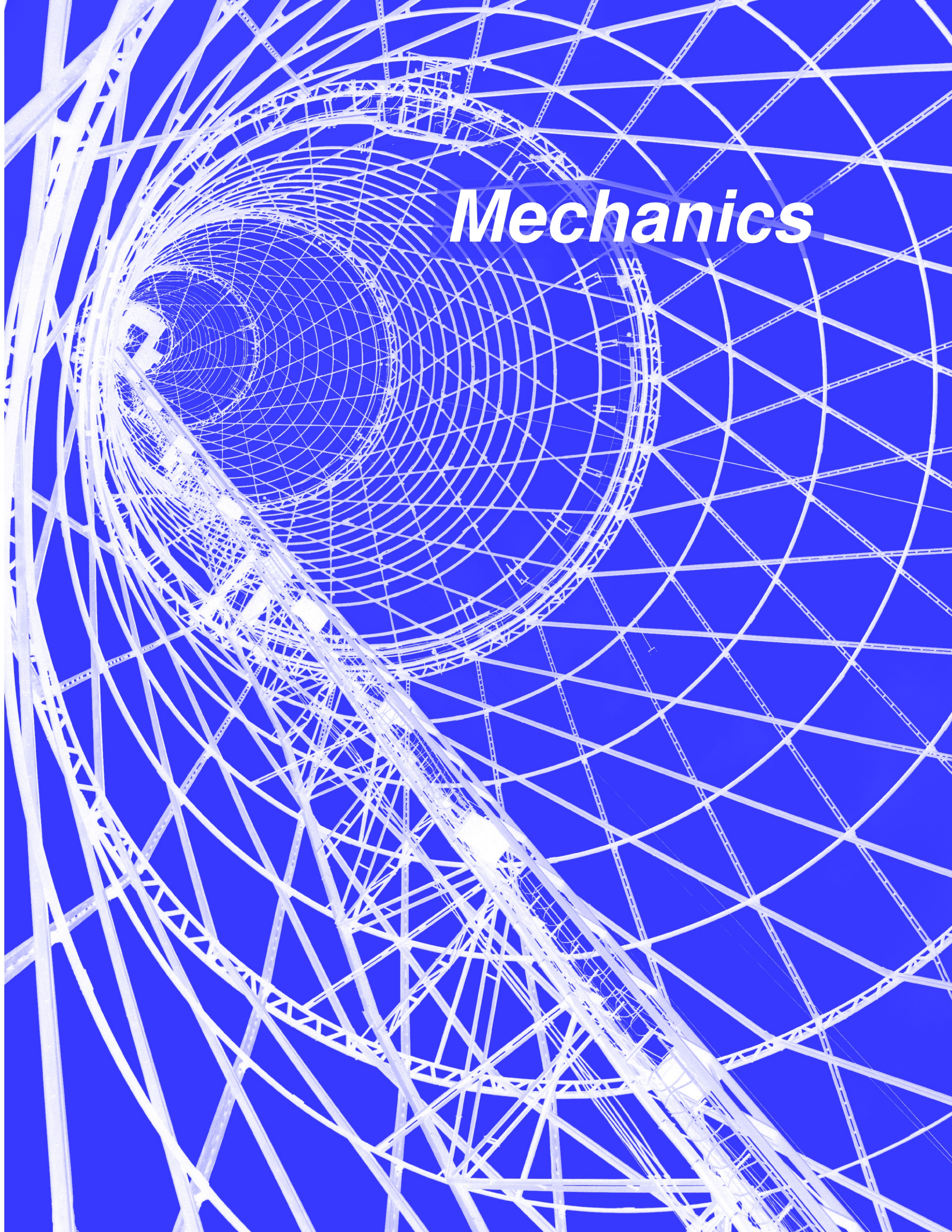
Mechanics



Mechanics

Light and Matter
Fullerton, California
www.lightandmatter.com

rev. February 22, 2017

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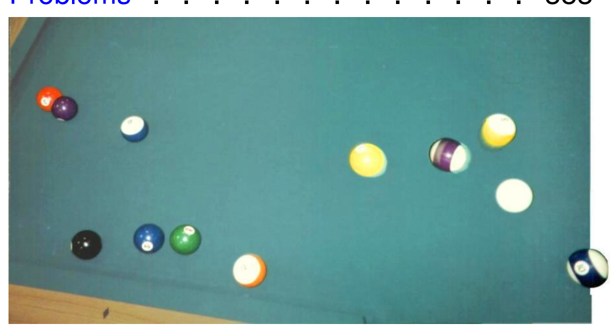

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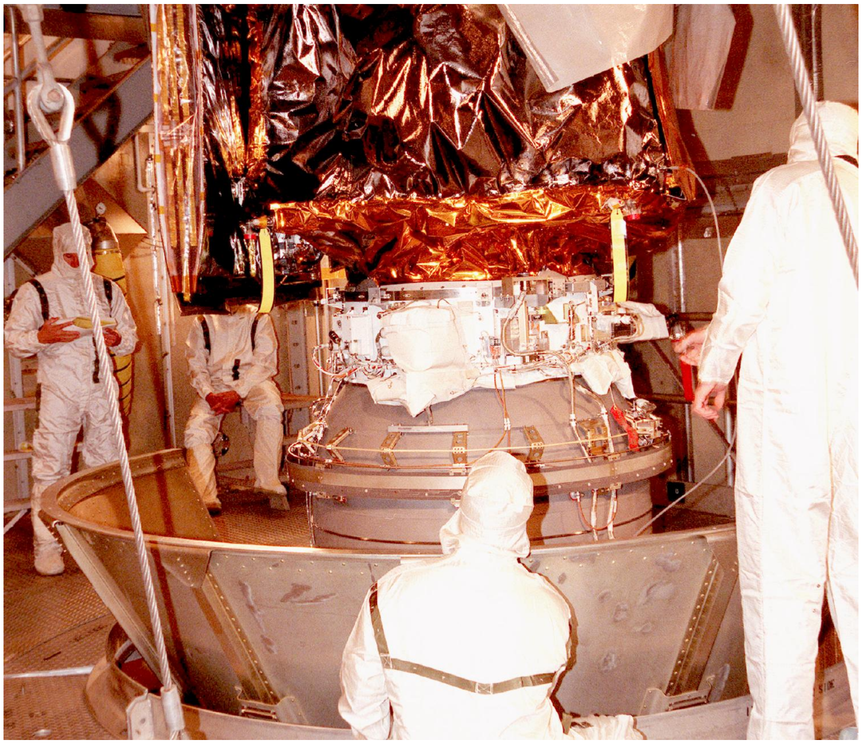


The Mars Climate Orbiter is prepared for its mission. The laws of physics are the same everywhere, even on Mars, so the probe could be designed based on the laws of physics as discovered on earth. There is unfortunately another reason why this spacecraft is relevant to the topics of this chapter: it was destroyed attempting to enter Mars' atmosphere because engineers at Lockheed Martin forgot to convert data on engine thrusts from pounds into the metric unit of force (newtons) before giving the information to NASA. Conversions are important!

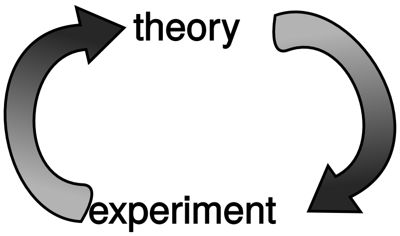
Chapter 0

Introduction and review

If you drop your shoe and a coin side by side, they hit the ground at the same time. Why doesn't the shoe get there first, since gravity is pulling harder on it? How does the lens of your eye work, and why do your eye's muscles need to squash its lens into different shapes in order to focus on objects nearby or far away? These are the kinds of questions that physics tries to answer about the behavior of light and matter, the two things that the universe is made of.

0.1 The scientific method

Until very recently in history, no progress was made in answering questions like these. Worse than that, the *wrong* answers written by thinkers like the ancient Greek physicist Aristotle were accepted without question for thousands of years. Why is it that scientific knowledge has progressed more since the Renaissance than it had in all the preceding millennia since the beginning of recorded history? Undoubtedly the industrial revolution is part of the answer. Building its centerpiece, the steam engine, required improved techniques for precise construction and measurement. (Early on, it was considered a major advance when English machine shops learned to build pistons and cylinders that fit together with a gap narrower than the thickness of a penny.) But even before the industrial revolution, the pace of discovery had picked up, mainly because of the introduction of the modern scientific method. Although it evolved over time, most scientists today would agree on something like the following list of the basic principles of the scientific method:



a / Science is a cycle of theory and experiment.

(1) *Science is a cycle of theory and experiment.* Scientific theories [1] are created to explain the results of experiments that were created under certain conditions. A successful theory will also make new predictions about new experiments under new conditions. Eventually, though, it always seems to happen that a new experiment comes along, showing that under certain conditions the theory is not a good approximation or is not valid at all. The ball is then back in the theorists' court. If an experiment disagrees with the current theory, the theory has to be changed, not the experiment.

(2) *Theories should both predict and explain.* The requirement of predictive power means that a theory is only meaningful if it predicts something that can be checked against experimental measurements that the theorist did not already have at hand. That is, a theory should be testable. Explanatory value means that many phenomena should be accounted for with few basic principles. If you answer every "why" question with "because that's the way it is," then your theory has no explanatory value. Collecting lots of data without being able to find any basic underlying principles is not science.

(3) *Experiments should be reproducible.* An experiment should be treated with suspicion if it only works for one person, or only in one part of the world. Anyone with the necessary skills and equipment should be able to get the same results from the same experiment. This implies that science transcends national and ethnic boundaries; you can be sure that nobody is doing actual science who claims that their work is "Aryan, not Jewish," "Marxist, not bourgeois," or "Christian, not atheistic." An experiment cannot be reproduced if it is secret, so science is necessarily a public enterprise.

b / A satirical drawing of an alchemist's laboratory. H. Cock, after a drawing by Peter Brueghel the Elder (16th century).

As an example of the cycle of theory and experiment, a vital step toward modern chemistry was the experimental observation that the chemical elements could not be transformed into each other, e.g., lead could not be turned into gold. This led to the theory that chemical reactions consisted of rearrangements of the elements in

[1] The term "theory" in science does not just mean "what someone thinks," or even "what a lot of scientists think." It means an interrelated set of statements that have predictive value, and that have survived a broad set of empirical tests. Thus, both Newton's law of gravity and Darwinian evolution are scientific *theories*. A "hypothesis," in contrast to a theory, is any statement of interest that can be empirically tested. That the moon is made of cheese is a hypothesis, which was empirically tested, for example, by the Apollo astronauts.

different combinations, without any change in the identities of the elements themselves. The theory worked for hundreds of years, and was confirmed experimentally over a wide range of pressures and temperatures and with many combinations of elements. Only in the twentieth century did we learn that one element could be transformed into one another under the conditions of extremely high pressure and temperature existing in a nuclear bomb or inside a star. That observation didn't completely invalidate the original theory of the immutability of the elements, but it showed that it was only an approximation, valid at ordinary temperatures and pressures.

self-check A

A psychic conducts seances in which the spirits of the dead speak to the participants. He says he has special psychic powers not possessed by other people, which allow him to "channel" the communications with the spirits. What part of the scientific method is being violated here?
▷ Answer, p. 557

The scientific method as described here is an idealization, and should not be understood as a set procedure for doing science. Scientists have as many weaknesses and character flaws as any other group, and it is very common for scientists to try to discredit other people's experiments when the results run contrary to their own favored point of view. Successful science also has more to do with luck, intuition, and creativity than most people realize, and the restrictions of the scientific method do not stifle individuality and self-expression any more than the fugue and sonata forms stifled Bach and Haydn. There is a recent tendency among social scientists to go even further and to deny that the scientific method even exists, claiming that science is no more than an arbitrary social system that determines what ideas to accept based on an in-group's criteria. I think that's going too far. If science is an arbitrary social ritual, it would seem difficult to explain its effectiveness in building such useful items as airplanes, CD players, and sewers. If alchemy and astrology were no less scientific in their methods than chemistry and astronomy, what was it that kept them from producing anything useful?

Discussion questions

Consider whether or not the scientific method is being applied in the following examples. If the scientific method is not being applied, are the people whose actions are being described performing a useful human activity, albeit an unscientific one?

A Acupuncture is a traditional medical technique of Asian origin in which small needles are inserted in the patient's body to relieve pain. Many doctors trained in the west consider acupuncture unworthy of experimental study because if it had therapeutic effects, such effects could not be explained by their theories of the nervous system. Who is being more scientific, the western or eastern practitioners?

B Goethe, a German poet, is less well known for his theory of color. He published a book on the subject, in which he argued that scientific apparatus for measuring and quantifying color, such as prisms, lenses and colored filters, could not give us full insight into the ultimate meaning of color, for instance the cold feeling evoked by blue and green or the heroic sentiments inspired by red. Was his work scientific?

C A child asks why things fall down, and an adult answers "because of gravity." The ancient Greek philosopher Aristotle explained that rocks fell because it was their nature to seek out their natural place, in contact with the earth. Are these explanations scientific?

D Buddhism is partly a psychological explanation of human suffering, and psychology is of course a science. The Buddha could be said to have engaged in a cycle of theory and experiment, since he worked by trial and error, and even late in his life he asked his followers to challenge his ideas. Buddhism could also be considered reproducible, since the Buddha told his followers they could find enlightenment for themselves if they followed a certain course of study and discipline. Is Buddhism a scientific pursuit?

0.2 What is physics?

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective positions of the things which compose it...nothing would be uncertain, and the future as the past would be laid out before its eyes.
>
> *Pierre Simon de Laplace*

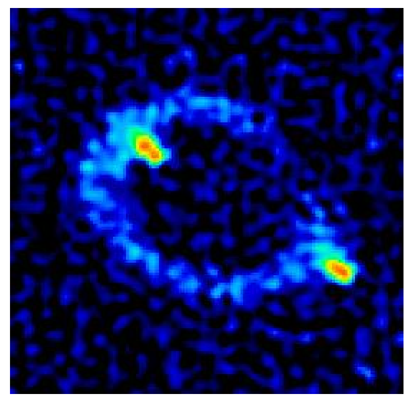
Physics is the use of the scientific method to find out the basic principles governing light and matter, and to discover the implications of those laws. Part of what distinguishes the modern outlook from the ancient mind-set is the assumption that there are rules by which the universe functions, and that those laws can be at least partially understood by humans. From the Age of Reason through the nineteenth century, many scientists began to be convinced that the laws of nature not only could be known but, as claimed by Laplace, those laws could in principle be used to predict everything about the universe's future if complete information was available about the present state of all light and matter. In subsequent sections, I'll describe two general types of limitations on prediction using the laws of physics, which were only recognized in the twentieth century.

Matter can be defined as anything that is affected by gravity, i.e., that has weight or would have weight if it was near the Earth or another star or planet massive enough to produce measurable gravity. Light can be defined as anything that can travel from one place to another through empty space and can influence matter, but has no weight. For example, sunlight can influence your body by heating it or by damaging your DNA and giving you skin cancer. The physicist's definition of light includes a variety of phenomena that are not visible to the eye, including radio waves, microwaves, x-rays, and gamma rays. These are the "colors" of light that do not happen to fall within the narrow violet-to-red range of the rainbow that we can see.

self-check B

At the turn of the 20th century, a strange new phenomenon was discovered in vacuum tubes: mysterious rays of unknown origin and nature. These rays are the same as the ones that shoot from the back of your TV's picture tube and hit the front to make the picture. Physicists in 1895 didn't have the faintest idea what the rays were, so they simply named them "cathode rays," after the name for the electrical contact from which they sprang. A fierce debate raged, complete with nationalistic overtones, over whether the rays were a form of light or of matter. What would they have had to do in order to settle the issue? ▷ Answer, p. 557

Many physical phenomena are not themselves light or matter, but are properties of light or matter or interactions between light and matter. For instance, motion is a property of all light and some matter, but it is not itself light or matter. The pressure that keeps a bicycle tire blown up is an interaction between the air and the tire. Pressure is not a form of matter in and of itself. It is as much a property of the tire as of the air. Analogously, sisterhood and employment are relationships among people but are not people themselves.



c / This telescope picture shows two images of the same distant object, an exotic, very luminous object called a quasar. This is interpreted as evidence that a massive, dark object, possibly a black hole, happens to be between us and it. Light rays that would otherwise have missed the earth on either side have been bent by the dark object's gravity so that they reach us. The actual direction to the quasar is presumably in the center of the image, but the light along that central line doesn't get to us because it is absorbed by the dark object. The quasar is known by its catalog number, MG1131+0456, or more informally as Einstein's Ring.

Some things that appear weightless actually do have weight, and so qualify as matter. Air has weight, and is thus a form of matter even though a cubic inch of air weighs less than a grain of sand. A helium balloon has weight, but is kept from falling by the force of the surrounding more dense air, which pushes up on it. Astronauts in orbit around the Earth have weight, and are falling along a curved arc, but they are moving so fast that the curved arc of their fall is broad enough to carry them all the way around the Earth in a circle. They perceive themselves as being weightless because their space capsule is falling along with them, and the floor therefore does not push up on their feet.

Optional Topic: Modern Changes in the Definition of Light and Matter

Einstein predicted as a consequence of his theory of relativity that light would after all be affected by gravity, although the effect would be extremely weak under normal conditions. His prediction was borne out by observations of the bending of light rays from stars as they passed close to the sun on their way to the Earth. Einstein's theory also implied the existence of black holes, stars so massive and compact that their intense gravity would not even allow light to escape. (These days there is strong evidence that black holes exist.)

Einstein's interpretation was that light doesn't really have mass, but that energy is affected by gravity just like mass is. The energy in a light beam is equivalent to a certain amount of mass, given by the famous equation $E = mc^2$, where c is the speed of light. Because the speed of light is such a big number, a large amount of energy is equivalent to only a very small amount of mass, so the gravitational force on a light ray can be ignored for most practical purposes.

There is however a more satisfactory and fundamental distinction between light and matter, which should be understandable to you if you have had a chemistry course. In chemistry, one learns that electrons obey the Pauli exclusion principle, which forbids more than one electron from occupying the same orbital if they have the same spin. The Pauli exclusion principle is obeyed by the subatomic particles of which matter is composed, but disobeyed by the particles, called photons, of which a beam of light is made.

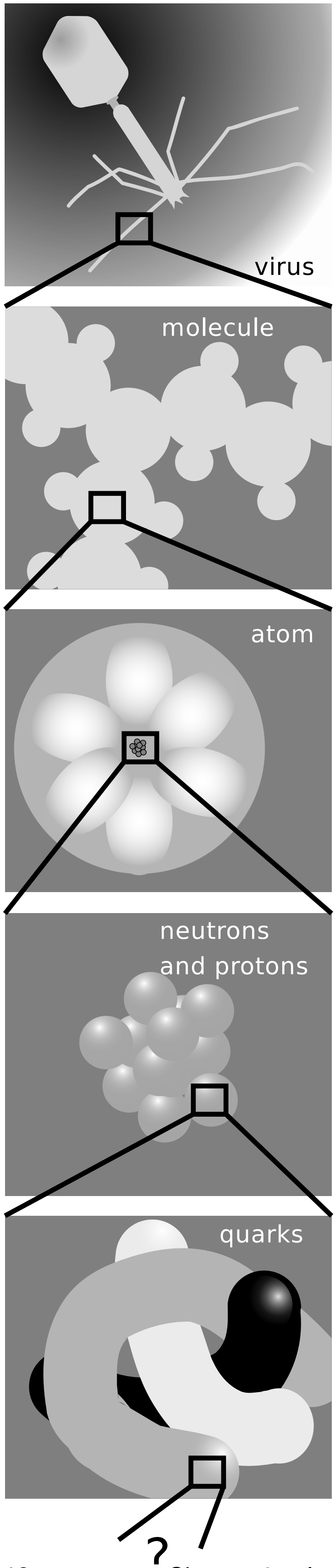
Einstein's theory of relativity is discussed more fully in book 6 of this series.

The boundary between physics and the other sciences is not always clear. For instance, chemists study atoms and molecules, which are what matter is built from, and there are some scientists who would be equally willing to call themselves physical chemists or chemical physicists. It might seem that the distinction between physics and biology would be clearer, since physics seems to deal with inanimate objects. In fact, almost all physicists would agree that the basic laws of physics that apply to molecules in a test tube work equally well for the combination of molecules that constitutes a bacterium. (Some might believe that something more happens in the minds of humans, or even those of cats and dogs.) What differentiates physics from biology is that many of the scientific theories that describe living things, while ultimately resulting from the fundamental laws of physics, cannot be rigorously derived from physical principles.

Isolated systems and reductionism

To avoid having to study everything at once, scientists isolate the things they are trying to study. For instance, a physicist who wants to study the motion of a rotating gyroscope would probably prefer that it be isolated from vibrations and air currents. Even in biology, where field work is indispensable for understanding how living things relate to their entire environment, it is interesting to note the vital historical role played by Darwin's study of the Galápagos Islands, which were conveniently isolated from the rest of the world. Any part of the universe that is considered apart from the rest can be called a "system."

Physics has had some of its greatest successes by carrying this process of isolation to extremes, subdividing the universe into smaller and smaller parts. Matter can be divided into atoms, and the behavior of individual atoms can be studied. Atoms can be split apart



d / Reductionism.

into their constituent neutrons, protons and electrons. Protons and neutrons appear to be made out of even smaller particles called quarks, and there have even been some claims of experimental evidence that quarks have smaller parts inside them. This method of splitting things into smaller and smaller parts and studying how those parts influence each other is called reductionism. The hope is that the seemingly complex rules governing the larger units can be better understood in terms of simpler rules governing the smaller units. To appreciate what reductionism has done for science, it is only necessary to examine a 19th-century chemistry textbook. At that time, the existence of atoms was still doubted by some, electrons were not even suspected to exist, and almost nothing was understood of what basic rules governed the way atoms interacted with each other in chemical reactions. Students had to memorize long lists of chemicals and their reactions, and there was no way to understand any of it systematically. Today, the student only needs to remember a small set of rules about how atoms interact, for instance that atoms of one element cannot be converted into another via chemical reactions, or that atoms from the right side of the periodic table tend to form strong bonds with atoms from the left side.

Discussion questions

A I've suggested replacing the ordinary dictionary definition of light with a more technical, more precise one that involves weightlessness. It's still possible, though, that the stuff a lightbulb makes, ordinarily called "light," does have some small amount of weight. Suggest an experiment to attempt to measure whether it does.

B Heat is weightless (i.e., an object becomes no heavier when heated), and can travel across an empty room from the fireplace to your skin, where it influences you by heating you. Should heat therefore be considered a form of light by our definition? Why or why not?

C Similarly, should sound be considered a form of light?

0.3 How to learn physics

> For as knowledges are now delivered, there is a kind of contract of error between the deliverer and the receiver; for he that delivereth knowledge desireth to deliver it in such a form as may be best believed, and not as may be best examined; and he that receiveth knowledge desireth rather present satisfaction than expectant inquiry.
>
> *Francis Bacon*

Many students approach a science course with the idea that they can succeed by memorizing the formulas, so that when a problem is assigned on the homework or an exam, they will be able to plug numbers in to the formula and get a numerical result on their calculator. Wrong! That's not what learning science is about! There is a big difference between memorizing formulas and understanding concepts. To start with, different formulas may apply in different situations. One equation might represent a definition, which is always true. Another might be a very specific equation for the speed of an object sliding down an inclined plane, which would not be true if the object was a rock drifting down to the bottom of the ocean. If you don't work to understand physics on a conceptual level, you won't know which formulas can be used when.

Most students taking college science courses for the first time also have very little experience with interpreting the meaning of an equation. Consider the equation $w = A/h$ relating the width of a rectangle to its height and area. A student who has not developed skill at interpretation might view this as yet another equation to memorize and plug in to when needed. A slightly more savvy student might realize that it is simply the familiar formula $A = wh$ in a different form. When asked whether a rectangle would have a greater or smaller width than another with the same area but a smaller height, the unsophisticated student might be at a loss, not having any numbers to plug in on a calculator. The more experienced student would know how to reason about an equation involving division — if h is smaller, and A stays the same, then w must be bigger. Often, students fail to recognize a sequence of equations as a derivation leading to a final result, so they think all the intermediate steps are equally important formulas that they should memorize.

When learning any subject at all, it is important to become as actively involved as possible, rather than trying to read through all the information quickly without thinking about it. It is a good idea to read and think about the questions posed at the end of each section of these notes as you encounter them, so that you know you have understood what you were reading.

Many students' difficulties in physics boil down mainly to difficulties with math. Suppose you feel confident that you have enough mathematical preparation to succeed in this course, but you are having trouble with a few specific things. In some areas, the brief review given in this chapter may be sufficient, but in other areas it probably will not. Once you identify the areas of math in which you are having problems, get help in those areas. Don't limp along through the whole course with a vague feeling of dread about something like scientific notation. The problem will not go away if you ignore it. The same applies to essential mathematical skills that you are learning in this course for the first time, such as vector addition.

Sometimes students tell me they keep trying to understand a certain topic in the book, and it just doesn't make sense. The worst thing you can possibly do in that situation is to keep on staring at the same page. Every textbook explains certain things badly — even mine! — so the best thing to do in this situation is to look at a different book. Instead of college textbooks aimed at the same mathematical level as the course you're taking, you may in some cases find that high school books or books at a lower math level give clearer explanations.

Finally, when reviewing for an exam, don't simply read back over the text and your lecture notes. Instead, try to use an active method of reviewing, for instance by discussing some of the discussion questions with another student, or doing homework problems you hadn't done the first time.

0.4 Self-evaluation

The introductory part of a book like this is hard to write, because every student arrives at this starting point with a different preparation. One student may have grown up outside the U.S. and so may be completely comfortable with the metric system, but may have had an algebra course in which the instructor passed too quickly over scientific notation. Another student may have already taken calculus, but may have never learned the metric system. The following self-evaluation is a checklist to help you figure out what you need to study to be prepared for the rest of the course.

If you disagree with this statement...	**you should study this section:**
I am familiar with the basic metric units of meters, kilograms, and seconds, and the most common metric prefixes: milli- (m), kilo- (k), and centi- (c).	section 0.5 Basic of the Metric System
I know about the newton, a unit of force	section 0.6 The newton, the Metric Unit of Force
I am familiar with these less common metric prefixes: mega- (M), micro- (μ), and nano- (n).	section 0.7 Less Common Metric Prefixes
I am comfortable with scientific notation.	section 0.8 Scientific Notation
I can confidently do metric conversions.	section 0.9 Conversions
I understand the purpose and use of significant figures.	section 0.10 Significant Figures

It wouldn't hurt you to skim the sections you think you already know about, and to do the self-checks in those sections.

0.5 Basics of the metric system

The metric system

Units were not standardized until fairly recently in history, so when the physicist Isaac Newton gave the result of an experiment with a pendulum, he had to specify not just that the string was 37 $^7/_8$ inches long but that it was "37 $^7/_8$ London inches long." The inch as defined in Yorkshire would have been different. Even after the British Empire standardized its units, it was still very inconvenient to do calculations involving money, volume, distance, time, or weight, because of all the odd conversion factors, like 16 ounces in a pound, and 5280 feet in a mile. Through the nineteenth century, schoolchildren squandered most of their mathematical education in preparing to do calculations such as making change when a customer in a shop offered a one-crown note for a book costing two pounds, thirteen shillings and tuppence. The dollar has always been decimal, and British money went decimal decades ago, but the United States is still saddled with the antiquated system of feet, inches, pounds, ounces and so on.

Every country in the world besides the U.S. uses a system of units known in English as the "metric system.[2]" This system is entirely decimal, thanks to the same eminently logical people who brought about the French Revolution. In deference to France, the system's official name is the Système International, or SI, meaning International System. (The phrase "SI system" is therefore redundant.)

The wonderful thing about the SI is that people who live in countries more modern than ours do not need to memorize how many ounces there are in a pound, how many cups in a pint, how many feet in a mile, etc. The whole system works with a single, consistent set of Greek and Latin prefixes that modify the basic units. Each prefix stands for a power of ten, and has an abbreviation that can be combined with the symbol for the unit. For instance, the meter is a unit of distance. The prefix kilo- stands for 10^3, so a kilometer, 1 km, is a thousand meters.

The basic units of the metric system are the meter for distance, the second for time, and the gram for mass.

The following are the most common metric prefixes. You should memorize them.

prefix		meaning	example	
kilo-	k	10^3	60 kg	= a person's mass
centi-	c	10^{-2}	28 cm	= height of a piece of paper
milli-	m	10^{-3}	1 ms	= time for one vibration of a guitar string playing the note D

[2]Liberia and Myanmar have not legally adopted metric units, but use them in everyday life.

The prefix centi-, meaning 10^{-2}, is only used in the centimeter; a hundredth of a gram would not be written as 1 cg but as 10 mg. The centi- prefix can be easily remembered because a cent is 10^{-2} dollars. The official SI abbreviation for seconds is "s" (not "sec") and grams are "g" (not "gm").

The second

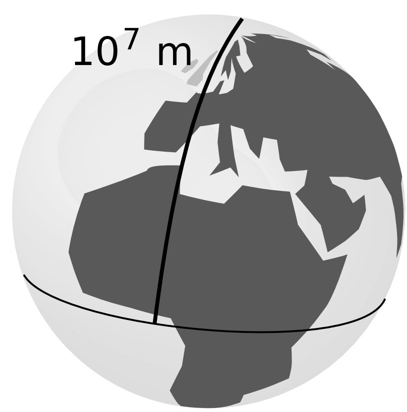
When I stated briefly above that the second was a unit of time, it may not have occurred to you that this was not much of a definition. We can make a dictionary-style definition of a term like "time," or give a general description like Isaac Newton's: "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external. . . " Newton's characterization sounds impressive, but physicists today would consider it useless as a definition of time. Today, the physical sciences are based on operational definitions, which means definitions that spell out the actual steps (operations) required to measure something numerically.

In an era when our toasters, pens, and coffee pots tell us the time, it is far from obvious to most people what is the fundamental operational definition of time. Until recently, the hour, minute, and second were defined operationally in terms of the time required for the earth to rotate about its axis. Unfortunately, the Earth's rotation is slowing down slightly, and by 1967 this was becoming an issue in scientific experiments requiring precise time measurements. The second was therefore redefined as the time required for a certain number of vibrations of the light waves emitted by a cesium atoms in a lamp constructed like a familiar neon sign but with the neon replaced by cesium. The new definition not only promises to stay constant indefinitely, but for scientists is a more convenient way of calibrating a clock than having to carry out astronomical measurements.

self-check C

What is a possible operational definition of how strong a person is? ▷ Answer, p. 557

The meter



e / The original definition of the meter.

The French originally defined the meter as 10^{-7} times the distance from the equator to the north pole, as measured through Paris (of course). Even if the definition was operational, the operation of traveling to the north pole and laying a surveying chain behind you was not one that most working scientists wanted to carry out. Fairly soon, a standard was created in the form of a metal bar with two scratches on it. This was replaced by an atomic standard in 1960, and finally in 1983 by the current definition, which is that the meter is the distance traveled by light in a vacuum over a period of (1/299792458) seconds.

The kilogram

The third base unit of the SI is the kilogram, a unit of mass. Mass is intended to be a measure of the amount of a substance, but that is not an operational definition. Bathroom scales work by measuring our planet's gravitational attraction for the object being weighed, but using that type of scale to define mass operationally would be undesirable because gravity varies in strength from place to place on the earth.

f / A duplicate of the Paris kilogram, maintained at the Danish National Metrology Institute.

There's a surprising amount of disagreement among physics textbooks about how mass should be defined, but here's how it's actually handled by the few working physicists who specialize in ultra-high-precision measurements. They maintain a physical object in Paris, which is the standard kilogram, a cylinder made of platinum-iridium alloy. Duplicates are checked against this mother of all kilograms by putting the original and the copy on the two opposite pans of a balance. Although this method of comparison depends on gravity, the problems associated with differences in gravity in different geographical locations are bypassed, because the two objects are being compared in the same place. The duplicates can then be removed from the Parisian kilogram shrine and transported elsewhere in the world. It would be desirable to replace this at some point with a universally accessible atomic standard rather than one based on a specific artifact, but as of 2010 the technology for automated counting of large numbers of atoms has not gotten good enough to make that work with the desired precision.

Combinations of metric units

Just about anything you want to measure can be measured with some combination of meters, kilograms, and seconds. Speed can be measured in m/s, volume in m^3, and density in kg/m^3. Part of what makes the SI great is this basic simplicity. No more funny units like a cord of wood, a bolt of cloth, or a jigger of whiskey. No more liquid and dry measure. Just a simple, consistent set of units. The SI measures put together from meters, kilograms, and seconds make up the mks system. For example, the mks unit of speed is m/s, not km/hr.

Checking units

A useful technique for finding mistakes in one's algebra is to analyze the units associated with the variables.

Checking units *example 1*

▷ Jae starts from the formula $V = \frac{1}{3}Ah$ for the volume of a cone, where A is the area of its base, and h is its height. He wants to find an equation that will tell him how tall a conical tent has to be in order to have a certain volume, given its radius. His algebra goes like this:

[1] $$V = \frac{1}{3}Ah$$

[2] $$A = \pi r^2$$

[3] $$V = \frac{1}{3}\pi r^2 h$$

[4] $$h = \frac{\pi r^2}{3V}$$

Is his algebra correct? If not, find the mistake.

▷ Line 4 is supposed to be an equation for the height, so the units of the expression on the right-hand side had better equal meters. The pi and the 3 are unitless, so we can ignore them. In terms of units, line 4 becomes

$$\mathrm{m} = \frac{\mathrm{m}^2}{\mathrm{m}^3} = \frac{1}{\mathrm{m}}.$$

This is false, so there must be a mistake in the algebra. The units of lines 1, 2, and 3 check out, so the mistake must be in the step from line 3 to line 4. In fact the result should have been

$$h = \frac{3V}{\pi r^2}.$$

Now the units check: $\mathrm{m} = \mathrm{m}^3/\mathrm{m}^2$.

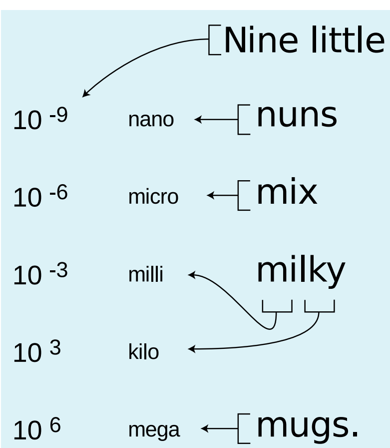
Discussion question

A Isaac Newton wrote, "...the natural days are truly unequal, though they are commonly considered as equal, and used for a measure of time... It may be that there is no such thing as an equable motion, whereby time may be accurately measured. All motions may be accelerated or retarded..." Newton was right. Even the modern definition of the second in terms of light emitted by cesium atoms is subject to variation. For instance, magnetic fields could cause the cesium atoms to emit light with a slightly different rate of vibration. What makes us think, though, that a pendulum clock is more accurate than a sundial, or that a cesium atom is a more accurate timekeeper than a pendulum clock? That is, how can one test experimentally how the accuracies of different time standards compare?

0.6 The Newton, the metric unit of force

A force is a push or a pull, or more generally anything that can change an object's speed or direction of motion. A force is required to start a car moving, to slow down a baseball player sliding in to home base, or to make an airplane turn. (Forces may fail to change an object's motion if they are canceled by other forces, e.g., the force of gravity pulling you down right now is being canceled by the force of the chair pushing up on you.) The metric unit of force is the Newton, defined as the force which, if applied for one second, will cause a 1-kilogram object starting from rest to reach a speed of 1 m/s. Later chapters will discuss the force concept in more detail. In fact, this entire book is about the relationship between force and motion.

In section 0.5, I gave a gravitational definition of mass, but by defining a numerical scale of force, we can also turn around and define a scale of mass without reference to gravity. For instance, if a force of two Newtons is required to accelerate a certain object from rest to 1 m/s in 1 s, then that object must have a mass of 2 kg. From this point of view, mass characterizes an object's resistance to a change in its motion, which we call inertia or inertial mass. Although there is no fundamental reason why an object's resistance to a change in its motion must be related to how strongly gravity affects it, careful and precise experiments have shown that the inertial definition and the gravitational definition of mass are highly consistent for a variety of objects. It therefore doesn't really matter for any practical purpose which definition one adopts.



g / This is a mnemonic to help you remember the most important metric prefixes. The word "little" is to remind you that the list starts with the prefixes used for small quantities and builds upward. The exponent changes by 3, except that of course that we do not need a special prefix for 10^0, which equals one.

Discussion question

A Spending a long time in weightlessness is unhealthy. One of the most important negative effects experienced by astronauts is a loss of muscle and bone mass. Since an ordinary scale won't work for an astronaut in orbit, what is a possible way of monitoring this change in mass? (Measuring the astronaut's waist or biceps with a measuring tape is not good enough, because it doesn't tell anything about bone mass, or about the replacement of muscle with fat.)

0.7 Less common metric prefixes

The following are three metric prefixes which, while less common than the ones discussed previously, are well worth memorizing.

prefix		meaning	example	
mega-	M	10^6	6.4 Mm	= radius of the earth
micro-	μ	10^{-6}	10 μm	= size of a white blood cell
nano-	n	10^{-9}	0.154 nm	= distance between carbon nuclei in an ethane molecule

Note that the abbreviation for micro is the Greek letter mu, μ — a common mistake is to confuse it with m (milli) or M (mega).

There are other prefixes even less common, used for extremely large and small quantities. For instance, 1 femtometer = 10^{-15} m is a convenient unit of distance in nuclear physics, and 1 gigabyte = 10^9 bytes is used for computers' hard disks. The international committee that makes decisions about the SI has recently even added some new prefixes that sound like jokes, e.g., 1 yoctogram = 10^{-24} g is about half the mass of a proton. In the immediate future, however, you're unlikely to see prefixes like "yocto-" and "zepto-" used except perhaps in trivia contests at science-fiction conventions or other geekfests.

self-check D

Suppose you could slow down time so that according to your perception, a beam of light would move across a room at the speed of a slow walk. If you perceived a nanosecond as if it was a second, how would you perceive a microsecond? ▷ Answer, p. 557

0.8 Scientific notation

Most of the interesting phenomena in our universe are not on the human scale. It would take about 1,000,000,000,000,000,000,000 bacteria to equal the mass of a human body. When the physicist Thomas Young discovered that light was a wave, it was back in the bad old days before scientific notation, and he was obliged to write that the time required for one vibration of the wave was 1/500 of a millionth of a millionth of a second. Scientific notation is a less awkward way to write very large and very small numbers such as these. Here's a quick review.

Scientific notation means writing a number in terms of a product of something from 1 to 10 and something else that is a power of ten. For instance,

$$32 = 3.2 \times 10^1$$
$$320 = 3.2 \times 10^2$$
$$3200 = 3.2 \times 10^3 \quad \ldots$$

Each number is ten times bigger than the previous one.

Since 10^1 is ten times smaller than 10^2 , it makes sense to use the notation 10^0 to stand for one, the number that is in turn ten times smaller than 10^1 . Continuing on, we can write 10^{-1} to stand for 0.1, the number ten times smaller than 10^0 . Negative exponents are used for small numbers:

$$3.2 = 3.2 \times 10^0$$
$$0.32 = 3.2 \times 10^{-1}$$
$$0.032 = 3.2 \times 10^{-2} \quad \ldots$$

A common source of confusion is the notation used on the displays of many calculators. Examples:

3.2×10^6	(written notation)
3.2E+6	(notation on some calculators)
3.2^6	(notation on some other calculators)

The last example is particularly unfortunate, because 3.2^6 really stands for the number $3.2 \times 3.2 \times 3.2 \times 3.2 \times 3.2 \times 3.2 = 1074$, a totally different number from $3.2 \times 10^6 = 3200000$. The calculator notation should never be used in writing. It's just a way for the manufacturer to save money by making a simpler display.

self-check E

A student learns that 10^4 bacteria, standing in line to register for classes at Paramecium Community College, would form a queue of this size:

The student concludes that 10^2 bacteria would form a line of this length:

Why is the student incorrect? ▷ Answer, p. 557

0.9 Conversions

Conversions are one of the three essential mathematical skills, summarized on pp.538-539, that you need for success in this course.

I suggest you avoid memorizing lots of conversion factors between SI units and U.S. units, but two that do come in handy are:

> 1 inch = 2.54 cm
>
> An object with a weight on Earth of 2.2 pounds-force has a mass of 1 kg.

The first one is the present definition of the inch, so it's exact. The second one is not exact, but is good enough for most purposes. (U.S. units of force and mass are confusing, so it's a good thing they're not used in science. In U.S. units, the unit of force is the pound-force, and the best unit to use for mass is the slug, which is about 14.6 kg.)

More important than memorizing conversion factors is understanding the right method for doing conversions. Even within the SI, you may need to convert, say, from grams to kilograms. Different people have different ways of thinking about conversions, but the method I'll describe here is systematic and easy to understand. The idea is that if 1 kg and 1000 g represent the same mass, then we can consider a fraction like

$$\frac{10^3 \text{ g}}{1 \text{ kg}}$$

to be a way of expressing the number one. This may bother you. For instance, if you type 1000/1 into your calculator, you will get 1000, not one. Again, different people have different ways of thinking about it, but the justification is that it helps us to do conversions, and it works! Now if we want to convert 0.7 kg to units of grams, we can multiply kg by the number one:

$$0.7 \text{ kg} \times \frac{10^3 \text{ g}}{1 \text{ kg}}$$

If you're willing to treat symbols such as "kg" as if they were variables as used in algebra (which they're really not), you can then cancel the kg on top with the kg on the bottom, resulting in

$$0.7 \cancel{\text{kg}} \times \frac{10^3 \text{ g}}{1 \cancel{\text{kg}}} = 700 \text{ g}.$$

To convert grams to kilograms, you would simply flip the fraction upside down.

One advantage of this method is that it can easily be applied to a series of conversions. For instance, to convert one year to units of seconds,

$$1 \cancel{\text{year}} \times \frac{365 \cancel{\text{days}}}{1 \cancel{\text{year}}} \times \frac{24 \cancel{\text{hours}}}{1 \cancel{\text{day}}} \times \frac{60 \cancel{\text{min}}}{1 \cancel{\text{hour}}} \times \frac{60 \text{ s}}{1 \cancel{\text{min}}} = \\ = 3.15 \times 10^7 \text{ s}.$$

Should that exponent be positive, or negative?

A common mistake is to write the conversion fraction incorrectly. For instance the fraction

$$\frac{10^3 \text{ kg}}{1 \text{ g}} \qquad \text{(incorrect)}$$

does not equal one, because 10^3 kg is the mass of a car, and 1 g is the mass of a raisin. One correct way of setting up the conversion factor would be

$$\frac{10^{-3} \text{ kg}}{1 \text{ g}} \qquad \text{(correct)}.$$

You can usually detect such a mistake if you take the time to check your answer and see if it is reasonable.

If common sense doesn't rule out either a positive or a negative exponent, here's another way to make sure you get it right. There are big prefixes and small prefixes:

big prefixes:	k	M	
small prefixes:	m	μ	n

(It's not hard to keep straight which are which, since "mega" and "micro" are evocative, and it's easy to remember that a kilometer is bigger than a meter and a millimeter is smaller.) In the example above, we want the top of the fraction to be the same as the bottom. Since k is a big prefix, we need to *compensate* by putting a small number like 10^{-3} in front of it, not a big number like 10^3.

▷ *Solved problem: a simple conversion* *page 35, problem 1*

▷ *Solved problem: the geometric mean* *page 35, problem 7*

Discussion question

A Each of the following conversions contains an error. In each case, explain what the error is.

(a) 1000 kg $\times \frac{1\ \text{kg}}{1000\ \text{g}}$ = 1 g

(b) 50 m $\times \frac{1\ \text{cm}}{100\ \text{m}}$ = 0.5 cm

(c) "Nano" is 10^{-9}, so there are 10^{-9} nm in a meter.

(d) "Micro" is 10^{-6}, so 1 kg is 10^6 μg.

0.10 Significant figures

The international governing body for football ("soccer" in the US) says the ball should have a circumference of 68 to 70 cm. Taking the middle of this range and dividing by π gives a diameter of approximately 21.96338214668155633610595934540698196 cm. The digits after the first few are completely meaningless. Since the circumference could have varied by about a centimeter in either direction, the diameter is fuzzy by something like a third of a centimeter. We say that the additional, random digits are not significant figures. If you write down a number with a lot of gratuitous insignificant figures, it shows a lack of scientific literacy and imples to other people a greater precision than you really have.

As a rule of thumb, the result of a calculation has as many significant figures, or "sig figs," as the least accurate piece of data that went in. In the example with the soccer ball, it didn't do us any good to know π to dozens of digits, because the bottleneck in the precision of the result was the figure for the circumference, which was two sig figs. The result is 22 cm. The rule of thumb works best for multiplication and division.

For calculations involving multiplication and division, a given fractional or "percent" error in one of the inputs causes the same fractional error in the output. The number of digits in a number provides a rough measure of its possible fractional error. These are called significant figures or "sig figs." Examples:

3.14	3 sig figs
3.1	2 sig figs
0.03	1 sig fig, because the zeroes are just placeholders
3.0×10^1	2 sig figs
30	could be 1 or 2 sig figs, since we can't tell if the 0 is a placeholder or a real sig fig

In such calculations, your result should not have more than the number of sig figs in the least accurate piece of data you started with.

Sig figs in the area of a triangle *example 2*

▷ A triangle has an area of 6.45 m^2 and a base with a width of 4.0138 m. Find its height.

▷ The area is related to the base and height by $A = bh/2$.

$$\begin{aligned} h &= \frac{2A}{b} \\ &= 3.21391200358762 \text{ m} \quad \text{(calculator output)} \\ &= 3.21 \text{ m} \end{aligned}$$

The given data were 3 sig figs and 5 sig figs. We're limited by the less accurate piece of data, so the final result is 3 sig figs. The additional digits on the calculator don't mean anything, and if we communicated them to another person, we would create the false impression of having determined h with more precision than we really obtained.

self-check F

The following quote is taken from an editorial by Norimitsu Onishi in the New York Times, August 18, 2002.

> Consider Nigeria. Everyone agrees it is Africa's most populous nation. But what is its population? The United Nations says 114 million; the State Department, 120 million. The World Bank says 126.9 million, while the Central Intelligence Agency puts it at 126,635,626.

What should bother you about this? ▷ Answer, p. 557

Dealing correctly with significant figures can save you time! Often, students copy down numbers from their calculators with eight significant figures of precision, then type them back in for a later calculation. That's a waste of time, unless your original data had that kind of incredible precision.

self-check G

How many significant figures are there in each of the following measurements?

(1) 9.937 m

(2) 4.0 s

(3) 0.0000000000000037 kg ▷ Answer, p. 557

The rules about significant figures are only rules of thumb, and are not a substitute for careful thinking. For instance, \$20.00 + \$0.05 is \$20.05. It need not and should not be rounded off to \$20. In general, the sig fig rules work best for multiplication and division, and we sometimes also apply them when doing a complicated calculation that involves many types of operations. For simple addition and subtraction, it makes more sense to maintain a fixed number of digits after the decimal point.

When in doubt, don't use the sig fig rules at all. Instead, intentionally change one piece of your initial data by the maximum amount by which you think it could have been off, and recalculate the final result. The digits on the end that are completely reshuffled are the ones that are meaningless, and should be omitted.

A nonlinear function *example 3*

▷ How many sig figs are there in $\sin 88.7°$?

▷ We're using a sine function, which isn't addition, subtraction, multiplication, or division. It would be reasonable to guess that since the input angle had 3 sig figs, so would the output. But if this was an important calculation and we really needed to know, we would do the following:

$$\sin 88.7° = 0.999742609322698$$
$$\sin 88.8° = 0.999780683474846$$

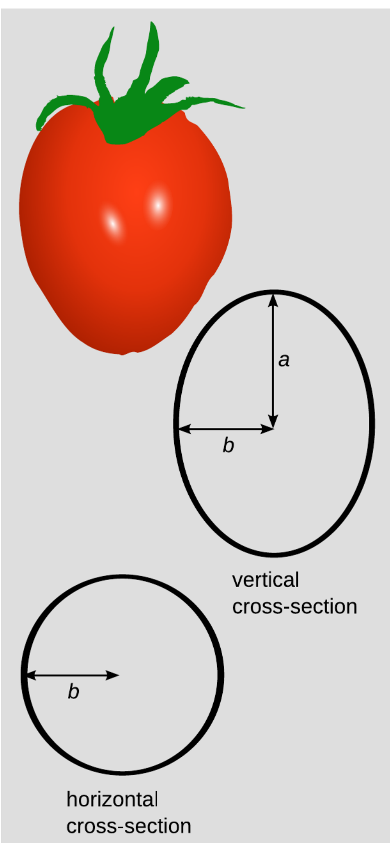
Surprisingly, the result appears to have as many as 5 sig figs, not just 3:

$$\sin 88.7° = 0.99974,$$

where the final 4 is uncertain but may have some significance. The unexpectedly high precision of the result is because the sine function is nearing its maximum at 90 degrees, where the graph flattens out and becomes insensitive to the input angle.

0.11 A note about diagrams

A quick note about diagrams. Often when you solve a problem, the best way to get started and organize your thoughts is by drawing a diagram. For an artist, it's desirable to be able to draw a recognizable, realistic, perspective picture of a tomato, like the one at the top of figure h. But in science and engineering, we usually don't draw solid figures in perspective, because that would make it difficult to label distances and angles. Usually we want views or cross-sections that project the object into its planes of symmetry, as in the line drawings in the figure.



h / A diagram of a tomato.

Summary

Selected vocabulary

matter	Anything that is affected by gravity.
light	Anything that can travel from one place to another through empty space and can influence matter, but is not affected by gravity.
operational definition	A definition that states what operations should be carried out to measure the thing being defined.
Système International	A fancy name for the metric system.
mks system	The use of metric units based on the meter, kilogram, and second. Example: meters per second is the mks unit of speed, not cm/s or km/hr.
mass	A numerical measure of how difficult it is to change an object's motion.
significant figures	Digits that contribute to the accuracy of a measurement.

Notation

m	meter, the metric distance unit
kg	kilogram, the metric unit of mass
s	second, the metric unit of time
M-	the metric prefix mega-, 10^6
k-	the metric prefix kilo-, 10^3
m-	the metric prefix milli-, 10^{-3}
μ-	the metric prefix micro-, 10^{-6}
n-	the metric prefix nano-, 10^{-9}

Summary

Physics is the use of the scientific method to study the behavior of light and matter. The scientific method requires a cycle of theory and experiment, theories with both predictive and explanatory value, and reproducible experiments.

The metric system is a simple, consistent framework for measurement built out of the meter, the kilogram, and the second plus a set of prefixes denoting powers of ten. The most systematic method for doing conversions is shown in the following example:

$$370 \text{ ms} \times \frac{10^{-3} \text{ s}}{1 \text{ ms}} = 0.37 \text{ s}$$

Mass is a measure of the amount of a substance. Mass can be defined gravitationally, by comparing an object to a standard mass on a double-pan balance, or in terms of inertia, by comparing the effect of a force on an object to the effect of the same force on a standard mass. The two definitions are found experimentally to be proportional to each other to a high degree of precision, so we

usually refer simply to "mass," without bothering to specify which type.

A force is that which can change the motion of an object. The metric unit of force is the Newton, defined as the force required to accelerate a standard 1-kg mass from rest to a speed of 1 m/s in 1 s.

Scientific notation means, for example, writing 3.2×10^5 rather than 320000.

Writing numbers with the correct number of significant figures correctly communicates how accurate they are. As a rule of thumb, the final result of a calculation is no more accurate than, and should have no more significant figures than, the least accurate piece of data.

Problems

Key

$\checkmark$ A computerized answer check is available online.
$\int$ A problem that requires calculus.
$\star$ A difficult problem.

1 Convert 134 mg to units of kg, writing your answer in scientific notation. ▷ Solution, p. 543

2 The speed of light is 3.0×10^8 m/s. Convert this to furlongs per fortnight. A furlong is 220 yards, and a fortnight is 14 days. An inch is 2.54 cm. $\checkmark$

3 Express each of the following quantities in micrograms: (a) 10 mg, (b) 10^4 g, (c) 10 kg, (d) 100×10^3 g, (e) 1000 ng. $\checkmark$

4 In the last century, the average age of the onset of puberty for girls has decreased by several years. Urban folklore has it that this is because of hormones fed to beef cattle, but it is more likely to be because modern girls have more body fat on the average and possibly because of estrogen-mimicking chemicals in the environment from the breakdown of pesticides. A hamburger from a hormone-implanted steer has about 0.2 ng of estrogen (about double the amount of natural beef). A serving of peas contains about 300 ng of estrogen. An adult woman produces about 0.5 mg of estrogen per day (note the different unit!). (a) How many hamburgers would a girl have to eat in one day to consume as much estrogen as an adult woman's daily production? (b) How many servings of peas? $\checkmark$

5 Compute the following things. If they don't make sense because of units, say so.
(a) 3 cm + 5 cm
(b) 1.11 m + 22 cm
(c) 120 miles + 2.0 hours
(d) 120 miles / 2.0 hours

6 Your backyard has brick walls on both ends. You measure a distance of 23.4 m from the inside of one wall to the inside of the other. Each wall is 29.4 cm thick. How far is it from the outside of one wall to the outside of the other? Pay attention to significant figures.

7 The usual definition of the mean (average) of two numbers a and b is $(a+b)/2$. This is called the arithmetic mean. The geometric mean, however, is defined as $(ab)^{1/2}$ (i.e., the square root of ab). For the sake of definiteness, let's say both numbers have units of mass.

(a) Compute the arithmetic mean of two numbers that have units of grams. Then convert the numbers to units of kilograms and recompute their mean. Is the answer consistent? (b) Do the same for the geometric mean. (c) If a and b both have units of grams, what should we call the units of ab? Does your answer make sense when you take the square root? (d) Suppose someone proposes to you a third kind of mean, called the superduper mean, defined as $(ab)^{1/3}$. Is this reasonable? ▷ Solution, p. 543

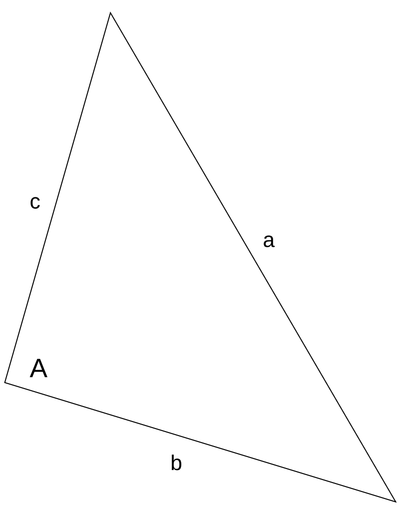
8 The distance to the horizon is given by the expression $\sqrt{2rh}$, where r is the radius of the Earth, and h is the observer's height above the Earth's surface. (This can be proved using the Pythagorean theorem.) Show that the units of this expression make sense. Don't try to prove the result, just check its units. (See example 1 on p. 25 for an example of how to do this.)

9 In an article on the SARS epidemic, the May 7, 2003 New York Times discusses conflicting estimates of the disease's incubation period (the average time that elapses from infection to the first symptoms). "The study estimated it to be 6.4 days. But other statistical calculations ... showed that the incubation period could be as long as 14.22 days." What's wrong here?

10 (a) Based on the definitions of the sine, cosine, and tangent, what units must they have? (b) A cute formula from trigonometry lets you find any angle of a triangle if you know the lengths of its sides. Using the notation shown in the figure, and letting $s = (a+b+c)/2$ be half the perimeter, we have

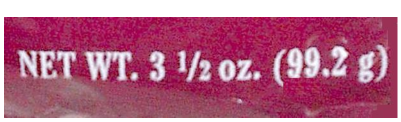
$$\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Show that the units of this equation make sense. In other words, check that the units of the right-hand side are the same as your answer to part a of the question. ▷ Solution, p. 543



Problem 10.

11 The photo shows the corner of a bag of pretzels. What's wrong here?



Problem 11.

12 Let the function x be defined by $x(t) = Ae^{bt}$, where t has units of seconds and x has units of meters. (For $b < 0$, this could be a fairly accurate model of the motion of a bullet shot into a tank of oil.) Show that the Taylor series of this function makes sense if and only if A and b have certain units.

Exercise 0: Models and idealization

Equipment:

coffee filters

ramps (one per group)

balls of various sizes

sticky tape

vacuum pump and "guinea and feather" apparatus (one)

The motion of falling objects has been recognized since ancient times as an important piece of physics, but the motion is inconveniently fast, so in our everyday experience it can be hard to tell exactly what objects are doing when they fall. In this exercise you will use several techniques to get around this problem and study the motion. Your goal is to construct a scientific *model* of falling. A model means an explanation that makes testable predictions. Often models contain simplifications or idealizations that make them easier to work with, even though they are not strictly realistic.

1. One method of making falling easier to observe is to use objects like feathers that we know from everyday experience will not fall as fast. You will use coffee filters, in stacks of various sizes, to test the following two hypotheses and see which one is true, or whether neither is true:

Hypothesis 1A: When an object is dropped, it rapidly speeds up to a certain natural falling speed, and then continues to fall at that speed. The falling speed is *proportional* to the object's weight. (A proportionality is not just a statement that if one thing gets bigger, the other does too. It says that if one becomes three times bigger, the other also gets three times bigger, etc.)

Hypothesis 1B: Different objects fall the same way, regardless of weight.

Test these hypotheses and discuss your results with your instructor.

2. A second way to slow down the action is to let a ball roll down a ramp. The steeper the ramp, the closer to free fall. Based on your experience in part 1, write a hypothesis about what will happen when you race a heavier ball against a lighter ball down the same ramp, starting them both from rest.

Hypothesis:______________________________

Show your hypothesis to your instructor, and then test it.

You have probably found that falling was more complicated than you thought! Is there more than one factor that affects the motion of a falling object? Can you imagine certain idealized situations that are simpler? Try to agree verbally with your group on an informal model of falling that can make predictions about the experiments described in parts 3 and 4.

3. You have three balls: a standard "comparison ball" of medium weight, a light ball, and a heavy ball. Suppose you stand on a chair and (a) drop the light ball side by side with the comparison ball, then (b) drop the heavy ball side by side with the comparison ball, then (c) join the light and heavy balls together with sticky tape and drop them side by side with the comparison ball.

Use your model to make a prediction:_________________________

Test your prediction.

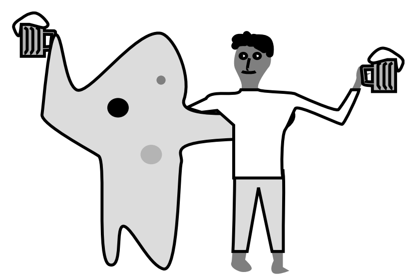
4. Your instructor will pump nearly all the air out of a chamber containing a feather and a heavier object, then let them fall side by side in the chamber.

Use your model to make a prediction:___________________________

Life would be very different if you were the size of an insect.

Chapter 1
Scaling and estimation

1.1 Introduction



a / Amoebas this size are seldom encountered.

Why can't an insect be the size of a dog? Some skinny stretched-out cells in your spinal cord are a meter tall — why does nature display no single cells that are not just a meter tall, but a meter wide, and a meter thick as well? Believe it or not, these are questions that can be answered fairly easily without knowing much more about physics than you already do. The only mathematical technique you really need is the humble conversion, applied to area and volume.

Area and volume

Area can be defined by saying that we can copy the shape of interest onto graph paper with 1 cm × 1 cm squares and count the number of squares inside. Fractions of squares can be estimated by eye. We then say the area equals the number of squares, in units of square cm. Although this might seem less "pure" than computing areas using formulae like $A = \pi r^2$ for a circle or $A = wh/2$ for a triangle, those formulae are not useful as definitions of area because they cannot be applied to irregularly shaped areas.

Units of square cm are more commonly written as cm^2 in science. Of course, the unit of measurement symbolized by "cm" is not an algebra symbol standing for a number that can be literally multiplied by itself. But it is advantageous to write the units of area that way and treat the units as if they were algebra symbols. For instance, if you have a rectangle with an area of 6m^2 and a width of 2 m, then calculating its length as $(6\ \text{m}^2)/(2\ \text{m}) = 3$ m gives a result that makes sense both numerically and in terms of units. This algebra-style treatment of the units also ensures that our methods of converting units work out correctly. For instance, if we accept the fraction

$$\frac{100\ \text{cm}}{1\ \text{m}}$$

as a valid way of writing the number one, then one times one equals one, so we should also say that one can be represented by

$$\frac{100\ \text{cm}}{1\ \text{m}} \times \frac{100\ \text{cm}}{1\ \text{m}},$$

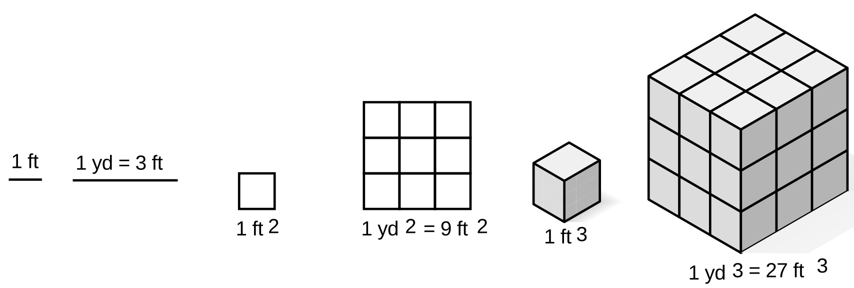
which is the same as

$$\frac{10000\ \text{cm}^2}{1\ \text{m}^2}.$$

That means the conversion factor from square meters to square centimeters is a factor of 10^4, i.e., a square meter has 10^4 square centimeters in it.

All of the above can be easily applied to volume as well, using one-cubic-centimeter blocks instead of squares on graph paper.

To many people, it seems hard to believe that a square meter equals 10000 square centimeters, or that a cubic meter equals a million cubic centimeters — they think it would make more sense if there were 100 cm^2 in 1 m^2, and 100 cm^3 in 1 m^3, but that would be incorrect. The examples shown in figure b aim to make the correct answer more believable, using the traditional U.S. units of feet and yards. (One foot is 12 inches, and one yard is three feet.)



b / Visualizing conversions of area and volume using traditional U.S. units.

self-check A

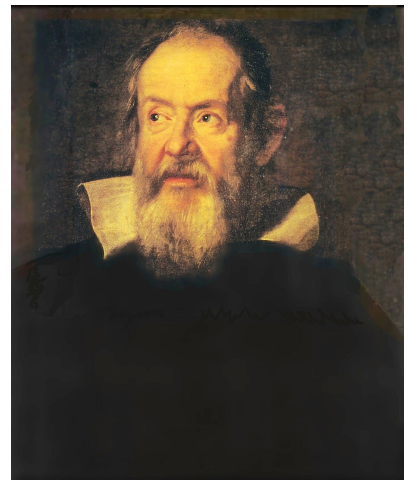
Based on figure b, convince yourself that there are 9 ft^2 in a square yard, and 27 ft^3 in a cubic yard, then demonstrate the same thing symbolically (i.e., with the method using fractions that equal one). ▷ Answer, p. 557

▷ *Solved problem: converting* mm^2 *to* cm^2 *page 54, problem 2*

▷ *Solved problem: scaling a liter* *page 54, problem 1*

Discussion question

A How many square centimeters are there in a square inch? (1 inch = 2.54 cm) First find an approximate answer by making a drawing, then derive the conversion factor more accurately using the symbolic method.



c / Galileo Galilei (1564-1642) was a Renaissance Italian who brought the scientific method to bear on physics, creating the modern version of the science. Coming from a noble but very poor family, Galileo had to drop out of medical school at the University of Pisa when he ran out of money. Eventually becoming a lecturer in mathematics at the same school, he began a career as a notorious troublemaker by writing a burlesque ridiculing the university's regulations — he was forced to resign, but found a new teaching position at Padua. He invented the pendulum clock, investigated the motion of falling bodies, and discovered the moons of Jupiter. The thrust of his life's work was to discredit Aristotle's physics by confronting it with contradictory experiments, a program that paved the way for Newton's discovery of the relationship between force and motion. In chapter 3 we'll come to the story of Galileo's ultimate fate at the hands of the Church.

1.2 Scaling of area and volume

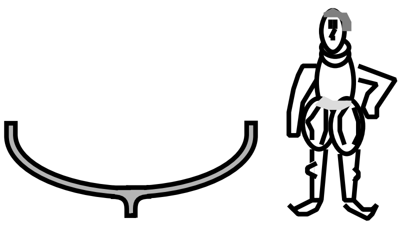
Great fleas have lesser fleas
Upon their backs to bite 'em.
And lesser fleas have lesser still,
And so ad infinitum.

Jonathan Swift

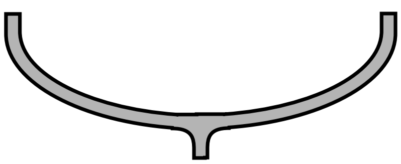
Now how do these conversions of area and volume relate to the questions I posed about sizes of living things? Well, imagine that you are shrunk like Alice in Wonderland to the size of an insect. One way of thinking about the change of scale is that what used to look like a centimeter now looks like perhaps a meter to you, because you're so much smaller. If area and volume scaled according to most people's intuitive, incorrect expectations, with 1 m^2 being the same as 100 cm^2, then there would be no particular reason why nature should behave any differently on your new, reduced scale. But nature does behave differently now that you're small. For instance, you will find that you can walk on water, and jump to many times your own height. The physicist Galileo Galilei had the basic insight that the scaling of area and volume determines how natural phenomena behave differently on different scales. He first reasoned about mechanical structures, but later extended his insights to living things, taking the then-radical point of view that at the fundamental level, a living organism should follow the same laws of nature as a machine. We will follow his lead by first discussing machines and then living things.

Galileo on the behavior of nature on large and small scales

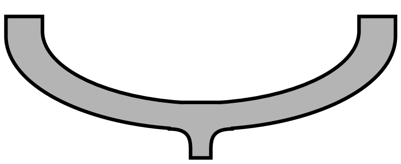
One of the world's most famous pieces of scientific writing is Galileo's Dialogues Concerning the Two New Sciences. Galileo was an entertaining writer who wanted to explain things clearly to laypeople, and he livened up his work by casting it in the form of a dialogue among three people. Salviati is really Galileo's alter ego. Simplicio is the stupid character, and one of the reasons Galileo got in trouble with the Church was that there were rumors that Simplicio represented the Pope. Sagredo is the earnest and intelligent student, with whom the reader is supposed to identify. (The following excerpts are from the 1914 translation by Crew and de Salvio.)



d / The small boat holds up just fine.



e / A larger boat built with the same proportions as the small one will collapse under its own weight.



f / A boat this large needs to have timbers that are thicker compared to its size.

