= \ln(4^2) && \text{Take } \ln \text{ of both sides.} \\
(x+2)\ln 5 &= x \ln 4 && \text{Use laws of logs.} \\
x\ln 5 + 2\ln 5 &= x \ln 4 && \text{Use the distributive law.} \\
x\ln 5 - x\ln 4 &= -2\ln 5 && \text{Get terms containing } x \text{on one side.} \\
x(\ln 5 - \ln 4) &= -2\ln 5 && \text{One the left-hand side, factor out an } x. \\
x\ln\left(\frac{5}{4}\right) &= \ln\left(\frac{1}{25}\right) && \text{Use the laws of logs.} \\
x &= \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)} && \text{Dive by the coefficient of } x.
\end{aligned}
$$

Answer

$$x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$$

Try It

Solve $2^x = 3^{x+1}$

Show Solution

$$x = \frac{\ln 3}{\ln(\frac{2}{3})}$$

Science Application

A lab tech is growing a culture of *E. coli*. The population grows according to the formula $P = P_0 e^{rt}$. If the initial population P_0 is 500 cfu/ml and the growth rate r is 0.4 per hour, how long (t) will it take for the population to reach 5,000 cfu/ml?

Sub values into the equation: $5,000 = 500e^{0.4t}$

Divide by sides by 500 to isolate the variable, take the natural log ($\ln$), then divide by 0.4.

Solution

$t \approx 5.76 \text{ hours}$

Equations Containing e

One common type of exponential equations is those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

How To

Given an equation of the form $y = Ae^{kt}$, solve for t.

1. Divide both sides of the equation by A.
2. Apply the natural logarithm to both sides of the equation.
3. Divide both sides of the equation by k.

Example 6

Solve:

$$100 = 20e^{2t}$$

Solution:

$$\begin{aligned} 100 &= 20e^{2t} \\ 5 &= e^{2t} && \text{Divide by the coefficient of the power.} \\ \ln 5 &= 2t && \text{Take } \ln \text{ of boths sides.} \\ & && \text{Use the fact that } \ln(x) \text{ and } e^x \text{ are inverse functions.} \\ t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of } t. \end{aligned}$$

Using laws of logs, we can also write this answer in the form $t = \ln(\sqrt{5})$.

If we want a decimal approximation of the answer, we use a calculator.

Answer

$$t = \frac{\ln 5}{2}$$

Try It

Solve: $3e^{0.5t} = 11$

Show Solution

$$t = 2\ln(\frac{11}{3})$$

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an extraneous solution, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. This can happen when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no solution.

Example 7

Solve:

$$e^{2x} - e^{x} = 56$$

Solution:

$$\begin{aligned} e^{2x} - e^{x} &= 56 && \\ e^{2x} - e^{x} - 56 &= 0 && \text{Get one side of the equation equal to zero} \\ (e^{x} + 7)(e^{x} - 8) &= 0 && \text{Factor by the FOIL method} \\ e^{x} + 7 = 0 \text{ or } e^{x} &= 8 && \text{Isolate the exponentials} \\ e^{8} &= 0 && \text{Reject the equation in which the power equals a negative number} \\ x &= \ln 8 && \text{Solve the equation in which the power equals a positive number} \end{aligned}$$

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^{x} = -7$ because a positive number never equals a negative number.

The solution $\ln(-7)$ is not a real number, and in the real number system, this solution is rejected as an extraneous solution.