> SAGREDO: Yes, that is what I mean; and I refer especially to his last assertion which I have always regarded as false. . . ; namely, that in speaking of these and other similar machines one cannot argue from the small to the large, because many devices which succeed on a small scale do not work on a large scale. Now, since mechanics has its foundations in geometry, where mere size [is unimportant], I do not see that the properties of circles, triangles, cylinders, cones and other solid figures will change with their size. If, therefore, a large machine be constructed in such a way that its parts bear to one another the same ratio as in a smaller one, and if the smaller is sufficiently strong for the purpose for which it is designed, I do not see why the larger should not be able to withstand any severe and destructive tests to which it may be subjected.

Salviati contradicts Sagredo:

> SALVIATI: . . . Please observe, gentlemen, how facts which at first seem improbable will, even on scant explanation, drop the cloak which has hidden them and stand forth in naked and simple beauty. Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon.

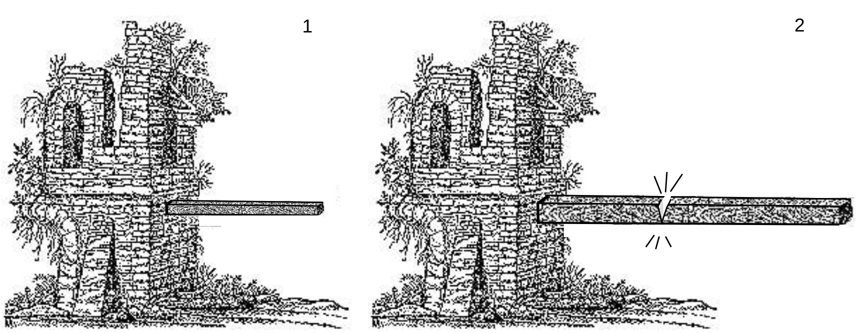
The point Galileo is making here is that small things are sturdier in proportion to their size. There are a lot of objections that could be raised, however. After all, what does it really mean for something to be "strong", to be "strong in proportion to its size," or to be strong "out of proportion to its size?" Galileo hasn't given operational definitions of things like "strength," i.e., definitions that spell out how to measure them numerically.

Also, a cat is shaped differently from a horse — an enlarged photograph of a cat would not be mistaken for a horse, even if the photo-doctoring experts at the National Inquirer made it look like a person was riding on its back. A grasshopper is not even a mammal, and it has an exoskeleton instead of an internal skeleton. The whole argument would be a lot more convincing if we could do some isolation of variables, a scientific term that means to change only one thing at a time, isolating it from the other variables that might have an effect. If size is the variable whose effect we're interested in seeing, then we don't really want to compare things that are different in size but also different in other ways.

> SALVIATI: ...we asked the reason why [shipbuilders] employed stocks, scaffolding, and bracing of larger dimensions for launching a big vessel than they do for a small one; and [an old man] answered that they did this in order to avoid the danger of the ship parting under its own heavy weight, a danger to which small boats are not subject?

After this entertaining but not scientifically rigorous beginning, Galileo starts to do something worthwhile by modern standards. He simplifies everything by considering the strength of a wooden plank. The variables involved can then be narrowed down to the type of wood, the width, the thickness, and the length. He also gives an operational definition of what it means for the plank to have a certain strength "in proportion to its size," by introducing the concept of a plank that is the longest one that would not snap under its own weight if supported at one end. If you increased its length by the slightest amount, without increasing its width or thickness, it would break. He says that if one plank is the same shape as another but a different size, appearing like a reduced or enlarged photograph of the other, then the planks would be strong "in proportion to their sizes" if both were just barely able to support their own weight.

g / Galileo discusses planks made of wood, but the concept may be easier to imagine with clay. All three clay rods in the figure were originally the same shape. The medium-sized one was twice the height, twice the length, and twice the width of the small one, and similarly the large one was twice as big as the medium one in all its linear dimensions. The big one has four times the linear dimensions of the small one, 16 times the cross-sectional area when cut perpendicular to the page, and 64 times the volume. That means that the big one has 64 times the weight to support, but only 16 times the strength compared to the smallest one.



h / 1. This plank is as long as it can be without collapsing under its own weight. If it was a hundredth of an inch longer, it would collapse. 2. This plank is made out of the same kind of wood. It is twice as thick, twice as long, and twice as wide. It will collapse under its own weight.

Also, Galileo is doing something that would be frowned on in modern science: he is mixing experiments whose results he has actually observed (building boats of different sizes), with experiments that he could not possibly have done (dropping an ant from the height of the moon). He now relates how he has done actual experiments with such planks, and found that, according to this operational definition, they are not strong in proportion to their sizes. The larger one breaks. He makes sure to tell the reader how important the result is, via Sagredo's astonished response:

> SAGREDO: My brain already reels. My mind, like a cloud momentarily illuminated by a lightning flash, is for an instant filled with an unusual light, which now beckons to me and which now suddenly mingles and obscures strange, crude ideas. From what you have said it appears to me impossible to build two similar structures of the same material, but of different sizes and have them proportionately strong.

In other words, this specific experiment, using things like wooden planks that have no intrinsic scientific interest, has very wide implications because it points out a general principle, that nature acts differently on different scales.

To finish the discussion, Galileo gives an explanation. He says that the strength of a plank (defined as, say, the weight of the heaviest boulder you could put on the end without breaking it) is proportional to its cross-sectional area, that is, the surface area of the fresh wood that would be exposed if you sawed through it in the middle. Its weight, however, is proportional to its volume.[1]

How do the volume and cross-sectional area of the longer plank compare with those of the shorter plank? We have already seen, while discussing conversions of the units of area and volume, that these quantities don't act the way most people naively expect. You might think that the volume and area of the longer plank would both be doubled compared to the shorter plank, so they would increase in proportion to each other, and the longer plank would be equally able to support its weight. You would be wrong, but Galileo knows that this is a common misconception, so he has Salviati address the point specifically:

> SALVIATI: ...Take, for example, a cube two inches on a side so that each face has an area of four square inches and the total area, i.e., the sum of the six faces, amounts to twenty-four square inches; now imagine this cube to be sawed through three times [with cuts in three perpendicular planes] so as to divide it into eight smaller cubes, each one inch on the side, each face one inch square, and the total

[1] Galileo makes a slightly more complicated argument, taking into account the effect of leverage (torque). The result I'm referring to comes out the same regardless of this effect.

surface of each cube six square inches instead of twenty-four in the case of the larger cube. It is evident therefore, that the surface of the little cube is only one-fourth that of the larger, namely, the ratio of six to twenty-four; but the volume of the solid cube itself is only one-eighth; the volume, and hence also the weight, diminishes therefore much more rapidly than the surface... You see, therefore, Simplicio, that I was not mistaken when ... I said that the surface of a small solid is comparatively greater than that of a large one.

The same reasoning applies to the planks. Even though they are not cubes, the large one could be sawed into eight small ones, each with half the length, half the thickness, and half the width. The small plank, therefore, has more surface area in proportion to its weight, and is therefore able to support its own weight while the large one breaks.

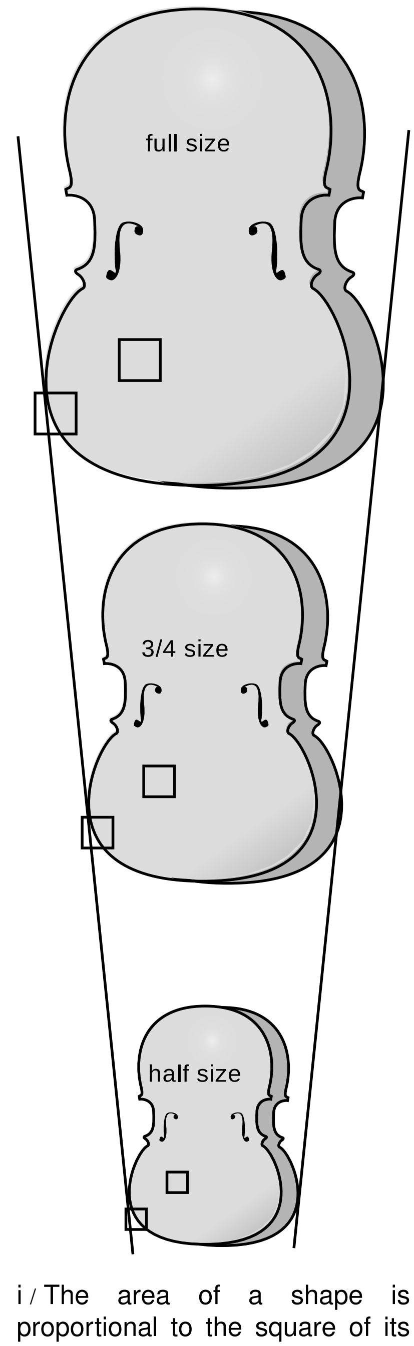
Scaling of area and volume for irregularly shaped objects

You probably are not going to believe Galileo's claim that this has deep implications for all of nature unless you can be convinced that the same is true for any shape. Every drawing you've seen so far has been of squares, rectangles, and rectangular solids. Clearly the reasoning about sawing things up into smaller pieces would not prove anything about, say, an egg, which cannot be cut up into eight smaller egg-shaped objects with half the length.

Is it always true that something half the size has one quarter the surface area and one eighth the volume, even if it has an irregular shape? Take the example of a child's violin. Violins are made for small children in smaller size to accomodate their small bodies. Figure i shows a full-size violin, along with two violins made with half and 3/4 of the normal length.[2] Let's study the surface area of the front panels of the three violins.

Consider the square in the interior of the panel of the full-size violin. In the 3/4-size violin, its height and width are both smaller by a factor of 3/4, so the area of the corresponding, smaller square becomes $3/4 \times 3/4 = 9/16$ of the original area, not 3/4 of the original area. Similarly, the corresponding square on the smallest violin has half the height and half the width of the original one, so its area is 1/4 the original area, not half.

The same reasoning works for parts of the panel near the edge, such as the part that only partially fills in the other square. The entire square scales down the same as a square in the interior, and in each violin the same fraction (about 70%) of the square is full, so the contribution of this part to the total area scales down just the same.



i / The area of a shape is proportional to the square of its linear dimensions, even if the shape is irregular.

[2]The customary terms "half-size" and "3/4-size" actually don't describe the sizes in any accurate way. They're really just standard, arbitrary marketing labels.

Since any small square region or any small region covering part of a square scales down like a square object, the entire surface area of an irregularly shaped object changes in the same manner as the surface area of a square: scaling it down by 3/4 reduces the area by a factor of 9/16, and so on.

In general, we can see that any time there are two objects with the same shape, but different linear dimensions (i.e., one looks like a reduced photo of the other), the ratio of their areas equals the ratio of the squares of their linear dimensions:

$$\frac{A_1}{A_2} = \left(\frac{L_1}{L_2}\right)^2 .$$

Note that it doesn't matter where we choose to measure the linear size, L, of an object. In the case of the violins, for instance, it could have been measured vertically, horizontally, diagonally, or even from the bottom of the left f-hole to the middle of the right f-hole. We just have to measure it in a consistent way on each violin. Since all the parts are assumed to shrink or expand in the same manner, the ratio L_1/L_2 is independent of the choice of measurement.

j / The muffin comes out of the oven too hot to eat. Breaking it up into four pieces increases its surface area while keeping the total volume the same. It cools faster because of the greater surface-to-volume ratio. In general, smaller things have greater surface-to-volume ratios, but in this example there is no easy way to compute the effect exactly, because the small pieces aren't the same shape as the original muffin.

It is also important to realize that it is completely unnecessary to have a formula for the area of a violin. It is only possible to derive simple formulas for the areas of certain shapes like circles, rectangles, triangles and so on, but that is no impediment to the type of reasoning we are using.

Sometimes it is inconvenient to write all the equations in terms of ratios, especially when more than two objects are being compared. A more compact way of rewriting the previous equation is

$$A \propto L^2 .$$

The symbol "$\propto$" means "is proportional to." Scientists and engineers often speak about such relationships verbally using the phrases "scales like" or "goes like," for instance "area goes like length squared."

All of the above reasoning works just as well in the case of volume. Volume goes like length cubed:

$$V \propto L^3 .$$

self-check B

When a car or truck travels over a road, there is wear and tear on the road surface, which incurs a cost. Studies show that the cost C per kilometer of travel is related to the weight per axle w by $C \propto w^4$. Translate this into a statement about ratios. ▷ Answer, p. 557

If different objects are made of the same material with the same density, $\rho = m/V$, then their masses, $m = \rho V$, are proportional to L^3. (The symbol for density is ρ, the lower-case Greek letter "rho.")

An important point is that all of the above reasoning about scaling only applies to objects that are the same shape. For instance, a piece of paper is larger than a pencil, but has a much greater surface-to-volume ratio.

Scaling of the area of a triangle — *example 1*

▷ In figure k, the larger triangle has sides twice as long. How many times greater is its area?

Correct solution #1: Area scales in proportion to the square of the linear dimensions, so the larger triangle has four times more area ($2^2 = 4$).

Correct solution #2: You could cut the larger triangle into four of the smaller size, as shown in fig. (b), so its area is four times greater. (This solution is correct, but it would not work for a shape like a circle, which can't be cut up into smaller circles.)

Correct solution #3: The area of a triangle is given by

$A = bh/2$, where b is the base and h is the height. The areas of the triangles are

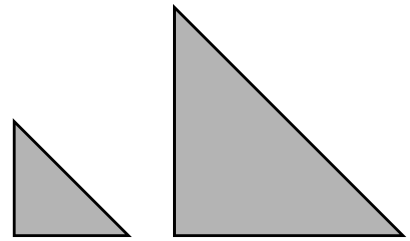
$$\begin{aligned} A_1 &= b_1 h_1/2 \\ A_2 &= b_2 h_2/2 \\ &= (2b_1)(2h_1)/2 \\ &= 2b_1 h_1 \\ A_2/A_1 &= (2b_1 h_1)/(b_1 h_1/2) \\ &= 4 \end{aligned}$$

(Although this solution is correct, it is a lot more work than solution #1, and it can only be used in this case because a triangle is a simple geometric shape, and we happen to know a formula for its area.)

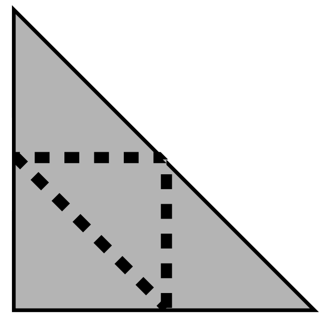
Correct solution #4: The area of a triangle is $A = bh/2$. The comparison of the areas will come out the same as long as the ratios of the linear sizes of the triangles is as specified, so let's just say $b_1 = 1.00$ m and $b_2 = 2.00$ m. The heights are then also $h_1 = 1.00$ m and $h_2 = 2.00$ m, giving areas $A_1 = 0.50\ \text{m}^2$ and $A_2 = 2.00\ \text{m}^2$, so $A_2/A_1 = 4.00$.

(The solution is correct, but it wouldn't work with a shape for whose area we don't have a formula. Also, the numerical calculation might make the answer of 4.00 appear inexact, whereas solution #1 makes it clear that it is exactly 4.)

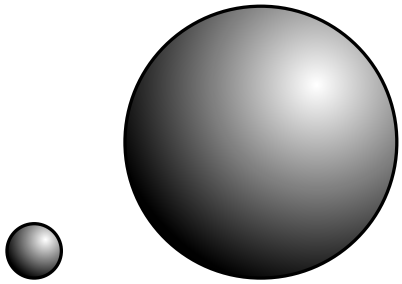
Incorrect solution: The area of a triangle is $A = bh/2$, and if you plug in $b = 2.00$ m and $h = 2.00$ m, you get $A = 2.00\ \text{m}^2$, so the bigger triangle has 2.00 times more area. (This solution is incorrect because no comparison has been made with the smaller triangle.)



k / Example 1. The big triangle has four times more area than the little one.



l / A tricky way of solving example 1, explained in solution #2.



m / Example 2. The big sphere has 125 times more volume than the little one.

Scaling of the volume of a sphere *example 2*

▷ In figure m, the larger sphere has a radius that is five times greater. How many times greater is its volume?

Correct solution #1: Volume scales like the third power of the linear size, so the larger sphere has a volume that is 125 times greater ($5^3 = 125$).

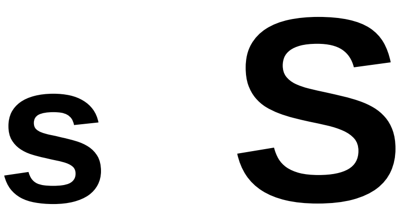
Correct solution #2: The volume of a sphere is $V = (4/3)\pi r^3$, so

$$\begin{aligned} V_1 &= \frac{4}{3}\pi r_1^3 \\ V_2 &= \frac{4}{3}\pi r_2^3 \\ &= \frac{4}{3}\pi(5r_1)^3 \\ &= \frac{500}{3}\pi r_1^3 \\ V_2/V_1 &= \left(\frac{500}{3}\pi r_1^3\right)/\left(\frac{4}{3}\pi r_1^3\right) &= 125 \end{aligned}$$

Incorrect solution: The volume of a sphere is $V = (4/3)\pi r^3$, so

$$\begin{aligned} V_1 &= \frac{4}{3}\pi r_1^3 \\ V_2 &= \frac{4}{3}\pi r_2^3 \\ &= \frac{4}{3}\pi\cdot 5r_1^3 \\ &= \frac{20}{3}\pi r_1^3 \\ V_2/V_1 &= \left(\frac{20}{3}\pi r_1^3\right)/\left(\frac{4}{3}\pi r_1^3\right) &= 5 \end{aligned}$$

(The solution is incorrect because $(5r_1)^3$ is not the same as $5r_1^3$.)



n / Example 3. The 48-point "S" has 1.78 times more area than the 36-point "S."

Scaling of a more complex shape *example 3*

▷ The first letter "S" in figure n is in a 36-point font, the second in 48-point. How many times more ink is required to make the larger "S"? (Points are a unit of length used in typography.)

Correct solution: The amount of ink depends on the area to be covered with ink, and area is proportional to the square of the linear dimensions, so the amount of ink required for the second "S" is greater by a factor of $(48/36)^2 = 1.78$.

Incorrect solution: The length of the curve of the second "S" is longer by a factor of $48/36 = 1.33$, so 1.33 times more ink is required.

(The solution is wrong because it assumes incorrectly that the width of the curve is the same in both cases. Actually both the width and the length of the curve are greater by a factor of 48/36, so the area is greater by a factor of $(48/36)^2 = 1.78$.)

Reasoning about ratios and proportionalities is one of the three essential mathematical skills, summarized on pp.538-539, that you need for success in this course.

▷ *Solved problem: a telescope gathers light* *page 54, problem 3*

▷ *Solved problem: distance from an earthquake* *page 54, problem 8*

Discussion questions

A A toy fire engine is 1/30 the size of the real one, but is constructed from the same metal with the same proportions. How many times smaller is its weight? How many times less red paint would be needed to paint it?

B Galileo spends a lot of time in his dialog discussing what really happens when things break. He discusses everything in terms of Aristotle's now-discredited explanation that things are hard to break, because if something breaks, there has to be a gap between the two halves with nothing in between, at least initially. Nature, according to Aristotle, "abhors a vacuum," i.e., nature doesn't "like" empty space to exist. Of course, air will rush into the gap immediately, but at the very moment of breaking, Aristotle imagined a vacuum in the gap. Is Aristotle's explanation of why it is hard to break things an experimentally testable statement? If so, how could it be tested experimentally?

1.3 Order-of-magnitude estimates

It is the mark of an instructed mind to rest satisfied with the degree of precision that the nature of the subject permits and not to seek an exactness where only an approximation of the truth is possible.

Aristotle

It is a common misconception that science must be exact. For instance, in the Star Trek TV series, it would often happen that Captain Kirk would ask Mr. Spock, "Spock, we're in a pretty bad situation. What do you think are our chances of getting out of here?" The scientific Mr. Spock would answer with something like, "Captain, I estimate the odds as 237.345 to one." In reality, he could not have estimated the odds with six significant figures of accuracy, but nevertheless one of the hallmarks of a person with a good education in science is the ability to make estimates that are likely to be at least somewhere in the right ballpark. In many such situations, it is often only necessary to get an answer that is off by no more than a factor of ten in either direction. Since things that differ by a factor of ten are said to differ by one order of magnitude, such an estimate is called an order-of-magnitude estimate. The tilde, $\sim$, is used to indicate that things are only of the same order of magnitude, but not exactly equal, as in

$$\text{odds of survival} \sim 100 \text{ to one.}$$

The tilde can also be used in front of an individual number to emphasize that the number is only of the right order of magnitude.

Although making order-of-magnitude estimates seems simple and natural to experienced scientists, it's a mode of reasoning that is completely unfamiliar to most college students. Some of the typical mental steps can be illustrated in the following example.

o / Can you guess how many jelly beans are in the jar? If you try to guess directly, you will almost certainly underestimate. The right way to do it is to estimate the linear dimensions, then get the volume indirectly. See problem 24, p. 57.

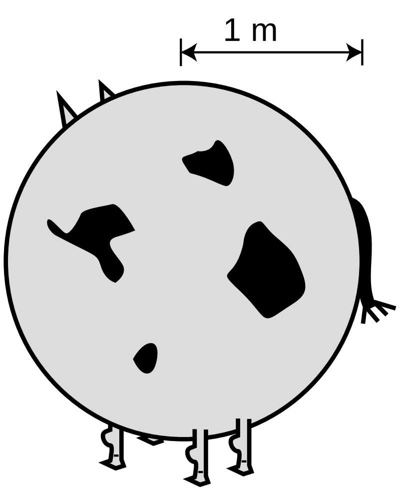
Cost of transporting tomatoes (incorrect solution) *example 4*

▷ Roughly what percentage of the price of a tomato comes from the cost of transporting it in a truck?

▷ The following incorrect solution illustrates one of the main ways you can go wrong in order-of-magnitude estimates.

Incorrect solution: Let's say the trucker needs to make a \$400 profit on the trip. Taking into account her benefits, the cost of gas, and maintenance and payments on the truck, let's say the total cost is more like \$2000. I'd guess about 5000 tomatoes would fit in the back of the truck, so the extra cost per tomato is 40 cents. That means the cost of transporting one tomato is comparable to the cost of the tomato itself. Transportation really adds a lot to the cost of produce, I guess.

The problem is that the human brain is not very good at estimating area or volume, so it turns out the estimate of 5000 tomatoes fitting in the truck is way off. That's why people have a hard time at those contests where you are supposed to estimate the number of jellybeans in a big jar. Another example is that most people think their families use about 10 gallons of water per day, but in reality the average is about 300 gallons per day. When estimating area or volume, you are much better off estimating linear dimensions, and computing volume from the linear dimensions. Here's a better solution to the problem about the tomato truck:



p / Consider a spherical cow.

Cost of transporting tomatoes (correct solution) *example 5*

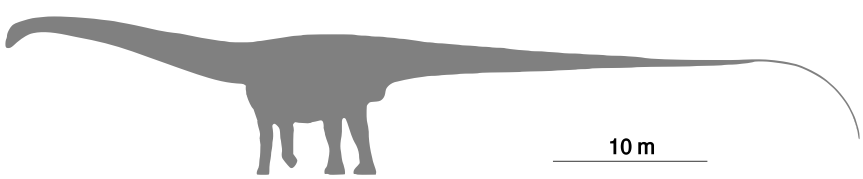
As in the previous solution, say the cost of the trip is \$2000. The dimensions of the bin are probably 4 m $\times$ 2 m $\times$ 1 m, for a volume of 8 m^3. Since the whole thing is just an order-of-magnitude estimate, let's round that off to the nearest power of ten, 10 m^3. The shape of a tomato is complicated, and I don't know any formula for the volume of a tomato shape, but since this is just an estimate, let's pretend that a tomato is a cube, 0.05 m $\times$ 0.05 m $\times$ 0.05 m, for a volume of 1.25×10^{-4} m^3. Since this is just a rough estimate, let's round that to $10^{-4} m^3$. We can find the total number of tomatoes by dividing the volume of the bin by the volume of one tomato: $10\ m^3/10^{-4}\ m^3 = 10^5$ tomatoes. The transportation cost per tomato is \$2000/10^5 tomatoes=\$0.02/tomato. That means that transportation really doesn't contribute very much to the cost of a tomato.

Approximating the shape of a tomato as a cube is an example of another general strategy for making order-of-magnitude estimates. A similar situation would occur if you were trying to estimate how many m^2 of leather could be produced from a herd of ten thousand cattle. There is no point in trying to take into account the shape of the cows' bodies. A reasonable plan of attack might be to consider a spherical cow. Probably a cow has roughly the same surface area as a sphere with a radius of about 1 m, which would be $4\pi(1\ \text{m})^2$. Using the well-known facts that pi equals three, and four times three equals about ten, we can guess that a cow has a surface area of about $10\ \text{m}^2$, so the herd as a whole might yield $10^5\ \text{m}^2$ of leather.

Estimating mass indirectly *example 6*

Usually the best way to estimate mass is to estimate linear dimensions, then use those to infer volume, and then get the mass based on the volume. For example, *Amphicoelias*, shown in the figure, may have been the largest land animal ever to live. Fossils tell us the linear dimensions of an animal, but we can only indirectly guess its mass. Given the length scale in the figure, let's estimate the mass of an *Amphicoelias*.

Its torso looks like it can be approximated by a rectangular box with dimensions $10\ \text{m} \times 5\ \text{m} \times 3\ \text{m}$, giving about $2 \times 10^2\ \text{m}^3$. Living things are mostly made of water, so we assume the animal to have the density of water, $1\ \text{g/cm}^3$, which converts to $10^3\ \text{kg/m}^3$. This gives a mass of about 2×10^5 kg, or 200 metric tons.



The following list summarizes the strategies for getting a good order-of-magnitude estimate.

1. Don't even attempt more than one significant figure of precision.

2. Don't guess area, volume, or mass directly. Guess linear dimensions and get area, volume, or mass from them.

3. When dealing with areas or volumes of objects with complex shapes, idealize them as if they were some simpler shape, a cube or a sphere, for example.

4. Check your final answer to see if it is reasonable. If you estimate that a herd of ten thousand cattle would yield 0.01 m^2 of leather, then you have probably made a mistake with conversion factors somewhere.

Summary

Notation

$\propto$ . . . . is proportional to
$\sim$ . . . . on the order of, is on the order of

Summary

Nature behaves differently on large and small scales. Galileo showed that this results fundamentally from the way area and volume scale. Area scales as the second power of length, $A \propto L^2$, while volume scales as length to the third power, $V \propto L^3$.

An order of magnitude estimate is one in which we do not attempt or expect an exact answer. The main reason why the uninitiated have trouble with order-of-magnitude estimates is that the human brain does not intuitively make accurate estimates of area and volume. Estimates of area and volume should be approached by first estimating linear dimensions, which one's brain has a feel for.

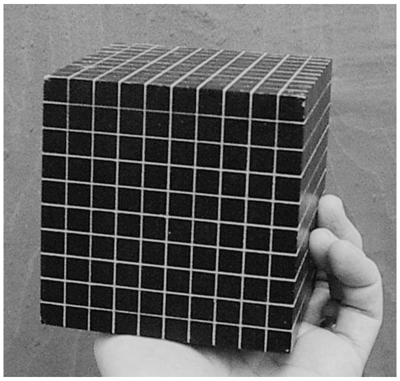
Problems

Key

√ A computerized answer check is available online.

∫ A problem that requires calculus.

⋆ A difficult problem.



Problem 1.

1 The one-liter cube in the photo has been marked off into smaller cubes, with linear dimensions one tenth those of the big one. What is the volume of each of the small cubes?
▹ Solution, p. 544

2 How many cm^2 is 1 mm^2? ▹ Solution, p. 544

3 Compare the light-gathering powers of a 3-cm-diameter telescope and a 30-cm telescope. ▹ Solution, p. 544

4 The traditional Martini glass is shaped like a cone with the point at the bottom. Suppose you make a Martini by pouring vermouth into the glass to a depth of 3 cm, and then adding gin to bring the depth to 6 cm. What are the proportions of gin and vermouth?
▹ Solution, p. 544

5 How many cubic inches are there in a cubic foot? The answer is not 12. √

6 Assume a dog's brain is twice as great in diameter as a cat's, but each animal's brain cells are the same size and their brains are the same shape. In addition to being a far better companion and much nicer to come home to, how many times more brain cells does a dog have than a cat? The answer is not 2.

7 The population density of Los Angeles is about 4000 people/km^2. That of San Francisco is about 6000 people/km^2. How many times farther away is the average person's nearest neighbor in LA than in San Francisco? The answer is not 1.5. √

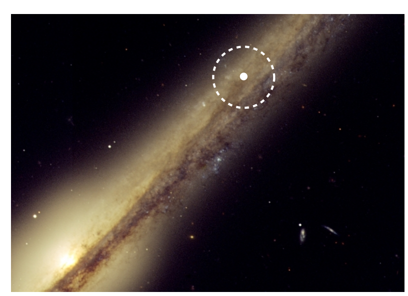
8 One step on the Richter scale corresponds to a factor of 100 in terms of the energy absorbed by something on the surface of the Earth, e.g., a house. For instance, a 9.3-magnitude quake would release 100 times more energy than an 8.3. The energy spreads out from the epicenter as a wave, and for the sake of this problem we'll assume we're dealing with seismic waves that spread out in three dimensions, so that we can visualize them as hemispheres spreading out under the surface of the earth. If a certain 7.6-magnitude earthquake and a certain 5.6-magnitude earthquake produce the same amount of vibration where I live, compare the distances from my house to the two epicenters. ▹ Solution, p. 544

9 The central portion of a CD is taken up by the hole and some surrounding clear plastic, and this area is unavailable for storing data. The radius of the central circle is about 35% of the outer radius of the data-storing area. What percentage of the CD's area is therefore lost? $\checkmark$

10 A taxon (plural taxa) is a group of living things. For example, *Homo sapiens* and *Homo neanderthalensis* are both taxa — specifically, they are two different species within the genus *Homo*. Surveys by botanists show that the number of plant taxa native to a given contiguous land area A is usually approximately proportional to $A^{1/3}$. (a) There are 70 different species of lupine native to Southern California, which has an area of about $200,000\ \text{km}^2$. The San Gabriel Mountains cover about $1,600\ \text{km}^2$. Suppose that you wanted to learn to identify all the species of lupine in the San Gabriels. Approximately how many species would you have to familiarize yourself with? ▷ Answer, p. 562 $\checkmark$
(b) What is the interpretation of the fact that the exponent, 1/3, is less than one?

11 X-ray images aren't only used with human subjects but also, for example, on insects and flowers. In 2003, a team of researchers at Argonne National Laboratory used x-ray imagery to find for the first time that insects, although they do not have lungs, do not necessarily breathe completely passively, as had been believed previously; many insects rapidly compress and expand their trachea, head, and thorax in order to force air in and out of their bodies. One difference between x-raying a human and an insect is that if a medical x-ray machine was used on an insect, virtually 100% of the x-rays would pass through its body, and there would be no contrast in the image produced. Less penetrating x-rays of lower energies have to be used. For comparison, a typical human body mass is about 70 kg, whereas a typical ant is about 10 mg. Estimate the ratio of the thicknesses of tissue that must be penetrated by x-rays in one case compared to the other. $\checkmark$

12 Radio was first commercialized around 1920, and ever since then, radio signals from our planet have been spreading out across our galaxy. It is possible that alien civilizations could detect these signals and learn that there is life on earth. In the 90 years that the signals have been spreading at the speed of light, they have created a sphere with a radius of 90 light-years. To show an idea of the size of this sphere, I've indicated it in the figure as a tiny white circle on an image of a spiral galaxy seen edge on. (We don't have similar photos of our own Milky Way galaxy, because we can't see it from the outside.) So far we haven't received answering signals from aliens within this sphere, but as time goes on, the sphere will expand as suggested by the dashed outline, reaching more and more stars that might harbor extraterrestrial life. Approximately what year will it be when the sphere has expanded to fill a volume 100 times greater than the volume it fills today in 2010? $\checkmark$



Problem 12.

13 The Earth's surface is about 70% water. Mars's diameter is about half the Earth's, but it has no surface water. Compare the land areas of the two planets. ✓

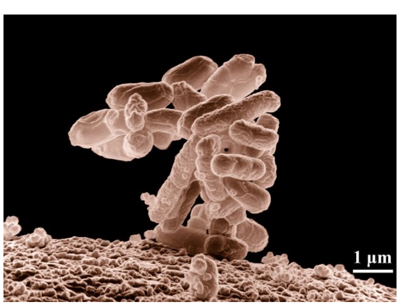
14 In Europe, a piece of paper of the standard size, called A4, is a little narrower and taller than its American counterpart. The ratio of the height to the width is the square root of 2, and this has some useful properties. For instance, if you cut an A4 sheet from left to right, you get two smaller sheets that have the same proportions. You can even buy sheets of this smaller size, and they're called A5. There is a whole series of sizes related in this way, all with the same proportions. (a) Compare an A5 sheet to an A4 in terms of area and linear size. (b) The series of paper sizes starts from an A0 sheet, which has an area of one square meter. Suppose we had a series of boxes defined in a similar way: the B0 box has a volume of one cubic meter, two B1 boxes fit exactly inside an B0 box, and so on. What would be the dimensions of a B0 box? ✓

15 Estimate the volume of a human body, in cm^3.

16 Estimate the number of blades of grass on a football field.

17 In a computer memory chip, each bit of information (a 0 or a 1) is stored in a single tiny circuit etched onto the surface of a silicon chip. The circuits cover the surface of the chip like lots in a housing development. A typical chip stores 64 Mb (megabytes) of data, where a byte is 8 bits. Estimate (a) the area of each circuit, and (b) its linear size.

18 Suppose someone built a gigantic apartment building, measuring 10 km × 10 km at the base. Estimate how tall the building would have to be to have space in it for the entire world's population to live.



Problem 19.

19 (a) Using the microscope photo in the figure, estimate the mass of a one cell of the *E. coli* bacterium, which is one of the most common ones in the human intestine. Note the scale at the lower right corner, which is 1 μm. Each of the tubular objects in the column is one cell. (b) The feces in the human intestine are mostly bacteria (some dead, some alive), of which *E. coli* is a large and typical component. Estimate the number of bacteria in your intestines, and compare with the number of human cells in your body, which is believed to be roughly on the order of 10^{13}. (c) Interpreting your result from part b, what does this tell you about the size of a typical human cell compared to the size of a typical bacterial cell?

20 A hamburger chain advertises that it has sold 10 billion Bongo Burgers. Estimate the total mass of feed required to raise the cows used to make the burgers.

21 Estimate the mass of one of the hairs in Albert Einstein's moustache, in units of kg.

22 Estimate the number of man-hours required for building the Great Wall of China. ▷ Solution, p. 544

23 According to folklore, every time you take a breath, you are inhaling some of the atoms exhaled in Caesar's last words. Is this true? If so, how many?

24 Estimate the number of jellybeans in figure o on p. 50. ▷ Solution, p. 544

25 At the grocery store you will see oranges packed neatly in stacks. Suppose we want to pack spheres as densely as possible, so that the greatest possible fraction of the space is filled by the spheres themselves, not by empty space. Let's call this fraction f. Mathematicians have proved that the best possible result is $f \approx 0.7405$, which requires a systematic pattern of stacking. If you buy ball bearings or golf balls, however, the seller is probably not going to go to the trouble of stacking them neatly. Instead they will probably pour the balls into a box and vibrate the box vigorously for a while to make them settle. This results in a random packing. The closest random packing has $f \approx 0.64$. Suppose that golf balls, with a standard diameter of 4.27 cm, are sold in bulk with the closest random packing. What is the diameter of the largest ball that could be sold in boxes of the same size, packed systematically, so that there would be the same number of balls per box? √

26 Plutonium-239 is one of a small number of important long-lived forms of high-level radioactive nuclear waste. The world's waste stockpiles have about 10^3 metric tons of plutonium. Drinking water is considered safe by U.S. government standards if it contains less than 2×10^{-13} g/cm^3 of plutonium. The amount of radioactivity to which you were exposed by drinking such water on a daily basis would be very small compared to the natural background radiation that you are exposed to every year. Suppose that the world's inventory of plutonium-239 were ground up into an extremely fine dust and then dispersed over the world's oceans, thereby becoming mixed uniformly into the world's water supplies over time. Estimate the resulting concentration of plutonium, and compare with the government standard.

Albert Einstein, and his moustache, problem 21.

Problem 25.

Exercise 1: Scaling applied to leaves

Equipment:

leaves of three sizes, having roughly similar proportions of length, width, and thickness

balance

Each group will have one leaf, and should measure its surface area and volume, and determine its surface-to-volume ratio. For consistency, every group should use units of cm^2 and cm^3, and should only find the area of one side of the leaf. The area can be found by tracing the area of the leaf on graph paper and counting squares. The volume can be found by weighing the leaf and assuming that its density is 1 g/cm^3 (the density of water). What implications do your results have for the plants' abilities to survive in different environments?

Motion in one dimension

Chapter 2
Velocity and relative motion

2.1 Types of motion

If you had to think consciously in order to move your body, you would be severely disabled. Even walking, which we consider to be no great feat, requires an intricate series of motions that your cerebrum would be utterly incapable of coordinating. The task of putting one foot in front of the other is controlled by the more primitive parts of your brain, the ones that have not changed much since the mammals and reptiles went their separate evolutionary ways. The thinking part of your brain limits itself to general directives such as "walk faster," or "don't step on her toes," rather than micromanaging every contraction and relaxation of the hundred or so muscles of your hips, legs, and feet.

Physics is all about the conscious understanding of motion, but we're obviously not immediately prepared to understand the most complicated types of motion. Instead, we'll use the divide-and-conquer technique. We'll first classify the various types of motion, and then begin our campaign with an attack on the simplest cases. To make it clear what we are and are not ready to consider, we need to examine and define carefully what types of motion can exist.

a / Rotation.

b / Simultaneous rotation and motion through space.

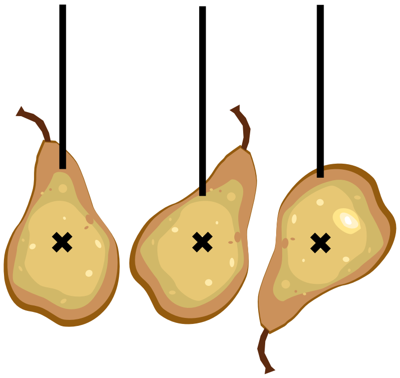
c / One person might say that the tipping chair was only rotating in a circle about its point of contact with the floor, but another could describe it as having both rotation and motion through space.

Rigid-body motion distinguished from motion that changes an object's shape

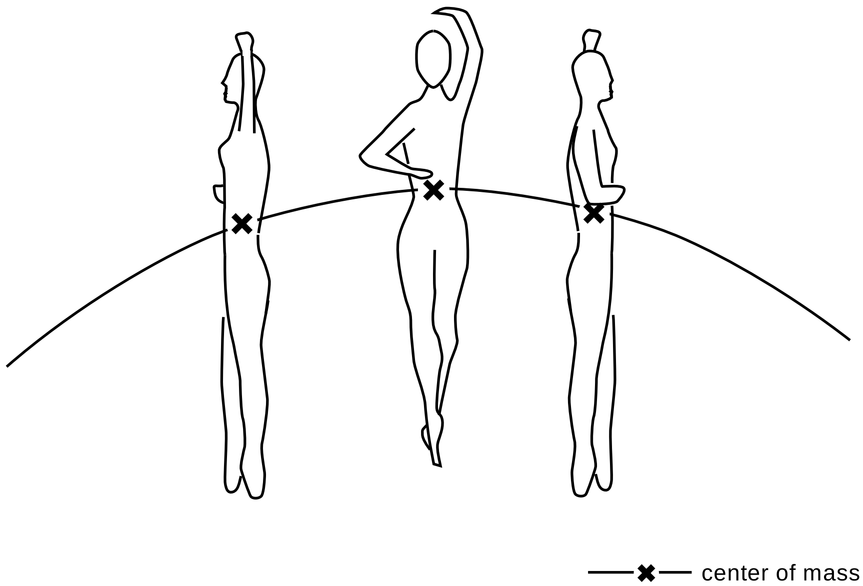
Nobody, with the possible exception of Fred Astaire, can simply glide forward without bending their joints. Walking is thus an example in which there is both a general motion of the whole object and a change in the shape of the object. Another example is the motion of a jiggling water balloon as it flies through the air. We are not presently attempting a mathematical description of the way in which the shape of an object changes. Motion without a change in shape is called rigid-body motion. (The word "body" is often used in physics as a synonym for "object.")

Center-of-mass motion as opposed to rotation

A ballerina leaps into the air and spins around once before landing. We feel intuitively that her rigid-body motion while her feet are off the ground consists of two kinds of motion going on simultaneously: a rotation and a motion of her body as a whole through space, along an arc. It is not immediately obvious, however, what is the most useful way to define the distinction between rotation and motion through space. Imagine that you attempt to balance a chair and it falls over. One person might say that the only motion was a rotation about the chair's point of contact with the floor, but another might say that there was both rotation and motion down and to the side.



e / No matter what point you hang the pear from, the string lines up with the pear's center of mass. The center of mass can therefore be defined as the intersection of all the lines made by hanging the pear in this way. Note that the X in the figure should not be interpreted as implying that the center of mass is on the surface — it is actually inside the pear.

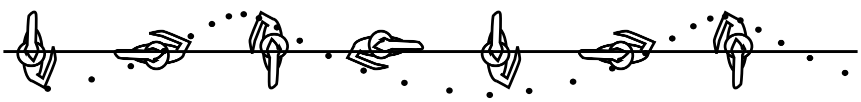


d / The leaping dancer's motion is complicated, but the motion of her center of mass is simple.

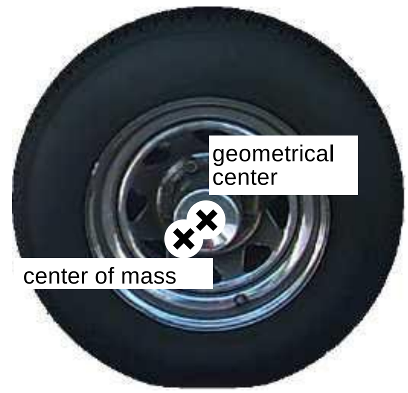
f / The circus performers hang with the ropes passing through their centers of mass.

It turns out that there is one particularly natural and useful way to make a clear definition, but it requires a brief digression. Every object has a balance point, referred to in physics as the *center of mass*. For a two-dimensional object such as a cardboard cutout, the center of mass is the point at which you could hang the object from a string and make it balance. In the case of the ballerina (who is likely to be three-dimensional unless her diet is particularly severe), it might be a point either inside or outside her body, depending on how she holds her arms. Even if it is not practical to attach a string to the balance point itself, the center of mass can be defined as shown in figure e.

Why is the center of mass concept relevant to the question of classifying rotational motion as opposed to motion through space? As illustrated in figures d and g, it turns out that the motion of an object's center of mass is nearly always far simpler than the motion of any other part of the object. The ballerina's body is a large object with a complex shape. We might expect that her motion would be much more complicated than the motion of a small, simply-shaped object, say a marble, thrown up at the same angle as the angle at which she leapt. But it turns out that the motion of the ballerina's center of mass is exactly the same as the motion of the marble. That is, the motion of the center of mass is the same as the motion the ballerina would have if all her mass was concentrated at a point. By restricting our attention to the motion of the center of mass, we can therefore simplify things greatly.

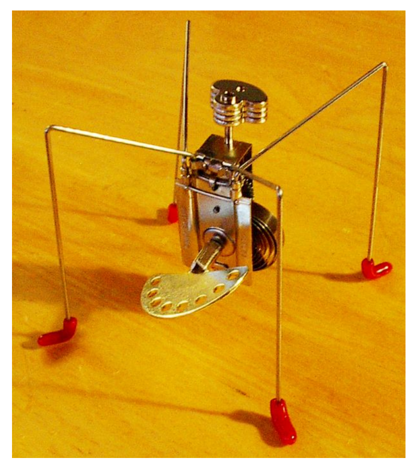


g / The same leaping dancer, viewed from above. Her center of mass traces a straight line, but a point away from her center of mass, such as her elbow, traces the much more complicated path shown by the dots.



h / An improperly balanced wheel has a center of mass that is not at its geometric center. When you get a new tire, the mechanic clamps little weights to the rim to balance the wheel.

We can now replace the ambiguous idea of "motion as a whole through space" with the more useful and better defined concept of "center-of-mass motion." The motion of any rigid body can be cleanly split into rotation and center-of-mass motion. By this definition, the tipping chair does have both rotational and center-of-mass motion. Concentrating on the center of mass motion allows us to make a simplified model of the motion, as if a complicated object like a human body was just a marble or a point-like particle. Science really never deals with reality; it deals with models of reality.

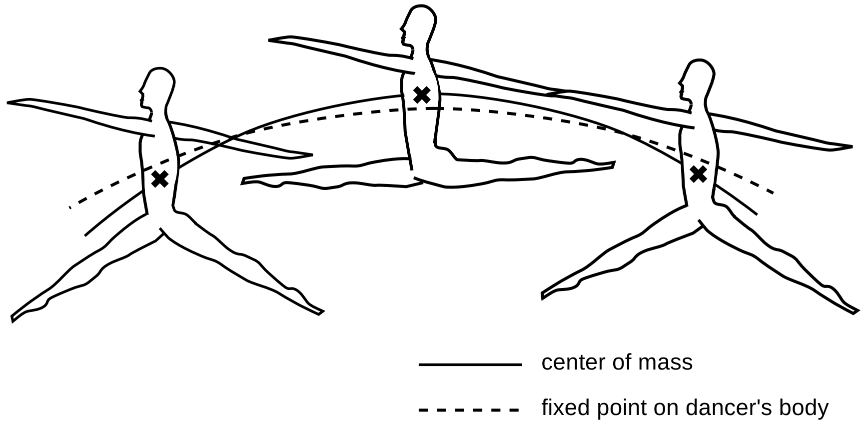


i / This toy was intentionally designed so that the mushroom-shaped piece of metal on top would throw off the center of mass. When you wind it up, the mushroom spins, but the center of mass doesn't want to move, so the rest of the toy tends to counter the mushroom's motion, causing the whole thing to jump around.

Note that the word "center" in "center of mass" is not meant to imply that the center of mass must lie at the geometrical center of an object. A car wheel that has not been balanced properly has a center of mass that does not coincide with its geometrical center. An object such as the human body does not even have an obvious geometrical center.

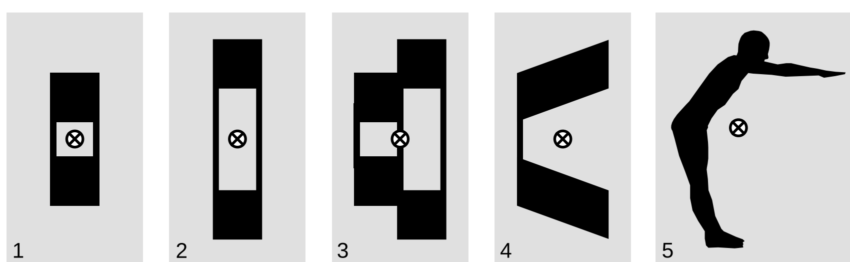
It can be helpful to think of the center of mass as the average location of all the mass in the object. With this interpretation, we can see for example that raising your arms above your head raises your center of mass, since the higher position of the arms' mass raises the average. We won't be concerned right now with calculating centers of mass mathematically; the relevant equations are in ch. 14.

Ballerinas and professional basketball players can create an illusion of flying horizontally through the air because our brains intuitively expect them to have rigid-body motion, but the body does not stay rigid while executing a grand jete or a slam dunk. The legs



j / A fixed point on the dancer's body follows a trajectory that is flatter than what we expect, creating an illusion of flight.

are low at the beginning and end of the jump, but come up higher at the middle. Regardless of what the limbs do, the center of mass will follow the same arc, but the low position of the legs at the beginning and end means that the torso is higher compared to the center of mass, while in the middle of the jump it is lower compared to the center of mass. Our eye follows the motion of the torso and tries to interpret it as the center-of-mass motion of a rigid body. But since the torso follows a path that is flatter than we expect, this attempted interpretation fails, and we experience an illusion that the person is flying horizontally.



k / Example 1.

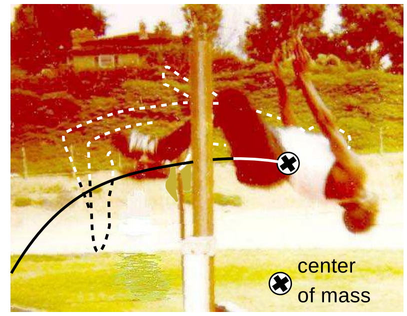
The center of mass as an average *example 1*

▷ Explain how we know that the center of mass of each object is at the location shown in figure k.

▷ The center of mass is a sort of average, so the height of the centers of mass in 1 and 2 has to be midway between the two squares, because that height is the average of the heights of the two squares. Example 3 is a combination of examples 1 and 2, so we can find its center of mass by averaging the horizontal positions of their centers of mass. In example 4, each square has been skewed a little, but just as much mass has been moved up as down, so the average vertical position of the mass hasn't changed. Example 5 is clearly not all that different from example

4, the main difference being a slight clockwise rotation, so just as in example 4, the center of mass must be hanging in empty space, where there isn't actually any mass. Horizontally, the center of mass must be between the heels and toes, or else it wouldn't be possible to stand without tipping over.

Another interesting example from the sports world is the high jump, in which the jumper's curved body passes over the bar, but the center of mass passes under the bar! Here the jumper lowers his legs and upper body at the peak of the jump in order to bring his waist higher compared to the center of mass.



l / The high-jumper's body passes over the bar, but his center of mass passes under it.

Later in this course, we'll find that there are more fundamental reasons (based on Newton's laws of motion) why the center of mass behaves in such a simple way compared to the other parts of an object. We're also postponing any discussion of numerical methods for finding an object's center of mass. Until later in the course, we will only deal with the motion of objects' centers of mass.

Center-of-mass motion in one dimension

In addition to restricting our study of motion to center-of-mass motion, we will begin by considering only cases in which the center of mass moves along a straight line. This will include cases such as objects falling straight down, or a car that speeds up and slows down but does not turn.

Note that even though we are not explicitly studying the more complex aspects of motion, we can still analyze the center-of-mass motion while ignoring other types of motion that might be occurring simultaneously . For instance, if a cat is falling out of a tree and is initially upside-down, it goes through a series of contortions that bring its feet under it. This is definitely not an example of rigid-body motion, but we can still analyze the motion of the cat's center of mass just as we would for a dropping rock.

self-check A

Consider a person running, a person pedaling on a bicycle, a person coasting on a bicycle, and a person coasting on ice skates. In which cases is the center-of-mass motion one-dimensional? Which cases are examples of rigid-body motion? ▷ Answer, p. 557

m / Self-check B.

self-check B

The figure shows a gymnast holding onto the inside of a big wheel. From inside the wheel, how could he make it roll one way or the other? ▷ Answer, p. 558

2.2 Describing distance and time

Center-of-mass motion in one dimension is particularly easy to deal with because all the information about it can be encapsulated in two variables: x, the position of the center of mass relative to the origin, and t, which measures a point in time. For instance, if someone supplied you with a sufficiently detailed table of x and t values, you would know pretty much all there was to know about the motion of the object's center of mass.

A point in time as opposed to duration

In ordinary speech, we use the word "time" in two different senses, which are to be distinguished in physics. It can be used, as in "a short time" or "our time here on earth," to mean a length or duration of time, or it can be used to indicate a clock reading, as in "I didn't know what time it was," or "now's the time." In symbols, t is ordinarily used to mean a point in time, while Δt signifies an interval or duration in time. The capital Greek letter delta, Δ, means "the change in...," i.e. a duration in time is the change or difference between one clock reading and another. The notation Δt does not signify the product of two numbers, Δ and t, but rather one single number, Δt. If a matinee begins at a point in time $t = 1$ o'clock and ends at $t = 3$ o'clock, the duration of the movie was the change in t,

$$\Delta t = 3 \text{ hours} - 1 \text{ hour} = 2 \text{ hours}.$$

To avoid the use of negative numbers for Δt, we write the clock reading "after" to the left of the minus sign, and the clock reading "before" to the right of the minus sign. A more specific definition of the delta notation is therefore that delta stands for "after minus before."

Even though our definition of the delta notation guarantees that Δt is positive, there is no reason why t can't be negative. If t could not be negative, what would have happened one second before $t = 0$? That doesn't mean that time "goes backward" in the sense that adults can shrink into infants and retreat into the womb. It just means that we have to pick a reference point and call it $t = 0$, and then times before that are represented by negative values of t. An example is that a year like 2007 A.D. can be thought of as a positive t value, while one like 370 B.C. is negative. Similarly, when you hear a countdown for a rocket launch, the phrase "t minus ten seconds" is a way of saying $t = -10$ s, where $t = 0$ is the time of blastoff, and $t > 0$ refers to times after launch.

Although a point in time can be thought of as a clock reading, it is usually a good idea to avoid doing computations with expressions such as "2:35" that are combinations of hours and minutes. Times can instead be expressed entirely in terms of a single unit, such as hours. Fractions of an hour can be represented by decimals rather than minutes, and similarly if a problem is being worked in terms of minutes, decimals can be used instead of seconds.

self-check C

Of the following phrases, which refer to points in time, which refer to time intervals, and which refer to time in the abstract rather than as a measurable number?

(1) "The time has come."

(2) "Time waits for no man."

(3) "The whole time, he had spit on his chin." ▷ Answer, p. 558

The Leibniz notation and infinitesimals

Δ is the Greek version of "D," suggesting that there is a relationship between Δt and the notation $\mathrm{d}t$ from calculus. The "d" notation was invented by Leibniz around 1675 to suggest the word "difference." The idea was that a $\mathrm{d}t$ would be like a Δt that was extremely small — smaller than any real number, and yet greater than zero. These infinitestimal numbers were the way the world's greatest mathematicians thought about calculus for the next two hundred years. For example, $\mathrm{d}y/\mathrm{d}x$ meant the number you got when you divided $\mathrm{d}y$ by $\mathrm{d}x$. The use of infinitesimal numbers was seen as a natural part of the process of generalization that had already seen the invention of fractions and irrational numbers by the ancient Greeks, zero and negative numbers in India, and complex numbers in Renaissance Italy. By the end of the 19th century, mathematicians had begun making formal mathematical descriptions of number systems, and they had succeeded in making nice tidy schemes out of all of these categories except for infinitesimals. Having run into a brick wall, they decided to rebuild calculus using the notion of a limit. Depending on when and where you got your education in calculus, you may have been warned severely that $\mathrm{d}y$ and $\mathrm{d}x$ were not numbers, and that $\mathrm{d}y/\mathrm{d}x$ didn't mean dividing one by another.

But in the 1960's, the logician Abraham Robinson at Yale proved that infinitesimals could be tamed and domesticated; they were no more self-contradictory than negative numbers or fractions. There is a handy rule for making sure that you don't come to incorrect conclusions by using infinitesimals. The rule is that you can apply any axiom of the real number system to infinitesimals, and the result will be correct, provided that the axiom can be put in a form like "for any number ...," but not "for any *set* of numbers ..." We carry over the axiom, reinterpreting "number" to mean any member of the enriched number system that includes both the real numbers and the infinitesimals.

Logic and infinitesimals — *example 2*

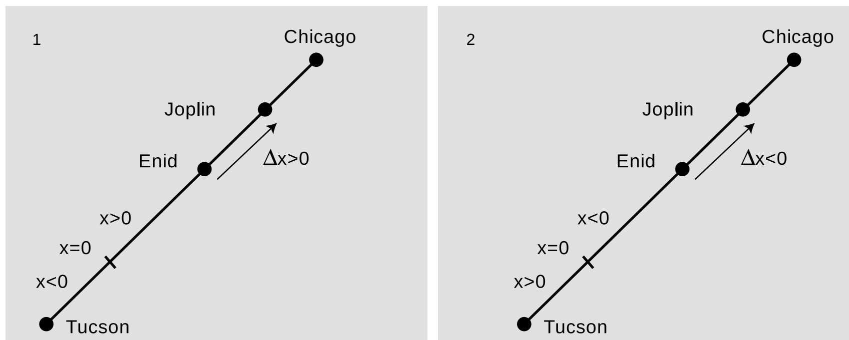
There is an axiom of the real number system that for any number t, $t + 0 = t$. This applies to infinitesimals as well, so that $\mathrm{d}t + 0 = \mathrm{d}t$.

Position as opposed to change in position

As with time, a distinction should be made between a point in space, symbolized as a coordinate x, and a change in position, symbolized as Δx.

As with t, x can be negative. If a train is moving down the tracks, not only do you have the freedom to choose any point along the tracks and call it $x = 0$, but it's also up to you to decide which side of the $x = 0$ point is positive x and which side is negative x.

Since we've defined the delta notation to mean "after minus before," it is possible that Δx will be negative, unlike Δt which is guaranteed to be positive. Suppose we are describing the motion of a train on tracks linking Tucson and Chicago. As shown in the figure, it is entirely up to you to decide which way is positive.



n / Two equally valid ways of describing the motion of a train from Tucson to Chicago. In example 1, the train has a positive Δx as it goes from Enid to Joplin. In 2, the same train going forward in the same direction has a negative Δx.

Note that in addition to x and Δx, there is a third quantity we could define, which would be like an odometer reading, or actual distance traveled. If you drive 10 miles, make a U-turn, and drive back 10 miles, then your Δx is zero, but your car's odometer reading has increased by 20 miles. However important the odometer reading is to car owners and used car dealers, it is not very important in physics, and there is not even a standard name or notation for it. The change in position, Δx, is more useful because it is so much easier to calculate: to compute Δx, we only need to know the beginning and ending positions of the object, not all the information about how it got from one position to the other.

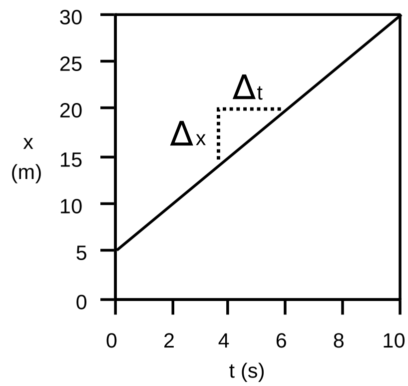
self-check D

A ball falls vertically, hits the floor, bounces to a height of one meter, falls, and hits the floor again. Is the Δx between the two impacts equal to zero, one, or two meters? ▷ Answer, p. 558

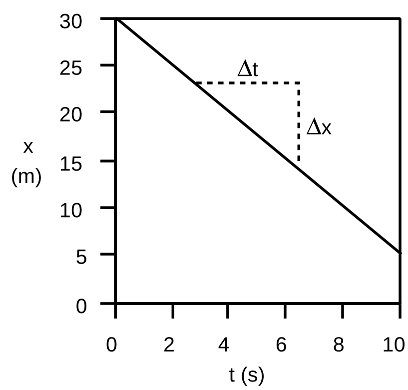
Frames of reference

The example above shows that there are two arbitrary choices you have to make in order to define a position variable, x. You have to decide where to put $x = 0$, and also which direction will be positive. This is referred to as choosing a coordinate system or choosing a frame of reference. (The two terms are nearly synonymous, but the first focuses more on the actual x variable, while the second is more of a general way of referring to one's point of view.) As long as you are consistent, any frame is equally valid. You just don't want to change coordinate systems in the middle of a calculation.

Have you ever been sitting in a train in a station when suddenly you notice that the station is moving backward? Most people would describe the situation by saying that you just failed to notice that the train was moving — it only seemed like the station was moving. But this shows that there is yet a third arbitrary choice that goes into choosing a coordinate system: valid frames of reference can differ from each other by moving relative to one another. It might seem strange that anyone would bother with a coordinate system that was moving relative to the earth, but for instance the frame of reference moving along with a train might be far more convenient for describing things happening inside the train.



o / Motion with constant velocity.



p / Motion that decreases *x* is represented with negative values of Δx and *v*.

2.3 Graphs of motion; velocity

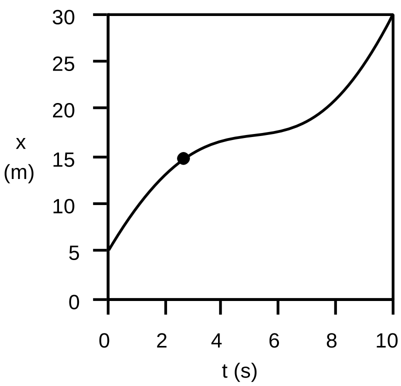
Motion with constant velocity

In example o, an object is moving at constant speed in one direction. We can tell this because every two seconds, its position changes by five meters.

In algebra notation, we'd say that the graph of x vs. t shows the same change in position, $\Delta x = 5.0$ m, over each interval of $\Delta t = 2.0$ s. The object's velocity or speed is obtained by calculating $v = \Delta x/\Delta t = (5.0\text{ m})/(2.0\text{ s}) = 2.5$ m/s. In graphical terms, the velocity can be interpreted as the slope of the line. Since the graph is a straight line, it wouldn't have mattered if we'd taken a longer time interval and calculated $v = \Delta x/\Delta t = (10.0\text{ m})/(4.0\text{ s})$. The answer would still have been the same, 2.5 m/s.

Note that when we divide a number that has units of meters by another number that has units of seconds, we get units of meters per second, which can be written m/s. This is another case where we treat units as if they were algebra symbols, even though they're not.

In example p, the object is moving in the opposite direction: as time progresses, its x coordinate decreases. Recalling the definition of the Δ notation as "after minus before," we find that Δt is still positive, but Δx must be negative. The slope of the line is therefore negative, and we say that the object has a negative velocity, $v = \Delta x/\Delta t = (-5.0\text{ m})/(2.0\text{ s}) = -2.5$ m/s. We've already seen that the plus and minus signs of Δx values have the interpretation of telling us which direction the object moved. Since Δt is always positive, dividing by Δt doesn't change the plus or minus sign, and the plus and minus signs of velocities are to be interpreted in the same way. In graphical terms, a positive slope characterizes a line that goes up as we go to the right, and a negative slope tells us that the line went down as we went to the right.



q / Motion with changing velocity. How can we find the velocity at the time indicated by the dot?

Motion with changing velocity

Now what about a graph like figure q? This might be a graph of a car's motion as the driver cruises down the freeway, then slows down to look at a car crash by the side of the road, and then speeds up again, disappointed that there is nothing dramatic going on such as flames or babies trapped in their car seats. (Note that we are still talking about one-dimensional motion. Just because the graph is curvy doesn't mean that the car's path is curvy. The graph is not like a map, and the horizontal direction of the graph represents the passing of time, not distance.)

If we apply the equation $v = \Delta x/\Delta t$ to this example, we will get the wrong answer, because the $\Delta x/\Delta t$ gives a single number, but the velocity is clearly changing. This is an example of a good general rule that tells you when you need to use your differential calculus. Any time a rate of change is measured by an expression of the form $\Delta \ldots/\Delta \ldots$, the result will only be right when the rate of change is constant. When the rate of change is varying, we need to generalize the expression by making it into a derivative. Just as an infinitesimally small[1] Δt is notated $\mathrm{d}t$, an infinitesimally small Δx is a $\mathrm{d}x$. The velocity is then the derivative $\mathrm{d}x/\,\mathrm{d}t$.

Units of velocity *example 3*

▷ Verify that the units of $v = \mathrm{d}x/\,\mathrm{d}t$ make sense.

▷ We expect the velocity to have units of meters per second, and it does come out to have those units, since dx has units of meters and dt seconds. This ability to check the units of derivatives is one of the main reasons that Leibniz designed his notation for derivatives the way he did.

An insect pest *example 4*

▷ An insect pest from the United States is inadvertently released in a village in rural China. The pests spread outward at a rate of s kilometers per year, forming a widening circle of contagion. Find the number of square kilometers per year that become newly infested. Check that the units of the result make sense. Interpret the result.

▷ Let t be the time, in years, since the pest was introduced. The radius of the circle is $r = st$, and its area is $a = \pi r^2 = \pi(st)^2$. The derivative is

$$\frac{\mathrm{d}a}{\mathrm{d}t} = (2\pi s^2)t$$

The units of s are km/year, so squaring it gives $\mathrm{km}^2/\mathrm{year}^2$. The 2 and the π are unitless, and multiplying by t gives units of $\mathrm{km}^2/\mathrm{year}$, which is what we expect for da/ dt, since it represents the number of square kilometers per year that become infested.

[1]see p. 67

Interpreting the result, we notice a couple of things. First, the rate of infestation isn't constant; it's proportional to t, so people might not pay so much attention at first, but later on the effort required to combat the problem will grow more and more quickly. Second, we notice that the result is proportional to s^2. This suggests that anything that could be done to reduce s would be very helpful. For instance, a measure that cut s in half would reduce $\mathrm{d}a/\mathrm{d}t$ by a factor of four.

2.4 The principle of inertia

Physical effects relate only to a change in velocity

Consider two statements of a kind that was at one time made with the utmost seriousness:

> People like Galileo and Copernicus who say the earth is rotating must be crazy. We know the earth can't be moving. Why, if the earth was really turning once every day, then our whole city would have to be moving hundreds of leagues in an hour. That's impossible! Buildings would shake on their foundations. Gale-force winds would knock us over. Trees would fall down. The Mediterranean would come sweeping across the east coasts of Spain and Italy. And furthermore, what force would be making the world turn?
>
> All this talk of passenger trains moving at forty miles an hour is sheer hogwash! At that speed, the air in a passenger compartment would all be forced against the back wall. People in the front of the car would suffocate, and people at the back would die because in such concentrated air, they wouldn't be able to expel a breath.

Some of the effects predicted in the first quote are clearly just based on a lack of experience with rapid motion that is smooth and free of vibration. But there is a deeper principle involved. In each case, the speaker is assuming that the mere fact of motion must have dramatic physical effects. More subtly, they also believe that a force is needed to keep an object in motion: the first person thinks a force would be needed to maintain the earth's rotation, and the second apparently thinks of the rear wall as pushing on the air to keep it moving.

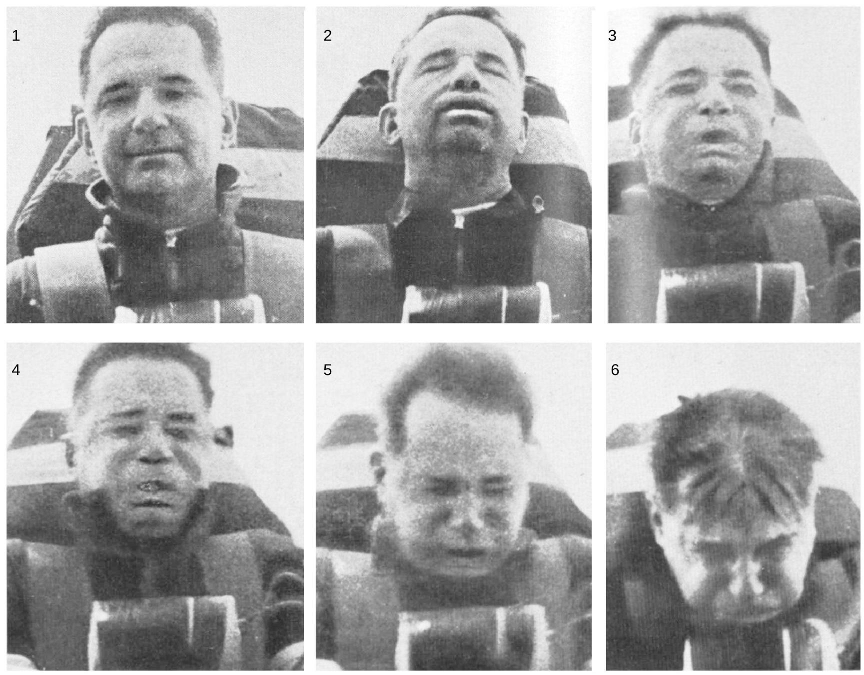
Common modern knowledge and experience tell us that these people's predictions must have somehow been based on incorrect reasoning, but it is not immediately obvious where the fundamental flaw lies. It's one of those things a four-year-old could infuriate you by demanding a clear explanation of. One way of getting at the fundamental principle involved is to consider how the modern concept of the universe differs from the popular conception at the time of the Italian Renaissance. To us, the word "earth" implies

r / Why does Aristotle look so sad? Has he realized that his entire system of physics is wrong?

s / The earth spins. People in Shanghai say they're at rest and people in Los Angeles are moving. Angelenos say the same about the Shanghainese.

t / The jets are at rest. The Empire State Building is moving.

a planet, one of the nine planets of our solar system, a small ball of rock and dirt that is of no significance to anyone in the universe except for members of our species, who happen to live on it. To Galileo's contemporaries, however, the earth was the biggest, most solid, most important thing in all of creation, not to be compared with the wandering lights in the sky known as planets. To us, the earth is just another object, and when we talk loosely about "how fast" an object such as a car "is going," we really mean the car-object's velocity relative to the earth-object.



u / This Air Force doctor volunteered to ride a rocket sled as a medical experiment. The obvious effects on his head and face are not because of the sled's speed but because of its rapid *changes* in speed: increasing in 2 and 3, and decreasing in 5 and 6. In 4 his speed is greatest, but because his speed is not increasing or decreasing very much at this moment, there is little effect on him.

Motion is relative

According to our modern world-view, it isn't reasonable to expect that a special force should be required to make the air in the train have a certain velocity relative to our planet. After all, the "moving" air in the "moving" train might just happen to have zero velocity relative to some other planet we don't even know about. Aristotle claimed that things "naturally" wanted to be at rest, lying on the surface of the earth. But experiment after experiment has shown that there is really nothing so special about being at rest relative to the earth. For instance, if a mattress falls out of the back of a truck on the freeway, the reason it rapidly comes to rest with respect to the planet is simply because of friction forces exerted by the asphalt, which happens to be attached to the planet.

Galileo's insights are summarized as follows:

The principle of inertia
No force is required to maintain motion with constant velocity in a straight line, and absolute motion does not cause any observable physical effects.

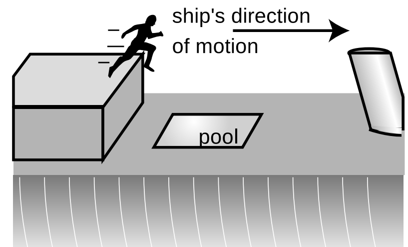
There are many examples of situations that seem to disprove the principle of inertia, but these all result from forgetting that friction is a force. For instance, it seems that a force is needed to keep a sailboat in motion. If the wind stops, the sailboat stops too. But the wind's force is not the only force on the boat; there is also a frictional force from the water. If the sailboat is cruising and the wind suddenly disappears, the backward frictional force still exists, and since it is no longer being counteracted by the wind's forward force, the boat stops. To disprove the principle of inertia, we would have to find an example where a moving object slowed down even though no forces whatsoever were acting on it. Over the years since Galileo's lifetime, physicists have done more and more precise experiments to search for such a counterexample, but the results have always been negative. Three such tests are described on pp. 117, 261, and 155.

self-check E
What is incorrect about the following supposed counterexamples to the principle of inertia?

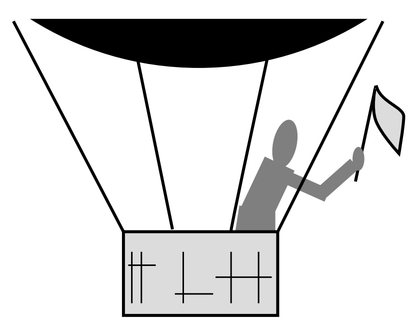
(1) When astronauts blast off in a rocket, their huge velocity does cause a physical effect on their bodies — they get pressed back into their seats, the flesh on their faces gets distorted, and they have a hard time lifting their arms.

(2) When you're driving in a convertible with the top down, the wind in your face is an observable physical effect of your absolute motion. ▷ Answer, p. 558

▷ *Solved problem: a bug on a wheel* *page 98, problem 13*



Discussion question A.

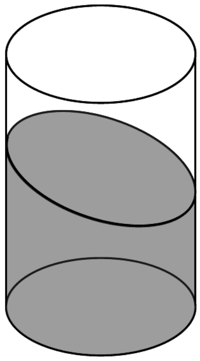

Discussion question B.

Discussion questions

A A passenger on a cruise ship finds, while the ship is docked, that he can leap off of the upper deck and just barely make it into the pool on the lower deck. If the ship leaves dock and is cruising rapidly, will this adrenaline junkie still be able to make it?

B You are a passenger in the open basket hanging under a helium balloon. The balloon is being carried along by the wind at a constant velocity. If you are holding a flag in your hand, will the flag wave? If so, which way? [Based on a question from PSSC Physics.]

C Aristotle stated that all objects naturally wanted to come to rest, with the unspoken implication that "rest" would be interpreted relative to the


Discussion question D.

surface of the earth. Suppose we go back in time and transport Aristotle to the moon. Aristotle knew, as we do, that the moon circles the earth; he said it didn't fall down because, like everything else in the heavens, it was made out of some special substance whose "natural" behavior was to go in circles around the earth. We land, put him in a space suit, and kick him out the door. What would he expect his fate to be in this situation? If intelligent creatures inhabited the moon, and one of them independently came up with the equivalent of Aristotelian physics, what would they think about objects coming to rest?

D The glass is sitting on a level table in a train's dining car, but the surface of the water is tilted. What can you infer about the motion of the train?

2.5 Addition of velocities

Addition of velocities to describe relative motion

Since absolute motion cannot be unambiguously measured, the only way to describe motion unambiguously is to describe the motion of one object relative to another. Symbolically, we can write v_{PQ} for the velocity of object P relative to object Q.

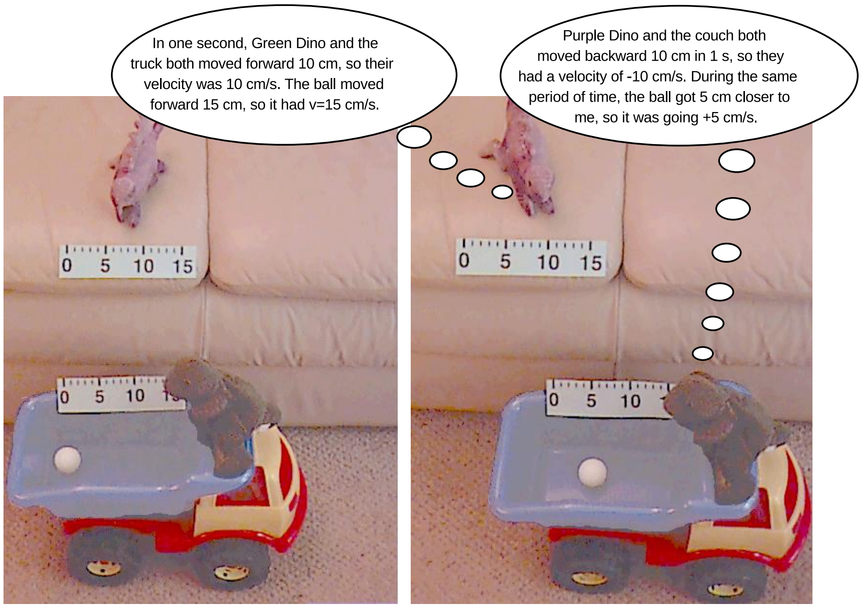
Velocities measured with respect to different reference points can be compared by addition. In the figure below, the ball's velocity relative to the couch equals the ball's velocity relative to the truck plus the truck's velocity relative to the couch:

$$\begin{aligned} v_{BC} &= v_{BT} + v_{TC} \\ &= 5\ \text{cm/s} + 10\ \text{cm/s} \\ &= 15\ \text{cm/s} \end{aligned}$$

The same equation can be used for any combination of three objects, just by substituting the relevant subscripts for B, T, and C. Just remember to write the equation so that the velocities being added have the same subscript twice in a row. In this example, if you read off the subscripts going from left to right, you get BC ... = ... BTTC. The fact that the two "inside" subscripts on the right are the same means that the equation has been set up correctly. Notice how subscripts on the left look just like the subscripts on the right, but with the two T's eliminated.

Negative velocities in relative motion

My discussion of how to interpret positive and negative signs of velocity may have left you wondering why we should bother. Why not just make velocity positive by definition? The original reason why negative numbers were invented was that bookkeepers decided it would be convenient to use the negative number concept for payments to distinguish them from receipts. It was just plain easier than writing receipts in black and payments in red ink. After adding up



v / These two highly competent physicists disagree on absolute velocities, but they would agree on relative velocities. Purple Dino considers the couch to be at rest, while Green Dino thinks of the truck as being at rest. They agree, however, that the truck's velocity relative to the couch is v_{TC} = 10 cm/s, the ball's velocity relative to the truck is v_{BT} = 5 cm/s, and the ball's velocity relative to the couch is $v_{BC} = v_{BT} + v_{TC}$ = 15 cm/s.

your month's positive receipts and negative payments, you either got a positive number, indicating profit, or a negative number, showing a loss. You could then show that total with a high-tech "+" or "−" sign, instead of looking around for the appropriate bottle of ink.

Nowadays we use positive and negative numbers for all kinds of things, but in every case the point is that it makes sense to add and subtract those things according to the rules you learned in grade school, such as "minus a minus makes a plus, why this is true we need not discuss." Adding velocities has the significance of comparing relative motion, and with this interpretation negative and positive velocities can be used within a consistent framework. For example, the truck's velocity relative to the couch equals the truck's velocity relative to the ball plus the ball's velocity relative to the couch:

$$\begin{aligned} v_{TC} &= v_{TB} + v_{BC} \\ &= -5\ \text{cm/s} + 15\ \text{cm/s} \\ &= 10\ \text{cm/s} \end{aligned}$$

If we didn't have the technology of negative numbers, we would have had to remember a complicated set of rules for adding velocities: (1) if the two objects are both moving forward, you add, (2) if one is moving forward and one is moving backward, you subtract, but (3) if they're both moving backward, you add. What a pain that would have been.

▷ *Solved problem: two dimensions* *page 98, problem 10*

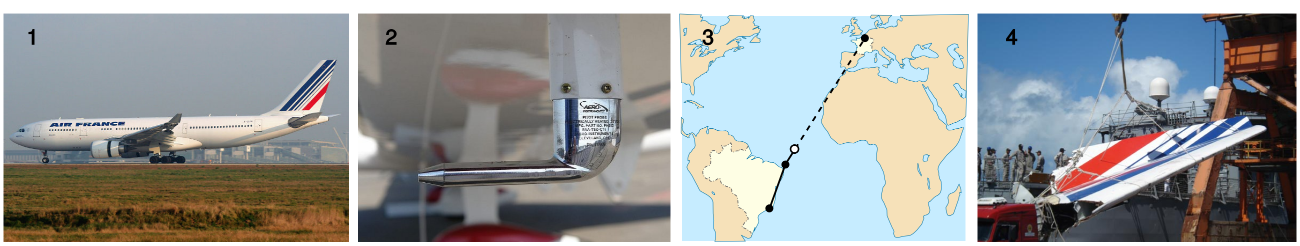
Airspeed *example 5*

On June 1, 2009, Air France flight 447 disappeared without warning over the Atlantic Ocean. All 232 people aboard were killed. Investigators believe the disaster was triggered because the pilots lost the ability to accurately determine their speed relative to the air. This is done using sensors called Pitot tubes, mounted outside the plane on the wing. Automated radio signals showed that these sensors gave conflicting readings before the crash, possibly because they iced up. For fuel efficiency, modern passenger jets fly at a very high altitude, but in the thin air they can only fly within a very narrow range of speeds. If the speed is too low, the plane stalls, and if it's too high, it breaks up. If the pilots can't tell what their airspeed is, they can't keep it in the safe range.

Many people's reaction to this story is to wonder why planes don't just use GPS to measure their speed. One reason is that GPS tells you your speed relative to the ground, not relative to the air. Letting P be the plane, A the air, and G the ground, we have

$$v_{PG} = v_{PA} + v_{AG},$$

where v_{PG} (the "true ground speed") is what GPS would measure, v_{PA} ("airspeed") is what's critical for stable flight, and v_{AG} is the velocity of the wind relative to the ground 9000 meters below. Knowing v_{PG} isn't enough to determine v_{PA} unless v_{AG} is also known.



w / Example 5. 1. The aircraft before the disaster. 2. A Pitot tube. 3. The flight path of flight 447. 4. Wreckage being recovered.

Discussion questions

A Interpret the general rule $v_{AB} = -v_{BA}$ in words.

B Wa-Chuen slips away from her father at the mall and walks up the down escalator, so that she stays in one place. Write this in terms of symbols.

2.6 ⋆ Relativity

Time is not absolute

So far we've been discussing relativity according to Galileo and Newton, but there is also relativity according to Einstein. When Einstein first began to develop the theory of relativity, around 1905, the only real-world observations he could draw on were ambiguous and indirect. Today, the evidence is part of everyday life. For example, every time you use a GPS receiver, x, you're using Einstein's theory of relativity. Somewhere between 1905 and today, technology became good enough to allow conceptually *simple* experiments that students in the early 20th century could only discuss in terms like "Imagine that we could..." A good jumping-on point is 1971. In that year, J.C. Hafele and R.E. Keating brought atomic clocks aboard commercial airliners, y, and went around the world, once from east to west and once from west to east. Hafele and Keating observed that there was a discrepancy between the times measured by the traveling clocks and the times measured by similar clocks that stayed home at the U.S. Naval Observatory in Washington. The east-going clock lost time, ending up off by -59 ± 10 nanoseconds, while the west-going one gained 273 ± 7 ns.

x / This Global Positioning System (GPS) system, running on a smartphone attached to a bike's handlebar, depends on Einstein's theory of relativity. Time flows at a different rate aboard a GPS satellite than it does on the bike, and the GPS software has to take this into account.

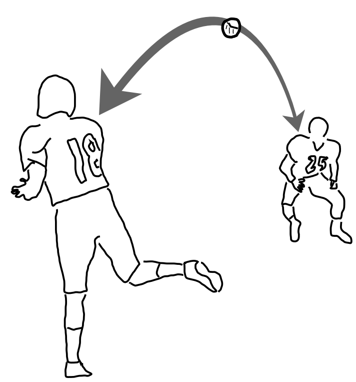
y / The clock took up two seats, and two tickets were bought for it under the name of "Mr. Clock."

The correspondence principle

This establishes that time doesn't work the way Newton believed it did when he wrote that "Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external..." We are used to thinking of time as absolute and universal, so it is disturbing to find that it can flow at a different rate for observers in different frames of reference. Nevertheless, the effects that Hafele and Keating observed were small. This makes sense: Newton's laws have already been thoroughly tested by experiments under a wide variety of conditions, so a new theory like relativity must agree with Newton's to a good approximation, within the Newtonian theory's realm of applicability. This requirement of backward-compatibility is known as the correspondence principle.

Causality

It's also reassuring that the effects on time were small compared to the three-day lengths of the plane trips. There was therefore no opportunity for paradoxical scenarios such as one in which the east-going experimenter arrived back in Washington before he left and then convinced himself not to take the trip. A theory that maintains this kind of orderly relationship between cause and effect is said to satisfy causality.

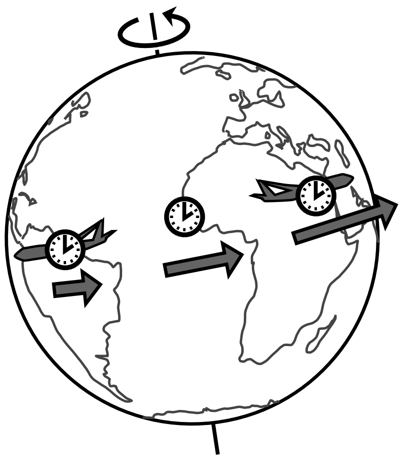


z / Newton's laws do not distinguish past from future. The football could travel in either direction while obeying Newton's laws.

Causality is like a water-hungry front-yard lawn in Los Angeles: we know we want it, but it's not easy to explain why. Even in plain old Newtonian physics, there is no clear distinction between past and future. In figure z, number 18 throws the football to number 25, and the ball obeys Newton's laws of motion. If we took a video of the pass and played it backward, we would see the ball flying from 25 to 18, and Newton's laws would still be satisfied. Nevertheless, we have a strong psychological impression that there is a forward arrow of time. I can remember what the stock market did last year, but I can't remember what it will do next year. Joan of Arc's military victories against England caused the English to burn her at the stake; it's hard to accept that Newton's laws provide an equally good description of a process in which her execution in 1431 caused her to win a battle in 1429. There is no consensus at this point among physicists on the origin and significance of time's arrow, and for our present purposes we don't need to solve this mystery. Instead, we merely note the empirical fact that, regardless of what causality really means and where it really comes from, its behavior is consistent. Specifically, experiments show that if an observer in a certain frame of reference observes that event A causes event B, then observers in other frames agree that A causes B, not the other way around. This is merely a generalization about a large body of experimental results, not a logically necessary assumption. If Keating had gone around the world and arrived back in Washington before he left, it would have disproved this statement about causality.

Time distortion arising from motion and gravity

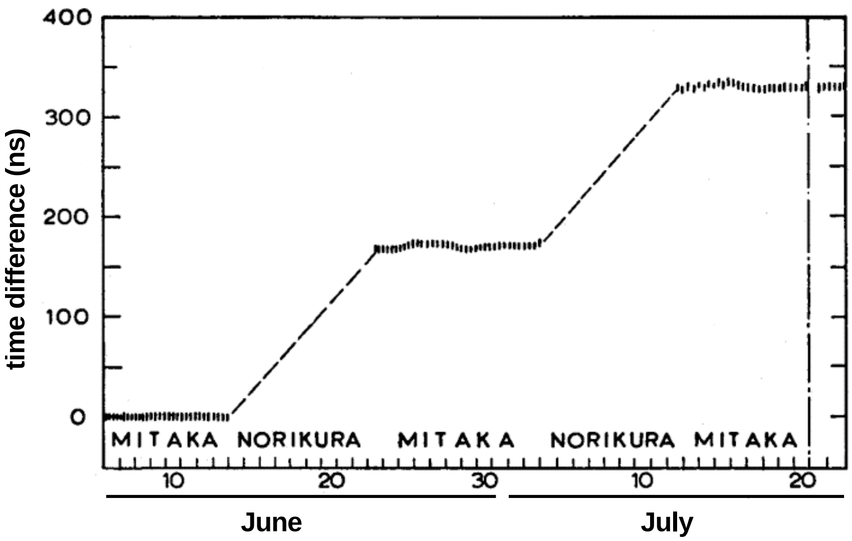
Hafele and Keating were testing specific quantitative predictions of relativity, and they verified them to within their experiment's error bars. Let's work backward instead, and inspect the empirical results for clues as to how time works.



aa / All three clocks are moving to the east. Even though the west-going plane is moving to the west relative to the air, the air is moving to the east due to the earth's rotation.

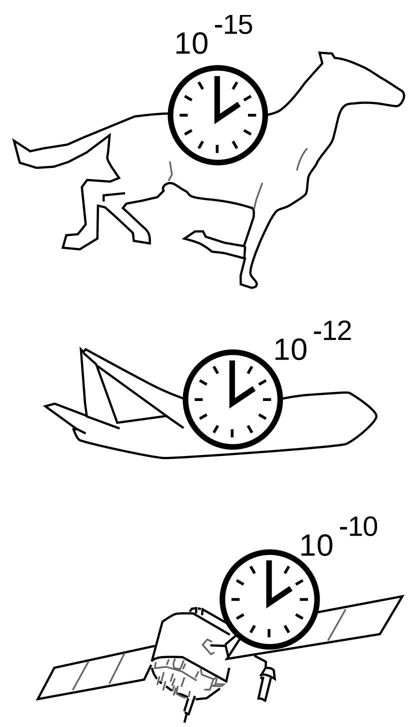
The two traveling clocks experienced effects in opposite directions, and this suggests that the rate at which time flows depends on the motion of the observer. The east-going clock was moving in the same direction as the earth's rotation, so its velocity relative to the earth's center was greater than that of the clock that remained in Washington, while the west-going clock's velocity was correspondingly reduced. The fact that the east-going clock fell behind, and the west-going one got ahead, shows that the effect of motion is to make time go more slowly. This effect of motion on time was predicted by Einstein in his original 1905 paper on relativity, written when he was 26.

If this had been the only effect in the Hafele-Keating experiment, then we would have expected to see effects on the two flying clocks that were equal in size. Making up some simple numbers to keep the arithmetic transparent, suppose that the earth rotates from west to east at 1000 km/hr, and that the planes fly at 300 km/hr. Then the speed of the clock on the ground is 1000 km/hr, the speed of the clock on the east-going plane is 1300 km/hr, and that of the west-going clock 700 km/hr. Since the speeds of 700, 1000, and 1300 km/hr have equal spacing on either side of 1000, we would expect the discrepancies of the moving clocks relative to the one in the lab to be equal in size but opposite in sign.



ab / A graph showing the time difference between two atomic clocks. One clock was kept at Mitaka Observatory, at 58 m above sea level. The other was moved back and forth to a second observatory, Norikura Corona Station, at the peak of the Norikura volcano, 2876 m above sea level. The plateaus on the graph are data from the periods when the clocks were compared side by side at Mitaka. The difference between one plateau and the next shows a gravitational effect on the rate of flow of time, accumulated during the period when the mobile clock was at the top of Norikura.

In fact, the two effects are unequal in size: −59 ns and 273 ns. This implies that there is a second effect involved, simply due to the planes' being up in the air. This was verified more directly in a 1978 experiment by Iijima and Fujiwara, figure ab, in which identical atomic clocks were kept at rest at the top and bottom of a mountain near Tokyo. This experiment, unlike the Hafele-Keating one, isolates one effect on time, the gravitational one: time's rate of flow increases with height in a gravitational field. Einstein didn't figure out how to incorporate gravity into relativity until 1915, after much frustration and many false starts. The simpler version of the theory without gravity is known as special relativity, the full version as general relativity. We'll restrict ourselves to special relativity, and that means that what we want to focus on right now is the distortion of time due to motion, not gravity.



ac / The correspondence principle requires that the relativistic distortion of time become small for small velocities.

We can now see in more detail how to apply the correspondence principle. The behavior of the three clocks in the Hafele-Keating experiment shows that the amount of time distortion increases as the speed of the clock's motion increases. Newton lived in an era when the fastest mode of transportation was a galloping horse, and the best pendulum clocks would accumulate errors of perhaps a minute over the course of several days. A horse is much slower than a jet plane, so the distortion of time would have had a relative size of only $\sim 10^{-15}$ — much smaller than the clocks were capable of detecting. At the speed of a passenger jet, the effect is about 10^{-12}, and state-of-the-art atomic clocks in 1971 were capable of measuring that. A GPS satellite travels much faster than a jet airplane, and the effect on the satellite turns out to be $\sim 10^{-10}$. The general idea here is that all physical laws are approximations, and approximations aren't simply right or wrong in different situations. Approximations are better or worse in different situations, and the question is whether a particular approximation is good enough in a given situation to serve a particular purpose. The faster the motion, the worse the Newtonian approximation of absolute time. Whether the approximation is good enough depends on what you're trying to accomplish. The correspondence principle says that the approximation must have been good enough to explain all the experiments done in the centuries before Einstein came up with relativity.

By the way, don't get an inflated idea of the importance of the Hafele-Keating experiment. Special relativity had already been confirmed by a vast and varied body of experiments decades before 1971. The only reason I'm giving such a prominent role to this experiment, which was actually more important as a test of general relativity, is that it is conceptually very direct.

Distortion of space and time

The Lorentz transformation

Relativity says that when two observers are in different frames of reference, each observer considers the other one's perception of time to be distorted. We'll also see that something similar happens to their observations of distances, so both space and time are distorted. What exactly is this distortion? How do we even conceptualize it?

The idea isn't really as radical as it might seem at first. We can visualize the structure of space and time using a graph with position and time on its axes. These graphs are familiar by now, but we're going to look at them in a slightly different way. Before, we used them to describe the motion of objects. The grid underlying the graph was merely the stage on which the actors played their parts. Now the background comes to the foreground: it's time and space themselves that we're studying. We don't necessarily need to have a line or a curve drawn on top of the grid to represent a particular object. We may, for example, just want to talk about events, depicted as points on the graph as in figure ad. A distortion of the Cartesian grid underlying the graph can arise for perfectly ordinary reasons that Isaac Newton would have readily accepted. For example, we can simply change the units used to measure time and position, as in figure ae.

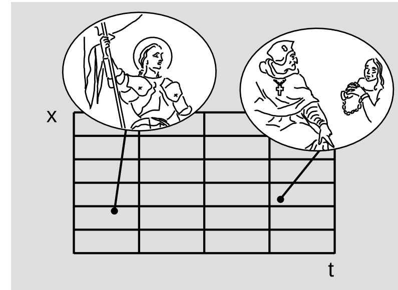
We're going to have quite a few examples of this type, so I'll adopt the convention shown in figure af for depicting them. Figure af summarizes the relationship between figures ad and ae in a more compact form. The gray rectangle represents the original coordinate grid of figure ad, while the grid of black lines represents the new version from figure ae. Omitting the grid from the gray rectangle makes the diagram easier to decode visually.

Our goal of unraveling the mysteries of special relativity amounts to nothing more than finding out how to draw a diagram like af in the case where the two different sets of coordinates represent measurements of time and space made by two different observers, each in motion relative to the other. Galileo and Newton thought they knew the answer to this question, but their answer turned out to be only approximately right. To avoid repeating the same mistakes, we need to clearly spell out what we think are the basic properties of time and space that will be a reliable foundation for our reasoning. I want to emphasize that there is no purely logical way of deciding on this list of properties. The ones I'll list are simply a summary of the patterns observed in the results from a large body of experiments. Furthermore, some of them are only approximate. For example, property 1 below is only a good approximation when the gravitational field is weak, so it is a property that applies to special relativity, not to general relativity.

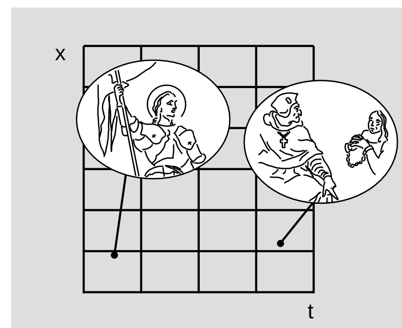
Experiments show that:

1. No point in time or space has properties that make it different from any other point.
2. Likewise, all directions in space have the same properties.
3. Motion is relative, i.e., all inertial frames of reference are equally valid.
4. Causality holds, in the sense described on page 77.
5. Time depends on the state of motion of the observer.

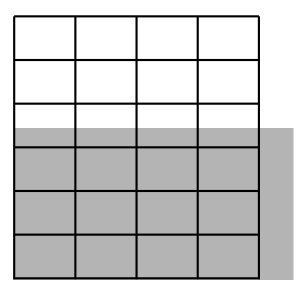
Most of these are not very subversive. Properties 1 and 2 date back to the time when Galileo and Newton started applying the same universal laws of motion to the solar system and to the earth; this contradicted Aristotle, who believed that, for example, a rock would naturally want to move in a certain special direction (down) in order to reach a certain special location (the earth's surface). Property 3 is the reason that Einstein called his theory "relativity," but Galileo and Newton believed exactly the same thing to be true,



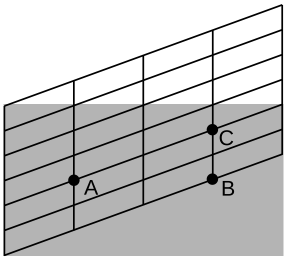
ad / Two events are given as points on a graph of position versus time. Joan of Arc helps to restore Charles VII to the throne. At a later time and a different position, Joan of Arc is sentenced to death.



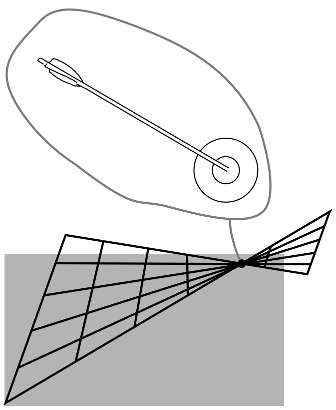
ae / A change of units distorts an x-t graph. This graph depicts exactly the same events as figure ad. The only change is that the x and t coordinates are measured using different units, so the grid is compressed in t and expanded in x.



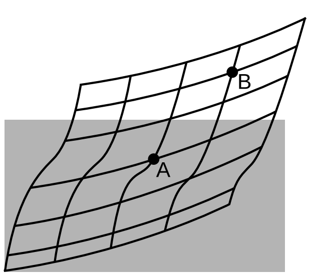
af / A convention we'll use to represent a distortion of time and space.



ag / A Galilean version of the relationship between two frames of reference. As in all such graphs in this chapter, the original coordinates, represented by the gray rectangle, have a time axis that goes to the right, and a position axis that goes straight up.



ah / A transformation that leads to disagreements about whether two events occur at the same time and place. This is not just a matter of opinion. Either the arrow hit the bull's-eye or it didn't.



ai / A nonlinear transformation.

as dramatized by Galileo's run-in with the Church over the question of whether the earth could really be in motion around the sun. Property 4 would probably surprise most people only because it asserts in such a weak and specialized way something that they feel deeply must be true. The only really strange item on the list is 5, but the Hafele-Keating experiment forces it upon us.

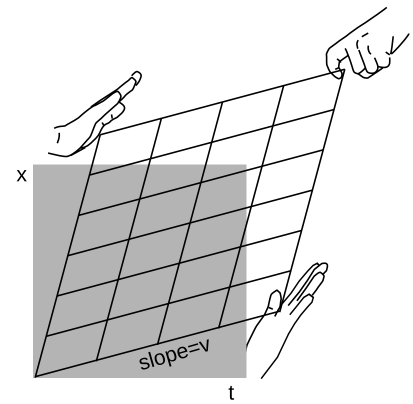
If it were not for property 5, we could imagine that figure ag would give the correct transformation between frames of reference in motion relative to one another. Let's say that observer 1, whose grid coincides with the gray rectangle, is a hitch-hiker standing by the side of a road. Event A is a raindrop hitting his head, and event B is another raindrop hitting his head. He says that A and B occur at the same location in space. Observer 2 is a motorist who drives by without stopping; to him, the passenger compartment of his car is at rest, while the asphalt slides by underneath. He says that A and B occur at different points in space, because during the time between the first raindrop and the second, the hitch-hiker has moved backward. On the other hand, observer 2 says that events A and C occur in the same place, while the hitch-hiker disagrees. The slope of the grid-lines is simply the velocity of the relative motion of each observer relative to the other.

Figure ag has familiar, comforting, and eminently sensible behavior, but it also happens to be wrong, because it violates property 5. The distortion of the coordinate grid has only moved the vertical lines up and down, so both observers agree that events like B and C are simultaneous. If this was really the way things worked, then all observers could synchronize all their clocks with one another for once and for all, and the clocks would never get out of sync. This contradicts the results of the Hafele-Keating experiment, in which all three clocks were initially synchronized in Washington, but later went out of sync because of their different states of motion.

It might seem as though we still had a huge amount of wiggle room available for the correct form of the distortion. It turns out, however, that properties 1-5 are sufficient to prove that there is only one answer, which is the one found by Einstein in 1905. To see why this is, let's work by a process of elimination.

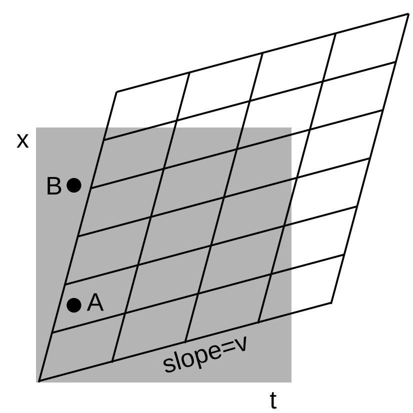
Figure ah shows a transformation that might seem at first glance to be as good a candidate as any other, but it violates property 3, that motion is relative, for the following reason. In observer 2's frame of reference, some of the grid lines cross one another. This means that observers 1 and 2 disagree on whether or not certain events are the same. For instance, suppose that event A marks the arrival of an arrow at the bull's-eye of a target, and event B is the location and time when the bull's-eye is punctured. Events A and B occur at the same location and at the same time. If one observer says that A and B coincide, but another says that they don't, we have a direct contradiction. Since the two frames of reference in figure ah give contradictory results, one of them is right and one is wrong. This violates property 3, because all inertial frames of reference are supposed to be equally valid. To avoid problems like this, we clearly need to make sure that none of the grid lines ever cross one another.

The next type of transformation we want to kill off is shown in figure ai, in which the grid lines curve, but never cross one another. The trouble with this one is that it violates property 1, the uniformity of time and space. The transformation is unusually "twisty" at A, whereas at B it's much more smooth. This can't be correct, because the transformation is only supposed to depend on the relative state of motion of the two frames of reference, and that given information doesn't single out a special role for any particular point in spacetime. If, for example, we had one frame of reference *rotating* relative to the other, then there would be something special about the axis of rotation. But we're only talking about *inertial* frames of reference here, as specified in property 3, so we can't have rotation; each frame of reference has to be moving in a straight line at constant speed. For frames related in this way, there is nothing that could single out an event like A for special treatment compared to B, so transformation ai violates property 1.



ak / In the units that are most convenient for relativity, the transformation has symmetry about a 45-degree diagonal line.

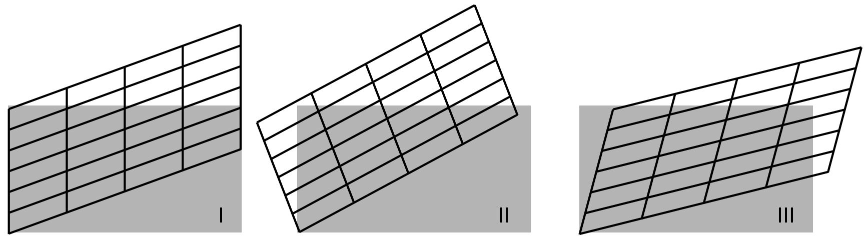
The examples in figures ah and ai show that the transformation we're looking for must be linear, meaning that it must transform lines into lines, and furthermore that it has to take parallel lines to parallel lines. Einstein wrote in his 1905 paper that "...on account of the property of homogeneity [property 1] which we ascribe to time and space, the [transformation] must be linear."[2] Applying this to our diagrams, the original gray rectangle, which is a special type of parallelogram containing right angles, must be transformed into another parallelogram. There are three types of transformations, figure aj, that have this property. Case I is the Galilean transformation of figure ag on page 82, which we've already ruled out.



al / Interpretation of the Lorentz transformation. The slope indicated in the figure gives the relative velocity of the two frames of reference. Events A and B that were simultaneous in frame 1 are not simultaneous in frame 2, where event A occurs to the right of the $t = 0$ line represented by the left edge of the grid, but event B occurs to its left.

Case II can also be discarded. Here every point on the grid rotates counterclockwise. What physical parameter would determine the amount of rotation? The only thing that could be relevant would be v, the relative velocity of the motion of the two frames of reference with respect to one another. But if the angle of rotation was proportional to v, then for large enough velocities the grid would have left and right reversed, and this would violate property 4, causality: one observer would say that event A caused a later event B, but another observer would say that B came first and caused A.

[2] A. Einstein, "On the Electrodynamics of Moving Bodies," *Annalen der Physik* 17 (1905), p. 891, tr. Saha and Bose.

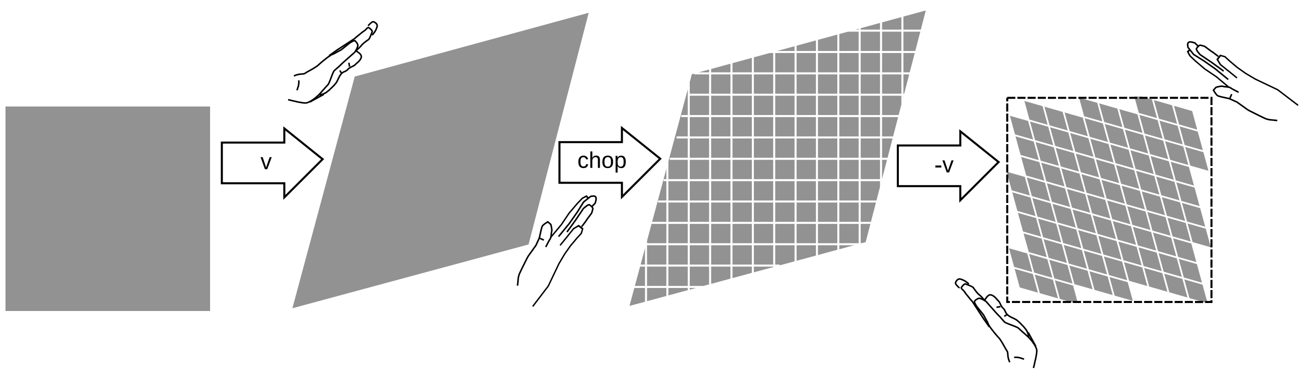


aj / Three types of transformations that preserve parallelism. Their distinguishing feature is what they do to simultaneity, as shown by what happens to the left edge of the original rectangle. In I, the left edge remains vertical, so simultaneous events remain simultaneous. In II, the left edge turns counterclockwise. In III, it turns clockwise.

The only remaining possibility is case III, which I've redrawn in figure ak with a couple of changes. This is the one that Einstein predicted in 1905. The transformation is known as the Lorentz transformation, after Hendrik Lorentz (1853-1928), who partially anticipated Einstein's work, without arriving at the correct interpretation. The distortion is a kind of smooshing and stretching, as suggested by the hands. Also, we've already seen in figures ad-af on page 81 that we're free to stretch or compress everything as much as we like in the horizontal and vertical directions, because this simply corresponds to choosing different units of measurement for time and distance. In figure ak I've chosen units that give the whole drawing a convenient symmetry about a 45-degree diagonal line. Ordinarily it wouldn't make sense to talk about a 45-degree angle on a graph whose axes had different units. But in relativity, the symmetric appearance of the transformation tells us that space and time ought to be treated on the same footing, and measured in the same units.

As in our discussion of the Galilean transformation, slopes are interpreted as velocities, and the slope of the near-horizontal lines in figure al is interpreted as the relative velocity of the two observers. The difference between the Galilean version and the relativistic one is that now there is smooshing happening from the other side as well. Lines that were vertical in the original grid, representing simultaneous events, now slant over to the right. This tells us that, as required by property 5, different observers do not agree on whether events that occur in different places are simultaneous. The Hafele-Keating experiment tells us that this non-simultaneity effect is fairly small, even when the velocity is as big as that of a passenger jet, and this is what we would have anticipated by the correspondence principle. The way that this is expressed in the graph is that if we pick the time unit to be the second, then the distance unit turns out to be hundreds of thousands of miles. In these units, the velocity of a passenger jet is an extremely small number, so the slope v in figure al is extremely small, and the amount of distortion is tiny — it would be much too small to see on this scale.

The only thing left to determine about the Lorentz transformation is the size of the transformed parallelogram relative to the size of the original one. Although the drawing of the hands in figure ak may suggest that the grid deforms like a framework made of rigid coat-hanger wire, that is not the case. If you look carefully at the figure, you'll see that the edges of the smooshed parallelogram are actually a little longer than the edges of the original rectangle. In fact what stays the same is not lengths but *areas*, as proved in the caption to figure am.



am / Proof that Lorentz transformations don't change area: We first subject a square to a transformation with velocity v, and this increases its area by a factor $R(v)$, which we want to prove equals 1. We chop the resulting parallelogram up into little squares and finally apply a $-v$ transformation; this changes each little square's area by a factor $R(-v)$, so the whole figure's area is also scaled by $R(-v)$. The final result is to restore the square to its original shape and area, so $R(v)R(-v) = 1$. But $R(v) = R(-v)$ by property 2 of spacetime on page 81, which states that all directions in space have the same properties, so $R(v) = 1$.

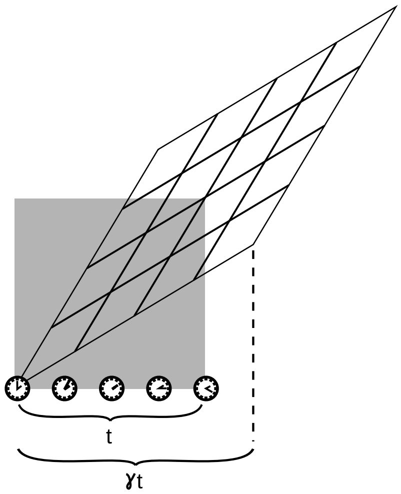
The γ factor

With a little algebra and geometry (homework problem 18, page 99), one can use the equal-area property to show that the factor γ (Greek letter gamma) defined in figure an is given by the equation

$$\gamma = \frac{1}{\sqrt{1-v^2}}.$$

If you've had good training in physics, the first thing you probably think when you look at this equation is that it must be nonsense, because its units don't make sense. How can we take something with units of velocity squared, and subtract it from a unitless 1? But remember that this is expressed in our special relativistic units, in which the same units are used for distance and time. In this system, velocities are always unitless. This sort of thing happens frequently in physics. For instance, before James Joule discovered conservation of energy, nobody knew that heat and mechanical energy were different forms of the same thing, so instead of measuring them both in units of joules as we would do now, they measured heat in one unit (such as calories) and mechanical energy in another (such as foot-pounds). In ordinary metric units, we just need an extra conversion factor c, and the equation becomes

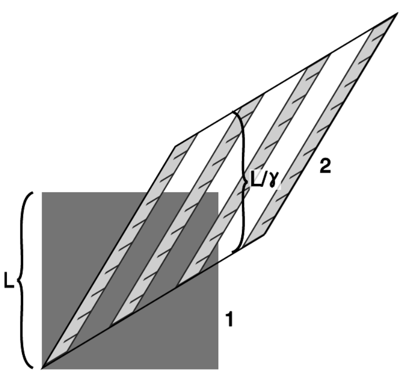
$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$



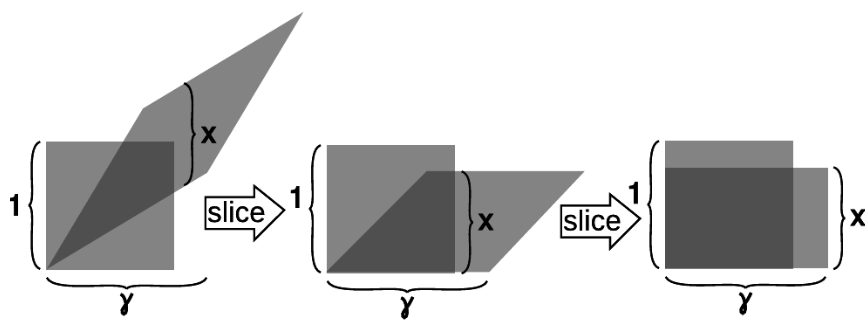
an / The γ factor.

Here's why we care about γ. Figure an defines it as the ratio of two times: the time between two events as expressed in one coordinate system, and the time between the same two events as measured in the other one. The interpretation is:

Time dilation
A clock runs fastest in the frame of reference of an observer who is at rest relative to the clock. An observer in motion relative to the clock at speed v perceives the clock as running more slowly by a factor of γ.



ao / The ruler is moving in frame 1, represented by a square, but at rest in frame 2, shown as a parallelogram. Each picture of the ruler is a snapshot taken at a certain moment as judged according to frame 2's notion of simultaneity. An observer in frame 1 judges the ruler's length instead according to frame 1's definition of simultaneity, i.e., using points that are lined up vertically on the graph. The ruler appears shorter in the frame in which it is moving. As proved in figure ap, the length contracts from L to L/γ.

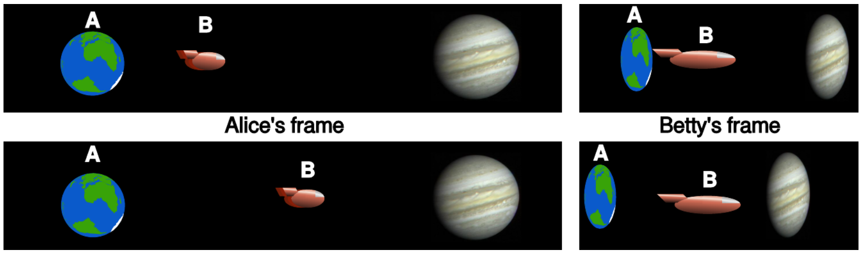


ap / This figure proves, as claimed in figure ao, that the length contraction is $x = 1/\gamma$. First we slice the parallelogram vertically like a salami and slide the slices down, making the top and bottom edges horizontal. Then we do the same in the horizontal direction, forming a rectangle with sides γ and x. Since both the Lorentz transformation and the slicing processes leave areas unchanged, the area γx of the rectangle must equal the area of the original square, which is 1.

As proved in figures ao and ap, lengths are also distorted:

Length contraction
A meter-stick appears longest to an observer who is at rest relative to it. An observer moving relative to the meter-stick at v observes the stick to be shortened by a factor of γ.

self-check F
What is γ when $v = 0$? What does this mean? ▷ Answer, p. 558



aq / Example 6.

An interstellar road trip — *example 6*

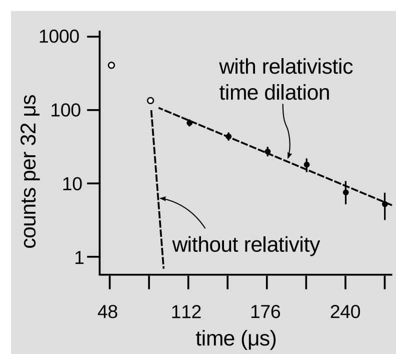
Alice stays on earth while her twin Betty heads off in a spaceship for Tau Ceti, a nearby star. Tau Ceti is 12 light-years away, so even though Betty travels at 87% of the speed of light, it will take her a long time to get there: 14 years, according to Alice.

Betty experiences time dilation. At this speed, her γ is 2.0, so that the voyage will only seem to her to last 7 years. But there is perfect symmetry between Alice's and Betty's frames of reference, so Betty agrees with Alice on their relative speed; Betty sees herself as being at rest, while the sun and Tau Ceti both move backward at 87% of the speed of light. How, then, can she observe Tau Ceti to get to her in only 7 years, when it should take 14 years to travel 12 light-years at this speed?

We need to take into account length contraction. Betty sees the distance between the sun and Tau Ceti to be shrunk by a factor of 2. The same thing occurs for Alice, who observes Betty and her spaceship to be foreshortened.

Large time dilation — *example 7*

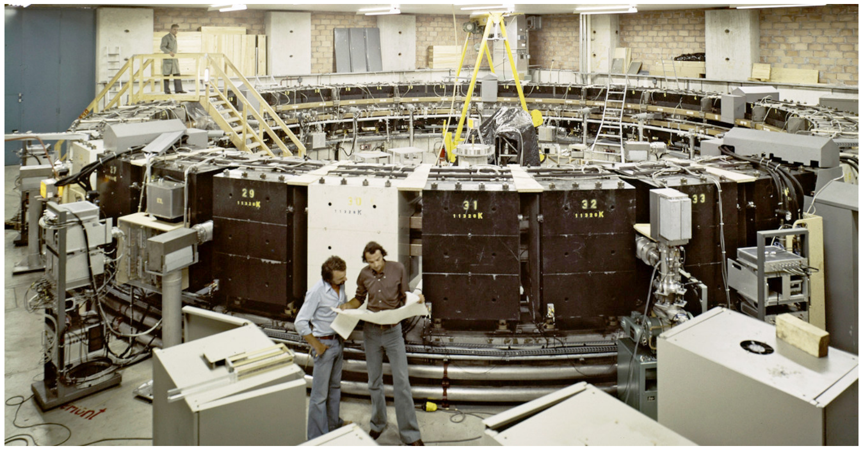
The time dilation effect in the Hafele-Keating experiment was very small. If we want to see a large time dilation effect, we can't do it with something the size of the atomic clocks they used; the kinetic energy would be greater than the total megatonnage of all the world's nuclear arsenals. We can, however, accelerate subatomic particles to speeds at which γ is large. For experimental particle physicists, relativity is something you do all day before heading home and stopping off at the store for milk. An early, low-precision experiment of this kind was performed by Rossi and Hall in 1941, using naturally occurring cosmic rays. Figure as shows a 1974 experiment[3] of a similar type which verified the time dilation predicted by relativity to a precision of about one part per thousand.



ar / Muons accelerated to nearly c undergo radioactive decay much more slowly than they would according to an observer at rest with respect to the muons. The first two data-points (unfilled circles) were subject to large systematic errors.

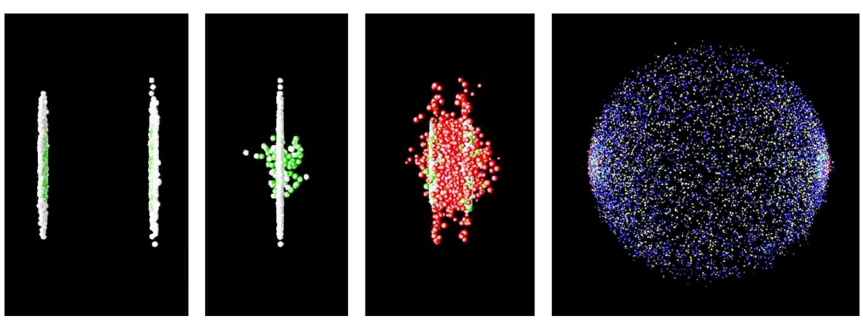
Particles called muons (named after the Greek letter μ, "myoo") were produced by an accelerator at CERN, near Geneva. A muon

[3] Bailey at al., Nucl. Phys. B150(1979) 1



as / Apparatus used for the test of relativistic time dilation described in example 7. The prominent black and white blocks are large magnets surrounding a circular pipe with a vacuum inside. (c) 1974 by CERN.

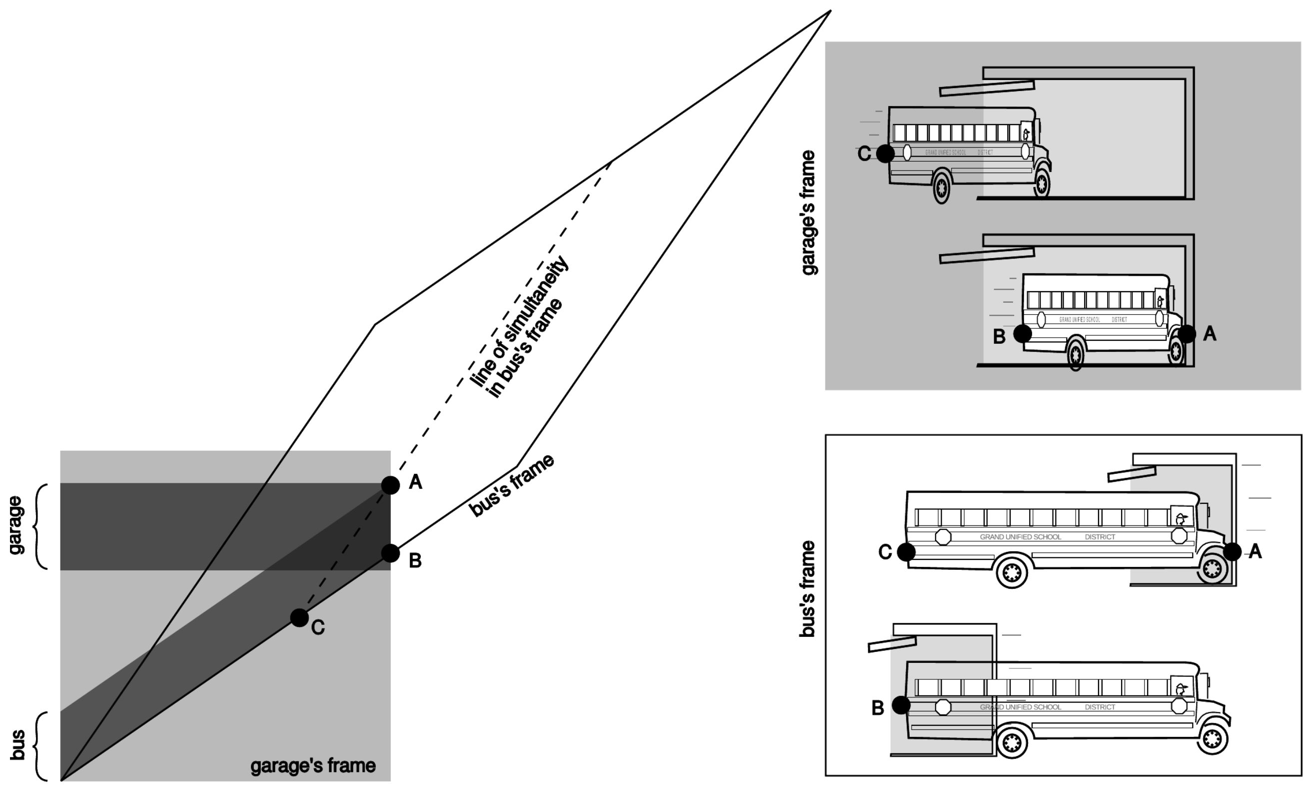
is essentially a heavier version of the electron. Muons undergo radioactive decay, lasting an average of only 2.197 μs before they evaporate into an electron and two neutrinos. The 1974 experiment was actually built in order to measure the magnetic properties of muons, but it produced a high-precision test of time dilation as a byproduct. Because muons have the same electric charge as electrons, they can be trapped using magnetic fields. Muons were injected into the ring shown in figure as, circling around it until they underwent radioactive decay. At the speed at which these muons were traveling, they had $\gamma = 29.33$, so on the average they lasted 29.33 times longer than the normal lifetime. In other words, they were like tiny alarm clocks that self-destructed at a randomly selected time. Figure ar shows the number of radioactive decays counted, as a function of the time elapsed after a given stream of muons was injected into the storage ring. The two dashed lines show the rates of decay predicted with and without relativity. The relativistic line is the one that agrees with experiment.



at / Colliding nuclei show relativistic length contraction.

An example of length contraction *example 8*

Figure at shows an artist's rendering of the length contraction for the collision of two gold nuclei at relativistic speeds in the RHIC accelerator in Long Island, New York, which went on line in 2000. The gold nuclei would appear nearly spherical (or just slightly lengthened like an American football) in frames moving along with them, but in the laboratory's frame, they both appear drastically foreshortened as they approach the point of collision. The later pictures show the nuclei merging to form a hot soup, in which experimenters hope to observe a new form of matter.

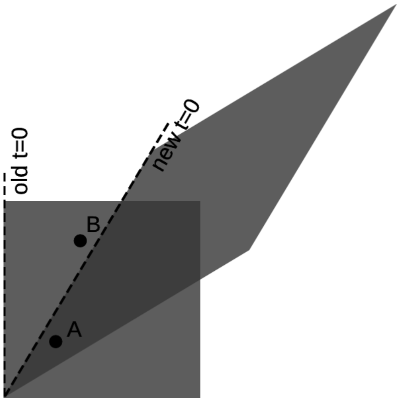


au / Example 9: In the garage's frame of reference, the bus is moving, and can fit in the garage due to its length contraction. In the bus's frame of reference, the garage is moving, and can't hold the bus due to *its* length contraction.

The garage paradox *example 9*

One of the most famous of all the so-called relativity paradoxes has to do with our incorrect feeling that simultaneity is well defined. The idea is that one could take a schoolbus and drive it at relativistic speeds into a garage of ordinary size, in which it normally would not fit. Because of the length contraction, the bus would supposedly fit in the garage. The driver, however, will perceive the *garage* as being contracted and thus even less able to contain the bus.

The paradox is resolved when we recognize that the concept of fitting the bus in the garage "all at once" contains a hidden assumption, the assumption that it makes sense to ask whether the front and back of the bus can *simultaneously* be in the garage. Observers in different frames of reference moving at high relative speeds do not necessarily agree on whether things happen simultaneously. As shown in figure au, the person in the garage's frame can shut the door at an instant B he perceives to be simultaneous with the front bumper's arrival A at the back wall of the garage, but the driver would not agree about the simultaneity of these two events, and would perceive the door as having shut long after she plowed through the back wall.



av / A proof that causality imposes a universal speed limit. In the original frame of reference, represented by the square, event A happens a little before event B. In the new frame, shown by the parallelogram, A happens after $t = 0$, but B happens before $t = 0$; that is, B happens before A. The time ordering of the two events has been reversed. This can only happen because events A and B are very close together in time and fairly far apart in space. The line segment connecting A and B has a slope greater than 1, meaning that if we wanted to be present at both events, we would have to travel at a speed greater than c (which equals 1 in the units used on this graph). You will find that if you pick any two points for which the slope of the line segment connecting them is less than 1, you can never get them to straddle the new $t = 0$ line in this funny, time-reversed way. Since different observers disagree on the time order of events like A and B, causality requires that information never travel from A to B or from B to A; if it did, then we would have time-travel paradoxes. The conclusion is that c is the maximum speed of cause and effect in relativity.

The universal speed c

Let's think a little more about the role of the 45-degree diagonal in the Lorentz transformation. Slopes on these graphs are interpreted as velocities. This line has a slope of 1 in relativistic units, but that slope corresponds to c in ordinary metric units. We already know that the relativistic distance unit must be extremely large compared to the relativistic time unit, so c must be extremely large. Now note what happens when we perform a Lorentz transformation: this particular line gets stretched, but the new version of the line lies right on top of the old one, and its slope stays the same. In other words, if one observer says that something has a velocity equal to c, every other observer will agree on that velocity as well. (The same thing happens with $-c$.)

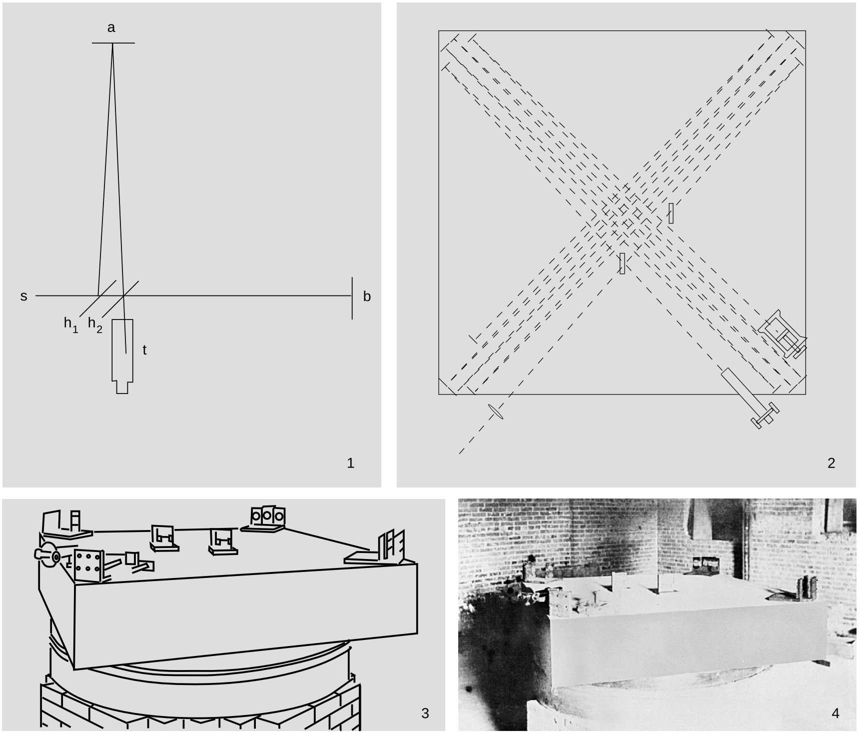
▷ Velocities don't simply add and subtract.

This is surprising, since we expect, as in section 2.5.1, that a velocity c in one frame should become $c + v$ in a frame moving at velocity v relative to the first one. But velocities are measured by dividing a distance by a time, and both distance and time are distorted by relativistic effects, so we actually shouldn't expect the ordinary arithmetic addition of velocities to hold in relativity; it's an approximation that's valid at velocities that are small compared to c. Problem 22 on p. 101 shows that relativistically, combining velocities u and v gives not $u + v$ but $(u + v)/(1 + uv)$ (in units where $c = 1$).

▷ A universal speed limit

For example, suppose Janet takes a trip in a spaceship, and accelerates until she is moving at $0.6c$ relative to the earth. She then launches a space probe in the forward direction at a speed relative to her ship of $0.6c$. We might think that the probe was then moving at a velocity of $1.2c$, but in fact the answer is still less than c (problem 21, page 100). This is an example of a more general fact about relativity, which is that c represents a universal speed limit. This is required by causality, as shown in figure av.

▷ Light travels at c.

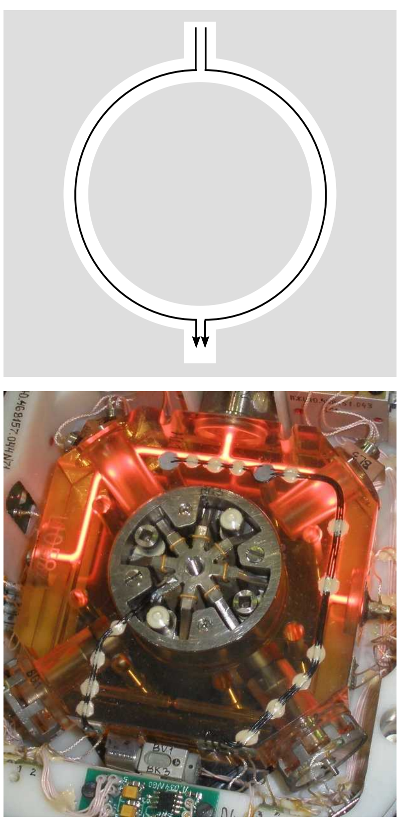


aw / The Michelson-Morley experiment, shown in photographs, and drawings from the original 1887 paper. 1. A simplified drawing of the apparatus. A beam of light from the source, s, is partially reflected and partially transmitted by the half-silvered mirror h_1. The two half-intensity parts of the beam are reflected by the mirrors at a and b, reunited, and observed in the telescope, t. If the earth's surface was supposed to be moving through the ether, then the times taken by the two light waves to pass through the moving ether would be unequal, and the resulting time lag would be detectable by observing the interference between the waves when they were reunited. 2. In the real apparatus, the light beams were reflected multiple times. The effective length of each arm was increased to 11 meters, which greatly improved its sensitivity to the small expected difference in the speed of light. 3. In an earlier version of the experiment, they had run into problems with its "extreme sensitiveness to vibration," which was "so great that it was impossible to see the interference fringes except at brief intervals . . . even at two o'clock in the morning." They therefore mounted the whole thing on a massive stone floating in a pool of mercury, which also made it possible to rotate it easily. 4. A photo of the apparatus.

Now consider a beam of light. We're used to talking casually about the "speed of light," but what does that really mean? Motion is relative, so normally if we want to talk about a velocity, we have to specify what it's measured relative to. A sound wave has a certain speed relative to the air, and a water wave has its own speed relative to the water. If we want to measure the speed of an ocean wave, for example, we should make sure to measure it in a frame of reference at rest relative to the water. But light isn't a vibration of a physical medium; it can propagate through the near-perfect vacuum of outer space, as when rays of sunlight travel to earth. This seems like a paradox: light is supposed to have a specific speed, but there is no way to decide what frame of reference to measure it in. The way out of the paradox is that light must travel at a velocity equal to c. Since all observers agree on a velocity of c, regardless of their frame of reference, everything is consistent.

▷ The Michelson-Morley experiment

The constancy of the speed of light had in fact already been observed when Einstein was an 8-year-old boy, but because nobody could figure out how to interpret it, the result was largely ignored. In 1887 Michelson and Morley set up a clever apparatus to measure any difference in the speed of light beams traveling east-west and north-south. The motion of the earth around the sun at 110,000 km/hour (about 0.01% of the speed of light) is to our west during the day. Michelson and Morley believed that light was a vibration of a mysterious medium called the ether, so they expected that the speed of light would be a fixed value relative to the ether. As the earth moved through the ether, they thought they would observe an effect on the velocity of light along an east-west line. For instance, if they released a beam of light in a westward direction during the day, they expected that it would move away from them at less than the normal speed because the earth was chasing it through the ether. They were surprised when they found that the expected 0.01% change in the speed of light did not occur.



ax / A ring laser gyroscope.

The ring laser gyroscope — *example 10*

If you've flown in a jet plane, you can thank relativity for helping you to avoid crashing into a mountain or an ocean. Figure ax shows a standard piece of navigational equipment called a ring laser gyroscope. A beam of light is split into two parts, sent around the perimeter of the device, and reunited. Since the speed of light is constant, we expect the two parts to come back together at the same time. If they don't, it's evidence that the device has been rotating. The plane's computer senses this and notes how much rotation has accumulated.

No frequency-dependence — *example 11*

Relativity has only one universal speed, so it requires that all light waves travel at the same speed, regardless of their frequency and wavelength. Presently the best experimental tests of the invariance of the speed of light with respect to wavelength come from astronomical observations of gamma-ray bursts, which are sudden outpourings of high-frequency light, believed to originate from a supernova explosion in another galaxy. One such observation, in 2009,[4] found that the times of arrival of all the different frequencies in the burst differed by no more than 2 seconds out

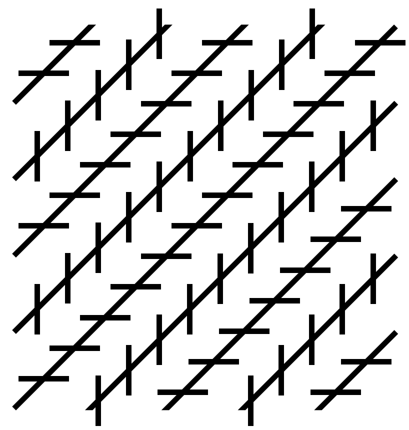
[4] http://arxiv.org/abs/0908.1832

of a total time in flight on the order of ten billion years!