Answer

$$x = \ln 8$$

Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use the rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$\begin{aligned}
\log_2(2 + \log_2(3x - 5) &= 3 && \\
\log_2(2(3x - 5)) &= 3 && \text{Condense the logarithms} \\
\log_2(6x - 10) &= 3 && \text{Distribute the 2} \\
2^3 &= 6x - 10 && \text{Apply the definition of a logarithm} \\
8 &= 6x - 10 && \text{Calculate } 2^3 \\
18 &= 6x && \text{Isolate the variable} \\
x &= 3 && \text{Divide by 6}
\end{aligned}$$

Example 8

Solve:

$$2\ln(x) + 3 = 7$$

Solution:

$$
\begin{aligned}
2\ln x + 3 &= 7 \\
2\ln x &= 4 && \text{Isolate x 3} \\
\ln x &= 2 && \text{Divide by 2} \\
x &= e^2 && \text{Rewrite in exponential form.}
\end{aligned}
$$

Answer

$x = e^2$

Try It

Solve 1: $6 + \ln(x) = 10$

Solve 2: $2\ln(x + 1) = 10$

Solve 3: Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Show Solution

1. $x = e^4$
2. $x = e^5 - 1$
3. $x \approx 9.97$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, if $\log_b(S) = \log_b(T)$, then $S = T$ (providing the numbers are real, the argument is > 0, and the base isn't one).

For example,

If $\log_2(x - 1) = \log_2(8)$, then $x - 1 = 8$. So, if $x - 1 = 8$, then we can solve for x, and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9 - 1) = \log_2(8) = 3$.

In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use the rules of logarithms to rewrite each side as a single logarithm.

Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2)-\log(2)=\log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:

$$\begin{aligned}
\log(3x-2)-\log(2)&=\log(x+4)\\
\log\left(\frac{3x-2}{2}\right)&=\log(x+4) && \text{Apply the quotient rule of logarithms}\\
\frac{3x-2}{2}&=x+4 && \text{Apply the one-to-one property of a logarithm.}\\
3x-2&=2x+8 && \text{Multiply both sides of the equation by 2}\\
x&=10 && \text{Subtract } 2x \text{ and add 2}
\end{aligned}$$

To check the result, substitute $x=10$ into $\log(3x-2)-\log(2)=\log(x+4)$.

$$\begin{aligned}
\log(3(10)-2)-\log(2)&=\log((10)+4)\\
(\log(28)-\log(2)&=\log(14)\\
(\log\left(\frac{28}{2}\right)&=\log(14) && \checkmark \text{The solution checks.}
\end{aligned}$$

For any algebraic expressions S and T and any positive real number b, where $b\neq 1$, $\log_b(S)=\log_b(T)$ if and only if $S=T$.

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

How To

Given an equation containing logarithms, solve it using the one-to-one property.

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b(S)=\log_b(T)$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S=T$, for the unknown.

Example 9

Solve:

$$\ln(x^2) = \ln(2x + 3)$$

Solution:

$\ln(x^2) = \ln(2x + 3)$	
$x^2 = 2x + 3$	Use the one-to-one property of the logarithm
$x^2 - 2x - 3 = 0$	Get zero on one side before factoring
$(x - 3)(x + 1) = 0$	Factor using FOIL
$x - 3 = 0$ or $x = -1$	If a product is zero, one of the factors must be zero
$x = 3$ or $x = -1$	Solve for x

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

Answer

$x = 3$ or $x = -1$

Try It

Solve: $\ln(x^2) = \ln(1)$

Show Solution

$x = 1 \text{or} x = -1$

Solving Applied Problems Using Exponential and Logarithmic Equations

We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. Table 14.1.1 lists the half-life for several of the more common radioactive substances.

Table 14.1.1.

Substance	Use	Half-life
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,730 years
uranium-235	atomic power	703,800,000 years

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A(t) = A_0 e^{\left(\frac{\ln(0.5)}{T}\right)t}$$
$$(A(t) = A_0 e^{\ln(0.5)\cdot\frac{t}{T}}$$
$$(A(t) = A_0 \left(e^{\ln(0.5)}\right)^{\frac{t}{T}}$$
$$(A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}}$$

where

- A_0 is the amount initially present
- T is the half-life of the substance
- t is the time period over which the substance is studied
- $A(t)$ is the amount of the substance present after time t

Science Application

An environmental tech is testing a water sample from a local stream. The formula for pH is $pH = -\log[H^+]$, where $[H^+]$ is the concentration of hydrogen ions. If the water sample has a pH of 5.8, find the hydrogen ion concentration.