Discussion questions

A A person in a spaceship moving at 99.99999999% of the speed of light relative to Earth shines a flashlight forward through dusty air, so the beam is visible. What does she see? What would it look like to an observer on Earth?

B A question that students often struggle with is whether time and space can really be distorted, or whether it just seems that way. Compare with optical illusions or magic tricks. How could you verify, for instance, that the lines in the figure are actually parallel? Are relativistic effects the same, or not?

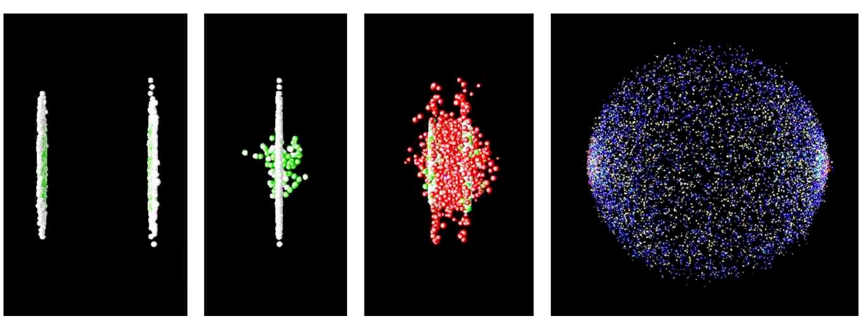


Discussion question B

C On a spaceship moving at relativistic speeds, would a lecture seem even longer and more boring than normal?

D Mechanical clocks can be affected by motion. For example, it was a significant technological achievement to build a clock that could sail aboard a ship and still keep accurate time, allowing longitude to be determined. How is this similar to or different from relativistic time dilation?

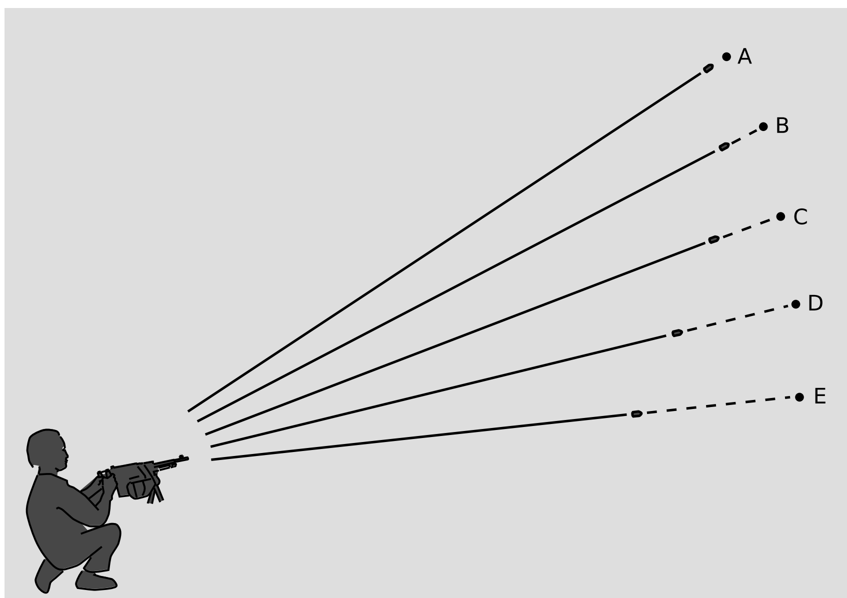
E Figure at from page 88, depicting the collision of two nuclei at the RHIC accelerator, is reproduced below. What would the shapes of the two nuclei look like to a microscopic observer riding on the left-hand nucleus? To an observer riding on the right-hand one? Can they agree on what is happening? If not, why not — after all, shouldn't they see the same thing if they both compare the two nuclei side-by-side at the same instant in time?



ay / Discussion question E: colliding nuclei show relativistic length contraction.

F If you stick a piece of foam rubber out the window of your car while driving down the freeway, the wind may compress it a little. Does it make sense to interpret the relativistic length contraction as a type of strain that pushes an object's atoms together like this? How does this relate to discussion question E?

G The machine-gunner in the figure sends out a spray of bullets. Suppose that the bullets are being shot into outer space, and that the distances traveled are trillions of miles (so that the human figure in the diagram is not to scale). After a long time, the bullets reach the points shown with dots which are all equally far from the gun. Their arrivals at those points are events A through E, which happen at different times. Sketch these events on a position-time graph. The chain of impacts extends across space at a speed greater than *c*. Does this violate special relativity?



Discussion question G.

Summary

Selected vocabulary

center of mass . .	the balance point of an object
velocity	the rate of change of position; the slope of the tangent line on an $x-t$ graph.

Notation

x	a point in space
t	a point in time, a clock reading
Δ	"change in;" the value of a variable afterwards minus its value before
Δx	a distance, or more precisely a change in x, which may be less than the distance traveled; its plus or minus sign indicates direction
Δt	a duration of time
v	velocity
v_{AB}	the velocity of object A relative to object B

Other terminology and notation

displacement . .	a name for the symbol Δx
speed	the absolute value of the velocity, i.e., the velocity stripped of any information about its direction

Summary

An object's center of mass is the point at which it can be balanced. For the time being, we are studying the mathematical description only of the motion of an object's center of mass in cases restricted to one dimension. The motion of an object's center of mass is usually far simpler than the motion of any of its other parts.

It is important to distinguish location, x, from distance, Δx, and clock reading, t, from time interval Δt. When an object's $x-t$ graph is linear, we define its velocity as the slope of the line, $\Delta x/\Delta t$. When the graph is curved, we generalize the definition so that the velocity is the derivative $\mathrm{d}x/\mathrm{d}t$.

Galileo's principle of inertia states that no force is required to maintain motion with constant velocity in a straight line, and absolute motion does not cause any observable physical effects. Things typically tend to reduce their velocity relative to the surface of our planet only because they are physically rubbing against the planet (or something attached to the planet), not because there is anything special about being at rest with respect to the earth's surface. When it seems, for instance, that a force is required to keep a book sliding across a table, in fact the force is only serving to cancel the contrary force of friction.

Absolute motion is not a well-defined concept, and if two observers are not at rest relative to one another they will disagree about the absolute velocities of objects. They will, however, agree about relative velocities. If object A is in motion relative to object B, and B is in motion relative to C, then A's velocity relative to C is given by $v_{AC} = v_{AB} + v_{BC}$. Positive and negative signs are used to indicate the direction of an object's motion.

Modern experiments show that space and time only approximately have the properties claimed by Galileo and Newton. Time and space as seen by one observer are distorted compared to another observer's perceptions if they are moving relative to each other. This distortion is quantified by the factor

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}},$$

where v is the relative velocity of the two observers, and c is a universal velocity that is the same in all frames of reference. Light travels at c. A clock appears to run fastest to an observer who is not in motion relative to it, and appears to run too slowly by a factor of γ to an observer who has a velocity v relative to the clock. Similarly, a meter-stick appears longest to an observer who sees it at rest, and appears shorter to other observers. Time and space are relative, not absolute. In particular, there is no well-defined concept of simultaneity. Velocities don't add according to $u + v$ but rather $(u + v)/(1 + uv)$ (in units where $c = 1$).

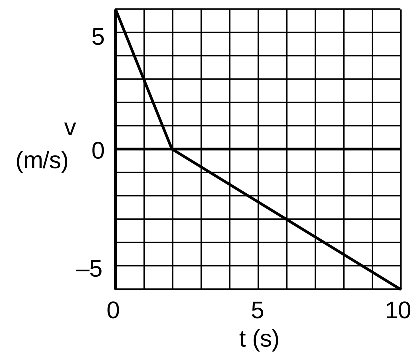
All of these strange effects, however, are very small when the relative velocities are small compared to c. This makes sense, because Newton's laws have already been thoroughly tested by experiments at such speeds, so a new theory like relativity must agree with the old one in their realm of common applicability. This requirement of backwards-compatibility is known as the correspondence principle.

Problems

Key

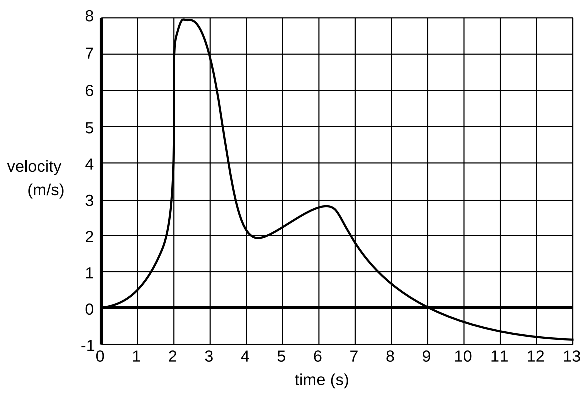
- $\checkmark$ A computerized answer check is available online.
- $\int$ A problem that requires calculus.
- $\star$ A difficult problem.

1 The graph represents the motion of a ball that rolls up a hill and then back down. When does the ball return to the location it had at $t = 0$? ▷ Solution, p. 544



Problem 1.

2 The graph represents the velocity of a bee along a straight line. At $t = 0$, the bee is at the hive. (a) When is the bee farthest from the hive? (b) How far is the bee at its farthest point from the hive? (c) At $t = 13$ s, how far is the bee from the hive? $\checkmark$



3 (a) Let θ be the latitude of a point on the Earth's surface. Derive an algebra equation for the distance, L, traveled by that point during one rotation of the Earth about its axis, i.e., over one day, expressed in terms of θ and R, the radius of the earth. Check: Your equation should give $L = 0$ for the North Pole.
(b) At what speed is Fullerton, at latitude $\theta = 34°$, moving with the rotation of the Earth about its axis? Give your answer in units of mi/h. [See the table in the back of the book for the relevant data.] $\checkmark$

4 A honeybee's position as a function of time is given by $x = 10t - t^3$, where t is in seconds and x in meters. What is its velocity at $t = 3.0$ s? $\checkmark$

5 Freddi Fish$^{(TM)}$ has a position as a function of time given by $x = a/(b + t^2)$. (a) Infer the units of the constants a and b. (b) Find her maximum speed. (c) Check that your answer has the right units. $\checkmark$

6 A metal square expands and contracts with temperature, the lengths of its sides varying according to the equation $\ell = (1+\alpha T)\ell_o$. Infer the units of α. Find the rate of change of its surface area with respect to temperature. That is, find dA/dT. Check that your answer has the right units, as in example 4 on page 70. $\checkmark$

7 Let t be the time that has elapsed since the Big Bang. In that time, one would imagine that light, traveling at speed c, has been able to travel a maximum distance ct. (In fact the distance is several times more than this, because according to Einstein's theory of general relativity, space itself has been expanding while the ray of light was in transit.) The portion of the universe that we can observe would then be a sphere of radius ct, with volume $v = (4/3)\pi r^3 = (4/3)\pi(ct)^3$. Compute the rate $\mathrm{d}v/\,\mathrm{d}t$ at which the volume of the observable universe is increasing, and check that your answer has the right units, as in example 4 on page 70. ✓

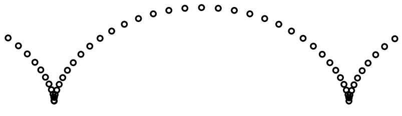
8 (a) Express the chain rule in Leibniz ("d") notation, and show that it always results in an answer whose units make sense.
(b) An object has a position as a function of time given by $x = A\cos(bt)$, where A and b are constants. Infer the units of A and b, and interpret their physical meanings.
(c) Find the velocity of this object, and check that the chain rule has indeed given an answer with the right units.
▷ Solution, p. 545

9 (a) Translate the following information into symbols, using the notation with two subscripts introduced in section 2.5. Eowyn is riding on her horse at a velocity of 11 m/s. She twists around in her saddle and fires an arrow backward. Her bow fires arrows at 25 m/s. (b) Find the velocity of the arrow relative to the ground.

10 Our full discussion of two- and three-dimensional motion is postponed until the second half of the book, but here is a chance to use a little mathematical creativity in anticipation of that generalization. Suppose a ship is sailing east at a certain speed v, and a passenger is walking across the deck at the same speed v, so that his track across the deck is perpendicular to the ship's center-line. What is his speed relative to the water, and in what direction is he moving relative to the water? ▷ Solution, p. 545

11 You're standing in a freight train, and have no way to see out. If you have to lean to stay on your feet, what, if anything, does that tell you about the train's velocity? Explain. ▷ Solution, p. 545

12 Driving along in your car, you take your foot off the gas, and your speedometer shows a reduction in speed. Describe a frame of reference in which your car was *speeding up* during that same period of time. (The frame of reference should be defined by an observer who, although perhaps in motion relative to the earth, is not changing her own speed or direction of motion.)



Problem 13.

13 The figure shows the motion of a point on the rim of a rolling wheel. (The shape is called a cycloid.) Suppose bug A is riding on the rim of the wheel on a bicycle that is rolling, while bug B is on the spinning wheel of a bike that is sitting upside down on the floor.

Bug A is moving along a cycloid, while bug B is moving in a circle. Both wheels are doing the same number of revolutions per minute. Which bug has a harder time holding on, or do they find it equally difficult? ▷ Solution, p. 545

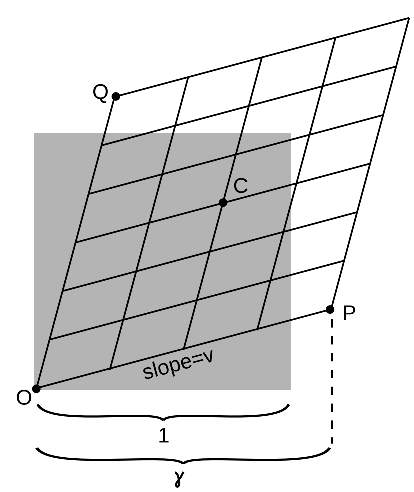
14 Astronauts in three different spaceships are communicating with each other. Those aboard ships A and B agree on the rate at which time is passing, but they disagree with the ones on ship C.
(a) Alice is aboard ship A. How does she describe the motion of her own ship, in its frame of reference?
(b) Describe the motion of the other two ships according to Alice.
(c) Give the description according to Betty, whose frame of reference is ship B.
(d) Do the same for Cathy, aboard ship C.

15 What happens in the equation for γ when you put in a negative number for v? Explain what this means physically, and why it makes sense.

16 The Voyager 1 space probe, launched in 1977, is moving faster relative to the earth than any other human-made object, at 17,000 meters per second.
(a) Calculate the probe's γ.
(b) Over the course of one year on earth, slightly less than one year passes on the probe. How much less? (There are 31 million seconds in a year.) ✓

17 The earth is orbiting the sun, and therefore is contracted relativistically in the direction of its motion. Compute the amount by which its diameter shrinks in this direction. ✓

18 In this homework problem, you'll fill in the steps of the algebra required in order to find the equation for γ on page 85. To keep the algebra simple, let the time t in figure an equal 1, as suggested in the figure accompanying this homework problem. The original square then has an area of 1, and the transformed parallelogram must also have an area of 1. (a) Prove that point P is at $x = v\gamma$, so that its (t, x) coordinates are $(\gamma, v\gamma)$. (b) Find the (t, x) coordinates of point Q. (c) Find the length of the short diagonal connecting P and Q. (d) Average the coordinates of P and Q to find the coordinates of the midpoint C of the parallelogram, and then find distance OC. (e) Find the area of the parallelogram by computing twice the area of triangle PQO. [Hint: You can take PQ to be the base of the triangle.] (f) Set this area equal to 1 and solve for γ to prove $\gamma = 1/\sqrt{1-v^2}$. ✓

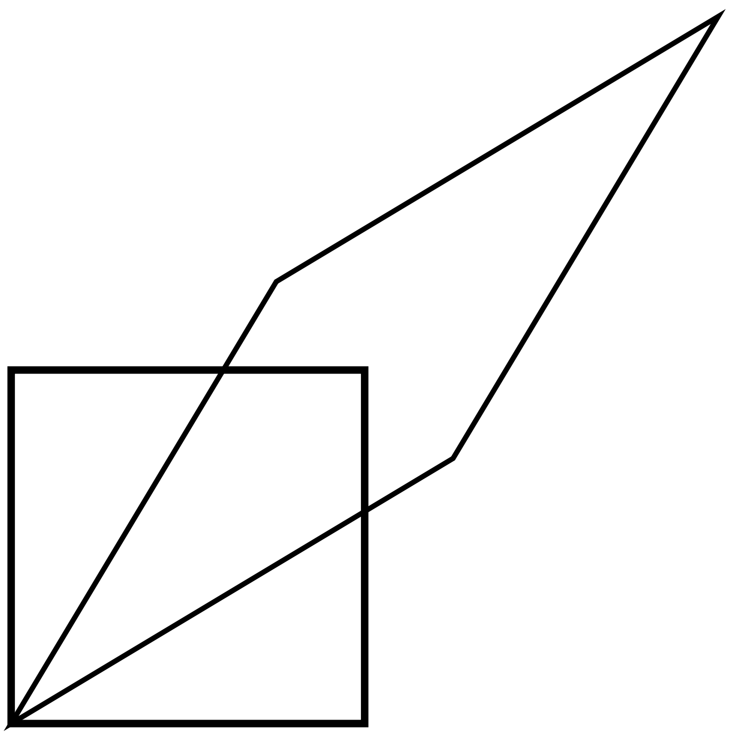


Problem 18.

19 (a) Show that for $v = (3/5)c$, γ comes out to be a simple fraction.
(b) Find another value of v for which γ is a simple fraction.

20 In the equation for the relativistic addition of velocities u and v, consider the limit in which u approaches 1, but v simultaneously approaches -1. Give both a physical and a mathematical interpretation.

21 The figure illustrates a Lorentz transformation using the conventions employed in section 2.6.2. For simplicity, the transformation chosen is one that lengthens one diagonal by a factor of 2. Since Lorentz transformations preserve area, the other diagonal is shortened by a factor of 2. Let the original frame of reference, depicted with the square, be A, and the new one B. (a) By measuring with a ruler on the figure, show that the velocity of frame B relative to frame A is $0.6c$. (b) Print out a copy of the page. With a ruler, draw a third parallelogram that represents a second successive Lorentz transformation, one that lengthens the long diagonal by another factor of 2. Call this third frame C. Use measurements with a ruler to determine frame C's velocity relative to frame A. Does it equal double the velocity found in part a? Explain why it should be expected to turn out the way it does. A general equation for this type of calculation is derived in problem 22. ✓

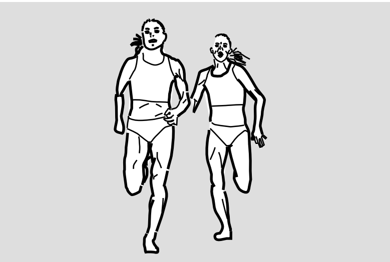


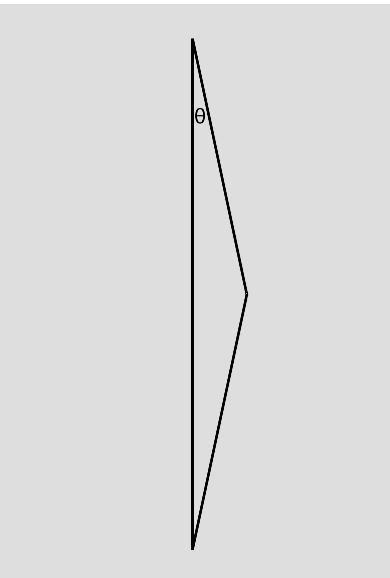
22 The purpose of this problem is to prove the general result $w = (u+v)/(1+uv)$ (in units where $c = 1$) for the kind of combination of velocities found graphically in problem 21. Suppose that we perform two Lorentz transformations, with velocities u and v, one after the other. Representing these transformations as distortions of parallelograms, we stretch the stretching diagonals by factors $S(u)$ and $S(v)$ (and compress the compressing ones by the inverses of these factors), so that the combined result is a stretching by $S(u)S(v)$. We want to prove that $S(w) = S(u)S(v)$ gives the expression claimed above for w. One can easily show by fiddling with the result of part c of problem 18 that $S(x) = \sqrt{(1+x)/(1-x)}$.
(a) Use these facts to write down an equation relating u, v, and w.
(b) Solve for w in terms of u and v. (c) Show that your answer to part b satisfies the correspondence principle. (d) Show that it is consistent with the constancy of c.

23 Sometimes doors are built with mechanisms that automatically close them after they have been opened. The designer can set both the strength of the spring and the amount of friction. If there is too much friction in relation to the strength of the spring, the door takes too long to close, but if there is too little, the door will oscillate. For an optimal design, we get motion of the form $x = cte^{-bt}$, where x is the position of some point on the door, and c and b are positive constants. (Similar systems are used for other mechanical devices, such as stereo speakers and the recoil mechanisms of guns.) In this example, the door moves in the positive direction up until a certain time, then stops and settles back in the negative direction, eventually approaching $x = 0$. This would be the type of motion we would get if someone flung a door open and the door closer then brought it back closed again. (a) Infer the units of the constants b and c.
(b) Find the door's maximum speed (i.e., the greatest absolute value of its velocity) as it comes back to the closed position. ✓
(c) Show that your answer has units that make sense.

24 At a picnic, someone hands you a can of beer. The ground is uneven, and you don't want to spill your drink. You reason that it will be more stable if you drink some of it first in order to lower its center of mass. How much should you drink in order to make the center of mass as low as possible? [Based on a problem by Walter van B. Roberts and Martin Gardner.]

25 (a) In a race, you run the first half of the distance at speed u, and the second half at speed v. Find the over-all speed, i.e., the total distance divided by the total time. ✓
(b) Check the units of your equation using the method shown in example 1 on p. 25.
(c) Check that your answer makes sense in the case where $u = v$.
(d) Show that the dependence of your result on u and v makes sense. That is, first check whether making u bigger makes the result bigger, or smaller. Then compare this with what you expect physically. [Problem by B. Shotwell.]





Problem 28.

26 (a) Let R be the radius of the Earth and T the time (one day) that it takes for one rotation. Find the speed at which a point on the equator moves due to the rotation of the earth. ✓
(b) Check the units of your equation using the method shown in example 1 on p. 25.
(c) Check that your answer to part a makes sense in the case where the Earth stops rotating completely, so that T is infinitely long.
(d) Nairobi, Kenya, is very close to the equator. Plugging in numbers to your answer from part a, find Nairobi's speed in meters per second. See the table in the back of the book for the relevant data. For comparison, the speed of sound is about 340 m/s. ✓

27 (a) Let θ be the latitude of a point on the Earth's surface. Derive an algebra equation for the distance, L, traveled by that point during one rotation of the Earth about its axis, i.e., over one day, expressed in terms of θ and R, the radius of the earth. You may find it helpful to draw one or more diagrams in the style of figure h on p. 32. ✓
(b) Generalize the result of problem 26a to points not necessarily on the equator. ✓
(c) Check the units of your equation using the method shown in example 1 on p. 25.
(d) Check that your equation in part b gives zero for the North Pole, and also that it agrees with problem 26a in the special case of a point on the equator.
(e) At what speed is Fullerton, California, at latitude $\theta = 34°$, moving with the rotation of the Earth about its axis? ✓

28 In running races at distances of 800 meters and longer, runners do not have their own lanes, so in order to pass, they have to go around their opponents. Suppose we adopt the simplified geometrical model suggested by the figure, in which the two runners take equal times to trace out the sides of an isoceles triangle, deviating from parallelism by the angle θ. The runner going straight runs at speed v, while the one who is passing must run at a greater speed. Let the difference in speeds be Δv.
(a) Find Δv in terms of v and θ. ✓
(b) Check the units of your equation using the method shown in example 1 on p. 25.
(c) Check that your answer makes sense in the special case where $\theta = 0$, i.e., in the case where the runners are on an extremely long straightaway.
(d) Suppose that $\theta = 1.0$ degrees, which is about the smallest value that will allow a runner to pass in the distance available on the straightaway of a track, and let $v = 7.06$ m/s, which is the women's world record pace at 800 meters. Plug numbers into your equation from part a to determine Δv, and comment on the result. ✓

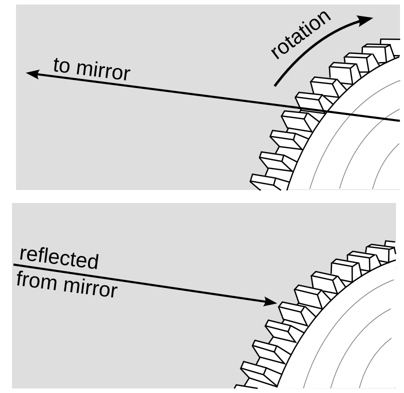
29 In 1849, Fizeau carried out the first terrestrial measurement of the speed of light; previous measurements by Roemer and Bradley had involved astronomical observation. The figure shows a simplified conceptual representation of Fizeau's experiment. A ray of light from a bright source was directed through the teeth at the edge of a spinning cogwheel. After traveling a distance L, it was reflected from a mirror and returned along the same path. The figure shows the case in which the ray passes between two teeth, but when it returns, the wheel has rotated by half the spacing of the teeth, so that the ray is blocked. When this condition is achieved, the observer looking through the teeth toward the far-off mirror sees it go completely dark. Fizeau adjusted the speed of the wheel to achieve this condition and recorded the rate of rotation to be f rotations per second. Let the number of teeth on the wheel be n.

(a) Find the speed of light c in terms of L, n, and f. √

(b) Check the units of your equation using the method shown in example 1 on p. 25. (Here f's units of rotations per second should be taken as inverse seconds, s^{-1}, since the number of rotations in a second is a unitless count.)

(c) Imagine that you are Fizeau trying to design this experiment. The speed of light is a huge number in ordinary units. Use your equation from part a to determine whether increasing c requires an increase in L, or a decrease. Do the same for n and f. Based on this, decide for each of these variables whether you want a value that is as big as possible, or as small as possible.

(d) Fizeau used $L = 8633$ m, $f = 12.6\ s^{-1}$, and $n = 720$. Plug in to your equation from part a and extract the speed of light from his data. √



Problem 29.

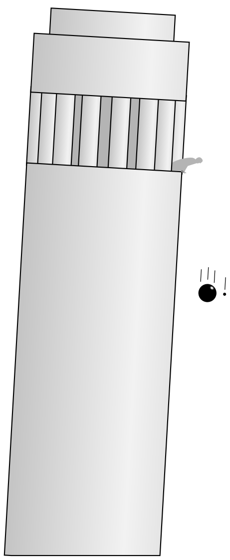
Galileo's contradiction of Aristotle had serious consequences. He was interrogated by the Church authorities and convicted of teaching that the earth went around the sun as a matter of fact and not, as he had promised previously, as a mere mathematical hypothesis. He was placed under permanent house arrest, and forbidden to write about or teach his theories. Immediately after being forced to recant his claim that the earth revolved around the sun, the old man is said to have muttered defiantly "and yet it does move." The story is dramatic, but there are some omissions in the commonly taught heroic version. There was a rumor that the Simplicio character represented the Pope. Also, some of the ideas Galileo advocated had controversial religious overtones. He believed in the existence of atoms, and atomism was thought by some people to contradict the Church's doctrine of transubstantiation, which said that in the Catholic mass, the blessing of the bread and wine literally transformed them into the flesh and blood of Christ. His support for a cosmology in which the earth circled the sun was also disreputable because one of its supporters, Giordano Bruno, had also proposed a bizarre synthesis of Christianity with the ancient Egyptian religion.

Chapter 3
Acceleration and free fall

3.1 The motion of falling objects

The motion of falling objects is the simplest and most common example of motion with changing velocity. The early pioneers of physics had a correct intuition that the way things drop was a message directly from Nature herself about how the universe worked. Other examples seem less likely to have deep significance. A walking person who speeds up is making a conscious choice. If one stretch of a river flows more rapidly than another, it may be only because the channel is narrower there, which is just an accident of the local geography. But there is something impressively consistent, universal, and inexorable about the way things fall.

Stand up now and simultaneously drop a coin and a bit of paper side by side. The paper takes much longer to hit the ground. That's why Aristotle wrote that heavy objects fell more rapidly. Europeans believed him for two thousand years.



a / According to Galileo's student Viviani, Galileo dropped a cannonball and a musketball simultaneously from the leaning tower of Pisa, and observed that they hit the ground at nearly the same time. This contradicted Aristotle's long-accepted idea that heavier objects fell faster.

Now repeat the experiment, but make it into a race between the coin and your shoe. My own shoe is about 50 times heavier than the nickel I had handy, but it looks to me like they hit the ground at exactly the same moment. So much for Aristotle! Galileo, who had a flair for the theatrical, did the experiment by dropping a bullet and a heavy cannonball from a tall tower. Aristotle's observations had been incomplete, his interpretation a vast oversimplification.

It is inconceivable that Galileo was the first person to observe a discrepancy with Aristotle's predictions. Galileo was the one who changed the course of history because he was able to assemble the observations into a coherent pattern, and also because he carried out systematic quantitative (numerical) measurements rather than just describing things qualitatively.

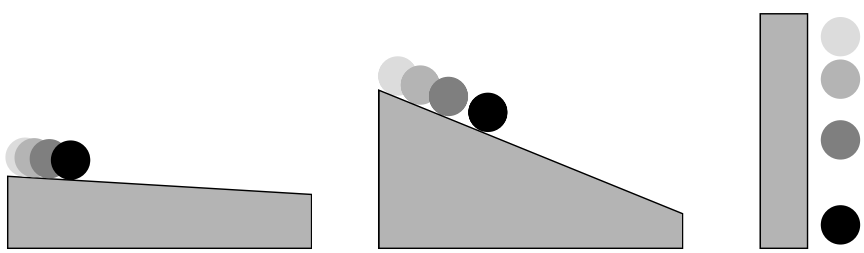
Why is it that some objects, like the coin and the shoe, have similar motion, but others, like a feather or a bit of paper, are different? Galileo speculated that in addition to the force that always pulls objects down, there was an upward force exerted by the air. Anyone can speculate, but Galileo went beyond speculation and came up with two clever experiments to probe the issue. First, he experimented with objects falling in water, which probed the same issues but made the motion slow enough that he could take time measurements with a primitive pendulum clock. With this technique, he established the following facts:

- All heavy, streamlined objects (for example a steel rod dropped point-down) reach the bottom of the tank in about the same amount of time, only slightly longer than the time they would take to fall the same distance in air.
- Objects that are lighter or less streamlined take a longer time to reach the bottom.

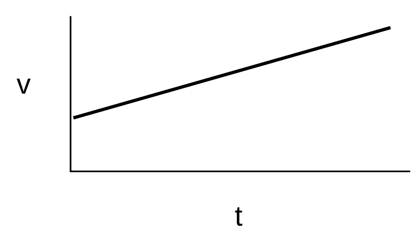
This supported his hypothesis about two contrary forces. He imagined an idealized situation in which the falling object did not have to push its way through any substance at all. Falling in air would be more like this ideal case than falling in water, but even a thin, sparse medium like air would be sufficient to cause obvious effects on feathers and other light objects that were not streamlined. Today, we have vacuum pumps that allow us to suck nearly all the air out of a chamber, and if we drop a feather and a rock side by side in a vacuum, the feather does not lag behind the rock at all.

How the speed of a falling object increases with time

Galileo's second stroke of genius was to find a way to make quantitative measurements of how the speed of a falling object increased as it went along. Again it was problematic to make sufficiently accurate time measurements with primitive clocks, and again he found a tricky way to slow things down while preserving the essential physical phenomena: he let a ball roll down a slope instead of dropping it vertically. The steeper the incline, the more rapidly the ball would gain speed. Without a modern video camera, Galileo had invented a way to make a slow-motion version of falling.



b / Velocity increases more gradually on the gentle slope, but the motion is otherwise the same as the motion of a falling object.

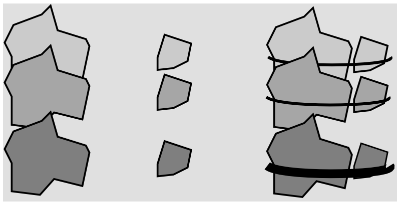


c / The $v-t$ graph of a falling object is a line.

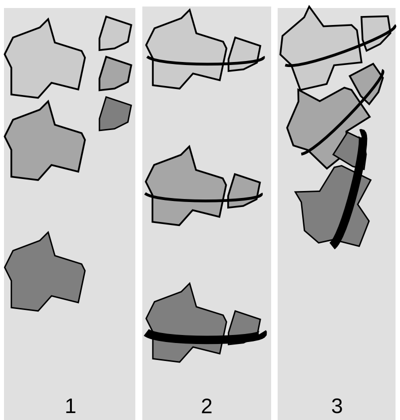
Although Galileo's clocks were only good enough to do accurate experiments at the smaller angles, he was confident after making a systematic study at a variety of small angles that his basic conclusions were generally valid. Stated in modern language, what he found was that the velocity-versus-time graph was a line. In the language of algebra, we know that a line has an equation of the form $y = ax + b$, but our variables are v and t, so it would be $v = at + b$. (The constant b can be interpreted simply as the initial velocity of the object, i.e., its velocity at the time when we started our clock, which we conventionally write as v_o.)

self-check A

An object is rolling down an incline. After it has been rolling for a short time, it is found to travel 13 cm during a certain one-second interval. During the second after that, it goes 16 cm. How many cm will it travel in the second after that? ▷ Answer, p. 558



d / Galileo's experiments show that all falling objects have the same motion if air resistance is negligible.



e / 1. Aristotle said that heavier objects fell faster than lighter ones. 2. If two rocks are tied together, that makes an extra-heavy rock, which should fall faster. 3. But Aristotle's theory would also predict that the light rock would hold back the heavy rock, resulting in a slower fall.

A contradiction in Aristotle's reasoning

Galileo's inclined-plane experiment disproved the long-accepted claim by Aristotle that a falling object had a definite "natural falling speed" proportional to its weight. Galileo had found that the speed just kept on increasing, and weight was irrelevant as long as air friction was negligible. Not only did Galileo prove experimentally that Aristotle had been wrong, but he also pointed out a logical contradiction in Aristotle's own reasoning. Simplicio, the stupid character, mouths the accepted Aristotelian wisdom:

> SIMPLICIO: There can be no doubt but that a particular body . . . has a fixed velocity which is determined by nature. . .
>
> SALVIATI: If then we take two bodies whose natural speeds are different, it is clear that, [according to Aristotle], on uniting the two, the more rapid one will be partly held back by the slower, and the slower will be somewhat hastened by the swifter. Do you not agree with me in this opinion?
>
> SIMPLICIO: You are unquestionably right.
>
> SALVIATI: But if this is true, and if a large stone moves with a speed of, say, eight [unspecified units] while a smaller moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. Thus you see how, from your assumption that the heavier body moves more rapidly than the lighter one, I infer that the heavier body moves more slowly.

What is gravity?

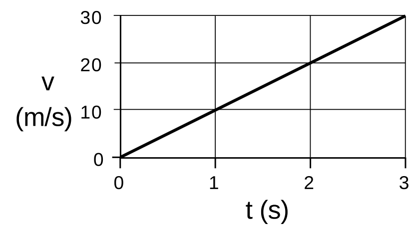
The physicist Richard Feynman liked to tell a story about how when he was a little kid, he asked his father, "Why do things fall?" As an adult, he praised his father for answering, "Nobody knows why things fall. It's a deep mystery, and the smartest people in the world don't know the basic reason for it." Contrast that with the average person's off-the-cuff answer, "Oh, it's because of gravity." Feynman liked his father's answer, because his father realized that simply giving a name to something didn't mean that you understood it. The radical thing about Galileo's and Newton's approach to science was that they concentrated first on describing mathematically what really did happen, rather than spending a lot of time on untestable speculation such as Aristotle's statement that "Things fall because they are trying to reach their natural place in contact with the earth." That doesn't mean that science can never answer the "why" questions. Over the next month or two as you delve deeper into physics, you will learn that there are more fundamental reasons why all falling objects have $v-t$ graphs with the same slope, regardless of their mass. Nevertheless, the methods of science always impose limits on how deep our explanation can go.

3.2 Acceleration

Definition of acceleration for linear $v - t$ graphs

Galileo's experiment with dropping heavy and light objects from a tower showed that all falling objects have the same motion, and his inclined-plane experiments showed that the motion was described by $v = at + v_o$. The initial velocity v_o depends on whether you drop the object from rest or throw it down, but even if you throw it down, you cannot change the slope, a, of the $v - t$ graph.

Since these experiments show that all falling objects have linear $v - t$ graphs with the same slope, the slope of such a graph is apparently an important and useful quantity. We use the word acceleration, and the symbol a, for the slope of such a graph. In symbols, $a = \Delta v / \Delta \mathrm{t}$. The acceleration can be interpreted as the amount of speed gained in every second, and it has units of velocity divided by time, i.e., "meters per second per second," or m/s/s. Continuing to treat units as if they were algebra symbols, we simplify "m/s/s" to read "m/s^2." Acceleration can be a useful quantity for describing other types of motion besides falling, and the word and the symbol "a" can be used in a more general context. We reserve the more specialized symbol "g" for the acceleration of falling objects, which on the surface of our planet equals 9.8 m/s^2. Often when doing approximate calculations or merely illustrative numerical examples it is good enough to use $g = 10$ m/s^2, which is off by only 2%.



f / Example 1.

Finding final speed, given time — *example 1*

▷ A despondent physics student jumps off a bridge, and falls for three seconds before hitting the water. How fast is he going when he hits the water?

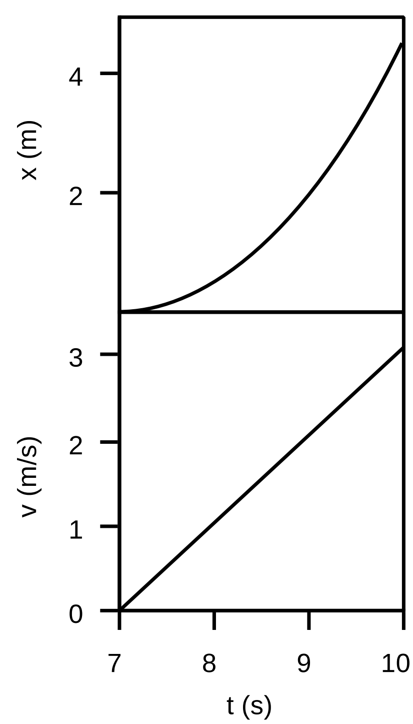
▷ Approximating g as 10 m/s^2, he will gain 10 m/s of speed each second. After one second, his velocity is 10 m/s, after two seconds it is 20 m/s, and on impact, after falling for three seconds, he is moving at 30 m/s.

Extracting acceleration from a graph — *example 2*

▷ The $x - t$ and $v - t$ graphs show the motion of a car starting from a stop sign. What is the car's acceleration?

▷ Acceleration is defined as the slope of the v-t graph. The graph rises by 3 m/s during a time interval of 3 s, so the acceleration is (3 m/s)/(3 s) = 1 m/s^2.

Incorrect solution #1: The final velocity is 3 m/s, and acceleration is velocity divided by time, so the acceleration is (3 m/s)/(10 s) = 0.3 m/s^2.



g / Example 2.

✗ The solution is incorrect because you can't find the slope of a graph from one point. This person was just using the point at the right end of the v-t graph to try to find the slope of the curve.

Incorrect solution #2: Velocity is distance divided by time so v = (4.5 m)/(3 s) = 1.5 m/s. Acceleration is velocity divided by time, so a = (1.5 m/s)/(3 s) = 0.5 m/s^2.

✗ The solution is incorrect because velocity is the slope of the tangent line. In a case like this where the velocity is changing, you can't just pick two points on the x-t graph and use them to find the velocity.

Converting g to different units *example 3*

▷ What is g in units of cm/s^2?

▷ The answer is going to be how many cm/s of speed a falling object gains in one second. If it gains 9.8 m/s in one second, then it gains 980 cm/s in one second, so g = 980 cm/s^2. Alternatively, we can use the method of fractions that equal one:

$$\frac{9.8\ \not{\text{m}}}{\text{s}^2} \times \frac{100\ \text{cm}}{1\ \not{\text{m}}} = \frac{980\ \text{cm}}{\text{s}^2}$$

▷ What is g in units of miles/hour^2?

▷

$$\frac{9.8\ \text{m}}{\text{s}^2} \times \frac{1\ \text{mile}}{1600\ \text{m}} \times \left(\frac{3600\ \text{s}}{1\ \text{hour}}\right)^2 = 7.9 \times 10^4\ \text{mile/hour}^2$$

This large number can be interpreted as the speed, in miles per hour, that you would gain by falling for one hour. Note that we had to square the conversion factor of 3600 s/hour in order to cancel out the units of seconds squared in the denominator.

▷ What is g in units of miles/hour/s?

▷

$$\frac{9.8\ \text{m}}{\text{s}^2} \times \frac{1\ \text{mile}}{1600\ \text{m}} \times \frac{3600\ \text{s}}{1\ \text{hour}} = 22\ \text{mile/hour/s}$$

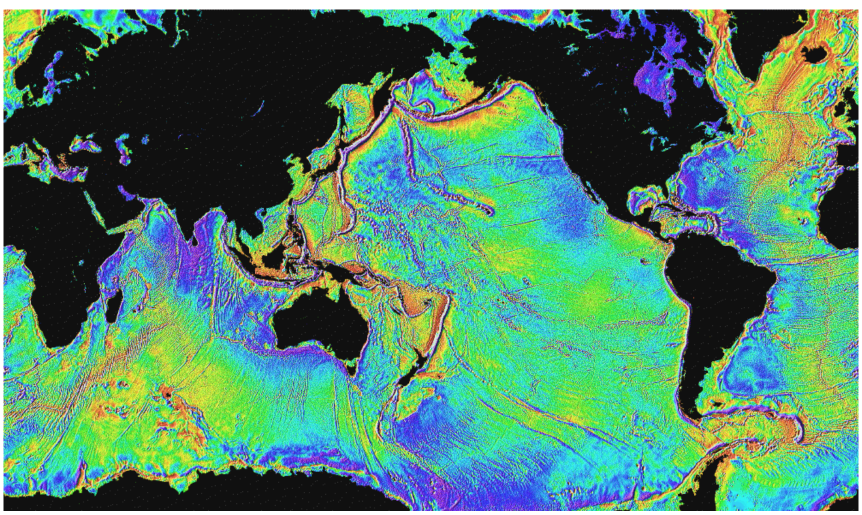
This is a figure that Americans will have an intuitive feel for. If your car has a forward acceleration equal to the acceleration of a falling object, then you will gain 22 miles per hour of speed every second. However, using mixed time units of hours and seconds like this is usually inconvenient for problem-solving. It would be like using units of foot-inches for area instead of ft^2 or in^2.

The acceleration of gravity is different in different locations.

Everyone knows that gravity is weaker on the moon, but actually it is not even the same everywhere on Earth, as shown by the sampling of numerical data in the following table.

location	latitude	elevation (m)	g (m/s^2)
north pole	90°N	0	9.8322
Reykjavik, Iceland	64°N	0	9.8225
Guayaquil, Ecuador	2°S	0	9.7806
Mt. Cotopaxi, Ecuador	1°S	5896	9.7624
Mt. Everest	28°N	8848	9.7643

The main variables that relate to the value of g on Earth are latitude and elevation. Although you have not yet learned how g would be calculated based on any deeper theory of gravity, it is not too hard to guess why g depends on elevation. Gravity is an attraction between things that have mass, and the attraction gets weaker with increasing distance. As you ascend from the seaport of Guayaquil to the nearby top of Mt. Cotopaxi, you are distancing yourself from the mass of the planet. The dependence on latitude occurs because we are measuring the acceleration of gravity relative to the earth's surface, but the earth's rotation causes the earth's surface to fall out from under you. (We will discuss both gravity and rotation in more detail later in the course.)



h / This false-color map shows variations in the strength of the earth's gravity. Purple areas have the strongest gravity, yellow the weakest. The overall trend toward weaker gravity at the equator and stronger gravity at the poles has been artificially removed to allow the weaker local variations to show up. The map covers only the oceans because of the technique used to make it: satellites look for bulges and depressions in the surface of the ocean. A very slight bulge will occur over an undersea mountain, for instance, because the mountain's gravitational attraction pulls water toward it. The US government originally began collecting data like these for military use, to correct for the deviations in the paths of missiles. The data have recently been released for scientific and commercial use (e.g., searching for sites for off-shore oil wells).

Much more spectacular differences in the strength of gravity can be observed away from the Earth's surface:

location	g (m/s^2)
asteroid Vesta (surface)	0.3
Earth's moon (surface)	1.6
Mars (surface)	3.7
Earth (surface)	9.8
Jupiter (cloud-tops)	26
Sun (visible surface)	270
typical neutron star (surface)	10^{12}
black hole (center)	infinite according to some theories, on the order of 10^{52} according to others

A typical neutron star is not so different in size from a large asteroid, but is orders of magnitude more massive, so the mass of a body definitely correlates with the g it creates. On the other hand, a neutron star has about the same mass as our Sun, so why is its g billions of times greater? If you had the misfortune of being on the surface of a neutron star, you'd be within a few thousand miles of all its mass, whereas on the surface of the Sun, you'd still be millions of miles from most of its mass.

Discussion questions

A What is wrong with the following definitions of g?

(1) "g is gravity."

(2) "g is the speed of a falling object."

(3) "g is how hard gravity pulls on things."

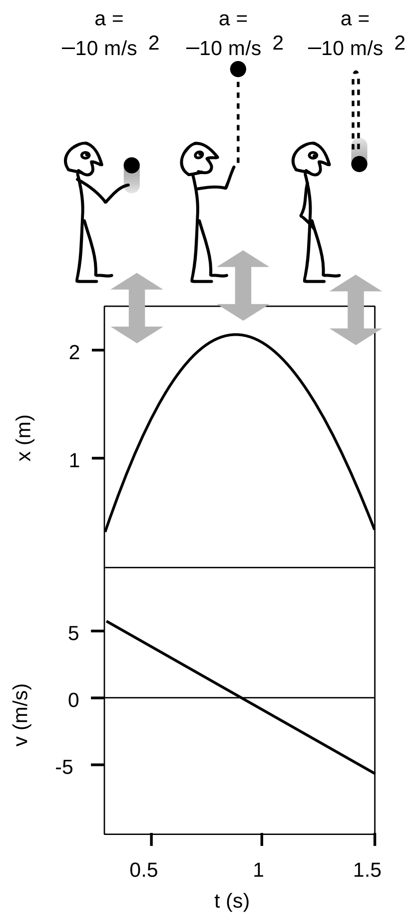
B When advertisers specify how much acceleration a car is capable of, they do not give an acceleration as defined in physics. Instead, they usually specify how many seconds are required for the car to go from rest to 60 miles/hour. Suppose we use the notation "a" for the acceleration as defined in physics, and "$a_{\text{car ad}}$" for the quantity used in advertisements for cars. In the US's non-metric system of units, what would be the units of a and $a_{\text{car ad}}$? How would the use and interpretation of large and small, positive and negative values be different for a as opposed to $a_{\text{car ad}}$?

C Two people stand on the edge of a cliff. As they lean over the edge, one person throws a rock down, while the other throws one straight up with an exactly opposite initial velocity. Compare the speeds of the rocks on impact at the bottom of the cliff.

3.3 Positive and negative acceleration

Gravity always pulls down, but that does not mean it always speeds things up. If you throw a ball straight up, gravity will first slow it down to $v = 0$ and then begin increasing its speed. When I took physics in high school, I got the impression that positive signs of acceleration indicated speeding up, while negative accelerations represented slowing down, i.e., deceleration. Such a definition would be inconvenient, however, because we would then have to say that the same downward tug of gravity could produce either a positive or a negative acceleration. As we will see in the following example, such a definition also would not be the same as the slope of the $v-t$ graph.

Let's study the example of the rising and falling ball. In the example of the person falling from a bridge, I assumed positive velocity values without calling attention to it, which meant I was assuming a coordinate system whose x axis pointed down. In this example, where the ball is reversing direction, it is not possible to avoid negative velocities by a tricky choice of axis, so let's make the more natural choice of an axis pointing up. The ball's velocity will initially be a positive number, because it is heading up, in the same direction as the x axis, but on the way back down, it will be a negative number. As shown in the figure, the $v-t$ graph does not do anything special at the top of the ball's flight, where v equals 0. Its slope is always negative. In the left half of the graph, there is a negative slope because the positive velocity is getting closer to zero. On the right side, the negative slope is due to a negative velocity that is getting farther from zero, so we say that the ball is speeding up, but its velocity is decreasing!



i / The ball's acceleration stays the same — on the way up, at the top, and on the way back down. It's always negative.

To summarize, what makes the most sense is to stick with the original definition of acceleration as the slope of the $v-t$ graph, $\Delta v/\Delta t$. By this definition, it just isn't necessarily true that things speeding up have positive acceleration while things slowing down have negative acceleration. The word "deceleration" is not used much by physicists, and the word "acceleration" is used unblushingly to refer to slowing down as well as speeding up: "There was a red light, and we accelerated to a stop."

Numerical calculation of a negative acceleration *example 4*

▷ In figure i, what happens if you calculate the acceleration between t = 1.0 and 1.5 s?

▷ Reading from the graph, it looks like the velocity is about −1 m/s at t = 1.0 s, and around −6 m/s at t = 1.5 s. The acceleration, figured between these two points, is

$$a = \frac{\Delta v}{\Delta t} = \frac{(-6\text{ m/s}) - (-1\text{ m/s})}{(1.5\text{ s}) - (1.0\text{ s})} = -10\text{ m/s}^2.$$

Even though the ball is speeding up, it has a negative acceleration.

Another way of convincing you that this way of handling the plus and minus signs makes sense is to think of a device that measures acceleration. After all, physics is supposed to use operational definitions, ones that relate to the results you get with actual measuring devices. Consider an air freshener hanging from the rear-view mirror of your car. When you speed up, the air freshener swings backward. Suppose we define this as a positive reading. When you slow down, the air freshener swings forward, so we'll call this a negative reading on our accelerometer. But what if you put the car in reverse and start speeding up backwards? Even though you're speeding up, the accelerometer responds in the same way as it did when you were going forward and slowing down. There are four possible cases:

motion of car	accelerometer swings	slope of v-t graph	direction of force acting on car
forward, speeding up	backward	$+$	forward
forward, slowing down	forward	$-$	backward
backward, speeding up	forward	$-$	backward
backward, slowing down	backward	$+$	forward

Note the consistency of the three right-hand columns — nature is trying to tell us that this is the right system of classification, not the left-hand column.

Because the positive and negative signs of acceleration depend on the choice of a coordinate system, the acceleration of an object under the influence of gravity can be either positive or negative. Rather than having to write things like "$g = 9.8\ \text{m/s}^2$ or $-9.8\ \text{m/s}^2$" every time we want to discuss g's numerical value, we simply define g as the absolute value of the acceleration of objects moving under the influence of gravity. We consistently let $g = 9.8\ \text{m/s}^2$, but we may have either $a = g$ or $a = -g$, depending on our choice of a coordinate system.

Acceleration with a change in direction of motion *example 5*

▷ A person kicks a ball, which rolls up a sloping street, comes to a halt, and rolls back down again. The ball has constant acceleration. The ball is initially moving at a velocity of 4.0 m/s, and after 10.0 s it has returned to where it started. At the end, it has sped back up to the same speed it had initially, but in the opposite direction. What was its acceleration?

▷ By giving a positive number for the initial velocity, the statement of the question implies a coordinate axis that points up the slope of the hill. The "same" speed in the opposite direction should therefore be represented by a negative number, -4.0 m/s. The acceleration is

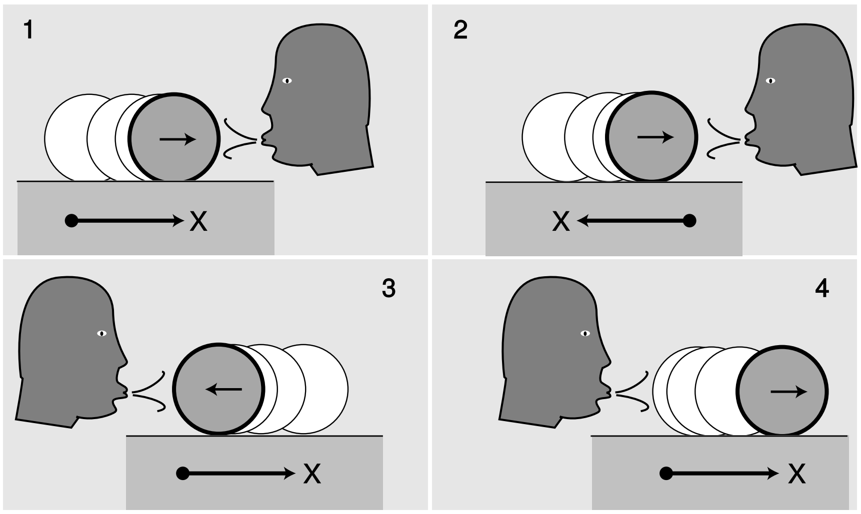
$$\begin{aligned} a &= \Delta v/\Delta t \\ &= (v_f - v_o)/10.0\ \text{s} \\ &= [(-4.0\ \text{m/s}) - (4.0\ \text{m/s})]/10.0s \\ &= -0.80\ \text{m/s}^2. \end{aligned}$$

The acceleration was no different during the upward part of the roll than on the downward part of the roll.

Incorrect solution: Acceleration is $\Delta v/\Delta$t, and at the end it's not moving any faster or slower than when it started, so Δv=0 and

$a = 0$.

✗ The velocity does change, from a positive number to a negative number.



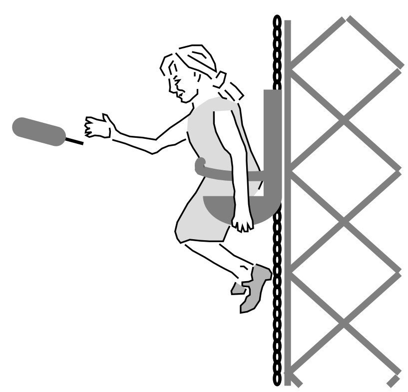
Discussion question B.

Discussion questions

A A child repeatedly jumps up and down on a trampoline. Discuss the sign and magnitude of his acceleration, including both the time when he is in the air and the time when his feet are in contact with the trampoline.

B The figure shows a refugee from a Picasso painting blowing on a rolling water bottle. In some cases the person's blowing is speeding the bottle up, but in others it is slowing it down. The arrow inside the bottle shows which direction it is going, and a coordinate system is shown at the bottom of each figure. In each case, figure out the plus or minus signs of the velocity and acceleration. It may be helpful to draw a $v-t$ graph in each case.

C Sally is on an amusement park ride which begins with her chair being hoisted straight up a tower at a constant speed of 60 miles/hour. Despite stern warnings from her father that he'll take her home the next time she misbehaves, she decides that as a scientific experiment she really needs to release her corndog over the side as she's on the way up. She does not throw it. She simply sticks it out of the car, lets it go, and watches it against the background of the sky, with no trees or buildings as reference points. What does the corndog's motion look like as observed by Sally? Does its speed ever appear to her to be zero? What acceleration does she observe it to have: is it ever positive? negative? zero? What would her enraged father answer if asked for a similar description of its motion as it appears to him, standing on the ground?



Discussion question C.

D Can an object maintain a constant acceleration, but meanwhile reverse the direction of its velocity?

E Can an object have a velocity that is positive and increasing at the same time that its acceleration is decreasing?

3.4 Varying acceleration

So far we have only been discussing examples of motion for which the acceleration is constant. As always, an expression of the form $\Delta \ldots / \Delta \ldots$ for a rate of change must be generalized to a derivative when the rate of change isn't constant. We therefore define the acceleration as $a = \mathrm{d}v / \mathrm{d}t$, which is the same as the second derivative, which Leibniz notated as

$$a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}.$$

The seemingly inconsistent placement of the twos on the top and bottom confuses all beginning calculus students. The motivation for this funny notation is that acceleration has units of $\mathrm{m/s^2}$, and the notation correctly suggests that: the top looks like it has units of meters, the bottom seconds^2. The notation is not meant, however, to suggest that t is really squared.

3.5 Algebraic results for constant acceleration

When an object is accelerating, the variables x, v, and t are all changing continuously. It is often of interest to eliminate one of these and relate the other two to each other.

Constant acceleration *example 6*

▹ How high does a diving board have to be above the water if the diver is to have as much as 1.0 s in the air?

▹ The diver starts at rest, and has an acceleration of 9.8 $\mathrm{m/s^2}$. We need to find a connection between the distance she travels and time it takes. In other words, we're looking for information about the function $x(t)$, given information about the acceleration. To go from acceleration to position, we need to integrate twice:

$$\begin{aligned}
x &= \int \int a \,\mathrm{d}t\,\mathrm{d}t \\
&= \int (at + v_o)\,\mathrm{d}t \qquad \text{[v_o is a constant of integration.]} \\
&= \int at\,\mathrm{d}t \qquad \text{[v_o is zero because she's dropping from rest.]} \\
&= \frac{1}{2}at^2 + x_o \qquad \text{[x_o is a constant of integration.]} \\
&= \frac{1}{2}at^2 \qquad \text{[x_o can be zero if we define it that way.]}
\end{aligned}$$

Note some of the good problem-solving habits demonstrated here. We solve the problem symbolically, and only plug in numbers at the very end, once all the algebra and calculus are done. One should also make a habit, after finding a symbolic result, of checking whether the dependence on the variables make sense. A greater value of t in this expression would lead to a greater value for x; that makes sense, because if you want more time in the air, you're going to have to jump from higher up. A greater acceleration also leads to a greater height; this also makes sense, because the stronger gravity is, the more height you'll need in order to stay in the air for a given amount of time. Now we plug in numbers.

$$\begin{aligned} x &= \frac{1}{2}\left(9.8\ \text{m/s}^2\right)(1.0\ \text{s})^2 \\ &= 4.9\ \text{m} \end{aligned}$$

Note that when we put in the numbers, we check that the units work out correctly, $\left(\text{m/s}^2\right)(\text{s})^2 = \text{m}$. We should also check that the result makes sense: 4.9 meters is pretty high, but not unreasonable.

Under conditions of constant acceleration, we can relate velocity and time,

$$a = \frac{\Delta v}{\Delta t},$$

or, as in the example 6, position and time,

$$x = \frac{1}{2}at^2 + v_o t + x_o.$$

It can also be handy to have a relation involving velocity and position, eliminating time. Straightforward algebra gives

$$v_f^2 = v_o^2 + 2a\Delta x,$$

where v_f is the final velocity, v_o the initial velocity, and Δx the distance traveled.

▷ *Solved problem: Dropping a rock on Mars* *page 122, problem 13*

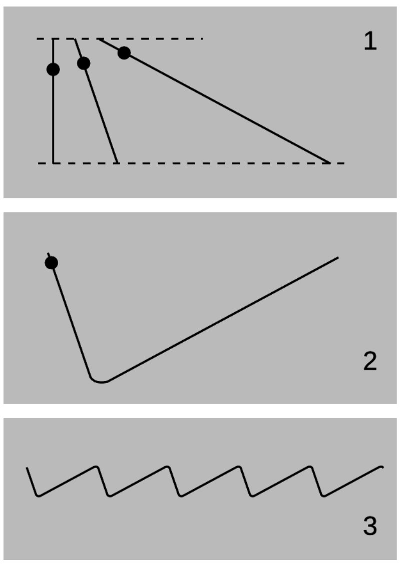
▷ *Solved problem: The Dodge Viper* *page 122, problem 11*

3.6 ⋆ A test of the principle of inertia

Historically, the first quantitative and well documented experimental test of the principle of inertia (p. 73) was performed by Galileo around 1590 and published decades later when he managed to find a publisher in the Netherlands that was beyond the reach of the Roman Inquisition.[1] It was ingenious but somewhat indirect, and required a layer of interpretation and extrapolation on top of the actual observations. As described on p. 107, he established that

[1]Galileo, *Discourses and Mathematical Demonstrations Relating to Two New Sciences*, 1638. The experiments are described in the Third Day, and their support for the principle of inertia is discussed in the Scholium following Theorems I-XIV. Another experiment involving a ship is described in Galileo's 1624 reply to a letter from Fr. Ingoli, but although Galileo vigorously asserts that he really did carry it out, no detailed description or quantitative results are given.

objects rolling on inclined planes moved according to mathematical laws that we would today describe as in section 3.5. He knew that his rolling balls were subject to friction, as well as random errors due to the limited precision of the water clock that he used, but he took the approximate agreement of his equations with experiment to indicate that they gave the results that would be exact in the absence of friction. He also showed, purely empirically, that when a ball went up or down a ramp inclined at an angle θ, its acceleration was proportional to $\sin\theta$. Again, this required extrapolation to idealized conditions of zero friction. He then reasoned that if a ball was rolled on a *horizontal* ramp, with $\theta = 0$, its acceleration would be zero. This is exactly what is required by the principle of inertia: in the absence of friction, motion continues indefinitely.



k / Example 7.

Reversing the logic *example 7*

If we assume that the principle of inertia holds, then we can reverse the direction of Galileo's reasoning and show that the acceleration of an object moving frictionlessly on an inclined plane is proportional to $\sin\theta$, which is a useful practical result for skateboarders or for mountaineers worried about sliding down an icy slope.

Figure k/1 shows beads sliding frictionlessly down three wires inclined at different angles, each losing the same amount of height as it travels from one end of its wire to the other. How do their speeds compare when they reach the bottom? It seems likely that the bead on the vertical wire will have the greatest acceleration, but that doesn't necessarily mean that its final speed will be greater, because it will also have less time available to accelerate. In fact, we will show that the final speed is the same regardless of the angle.

To demonstrate this, we assume that it is *not* true and show that it leads to an impossible result. Suppose that the final speed is greater, for the same vertical drop, when the slope is steeper (which is probably what most people intuitively expect, and is true when there is friction). In figure k/2, we join together two pieces of wire with different slopes, and let a bead ride down the steep side and then coast back *up* the less steep one. Under our assumption, the *loss* of speed on the way back up must be smaller than the gain on the way down, so that there is a net gain in speed. If we released the bead on the right and let it travel to the left, there would be a net loss.

Now suppose that we join together a series of these shapes, making a kind of asymmetric sawtooth pattern, k/3. A bead released on the left near the top of one of the steep sections will speed up indefinitely as it travels to the right. There is no limit to how much speed can be gained if we extend the wire far enough. We could use this arrangement to launch satellites into orbit, without the need for fuel or rocket engines. If we instead start the bead heading to the left with an initial push, it will gradually slow down, turn around, and then speed up indefinitely to the right as in the case considered earlier.

This is not just an absurd conclusion but one that in some sense violates the principle of inertia, if we state the principle loosely as a rule that motion doesn't naturally speed up or slow down in the absence of friction. We therefore conclude that the assumption was false: an object moving frictionlessly down an inclined plane does *not* gain a different amount of speed, for a fixed loss of height, depending on the slope.

It is now straightforward to show that the acceleration equals $g \sin\theta$. Applying the appropriate constant-acceleration equation, we have $v_f^2 = 2a\ell$, where ℓ is the distance traveled along the slope. Since v_f is the same regardless of angle, we have $a \propto \ell^{-1}$, but for a change in height h, we have $h/\ell = \sin\theta$, so $a \propto \sin\theta$, which is what we wanted to prove.

For the bead on the wire, or for an object sliding down a ramp, we can fix the constant of proportionality because $\theta = 90°$ should give $a = g$, so we have $a = g \sin\theta$.

For an object such as a sphere or a cylinder that rolls down a ramp without slipping, a similar proportionality holds between a and $\sin\theta$, but the constant of proportionality is smaller by a unitless factor that depends on the object's shape and how its mass is distributed. The difference comes about because of the frictional force (the "traction") between the object and the ramp.

Summary

Selected vocabulary

gravity . . . . . . A general term for the phenomenon of attraction between things having mass. The attraction between our planet and a human-sized object causes the object to fall.

acceleration . . . The rate of change of velocity; the slope of the tangent line on a $v-t$ graph.

Notation

v_o . . . . initial velocity

v_f . . . . final velocity

a . . . . acceleration

g . . . . the acceleration of objects in free fall; the strength of the local gravitational field

Summary

Galileo showed that when air resistance is negligible all falling bodies have the same motion regardless of mass. Moreover, their $v-t$ graphs are straight lines. We therefore define a quantity called acceleration as the derivative $\mathrm{d}v/\,\mathrm{d}t$. This definition has the advantage that a force with a given sign, representing its direction, always produces an acceleration with the same sign. The acceleration of objects in free fall varies slightly across the surface of the earth, and greatly on other planets.

For motion with constant acceleration, the following three equations hold:

$$\Delta x = v_\text{o}\Delta t + \frac{1}{2}a\Delta t^2$$
$$v_f^2 = v_\text{o}^2 + 2a\Delta x$$
$$a = \frac{\Delta v}{\Delta t}$$

They are not valid if the acceleration is changing.

Problems

Key

- √ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ⋆ A difficult problem.

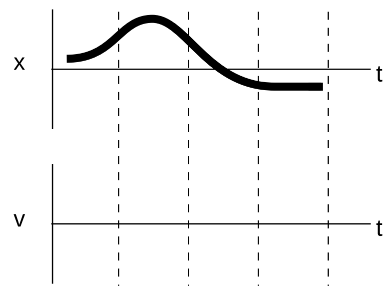
1 On New Year's Eve, a stupid person fires a pistol straight up. The bullet leaves the gun at a speed of 100 m/s. How long does it take before the bullet hits the ground? ▷ Solution, p. 545

2 What is the acceleration of a car that moves at a steady velocity of 100 km/h for 100 seconds? Explain your answer. [Based on a problem by Hewitt.]

3 You are looking into a deep well. It is dark, and you cannot see the bottom. You want to find out how deep it is, so you drop a rock in, and you hear a splash 3.0 seconds later. How deep is the well? √

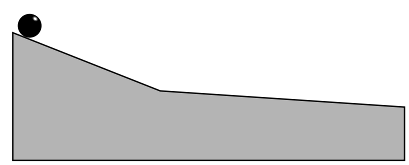
4 A honeybee's position as a function of time is given by $x = 10t - t^3$, where t is in seconds and x in meters. What is its acceleration at $t = 3.0$ s? ▷ Solution, p. 545

5 Alice drops a rock off a cliff. Bubba shoots a gun straight down from the edge of the same cliff. Compare the accelerations of the rock and the bullet while they are in the air on the way down. [Based on a problem by Serway and Faughn.]



Problem 6.

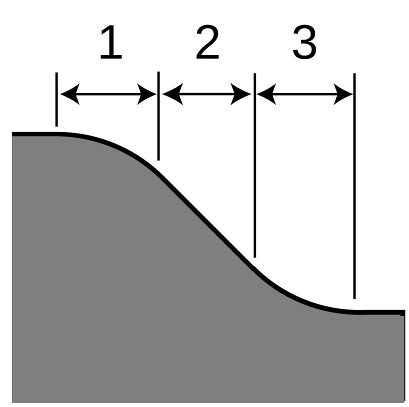
6 The top part of the figure shows the position-versus-time graph for an object moving in one dimension. On the bottom part of the figure, sketch the corresponding v-versus-t graph.
▷ Solution, p. 546



Problem 7.

7 (a) The ball is released at the top of the ramp shown in the figure. Friction is negligible. Use physical reasoning to draw $v - t$ and $a - t$ graphs. Assume that the ball doesn't bounce at the point where the ramp changes slope. (b) Do the same for the case where the ball is rolled up the slope from the right side, but doesn't quite have enough speed to make it over the top. ▷ Solution, p. 546

8 You throw a rubber ball up, and it falls and bounces several times. Draw graphs of position, velocity, and acceleration as functions of time. ▷ Solution, p. 546



Problem 9.

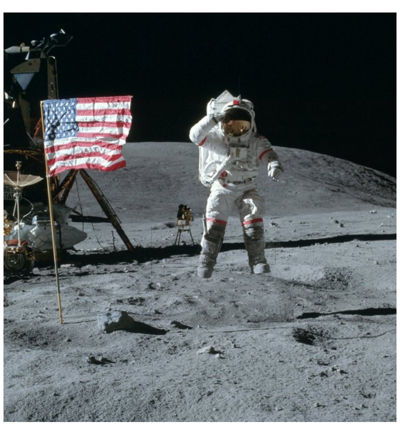
9 A ball rolls down the ramp shown in the figure, consisting of a curved knee, a straight slope, and a curved bottom. For each part of the ramp, tell whether the ball's velocity is increasing, decreasing, or constant, and also whether the ball's acceleration is increasing, decreasing, or constant. Explain your answers. Assume there is no air friction or rolling resistance.

10 Consider the following passage from Alice in Wonderland, in which Alice has been falling for a long time down a rabbit hole:

Down, down, down. Would the fall *never* come to an end? "I wonder how many miles I've fallen by this time?" she said aloud. "I must be getting somewhere near the center of the earth. Let me see: that would be four thousand miles down, I think" (for, you see, Alice had learned several things of this sort in her lessons in the schoolroom, and though this was not a *very* good opportunity for showing off her knowledge, as there was no one to listen to her, still it was good practice to say it over)...

Alice doesn't know much physics, but let's try to calculate the amount of time it would take to fall four thousand miles, starting from rest with an acceleration of 10 m/s^2. This is really only a lower limit; if there really was a hole that deep, the fall would actually take a longer time than the one you calculate, both because there is air friction and because gravity gets weaker as you get deeper (at the center of the earth, g is zero, because the earth is pulling you equally in every direction at once). ✓

11 In July 1999, Popular Mechanics carried out tests to find which car sold by a major auto maker could cover a quarter mile (402 meters) in the shortest time, starting from rest. Because the distance is so short, this type of test is designed mainly to favor the car with the greatest acceleration, not the greatest maximum speed (which is irrelevant to the average person). The winner was the Dodge Viper, with a time of 12.08 s. The car's top (and presumably final) speed was 118.51 miles per hour (52.98 m/s). (a) If a car, starting from rest and moving with *constant* acceleration, covers a quarter mile in this time interval, what is its acceleration? (b) What would be the final speed of a car that covered a quarter mile with the constant acceleration you found in part a? (c) Based on the discrepancy between your answer in part b and the actual final speed of the Viper, what do you conclude about how its acceleration changed over time? ▷ Solution, p. 547



Problem 12.

12 The photo shows Apollo 16 astronaut John Young jumping on the moon and saluting at the top of his jump. The video footage of the jump shows him staying aloft for 1.45 seconds. Gravity on the moon is 1/6 as strong as on the earth. Compute the height of the jump. ✓

13 If the acceleration of gravity on Mars is 1/3 that on Earth, how many times longer does it take for a rock to drop the same distance on Mars? Ignore air resistance. ▷ Solution, p. 547

14 You climb half-way up a tree, and drop a rock. Then you climb to the top, and drop another rock. How many times greater is the velocity of the second rock on impact? Explain. (The answer is not two times greater.)

15 Starting from rest, a ball rolls down a ramp, traveling a distance L and picking up a final speed v. How much of the distance did the ball have to cover before achieving a speed of $v/2$? [Based on a problem by Arnold Arons.] ▷ Solution, p. 547

16 A toy car is released on one side of a piece of track that is bent into an upright U shape. The car goes back and forth. When the car reaches the limit of its motion on one side, its velocity is zero. Is its acceleration also zero? Explain using a $v-t$ graph. [Based on a problem by Serway and Faughn.]

17 A physics homework question asks, "If you start from rest and accelerate at 1.54 m/s^2 for 3.29 s, how far do you travel by the end of that time?" A student answers as follows:

$$1.54 \times 3.29 = 5.07 \text{ m}$$

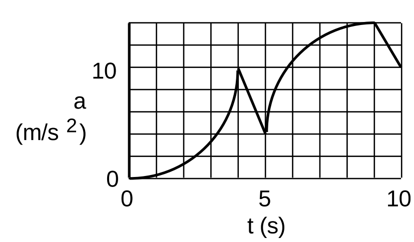
His Aunt Wanda is good with numbers, but has never taken physics. She doesn't know the formula for the distance traveled under constant acceleration over a given amount of time, but she tells her nephew his answer cannot be right. How does she know?

18 Find the error in the following calculation. A student wants to find the distance traveled by a car that accelerates from rest for 5.0 s with an acceleration of 2.0 m/s^2. First he solves $a = \Delta v/\Delta t$ for $\Delta v = 10$ m/s. Then he multiplies to find (10 m/s)(5.0 s) = 50 m. Do not just recalculate the result by a different method; if that was all you did, you'd have no way of knowing which calculation was correct, yours or his.

19 Acceleration could be defined either as $\Delta v/\Delta t$ or as the slope of the tangent line on the $v-t$ graph. Is either one superior as a definition, or are they equivalent? If you say one is better, give an example of a situation where it makes a difference which one you use.

20 If an object starts accelerating from rest, we have $v^2 = 2a\Delta x$ for its speed after it has traveled a distance Δx. Explain in words why it makes sense that the equation has velocity squared, but distance only to the first power. Don't recapitulate the derivation in the book, or give a justification based on units. The point is to explain what this feature of the equation tells us about the way speed increases as more distance is covered.

21 The graph shows the acceleration of a chipmunk in a TV cartoon. It consists of two circular arcs and two line segments. At $t = 0.00$ s, the chipmunk's velocity is -3.10 m/s. What is its velocity at $t = 10.00$ s? ✓



Problem 21.

22 You take a trip in your spaceship to another star. Setting off, you increase your speed at a constant acceleration. Once you get half-way there, you start decelerating, at the same rate, so that by the time you get there, you have slowed down to zero speed. You see the tourist attractions, and then head home by the same method.
(a) Find a formula for the time, T, required for the round trip, in terms of d, the distance from our sun to the star, and a, the magnitude of the acceleration. Note that the acceleration is not constant over the whole trip, but the trip can be broken up into constant-acceleration parts.
(b) The nearest star to the Earth (other than our own sun) is Proxima Centauri, at a distance of $d = 4 \times 10^{16}$ m. Suppose you use an acceleration of $a = 10\ \text{m/s}^2$, just enough to compensate for the lack of true gravity and make you feel comfortable. How long does the round trip take, in years?
(c) Using the same numbers for d and a, find your maximum speed. Compare this to the speed of light, which is 3.0×10^8 m/s. (Later in this course, you will learn that there are some new things going on in physics when one gets close to the speed of light, and that it is impossible to exceed the speed of light. For now, though, just use the simpler ideas you've learned so far.) $\checkmark$ $\star$

Problem 23. This spectacular series of photos from a 2011 paper by Burrows and Sutton ("Biomechanics of jumping in the flea," J. Exp. Biology 214:836) shows the flea jumping at about a 45-degree angle, but for the sake of this estimate just consider the case of a flea jumping vertically.

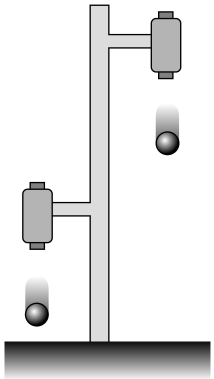
23 Some fleas can jump as high as 30 cm. The flea only has a short time to build up speed — the time during which its center of mass is accelerating upward but its feet are still in contact with the ground. Make an order-of-magnitude estimate of the acceleration the flea needs to have while straightening its legs, and state your answer in units of g, i.e., how many "g's it pulls." (For comparison, fighter pilots black out or die if they exceed about 5 or 10 g's.)

24 The speed required for a low-earth orbit is 7.9×10^3 m/s. When a rocket is launched into orbit, it goes up a little at first to get above almost all of the atmosphere, but then tips over horizontally to build up to orbital speed. Suppose the horizontal acceleration is limited to $3g$ to keep from damaging the cargo (or hurting the crew, for a crewed flight). (a) What is the minimum distance the rocket must travel downrange before it reaches orbital speed? How much does it matter whether you take into account the initial eastward velocity due to the rotation of the earth? (b) Rather than a rocket ship, it might be advantageous to use a railgun design, in which the craft would be accelerated to orbital speeds along a railroad track. This has the advantage that it isn't necessary to lift a large mass of fuel, since the energy source is external. Based on your answer to part a, comment on the feasibility of this design for crewed launches from the earth's surface.

25 A person is parachute jumping. During the time between when she leaps out of the plane and when she opens her chute, her altitude is given by an equation of the form

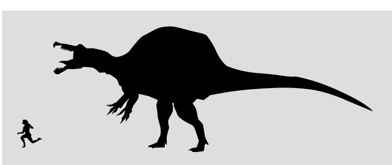
$$y = b - c\left(t + ke^{-t/k}\right),$$

where e is the base of natural logarithms, and b, c, and k are constants. Because of air resistance, her velocity does not increase at a steady rate as it would for an object falling in vacuum.
(a) What units would b, c, and k have to have for the equation to make sense?
(b) Find the person's velocity, v, as a function of time. [You will need to use the chain rule, and the fact that $\mathrm{d}(e^x)/\,\mathrm{d}x = e^x$.] ✓
(c) Use your answer from part (b) to get an interpretation of the constant c. [Hint: e^{-x} approaches zero for large values of x.]
(d) Find the person's acceleration, a, as a function of time. ✓
(e) Use your answer from part (d) to show that if she waits long enough to open her chute, her acceleration will become very small.



Problem 26.

26 The figure shows a practical, simple experiment for determining g to high precision. Two steel balls are suspended from electromagnets, and are released simultaneously when the electric current is shut off. They fall through unequal heights Δx_1 and Δx_2. A computer records the sounds through a microphone as first one ball and then the other strikes the floor. From this recording, we can accurately determine the quantity T defined as $T = \Delta t_2 - \Delta t_1$, i.e., the time lag between the first and second impacts. Note that since the balls do not make any sound when they are released, we have no way of measuring the individual times Δt_2 and Δt_1.
(a) Find an equation for g in terms of the measured quantities T, Δx_1 and Δx_2. ✓
(b) Check the units of your equation.
(c) Check that your equation gives the correct result in the case where Δx_1 is very close to zero. However, is this case realistic?
(d) What happens when $\Delta x_1 = \Delta x_2$? Discuss this both mathematically and physically.



Problem 27.

27 Most people don't know that *Spinosaurus aegyptiacus*, not *Tyrannosaurus rex*, was the biggest theropod dinosaur. We can't put a dinosaur on a track and time it in the 100 meter dash, so we can only infer from physical models how fast it could have run. When an animal walks at a normal pace, typically its legs swing more or less like pendulums of the same length ℓ. As a further simplification of this model, let's imagine that the leg simply moves at a fixed acceleration as it falls to the ground. That is, we model the time for a quarter of a stride cycle as being the same as the time required for free fall from a height ℓ. *S. aegyptiacus* had legs about four times longer than those of a human. (a) Compare the time required for a human's stride cycle to that for *S. aegyptiacus*. ✓
(b) Compare their running speeds. ✓

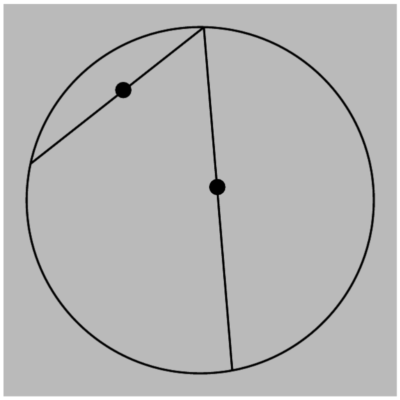
28 Engineering professor Qingming Li used sensors and video cameras to study punches delivered in the lab by British former welterweight boxing champion Ricky "the Hitman" Hatton. For comparison, Li also let a TV sports reporter put on the gloves and throw punches. The time it took for Hatton's best punch to arrive, i.e., the time his opponent would have had to react, was about 0.47 of that for the reporter. Let's assume that the fist starts from rest and moves with constant acceleration all the way up until impact, at some fixed distance (arm's length). Compare Hatton's acceleration to the reporter's. ✓

29 Aircraft carriers originated in World War I, and the first landing on a carrier was performed by E.H. Dunning in a Sopwith Pup biplane, landing on HMS Furious. (Dunning was killed the second time he attempted the feat.) In such a landing, the pilot slows down to just above the plane's stall speed, which is the minimum speed at which the plane can fly without stalling. The plane then lands and is caught by cables and decelerated as it travels the length of the flight deck. Comparing a modern US F-14 fighter jet landing on an Enterprise-class carrier to Dunning's original exploit, the stall speed is greater by a factor of 4.8, and to accomodate this, the length of the flight deck is greater by a factor of 1.9. Which deceleration is greater, and by what factor? √

30 In college-level women's softball in the U.S., typically a pitcher is expected to be at least 1.75 m tall, but Virginia Tech pitcher Jasmin Harrell is 1.62 m. Although a pitcher actually throws by stepping forward and swinging her arm in a circle, let's make a simplified physical model to estimate how much of a disadvantage Harrell has had to overcome due to her height. We'll pretend that the pitcher gives the ball a constant acceleration in a straight line, and that the length of this line is proportional to the pitcher's height. Compare the acceleration Harrell would have to supply with the acceleration that would suffice for a pitcher of the nominal minimum height, if both were to throw a pitch at the same speed. √

31 When the police engage in a high-speed chase on city streets, it can be extremely dangerous both to the police and to other motorists and pedestrians. Suppose that the police car must travel at a speed that is limited by the need to be able to stop before hitting a baby carriage, and that the distance at which the driver first sees the baby carriage is fixed. Tests show that in a panic stop from high speed, a police car based on a Chevy Impala has a deceleration 9% greater than that of a Dodge Intrepid. Compare the maximum safe speeds for the two cars. √

32 When an object slides frictionlessly down a plane inclined at an angle θ, its acceleration equals $g \sin\theta$ (example 7, p. 118). Suppose that a person on a bike is to coast down a ramp, starting from rest, and then coast back up an identical ramp, tracing a "V." The horizontal distance is fixed to be $2w$, and we want to set the depth of the "V" so as to achieve the minimal possible value t^* for the total time.
(a) Based only on units, infer the form of the expression for t^* in terms of w, up to a unitless multiplicative constant.
(b) Find the angle that minimizes the time.
(c) Complete the determination of t^* by finding the unitless constant. ▷ Solution, p. 547 ⋆



Problem 33.

33 The figure shows a circle in a vertical plane, with two wires positioned along chords of the circle. The top of each wire coincides with the top of the circle. Beads slide frictionlessly on the wires. If the beads are released simultaneously at the top, which one wins the race? You will need the fact that the acceleration equals $g \sin\theta$ (example 7, p. 118). ⋆

34 You shove a box with initial velocity 2.0 m/s, and it stops after sliding 1.3 m. What is the magnitude of the deceleration, assuming it is constant? √ *[problem by B. Shotwell]*

35 You're an astronaut, and you've arrived on planet X, which is airless. You drop a hammer from a height of 1.00 m and find that it takes 350 ms to fall to the ground. What is the acceleration due to gravity on planet X? √ *[problem by B. Shotwell]*

36 A naughty child drops a golf ball from the roof of your apartment building, and you see it drop past your window. It takes the ball time T to traverse the window's height H. Find the initial speed of the ball when it first came into view.
√ *[problem by B. Shotwell]*

Isaac Newton

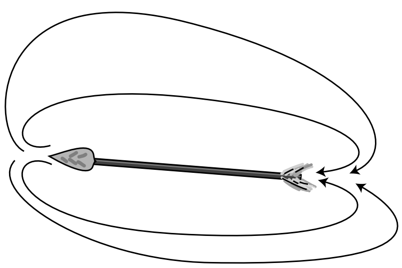
Chapter 4
Force and motion

> If I have seen farther than others, it is because I have stood on the shoulders of giants.
>
> *Newton, referring to Galileo*

Even as great and skeptical a genius as Galileo was unable to make much progress on the causes of motion. It was not until a generation later that Isaac Newton (1642-1727) was able to attack the problem successfully. In many ways, Newton's personality was the opposite of Galileo's. Where Galileo agressively publicized his ideas, Newton had to be coaxed by his friends into publishing a book on his physical discoveries. Where Galileo's writing had been popular and dramatic, Newton originated the stilted, impersonal style that most people think is standard for scientific writing. (Scientific journals today encourage a less ponderous style, and papers are often written in the first person.) Galileo's talent for arousing animosity among the rich and powerful was matched by Newton's skill at making himself a popular visitor at court. Galileo narrowly escaped being burned at the stake, while Newton had the good fortune of being on the winning side of the revolution that replaced King James II with William and Mary of Orange, leading to a lucrative post running the English royal mint.

Newton discovered the relationship between force and motion, and revolutionized our view of the universe by showing that the same physical laws applied to all matter, whether living or nonliving, on or off of our planet's surface. His book on force and motion, the **Mathematical Principles of Natural Philosophy**, was uncontradicted by experiment for 200 years, but his other main work, **Optics**, was on the wrong track, asserting that light was composed of particles rather than waves. Newton was also an avid alchemist, a fact that modern scientists would like to forget.

4.1 Force



a / Aristotle said motion had to be caused by a force. To explain why an arrow kept flying after the bowstring was no longer pushing on it, he said the air rushed around behind the arrow and pushed it forward. We know this is wrong, because an arrow shot in a vacuum chamber does not instantly drop to the floor as it leaves the bow. Galileo and Newton realized that a force would only be needed to change the arrow's motion, not to make its motion continue.

We need only explain changes in motion, not motion itself.

So far you've studied the measurement of motion in some detail, but not the reasons why a certain object would move in a certain way. This chapter deals with the "why" questions. Aristotle's ideas about the causes of motion were completely wrong, just like all his other ideas about physical science, but it will be instructive to start with them, because they amount to a road map of modern students' incorrect preconceptions.

Aristotle thought he needed to explain both why motion occurs and why motion might change. Newton inherited from Galileo the important counter-Aristotelian idea that motion needs no explanation, that it is only *changes* in motion that require a physical cause. Aristotle's needlessly complex system gave three reasons for motion:

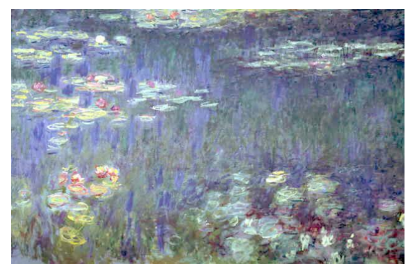
> Natural motion, such as falling, came from the tendency of objects to go to their "natural" place, on the ground, and come to rest.
>
> Voluntary motion was the type of motion exhibited by animals, which moved because they chose to.
>
> Forced motion occurred when an object was acted on by some other object that made it move.

Motion changes due to an interaction between two objects.

In the Aristotelian theory, natural motion and voluntary motion are one-sided phenomena: the object causes its own motion. Forced motion is supposed to be a two-sided phenomenon, because one object imposes its "commands" on another. Where Aristotle conceived of some of the phenomena of motion as one-sided and others as two-sided, Newton realized that a change in motion was always a two-sided relationship of a force acting between two physical objects.

The one-sided "natural motion" description of falling makes a crucial omission. The acceleration of a falling object is not caused by its own "natural" tendencies but by an attractive force between it and the planet Earth. Moon rocks brought back to our planet do not "want" to fly back up to the moon because the moon is their "natural" place. They fall to the floor when you drop them, just like our homegrown rocks. As we'll discuss in more detail later in this course, gravitational forces are simply an attraction that occurs between any two physical objects. Minute gravitational forces can even be measured between human-scale objects in the laboratory.

The idea of natural motion also explains incorrectly why things come to rest. A basketball rolling across a beach slows to a stop because it is interacting with the sand via a frictional force, not because of its own desire to be at rest. If it was on a frictionless surface, it would never slow down. Many of Aristotle's mistakes stemmed from his failure to recognize friction as a force.



b / "Our eyes receive blue light reflected from this painting because Monet wanted to represent water with the color blue." This is a valid statement at one level of explanation, but physics works at the physical level of explanation, in which blue light gets to your eyes because it is reflected by blue pigments in the paint.

The concept of voluntary motion is equally flawed. You may have been a little uneasy about it from the start, because it assumes a clear distinction between living and nonliving things. Today, however, we are used to having the human body likened to a complex machine. In the modern world-view, the border between the living and the inanimate is a fuzzy no-man's land inhabited by viruses, prions, and silicon chips. Furthermore, Aristotle's statement that you can take a step forward "because you choose to" inappropriately mixes two levels of explanation. At the physical level of explanation, the reason your body steps forward is because of a frictional force acting between your foot and the floor. If the floor was covered with a puddle of oil, no amount of "choosing to" would enable you to take a graceful stride forward.

Forces can all be measured on the same numerical scale.

In the Aristotelian-scholastic tradition, the description of motion as natural, voluntary, or forced was only the broadest level of classification, like splitting animals into birds, reptiles, mammals, and amphibians. There might be thousands of types of motion, each of which would follow its own rules. Newton's realization that all changes in motion were caused by two-sided interactions made it seem that the phenomena might have more in common than had been apparent. In the Newtonian description, there is only one cause for a change in motion, which we call force. Forces may be of different types, but they all produce changes in motion according to the same rules. Any acceleration that can be produced by a magnetic force can equally well be produced by an appropriately controlled stream of water. We can speak of two forces as being equal if they produce the same change in motion when applied in the same situation, which means that they pushed or pulled equally hard in the same direction.

The idea of a numerical scale of force and the newton unit were introduced in chapter 0. To recapitulate briefly, a force is when a pair of objects push or pull on each other, and one newton is the force required to accelerate a 1-kg object from rest to a speed of 1 m/s in 1 second.

More than one force on an object

As if we hadn't kicked poor Aristotle around sufficiently, his theory has another important flaw, which is important to discuss because it corresponds to an extremely common student misconception. Aristotle conceived of forced motion as a relationship in which one object was the boss and the other "followed orders." It therefore would only make sense for an object to experience one force at a time, because an object couldn't follow orders from two sources at once. In the Newtonian theory, forces are numbers, not orders, and if more than one force acts on an object at once, the result is found by adding up all the forces. It is unfortunate that the use of the English word "force" has become standard, because to many people it suggests that you are "forcing" an object to do something. The force of the earth's gravity cannot "force" a boat to sink, because there are other forces acting on the boat. Adding them up gives a total of zero, so the boat accelerates neither up nor down.

Objects can exert forces on each other at a distance.

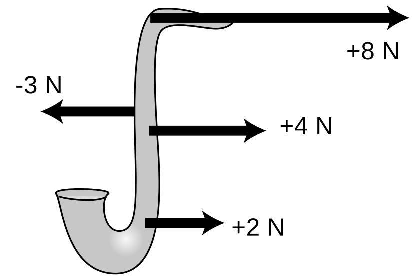
Aristotle declared that forces could only act between objects that were touching, probably because he wished to avoid the type of occult speculation that attributed physical phenomena to the influence of a distant and invisible pantheon of gods. He was wrong, however, as you can observe when a magnet leaps onto your refrigerator or when the planet earth exerts gravitational forces on objects that are in the air. Some types of forces, such as friction, only operate between objects in contact, and are called contact forces. Magnetism, on the other hand, is an example of a noncontact force. Although the magnetic force gets stronger when the magnet is closer to your refrigerator, touching is not required.

Weight

In physics, an object's weight, F_W, is defined as the earth's gravitational force on it. The SI unit of weight is therefore the Newton. People commonly refer to the kilogram as a unit of weight, but the kilogram is a unit of mass, not weight. Note that an object's weight is not a fixed property of that object. Objects weigh more in some places than in others, depending on the local strength of gravity. It is their mass that always stays the same. A baseball pitcher who can throw a 90-mile-per-hour fastball on earth would not be able to throw any faster on the moon, because the ball's inertia would still be the same.

Positive and negative signs of force

We'll start by considering only cases of one-dimensional center-of-mass motion in which all the forces are parallel to the direction of motion, i.e., either directly forward or backward. In one dimension, plus and minus signs can be used to indicate directions of forces, as shown in figure c. We can then refer generically to addition of forces, rather than having to speak sometimes of addition and sometimes of subtraction. We add the forces shown in the figure and get 11 N. In general, we should choose a one-dimensional coordinate system with its x axis parallel the direction of motion. Forces that point along the positive x axis are positive, and forces in the opposite direction are negative. Forces that are not directly along the x axis cannot be immediately incorporated into this scheme, but that's OK, because we're avoiding those cases for now.



c / Forces are applied to a saxophone. In this example, positive signs have been used consistently for forces to the right, and negative signs for forces to the left. (The forces are being applied to different places on the saxophone, but the numerical value of a force carries no information about that.)

Discussion questions

A In chapter 0, I defined 1 N as the force that would accelerate a 1-kg mass from rest to 1 m/s in 1 s. Anticipating the following section, you might guess that 2 N could be defined as the force that would accelerate the same mass to twice the speed, or twice the mass to the same speed. Is there an easier way to define 2 N based on the definition of 1 N?

4.2 Newton's first law

We are now prepared to make a more powerful restatement of the principle of inertia.[1]

> **Newton's first law**
> If the total force acting on an object is zero, its center of mass continues in the same state of motion.

In other words, an object initially at rest is predicted to remain at rest if the total force acting on it is zero, and an object in motion

[1] Page 73 lists places in this book where we describe experimental tests of the principle of inertia and Newton's first law.

remains in motion with the same velocity in the same direction. The converse of Newton's first law is also true: if we observe an object moving with constant velocity along a straight line, then the total force on it must be zero.

In a future physics course or in another textbook, you may encounter the term "net force," which is simply a synonym for total force.

What happens if the total force on an object is not zero? It accelerates. Numerical prediction of the resulting acceleration is the topic of Newton's second law, which we'll discuss in the following section.

This is the first of Newton's three laws of motion. It is not important to memorize which of Newton's three laws are numbers one, two, and three. If a future physics teacher asks you something like, "Which of Newton's laws are you thinking of?," a perfectly acceptable answer is "The one about constant velocity when there's zero total force." The concepts are more important than any specific formulation of them. Newton wrote in Latin, and I am not aware of any modern textbook that uses a verbatim translation of his statement of the laws of motion. Clear writing was not in vogue in Newton's day, and he formulated his three laws in terms of a concept now called momentum, only later relating it to the concept of force. Nearly all modern texts, including this one, start with force and do momentum later.

An elevator *example 1*

▷ An elevator has a weight of 5000 N. Compare the forces that the cable must exert to raise it at constant velocity, lower it at constant velocity, and just keep it hanging.

▷ In all three cases the cable must pull up with a force of exactly 5000 N. Most people think you'd need at least a little more than 5000 N to make it go up, and a little less than 5000 N to let it down, but that's incorrect. Extra force from the cable is only necessary for speeding the car up when it starts going up or slowing it down when it finishes going down. Decreased force is needed to speed the car up when it gets going down and to slow it down when it finishes going up. But when the elevator is cruising at constant velocity, Newton's first law says that you just need to cancel the force of the earth's gravity.

To many students, the statement in the example that the cable's upward force "cancels" the earth's downward gravitational force implies that there has been a contest, and the cable's force has won, vanquishing the earth's gravitational force and making it disappear. That is incorrect. Both forces continue to exist, but because they add up numerically to zero, the elevator has no center-of-mass acceleration. We know that both forces continue to exist because they both have side-effects other than their effects on the car's center-of-mass motion. The force acting between the cable and the car continues to produce tension in the cable and keep the cable taut. The earth's gravitational force continues to keep the passengers (whom we are considering as part of the elevator-object) stuck to the floor and to produce internal stresses in the walls of the car, which must hold up the floor.

Terminal velocity for falling objects *example 2*

▷ An object like a feather that is not dense or streamlined does not fall with constant acceleration, because air resistance is nonnegligible. In fact, its acceleration tapers off to nearly zero within a fraction of a second, and the feather finishes dropping at constant speed (known as its terminal velocity). Why does this happen?

▷ Newton's first law tells us that the total force on the feather must have been reduced to nearly zero after a short time. There are two forces acting on the feather: a downward gravitational force from the planet earth, and an upward frictional force from the air. As the feather speeds up, the air friction becomes stronger and stronger, and eventually it cancels out the earth's gravitational force, so the feather just continues with constant velocity without speeding up any more.

The situation for a skydiver is exactly analogous. It's just that the skydiver experiences perhaps a million times more gravitational force than the feather, and it is not until she is falling very fast that the force of air friction becomes as strong as the gravitational force. It takes her several seconds to reach terminal velocity, which is on the order of a hundred miles per hour.

More general combinations of forces

It is too constraining to restrict our attention to cases where all the forces lie along the line of the center of mass's motion. For one thing, we can't analyze any case of horizontal motion, since any object on earth will be subject to a vertical gravitational force! For instance, when you are driving your car down a straight road, there are both horizontal forces and vertical forces. However, the vertical forces have no effect on the center of mass motion, because the road's upward force simply counteracts the earth's downward gravitational force and keeps the car from sinking into the ground.

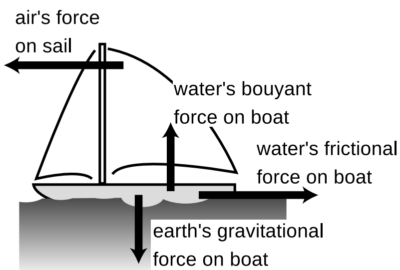
Later in the book we'll deal with the most general case of many forces acting on an object at any angles, using the mathematical technique of vector addition, but the following slight generalization of Newton's first law allows us to analyze a great many cases of interest:

Suppose that an object has two sets of forces acting on it, one set along the line of the object's initial motion and another set perpendicular to the first set. If both sets of forces cancel, then the object's center of mass continues in the same state of motion.

A passenger riding the subway *example 3*

▷ Describe the forces acting on a person standing in a subway train that is cruising at constant velocity.

▷ No force is necessary to keep the person moving relative to the ground. He will not be swept to the back of the train if the floor is slippery. There are two vertical forces on him, the earth's downward gravitational force and the floor's upward force, which cancel. There are no horizontal forces on him at all, so of course the total horizontal force is zero.



d / Example 4.

Forces on a sailboat *example 4*

▷ If a sailboat is cruising at constant velocity with the wind coming from directly behind it, what must be true about the forces acting on it?

▷ The forces acting on the boat must be canceling each other out. The boat is not sinking or leaping into the air, so evidently the vertical forces are canceling out. The vertical forces are the downward gravitational force exerted by the planet earth and an upward force from the water.

The air is making a forward force on the sail, and if the boat is not accelerating horizontally then the water's backward frictional force must be canceling it out.

Contrary to Aristotle, more force is not needed in order to maintain a higher speed. Zero total force is always needed to maintain constant velocity. Consider the following made-up numbers:

	boat moving at a low, constant velocity	boat moving at a high, constant velocity
forward force of the wind on the sail . . .	10,000 N	20,000 N
backward force of the water on the hull . . .	$-10,000$ N	$-20,000$ N
total force on the boat . . .	0 N	0 N

The faster boat still has zero total force on it. The forward force on it is greater, and the backward force smaller (more negative), but that's irrelevant because Newton's first law has to do with the total force, not the individual forces.

This example is quite analogous to the one about terminal velocity of falling objects, since there is a frictional force that increases with speed. After casting off from the dock and raising the sail, the boat will accelerate briefly, and then reach its terminal velocity, at which the water's frictional force has become as great as the wind's force on the sail.

A car crash *example 5*

▷ If you drive your car into a brick wall, what is the mysterious force that slams your face into the steering wheel?

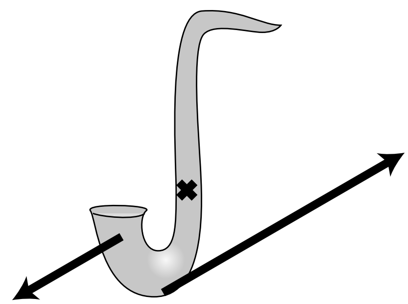
▷ Your surgeon has taken physics, so she is not going to believe your claim that a mysterious force is to blame. She knows that your face was just following Newton's first law. Immediately after your car hit the wall, the only forces acting on your head were the same canceling-out forces that had existed previously: the earth's downward gravitational force and the upward force from your neck. There were no forward or backward forces on your head, but the car did experience a backward force from the wall, so the car slowed down and your face caught up.

Discussion questions

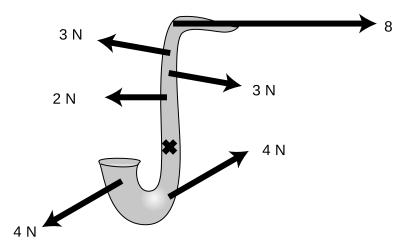
A Newton said that objects continue moving if no forces are acting on them, but his predecessor Aristotle said that a force was necessary to keep an object moving. Why does Aristotle's theory seem more plausible, even though we now believe it to be wrong? What insight was Aristotle missing about the reason why things seem to slow down naturally? Give an example.

B In the figure what would have to be true about the saxophone's initial motion if the forces shown were to result in continued one-dimensional motion of its center of mass?

C This figure requires an ever further generalization of the preceding discussion. After studying the forces, what does your physical intuition tell you will happen? Can you state in words how to generalize the conditions for one-dimensional motion to include situations like this one?



Discussion question B.



Discussion question C.

4.3 Newton's second law

What about cases where the total force on an object is not zero, so that Newton's first law doesn't apply? The object will have an acceleration. The way we've defined positive and negative signs of force and acceleration guarantees that positive forces produce positive accelerations, and likewise for negative values. How much acceleration will it have? It will clearly depend on both the object's mass and on the amount of force.

Experiments with any particular object show that its acceleration is directly proportional to the total force applied to it. This may seem wrong, since we know of many cases where small amounts of force fail to move an object at all, and larger forces get it going. This apparent failure of proportionality actually results from forgetting that there is a frictional force in addition to the force we apply to move the object. The object's acceleration is exactly proportional to the total force on it, not to any individual force on it. In the absence of friction, even a very tiny force can slowly change the velocity of a very massive object.

Experiments (e.g., the one described in example 12 on p. 142) also show that the acceleration is inversely proportional to the object's mass, and combining these two proportionalities gives the following way of predicting the acceleration of any object:

Newton's second law

$$a = F_{total}/m,$$

where

m is an object's mass, a measure of its resistance to changes in its motion

F_{total} is the sum of the forces acting on it, and

a is the acceleration of the object's center of mass.

We are presently restricted to the case where the forces of interest are parallel to the direction of motion.

We have already encountered the SI unit of force, which is the newton (N). It is designed so that the units in Newton's second law all work out if we use SI units: $\mathrm{m/s^2}$ for acceleration and kg (*not* grams!) for mass.

Rocket science *example 6*

▷ The Falcon 9 launch vehicle, built and operated by the private company SpaceX, has mass $m = 5.1\times10^5$ kg. At launch, it has two forces acting on it: an upward thrust $F_t = 5.9\times10^6$ N and a downward gravitational force of $F_g = 5.0\times10^6$ N. Find its acceleration.

e / Example 6

▷ Let's choose our coordinate system such that positive is up. Then the downward force of gravity is considered negative. Using Newton's second law,

$$\begin{aligned}
a &= \frac{F_{total}}{m} \\
&= \frac{F_t - F_g}{m} \\
&= \frac{(5.9\times10^6\ \mathrm{N}) - (5.0\times10^6\ \mathrm{N})}{5.1\times10^5\ \mathrm{kg}} \\
&= 1.6\ \mathrm{m/s^2},
\end{aligned}$$

where as noted above, units of N/kg (newtons per kilogram) are the same as $\mathrm{m/s^2}$.

An accelerating bus *example 7*

▷ A VW bus with a mass of 2000 kg accelerates from 0 to 25 m/s (freeway speed) in 34 s. Assuming the acceleration is constant, what is the total force on the bus?

▷ We solve Newton's second law for $F_{total} = ma$, and substitute $\Delta v/\Delta t$ for a, giving

$$\begin{aligned} F_{total} &= m\Delta v/\Delta t \\ &= (2000\text{ kg})(25\text{ m/s} - 0\text{ m/s})/(34\text{ s}) \\ &= 1.5\text{ kN.} \end{aligned}$$

Some applications of calculus

Newton doesn't care what frame of reference you use his laws in, and this makes him different from Aristotle, who says there is something special about the frame of reference attached firmly to the dirt underfoot. Suppose that an object obeys Newton's second law in the dirt's frame. It has some velocity that is a function of time, and differentiating this function gives $\mathrm{d}v/\mathrm{d}t = F/m$. Suppose we change to the frame of reference of a train that is in motion relative to the dirt at constant velocity c. Looking out the window of the train, we see the object moving with velocity $v - c$. But the derivative of a constant is zero, so when we differentiate $v - c$, the constant goes away, and we get exactly the same result. Newton is still happy, although Aristotle feels a great disturbance in the force.

Often we know the forces acting on an object, and we want to find its motion, i.e., its position as a function of time, $x(t)$. Since Newton's second law predicts the acceleration $\mathrm{d}^2x/\mathrm{d}t^2$, we need to integrate twice to find x. The first integration gives the velocity, and the constant of integration is also a velocity, which can be fixed by giving the object's velocity at some initial time. In the second integration we pick up a second constant of integration, this one related to an initial position.

A force that tapers off to zero *example 8*

▷ An object of mass m starts at rest at $t = t_o$. A force varying as $F = bt^{-2}$, where b is a constant, begins acting on it. Find the greatest speed it will ever have.

▷

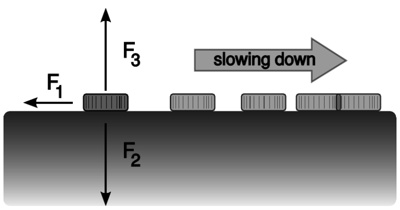
$$\begin{aligned} F &= m\frac{\mathrm{d}v}{\mathrm{d}t} \\ \mathrm{d}v &= \frac{F}{m}\mathrm{d}t \\ \int \mathrm{d}v &= \int \frac{F}{m}\mathrm{d}t \\ v &= -\frac{b}{m}t^{-1} + v^*, \end{aligned}$$

where v^* is a constant of integration with units of velocity. The given initial condition is that $v = 0$ at $t = t_0$, so we find that $v^* = b/mt_0$. The negative term gets closer to zero with increasing time, so the maximum velocity is achieved by letting t approach infinity. That is, the object will never stop speeding up, but it will also never surpass a certain speed. In the limit $t \to \infty$, we identify v^* as the velocity that the object will approach asymptotically.

A generalization

As with the first law, the second law can be easily generalized to include a much larger class of interesting situations:

> Suppose an object is being acted on by two sets of forces, one set lying parallel to the object's initial direction of motion and another set acting along a perpendicular line. If the forces perpendicular to the initial direction of motion cancel out, then the object accelerates along its original line of motion according to $a = F_\parallel/m$, where $F_\parallel$ is the sum of the forces parallel to the line.



f / A coin slides across a table. Even for motion in one dimension, some of the forces may not lie along the line of the motion.

A coin sliding across a table — *example 9*

Suppose a coin is sliding to the right across a table, f, and let's choose a positive x axis that points to the right. The coin's velocity is positive, and we expect based on experience that it will slow down, i.e., its acceleration should be negative.

Although the coin's motion is purely horizontal, it feels both vertical and horizontal forces. The Earth exerts a downward gravitational force F_2 on it, and the table makes an upward force F_3 that prevents the coin from sinking into the wood. In fact, without these vertical forces the horizontal frictional force wouldn't exist: surfaces don't exert friction against one another unless they are being pressed together.

Although F_2 and F_3 contribute to the physics, they do so only indirectly. The only thing that directly relates to the acceleration along the horizontal direction is the horizontal force: $a = F_1/m$.

The relationship between mass and weight

Mass is different from weight, but they're related. An apple's mass tells us how hard it is to change its motion. Its weight measures the strength of the gravitational attraction between the apple and the planet earth. The apple's weight is less on the moon, but its mass is the same. Astronauts assembling the International Space Station in zero gravity couldn't just pitch massive modules back and forth with their bare hands; the modules were weightless, but not massless.

We have already seen the experimental evidence that when weight (the force of the earth's gravity) is the only force acting on an object, its acceleration equals the constant g, and g depends on where you are on the surface of the earth, but not on the mass of the object. Applying Newton's second law then allows us to calculate the magnitude of the gravitational force on any object in terms of its mass:

$$|F_W| = mg.$$

(The equation only gives the magnitude, i.e. the absolute value, of F_W, because we're defining g as a positive number, so it equals the absolute value of a falling object's acceleration.)

▷ *Solved problem: Decelerating a car* *page 159, problem 1*

Weight and mass *example 10*

▷ Figure h shows masses of one and two kilograms hung from a spring scale, which measures force in units of newtons. Explain the readings.

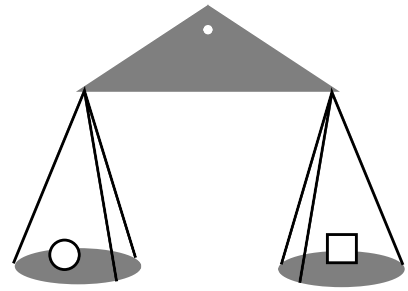
▷ Let's start with the single kilogram. It's not accelerating, so evidently the total force on it is zero: the spring scale's upward force on it is canceling out the earth's downward gravitational force. The spring scale tells us how much force it is being obliged to supply, but since the two forces are equal in strength, the spring scale's reading can also be interpreted as measuring the strength of the gravitational force, i.e., the weight of the one-kilogram mass. The weight of a one-kilogram mass should be

$$\begin{aligned} F_W &= mg \\ &= (1.0\ \text{kg})(9.8\ \text{m/s}^2) = 9.8\ \text{N}, \end{aligned}$$

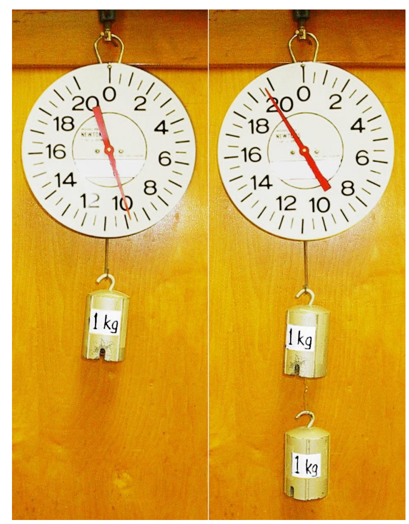
and that's indeed the reading on the spring scale.

Similarly for the two-kilogram mass, we have

$$\begin{aligned} F_W &= mg \\ &= (2.0\ \text{kg})(9.8\ \text{m/s}^2) = 19.6\ \text{N}. \end{aligned}$$



g / A simple double-pan balance works by comparing the weight forces exerted by the earth on the contents of the two pans. Since the two pans are at almost the same location on the earth's surface, the value of g is essentially the same for each one, and equality of weight therefore also implies equality of mass.



h / Example 10.

Calculating terminal velocity *example 11*

▷ Experiments show that the force of air friction on a falling object such as a skydiver or a feather can be approximated fairly well with the equation $|F_{air}| = c\rho Av^2$, where c is a constant, ρ is the density of the air, A is the cross-sectional area of the object as seen from below, and v is the object's velocity. Predict the object's terminal velocity, i.e., the final velocity it reaches after a long time.

▷ As the object accelerates, its greater v causes the upward force of the air to increase until finally the gravitational force and the force of air friction cancel out, after which the object continues at constant velocity. We choose a coordinate system in which positive is up, so that the gravitational force is negative and the force of air friction is positive. We want to find the velocity at which

$$F_{air} + F_W = 0, \quad i.e.,$$
$$c\rho A v^2 - mg = 0.$$

Solving for v gives

$$v_{terminal} = \sqrt{\frac{mg}{c\rho A}}$$

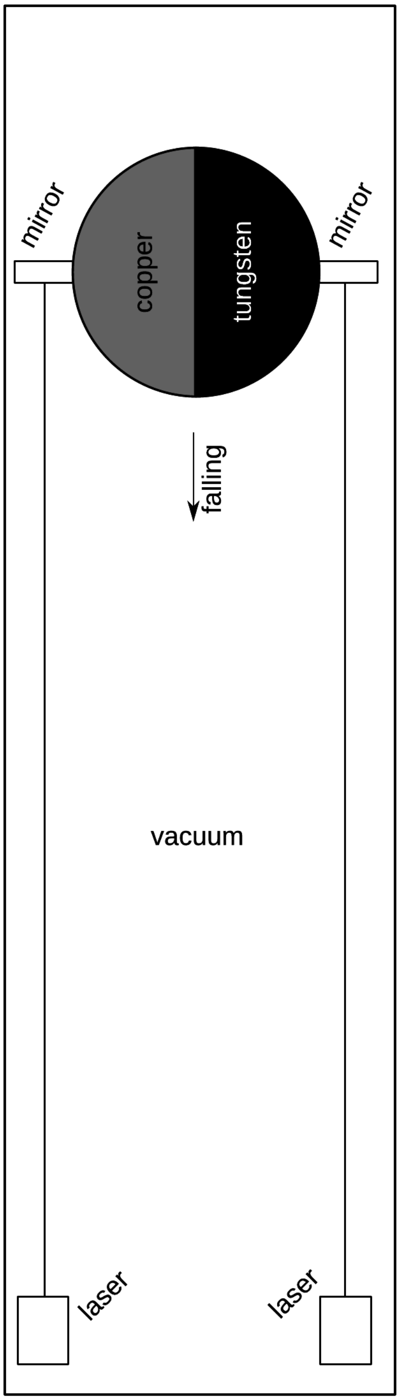
self-check A

It is important to get into the habit of interpreting equations. This may be difficult at first, but eventually you will get used to this kind of reasoning.

(1) Interpret the equation $v_{terminal} = \sqrt{mg/c\rho A}$ in the case of ρ=0.

(2) How would the terminal velocity of a 4-cm steel ball compare to that of a 1-cm ball?

(3) In addition to teasing out the *mathematical* meaning of an equation, we also have to be able to place it in its *physical* context. How generally important is this equation? ▷ Answer, p. 558



i / A simplified diagram of the experiment described in example 12.

A test of the second law *example 12*

Because the force mg of gravity on an object of mass m is proportional to m, the acceleration predicted by Newton's second law is $a = F/m = mg/m = g$, in which the mass cancels out. It is therefore an ironclad prediction of Newton's laws of motion that free fall is universal: in the absence of other forces such as air resistance, heavier objects do not fall with a greater acceleration than lighter ones. The experiment by Galileo at the Leaning Tower of Pisa (p. 106) is therefore consistent with Newton's second law. Since Galileo's time, experimental methods have had several centuries in which to improve, and the second law has been subjected to similar tests with exponentially improving precision. For such an experiment in 1993,[2] physicists at the University of Pisa (!) built a metal disk out of copper and tungsten semicircles joined together at their flat edges. They evacuated the air from a vertical shaft and dropped the disk down it 142 times, using lasers to measure any tiny rotation that would result if the accelerations of the copper and tungsten were very slightly different. The results were statistically consistent with zero rotation, and put an upper limit of 1×10^{-9} on the fractional difference in acceleration $|g_{\text{copper}} - g_{\text{tungsten}}|/g$.

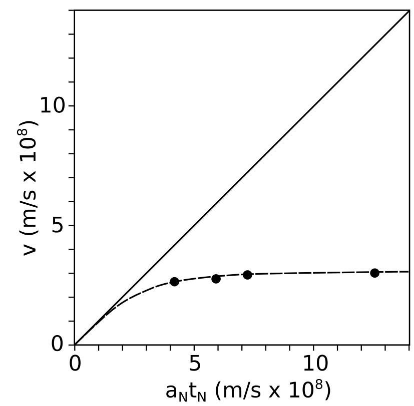
[2] Carusotto *et al.*, "Limits on the violation of g-universality with a Galileo-type experiment," Phys Lett A183 (1993) 355. Freely available online at researchgate.net.

A failure of the second law *example 13*

The graph in the figure displays data from a 1964 experiment by Bertozzi that shows how Newton's second law fails if you keep on applying a force to an object indefinitely. Electrons were accelerated by a constant electrical force through a certain distance. Applying Newton's laws gives Newtonian predictions a_N for the acceleration and t_N for the time required. The electrons were then allowed to fly down a pipe for a further distance of 8.4 m without being acted on by any force. The time of flight for this second distance was used to find the final velocity v to which they had actually been accelerated.

According to Newton, an acceleration a_N acting for a time t_N should produce a final velocity $a_N t_N$. The solid line in the graph shows the prediction of Newton's laws, which is that a constant force exerted steadily over time will produce a velocity that rises linearly and without limit.

The experimental data, shown as black dots, clearly tell a different story. The velocity never goes above a certain maximum value, which turns out to be the speed of light. The dashed line shows the predictions of Einstein's theory of special relativity, which are in good agreement with the experimental results. This experiment is an example of a general fact, which is that Newton's laws are only good approximations when objects move at velocities that are small compared to the speed of light. This is discussed further on p. 156.



j / Example 13.

Discussion questions

A Show that the Newton can be reexpressed in terms of the three basic mks units as the combination kg$\cdot$m/s^2.

B What is wrong with the following statements?

(1) "g is the force of gravity."

(2) "Mass is a measure of how much space something takes up."

C Criticize the following incorrect statement:

"If an object is at rest and the total force on it is zero, it stays at rest. There can also be cases where an object is moving and keeps on moving without having any total force on it, but that can only happen when there's no friction, like in outer space."

D Table k gives laser timing data for Ben Johnson's 100 m dash at the 1987 World Championship in Rome. (His world record was later revoked because he tested positive for steroids.) How does the total force on him change over the duration of the race?

x (m)	t (s)
10	1.84
20	2.86
30	3.80
40	4.67
50	5.53
60	6.38
70	7.23
80	8.10
90	8.96
100	9.83

k / Discussion question D.

4.4 What force is not

Violin teachers have to endure their beginning students' screeching. A frown appears on the woodwind teacher's face as she watches her student take a breath with an expansion of his ribcage but none in his belly. What makes physics teachers cringe is their students' verbal statements about forces. Below I have listed six dicta about what force is not.

1. Force is not a property of one object.

A great many of students' incorrect descriptions of forces could be cured by keeping in mind that a force is an interaction of two objects, not a property of one object.

Incorrect statement: "That magnet has a lot of force."

✗ If the magnet is one millimeter away from a steel ball bearing, they may exert a very strong attraction on each other, but if they were a meter apart, the force would be virtually undetectable. The magnet's strength can be rated using certain electrical units ($\text{ampere} - \text{meters}^2$), but not in units of force.

2. Force is not a measure of an object's motion.

If force is not a property of a single object, then it cannot be used as a measure of the object's motion.

Incorrect statement: "The freight train rumbled down the tracks with awesome force."

✗ Force is not a measure of motion. If the freight train collides with a stalled cement truck, then some awesome forces will occur, but if it hits a fly the force will be small.

3. Force is not energy.

There are two main approaches to understanding the motion of objects, one based on force and one on a different concept, called energy. The SI unit of energy is the Joule, but you are probably more familiar with the calorie, used for measuring food's energy, and the kilowatt-hour, the unit the electric company uses for billing you. Physics students' previous familiarity with calories and kilowatt-hours is matched by their universal unfamiliarity with measuring forces in units of Newtons, but the precise operational definitions of the energy concepts are more complex than those of the force concepts, and textbooks, including this one, almost universally place the force description of physics before the energy description. During the long period after the introduction of force and before the careful definition of energy, students are therefore vulnerable to situations in which, without realizing it, they are imputing the properties of energy to phenomena of force.

Incorrect statement: "How can my chair be making an upward force on my rear end? It has no power!"

✗ Power is a concept related to energy, e.g., a 100-watt lightbulb uses up 100 joules per second of energy. When you sit in a chair, no energy is used up, so forces can exist between you and the chair without any need for a source of power.

4. Force is not stored or used up.

Because energy can be stored and used up, people think force also can be stored or used up.

Incorrect statement: "If you don't fill up your tank with gas, you'll run out of force."

✗ Energy is what you'll run out of, not force.

5. Forces need not be exerted by living things or machines.

Transforming energy from one form into another usually requires some kind of living or mechanical mechanism. The concept is not applicable to forces, which are an interaction between objects, not a thing to be transferred or transformed.

Incorrect statement: "How can a wooden bench be making an upward force on my rear end? It doesn't have any springs or anything inside it."

✗ No springs or other internal mechanisms are required. If the bench didn't make any force on you, you would obey Newton's second law and fall through it. Evidently it does make a force on you!

6. A force is the direct cause of a change in motion.

I can click a remote control to make my garage door change from being at rest to being in motion. My finger's force on the button, however, was not the force that acted on the door. When we speak of a force on an object in physics, we are talking about a force that acts directly. Similarly, when you pull a reluctant dog along by its leash, the leash and the dog are making forces on each other, not your hand and the dog. The dog is not even touching your hand.

self-check B

Which of the following things can be correctly described in terms of force?

(1) A nuclear submarine is charging ahead at full steam.

(2) A nuclear submarine's propellers spin in the water.

(3) A nuclear submarine needs to refuel its reactor periodically. ▷ Answer, p. 558

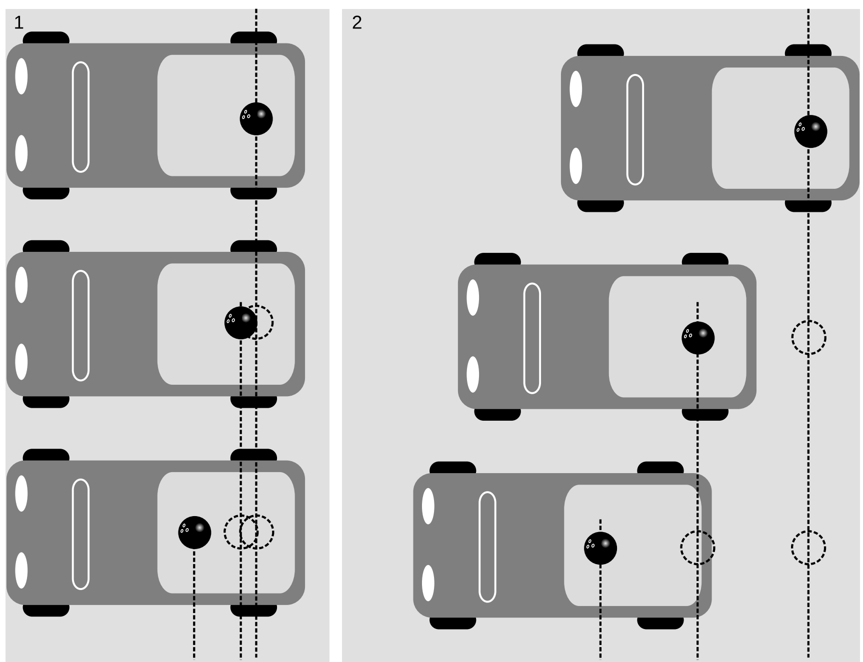
Discussion questions

A Criticize the following incorrect statement: "If you shove a book across a table, friction takes away more and more of its force, until finally it stops."

B You hit a tennis ball against a wall. Explain any and all incorrect ideas in the following description of the physics involved: "The ball gets some force from you when you hit it, and when it hits the wall, it loses part of that force, so it doesn't bounce back as fast. The muscles in your arm are the only things that a force can come from."

4.5 Inertial and noninertial frames of reference

One day, you're driving down the street in your pickup truck, on your way to deliver a bowling ball. The ball is in the back of the truck, enjoying its little jaunt and taking in the fresh air and sunshine. Then you have to slow down because a stop sign is coming up. As you brake, you glance in your rearview mirror, and see your trusty companion accelerating toward you. Did some mysterious force push it forward? No, it only seems that way because you and the car are slowing down. The ball is faithfully obeying Newton's first law, and as it continues at constant velocity it gets ahead relative to the slowing truck. No forces are acting on it (other than the same canceling-out vertical forces that were always acting on it).[3] The ball only appeared to violate Newton's first law because there was something wrong with your frame of reference, which was based on the truck.



l / 1. In a frame of reference that moves with the truck, the bowling ball appears to violate Newton's first law by accelerating despite having no horizontal forces on it. 2. In an inertial frame of reference, which the surface of the earth approximately is, the bowling ball obeys Newton's first law. It moves equal distances in equal time intervals, i.e., maintains constant velocity. In this frame of reference, it is the truck that appears to have a change in velocity, which makes sense, since the road is making a horizontal force on it.

How, then, are we to tell in which frames of reference Newton's laws are valid? It's no good to say that we should avoid moving frames of reference, because there is no such thing as absolute rest or absolute motion. All frames can be considered as being either at rest or in motion. According to an observer in India, the strip mall that constituted the frame of reference in panel (b) of the figure was moving along with the earth's rotation at hundreds of miles per hour.

The reason why Newton's laws fail in the truck's frame of refer-

[3]Let's assume for simplicity that there is no friction.

ence is not because the truck is *moving* but because it is *accelerating*. (Recall that physicists use the word to refer either to speeding up or slowing down.) Newton's laws were working just fine in the moving truck's frame of reference as long as the truck was moving at constant velocity. It was only when its speed changed that there was a problem. How, then, are we to tell which frames are accelerating and which are not? What if you claim that your truck is not accelerating, and the sidewalk, the asphalt, and the Burger King are accelerating? The way to settle such a dispute is to examine the motion of some object, such as the bowling ball, which we know has zero total force on it. Any frame of reference in which the ball appears to obey Newton's first law is then a valid frame of reference, and to an observer in that frame, Mr. Newton assures us that all the other objects in the universe will obey his laws of motion, not just the ball.

Valid frames of reference, in which Newton's laws are obeyed, are called inertial frames of reference. Frames of reference that are not inertial are called noninertial frames. In those frames, objects violate the principle of inertia and Newton's first law. While the truck was moving at constant velocity, both it and the sidewalk were valid inertial frames. The truck became an invalid frame of reference when it began changing its velocity.

You usually assume the ground under your feet is a perfectly inertial frame of reference, and we made that assumption above. It isn't perfectly inertial, however. Its motion through space is quite complicated, being composed of a part due to the earth's daily rotation around its own axis, the monthly wobble of the planet caused by the moon's gravity, and the rotation of the earth around the sun. Since the accelerations involved are numerically small, the earth is approximately a valid inertial frame.

Noninertial frames are avoided whenever possible, and we will seldom, if ever, have occasion to use them in this course. Sometimes, however, a noninertial frame can be convenient. Naval gunners, for instance, get all their data from radars, human eyeballs, and other detection systems that are moving along with the earth's surface. Since their guns have ranges of many miles, the small discrepancies between their shells' actual accelerations and the accelerations predicted by Newton's second law can have effects that accumulate and become significant. In order to kill the people they want to kill, they have to add small corrections onto the equation $a = F_{total}/m$. Doing their calculations in an inertial frame would allow them to use the usual form of Newton's second law, but they would have to convert all their data into a different frame of reference, which would require cumbersome calculations.

Discussion question

A If an object has a linear $x - t$ graph in a certain inertial frame, what is the effect on the graph if we change to a coordinate system with a different origin? What is the effect if we keep the same origin but reverse the positive direction of the x axis? How about an inertial frame moving alongside the object? What if we describe the object's motion in a noninertial frame?

4.6 ⋆ Numerical techniques

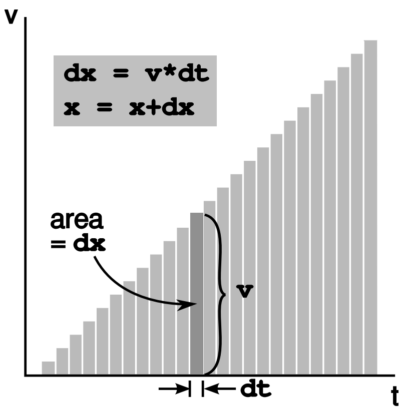
Engineering majors are a majority of the students in the kind of physics course for which this book is designed, so most likely you fall into that category. Although you surely recognize that physics is an important part of your training, if you've had any exposure to how engineers really work, you're probably skeptical about the flavor of problem-solving taught in most science courses. You realize that not very many practical engineering calculations fall into the narrow range of problems for which an exact solution can be calculated with a piece of paper and a sharp pencil. Real-life problems are usually complicated, and typically they need to be solved by number-crunching on a computer, although we can often gain insight by working simple approximations that have algebraic solutions. Not only is numerical problem-solving more useful in real life, it's also educational; as a beginning physics student, I really only felt like I understood projectile motion after I had worked it both ways, using algebra and then a computer program.

In this section, we'll start by seeing how to apply numerical techniques to some simple problems for which we know the answer in "closed form," i.e., a single algebraic expression without any calculus or infinite sums. After that, we'll solve a problem that would have made you world-famous if you could have done it in the seventeenth century using paper and a quill pen! Before you continue, you should read Appendix 1 on page 540 that introduces you to the Python programming language.

First let's solve the trivial problem of finding the distance traveled by an object moving at speed v to in time t. This closed-form answer is, of course, vt, but the point is to introduce the techniques we can use to solve other problems of this type. The basic idea is to divide the time up into n equal parts, and add up the distances traveled in all the parts. The following Python function does the job. Note that you shouldn't type in the line numbers on the left, and you don't need to type in the comments, either.

```
1   import math
2   def dist(n):
3     t = 1.0                     # seconds
4     v = 1.0                     # m/s
5     x=0                         # Initialize the position.
6     dt = t/n                    # Divide t into n equal parts.
7     for i in range(n):
8       dx = v*dt                 # tiny distance traveled in dt
9       x = x+dx                  # Change x.
10    return x
```

Of course line 8 shows how silly this example is — if we knew $\mathrm{d}x = v\,\mathrm{d}t$, then presumably we knew $x = vt$, which was the answer to



m / Through what distance does an object fall in 1.0 s, starting from rest? We calculate the area of the rectangle as `dx = v*dt`, then add this rectangle into the accumulated area under the curve using `x = x+dx`.

the whole problem — but the point is to illustrate the technique with the simplest possible example. How far do we move in 1 s at a constant speed of 1 m/s? If we do this,

```
>>> print(dist(10))
1.0
```

Python produces the expected answer by dividing the time into ten equal 0.1-second intervals, and adding up the ten 0.1-meter segments traversed in them. Since the object moves at constant speed, it doesn't even matter whether we set `n` to 10, 1, or a million:

```
>>> print(dist(1))
1.0
```

Now let's do an example where the answer isn't obvious to people who don't know calculus: through what distance does an object fall in 1.0 s, starting from rest? By integrating $a = g$ to find $v = gt$ and the integrating again to get $x = (1/2)gt^2$, we know that the exact answer is 4.9 m. Let's see if we can reproduce that answer numerically, as suggested by figure m. The main difference between this program and the previous one is that now the velocity isn't constant, so we need to update it as we go along.

```
import math
def dist2(n):
  t = 1.0                       # seconds
  g=9.8                         # strength of gravity, in m/s2
  x=0                           # Initialize the distance fallen.
  v=0                           # Initialize the velocity.
  dt = t/n                      # Divide t into n equal parts.
  for i in range(n):
    dx = v*dt                   # tiny distance traveled during tiny ti
    x = x+dx                    # Change x.
    dv = g*dt                   # tiny change in vel. during tiny time
    v = v+dv
  return x
```

With the drop split up into only 10 equal height intervals, the numerical technique provides a decent approximation:

```
>>> print(dist2(10))
4.41
```

By increasing `n` to ten thousand, we get an answer that's as close as we need, given the limited accuracy of the raw data:

```
>>> print(dist2(10000))
4.89951
```

Now let's use these techniques to solve the following somewhat whimsical problem, which cannot be solved in closed form using elementary functions such as sines, exponentials, roots, etc.

Ann E. Hodges of Sylacauga, Alabama is the only person ever known to have been injured by a meteorite. In 1954, she was struck in the hip by a meteorite that crashed through the roof of her house while she was napping on the couch. Since Hodges was asleep, we do not have direct evidence on the following silly trivia question: if you're going to be hit by a meteorite, will you hear it coming, or will it approach at more than the speed of sound? To answer this question, we start by constructing a physical model that is as simple as possible. We take the meteor as entering the earth's atmosphere directly along the vertical. The atmosphere does not cut off suddenly at a certain height; its density can be approximated as being proportional to $e^{-x/H}$, where x is the altitude and $H \approx 7.6$ km is called the scale height. The force of air friction is proportional to the density and to the square of the velocity, so

$$F = bv^2 e^{-x/H}$$

where b is a constant and F is positive in the coordinate system we've chosen, where $+x$ is up. The constant b depends on the size of the object, and its mass also affects the acceleration through Newton'a second law, $a = F/m$. The answer to the question therefore depends on the size of the meteorite. However, it is reasonable to take the results for the Sylacauga meteorite as constituting a general answer to our question, since larger ones are very rare, while the much more common pebble-sized ones do not make it through the atmosphere before they disintegrate. The object's initial velocity as it entered the atmosphere is not known, so we assume a typical value of 20 km/s. The Sylacauga meteorite was seen breaking up into three pieces, only two of which were recovered. The complete object's mass was probably about 7 kg and its radius about 9 cm. For an object with this radius, we expect $b \approx 1.5 \times 10^{-3}$ kg/m. Using Newton's second law, we find

$$\begin{aligned} a &= \frac{F_{total}}{m} \\ &= \frac{bv^2 e^{-x/H} - mg}{m}. \end{aligned}$$

I don't know of any way to solve this to find the function $x(t)$ closed form, so let's solve it numerically.