Sub values into the equation: $5.8 = -\log[H^+]$

Multiply by -1, then convert to exponential form: $10^{-5.8} = [H^+]$.

Solution

$$[H^+] \approx 1.58 \times 10^{-6}\ \text{mol/L}$$

Example 10

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

Solution:

$$y = 1000e^{\left(\frac{\ln(0.5)}{703{,}800{,}000}\right)t}$$

$$(900 = 1000e^{\left(\frac{\ln(0.5)}{703{,}800{,}000}\right)t} \quad \text{After } 10\% \text{ decays, 900 grams are left.}$$

$$0.9 = e^{\left(\frac{\ln(0.5)}{703{,}800{,}000}\right)t} \quad \text{Divide by 1000.}$$

$$\ln(0.9) = \ln\left(e^{\left(\frac{\ln(0.5)}{703{,}800{,}000}\right)t}\right) \quad \text{Take } \ln \text{ of both sides.}$$

$$\ln(0.9) = \ln\left(e^{\frac{\ln(0.5}{703{,}800{,}000}t}\right) \quad \ln(e^M) = M$$

$$t = 703{,}800{,}000 \cdot \frac{\ln(0.9)}{\ln(0.5)} \text{ years} \quad \text{Solve for } t.$$

$$t \approx 106{,}979{,}777 \text{ years}$$

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

Answer

$t \approx 106{,}979{,}777$ years

Try It

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

Show Solution

$\approx 226,573,000$ years.

Media

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

Key Takeaways

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are not explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side.
- We can solve exponential equations with base e by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions.
- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown.
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the x-value of the intersecting point.
- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown.
- Combining the skills learned in this and previous sections, we can solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

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14.2 Exercises

14.1 Exponential and Log Equations

For the following exercises, use like bases to solve the exponential equation.

1. $64 \cdot 4^{3x} = 16$
2. $3^{2x+1} \cdot 3^{x} = 243$
3. $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$
4. $625 \cdot 5^{3x+3} = 125$
5. $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$
6. $\left(\frac{1}{4}\right)^{3n} \cdot 8 = 2^{6}$

For the following exercises, use logarithms to solve.

7. $9^{x-10} = 1$
8. $2e^{6x} = 13$
9. $e^{r+10} - 10 = -42$
10. $2 \cdot 10^{9a} = 29$
11. $-8 \cdot 10^{p+7} - 7 = -24$
12. $7e^{3h-5} + 5 = -89$
13. $e^{-3k} + 6 = 44$
14. $-5e^{9x-8} - 8 = -62$

15. $-6e^{9x+8}+2=-74$

16. $2^{x+1}=5^{2x-1}$

17. $e^{2x}-e^{x}-132=0$

18. $7e^{8x+8}-5=-95$

19. $10e^{8x+3}+2=8$

20. $4e^{3x+3}-7=53$

21. $8e^{-5x-2}-4=-90$

22. $3^{2x+1}=7^{x-2}$

23. $e^{2x}-e^{x}-6=0$

24. $3e^{3-3x}+6=-31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

25. $\log\left(\frac{1}{100}\right)=-2$

26. $\log_{324}(18)=\frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

27. $5\log_{7} n=10$

28. $-8\log_{9} x=16$

29. $4+\log_{2}(9k)=2$

30. $2\log(8n+4)+6=10$

31. $10-4\ln(9-8x)=6$

For the following exercises, use the one-to-one property of logarithms to solve.