This problem is of a slightly different form than the ones above, where we knew we wanted to follow the motion up until a certain time. This problem is more of an "Are we there yet?" We want to stop the calculation where the altitude reaches zero. If it started at an initial position x at velocity v ($v < 0$) and maintained that velocity all the way down to sea level, the time interval would be

$\Delta t = \Delta x/v = (0 - x)/v = -x/v$. Since it actually slows down, Δt will be greater than that. We guess ten times that as a maximum, and then have the program check each time through the loop to see if we've hit the ground yet. When this happens, we bail out of the loop at line 15 before completing all n iterations.

```
1   import math
2   def meteor(n):
3     r = .09
4     m=7                # mass in kg
5     b=1.5e-3           # const. of prop. for friction, kg/m
6     x = 200.*1000.     # start at 200 km altitude, far above air
7     v = -20.*1000.     # 20 km/s
8     H = 7.6*1000.      # scale height in meters
9     g = 9.8            # m/s2
10    t_max = -x/v*10.   # guess the longest time it could take
11    dt = t_max/n       # Divide t into n equal parts.
12    for i in range(n):
13      dx = v*dt
14      x = x+dx         # Change x.
15      if x<0.:         # If we've hit the ground...
16        return v       # ...quit.
17      F = b*v**2*math.exp(-x/H)-m*g
18      a = F/m
19      dv = a*dt
20      v = v+dv
21    return -999.       # If we get here, t_max was too short.
```

The result is:

```
>>> print(meteor(100000))
-3946.95754982
```

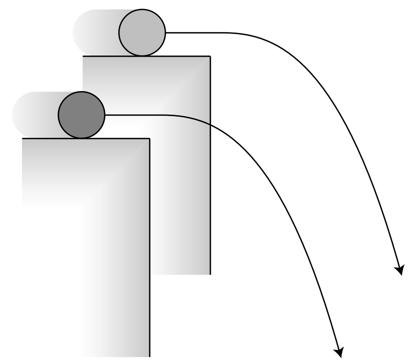
For comparison, the speed of sound is about 340 m/s. The answer is that if you are hit by a meteorite, you will not be able to hear its sound before it hits you.

4.7 ⋆ Do Newton's laws mean anything, and if so, are they true?

On your first encounter with Newton's first and second laws, you probably had a hard enough time just figuring out what they really meant and reconciling them with the whispers in your ear from the little Aristotelian devil sitting on your shoulder. This optional section is more likely to be of interest to you if you're already beyond that point and are starting to worry about deeper questions. It addresses the logical foundations of Newton's laws and sketches some of the empirical evidence for and against them. Section 5.7 gives a similar discussion for the third law, which we haven't yet encountered.

Newton's first law

Similar ideas are expressed by the principle of inertia (p. 73) and Newton's first law (p. 133). Both of these assertions are false in a noninertial frame (p. 146). Let's repackage all of these ideas as follows:

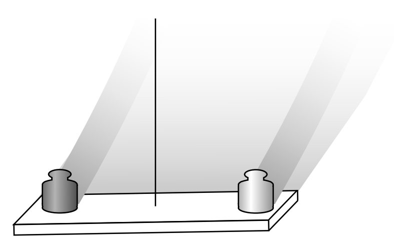


n / If the cylinders have slightly unequal ratios of inertial to gravitational mass, their trajectories will be a little different.

Newton's first law (repackaged)
When we find ourselves at any time and place in the universe, and we want to describe our immediate surroundings, we can always find some frame of reference that is inertial. An inertial frame is one in which an object acted on by zero total force responds by moving in a straight line at constant velocity.

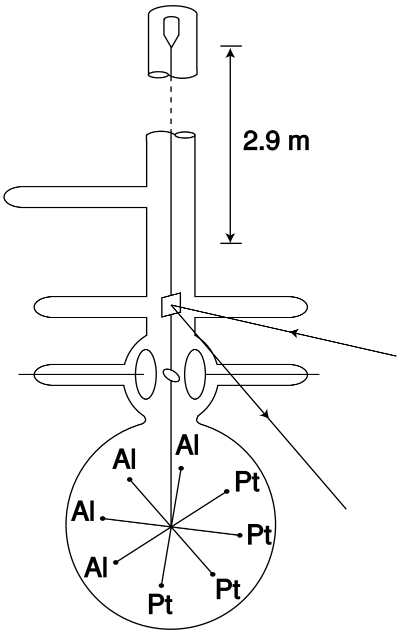
A corollary of this definition of an inertial frame is that given any inertial frame F, any other frame F′ moving relative to it at constant velocity is also inertial. That is, we only need to find one inertial frame, and then we get infinitely many others for free.

Ambiguities due to gravity



o / A simplified drawing of an Eötvös-style experiment. If the two masses, made out of two different substances, have slightly different ratios of inertial to gravitational mass, then the apparatus will twist slightly as the earth spins.

But finding that first inertial frame can be as difficult as knowing when you've found your first true love. Suppose that Alice is doing experiments inside a certain laboratory (the "immediate surroundings"), and unknown to her, her lab happens to be an elevator that is in a state of free fall. (We assume that someone will gently decelerate the lab before it hits the bottome of the shaft, so she isn't doomed.) Meanwhile, her twin sister Betty is doing similar experiments sealed inside a lab somewhere in the depths of outer space, where there is no gravity. Every experiment comes out exactly the same, regardless of whether it is performed by Alice or by Betty. Alice releases a pencil and sees it float in front of her; Newton says this is because she and the pencil are both falling with the same acceleration. Betty does the same experiment and gets the same result, but according to Newton the reason is now completely different: Betty and the pencil are not accelerating at all, because there is no gravity. It appears, then, that the distinction between inertial and noninertial frames is not always possible to make.



p / A more realistic drawing of Braginskii and Panov's experiment. The whole thing was encased in a tall vacuum tube, which was placed in a sealed basement whose temperature was controlled to within 0.02°C. The total mass of the platinum and aluminum test masses, plus the tungsten wire and the balance arms, was only 4.4 g. To detect tiny motions, a laser beam was bounced off of a mirror attached to the wire. There was so little friction that the balance would have taken on the order of several years to calm down completely after being put in place; to stop these vibrations, static electrical forces were applied through the two circular plates to provide very gentle twists on the ellipsoidal mass between them. After Braginskii and Panov.

One way to recover this distinction would be if we had access to some exotic matter — call it FloatyStuff^{TM} — that had the ordinary amount of inertia, but was completely unaffected by gravity. Normally when we release a material object in a gravitational field, it experiences a force mg, and then by Newton's second law its acceleration is $a = F/m = mg/m = g$. The m's cancel, which is the reason that everything falls with the same acceleration (in the absence of other forces such as air resistance). If Alice and Betty both release a blob of FloatyStuff, they observe different results. Unfortunately, nobody has ever found anything like FloatyStuff. In fact, extremely delicate experiments have shown that the proportionality between weight and inertia holds to the incredible precision of one part in 10^{12}. Figure n shows a crude test of this type, figure o a concept better suited to high-precision tests, and p a diagram of the actual apparatus used in one such experiment.[4]

If we could tell Newton the story of Alice and Betty, he would probably propose a different solution: don't seal the twins in boxes. Let them look around at all the nearby objects that could be making gravitational forces on their pencils. Alice will see such an object (the planet earth), so she'll know that her pencil is subject to a nonzero force and that her frame is noninertial. Betty will not see any planet, so she'll know that her frame is inertial.

The problem with Newton's solution is that gravity can act from very far away. For example, Newton didn't know that our solar system was embedded in the Milky Way Galaxy, so he imagined that the gravitational forces it felt from the uniform background of stars would almost perfectly cancel out by symmetry. But in reality, the galactic core is off in the direction of the constellation Sagittarius, and our solar system experiences a nonzero force in that direction, which keeps us from flying off straight and leaving

[4] V.B. Braginskii and V.I. Panov, Soviet Physics JETP 34, 463 (1972).

the galaxy. No problem, says Newton, that just means we should have taken our galaxy's center of mass to define an inertial frame. But our galaxy turns out to be free-falling toward a distant-future collision with the Andromeda Galaxy. We can keep on zooming out, and the residual gravitational accelerations get smaller and smaller, but there is no guarantee that the process will ever terminate with a perfect inertial frame. We do find, however, that the accelerations seem to get pretty small on large scales. For example, our galaxy's acceleration due to the gravitational attraction of the Andromeda Galaxy is only about 10^{-13} m/s^2.

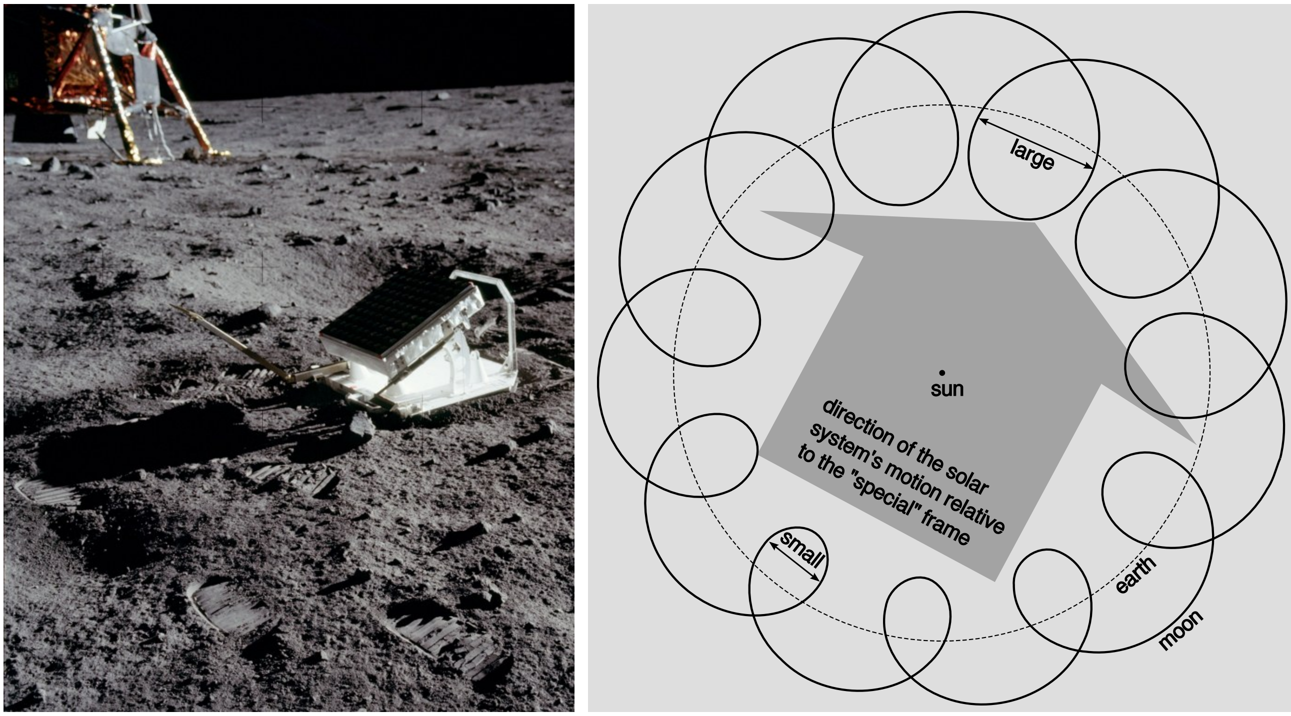
Furthermore, these accelerations don't necessarily hurt us, even if we don't know about them and fail to take them into account. Alice, free-falling in her elevator, gets perfectly valid experimental results, identical to Betty's in outer space. The only real problem would be if Alice did an experiment sensitive enough to be affected by the tiny *difference* in gravity between the floor and ceiling of the elevator. (The ocean tides are caused by small differences of this type in the moon's gravity.) For a small enough laboratory, i.e., on a *local* scale, we expect such effects to be negligible for most purposes.

An example of an empirical test

These ambiguities in defining an inertial frame are not severe enough to prevent us from performing highly precise tests of the first law. One important type of test comes from observations in which the "laboratory" is our solar system. External bodies do produce gravitational forces that intrude into the solar system, but these forces are quite weak, and their differences from one side of the solar system to another are weaker still. Therefore the workings of the solar system can be considered as a local experiment.

The left panel of figure q shows a mirror on the moon. By reflecting laser pulses from the mirror, the distance from the earth to the moon has been measured to the phenomenal precision of a few centimeters, or about one part in 10^{10}. This distance changes for a variety of known reasons. The biggest effect is that the moon's orbit is not a circle but an ellipse (see ch. 10), with its long axis about 11% longer than its short one. A variety of other effects can also be accounted for, including such exotic phenomena as the slightly nonspherical shape of the earth, and the gravitational forces of bodies as small and distant as Pluto. Suppose for simplicity that all these effects had never existed, so that the moon was initially placed in a perfectly circular orbit around the earth, and the earth in a perfectly circular orbit around the sun.

If we then observed something like what is shown in the right panel of figure q, Newton's first law would be disproved. If space itself is symmetrical in all directions, then there is no reason for the moon's orbit to poof up near the top of the diagram and con-



q / Left: The Apollo 11 mission left behind a mirror, which in this photo shows the reflection of the black sky. Right: A highly exaggerated example of an observation that would disprove Newton's first law. The radius of the moon's orbit gets bigger and smaller over the course of a year.

tract near the bottom. The only possible explanation would be that there was some "special" or "preferred" frame of reference of the type envisioned by Aristotle, and that our solar system was moving relative to it. One could then imagine that the gravitational force of the earth on the moon could be affected by the moon's motion relative to this frame. The lunar laser ranging data[5] contain no measurable effect of the type shown in figure q, so that if the moon's orbit is distorted in this way (or in a variety of other ways), the distortion must be less than a few centimeters. This constitutes a very strict upper limit on violation of Newton's first law by gravitational forces. If the first law is violated, and the violation causes a fractional change in gravity that is proportional to the velocity relative to the hypothetical preferred frame, then the change is no more than about one part in 10^7, even if the velocity is comparable to the speed of light. This is only one particular experiment involving gravity, but many different types of experiments have been done, and none have given any evidence for a preferred frame.

Newton's second law

Newton's second law, $a = F/m$, is false in general.

It fails at the microscopic level because particles are not just par-

[5] Battat, Chandler, and Stubbs, `http://arxiv.org/abs/0710.0702`

ticles, they're also waves. One consequence is that they do not have exactly well defined positions, so that the acceleration a appearing in $a = F/m$ is not even well defined.

Example 13 on p. 143 shows an example of the failure of the second law as electrons approach the speed of light. We've seen in section 2.6 that relativity forbids objects from moving at speeds faster than the speed of light. We will see in section 14.7 that an object's inertia F/a is larger for an object moving closer to the speed of light (relative to the observer who measures F and a). It is not a velocity-independent constant m as in the usual interpretation of the second law. The second law is nevertheless highly accurate within its domain of validity, i.e., small relative speeds.

It is difficult in general to design an unambiguous test of the second law because if we like, we can take $F = ma$ to be a definition of force. For example, in the experiment with the electrons, we could simply say that the force must have been mysteriously decreasing, despite our best efforts to keep it constant. One way around this is to use the fact that the second law should be *reproducible*. For example, if the earth makes a certain gravitational force on the moon at a certain point in the moon's orbit, then it is to be expected that one month later, when the moon has revolved once around the earth and is back at the same point, we should again obtain the same force and acceleration. If, for example, we saw that the moon had a slightly larger acceleration this time around, we could interpret this as evidence for a gradual trend of reducing mass, or increasing force. But either way, this trend over time would be a violation of Newton's laws, because it would not have been caused by any change in the conditions to which the moon was subjected. Lunar laser ranging experiments of the type described above show that if any such trend in acceleration exists, it must be less than about one part in 10^{12} per year.

Newton's second law refers to the total force acting on an object, so it predicts that forces are exactly additive. If we apply two forces of exactly 1 N to an object, the result is supposed to be exactly 2 N, not 1.999997 N or 2.00002 N. An alternative physical theory called MOND has been proposed, in an attempt to avoid the need for invoking exotic "dark matter" in order to correctly describe the rotation of galaxies. MOND is only approximately additive, so one of its predictions is that the gravitational forces exerted by the galaxy on the planets of the solar system would interact in complicated ways with the solar system's internal gravitational forces, producing tiny discrepancies with the predictions of Newton's laws. High-precision data from spacecraft have failed to detect this effect,[6] and have limited any such anomalous accelerations of the planets to no more than about 10^{-14} m/s^2.

[6]Iorio, arxiv.org/abs/0906.2937

Summary

Selected vocabulary

weight	the force of gravity on an object, equal to mg
inertial frame . .	a frame of reference that is not accelerating, one in which Newton's first law is true
noninertial frame	an accelerating frame of reference, in which Newton's first law is violated

Notation

F_W	weight

Other terminology and notation

net force	another way of saying "total force"

Summary

Newton's first law of motion states that if all the forces acting on an object cancel each other out, then the object continues in the same state of motion. This is essentially a more refined version of Galileo's principle of inertia, which did not refer to a numerical scale of force.

Newton's second law of motion allows the prediction of an object's acceleration given its mass and the total force on it, $a_{cm} = F_{total}/m$. This is only the one-dimensional version of the law; the full-three dimensional treatment will come in chapter 8, Vectors. Without the vector techniques, we can still say that the situation remains unchanged by including an additional set of vectors that cancel among themselves, even if they are not in the direction of motion.

Newton's laws of motion are only true in frames of reference that are not accelerating, known as inertial frames.

Even in one-dimensional motion, it is seldom possible to solve real-world problems and predict the motion of an object in closed form. However, there are straightforward numerical techniques for solving such problems.

Problems

Key

√ A computerized answer check is available online.

∫ A problem that requires calculus.

⋆ A difficult problem.

1 A car is accelerating forward along a straight road. If the force of the road on the car's wheels, pushing it forward, is a constant 3.0 kN, and the car's mass is 1000 kg, then how long will the car take to go from 20 m/s to 50 m/s? ▷ Solution, p. 548

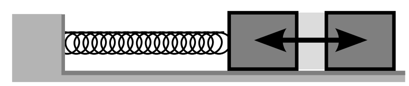
2 (a) Let T be the maximum tension that an elevator's cable can withstand without breaking, i.e., the maximum force it can exert. If the motor is programmed to give the car an acceleration a ($a > 0$ is upward), what is the maximum mass that the car can have, including passengers, if the cable is not to break? √
(b) Interpret the equation you derived in the special cases of $a = 0$ and of a downward acceleration of magnitude g.

("Interpret" means to analyze the behavior of the equation, and connect that to reality, as in the self-check on page 142.)

3 An object is observed to be moving at constant speed in a certain direction. Can you conclude that no forces are acting on it? Explain. [Based on a problem by Serway and Faughn.]

4 You are given a large sealed box, and are not allowed to open it. Which of the following experiments measure its mass, and which measure its weight? [Hint: Which experiments would give different results on the moon?]
(a) Put it on a frozen lake, throw a rock at it, and see how fast it scoots away after being hit.
(b) Drop it from a third-floor balcony, and measure how loud the sound is when it hits the ground.
(c) As shown in the figure, connect it with a spring to the wall, and watch it vibrate.

▷ Solution, p. 548



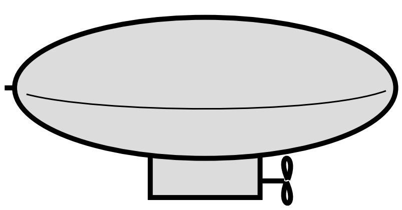
Problem 4, part c.

5 While escaping from the palace of the evil Martian emperor, Sally Spacehound jumps from a tower of height h down to the ground. Ordinarily the fall would be fatal, but she fires her blaster rifle straight down, producing an upward force of magnitude F_B. This force is insufficient to levitate her, but it does cancel out some of the force of gravity. During the time t that she is falling, Sally is unfortunately exposed to fire from the emperor's minions, and can't dodge their shots. Let m be her mass, and g the strength of gravity on Mars.
(a) Find the time t in terms of the other variables.
(b) Check the units of your answer to part a.
(c) For sufficiently large values of F_B, your answer to part a becomes nonsense — explain what's going on. √

6 At low speeds, every car's acceleration is limited by traction, not by the engine's power. Suppose that at low speeds, a certain car is normally capable of an acceleration of 3 m/s^2. If it is towing a trailer with half as much mass as the car itself, what acceleration can it achieve? [Based on a problem from PSSC Physics.]

7 A helicopter of mass m is taking off vertically. The only forces acting on it are the earth's gravitational force and the force, F_{air}, of the air pushing up on the propeller blades.
(a) If the helicopter lifts off at $t = 0$, what is its vertical speed at time t?
(b) Check that the units of your answer to part a make sense.
(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you've figured out this *mathematical* relationship, show that it makes sense *physically*.
(d) Plug numbers into your equation from part a, using $m = 2300$ kg, $F_{air} = 27000$ N, and $t = 4.0$ s. ✓

8 A uranium atom deep in the earth spits out an alpha particle. An alpha particle is a fragment of an atom. This alpha particle has initial speed v, and travels a distance d before stopping in the earth.
(a) Find the force, F, from the dirt that stopped the particle, in terms of v, d, and its mass, m. Don't plug in any numbers yet. Assume that the force was constant. ✓
(b) Show that your answer has the right units.
(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you've figured out this *mathematical* relationship, show that it makes sense *physically*.
(d) Evaluate your result for $m = 6.7 \times 10^{-27}$ kg, $v = 2.0 \times 10^4$ km/s, and $d = 0.71$ mm. ✓


Problem 9.

9 A blimp is initially at rest, hovering, when at $t = 0$ the pilot turns on the engine driving the propeller. The engine cannot instantly get the propeller going, but the propeller speeds up steadily. The steadily increasing force between the air and the propeller is given by the equation $F = kt$, where k is a constant. If the mass of the blimp is m, find its position as a function of time. (Assume that during the period of time you're dealing with, the blimp is not yet moving fast enough to cause a significant backward force due to air resistance.) ✓

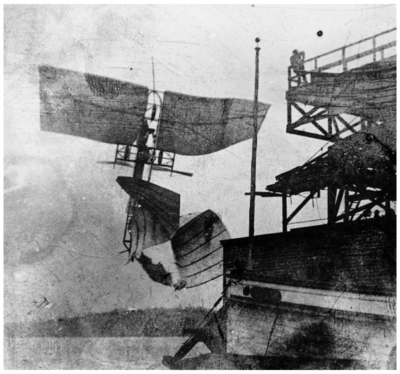
10 Some garden shears are like a pair of scissors: one sharp blade slices past another. In the "anvil" type, however, a sharp blade presses against a flat one rather than going past it. A gardening book says that for people who are not very physically strong, the anvil type can make it easier to cut tough branches, because it concentrates the force on one side. Evaluate this claim based on Newton's laws. [Hint: Consider the forces acting on the branch, and the motion of the branch.]

11 In the 1964 Olympics in Tokyo, the best men's high jump was 2.18 m. Four years later in Mexico City, the gold medal in the same event was for a jump of 2.24 m. Because of Mexico City's altitude (2400 m), the acceleration of gravity there is lower than that in Tokyo by about 0.01 m/s^2. Suppose a high-jumper has a mass of 72 kg.
(a) Compare his mass and weight in the two locations.
(b) Assume that he is able to jump with the same initial vertical velocity in both locations, and that all other conditions are the same except for gravity. How much higher should he be able to jump in Mexico City? ✓
(Actually, the reason for the big change between '64 and '68 was the introduction of the "Fosbury flop.") ⋆

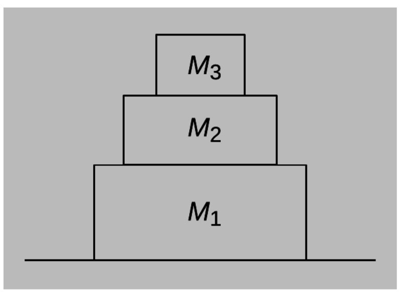
12 The factorial of an integer n, written $n!$, is defined as the product of all the positive integers less than or equal to n. For example, $3! = 1\times2\times3 = 6$. Write a Python program to compute the factorial of a number. Test it with a small number whose factorial you can check by hand. Then use it to compute 30!. (Python computes integer results with unlimited precision, so you won't get any problems with rounding or overflows.) Turn in a printout of your program and its output, including the test.

13 A ball falls from a height h. Without air resistance, the time it takes to reach the floor is $\sqrt{2h/g}$. Now suppose that air resistance is not negligible. For a smooth sphere of radius r, moving at speed v through air of density ρ, the force of air resistance is $(\pi/4)\rho v^2 r^2$. Modify the program `meteor` on page 152 to handle this problem, and find the resulting change in the fall time in the case of a 21 g ball of radius 1.0 cm, falling from a height of 1.0 m. The density of air at sea level is about 1.2 kg/m^3. You will need to use a very large value of `n` to achieve the required precision. Turn in a printout of both your program and its output. Answer: 0.34 ms.

14 The tires used in Formula 1 race cars can generate traction (i.e., force from the road) that is as much as 1.9 times greater than with the tires typically used in a passenger car. Suppose that we're trying to see how fast a car can cover a fixed distance starting from rest, and traction is the limiting factor. By what factor is this time reduced when switching from ordinary tires to Formula 1 tires? ✓



Problem 15. The rear wings of the plane collapse under the stress of the catapult launch.



Problem 17.

15 At the turn of the 20th century, Samuel Langley engaged in a bitter rivalry with the Wright brothers to develop human flight. Langley's design used a catapult for launching. For safety, the catapult was built on the roof of a houseboat, so that any crash would be into the water. This design required reaching cruising speed within a fixed, short distance, so large accelerations were required, and the forces frequently damaged the craft, causing dangerous and embarrassing accidents. Langley achieved several uncrewed, unguided flights, but never succeeded with a human pilot. If the force of the catapult is fixed by the structural strength of the plane, and the distance for acceleration by the size of the houseboat, by what factor is the launch velocity reduced when the plane's 340 kg is augmented by the 60 kg mass of a small man? √

16 A bullet of mass m is fired from a pistol, accelerating from rest to a speed v in the barrel's length L.
(a) What is the force on the bullet? (Assume this force is constant.) √
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence of your answer on each of the three variables makes sense. *[problem by B. Shotwell]*

17 Blocks of mass M_1, M_2, and M_3 are stacked on a table as shown in the figure. Let the upward direction be positive.
(a) What is the force on block 2 from block 3? √
(b) What is the force on block 2 from block 1? √ *[problem by B. Shotwell]*

Exercise 4: Force and motion

Equipment:

2-meter pieces of butcher paper

wood blocks with hooks

string

masses to put on top of the blocks to increase friction

spring scales (preferably calibrated in Newtons)

Suppose a person pushes a crate, sliding it across the floor at a certain speed, and then repeats the same thing but at a higher speed. This is essentially the situation you will act out in this exercise. What do you think is different about her force on the crate in the two situations? Discuss this with your group and write down your hypothesis:

____

1. First you will measure the amount of friction between the wood block and the butcher paper when the wood and paper surfaces are slipping over each other. The idea is to attach a spring scale to the block and then slide the butcher paper under the block while using the scale to keep the block from moving with it. Depending on the amount of force your spring scale was designed to measure, you may need to put an extra mass on top of the block in order to increase the amount of friction. It is a good idea to use long piece of string to attach the block to the spring scale, since otherwise one tends to pull at an angle instead of directly horizontally.

First measure the amount of friction force when sliding the butcher paper as slowly as possible:_________

Now measure the amount of friction force at a significantly higher speed, say 1 meter per second. (If you try to go too fast, the motion is jerky, and it is impossible to get an accurate reading.)

____

Discuss your results. Why are we justified in assuming that the string's force on the block (i.e., the scale reading) is the same amount as the paper's frictional force on the block?

2. Now try the same thing but with the block moving and the paper standing still. Try two different speeds.

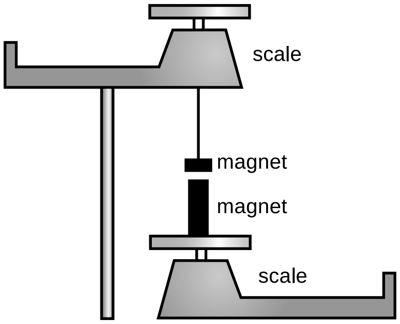
Do your results agree with your original hypothesis? If not, discuss what's going on. How does the block "know" how fast to go?

What forces act on the girl?

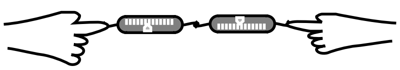
Chapter 5
Analysis of forces

5.1 Newton's third law

Newton created the modern concept of force starting from his insight that all the effects that govern motion are interactions between two objects: unlike the Aristotelian theory, Newtonian physics has no phenomena in which an object changes its own motion.



a / Two magnets exert forces on each other.



b / Two people's hands exert forces on each other.

c / Rockets work by pushing exhaust gases out the back. Newton's third law says that if the rocket exerts a backward force on the gases, the gases must make an equal forward force on the rocket. Rocket engines can function above the atmosphere, unlike propellers and jets, which work by pushing against the surrounding air.

Is one object always the "order-giver" and the other the "order-follower"? As an example, consider a batter hitting a baseball. The bat definitely exerts a large force on the ball, because the ball accelerates drastically. But if you have ever hit a baseball, you also know that the ball makes a force on the bat — often with painful results if your technique is as bad as mine!

How does the ball's force on the bat compare with the bat's force on the ball? The bat's acceleration is not as spectacular as the ball's, but maybe we shouldn't expect it to be, since the bat's mass is much greater. In fact, careful measurements of both objects' masses and accelerations would show that $m_{ball}a_{ball}$ is very nearly equal to $-m_{bat}a_{bat}$, which suggests that the ball's force on the bat is of the same magnitude as the bat's force on the ball, but in the opposite direction.

Figures a and b show two somewhat more practical laboratory experiments for investigating this issue accurately and without too much interference from extraneous forces.

In experiment a, a large magnet and a small magnet are weighed separately, and then one magnet is hung from the pan of the top balance so that it is directly above the other magnet. There is an attraction between the two magnets, causing the reading on the top scale to increase and the reading on the bottom scale to decrease. The large magnet is more "powerful" in the sense that it can pick up a heavier paperclip from the same distance, so many people have a strong expectation that one scale's reading will change by a far different amount than the other. Instead, we find that the two changes are equal in magnitude but opposite in direction: the force of the bottom magnet pulling down on the top one has the same strength as the force of the top one pulling up on the bottom one.

In experiment b, two people pull on two spring scales. Regardless of who tries to pull harder, the two forces as measured on the spring scales are equal. Interposing the two spring scales is necessary in order to measure the forces, but the outcome is not some artificial result of the scales' interactions with each other. If one person slaps another hard on the hand, the slapper's hand hurts just as much as the slappee's, and it doesn't matter if the recipient of the slap tries to be inactive. (Punching someone in the mouth causes just as much force on the fist as on the lips. It's just that the lips are more delicate. The forces are equal, but not the levels of pain and injury.)

Newton, after observing a series of results such as these, decided that there must be a fundamental law of nature at work:

Newton's third law

Forces occur in equal and opposite pairs: whenever object A exerts a force on object B, object B must also be exerting a force on object A. The two forces are equal in magnitude and opposite in direction.

d / A swimmer doing the breast stroke pushes backward against the water. By Newton's third law, the water pushes forward on him.

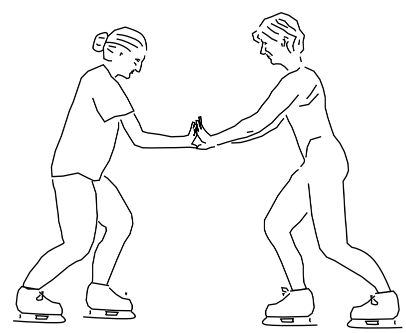
Two modern, high-precision tests of the third law are described on p. 191.

In one-dimensional situations, we can use plus and minus signs to indicate the directions of forces, and Newton's third law can be written succinctly as $F_{\text{A on B}} = -F_{\text{B on A}}$. Section 5.7 gives a more detailed discussion of the logical and empirical underpinnings of the third law.

self-check A

Figure d analyzes swimming using Newton's third law. Do a similar analysis for a sprinter leaving the starting line. ▷ Answer, p. 558

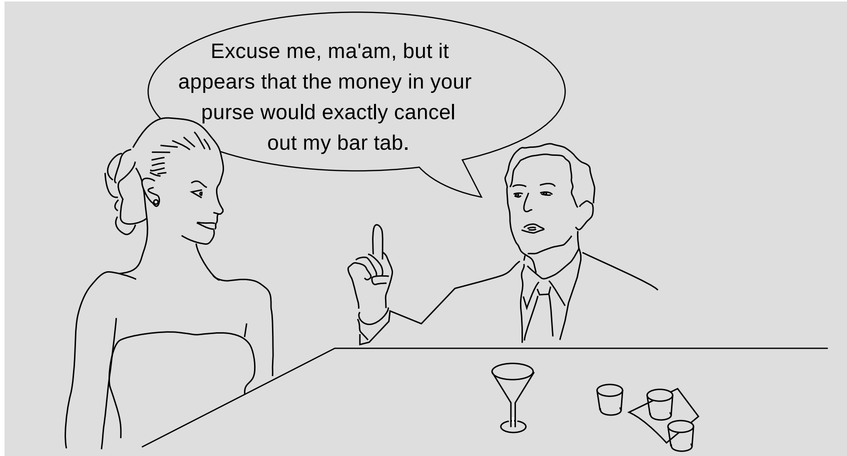
There is no cause and effect relationship between the two forces in Newton's third law. There is no "original" force, and neither one is a response to the other. The pair of forces is a relationship, like marriage, not a back-and-forth process like a tennis match. Newton came up with the third law as a generalization about all the types of forces with which he was familiar, such as frictional and gravitational forces. When later physicists discovered a new type of force, such as the force that holds atomic nuclei together, they had to check whether it obeyed Newton's third law. So far, no violation of the third law has ever been discovered, whereas the first and second laws were shown to have limitations by Einstein and the pioneers of atomic physics.



e / Newton's third law does not mean that forces always cancel out so that nothing can ever move. If these two ice skaters, initially at rest, push against each other, they will both move.

The English vocabulary for describing forces is unfortunately rooted in Aristotelianism, and often implies incorrectly that forces are one-way relationships. It is unfortunate that a half-truth such as "the table exerts an upward force on the book" is so easily expressed, while a more complete and correct description ends up sounding awkward or strange: "the table and the book interact via a force," or "the table and book participate in a force."

To students, it often sounds as though Newton's third law implies nothing could ever change its motion, since the two equal and opposite forces would always cancel. The two forces, however, are always on two different objects, so it doesn't make sense to add them in the first place — we only add forces that are acting on the same object. If two objects are interacting via a force and no other forces are involved, then *both* objects will accelerate — in opposite directions!



f / It doesn't make sense for the man to talk about using the woman's money to cancel out his bar tab, because there is no good reason to combine his debts and her assets. Similarly, it doesn't make sense to refer to the equal and opposite forces of Newton's third law as canceling. It only makes sense to add up forces that are acting on the *same* object, whereas two forces related to each other by Newton's third law are always acting on two *different* objects.

A mnemonic for using Newton's third law correctly

Mnemonics are tricks for memorizing things. For instance, the musical notes that lie between the lines on the treble clef spell the word FACE, which is easy to remember. Many people use the mnemonic "SOHCAHTOA" to remember the definitions of the sine, cosine, and tangent in trigonometry. I have my own modest offering, POFOSTITO, which I hope will make it into the mnemonics hall of fame. It's a way to avoid some of the most common problems with applying Newton's third law correctly:

P air of
O pposite
F orces

O f the
S ame
T ype

I nvolving
T wo
O bjects

A book lying on a table *example 1*

▷ A book is lying on a table. What force is the Newton's-third-law partner of the earth's gravitational force on the book?

Answer: Newton's third law works like "B on A, A on *B*," so the partner must be the book's gravitational force pulling upward on the planet earth. Yes, there is such a force! No, it does not cause the earth to do anything noticeable.

Incorrect answer: The table's upward force on the book is the Newton's-third-law partner of the earth's gravitational force on the book.

✗ This answer violates two out of three of the commandments of POFOSTITO. The forces are not of the same type, because the table's upward force on the book is not gravitational. Also, three objects are involved instead of two: the book, the table, and the planet earth.

Pushing a box up a hill *example 2*

▷ A person is pushing a box up a hill. What force is related by Newton's third law to the person's force on the box?

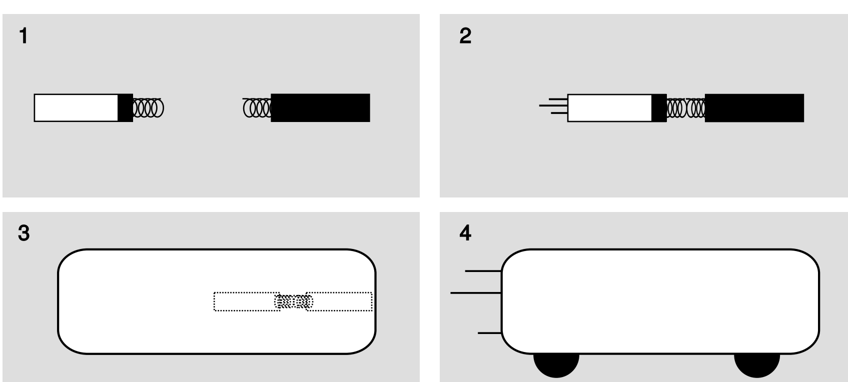
▷ The box's force on the person.

Incorrect answer: The person's force on the box is opposed by friction, and also by gravity.

✗ This answer fails all three parts of the POFOSTITO test, the most obvious of which is that three forces are referred to instead of a pair.

If we could violate Newton's third law... *example 3*

If we could violate Newton's third law, we could do strange and wonderful things. Newton's third laws says that the unequal magnets in figure a on p. 166 should exert equal forces on each other, and this is what we actually find when we do the experiment shown in that figure. But suppose instead that it worked as most people intuitively expect. What if the third law was violated, so that the big magnet made more force on the small one than the small one made on the big one? To make the analysis simple, we add some extra nonmagnetic material to the small magnet in figure g/1, so that it has the same mass and size as the big one. We also attach springs. When we release the magnets, g/2, the weak one is accelerated strongly, while the strong one barely moves. If we put them inside a box, g/3, the recoiling strong magnet bangs hard against the side of the box, and the box mysteriously accelerates itself. The process can be repeated indefinitely for free, so we have a magic box that propels itself without needing fuel. We can make it into a perpetual-motion car, g/4. If Newton's third law was violated, we'd never have to pay for gas!



g / Example 3. This doesn't actually happen!

Optional topic: Newton's third law and action at a distance

Newton's third law is completely symmetric in the sense that neither force constitutes a delayed response to the other. Newton's third law does not even mention time, and the forces are supposed to agree at any given instant. This creates an interesting situation when it comes to noncontact forces. Suppose two people are holding magnets, and when one person waves or wiggles her magnet, the other person feels an effect on his. In this way they can send signals to each other from opposite sides of a wall, and if Newton's third law is correct, it would seem that the signals are transmitted instantly, with no time lag. The signals are indeed transmitted quite quickly, but experiments with electrically controlled magnets show that the signals do not leap the gap instantly: they travel at the same speed as light, which is an extremely high speed but not an infinite one.

Is this a contradiction to Newton's third law? Not really. According to current theories, there are no true noncontact forces. Action at a distance does not exist. Although it appears that the wiggling of one magnet affects the other with no need for anything to be in contact with anything, what really happens is that wiggling a magnet creates a ripple in the magnetic field pattern that exists even in empty space. The magnet shoves the ripples out with a kick and receives a kick in return, in strict obedience to Newton's third law. The ripples spread out in all directions, and the ones that hit the other magnet then interact with it, again obeying Newton's third law.

▷ *Solved problem: More about example 2* *page 195, problem 2*

▷ *Solved problem: Why did it accelerate?* *page 195, problem 1*

Discussion questions

A When you fire a gun, the exploding gases push outward in all directions, causing the bullet to accelerate down the barrel. What third-law pairs are involved? [Hint: Remember that the gases themselves are an object.]

B Tam Anh grabs Sarah by the hand and tries to pull her. She tries to remain standing without moving. A student analyzes the situation as follows. "If Tam Anh's force on Sarah is greater than her force on him, he can get her to move. Otherwise, she'll be able to stay where she is." What's wrong with this analysis?

C You hit a tennis ball against a wall. Explain any and all incorrect ideas in the following description of the physics involved: "According to Newton's third law, there has to be a force opposite to your force on the ball. The opposite force is the ball's mass, which resists acceleration, and also air resistance."

5.2 Classification and behavior of forces

One of the most basic and important tasks of physics is to classify the forces of nature. I have already referred informally to "types" of forces such as friction, magnetism, gravitational forces, and so on. Classification systems are creations of the human mind, so there is always some degree of arbitrariness in them. For one thing, the level of detail that is appropriate for a classification system depends on what you're trying to find out. Some linguists, the "lumpers," like to emphasize the similarities among languages, and a few extremists have even tried to find signs of similarities between words in languages as different as English and Chinese, lumping the world's languages into only a few large groups. Other linguists, the "splitters," might be more interested in studying the differences in pronunciation between English speakers in New York and Connecticut. The splitters call the lumpers sloppy, but the lumpers say that science isn't worthwhile unless it can find broad, simple patterns within the seemingly complex universe.

Scientific classification systems are also usually compromises between practicality and naturalness. An example is the question of how to classify flowering plants. Most people think that biological classification is about discovering new species, naming them, and classifying them in the class-order-family-genus-species system according to guidelines set long ago. In reality, the whole system is in a constant state of flux and controversy. One very practical way of classifying flowering plants is according to whether their petals are separate or joined into a tube or cone — the criterion is so clear that it can be applied to a plant seen from across the street. But here practicality conflicts with naturalness. For instance, the begonia has separate petals and the pumpkin has joined petals, but they are so similar in so many other ways that they are usually placed within the same order. Some taxonomists have come up with classification criteria that they claim correspond more naturally to the apparent relationships among plants, without having to make special exceptions, but these may be far less practical, requiring for instance the examination of pollen grains under an electron microscope.

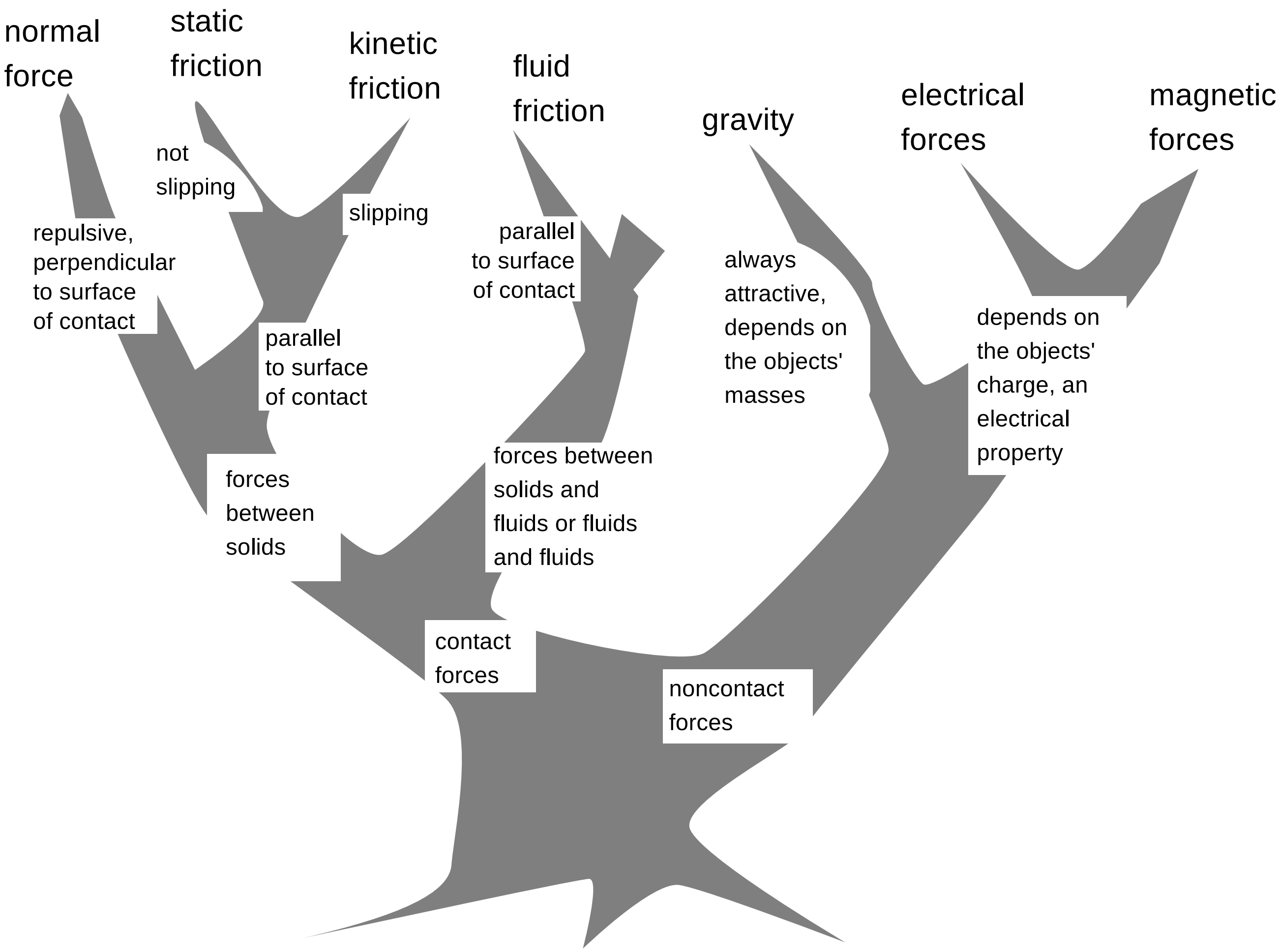
In physics, there are two main systems of classification for forces. At this point in the course, you are going to learn one that is very practical and easy to use, and that splits the forces up into a relatively large number of types: seven very common ones that we'll discuss explicitly in this chapter, plus perhaps ten less important ones such as surface tension, which we will not bother with right now.

Physicists, however, are obsessed with finding simple patterns, so recognizing as many as fifteen or twenty types of forces strikes them as distasteful and overly complex. Since about the year 1900, physics has been on an aggressive program to discover ways in which these many seemingly different types of forces arise from a smaller number of fundamental ones. For instance, when you press your hands together, the force that keeps them from passing through each other may seem to have nothing to do with electricity, but at the atomic level, it actually does arise from electrical repulsion between atoms. By about 1950, all the forces of nature had been explained as arising from four fundamental types of forces at the atomic and nuclear level, and the lumping-together process didn't stop there. By the 1960's the length of the list had been reduced to three, and some theorists even believe that they may be able to reduce it to two or one. Although the unification of the forces of nature is one of the most beautiful and important achievements of physics, it makes much more sense to start this course with the more practical and easy system of classification. The unified system of four forces will be one of the highlights of the end of your introductory physics sequence.

h / A scientific classification system.

The practical classification scheme which concerns us now can be laid out in the form of the tree shown in figure i. The most specific types of forces are shown at the tips of the branches, and it is these types of forces that are referred to in the POFOSTITO mnemonic. For example, electrical and magnetic forces belong to the same general group, but Newton's third law would never relate an electrical force to a magnetic force.

The broadest distinction is that between contact and noncontact forces, which has been discussed in ch. 4. Among the contact forces, we distinguish between those that involve solids only and those that have to do with fluids, a term used in physics to include both gases and liquids.



i / A practical classification scheme for forces.

It should not be necessary to memorize this diagram by rote. It is better to reinforce your memory of this system by calling to mind your commonsense knowledge of certain ordinary phenomena. For instance, we know that the gravitational attraction between us and the planet earth will act even if our feet momentarily leave the ground, and that although magnets have mass and are affected by gravity, most objects that have mass are nonmagnetic.

Hitting a wall *example 4*

▷ A bullet, flying horizontally, hits a steel wall. What type of force is there between the bullet and the wall?

▷ Starting at the bottom of the tree, we determine that the force is a contact force, because it only occurs once the bullet touches the wall. Both objects are solid. The wall forms a vertical plane. If the nose of the bullet was some shape like a sphere, you might imagine that it would only touch the wall at one point. Realistically, however, we know that a lead bullet will flatten out a lot on impact, so there is a surface of contact between the two, and its orientation is vertical. The effect of the force on the bullet is to stop the horizontal motion of the bullet, and this horizontal acceleration must be produced by a horizontal force. The force is therefore perpendicular to the surface of contact, and it's also repulsive (tending to keep the bullet from entering the wall), so it must be a normal force.

Diagram i is meant to be as simple as possible while including most of the forces we deal with in everyday life. If you were an insect, you would be much more interested in the force of surface tension, which allowed you to walk on water. I have not included the nuclear forces, which are responsible for holding the nuclei of atoms, because they are not evident in everyday life.

You should not be afraid to invent your own names for types of forces that do not fit into the diagram. For instance, the force that holds a piece of tape to the wall has been left off of the tree, and if you were analyzing a situation involving scotch tape, you would be absolutely right to refer to it by some commonsense name such as "sticky force."

On the other hand, if you are having trouble classifying a certain force, you should also consider whether it is a force at all. For instance, if someone asks you to classify the force that the earth has because of its rotation, you would have great difficulty creating a place for it on the diagram. That's because it's a type of motion, not a type of force!

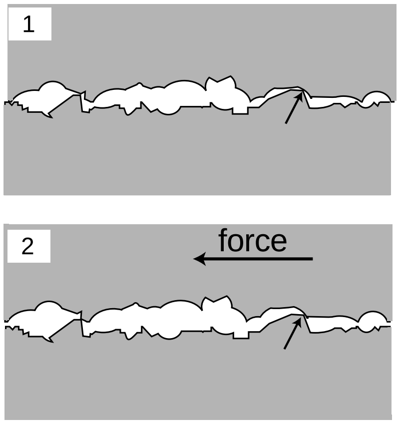
Normal forces

A normal force, F_N, is a force that keeps one solid object from passing through another. "Normal" is simply a fancy word for "perpendicular," meaning that the force is perpendicular to the surface of contact. Intuitively, it seems the normal force magically adjusts itself to provide whatever force is needed to keep the objects from occupying the same space. If your muscles press your hands together gently, there is a gentle normal force. Press harder, and the normal force gets stronger. How does the normal force know how strong to be? The answer is that the harder you jam your hands together, the more compressed your flesh becomes. Your flesh is acting like a spring: more force is required to compress it more. The same is true when you push on a wall. The wall flexes imperceptibly in proportion to your force on it. If you exerted enough force, would it be possible for two objects to pass through each other? No, typically the result is simply to strain the objects so much that one of them breaks.

Gravitational forces

As we'll discuss in more detail later in the course, a gravitational force exists between any two things that have mass. In everyday life, the gravitational force between two cars or two people is negligible, so the only noticeable gravitational forces are the ones between the earth and various human-scale objects. We refer to these planet-earth-induced gravitational forces as weight forces, and as we have already seen, their magnitude is given by $|F_W| = mg$.

▷ *Solved problem: Weight and mass* *page 195, problem 3*



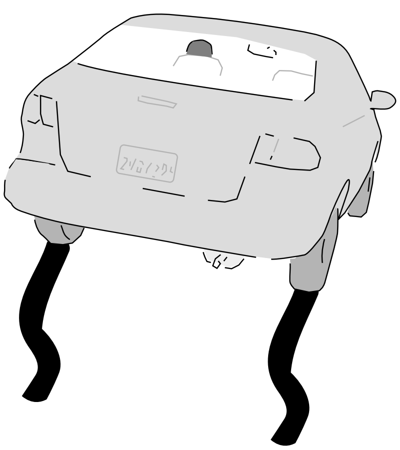
j / A model that correctly explains many properties of friction. The microscopic bumps and holes in two surfaces dig into each other.

Static and kinetic friction

If you have pushed a refrigerator across a kitchen floor, you have felt a certain series of sensations. At first, you gradually increased your force on the refrigerator, but it didn't move. Finally, you supplied enough force to unstick the fridge, and there was a sudden jerk as the fridge started moving. Once the fridge was unstuck, you could reduce your force significantly and still keep it moving.


k / Static friction: the tray doesn't slip on the waiter's fingers.

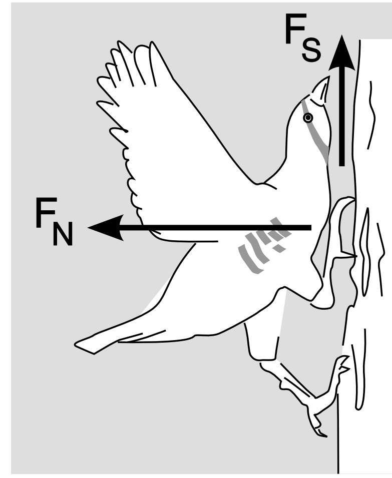
While you were gradually increasing your force, the floor's frictional force on the fridge increased in response. The two forces on the fridge canceled, and the fridge didn't accelerate. How did the floor know how to respond with just the right amount of force? Figure j shows one possible *model* of friction that explains this behavior. (A scientific model is a description that we expect to be incomplete, approximate, or unrealistic in some ways, but that nevertheless succeeds in explaining a variety of phenomena.) Figure j/1 shows a microscopic view of the tiny bumps and holes in the surfaces of the floor and the refrigerator. The weight of the fridge presses the two surfaces together, and some of the bumps in one surface will settle as deeply as possible into some of the holes in the other surface. In j/2, your leftward force on the fridge has caused it to ride up a little higher on the bump in the floor labeled with a small arrow. Still more force is needed to get the fridge over the bump and allow it to start moving. Of course, this is occurring simultaneously at millions of places on the two surfaces.


l / Kinetic friction: the car skids.

Once you had gotten the fridge moving at constant speed, you found that you needed to exert less force on it. Since zero total force is needed to make an object move with constant velocity, the floor's rightward frictional force on the fridge has apparently decreased somewhat, making it easier for you to cancel it out. Our model also gives a plausible explanation for this fact: as the surfaces slide past each other, they don't have time to settle down and mesh with one another, so there is less friction.

Even though this model is intuitively appealing and fairly successful, it should not be taken too seriously, and in some situations it is misleading. For instance, fancy racing bikes these days are made with smooth tires that have no tread — contrary to what we'd expect from our model, this does not cause any decrease in friction. Machinists know that two very smooth and clean metal surfaces may stick to each other firmly and be very difficult to slide apart. This cannot be explained in our model, but makes more sense in terms of a model in which friction is described as arising from chemical bonds between the atoms of the two surfaces at their points of contact: very flat surfaces allow more atoms to come in contact.

Since friction changes its behavior dramatically once the surfaces come unstuck, we define two separate types of frictional forces. *Static friction* is friction that occurs between surfaces that are not slipping over each other. Slipping surfaces experience *kinetic friction.* The forces of static and kinetic friction, notated F_s and F_k, are always parallel to the surface of contact between the two objects.



m / Many landfowl, even those that are competent fliers, prefer to escape from a predator by running upward rather than by flying. This partridge is running up a vertical tree trunk. Humans can't walk up walls because there is no normal force and therefore no frictional force; when $F_N = 0$, the maximum force of static friction $F_{s,max} = \mu_s F_N$ is also zero. The partridge, however, has wings that it can flap in order to create a force between it and the air. Typically when a bird flaps its wings, the resulting force from the air is in the direction that would tend to lift the bird up. In this situation, however, the partridge changes its style of flapping so that the direction is reversed. The normal force between the feet and the tree allows a nonzero static frictional force. The mechanism is similar to that of a spoiler fin on a racing car. Some evolutionary biologists believe that when vertebrate flight first evolved, in dinosaurs, there was first a stage in which the wings were used only as an aid in running up steep inclines, and only later a transition to flight. (Redrawn from a figure by K.P. Dial.)

self-check B

1. When a baseball player slides in to a base, is the friction static, or kinetic?

2. A mattress stays on the roof of a slowly accelerating car. Is the friction static, or kinetic?

3. Does static friction create heat? Kinetic friction? ▷ Answer, p. 558

The maximum possible force of static friction depends on what kinds of surfaces they are, and also on how hard they are being pressed together. The approximate mathematical relationships can be expressed as follows:

$$F_{s,max} = \mu_s F_N,$$

where μ_s is a unitless number, called the coefficient of static friction, which depends on what kinds of surfaces they are. The maximum force that static friction can supply, $\mu_s F_N$, represents the boundary between static and kinetic friction. It depends on the normal force, which is numerically equal to whatever force is pressing the two surfaces together. In terms of our model, if the two surfaces are being pressed together more firmly, a greater sideways force will be required in order to make the irregularities in the surfaces ride up and over each other.

Note that just because we use an adjective such as "applied" to refer to a force, that doesn't mean that there is some special type of force called the "applied force." The applied force could be any type of force, or it could be the sum of more than one force trying to make an object move.

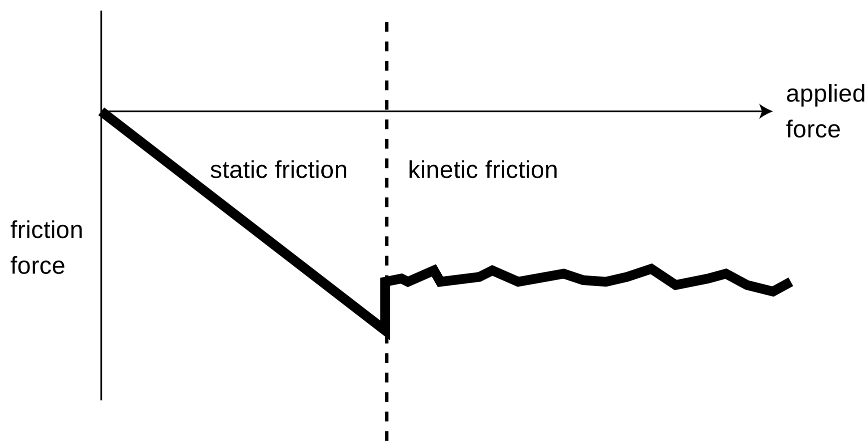
self-check C

The arrows in figure m show the forces of the tree trunk on the partridge. Describe the forces the bird makes on the tree. ▷ Answer, p. 558

The force of kinetic friction on each of the two objects is in the direction that resists the slippage of the surfaces. Its magnitude is usually well approximated as

$$F_k = \mu_k F_N$$

where μ_k is the coefficient of kinetic friction. Kinetic friction is usually more or less independent of velocity.



n / We choose a coordinate system in which the applied force, i.e., the force trying to move the objects, is positive. The friction force is then negative, since it is in the opposite direction. As you increase the applied force, the force of static friction increases to match it and cancel it out, until the maximum force of static friction is surpassed. The surfaces then begin slipping past each other, and the friction force becomes smaller in absolute value.

self-check D

Can a frictionless surface exert a normal force? Can a frictional force exist without a normal force? ▷ Answer, p. 559

If you try to accelerate or decelerate your car too quickly, the forces between your wheels and the road become too great, and they begin slipping. This is not good, because kinetic friction is weaker than static friction, resulting in less control. Also, if this occurs while you are turning, the car's handling changes abruptly because the kinetic friction force is in a different direction than the static friction force had been: contrary to the car's direction of motion, rather than contrary to the forces applied to the tire.

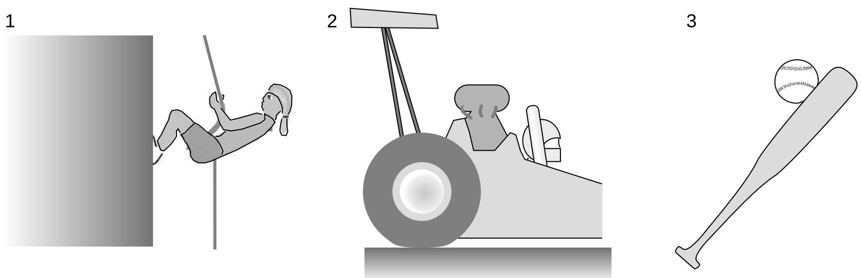
Most people respond with disbelief when told of the experimental evidence that both static and kinetic friction are approximately independent of the amount of surface area in contact. Even after doing a hands-on exercise with spring scales to show that it is true, many students are unwilling to believe their own observations, and insist that bigger tires "give more traction." In fact, the main reason why you would not want to put small tires on a big heavy car is that the tires would burst!

Although many people expect that friction would be proportional to surface area, such a proportionality would make predictions contrary to many everyday observations. A dog's feet, for example, have very little surface area in contact with the ground compared to a human's feet, and yet we know that a dog can often win a tug-of-war with a person.

The reason a smaller surface area does not lead to less friction is that the force between the two surfaces is more concentrated, causing their bumps and holes to dig into each other more deeply.

self-check E

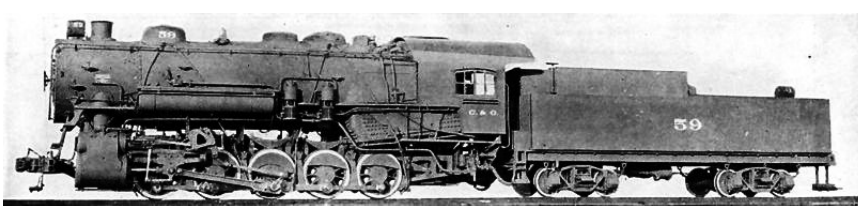
Find the direction of each of the forces in figure o. ▷ Answer, p. 559



o / 1. The cliff's normal force on the climber's feet. 2. The track's static frictional force on the wheel of the accelerating dragster. 3. The ball's normal force on the bat.

Locomotives *example 5*

Looking at a picture of a locomotive, p, we notice two obvious things that are different from an automobile. Where a car typically has two drive wheels, a locomotive normally has many — ten in this example. (Some also have smaller, unpowered wheels in front of and behind the drive wheels, but this example doesn't.) Also, cars these days are generally built to be as light as possible for their size, whereas locomotives are very massive, and no effort seems to be made to keep their weight low. (The steam locomotive in the photo is from about 1900, but this is true even for modern diesel and electric trains.)

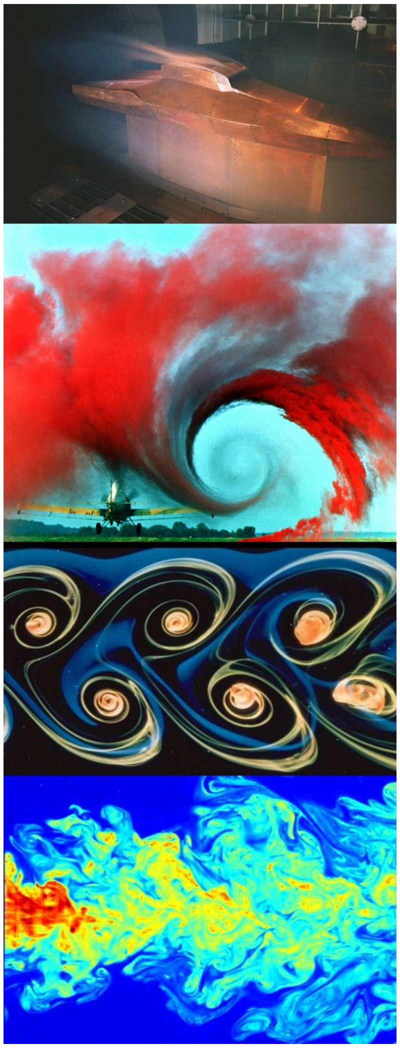


p / Example 5.

The reason locomotives are built to be so heavy is for traction. The upward normal force of the rails on the wheels, F_N, cancels the downward force of gravity, F_W, so ignoring plus and minus signs, these two forces are equal in absolute value, $F_N = F_W$. Given this amount of normal force, the maximum force of static friction is $F_s = \mu_s F_N = \mu_s F_W$. This static frictional force, of the rails pushing forward on the wheels, is the only force that can accelerate the train, pull it uphill, or cancel out the force of air resistance while cruising at constant speed. The coefficient of static friction for steel on steel is about 1/4, so no locomotive can pull with a force greater than about 1/4 of its own weight. If the engine is capable of supplying more than that amount of force, the result will be simply to break static friction and spin the wheels.

The reason this is all so different from the situation with a car is that a car isn't pulling something else. If you put extra weight in a car, you improve the traction, but you also increase the inertia of the car, and make it just as hard to accelerate. In a train, the inertia is almost all in the cars being pulled, not in the locomotive.

The other fact we have to explain is the large number of driving wheels. First, we have to realize that increasing the number of driving wheels neither increases nor decreases the total amount of static friction, because static friction is independent of the amount of surface area in contact. (The reason four-wheel-drive is good in a car is that if one or more of the wheels is slipping on ice or in mud, the other wheels may still have traction. This isn't typically an issue for a train, since all the wheels experience the same conditions.) The advantage of having more driving wheels on a train is that it allows us to increase the weight of the locomotive without crushing the rails, or damaging bridges.



q / Fluid friction depends on the fluid's pattern of flow, so it is more complicated than friction between solids, and there are no simple, universally applicable formulas to calculate it. From top to bottom: supersonic wind tunnel, vortex created by a crop duster, series of vortices created by a single object, turbulence.

Fluid friction

Try to drive a nail into a waterfall and you will be confronted with the main difference between solid friction and fluid friction. Fluid friction is purely kinetic; there is no static fluid friction. The nail in the waterfall may tend to get dragged along by the water flowing past it, but it does not stick in the water. The same is true for gases such as air: recall that we are using the word "fluid" to include both gases and liquids.

Unlike kinetic friction between solids, fluid friction increases rapidly with velocity. It also depends on the shape of the object, which is why a fighter jet is more streamlined than a Model T. For objects of the same shape but different sizes, fluid friction typically scales up with the cross-sectional area of the object, which is one of the main reasons that an SUV gets worse mileage on the freeway than a compact car.

Discussion questions

A A student states that when he tries to push his refrigerator, the reason it won't move is because Newton's third law says there's an equal and opposite frictional force pushing back. After all, the static friction force is equal and opposite to the applied force. How would you convince him he is wrong?

B Kinetic friction is usually more or less independent of velocity. However, inexperienced drivers tend to produce a jerk at the last moment of deceleration when they stop at a stop light. What does this tell you about the kinetic friction between the brake shoes and the brake drums?

C Some of the following are correct descriptions of types of forces that could be added on as new branches of the classification tree. Others are not really types of forces, and still others are not force phenomena at all. In each case, decide what's going on, and if appropriate, figure out how you would incorporate them into the tree.

sticky force	makes tape stick to things
opposite force	the force that Newton's third law says relates to every force you make
flowing force	the force that water carries with it as it flows out of a hose
surface tension	lets insects walk on water
horizontal force	a force that is horizontal
motor force	the force that a motor makes on the thing it is turning
canceled force	a force that is being canceled out by some other force

r / What do the golf ball and the shark have in common? Both use the same trick to reduce fluid friction. The dimples on the golf ball modify the pattern of flow of the air around it, counterintuitively *reducing* friction. Recent studies have shown that sharks can accomplish the same thing by raising, or "bristling," the scales on their skin at high speeds.

s / The wheelbases of the Hummer H3 and the Toyota Prius are surprisingly similar, differing by only 10%. The main difference in shape is that the Hummer is much taller and wider. It presents a much greater cross-sectional area to the wind, and this is the main reason that it uses about 2.5 times more gas on the freeway.

5.3 Analysis of forces

Newton's first and second laws deal with the total of all the forces exerted on a specific object, so it is very important to be able to figure out what forces there are. Once you have focused your attention on one object and listed the forces on it, it is also helpful to describe all the corresponding forces that must exist according to Newton's third law. We refer to this as "analyzing the forces" in which the object participates.