32. $\ln(10-3x)=\ln(-4x)$

33. $\log_{13}(5n-2) = \log_{13}(8-5n)$

34. $\log(x+3) - \log(x) = \log(74)$

35. $\ln(-3x) = \ln(x^2 - 6x)$

36. $\log_4(6-m) = \log_4 3m$

37. $\ln(x-2) - \ln(x) = \ln(54)$

38. $\log_9(2n^2 - 14n) = \log_9(-45 + n^2)$

39. $\ln(x^2 - 10) + \ln(9) = \ln(10)$

For the following exercises, solve each equation for x

40. $\ln(10 - 3x) = \ln(-4x)$

41. $\log_{13}(5n-2) = \log_{13}(8-5n)$

42. $\log(x+3) - \log(x) = \log(74)$

43. $\ln(-3x) = \ln(x^2 - 6x)$

44. $\log_4(6-m) = \log_4 3m$

45. $\ln(x-2) - \ln(x) = \ln(54)$

46. $\log_9(2n^2 - 14n) = \log_9(-45 + n^2)$

47. $\ln(x^2 - 10) + \ln(9) = \ln(10)$

Show Solutions

1. $x = -\frac{1}{3}$

3. $n = -1$

5. $b = \frac{6}{5}$

7. $x = 10$

9. No solution

11. $p = \log\left(\frac{17}{8}\right) - 7$

13. $k = \frac{-\ln(38)}{3}$

15. $x = \frac{\ln\left(\frac{38}{3}\right) - 8}{9}$

17. $x = \ln 12$

19. $x = \frac{\ln\left(\frac{3}{5}\right) - 3}{8}$

21. No solution

23. $x = \ln(3)$

25. $10^{-2} = \frac{1}{100}$

27. $n = 49$

29. $k = \dfrac{1}{36}$

31. $x = \dfrac{9 - e}{8}$

33. $n = 1$

35. No solution

37. No solution

39. $x = \pm\dfrac{10}{3}$

41. $x = 10$

43. $x = 0$

45. $x = \dfrac{3}{4}$

47. $x = 9$

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Glossary

Key Terms

Absolute value: The absolute value of a number is the distance from the graph of the number to zero on a number line. 1.2

AC method: Method for factoring trinomials by replacing the middle term with two terms that allow us to factor the resulting four-term polynomial by grouping. 6.2

Additive identity property: Given any real number a, $a + 0 = 0 + a = a$. 4.1

Algebraic expressions: Combinations of variables and numbers along with mathematical operations used to generalize specific arithmetic operations. 4.1

Algebraic fraction: A term used when referring to a rational expression. 6.6

Base: The number or expression being repeatedly multiplied in an exponential expression (the a in a^n). 2.1

Base-ten (decimal) system: A number system where place values are powers of 10, which makes metric conversions convenient because they involve multiplying/dividing by 10, 100, 1000, etc. 2.1

Binomial: A polynomial with two terms. 3.1

Caret: The symbol ^ that indicates exponents on many calculators. 2.2

Cartesian coordinate system: Used in honour of René Descartes when referring to the rectangular coordinate system. 5.1

Circumference: The perimeter of a circle given by $C = 2\pi r$, where r represents the radius of the circle and $\pi \approx 3.14159$. 4.2

Collinear: Describes points that lie on the same line. 5.1

Combining like terms: Adding or subtracting like terms within an algebraic expression to obtain a single term with the same variable part. 4.1

Common denominator: A denominator that is shared by more than one fraction. 1.3

Common factor: A factor that is shared by more than one real number. 6.3

Commutative property: Given real numbers a and b, $a + b = b + a$. 11.1

Completing the square: The process of rewriting a quadratic equation in the form $(x - p)^2 = q$. 9.2

Complex fraction: A fraction where the numerator or denominator consists of one or more fractions. 1.3

Complex rational expression: A rational expression where the numerator or denominator consists of one or more rational expressions. 8.1

Conjugates: The factors (a+b) and (a-b) are conjugates. 11.1

Conjugate binomials: The binomials $(a+b)$ and $(a-b)$. 3.3

Constant term: A term written without a variable factor. 3.1

Contradiction: An equation that is never true and has no solution. 8.2

Conversion factor: A ratio made from two equal quantities (so it equals 1), used to convert units (e.g., $\frac{1\text{km}}{1000\text{m}}$). 2.1

Cube: The result when the exponent of any real number is 3. 2.2

Cube root: A number that, when multiplied by itself three times, yields the original number. 2.3

Degree of a polynomial: The largest degree of all of its terms. 3.1

Degree of a term: The exponent of the variable; if there is more than one variable in the term, the degree of the term is the sum of their exponents. 3.1

Denominator: The integer below the fraction bar. 1.3

Dependent variable: The variable whose value is determined by the value of the independent variable. Usually, we think of the y-value as the dependent variable. 5.2

Difference of cubes: $a^3 - b^3 = (a-b)(a^2 = ab = b^2)$, where a and b represent algebraic expressions. 6.4

Difference of squares: $a^2 - b^2 = (a+b)(a-b)$, where a and b represent algebraic expressions. 3.3

Discriminant: The expression inside the radical of the quadratic formula, $b^2 - 4ac$. 9.4

Distance formula: Given two points (x_1, y_1) and (x_2, y_2) calculate the distance d between them using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. 10.2

Distributive property: Given any real numbers a, b, and c, $a(b+c) = ab + ac$ or $(b+c)a = ba + ca$. 3.2

Dividend: The numerator of a quotient. 3.4

Divisor: The denominator of a quotient. 3.4

Double root: A root that is repeated twice. 9.1

Equivalent equations: Equations with the same solution set. 4.1

Equivalent fractions: Two equal fractions expressed using different numerators and denominators. 1.3

Even integers: Integers that are divisible by two or are multiples of two. 4.2

Exponent: The positive integer n in the exponential notation that indicates the number of times the base is used as a factor. 1.2

Exponential notation: The compact notation $ax^2 + bx + c = 0$ is used when a factor is repeated multiple times. 2.2

Extracting the roots: Applying the square root property as a means of solving a quadratic equation. 9.2

Extraneous solutions: A solution that does not solve the original equation. 8.2

Factors: Any of the numbers or expressions that form a product. 2.3

Factor by grouping: A technique for factoring polynomials with four terms. 6.1

Factoring out the GCF: The process of rewriting a polynomial as a product using the GCF of all of its terms. 9.1

FOIL method: When multiplying binomials, we apply the distributive property multiple times in such a way as to multiply the first terms, outer terms, inner terms, and last terms. 3.3

Fraction: a real number written as a quotient, or ratio, of two integers a and b, where $b \neq 0$. 1.3

Greatest common factor (GCF): The largest shared factor of any number of integers. 1.3

Grouping symbols: Parentheses, brackets, braces, and the fraction bar are the common symbols used to group expressions and mathematical operations within a computation. 1.2

Guess and check: Used when referring to the trial and error method for factoring trinomials. 6.2

Horizontal line: Any line whose equation can be written in the form $y = k$, where k is a real number. 5.3

Improper fraction: one where the numerator is larger than the denominator. 1.3

Integers: the set consisting of the natural numbers, their opposites, and 0: $\{\cdots -3, -2, -1, 01, 2, 3 \ldots\}$. 1.1

Index: The positive integer n in the notation $\sqrt[n]{a}$ that is used to indicate an nth root. 2.3

Intersection: The set formed by the shared values of the individual solution sets that is indicated by the logical use of the word “and,” denoted with the symbol ∩. 5.1

Irrational numbers: the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers. 1.1

Leading coefficient: The coefficient of the term with the largest degree. 3.1

Least common denominator (LCD): The least common multiple of a set of denominators. 1.3

Least common multiple (LCM): The smallest number that is evenly divisible by a set of numbers. 1.3

Linear equation with one variable: An equation that can be written in the general form $ax + b = 0$, where a and b are real numbers and $a \neq 0$. 4.1