A barge *example 6*

A barge is being pulled to the right along a canal by teams of horses on the shores. Analyze all the forces in which the barge participates.

force acting on barge	*force related to it by Newton's third law*
ropes' normal forces on barge, →	barge's normal force on ropes, ←
water's fluid friction force on barge, ←	barge's fluid friction force on water, →
planet earth's gravitational force on barge, ↓	barge's gravitational force on earth, ↑
water's "floating" force on barge, ↑	barge's "floating" force on water, ↓

Here I've used the word "floating" force as an example of a sensible invented term for a type of force not classified on the tree on p. 172. A more formal technical term would be "hydrostatic force."

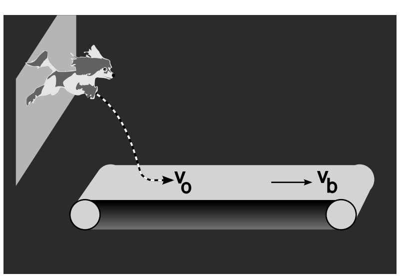
Note how the pairs of forces are all structured as "A's force on B, B's force on A": ropes on barge and barge on ropes; water on barge and barge on water. Because all the forces in the left column are forces acting on the barge, all the forces in the right column are forces being exerted by the barge, which is why each entry in the column begins with "barge."

Often you may be unsure whether you have forgotten one of the forces. Here are three strategies for checking your list:

1. See what physical result would come from the forces you've found so far. Suppose, for instance, that you'd forgotten the "floating" force on the barge in the example above. Looking at the forces you'd found, you would have found that there was a downward gravitational force on the barge which was not canceled by any upward force. The barge isn't supposed to sink, so you know you need to find a fourth, upward force.

2. Another technique for finding missing forces is simply to go through the list of all the common types of forces and see if any of them apply.

3. Make a drawing of the object, and draw a dashed boundary line around it that separates it from its environment. Look for points on the boundary where other objects come in contact with your object. This strategy guarantees that you'll find every contact force that acts on the object, although it won't help you to find non-contact forces.



t / Example 7.

Fifi *example 7*

▷ Fifi is an industrial espionage dog who loves doing her job and looks great doing it. She leaps through a window and lands at initial horizontal speed v_o on a conveyor belt which is itself moving at the greater speed v_b. Unfortunately the coefficient of kinetic friction μ_k between her foot-pads and the belt is fairly low, so she skids for a time Δt, during which the effect on her coiffure is *un désastre*. Find Δt.

▷ We analyze the forces:

force acting on Fifi	*force related to it by Newton's third law*
planet earth's gravitational force $F_W = mg$ on Fifi, ↓	Fifi's gravitational force on earth, ↑
belt's kinetic frictional force F_k on Fifi, →	Fifi's kinetic frictional force on belt, ←
belt's normal force F_N on Fifi, ↑	Fifi's normal force on belt, ↓

Checking the analysis of the forces as described on p. 180:

(1) The physical result makes sense. The left-hand column consists of forces ↓→↑. We're describing the time when she's moving horizontally on the belt, so it makes sense that we have two vertical forces that could cancel. The rightward force is what will accelerate her until her speed matches that of the belt.

(2) We've included every relevant type of force from the tree on p. 172.

(3) We've included forces from the belt, which is the only object in contact with Fifi.

The purpose of the analysis is to let us set up equations containing enough information to solve the problem. Using the generalization of Newton's second law given on p. 140, we use the horizontal force to determine the horizontal acceleration, and separately require the vertical forces to cancel out.

Let positive x be to the right. Newton's second law gives

$$(\rightarrow) \qquad a = F_k/m$$

Although it's the horizontal motion we care about, the only way to find F_k is via the relation $F_k = \mu_k F_N$, and the only way to find F_N is from the ↑↓ forces. The two vertical forces must cancel, which means they have to be of equal strength:

$$(\uparrow\downarrow) \qquad F_N - mg = 0.$$

Using the constant-acceleration equation $a = \Delta v/\Delta t$, we have

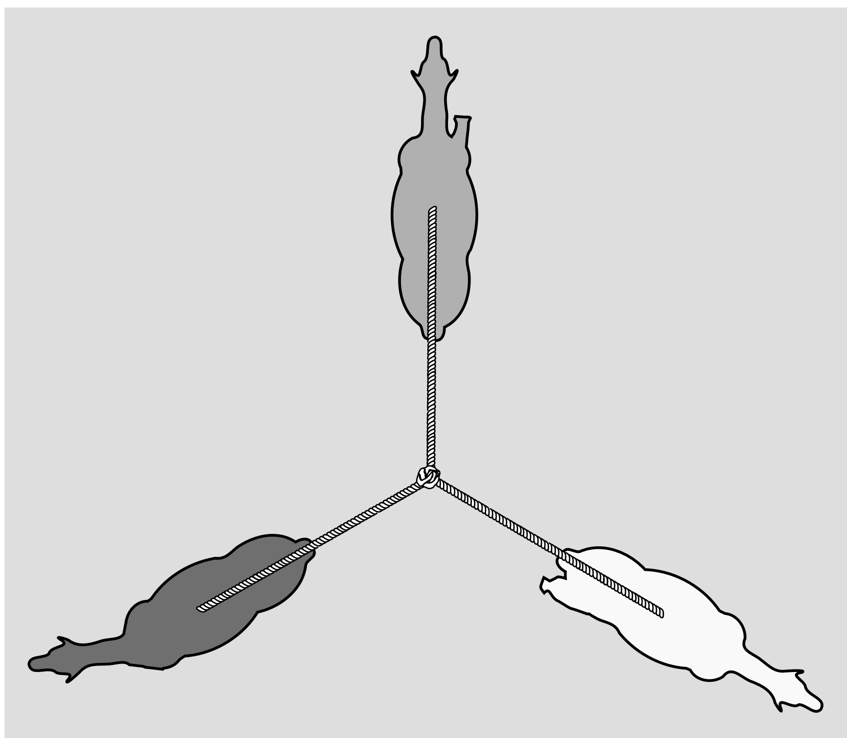
$$\begin{aligned}\Delta t &= \frac{\Delta v}{a} \\ &= \frac{v_b - v_o}{\mu_k mg/m} \\ &= \frac{v_b - v_o}{\mu_k g}.\end{aligned}$$

The units check out:

$$\text{s} = \frac{\text{m/s}}{\text{m/s}^2},$$

where μ_k is omitted as a factor because it's unitless.

We should also check that the dependence on the variables makes sense. If Fifi puts on her rubber ninja booties, increasing μ_k, then dividing by a larger number gives a smaller result for Δt; this makes sense physically, because the greater friction will cause her to come up to the belt's speed more quickly. The dependence on g is similar; more gravity would press her harder against the belt, improving her traction. Increasing v_b increases Δt, which makes sense because it will take her longer to get up to a bigger speed. Since v_o is subtracted, the dependence of Δt on it is the other way around, and that makes sense too, because if she can land with a greater speed, she has less speeding up left to do.



u / Example 8.

Forces don't have to be in pairs or at right angles *example 8*
In figure u, the three horses are arranged symmetrically at 120 degree intervals, and are all pulling on the central knot. Let's say the knot is at rest and at least momentarily in equilibrium. The analysis of forces on the knot is as follows.

force acting on knot	*force related to it by Newton's third law*
top rope's normal force on knot, ↑	knot's normal force on top rope, ↓
left rope's normal force on knot, ↙	knot's normal force on left rope, ↗
right rope's normal force on knot, ↘	knot's normal force on right rope, ↖

In our previous examples, the forces have all run along two perpendicular lines, and they often canceled in pairs. This example shows that neither of these always happens. Later in the book we'll see how to handle forces that are at arbitrary angles, using mathematical objects called vectors. But even without knowing about vectors, we already know what directions to draw the arrows in the table, since a rope can only pull parallel to itself at its ends. And furthermore, we can say something about the forces: by symmetry, we expect them all to be equal in strength. (If the knot was not in equilibrium, then this symmetry would be broken.)

This analysis also demonstrates that it's all right to leave out details if they aren't of interest and we don't intend to include them in our model. We called the forces normal forces, but we can't actually tell whether they are normal forces or frictional forces. They are probably some combination of those, but we don't include such details in this model, since aren't interested in describing the internal physics of the knot. This is an example of a more general fact about science, which is that science doesn't describe reality. It describes simplified *models* of reality, because reality is always too complex to model exactly.

Discussion questions

A In the example of the barge going down the canal, I referred to a "floating" or "hydrostatic" force that keeps the boat from sinking. If you were adding a new branch on the force-classification tree to represent this force, where would it go?

B The earth's gravitational force on you, i.e., your weight, is always equal to mg, where m is your mass. So why can you get a shovel to go deeper into the ground by jumping onto it? Just because you're jumping, that doesn't mean your mass or weight is any greater, does it?

5.4 Transmission of forces by low-mass objects

You're walking your dog. The dog wants to go faster than you do, and the leash is taut. Does Newton's third law guarantee that your force on your end of the leash is equal and opposite to the dog's force on its end? If they're not exactly equal, is there any reason why they should be approximately equal?

If there was no leash between you, and you were in direct contact with the dog, then Newton's third law would apply, but Newton's third law cannot relate your force on the leash to the dog's force on the leash, because that would involve three separate objects. Newton's third law only says that your force on the leash is equal and opposite to the leash's force on you,

$$F_{yL} = -F_{Ly},$$

and that the dog's force on the leash is equal and opposite to its force on the dog

$$F_{dL} = -F_{Ld}.$$

Still, we have a strong intuitive expectation that whatever force we make on our end of the leash is transmitted to the dog, and vice-versa. We can analyze the situation by concentrating on the forces that act on the leash, F_{dL} and F_{yL}. According to Newton's second law, these relate to the leash's mass and acceleration:

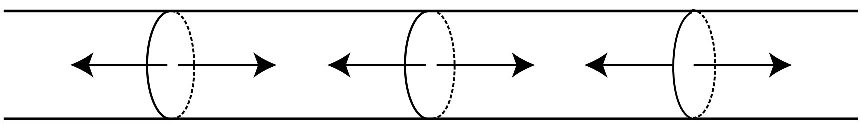
$$F_{dL} + F_{yL} = m_L a_L.$$

The leash is far less massive then any of the other objects involved, and if m_L is very small, then apparently the total force on the leash is also very small, $F_{dL} + F_{yL} \approx 0$, and therefore

$$F_{dL} \approx -F_{yL}.$$

Thus even though Newton's third law does not apply directly to these two forces, we can approximate the low-mass leash as if it was not intervening between you and the dog. It's at least approximately as if you and the dog were acting directly on each other, in which case Newton's third law would have applied.

In general, low-mass objects can be treated approximately as if they simply transmitted forces from one object to another. This can be true for strings, ropes, and cords, and also for rigid objects such as rods and sticks.

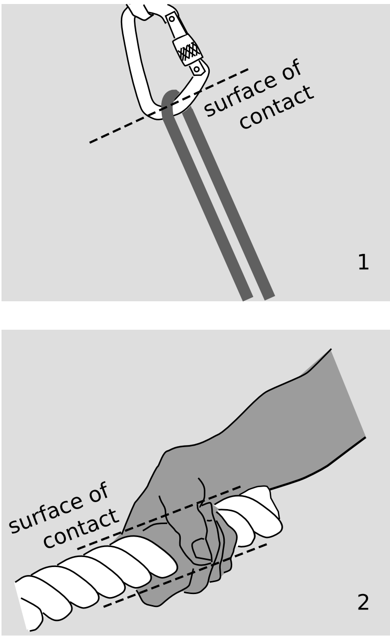


v / If we imagine dividing a taut rope up into small segments, then any segment has forces pulling outward on it at each end. If the rope is of negligible mass, then all the forces equal $+T$ or $-T$, where T, the tension, is a single number.

If you look at a piece of string under a magnifying glass as you pull on the ends more and more strongly, you will see the fibers straightening and becoming taut. Different parts of the string are apparently exerting forces on each other. For instance, if we think of the two halves of the string as two objects, then each half is exerting a force on the other half. If we imagine the string as consisting of many small parts, then each segment is transmitting a force to the next segment, and if the string has very little mass, then all the forces are equal in magnitude. We refer to the magnitude of the forces as the tension in the string, T.

The term "tension" refers only to internal forces within the string. If the string makes forces on objects at its ends, then those forces are typically normal or frictional forces (example 9).

w / The Golden Gate Bridge's roadway is held up by the tension in the vertical cables.



x / Example 9. The forces between the rope and other objects are normal and frictional forces.

Types of force made by ropes *example 9*

▷ Analyze the forces in figures x/1 and x/2.

▷ In all cases, a rope can only make "pulling" forces, i.e., forces that are parallel to its own length and that are toward itself, not away from itself. You can't push with a rope!

In x/1, the rope passes through a type of hook, called a carabiner, used in rock climbing and mountaineering. Since the rope can only pull along its own length, the direction of its force on the carabiner must be down and to the right. This is perpendicular to the surface of contact, so the force is a normal force.

force acting on carabiner	*force related to it by Newton's third law*
rope's normal force on carabiner ↘	carabiner's normal force on rope ↖

(There are presumably other forces acting on the carabiner from other hardware above it.)

In figure x/2, the rope can only exert a net force at its end that is parallel to itself and in the pulling direction, so its force on the hand is down and to the left. This is parallel to the surface of contact, so it must be a frictional force. If the rope isn't slipping through the hand, we have static friction. Friction can't exist without normal forces. These forces are perpendicular to the surface of contact. For simplicity, we show only two pairs of these normal forces, as if the hand were a pair of pliers.

force acting on person	*force related to it by Newton's third law*
rope's static frictional force on person ↙	person's static frictional force on rope ↗
rope's normal force on person ↖	person's normal force on rope ↘
rope's normal force on person ↘	person's normal force on rope ↖

(There are presumably other forces acting on the person as well, such as gravity.)

If a rope goes over a pulley or around some other object, then the tension throughout the rope is approximately equal so long as the pulley has negligible mass and there is not too much friction. A rod or stick can be treated in much the same way as a string, but it is possible to have either compression or tension.

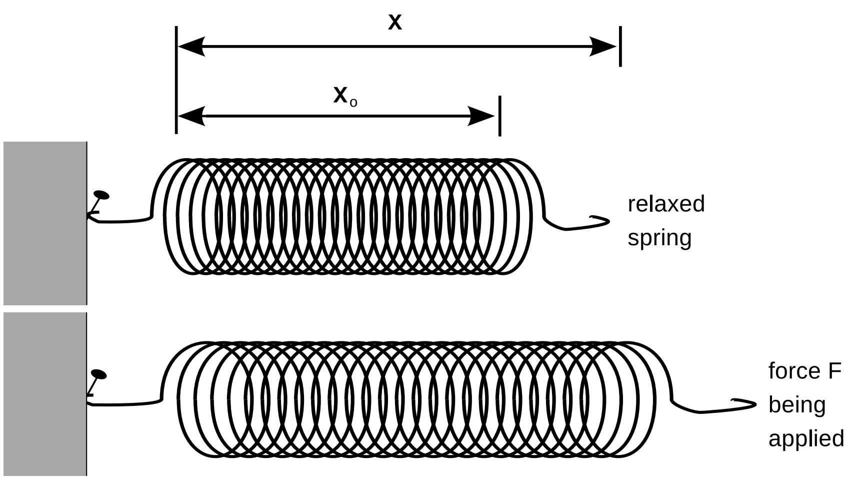
Discussion question

A When you step on the gas pedal, is your foot's force being transmitted in the sense of the word used in this section?

5.5 Objects under strain

A string lengthens slightly when you stretch it. Similarly, we have already discussed how an apparently rigid object such as a wall is actually flexing when it participates in a normal force. In other cases, the effect is more obvious. A spring or a rubber band visibly elongates when stretched.

Common to all these examples is a change in shape of some kind: lengthening, bending, compressing, etc. The change in shape can be measured by picking some part of the object and measuring its position, x. For concreteness, let's imagine a spring with one end attached to a wall. When no force is exerted, the unfixed end of the spring is at some position x_o. If a force acts at the unfixed end, its position will change to some new value of x. The more force, the greater the departure of x from x_o.



y / Defining the quantities F, x, and x_o in Hooke's law.

Back in Newton's time, experiments like this were considered cutting-edge research, and his contemporary Hooke is remembered today for doing them and for coming up with a simple mathematical generalization called Hooke's law:

$$F \approx k(x - x_o). \qquad \text{[force required to stretch a spring; valid for small forces only]}$$

Here k is a constant, called the spring constant, that depends on how stiff the object is. If too much force is applied, the spring exhibits more complicated behavior, so the equation is only a good approximation if the force is sufficiently small. Usually when the force is so large that Hooke's law is a bad approximation, the force ends up permanently bending or breaking the spring.

Although Hooke's law may seem like a piece of trivia about springs, it is actually far more important than that, because all solid objects exert Hooke's-law behavior over some range of sufficiently small forces. For example, if you push down on the hood of a car, it dips by an amount that is directly proportional to the force. (But the car's behavior would not be as mathematically simple if you dropped a boulder on the hood!)

▷ *Solved problem: Combining springs* *page 200, problem 26*

▷ *Solved problem: Young's modulus* *page 200, problem 28*

Discussion question

A A car is connected to its axles through big, stiff springs called shock absorbers, or "shocks." Although we've discussed Hooke's law above only in the case of stretching a spring, a car's shocks are continually going through both stretching and compression. In this situation, how would you interpret the positive and negative signs in Hooke's law?

5.6 Simple Machines: the pulley

Even the most complex machines, such as cars or pianos, are built out of certain basic units called *simple machines*. The following are some of the main functions of simple machines:

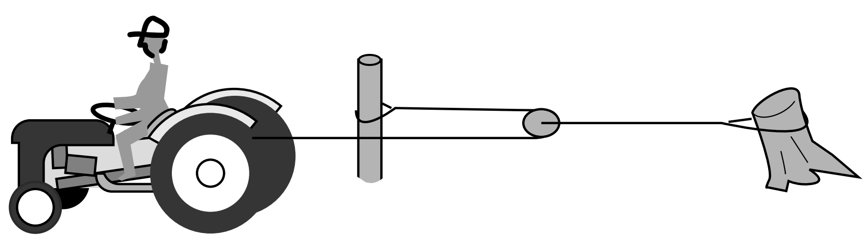
transmitting a force: The chain on a bicycle transmits a force from the crank set to the rear wheel.

changing the direction of a force: If you push down on a seesaw, the other end goes up.

changing the speed and precision of motion: When you make the "come here" motion, your biceps only moves a couple of centimeters where it attaches to your forearm, but your arm moves much farther and more rapidly.

changing the amount of force: A lever or pulley can be used to increase or decrease the amount of force.

You are now prepared to understand one-dimensional simple machines, of which the pulley is the main example.



z / Example 10.

A pulley *example 10*

▷ Farmer Bill says this pulley arrangement doubles the force of his tractor. Is he just a dumb hayseed, or does he know what he's doing?

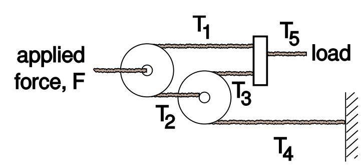
▷ To use Newton's first law, we need to pick an object and consider the sum of the forces on it. Since our goal is to relate the tension in the part of the cable attached to the stump to the tension in the part attached to the tractor, we should pick an object to which both those cables are attached, i.e., the pulley itself. The tension in a string or cable remains approximately constant as it passes around an idealized pulley. [1] There are therefore two leftward forces acting on the pulley, each equal to the force exerted by the tractor. Since the acceleration of the pulley is essentially zero, the forces on it must be canceling out, so the rightward force of the pulley-stump cable on the pulley must be double the force exerted by the tractor. Yes, Farmer Bill knows what he's talking about.

More complicated pulley systems can be constructed to give greater amplification of forces or to redirect forces in different directions. For an idealized system,[2] the fundamental principles are:

1. The total force acting on any pulley is zero.[3]
2. The tension in any given piece of rope is constant throughout its length.
3. The length of every piece of rope remains the same.

A compound pulley — *example 11*

▷ Find the mechanical advantage T_5/F of the pulley system. The bar is massless.



aa / Example 11.

▷ By rule 2, $T_1 = T_2$, and by rule 1, $F = T_1 + T_2$, so $T_1 = T_2 = F/2$. Similarly, $T_3 = T_4 = F/4$. Since the bar is massless, the same reasoning that led to rule 1 applies to the bar as well, and $T_5 = T_1 + T_3$. The mechanical advantage is $T_5/F = 3/4$, i.e., this pulley system *reduces* the input force.

How far does the tractor go compared to the stump? — *example 12*

▷ To move the stump in figure z by 1 cm, how far must the tractor move?

▷ Applying rule 3 to the the right-hand piece of rope, we find that the pulley moves 1 cm. The upper leg of the U-shaped rope therefore shortens by 1 cm, so the lower leg must lengthen by 1 cm. Since the pulley moves 1 cm to the left, and the lower leg extending from it also lengths by 1 cm, the tractor must move 2 cm.

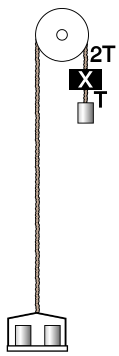
[1] This was asserted in section 5.4 without proof. Essentially it holds because of symmetry. E.g., if the U-shaped piece of rope in figure z had unequal tension in its two legs, then this would have to be caused by some asymmetry between clockwise and counterclockwise rotation. But such an asymmetry can only be caused by friction or inertia, which we assume don't exist.

[2] In such a system: (1) The ropes and pulleys have negligible mass. (2) Friction in the pulleys' bearings is negligible. (3) The ropes don't stretch.

[3] $F = ma$, and $m = 0$ since the pulley's mass is assumed to be negligible.

Examples 10 and 12 showed that the pulley system in figure z amplifies the force by a factor of 2, but it reduces the motion by 1/2. This is an example of a more general inverse proportionality for all such systems. Superficially, it follows from rules 1-3 above. If, for example, we try to construct a pulley system that doubles the force while keeping the motion the same, we will find that the rules seem to mysteriously conspire against us, and every attempt ends in failure. We could in fact prove as a mathematical theorem that the inverse proportionality always holds if we assume these rules.

But these rules are only an idealized mathematical model of a specific type of simple machine. What about other machines built out of other parts such as levers, screws, or gears? Through trial and error we will find that the inverse proportionality holds for them as well, so there must be some more fundamental principles involved. These principles, which we won't discuss formally until ch. 11 and 13, are conservation of energy and the equation for mechanical work. Informally, imagine that we had a machine that violated this rule. We could then insert it into a setup like the one in figure ab. When we release the single weight at the top, it drops to the ground while lifting the pan, which holds double the weight, all the way to the top. This is the ultimate free lunch. Once the pair of weights is up at the top, we can use them to hoist four more, then 8, 16, and so on. This is known as a perpetual motion machine.



ab / The black box marked with an X is a machine that doubles force while leaving the amount of motion unchanged. If 1 cm of rope is pulled out through the input on the bottom at tension T, the amount of rope consumed at tension $2T$ on top is not 1/2 cm, as we would normally expect, but 1 cm. This machine is impossible.

If this seems to be too good to be true, it is. Just as small machines can be put together to make bigger ones, any machine can also be broken down into smaller and smaller ones. This process can be continued until we get down to the level of atoms. The law of conservation of energy essentially says that atoms don't act like perpetual motion machines, and therefore any machine built out of atoms also fails to be a perpetual motion machine.

5.7 ⋆ Does Newton's third law mean anything, and if so, is it true?

This section discusses Newton's third law in the same spirit as section 4.7 on the first and second laws.

Ernst Mach gave a cogent critique of the third laws's logical assumptions in his book *The Science of Mechanics*. The book is available online for free at `archive.org`, and is very readable. To understand Mach's criticism, consider the experiment illustrated in figure a on p. 166, in which a large magnet and a small magnet are found to exert equal forces on one another. I use this as a student lab, and I find that most students are surprised by the result. Nevertheless, the lab can be considered a swindle, for the following reason. If we wanted to, we could cut the large magnet apart into smaller pieces, each of which was the same size as the small magnet. In fact, the large magnets I use for this lab were constructed simply by taking six small ones, stacking them together, and wrapping them in plastic. To represent this symbolically, let the small magnet be [A] and the large one [BCDEFG]. Since A and B are identical, and they are oriented in the same way, it follows simply by symmetry that A's force on B and B's on A obey the third law. The same holds for A on C and C on A, and so on. Since Newton claims that forces combine by addition, it follows that the result of the experiment must be in accord with the third law, despite the superficial asymmetry.

ac / Ernst Mach (1838-1916) is mainly known for having proposed a radical extension of the principle of inertia to state that all motion, not just constant-velocity motion, was relative. His ideas strongly influenced Einstein. The Mach factor (used, e.g., when we describe a jet as traveling at "Mach 2") is named after him.

Now suppose that material objects 1 and 2 have the same chemical composition. By a similar argument it seems likely that F_{12} and F_{21} obey Newton's third law.

This argument shows how pointless it can be to attempt to test a scientific theory unless you have in your possession a sensible alternative theory that predicts something different. One could spend decades doing experiments of the kind described above without realizing that the tests were all trivially guaranteed to give null results, even if nature was really described by a theory that violated Newton's third law.

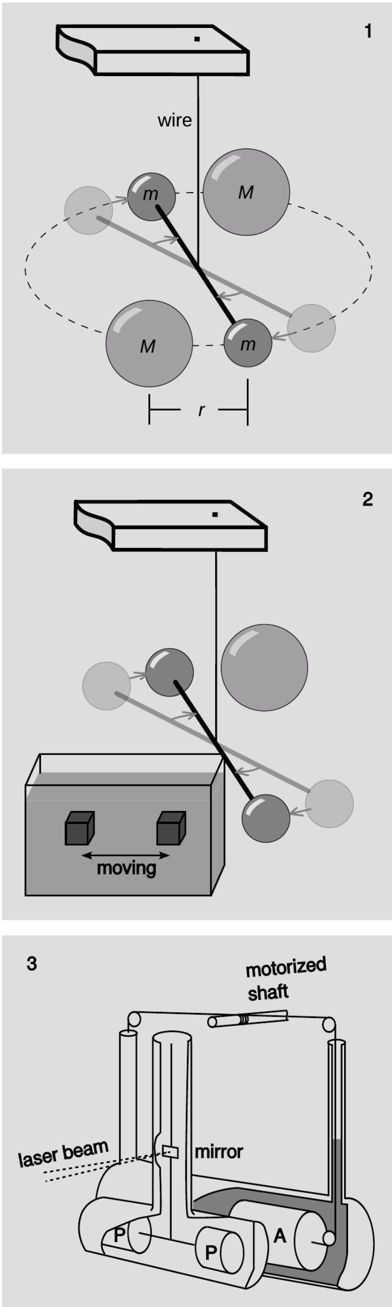
Here is an example of a fairly sane theory that could violate Newton's third law. Einstein's famous $E = mc^2$ states that a certain amount of energy E is equivalent to a certain amount of mass m, with c being the speed of light. (We won't formally encounter energy until ch. 11, or the reasons for $E = mc^2$ until section 12.5, but for now just think of energy as the kind of thing you intuitively associate with food calories or a tank full of gasoline, and take $E = mc^2$ for granted.) Einstein claimed that this would hold for three different kinds of mass: the mass measured by an object's inertia, the "active" gravitational mass m_a that determines the gravitational forces it makes on other objects, and the "passive" gravitational mass m_p

that measures how strongly it feels gravity. Einstein's reason for predicting similar behavior for m_a and m_p was that anything else would have violated Newton's third law for gravitational forces.

Suppose instead that an object's energy content contributes only to m_p, not to m_a. Atomic nuclei get something like 1% of their mass from the energy of the electric fields inside their nuclei, but this percentage varies with the number of protons, so if we have objects m and M with different chemical compositions, it follows that in this theory m_p/m_a will not be the same as M_p/M_a, and in this non-Einsteinian version of relativity, Newton's third law is violated.

This was tested in a Princeton PhD-thesis experiment by Kreuzer[4] in 1966. Kreuzer carried out an experiment, figure ad, using masses made of two different substances. The first substance was teflon. The second substance was a mixture of the liquids trichloroethylene and dibromoethane, with the proportions chosen so as to give a passive-mass density as close as possible to that of teflon, as determined by the neutral buoyancy of the teflon masses suspended inside the liquid. If the active-mass densities of these substances are not strictly proportional to their passive-mass densities, then moving the chunk of teflon back and forth in figure ad/2 would change the gravitational force acting on the nearby small sphere. No such change was observed, and the results verified $m_p/m_a = M_p/M_a$ to within one part in 10^6, in agreement with Einstein and Newton. If electrical energy had not contributed at all to active mass, then a violation of the third law would have been detected at the level of about one part in 10^2.

The Kreuzer result was improved in 1986 by Bartlett and van Buren[5] using lunar laser ranging data similar to those described in section 4.7. Since the moon has an asymmetrical distribution of iron and aluminum, a theory with $m_p/m_a \neq M_p/M_a$ would cause it to have an anomalous acceleration along a certain line. The lack of any such observed acceleration limits violations of Newton's third law to about one part in 10^{10}.



ad / 1. A balance that measures the gravitational attraction between masses M and m. (See section 10.5 for a more detailed description.) When the two masses M are inserted, the fiber twists. 2. A simplified diagram of Kreuzer's modification. The moving teflon mass is submerged in a liquid with nearly the same density. 3. Kreuzer's actual apparatus.

[4] Kreuzer, Phys. Rev. 169 (1968) 1007
[5] Phys. Rev. Lett. 57 (1986) 21

Summary

Selected vocabulary

repulsive	describes a force that tends to push the two participating objects apart
attractive	describes a force that tends to pull the two participating objects together
oblique	describes a force that acts at some other angle, one that is not a direct repulsion or attraction
normal force	the force that keeps two objects from occupying the same space
static friction	a friction force between surfaces that are not slipping past each other
kinetic friction	a friction force between surfaces that are slipping past each other
fluid	a gas or a liquid
fluid friction	a friction force in which at least one of the object is is a fluid
spring constant	the constant of proportionality between force and elongation of a spring or other object under strain

Notation

F_N	a normal force
F_s	a static frictional force
F_k	a kinetic frictional force
μ_s	the coefficient of static friction; the constant of proportionality between the maximum static frictional force and the normal force; depends on what types of surfaces are involved
μ_k	the coefficient of kinetic friction; the constant of proportionality between the kinetic frictional force and the normal force; depends on what types of surfaces are involved
k	the spring constant; the constant of proportionality between the force exerted on an object and the amount by which the object is lengthened or compressed

Summary

Newton's third law states that forces occur in equal and opposite pairs. If object A exerts a force on object B, then object B must simultaneously be exerting an equal and opposite force on object A. Each instance of Newton's third law involves exactly two objects, and exactly two forces, which are of the same type.

There are two systems for classifying forces. We are presently using the more practical but less fundamental one. In this system, forces are classified by whether they are repulsive, attractive, or oblique; whether they are contact or noncontact forces; and whether the two objects involved are solids or fluids.

Static friction adjusts itself to match the force that is trying to make the surfaces slide past each other, until the maximum value is reached,

$$F_{s,max} = \mu_s F_N.$$

Once this force is exceeded, the surfaces slip past one another, and kinetic friction applies,

$$F_k = \mu_k F_N.$$

Both types of frictional force are nearly independent of surface area, and kinetic friction is usually approximately independent of the speed at which the surfaces are slipping. The direction of the force is in the direction that would tend to stop or prevent slipping.

A good first step in applying Newton's laws of motion to any physical situation is to pick an object of interest, and then to list all the forces acting on that object. We classify each force by its type, and find its Newton's-third-law partner, which is exerted by the object on some other object.

When two objects are connected by a third low-mass object, their forces are transmitted to each other nearly unchanged.

Objects under strain always obey Hooke's law to a good approximation, as long as the force is small. Hooke's law states that the stretching or compression of the object is proportional to the force exerted on it,

$$F \approx k(x - x_\text{o}).$$

Problems

Key

√ A computerized answer check is available online.

∫ A problem that requires calculus.

⋆ A difficult problem.

1 In each case, identify the force that causes the acceleration, and give its Newton's-third-law partner. Describe the effect of the partner force. (a) A swimmer speeds up. (b) A golfer hits the ball off of the tee. (c) An archer fires an arrow. (d) A locomotive slows down. ▷ Solution, p. 548

2 Example 2 on page 169 involves a person pushing a box up a hill. The incorrect answer describes three forces. For each of these three forces, give the force that it is related to by Newton's third law, and state the type of force. ▷ Solution, p. 548

3 (a) Compare the mass of a one-liter water bottle on earth, on the moon, and in interstellar space. ▷ Solution, p. 548
(b) Do the same for its weight.

In problems 4-8, analyze the forces using a table in the format shown in section 5.3. Analyze the forces in which the italicized object participates.

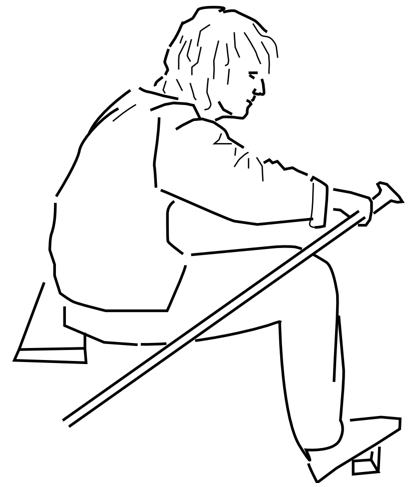
4 Some people put a spare car key in a little magnetic *box* that they stick under the chassis of their car. Let's say that the box is stuck directly underneath a horizontal surface, and the car is parked. (See instructions above.)

5 Analyze two examples of *objects* at rest relative to the earth that are being kept from falling by forces other than the normal force. Do not use objects in outer space, and do not duplicate problem 4 or 8. (See instructions above.)

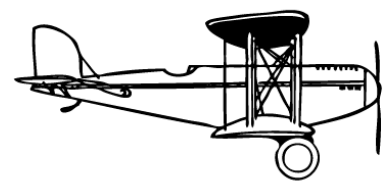
6 A *person* is rowing a boat, with her feet braced. She is doing the part of the stroke that propels the boat, with the ends of the oars in the water (not the part where the oars are out of the water). (See instructions above.)

7 A *farmer* is in a stall with a cow when the cow decides to press him against the wall, pinning him with his feet off the ground. Analyze the forces in which the farmer participates. (See instructions above.)

8 A propeller *plane* is cruising east at constant speed and altitude. (See instructions above.)

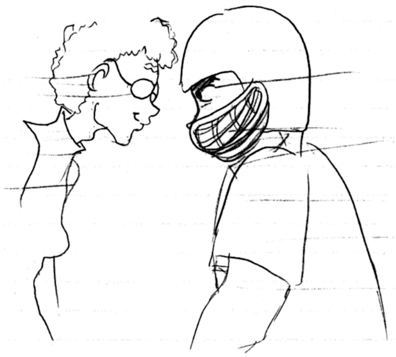


Problem 6.

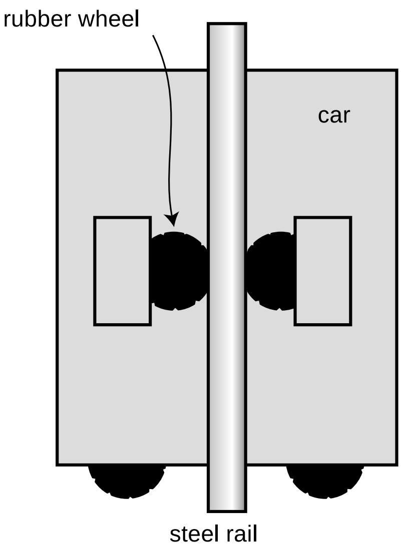


Problem 8.

Problem 7.



Problem 9.



Problem 12.

9 A little old lady and a pro football player collide head-on. Compare their forces on each other, and compare their accelerations. Explain.

10 The earth is attracted to an object with a force equal and opposite to the force of the earth on the object. If this is true, why is it that when you drop an object, the earth does not have an acceleration equal and opposite to that of the object?

11 When you stand still, there are two forces acting on you, the force of gravity (your weight) and the normal force of the floor pushing up on your feet. Are these forces equal and opposite? Does Newton's third law relate them to each other? Explain.

12 Today's tallest buildings are really not that much taller than the tallest buildings of the 1940's. One big problem with making an even taller skyscraper is that every elevator needs its own shaft running the whole height of the building. So many elevators are needed to serve the building's thousands of occupants that the elevator shafts start taking up too much of the space within the building. An alternative is to have elevators that can move both horizontally and vertically: with such a design, many elevator cars can share a few shafts, and they don't get in each other's way too much because they can detour around each other. In this design, it becomes impossible to hang the cars from cables, so they would instead have to ride on rails which they grab onto with wheels. Friction would keep them from slipping. The figure shows such a frictional elevator in its vertical travel mode. (The wheels on the bottom are for when it needs to switch to horizontal motion.)
(a) If the coefficient of static friction between rubber and steel is μ_s, and the maximum mass of the car plus its passengers is M, how much force must there be pressing each wheel against the rail in order to keep the car from slipping? (Assume the car is not accelerating.) ✓
(b) Show that your result has physically reasonable behavior with respect to μ_s. In other words, if there was less friction, would the wheels need to be pressed more firmly or less firmly? Does your equation behave that way?

13 An ice skater builds up some speed, and then coasts across the ice passively in a straight line. (a) Analyze the forces, using a table in the format shown in section 5.3.
(b) If his initial speed is v, and the coefficient of kinetic friction is μ_k, find the maximum theoretical distance he can glide before coming to a stop. Ignore air resistance. ✓
(c) Show that your answer to part b has the right units.
(d) Show that your answer to part b depends on the variables in a way that makes sense physically.
(e) Evaluate your answer numerically for $\mu_k = 0.0046$, and a world-record speed of 14.58 m/s. (The coefficient of friction was measured by De Koning et al., using special skates worn by real speed skaters.) ✓

(f) Comment on whether your answer in part e seems realistic. If it doesn't, suggest possible reasons why.

14 A cop investigating the scene of an accident measures the length L of a car's skid marks in order to find out its speed v at the beginning of the skid. Express v in terms of L and any other relevant variables. ✓

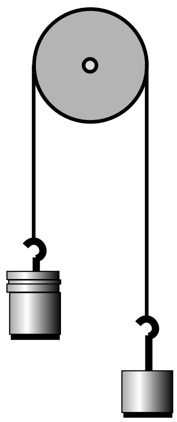
15 Someone tells you she knows of a certain type of Central American earthworm whose skin, when rubbed on polished diamond, has $\mu_k > \mu_s$. Why is this not just empirically unlikely but logically suspect?

16 When I cook rice, some of the dry grains always stick to the measuring cup. To get them out, I turn the measuring cup upside-down and hit the "roof" with my hand so that the grains come off of the "ceiling." (a) Explain why static friction is irrelevant here. (b) Explain why gravity is negligible. (c) Explain why hitting the cup works, and why its success depends on hitting the cup hard enough.

17 Pick up a heavy object such as a backpack or a chair, and stand on a bathroom scale. Shake the object up and down. What do you observe? Interpret your observations in terms of Newton's third law.

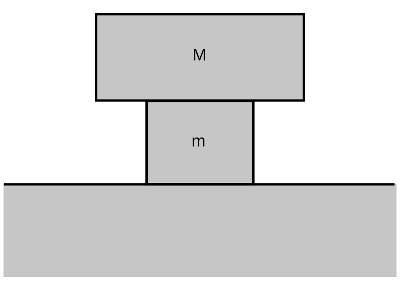
18 The following reasoning leads to an apparent paradox; explain what's wrong with the logic. A baseball player hits a ball. The ball and the bat spend a fraction of a second in contact. During that time they're moving together, so their accelerations must be equal. Newton's third law says that their forces on each other are also equal. But $a = F/m$, so how can this be, since their masses are unequal? (Note that the paradox isn't resolved by considering the force of the batter's hands on the bat. Not only is this force very small compared to the ball-bat force, but the batter could have just thrown the bat at the ball.)

19 A tugboat of mass m pulls a ship of mass M, accelerating it. The speeds are low enough that you can ignore fluid friction acting on their hulls, although there will of course need to be fluid friction acting on the tug's propellers.
(a) Analyze the forces in which the tugboat participates, using a table in the format shown in section 5.3. Don't worry about vertical forces.
(b) Do the same for the ship.
(c) If the force acting on the tug's propeller is F, what is the tension, T, in the cable connecting the two ships? [Hint: Write down two equations, one for Newton's second law applied to each object. Solve these for the two unknowns T and a.] ✓
(d) Interpret your answer in the special cases of $M = 0$ and $M = \infty$.



Problem 20.

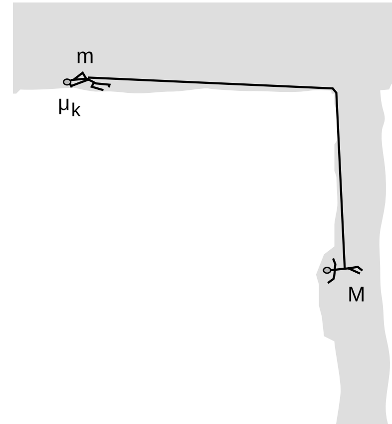
20 Unequal masses M and m are suspended from a pulley as shown in the figure.
(a) Analyze the forces in which mass m participates, using a table in the format shown in section 5.3. [The forces in which the other mass participates will of course be similar, but not numerically the same.]
(b) Find the magnitude of the accelerations of the two masses. [Hints: (1) Pick a coordinate system, and use positive and negative signs consistently to indicate the directions of the forces and accelerations. (2) The two accelerations of the two masses have to be equal in magnitude but of opposite signs, since one side eats up rope at the same rate at which the other side pays it out. (3) You need to apply Newton's second law twice, once to each mass, and then solve the two equations for the unknowns: the acceleration, a, and the tension in the rope, T.] √
(c) Many people expect that in the special case of $M = m$, the two masses will naturally settle down to an equilibrium position side by side. Based on your answer from part b, is this correct?
(d) Find the tension in the rope, T. √
(e) Interpret your equation from part d in the special case where one of the masses is zero. Here "interpret" means to figure out what happens mathematically, figure out what should happen physically, and connect the two.



Problem 21

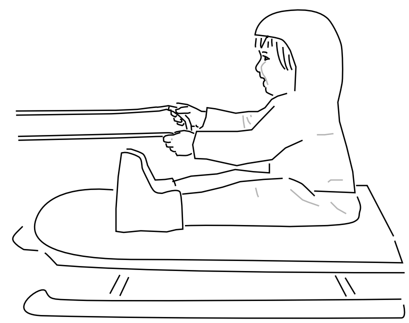
21 The figure shows a stack of two blocks, sitting on top of a table that is bolted to the floor. All three objects are made from identical wood, with their surfaces finished identically using the same sandpaper. We tap the middle block, giving it an initial velocity v to the right. The tap is executed so rapidly that almost no initial velocity is imparted to the top block.
(a) Find the time that will elapse until the slipping between the top and middle blocks stops. Express your answer in terms of v, m, M, g, and the relevant coefficient of friction. √
(b) Show that your answer makes sense in terms of units.
(c) Check that your result has the correct behavior when you make m bigger or smaller. Explain. This means that you should discuss the mathematical behavior of the result, and then explain how this corresponds to what would really happen physically.
(d) Similarly, discuss what happens when you make M bigger or smaller.
(e) Similarly, discuss what happens when you make g bigger or smaller.

22 Mountain climbers with masses m and M are roped together while crossing a horizontal glacier when a vertical crevasse opens up under the climber with mass M. The climber with mass m drops down on the snow and tries to stop by digging into the snow with the pick of an ice ax. Alas, this story does not have a happy ending, because this doesn't provide enough friction to stop. Both m and M continue accelerating, with M dropping down into the crevasse and m being dragged across the snow, slowed only by the kinetic friction with coefficient μ_k acting between the ax and the snow. There is no significant friction between the rope and the lip of the crevasse.
(a) Find the acceleration a. ✓
(b) Check the units of your result.
(c) Check the dependence of your equation on the variables. That means that for each variable, you should determine what its effect on a should be physically, and then what your answer from part a says its effect would be mathematically.



Problem 22.

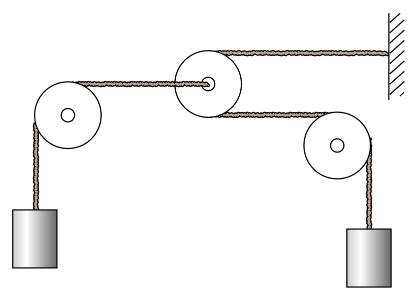
23 Ginny has a plan. She is going to ride her sled while her dog Foo pulls her, and she holds on to his leash. However, Ginny hasn't taken physics, so there may be a problem: she may slide right off the sled when Foo starts pulling.
(a) Analyze all the forces in which Ginny participates, making a table as in section 5.3.
(b) Analyze all the forces in which the sled participates.
(c) The sled has mass m, and Ginny has mass M. The coefficient of static friction between the sled and the snow is μ_1, and μ_2 is the corresponding quantity for static friction between the sled and her snow pants. Ginny must have a certain minimum mass so that she will not slip off the sled. Find this in terms of the other three variables. ✓
(d) Interpreting your equation from part c, under what conditions will there be no physically realistic solution for M? Discuss what this means physically.



Problem 23.

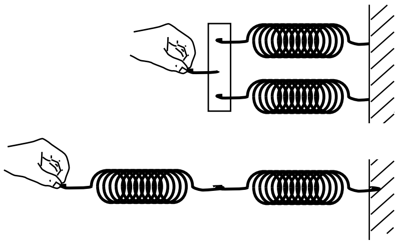
24 In the system shown in the figure, the pulleys on the left and right are fixed, but the pulley in the center can move to the left or right. The two masses are identical. Find the upward acceleration of the mass on the left, in terms of g only. Assume all the ropes and pulleys are massless and frictionless. Hints: (1) Use rules 1-3 on p. 189. (2) The approach is similar to the one in problem 20, but the ratio of the accelerations isn't 1:1.

✓



Problem 24.

25 Example 10 on page 188 describes a force-doubling setup involving a pulley. Make up a more complicated arrangement, using two pulleys, that would multiply the force by four. The basic idea is to take the output of one force doubler and feed it into the input of a second one.

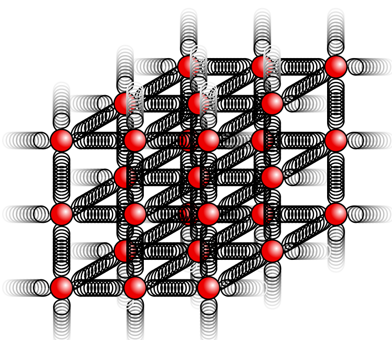


Problem 26.

26 The figure shows two different ways of combining a pair of identical springs, each with spring constant k. We refer to the top setup as parallel, and the bottom one as a series arrangement.
(a) For the parallel arrangement, analyze the forces acting on the connector piece on the left, and then use this analysis to determine the equivalent spring constant of the whole setup. Explain whether the combined spring constant should be interpreted as being stiffer or less stiff.
(b) For the series arrangement, analyze the forces acting on each spring and figure out the same things. ▷ Solution, p. 548

27 Generalize the results of problem 26 to the case where the two spring constants are unequal.

28 (a) Using the solution of problem 26, which is given in the back of the book, predict how the spring constant of a fiber will depend on its length and cross-sectional area.
(b) The constant of proportionality is called the Young's modulus, E, and typical values of the Young's modulus are about 10^{10} to 10^{11}. What units would the Young's modulus have in the SI (meter-kilogram-second) system? ▷ Solution, p. 549



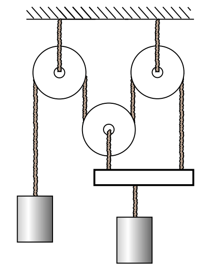
Problem 29.

29 This problem depends on the results of problems 26 and 28, whose solutions are in the back of the book. When atoms form chemical bonds, it makes sense to talk about the spring constant of the bond as a measure of how "stiff" it is. Of course, there aren't really little springs — this is just a mechanical model. The purpose of this problem is to estimate the spring constant, k, for a single bond in a typical piece of solid matter. Suppose we have a fiber, like a hair or a piece of fishing line, and imagine for simplicity that it is made of atoms of a single element stacked in a cubical manner, as shown in the figure, with a center-to-center spacing b. A typical value for b would be about 10^{-10} m.
(a) Find an equation for k in terms of b, and in terms of the Young's modulus, E, defined in problem 16 and its solution.
(b) Estimate k using the numerical data given in problem 28.
(c) Suppose you could grab one of the atoms in a diatomic molecule like H_2 or O_2, and let the other atom hang vertically below it. Does the bond stretch by any appreciable fraction due to gravity?

30 A cross-country skier is gliding on a level trail, with negligible friction. Then, when he is at position $x = 0$, the tip of his skis enters a patch of dirt. As he rides onto the dirt, more and more of his weight is being supported by the dirt. The skis have length ℓ, so if he reached $x = \ell$ without stopping, his weight would be completely on the dirt. This problem deals with the motion for $x < \ell$.
(a) Find the acceleration in terms of x, as well as any other relevant constants.
(b) This is a second-order differential equation. You should be able to find the solution simply by thinking about some commonly oc-

curing functions that you know about, and finding two that have the right properties. If these functions are $x = f(t)$ and $x = g(t)$, then the most general solution to the equations of motion will be of the form $x = af + bg$, where a and b are constants to be determined from the initial conditions.
(c) Suppose that the initial velocity v_o at $x = 0$ is such that he stops at $x < \ell$. Find the time until he stops, and show that, counterintuitively, this time is independent of v_o. Explain physically why this is true. √

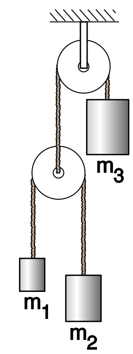
⋆



Problem 31.

31 The two masses are identical. Find the upward acceleration of the mass on the right, in terms of g only. Assume all the ropes and pulleys, as well as the cross-bar, are massless, and the pulleys are frictionless. The right-hand mass has been positioned away from the bar's center, so that the bar will not twist. Hints: (1) Use rules 1-3 on p. 189. (2) The approach is similar to the one in problem 20, but the ratio of the accelerations isn't 1:1.

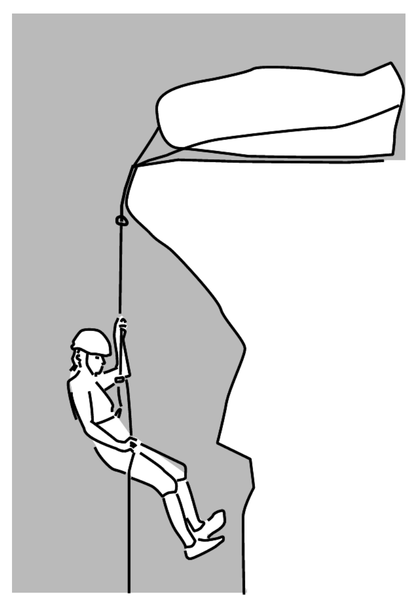
√



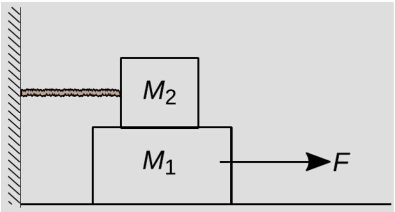
Problem 32.

32 Find the upward acceleration of mass m_1 in the figure. √ ⋆

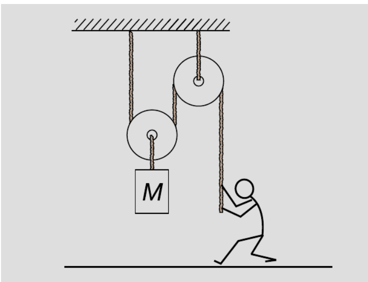
33 The figure shows a mountaineer doing a vertical rappel. Her anchor is a big boulder. The American Mountain Guides Association suggests as a rule of thumb that in this situation, the boulder should be at least as big as a refrigerator, and should be sitting on a surface that is horizontal rather than sloping. The goal of this problem is to estimate what coefficient of static friction μ_s between the boulder and the ledge is required if this setup is to hold the person's body weight. For comparison, reference books meant for civil engineers building walls out of granite blocks state that granite on granite typically has a $\mu_s \approx 0.6$. We expect the result of our calculation to be much less than this, both because a large margin of safety is desired and because the coefficient could be much lower if, for example, the surface was sandy rather than clean. We will assume that there is no friction where the rope goes over the lip of the cliff, although in reality this friction significantly reduces the load on the boulder.
(a) Let m be the mass of the climber, V the volume of the boulder, ρ its density, and g the strength of the gravitational field. Find the minimum value of μ_s. √
(b) Show that the units of your answer make sense.
(c) Check that its dependence on the variables makes sense.
(d) Evaluate your result numerically. The volume of my refrigerator is about 0.7 m^3, the density of granite is about 2.7 g/cm^3, and standards bodies use a body mass of 80 kg for testing climbing equipment. √



Problem 33.



Problem 35.



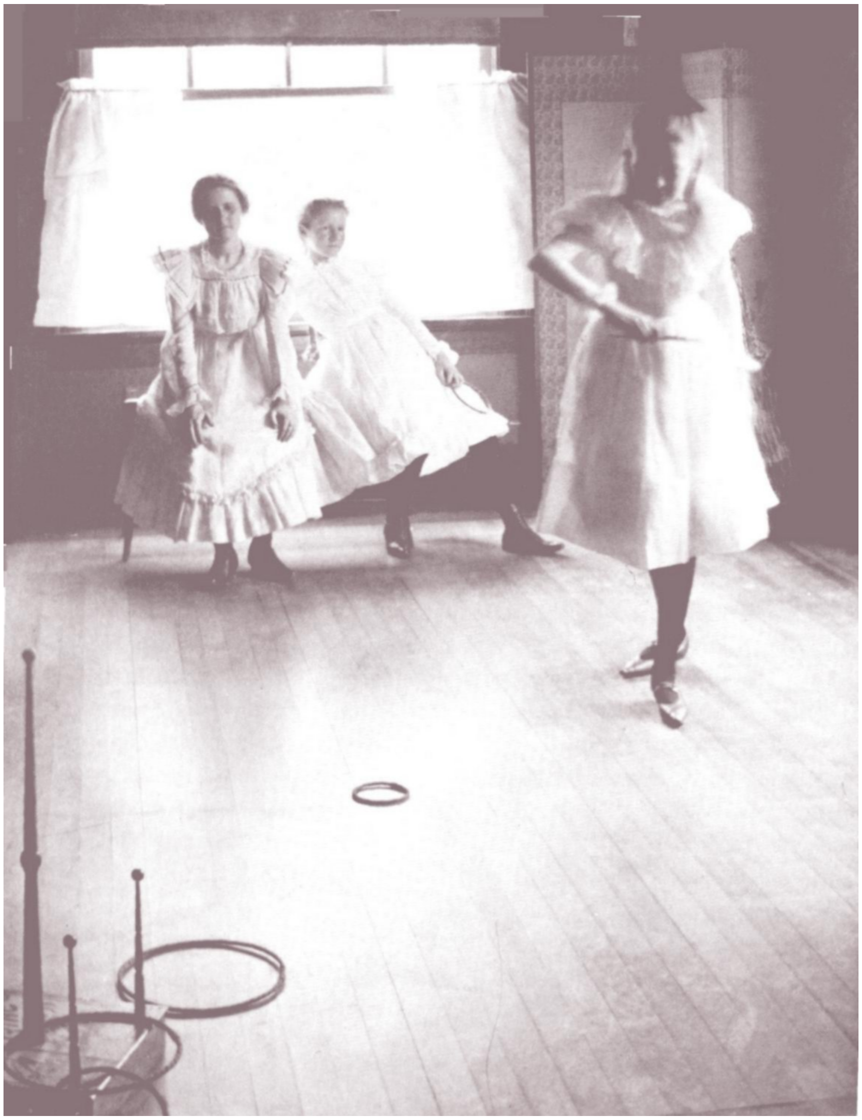
Problem 36.

34 A toy manufacturer is playtesting teflon booties that slip on over your shoes. In the parking lot, giggling engineers find that when they start with an initial speed of 1.2 m/s, they glide for 2.0 m before coming to a stop. What is the coefficient of friction between the asphalt and the booties? √ *[problem by B. Shotwell]*

35 Blocks M_1 and M_2 are stacked as shown, with M_2 on top. M_2 is connected by a string to the wall, and M_1 is pulled to the right with a force F big enough to get M_1 to move. The coefficient of kinetic friction has the same value μ_k among all surfaces (i.e., the block-block and ground-block interfaces).
(a) Analyze the forces in which each block participates, as in section 5.3.
(b) Determine the tension in the string. √
(c) Find the acceleration of the block of mass M_1.
√ *[problem by B. Shotwell]*

36 A person can pull with a maximum force F. What is the maximum mass that the person can lift with the pulley setup shown in the figure? √ *[problem by B. Shotwell]*

Motion in three dimensions

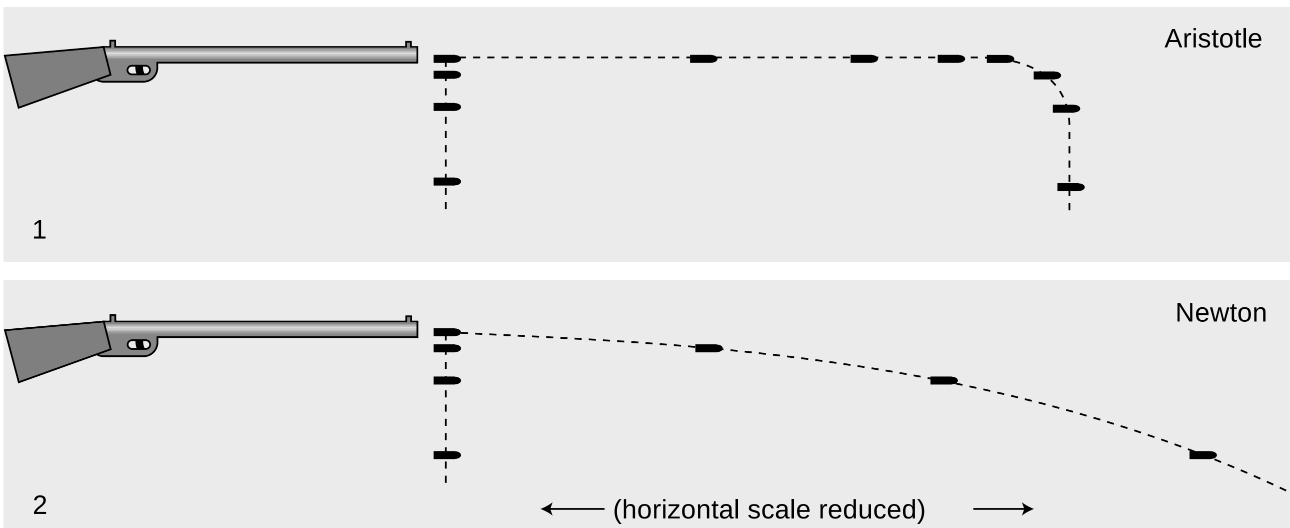


Chapter 6

Newton's laws in three dimensions

6.1 Forces have no perpendicular effects

Suppose you could shoot a rifle and arrange for a second bullet to be dropped from the same height at the exact moment when the first left the barrel. Which would hit the ground first? Nearly everyone expects that the dropped bullet will reach the dirt first, and Aristotle would have agreed. Aristotle would have described it like this. The shot bullet receives some forced motion from the gun. It travels forward for a split second, slowing down rapidly because there is no longer any force to make it continue in motion. Once it is done with its forced motion, it changes to natural motion, i.e. falling straight down. While the shot bullet is slowing down, the dropped bullet gets on with the business of falling, so according to Aristotle it will hit the ground first.



a / A bullet is shot from a gun, and another bullet is simultaneously dropped from the same height. 1. Aristotelian physics says that the horizontal motion of the shot bullet delays the onset of falling, so the dropped bullet hits the ground first. 2. Newtonian physics says the two bullets have the same vertical motion, regardless of their different horizontal motions.

Luckily, nature isn't as complicated as Aristotle thought! To convince yourself that Aristotle's ideas were wrong and needlessly complex, stand up now and try this experiment. Take your keys out of your pocket, and begin walking briskly forward. Without speeding up or slowing down, release your keys and let them fall while you continue walking at the same pace.

You have found that your keys hit the ground right next to your feet. Their horizontal motion never slowed down at all, and the whole time they were dropping, they were right next to you. The horizontal motion and the vertical motion happen at the same time, and they are independent of each other. Your experiment proves that the horizontal motion is unaffected by the vertical motion, but it's also true that the vertical motion is not changed in any way by the horizontal motion. The keys take exactly the same amount of time to get to the ground as they would have if you simply dropped them, and the same is true of the bullets: both bullets hit the ground simultaneously.

These have been our first examples of motion in more than one dimension, and they illustrate the most important new idea that is required to understand the three-dimensional generalization of Newtonian physics:

Forces have no perpendicular effects.
When a force acts on an object, it has no effect on the part of the object's motion that is perpendicular to the force.

In the examples above, the vertical force of gravity had no effect on the horizontal motions of the objects. These were examples of projectile motion, which interested people like Galileo because of its military applications. The principle is more general than that, however. For instance, if a rolling ball is initially heading straight for a wall, but a steady wind begins blowing from the side, the ball does not take any longer to get to the wall. In the case of projectile motion, the force involved is gravity, so we can say more specifically that the vertical acceleration is $9.8\ \text{m/s}^2$, regardless of the horizontal motion.

self-check A

In the example of the ball being blown sideways, why doesn't the ball take longer to get there, since it has to travel a greater distance? ▷ Answer, p. 559

Relationship to relative motion

These concepts are directly related to the idea that motion is relative. Galileo's opponents argued that the earth could not possibly be rotating as he claimed, because then if you jumped straight up in the air you wouldn't be able to come down in the same place. Their argument was based on their incorrect Aristotelian assumption that once the force of gravity began to act on you and bring you back down, your horizontal motion would stop. In the correct Newtonian theory, the earth's downward gravitational force is acting before, during, and after your jump, but has no effect on your motion in the perpendicular (horizontal) direction.

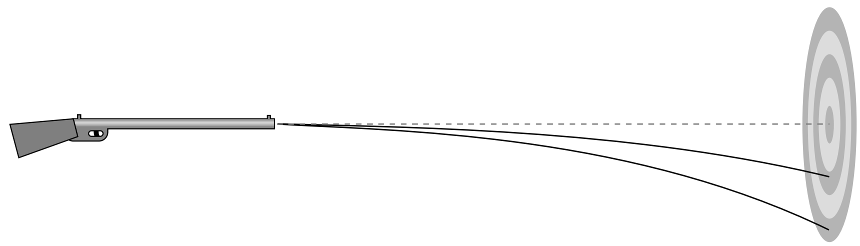
If Aristotle had been correct, then we would have a handy way to determine absolute motion and absolute rest: jump straight up in the air, and if you land back where you started, the surface from which you jumped must have been in a state of rest. In reality, this test gives the same result as long as the surface under you is an inertial frame. If you try this in a jet plane, you land back on the same spot on the deck from which you started, regardless of whether the plane is flying at 500 miles per hour or parked on the runway. The method would in fact only be good for detecting whether the plane was accelerating.

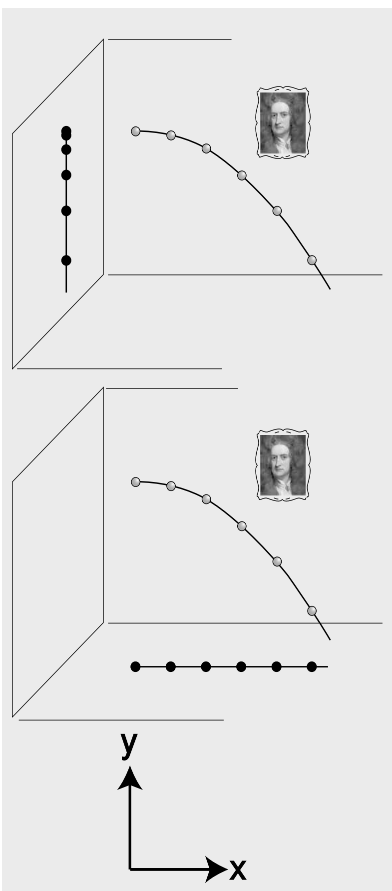
Discussion questions

A The following is an incorrect explanation of a fact about target shooting:

"Shooting a high-powered rifle with a high muzzle velocity is different from shooting a less powerful gun. With a less powerful gun, you have to aim quite a bit above your target, but with a more powerful one you don't have to aim so high because the bullet doesn't drop as fast."

Explain why it's incorrect. What is the correct explanation?





c / The shadow on the wall shows the ball's *y* motion, the shadow on the floor its *x* motion.

B You have thrown a rock, and it is flying through the air in an arc. If the earth's gravitational force on it is always straight down, why doesn't it just go straight down once it leaves your hand?

C Consider the example of the bullet that is dropped at the same moment another bullet is fired from a gun. What would the motion of the two bullets look like to a jet pilot flying alongside in the same direction as the shot bullet and at the same horizontal speed?

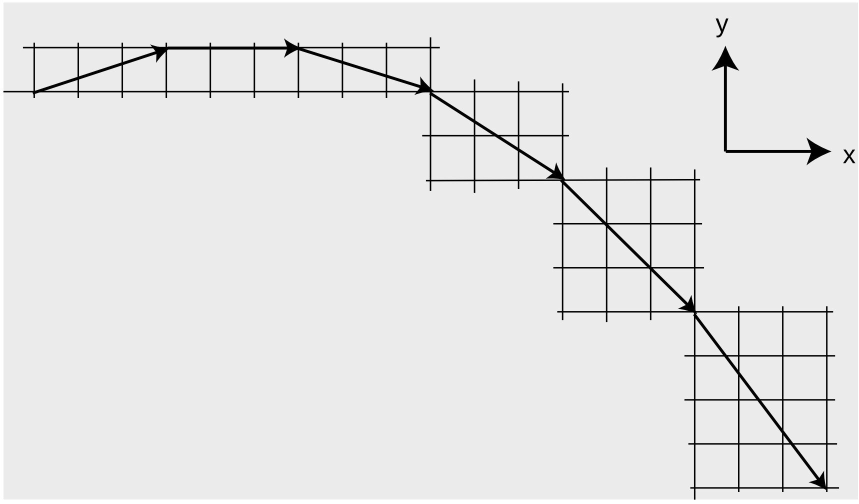
6.2 Coordinates and components

'Cause we're all
Bold as love,
Just ask the axis.

Jimi Hendrix

How do we convert these ideas into mathematics? Figure b shows a good way of connecting the intuitive ideas to the numbers. In one dimension, we impose a number line with an x coordinate on a certain stretch of space. In two dimensions, we imagine a grid of squares which we label with x and y values, as shown in figure b.