Linear equation with two variables: An equation with two variables that can be written in the standard form $ax + by = c$, where the real numbers a and b are not both zero. 5.2

Literal equations: A formula that summarizes whole classes of problems. 8.2

Metric system: A standardized measurement system commonly used in science that is built on the base-ten (decimal) number system. 2.1

Monomial: A polynomial with one term. 3.1

Mixed number: A number that represents the sum of a whole number and a fraction. 1.3

Natural numbers: the set of counting numbers: $\{1, 2, 3 \ldots\}$. 1.1

Numerator: The integer above the fraction bar. 1.3

Odd integers: Integers that are not divisible by 2. 4.2

Opposite binomial property: If given a binomial $a - b$, then the opposite is $-(a - b) = b - a$. 7.2

Opposite-side like terms: Like terms of an equation on opposite sides of the equal sign. 4.1

Order of precedence: To ensure a single correct result, perform mathematical operations in a specific order. 1.2

Origin: The point where the x– and y-axes cross, denoted by (0, 0). 5.1

Perfect cube: The result of cubing an integer. 2.2

Perfect square: The result of squaring an integer. 2.2

Perfect square trinomials: The trinomials obtained by squaring the binomials $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$. 3.3

Perimeter: The sum of the lengths of all the outside edges of a polygon. 4.2

Placeholders: Terms with zero coefficients are used to fill in all missing exponents within a polynomial. 3.4

Polynomial: An algebraic expression consisting of terms with real number coefficients and variables with whole number exponents. 3.1

Polynomial long division: The process of dividing two polynomials using the division algorithm. 3.4

Power: Another name for an exponential expression, such as a^n, read as "a raised to the n^{th} power." 2.1

Power property of equality: Given any positive integer n and real numbers a and b, where $a = b$, then $a^n = b^n$. 11.2

Power rule for exponents: $(x^m)^n = x^{mn}$; a power raised to a power can be simplified by multiplying the exponents. 6.4

Prime factorization: The unique factorization of a natural number written as a product of primes. 6.1

Prime number: Integers greater than 1 that are divisible only by 1 and itself. 6.1

Prime polynomial: A polynomial with integer coefficients that cannot be factored as a product of polynomials with integer coefficients other than 1 and itself. 6.1

Principal (nonnegative) square root: The positive square root of a real number, denoted by the symbol $\sqrt{}$. 2.3

Power Property: According to the Power Property a to the m to the n equals a to the m times n. 13.1

Product rule for exponents: $x^m \cdot x^n = x^{m+n}$; the product of two expressions with the same base can be simplified by adding the exponents. 3.3

Product rule for radicals: $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, where a and b represent positive real numbers. 10.2

Properties of equality: Properties that allow us to obtain equivalent equations by adding, subtracting, multiplying, and dividing both sides of an equation by nonzero real numbers. 4.1

Pythagorean theorem: Given any right triangle with legs measuring a and b units and hypotenuse measuring c units, then $a^2 + b^2 = c^2$. 10.2

Quadrants: The four regions of a rectangular coordinate plane, partly bounded by the x– and y-axes and numbered using the roman numerals I, II, III, and IV. 5.1

Quadratic formula: The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which gives the solutions to any quadratic equation in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. 9.3

Quotient: The result after dividing. 3.4

Quotient Property: According to the Quotient Property, a to the m divided by a to the n equals a to the m minus n as long as a is not zero. 13.1

Quotient rule for exponents: $\frac{x^m}{x^n} = x^{m-n}$; the quotient of two expressions with the same base can be simplified by subtracting the exponents. 3.4

Quotient rule for radicals: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, where a and b represent positive real numbers. 10.1

Radical: Used when referring to an expression of the form $\sqrt[n]{a}$. 2.3

Radical expression: An algebraic expression that contains radicals. 10.2

Radicand: The expression a within a radical sign, $\sqrt[n]{a}$. 2.3

Rate: A ratio where the units for the numerator and the denominator are different. 5.3

Ratio: Relationship between two numbers or quantities usually expressed as a quotient. 5.3

Rational (or fractional) exponents: The fractional exponent m/n that indicates a radical with index n and exponent m: $a^{m/n} = \sqrt[n]{a^m}$. 10.1

Rational numbers: the set of all numbers of the form $\frac{m}{n}$ where m and n are integers and $n \neq 0$. Any rational number may be written as a fraction or a terminating or repeating decimal. 1.1

Rationalizing the denominator: The process of determining an equivalent radical expression with a rational denominator. 11.1

Real number line: a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left. 1.1

Real numbers: the sets of rational numbers and irrational numbers taken together. 1.1

Reciprocals: The reciprocal of a nonzero number n is $\frac{1}{n}$. 1.3

Reducing to lowest terms: Finding equivalent fractions where the numerator and the denominator share no common integer factor other than 1. 1.3

Relatively prime: Expressions that share no common factors other than 1. 6.1

Remainder: The expression that is left after the division algorithm ends. 3.4

Rise: The vertical change between any two points on a line. 5.3

Root: A solution to a quadratic equation in standard form. 9.1

Run: The horizontal change between any two points on a line. 5.3

Same-side like terms: Like terms of an equation on the same side of the equal sign. 4.1

Satisfy the equation: After replacing the variable with a solution and simplifying, it produces a true statement. 4.1

Scientific notation: A way to write numbers as $a \times 10^n$ where $1 \leq a < 10$ and n is an integer. 2.2

Simplified: A radical where the radicand does not consist of any factor that can be written as a perfect power of the index. 4.1

Solution: Any value that can replace the variable in an equation to produce a true statement. 4.1

Slope formula: Given two points (x_1, y_1) and (x_2, y_2), then the slope of the line is given by the formula $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{y_2 - y_1}{x_2 - x_1}$. 5.3

Slope-intercept form: Any nonvertical line can be written in the form $y = mx + b$, where m is the slope and (0, b) is the y-intercept. 5.3

Square: The result when the exponent of any real number is 2. 2.2

Square root: The number that, when multiplied by itself, yields the original number. 2.3

Squaring property of equality: Given real numbers a and b, where $a = b$, then $a^2 = b^2$. 11.2

Standard form: A quadratic equation written in the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$. 3.1

Symmetric property: Allows you to solve for the variable on either side of the equal sign, because $5 = x$ is equivalent to $x = 5$. 4.1

Trinomial: A polynomial with three terms. 3.1

Undefined: A quotient such as $\dfrac{5}{0}$, which is left without meaning and is not assigned an interpretation. 4.1

Unlike denominators: Denominators of fractions that are not the same. 1.3

Vertical line: Any line whose equation can be written in the form $x = k$, where k is a real number. 5.3

Whole numbers: the set consisting of 0 plus the natural numbers: $\{0, 1, 2, 3 \ldots\}$. 1.1

Work rate: The rate at which a task can be performed. 8.3

x-intercept: The point (or points) where a graph intersects the x-axis, expressed as an ordered pair (x, 0). 5.3

y-intercept: The point (or points) where a graph intersects the y-axis, expressed as an ordered pair (0, y). 5.3

Zero-product property: Any product is equal to zero if and only if at least one of the factors is zero. 9.1

References

Gram. (2025, November 2). In *Wikipedia*. en.wikipedia.org/wiki/Gram

History of the metre. (2025, December 23). In *Wikipedia*. https://en.wikipedia.org/wiki/History_of_the_metre

Litre. (2026, January 18). In *Wikipedia*. https://en.wikipedia.org/wiki/Litre

Version History

This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.1. If the edits involve a number of changes, the version number increases to the next full number.

The files posted alongside this book always reflect the most recent version.

Version	Date	Change	Affected Web Page
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