But of course motion doesn't really occur in a series of discrete hops like in chess or checkers. Figure c shows a way of conceptualizing the smooth variation of the x and y coordinates. The ball's shadow on the wall moves along a line, and we describe its position with a single coordinate, y, its height above the floor. The wall shadow has a constant acceleration of -9.8 m/s^2. A shadow on the floor, made by a second light source, also moves along a line, and we describe its motion with an x coordinate, measured from the wall.



b / This object experiences a force that pulls it down toward the bottom of the page. In each equal time interval, it moves three units to the right. At the same time, its vertical motion is making a simple pattern of +1, 0, −1, −2, −3, −4, ... units. Its motion can be described by an *x* coordinate that has zero acceleration and a *y* coordinate with constant acceleration. The arrows labeled *x* and *y* serve to explain that we are defining increasing *x* to the right and increasing *y* as upward.

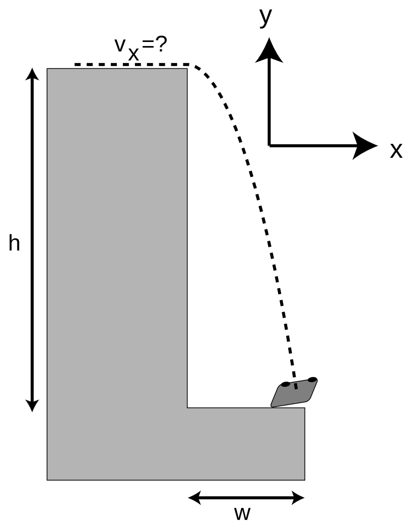
The velocity of the floor shadow is referred to as the x component of the velocity, written v_x. Similarly we can notate the acceleration of the floor shadow as a_x. Since v_x is constant, a_x is zero.

Similarly, the velocity of the wall shadow is called v_y, its acceleration a_y. This example has $a_y = -9.8\ \text{m/s}^2$.

Because the earth's gravitational force on the ball is acting along the y axis, we say that the force has a negative y component, F_y, but $F_x = F_z = 0$.

The general idea is that we imagine two observers, each of whom perceives the entire universe as if it was flattened down to a single line. The y-observer, for instance, perceives y, v_y, and a_y, and will infer that there is a force, F_y, acting downward on the ball. That is, a y component means the aspect of a physical phenomenon, such as velocity, acceleration, or force, that is observable to someone who can only see motion along the y axis.

All of this can easily be generalized to three dimensions. In the example above, there could be a z-observer who only sees motion toward or away from the back wall of the room.



d / Example 1.

A car going over a cliff — *example 1*

▷ The police find a car at a distance w = 20 m from the base of a cliff of height h = 100 m. How fast was the car going when it went over the edge? Solve the problem symbolically first, then plug in the numbers.

▷ Let's choose y pointing up and x pointing away from the cliff. The car's vertical motion was independent of its horizontal motion, so we know it had a constant vertical acceleration of $a = -g = -9.8\ \text{m}/s^2$. The time it spent in the air is therefore related to the vertical distance it fell by the constant-acceleration equation

$$\Delta y = \frac{1}{2}a_y\Delta t^2,$$

or

$$-h = \frac{1}{2}(-g)\Delta t^2.$$

Solving for Δt gives

$$\Delta t = \sqrt{\frac{2h}{g}}.$$

Since the vertical force had no effect on the car's horizontal motion, it had $a_x = 0$, i.e., constant horizontal velocity. We can apply the constant-velocity equation

$$v_x = \frac{\Delta x}{\Delta t},$$

i.e.,

$$v_x = \frac{w}{\Delta t}.$$

We now substitute for Δt to find

$$v_x = w/\sqrt{\frac{2h}{g}},$$

which simplifies to

$$v_x = w\sqrt{\frac{g}{2h}}.$$

Plugging in numbers, we find that the car's speed when it went over the edge was 4 m/s, or about 10 mi/hr.

Projectiles move along parabolas.

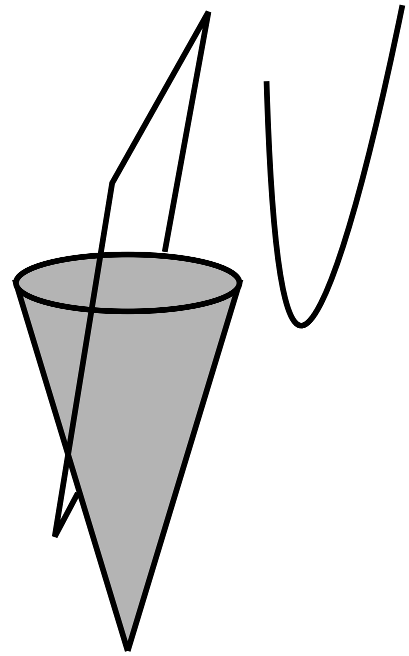
What type of mathematical curve does a projectile follow through space? To find out, we must relate x to y, eliminating t. The reasoning is very similar to that used in the example above. Arbitrarily choosing $x = y = t = 0$ to be at the top of the arc, we conveniently have $x = \Delta x$, $y = \Delta y$, and $t = \Delta t$, so

$$y = \frac{1}{2}a_y t^2 \qquad (a_y < 0)$$
$$x = v_x t$$

We solve the second equation for $t = x/v_x$ and eliminate t in the first equation:

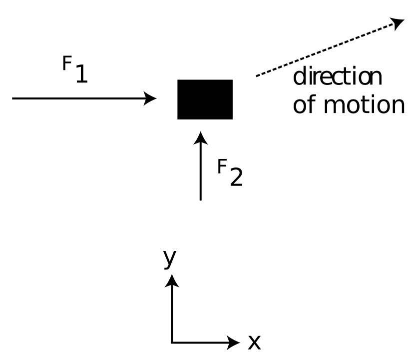
$$y = \frac{1}{2}a_y\left(\frac{x}{v_x}\right)^2 .$$

Since everything in this equation is a constant except for x and y, we conclude that y is proportional to the square of x. As you may or may not recall from a math class, $y \propto x^2$ describes a parabola.



e / A parabola can be defined as the shape made by cutting a cone parallel to its side. A parabola is also the graph of an equation of the form $y \propto x^2$.

f / Each water droplet follows a parabola. The faster drops' parabolas are bigger.



g / Example 2.

Discussion question

A At the beginning of this section I represented the motion of a projectile on graph paper, breaking its motion into equal time intervals. Suppose instead that there is no force on the object at all. It obeys Newton's first law and continues without changing its state of motion. What would the corresponding graph-paper diagram look like? If the time interval represented by each arrow was 1 second, how would you relate the graph-paper diagram to the velocity components v_x and v_y?

B Make up several different coordinate systems oriented in different ways, and describe the a_x and a_y of a falling object in each one.

6.3 Newton's laws in three dimensions

It is now fairly straightforward to extend Newton's laws to three dimensions:

Newton's first law
If all three components of the total force on an object are zero, then it will continue in the same state of motion.

Newton's second law
The components of an object's acceleration are predicted by the equations

$$a_x = F_{x,total}/m,$$
$$a_y = F_{y,total}/m, \qquad \text{and}$$
$$a_z = F_{z,total}/m.$$

Newton's third law
If two objects A and B interact via forces, then the components of their forces on each other are equal and opposite:

$$F_{\text{A on B},x} = -F_{\text{B on A},x},$$
$$F_{\text{A on B},y} = -F_{\text{B on A},y}, \qquad \text{and}$$
$$F_{\text{A on B},z} = -F_{\text{B on A},z}.$$

Forces in perpendicular directions on the same object *example 2*

▷ An object is initially at rest. Two constant forces begin acting on it, and continue acting on it for a while. As suggested by the two arrows, the forces are perpendicular, and the rightward force is stronger. What happens?

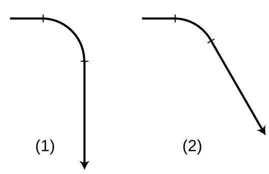
▷ Aristotle believed, and many students still do, that only one force can "give orders" to an object at one time. They therefore think that the object will begin speeding up and moving in the direction of the stronger force. In fact the object will move along a diagonal. In the example shown in the figure, the object will respond to the large rightward force with a large acceleration component to the right, and the small upward force will give it a small acceleration component upward. The stronger force does not overwhelm the weaker force, or have any effect on the upward motion at all. The force components simply add together:

$$F_{x,total} = F_{1,x} + \cancelto{0}{F_{2,x}}$$

$$F_{y,total} = \cancelto{0}{F_{1,y}} + F_{2,y}$$

Discussion question

A The figure shows two trajectories, made by splicing together lines and circular arcs, which are unphysical for an object that is only being acted on by gravity. Prove that they are impossible based on Newton's laws.



Summary

Selected vocabulary

component	the part of a velocity, acceleration, or force that would be perceptible to an observer who could only see the universe projected along a certain one-dimensional axis
parabola	the mathematical curve whose graph has y proportional to x^2

Notation

x, y, z	an object's positions along the x, y, and z axes
v_x, v_y, v_z	the x, y, and z components of an object's velocity; the rates of change of the object's x, y, and z coordinates
a_x, a_y, a_z	the x, y, and z components of an object's acceleration; the rates of change of v_x, v_y, and v_z

Summary

A force does not produce any effect on the motion of an object in a perpendicular direction. The most important application of this principle is that the horizontal motion of a projectile has zero acceleration, while the vertical motion has an acceleration equal to g. That is, an object's horizontal and vertical motions are independent. The arc of a projectile is a parabola.

Motion in three dimensions is measured using three coordinates, x, y, and z. Each of these coordinates has its own corresponding velocity and acceleration. We say that the velocity and acceleration both have x, y, and z components

Newton's second law is readily extended to three dimensions by rewriting it as three equations predicting the three components of the acceleration,

$$a_x = F_{x,total}/m,$$
$$a_y = F_{y,total}/m,$$
$$a_z = F_{z,total}/m,$$

and likewise for the first and third laws.

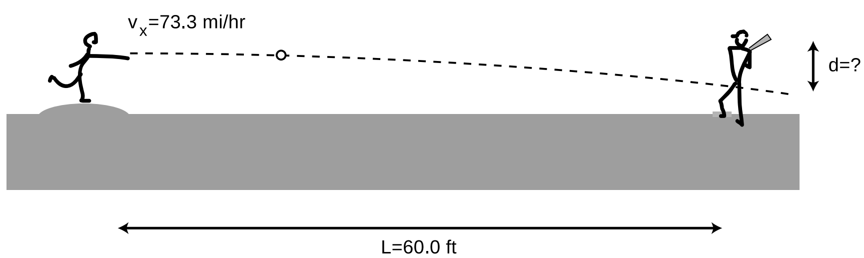
Problems

Key

- √ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ⋆ A difficult problem.

1 Two daredevils, Wendy and Bill, go over Niagara Falls. Wendy sits in an inner tube, and lets the 30 km/hr velocity of the river throw her out horizontally over the falls. Bill paddles a kayak, adding an extra 10 km/hr to his velocity. They go over the edge of the falls at the same moment, side by side. Ignore air friction. Explain your reasoning.
(a) Who hits the bottom first?
(b) What is the horizontal component of Wendy's velocity on impact?
(c) What is the horizontal component of Bill's velocity on impact?
(d) Who is going faster on impact?

2 At the 2010 Salinas Lettuce Festival Parade, the Lettuce Queen drops her bouquet while riding on a float moving toward the right. Sketch the shape of its trajectory in her frame of reference, and compare with the shape seen by one of her admirers standing on the sidewalk.



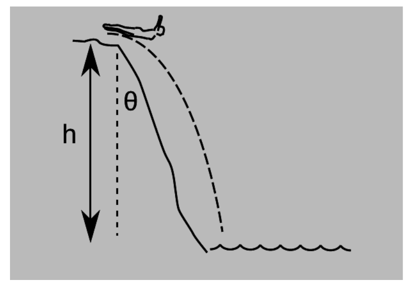
Problem 3.

3 A baseball pitcher throws a pitch clocked at $v_x = 73.3$ miles/hour. He throws horizontally. By what amount, d, does the ball drop by the time it reaches home plate, $L = 60.0$ feet away?
(a) First find a symbolic answer in terms of L, v_x, and g. √
(b) Plug in and find a numerical answer. Express your answer in units of ft. (Note: 1 foot=12 inches, 1 mile=5280 feet, and 1 inch=2.54 cm) √

4 Two cars go over the same speed bump in a parking lot, Maria's Maserati at 25 miles per hour and Park's Porsche at 37. How many times greater is the vertical acceleration of the Porsche? Hint: Remember that acceleration depends both on how much the velocity changes and on how much time it takes to change. √

5 A batter hits a baseball at speed v, at an angle θ above horizontal.
(a) Find an equation for the range (horizontal distance to where the ball falls), R, in terms of the relevant variables. Neglect air friction and the height of the ball above the ground when it is hit.
▷ Answer, p. 562
(b) Interpret your equation in the cases of θ=0 and $\theta = 90°$.
(c) Find the angle that gives the maximum range.
▷ Answer, p. 562

6 (a) A ball is thrown straight up with velocity v. Find an equation for the height to which it rises. ✓
(b) Generalize your equation for a ball thrown at an angle θ above horizontal, in which case its initial velocity components are $v_x = v\cos\theta$ and $v_y = v\sin\theta$. ✓

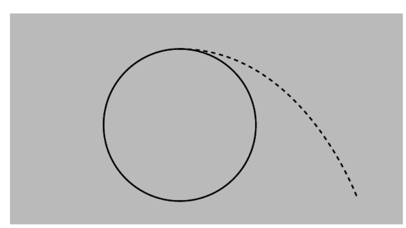


Problem 7.

7 You're running off a cliff into a pond. The cliff is $h = 5.0$ m above the water, but the cliff is not strictly vertical; it slopes down to the pond at an angle of $\theta = 20°$ with respect to the vertical. You want to find the minimum speed you need to jump off the cliff in order to land in the water.
(a) Find a symbolic answer in terms of h, θ, and g. ✓
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence on the variables g, h, and θ makes sense, and check the special cases $\theta = 0$ and $\theta = 90°$.
(d) Plug in numbers to find the numerical result. ✓
[problem by B. Shotwell]

8 Two footballs, one white and one green, are on the ground and kicked by two different footballers. The white ball, which is kicked straight upward with initial speed v_0, rises to height H. The green ball is hit with twice the initial speed but reaches the same height.
(a) What is the y-component of the green ball's initial velocity vector? Give your answer in terms of v_0 alone. ✓
(b) Which ball is in the air for a longer amount of time?
(c) What is the range of the green ball? Your answer should only depend on H. ✓ *[problem by B. Shotwell]*

9 You throw a rock horizontally from the edge of the roof of a building of height h with speed v_0.
(a) What is its speed when it hits the ground? ✓
(b) What is the (positive) angle between the final velocity vector and the horizontal when the rock hits the ground?
✓ *[problem by B. Shotwell]*


Problem 10.

10 The figure shows a vertical cross-section of a cylinder. A gun at the top shoots a bullet horizontally. What is the minimum speed at which the bullet must be shot in order to completely clear the cylinder? ⋆

a / Vectors are used in aerial navigation.

Chapter 7
Vectors

7.1 Vector notation

The idea of components freed us from the confines of one-dimensional physics, but the component notation can be unwieldy, since every one-dimensional equation has to be written as a set of three separate equations in the three-dimensional case. Newton was stuck with the component notation until the day he died, but eventually someone sufficiently lazy and clever figured out a way of abbreviating three equations as one.

(a)	$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}$	stands for	$F_{\text{A on B},x} = -F_{\text{B on A},x}$ $F_{\text{A on B},y} = -F_{\text{B on A},y}$ $F_{\text{A on B},z} = -F_{\text{B on A},z}$
(b)	$\vec{F}_{\text{total}} = \vec{F}_1 + \vec{F}_2 + \ldots$	stands for	$F_{\text{total},x} = F_{1,x} + F_{2,x} + \ldots$ $F_{\text{total},y} = F_{1,y} + F_{2,y} + \ldots$ $F_{\text{total},z} = F_{1,z} + F_{2,z} + \ldots$
(c)	$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$	stands for	$a_x = \Delta v_x/\Delta t$ $a_y = \Delta v_y/\Delta t$ $a_z = \Delta v_z/\Delta t$

Example (a) shows both ways of writing Newton's third law. Which would you rather write?

The idea is that each of the algebra symbols with an arrow written on top, called a vector, is actually an abbreviation for three different numbers, the x, y, and z components. The three components are referred to as the components of the vector, e.g., F_x is the x component of the vector $\vec{F}$. The notation with an arrow on top is good for handwritten equations, but is unattractive in a printed book, so books use boldface, **F**, to represent vectors. After this point, I'll use boldface for vectors throughout this book.

Quantities can be classified as vectors or scalars. In a phrase like "a _____ to the northeast," it makes sense to fill in the blank with "force" or "velocity," which are vectors, but not with "mass" or "time," which are scalars. Any nonzero vector has both a direction and an amount. The amount is called its magnitude. The notation for the magnitude of a vector **A** is $|\mathbf{A}|$, like the absolute value sign used with scalars.

Often, as in example (b), we wish to use the vector notation to represent adding up all the x components to get a total x component, etc. The plus sign is used between two vectors to indicate this type of component-by-component addition. Of course, vectors are really triplets of numbers, not numbers, so this is not the same as the use of the plus sign with individual numbers. But since we don't want to have to invent new words and symbols for this operation on vectors, we use the same old plus sign, and the same old addition-related words like "add," "sum," and "total." Combining vectors this way is called vector addition.

Similarly, the minus sign in example (a) was used to indicate negating each of the vector's three components individually. The equals sign is used to mean that all three components of the vector on the left side of an equation are the same as the corresponding components on the right.

Example (c) shows how we abuse the division symbol in a similar manner. When we write the vector $\Delta\mathbf{v}$ divided by the scalar Δt, we mean the new vector formed by dividing each one of the velocity components by Δt.

It's not hard to imagine a variety of operations that would combine vectors with vectors or vectors with scalars, but only four of them are required in order to express Newton's laws:

operation	definition
vector + **vector**	Add component by component to make a new set of three numbers.
vector − **vector**	Subtract component by component to make a new set of three numbers.
vector · scalar	Multiply each component of the vector by the scalar.
vector/scalar	Divide each component of the vector by the scalar.

As an example of an operation that is not useful for physics, there just aren't any useful physics applications for dividing a vector by another vector component by component. In optional section 7.5, we discuss in more detail the fundamental reasons why some vector operations are useful and others useless.

We can do algebra with vectors, or with a mixture of vectors and scalars in the same equation. Basically all the normal rules of algebra apply, but if you're not sure if a certain step is valid, you should simply translate it into three component-based equations and see if it works.

Order of addition *example 1*

▷ If we are adding two force vectors, **F** + **G**, is it valid to assume as in ordinary algebra that **F** + **G** is the same as **G** + **F**?

▷ To tell if this algebra rule also applies to vectors, we simply translate the vector notation into ordinary algebra notation. In terms of ordinary numbers, the components of the vector **F** + **G** would be $F_x + G_x$, $F_y + G_y$, and $F_z + G_z$, which are certainly the same three numbers as $G_x + F_x$, $G_y + F_y$, and $G_z + F_z$. Yes, **F** + **G** is the same as **G** + **F**.

It is useful to define a symbol **r** for the vector whose components are x, y, and z, and a symbol $\Delta\mathbf{r}$ made out of Δx, Δy, and Δz.

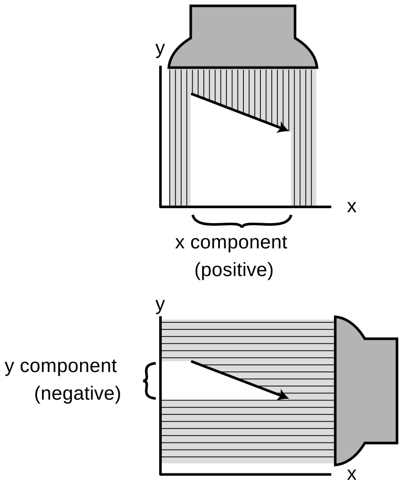
Although this may all seem a little formidable, keep in mind that it amounts to nothing more than a way of abbreviating equations! Also, to keep things from getting too confusing the remainder of this chapter focuses mainly on the $\Delta\mathbf{r}$ vector, which is relatively easy to visualize.

self-check A

Translate the equations $v_x = \Delta x/\Delta t$, $v_y = \Delta y/\Delta t$, and $v_z = \Delta z/\Delta t$ for motion with constant velocity into a single equation in vector notation.
▷ Answer, p. 559

Drawing vectors as arrows

A vector in two dimensions can be easily visualized by drawing an arrow whose length represents its magnitude and whose direction represents its direction. The x component of a vector can then be visualized as the length of the shadow it would cast in a beam of light projected onto the x axis, and similarly for the y component. Shadows with arrowheads pointing back against the direction of the positive axis correspond to negative components.

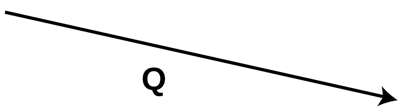


b / The x and y components of a vector can be thought of as the shadows it casts onto the x and y axes.

In this type of diagram, the negative of a vector is the vector with the same magnitude but in the opposite direction. Multiplying a vector by a scalar is represented by lengthening the arrow by that factor, and similarly for division.

self-check B

Given vector **Q** represented by an arrow in figure c, draw arrows representing the vectors 1.5**Q** and −**Q**. ▷ Answer, p. 559



c / Self-check B.

This leads to a way of defining vectors and scalars that reflects how physicists think in general about these things:

definition of vectors and scalars

A general type of measurement (force, velocity, ...) is a vector if it can be drawn as an arrow so that rotating the paper produces the same result as rotating the actual quantity. A type of quantity that never changes at all under rotation is a scalar.

For example, a force reverses itself under a 180-degree rotation, but a mass doesn't. We could have defined a vector as something that had both a magnitude and a direction, but that would have left out zero vectors, which don't have a direction. A zero vector is a legitimate vector, because it behaves the same way under rotations as a zero-length arrow, which is simply a dot.

d / A playing card returns to its original state when rotated by 180 degrees.

A remark for those who enjoy brain-teasers: not everything is a vector or a scalar. An American football is distorted compared to a sphere, and we can measure the orientation and amount of that distortion quantitatively. The distortion is not a vector, since a 180-degree rotation brings it back to its original state. Something similar happens with playing cards, figure d. For some subatomic particles, such as electrons, 360 degrees isn't even *enough*; a 720-degree rotation is needed to put them back the way they were!

Discussion questions

A You drive to your friend's house. How does the magnitude of your Δ**r** vector compare with the distance you've added to the car's odometer?

7.2 Calculations with magnitude and direction

If you ask someone where Las Vegas is compared to Los Angeles, they are unlikely to say that the Δx is 290 km and the Δy is 230 km, in a coordinate system where the positive x axis is east and the y axis points north. They will probably say instead that it's 370 km to the northeast. If they were being precise, they might give the direction as 38° counterclockwise from east. In two dimensions, we can always specify a vector's direction like this, using a single angle. A magnitude plus an angle suffice to specify everything about the vector. The following two examples show how we use trigonometry and the Pythagorean theorem to go back and forth between the $x-y$ and magnitude-angle descriptions of vectors.

Finding magnitude and angle from components *example 2*

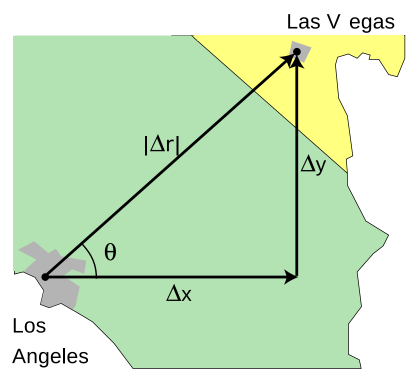
▷ Given that the $\Delta\mathbf{r}$ vector from LA to Las Vegas has Δx = 290 km and Δy = 230 km, how would we find the magnitude and direction of Δr?

▷ We find the magnitude of $\Delta\mathbf{r}$ from the Pythagorean theorem:

$$\begin{aligned}|\Delta\mathbf{r}| &= \sqrt{\Delta x^2 + \Delta y^2}\\ &= 370\text{ km}\end{aligned}$$

We know all three sides of the triangle, so the angle θ can be found using any of the inverse trig functions. For example, we know the opposite and adjacent sides, so

$$\begin{aligned}\theta &= \tan^{-1}\frac{\Delta y}{\Delta x}\\ &= 38°.\end{aligned}$$



e / Examples 2 and 3.

Finding components from magnitude and angle *example 3*

▷ Given that the straight-line distance from Los Angeles to Las Vegas is 370 km, and that the angle θ in the figure is 38°, how can the x and y components of the $\Delta\mathbf{r}$ vector be found?

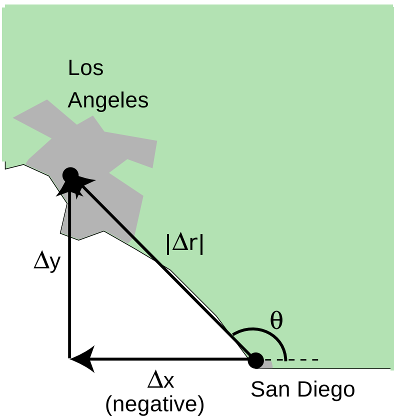
▷ The sine and cosine of θ relate the given information to the information we wish to find:

$$\begin{aligned}\cos\theta &= \frac{\Delta x}{|\Delta\mathbf{r}|}\\ \sin\theta &= \frac{\Delta y}{|\Delta\mathbf{r}|}\end{aligned}$$

Solving for the unknowns gives

$$\begin{aligned}\Delta x &= |\Delta\mathbf{r}|\cos\theta\\ &= 290\text{ km} \qquad \text{and}\\ \Delta y &= |\Delta\mathbf{r}|\sin\theta\\ &= 230\text{ km.}\end{aligned}$$

The following example shows the correct handling of the plus and minus signs, which is usually the main cause of mistakes.



f / Example 4.

Negative components *example 4*

▷ San Diego is 120 km east and 150 km south of Los Angeles. An airplane pilot is setting course from San Diego to Los Angeles. At what angle should she set her course, measured counterclockwise from east, as shown in the figure?

▷ If we make the traditional choice of coordinate axes, with x pointing to the right and y pointing up on the map, then her Δx is negative, because her final x value is less than her initial x value. Her Δy is positive, so we have

$$\Delta x = -120 \text{ km}$$
$$\Delta y = 150 \text{ km}.$$

If we work by analogy with example 2, we get

$$\begin{aligned}\theta &= \tan^{-1} \frac{\Delta y}{\Delta x} \\ &= \tan^{-1}(-1.25) \\ &= -51^\circ.\end{aligned}$$

According to the usual way of defining angles in trigonometry, a negative result means an angle that lies clockwise from the x axis, which would have her heading for the Baja California. What went wrong? The answer is that when you ask your calculator to take the arctangent of a number, there are always two valid possibilities differing by 180°. That is, there are two possible angles whose tangents equal -1.25:

$$\tan 129^\circ = -1.25$$
$$\tan -51^\circ = -1.25$$

You calculator doesn't know which is the correct one, so it just picks one. In this case, the one it picked was the wrong one, and it was up to you to add 180°to it to find the right answer.

A shortcut *example 5*

▷ A split second after nine o'clock, the hour hand on a clock dial has moved clockwise past the nine-o'clock position by some imperceptibly small angle ϕ. Let positive x be to the right and positive y up. If the hand, with length ℓ, is represented by a $\Delta\mathbf{r}$ vector going from the dial's center to the tip of the hand, find this vector's Δx.

▷ The following shortcut is the easiest way to work out examples like these, in which a vector's direction is known relative to one of the axes. We can tell that $\Delta\mathbf{r}$ will have a large, negative x component and a small, positive y. Since $\Delta x < 0$, there are really only two logical possibilities: either $\Delta x = -\ell\cos\phi$, or $\Delta x = -\ell\sin\phi$. Because ϕ is small, $\cos\phi$ is large and $\sin\phi$ is small. We conclude that $\Delta x = -\ell\cos\phi$.

A typical application of this technique to force vectors is given in example 7 on p. 241.

Discussion question

A In example 4, we dealt with *components* that were negative. Does it make sense to classify *vectors* as positive and negative?

7.3 Techniques for adding vectors

Vector addition is one of the three essential mathematical skills, summarized on pp.538-539, that you need for success in this course.

Addition of vectors given their components

The easiest type of vector addition is when you are in possession of the components, and want to find the components of their sum.

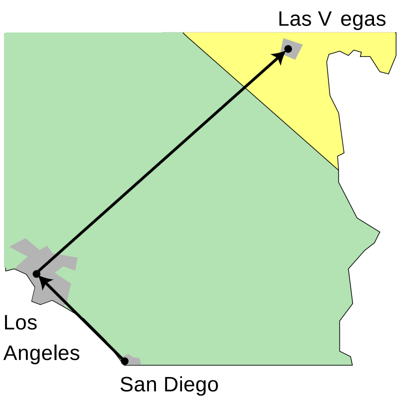
Adding components *example 6*

▷ Given the Δx and Δy values from the previous examples, find the Δx and Δy from San Diego to Las Vegas.

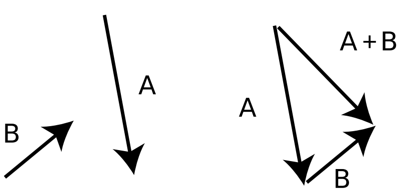
▷

$$\begin{aligned}\Delta x_{total} &= \Delta x_1 + \Delta x_2 \\ &= -120 \text{ km} + 290 \text{ km} \\ &= 170 \text{ km} \\ \Delta y_{total} &= \Delta y_1 + \Delta y_2 \\ &= 150 \text{ km} + 230 \text{ km} \\ &= 380\end{aligned}$$

Note how the signs of the x components take care of the westward and eastward motions, which partially cancel.



g / Example 6.



h / Vectors can be added graphically by placing them tip to tail, and then drawing a vector from the tail of the first vector to the tip of the second vector.

Addition of vectors given their magnitudes and directions

In this case, you must first translate the magnitudes and directions into components, and the add the components. In our San Diego-Los Angeles-Las Vegas example, we can simply string together the preceding examples; this is done on p. 539.

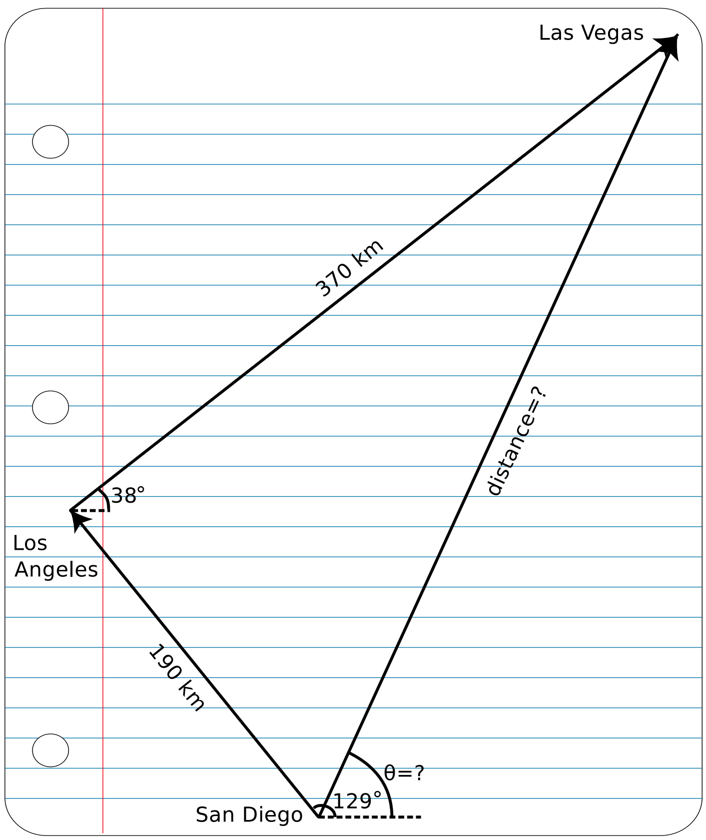
Graphical addition of vectors

Often the easiest way to add vectors is by making a scale drawing on a piece of paper. This is known as graphical addition, as opposed to the analytic techniques discussed previously. (It has nothing to do with $x - y$ graphs or graph paper. “Graphical” here simply means drawing. It comes from the Greek verb “*grapho*,” to write, like related English words including “graphic.”)

LA to Vegas, graphically *example 7*

▷ Given the magnitudes and angles of the $\Delta\mathbf{r}$ vectors from San Diego to Los Angeles and from Los Angeles to Las Vegas, find the magnitude and angle of the $\Delta\mathbf{r}$ vector from San Diego to Las Vegas.

▷ Using a protractor and a ruler, we make a careful scale drawing, as shown in figure i. The protractor can be conveniently aligned with the blue rules on the notebook paper. A scale of 1 mm $\rightarrow$ 2 km was chosen for this solution because it was as big as possible (for accuracy) without being so big that the drawing wouldn't fit on the page. With a ruler, we measure the distance from San Diego to Las Vegas to be 206 mm, which corresponds to 412 km. With a protractor, we measure the angle θ to be 65°.



i / Example 7.

Even when we don't intend to do an actual graphical calculation with a ruler and protractor, it can be convenient to diagram the addition of vectors in this way. With $\Delta\mathbf{r}$ vectors, it intuitively makes sense to lay the vectors tip-to-tail and draw the sum vector from the tail of the first vector to the tip of the second vector. We can do the same when adding other vectors such as force vectors.

self-check C

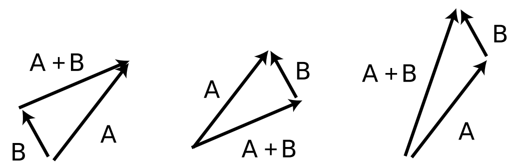
How would you subtract vectors graphically? ▷ Answer, p. 559

Discussion questions

A If you're doing *graphical* addition of vectors, does it matter which vector you start with and which vector you start from the other vector's tip?

B If you add a vector with magnitude 1 to a vector of magnitude 2, what magnitudes are possible for the vector sum?

C Which of these examples of vector addition are correct, and which are incorrect?



7.4 ⋆ Unit vector notation

When we want to specify a vector by its components, it can be cumbersome to have to write the algebra symbol for each component:

$$\Delta x = 290 \text{ km},\ \Delta y = 230 \text{ km}$$

A more compact notation is to write

$$\Delta \mathbf{r} = (290 \text{ km})\hat{\mathbf{x}} + (230 \text{ km})\hat{\mathbf{y}},$$

where the vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, called the unit vectors, are defined as the vectors that have magnitude equal to 1 and directions lying along the x, y, and z axes. In speech, they are referred to as "x-hat" and so on.

A slightly different, and harder to remember, version of this notation is unfortunately more prevalent. In this version, the unit vectors are called $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$:

$$\Delta \mathbf{r} = (290 \text{ km})\hat{\mathbf{i}} + (230 \text{ km})\hat{\mathbf{j}}.$$

7.5 ⋆ Rotational invariance

Let's take a closer look at why certain vector operations are useful and others are not. Consider the operation of multiplying two vectors component by component to produce a third vector:

$$R_x = P_x Q_x$$
$$R_y = P_y Q_y$$
$$R_z = P_z Q_z$$

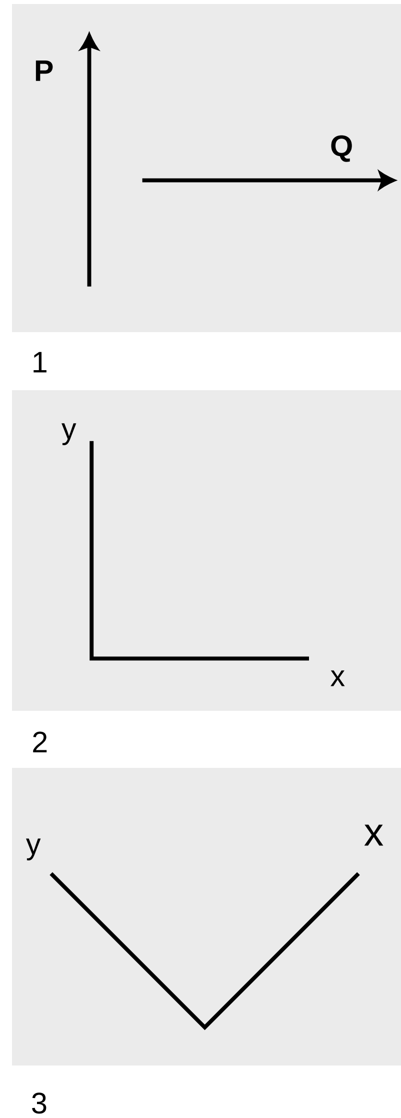
As a simple example, we choose vectors **P** and **Q** to have length 1, and make them perpendicular to each other, as shown in figure j/1. If we compute the result of our new vector operation using the coordinate system in j/2, we find:

$$R_x = 0$$
$$R_y = 0$$
$$R_z = 0.$$

The x component is zero because $P_x = 0$, the y component is zero because $Q_y = 0$, and the z component is of course zero because both vectors are in the $x - y$ plane. However, if we carry out the same operations in coordinate system j/3, rotated 45 degrees with respect to the previous one, we find

$$R_x = 1/2$$
$$R_y = -1/2$$
$$R_z = 0.$$

The operation's result depends on what coordinate system we use, and since the two versions of **R** have different lengths (one being zero and the other nonzero), they don't just represent the same answer expressed in two different coordinate systems. Such an operation will never be useful in physics, because experiments show physics works the same regardless of which way we orient the laboratory building! The *useful* vector operations, such as addition and scalar multiplication, are rotationally invariant, i.e., come out the same regardless of the orientation of the coordinate system.



j / Component-by-component multiplication of the vectors in 1 would produce different vectors in coordinate systems 2 and 3.

Calibrating an electronic compass *example 8*

Some smart phones and GPS units contain electronic compasses that can sense the direction of the earth's magnetic field vector, notated **B**. Because all vectors work according to the same rules, you don't need to know anything special about magnetism in order to understand this example. Unlike a traditional compass that uses a magnetized needle on a bearing, an electronic compass has no moving parts. It contains two sensors oriented perpendicular to one another, and each sensor is only sensitive to the component of the earth's field that lies along its own axis. Because a choice of coordinates is arbitrary, we can take one of these sensors as defining the x axis and the other the y. Given the two components B_x and B_y, the device's computer chip can compute the angle of magnetic north relative to its sensors, $\tan^{-1}(B_y/B_x)$.

All compasses are vulnerable to errors because of nearby magnetic materials, and in particular it may happen that some part of the compass's own housing becomes magnetized. In an electronic compass, rotational invariance provides a convenient way of calibrating away such effects by having the user rotate the device in a horizontal circle.

Suppose that when the compass is oriented in a certain way, it measures $B_x = 1.00$ and $B_y = 0.00$ (in certain units). We then expect that when it is rotated 90 degrees clockwise, the sensors will detect $B_x = 0.00$ and $B_y = 1.00$.

But imagine instead that we get $B_x = 0.20$ and $B_y = 0.80$. This would violate rotational invariance, since rotating the coordinate system is supposed to give a different description of the *same* vector. The magnitude appears to have changed from 1.00 to $\sqrt{0.20^2 + 0.80^2} = 0.82$, and a vector can't change its magnitude just because you rotate it. The compass's computer chip figures out that some effect, possibly a slight magnetization of its housing, must be adding an erroneous 0.2 units to all the B_x readings, because subtracting this amount from all the B_x values gives vectors that have the same magnitude, satisfying rotational invariance.

Summary

Selected vocabulary

vector . . . . a quantity that has both an amount (magnitude) and a direction in space
magnitude . . . . the "amount" associated with a vector
scalar . . . . a quantity that has no direction in space, only an amount

Notation

A . . . . a vector with components A_x, A_y, and A_z
$\overrightarrow{A}$ . . . . handwritten notation for a vector
$|\mathbf{A}|$ . . . . the magnitude of vector **A**
r . . . . the vector whose components are x, y, and z
$\Delta\mathbf{r}$ . . . . the vector whose components are Δx, Δy, and Δz
$\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ . . . . . . (optional topic) unit vectors; the vectors with magnitude 1 lying along the x, y, and z axes
$\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ . . . . a harder to remember notation for the unit vectors

Other terminology and notation

displacement vector . . . . a name for the symbol $\Delta\mathbf{r}$
speed . . . . the magnitude of the velocity vector, i.e., the velocity stripped of any information about its direction

Summary

A vector is a quantity that has both a magnitude (amount) and a direction in space, as opposed to a scalar, which has no direction. The vector notation amounts simply to an abbreviation for writing the vector's three components.

In two dimensions, a vector can be represented either by its two components or by its magnitude and direction. The two ways of describing a vector can be related by trigonometry.

The two main operations on vectors are addition of a vector to a vector, and multiplication of a vector by a scalar.

Vector addition means adding the components of two vectors to form the components of a new vector. In graphical terms, this corresponds to drawing the vectors as two arrows laid tip-to-tail and drawing the sum vector from the tail of the first vector to the tip of the second one. Vector subtraction is performed by negating the vector to be subtracted and then adding.

Multiplying a vector by a scalar means multiplying each of its components by the scalar to create a new vector. Division by a scalar is defined similarly.

Differentiation and integration of vectors is defined component by component.

Problems

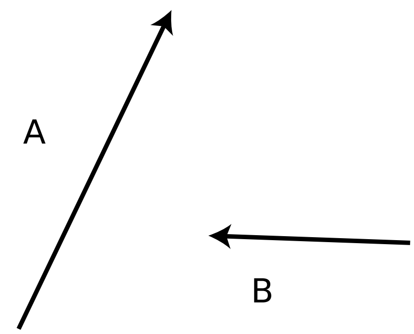
Key

- $\checkmark$ A computerized answer check is available online.
- $\int$ A problem that requires calculus.
- $\star$ A difficult problem.

1 The figure shows vectors **A** and **B**. Graphically calculate the following, as in figure h on p. 224, self-check C on p. 225, and self-check B on p. 220.

$\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, $\mathbf{B}-\mathbf{A}$, $-2\mathbf{B}$, $\mathbf{A}-2\mathbf{B}$

No numbers are involved.

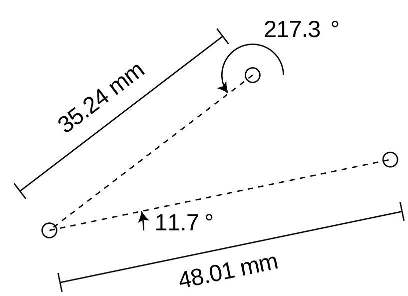


Problem 1.

2 Phnom Penh is 470 km east and 250 km south of Bangkok. Hanoi is 60 km east and 1030 km north of Phnom Penh.
(a) Choose a coordinate system, and translate these data into Δx and Δy values with the proper plus and minus signs.
(b) Find the components of the $\Delta\mathbf{r}$ vector pointing from Bangkok to Hanoi. $\checkmark$

3 If you walk 35 km at an angle 25° counterclockwise from east, and then 22 km at 230° counterclockwise from east, find the distance and direction from your starting point to your destination. $\checkmark$

4 A machinist is drilling holes in a piece of aluminum according to the plan shown in the figure. She starts with the top hole, then moves to the one on the left, and then to the one on the right. Since this is a high-precision job, she finishes by moving in the direction and at the angle that should take her back to the top hole, and checks that she ends up in the same place. What are the distance and direction from the right-hand hole to the top one? $\checkmark$



Problem 4.

5 Suppose someone proposes a new operation in which a vector **A** and a scalar B are added together to make a new vector **C** like this:

$$C_x = A_x + B$$
$$C_y = A_y + B$$
$$C_z = A_z + B$$

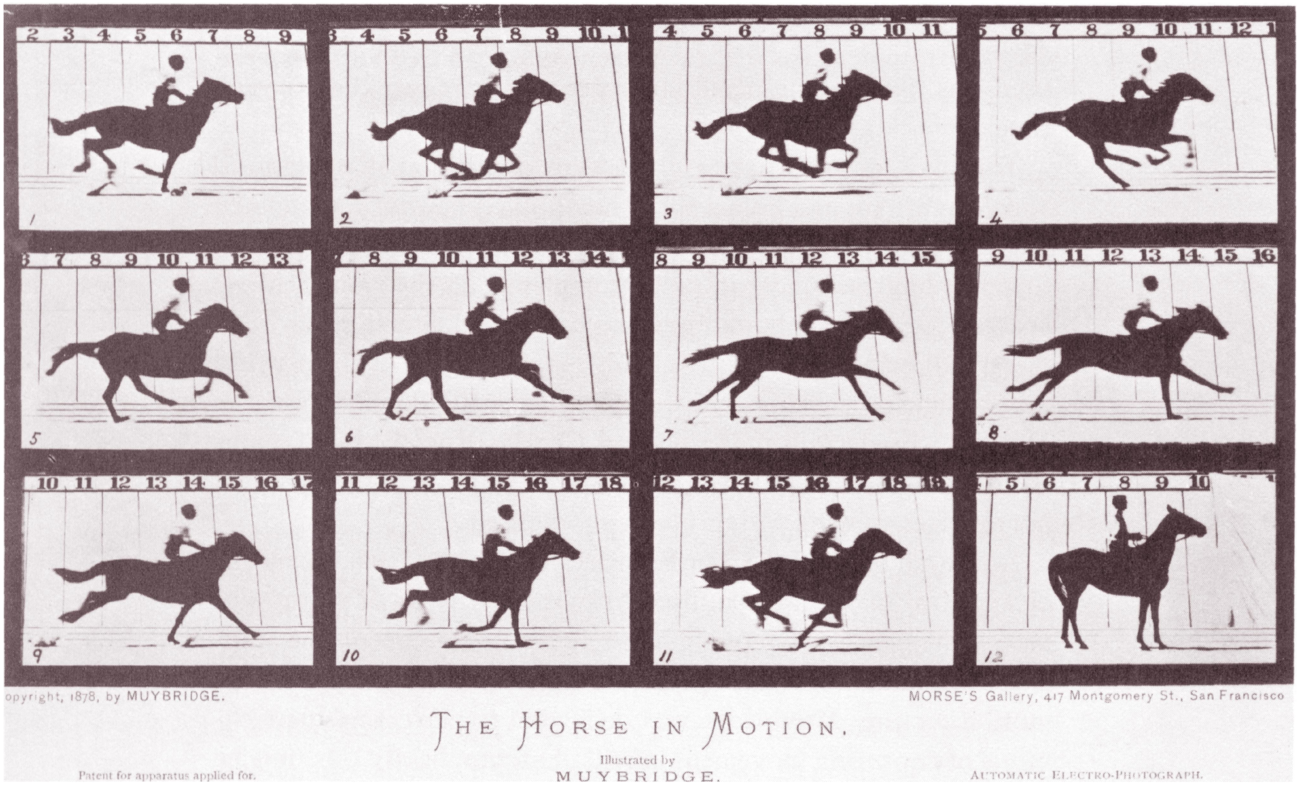
Prove that this operation won't be useful in physics, because it's not rotationally invariant.

6 In this problem you'll extend the analysis in problem 5 on p. 215 to include air friction by writing a computer program. For a game played at sea level, the force due to air friction is approximately $(7 \times 10^{-4}\ \text{N}{\cdot}\text{s}^2/\text{m}^2)v^2$, in the direction opposite to the motion of the ball.The mass of a baseball is 0.146 kg.
(a) For a ball hit at a speed of 45.0 m/s from a height of 1.0 m, find the optimal angle and the resulting range. ▷ Answer, p. 562
(b) How much farther would the ball fly at the Colorado Rockies' stadium, where the thinner air gives 18 percent less air friction? ▷ Answer, p. 562

7 Suppose someone proposes a new operation in which a vector **A** is used to produce a new vector **B** like this:

$$B_x = \sin A_x$$
$$B_y = \sin A_y$$
$$B_z = \sin A_z$$

Prove that this operation won't be useful in physics, because it's not rotationally invariant. ▷ Solution, p. 549



Chapter 8
Vectors and motion

In 1872, capitalist and former California governor Leland Stanford asked photographer Eadweard Muybridge if he would work for him on a project to settle a \$25,000 bet (a princely sum at that time). Stanford's friends were convinced that a trotting horse always had at least one foot on the ground, but Stanford claimed that there was a moment during each cycle of the motion when all four feet were in the air. The human eye was simply not fast enough to settle the question. In 1878, Muybridge finally succeeded in producing what amounted to a motion picture of the horse, showing conclusively that all four feet did leave the ground at one point. (Muybridge was a colorful figure in San Francisco history, and his acquittal for the murder of his wife's lover was considered the trial of the century in California.)

The losers of the bet had probably been influenced by Aristotelian reasoning, for instance the expectation that a leaping horse would lose horizontal velocity while in the air with no force to push it forward, so that it would be more efficient for the horse to run without leaping. But even for students who have converted wholeheartedly to Newtonianism, the relationship between force and acceleration leads to some conceptual difficulties, the main one being a problem with the true but seemingly absurd statement that an object can have an acceleration vector whose direction is not the same as the direction of motion. The horse, for instance, has nearly constant horizontal velocity, so its a_x is zero. But as anyone can tell you who has ridden a galloping horse, the horse accelerates up and down. The horse's acceleration vector therefore changes back and forth between the up and down directions, but is never in the same direction as the horse's motion. In this chapter, we will examine more carefully the properties of the velocity, acceleration, and force vectors. No new principles are introduced, but an attempt is made to tie things together and show examples of the power of the vector formulation of Newton's laws.

8.1 The velocity vector

For motion with constant velocity, the velocity vector is

$$\mathbf{v} = \Delta\mathbf{r}/\Delta t. \qquad \text{[only for constant velocity]}$$

The $\Delta\mathbf{r}$ vector points in the direction of the motion, and dividing it by the scalar Δt only changes its length, not its direction, so the velocity vector points in the same direction as the motion. When the velocity vector is not constant, we form it from the components $v_x = \mathrm{d}x/\mathrm{d}t$, $v_y = \mathrm{d}y/\mathrm{d}t$, and $v_z = \mathrm{d}z/\mathrm{d}t$. This set of three equations can be notated more compactly as

$$\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t.$$

This is an example of a more general rule about differentiating vectors: to differentiate a vector, take the derivative component by component. Even when the velocity vector is not constant, it still points along the direction of motion.

A car bouncing on its shock absorbers *example 1*

▷ A car bouncing on its shock absorbers has a position as a function of time given by

$$\mathbf{r} = bt\hat{\mathbf{x}} + (c\sin\omega t)\hat{\mathbf{y}},$$

where b, c, and ω (Greek letter omega) are constants. Infer the units of the constants, find the velocity, and check the units of the result.

▷ The components of the position vector are bt and $c\sin\omega t$, and if these are both to have units of meters, then b must have units of m/s and c units of meters. The sine function requires a unitless input, so ω must have units of s^{-1} (interpreted as radians per second, e.g., if $c = 2\pi$ rad/s, then the car completes one cycle of vertical oscillation in one second).

Differentiating component by component, we find

$$\mathbf{v} = b\hat{\mathbf{x}} + (c\omega \cos \omega t)\hat{\mathbf{y}}.$$

The units of the first component are simply the units of b, m/s, which makes sense. The units of the second component are $\text{m} \cdot \text{s}^{-1}$, which also checks out.

Vector addition is the correct way to generalize the one-dimensional concept of adding velocities in relative motion, as shown in the following example:

Velocity vectors in relative motion *example 2*

▷ You wish to cross a river and arrive at a dock that is directly across from you, but the river's current will tend to carry you downstream. To compensate, you must steer the boat at an angle. Find the angle θ, given the magnitude, $|\mathbf{v}_{WL}|$, of the water's velocity relative to the land, and the maximum speed, $|\mathbf{v}_{BW}|$, of which the boat is capable relative to the water.

▷ The boat's velocity relative to the land equals the vector sum of its velocity with respect to the water and the water's velocity with respect to the land,

$$\mathbf{v}_{BL} = \mathbf{v}_{BW} + \mathbf{v}_{WL}.$$

If the boat is to travel straight across the river, i.e., along the y axis, then we need to have $\mathbf{v}_{BL,x} = 0$. This x component equals the sum of the x components of the other two vectors,

$$\mathbf{v}_{BL,x} = \mathbf{v}_{BW,x} + \mathbf{v}_{WL,x},$$

or

$$0 = -|\mathbf{v}_{BW}| \sin \theta + |\mathbf{v}_{WL}|.$$

Solving for θ, we find $\sin \theta = |\mathbf{v}_{WL}|/|\mathbf{v}_{BW}|$, so

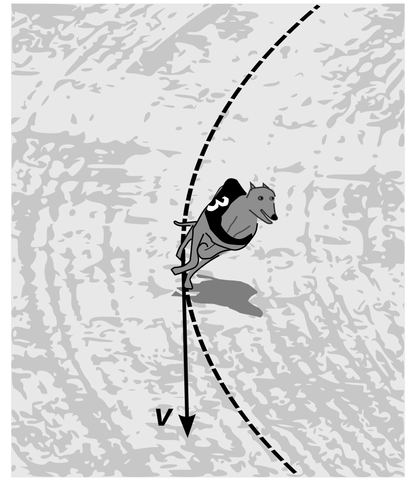
$$\theta = \sin^{-1} \frac{|\mathbf{v}_{WL}|}{|\mathbf{v}_{BW}|}.$$

▷ *Solved problem: Annie Oakley* *page 247, problem 3*

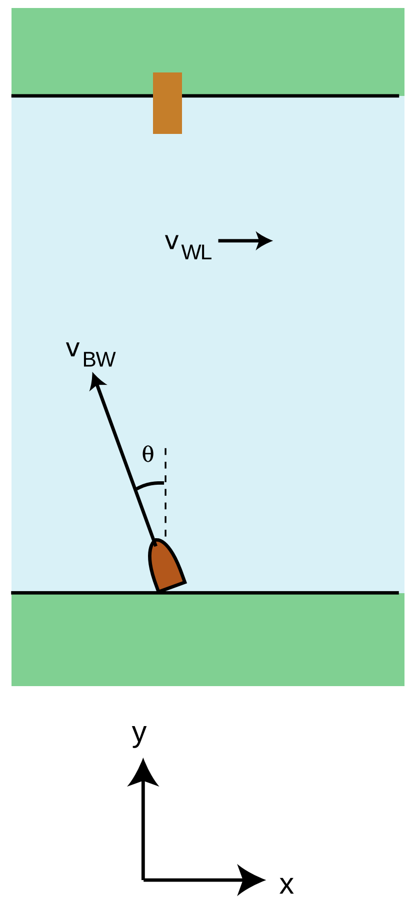
Discussion questions

A Is it possible for an airplane to maintain a constant velocity vector but not a constant $|\mathbf{v}|$? How about the opposite – a constant $|\mathbf{v}|$ but not a constant velocity vector? Explain.

B New York and Rome are at about the same latitude, so the earth's rotation carries them both around nearly the same circle. Do the two cities have the same velocity vector (relative to the center of the earth)? If not, is there any way for two cities to have the same velocity vector?



a / The racing greyhound's velocity vector is in the direction of its motion, i.e., tangent to its curved path.

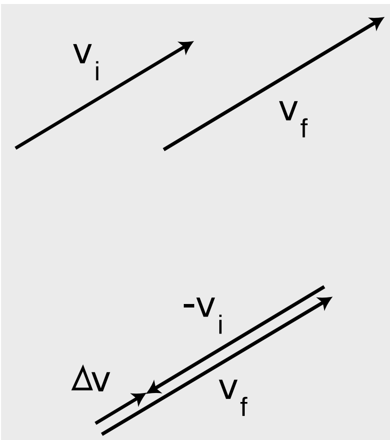


b / Example 2.

8.2 The acceleration vector

When the acceleration is constant, we have

$$\mathbf{a} = \Delta\mathbf{v}/\Delta t, \qquad \text{[only for constant acceleration]}$$



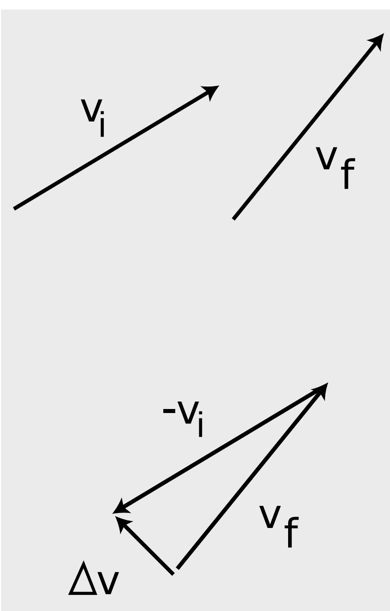
c / A change in the magnitude of the velocity vector implies an acceleration.

which can be written in terms of initial and final velocities as

$$\mathbf{a} = (\mathbf{v}_f - \mathbf{v}_i)/\Delta t. \qquad \text{[only for constant acceleration]}$$

In general, we define the acceleration vector as the derivative

$$\mathbf{a} = \mathrm{d}\mathbf{v}/\,\mathrm{d}t.$$



d / A change in the direction of the velocity vector also produces a nonzero $\Delta\mathbf{v}$ vector, and thus a nonzero acceleration vector, $\Delta\mathbf{v}/\Delta t$.

Now there are two ways in which we could have a nonzero acceleration. Either the magnitude or the direction of the velocity vector could change. This can be visualized with arrow diagrams as shown in figures c and d. Both the magnitude and direction can change simultaneously, as when a car accelerates while turning. Only when the magnitude of the velocity changes while its direction stays constant do we have a Δv vector and an acceleration vector along the same line as the motion.

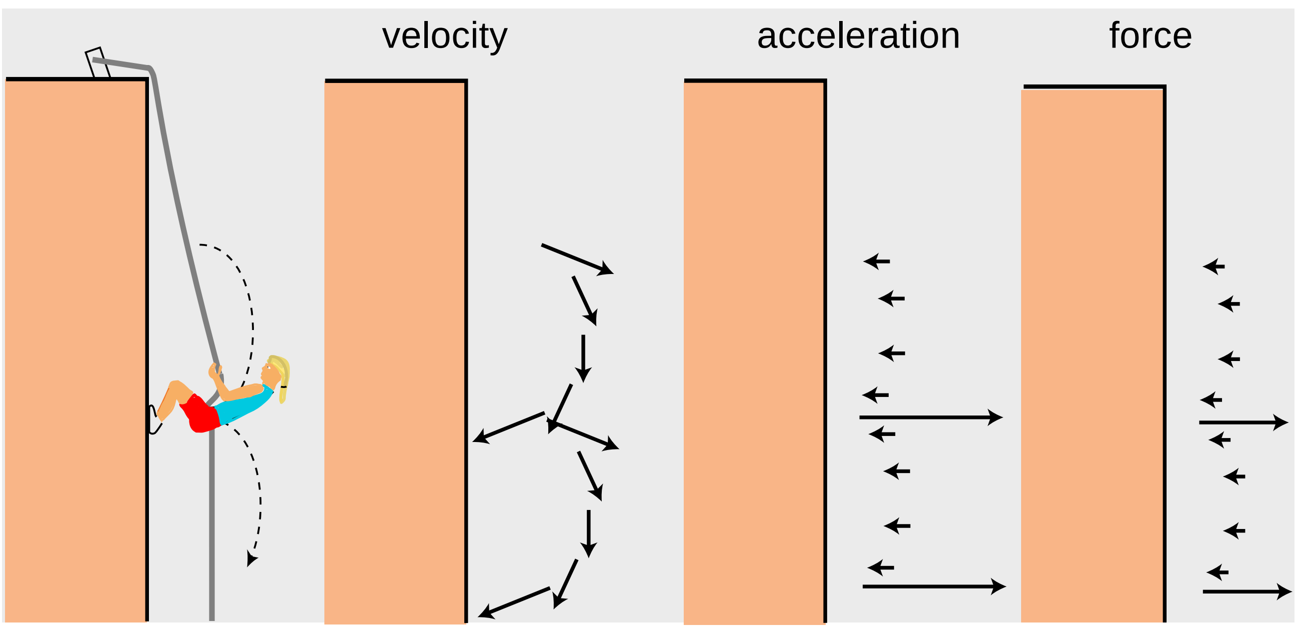
self-check A

(1) In figure c, is the object speeding up, or slowing down? (2) What would the diagram look like if $\mathbf{v}_i$ was the same as $\mathbf{v}_f$? (3) Describe how the $\Delta\mathbf{v}$ vector is different depending on whether an object is speeding up or slowing down. ▷ Answer, p. 559

The acceleration vector points in the direction that an accelerometer would point, as in figure e.

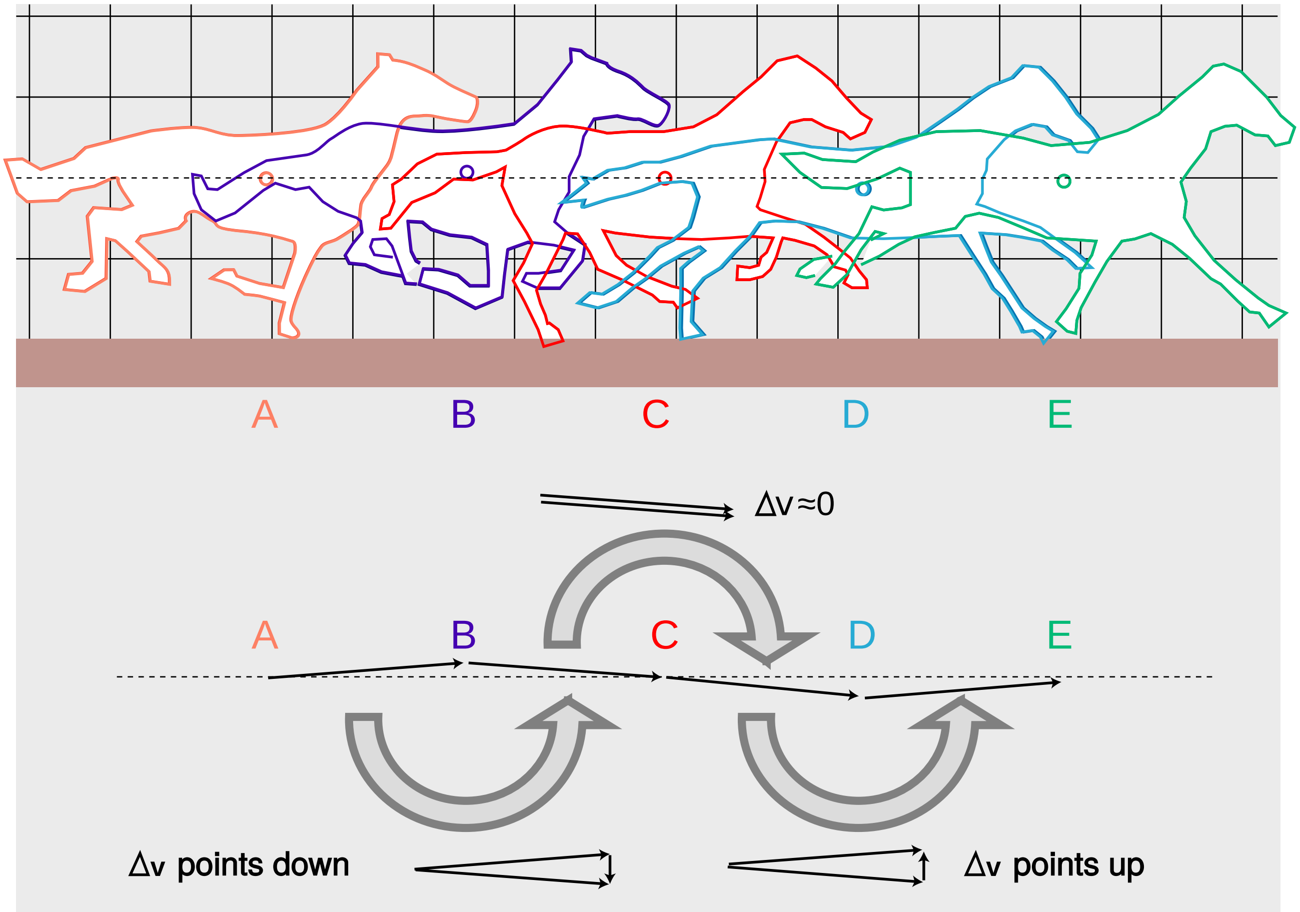
e / The car has just swerved to the right. The air freshener hanging from the rear-view mirror acts as an accelerometer, showing that the acceleration vector is to the right.

self-check B
In projectile motion, what direction does the acceleration vector have?
▷ Answer, p. 560



f / Example 3.

Rappelling *example 3*
In figure f, the rappeller's velocity has long periods of gradual change interspersed with short periods of rapid change. These correspond to periods of small acceleration and force, and periods of large acceleration and force.



g / Example 4.

The galloping horse *example 4*

Figure g on page 238 shows outlines traced from the first, third, fifth, seventh, and ninth frames in Muybridge's series of photographs of the galloping horse. The estimated location of the horse's center of mass is shown with a circle, which bobs above and below the horizontal dashed line.

If we don't care about calculating velocities and accelerations in any particular system of units, then we can pretend that the time between frames is one unit. The horse's velocity vector as it moves from one point to the next can then be found simply by drawing an arrow to connect one position of the center of mass to the next. This produces a series of velocity vectors which alternate between pointing above and below horizontal.

The $\Delta\mathbf{v}$ vector is the vector which we would have to add onto one velocity vector in order to get the next velocity vector in the series. The $\Delta\mathbf{v}$ vector alternates between pointing down (around the time when the horse is in the air, B) and up (around the time when the horse has two feet on the ground, D).

Discussion questions

A When a car accelerates, why does a bob hanging from the rearview mirror swing toward the back of the car? Is it because a force throws it backward? If so, what force? Similarly, describe what happens in the other cases described above.

B Superman is guiding a crippled spaceship into port. The ship's engines are not working. If Superman suddenly changes the direction of his force on the ship, does the ship's velocity vector change suddenly? Its acceleration vector? Its direction of motion?

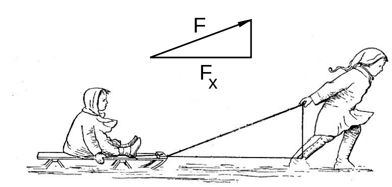
8.3 The force vector and simple machines

Force is relatively easy to intuit as a vector. The force vector points in the direction in which it is trying to accelerate the object it is acting on.

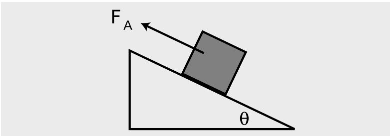
Since force vectors are so much easier to visualize than acceleration vectors, it is often helpful to first find the direction of the (total) force vector acting on an object, and then use that to find the direction of the acceleration vector. Newton's second law tells us that the two must be in the same direction.

A component of a force vector *example 5*

Figure h, redrawn from a classic 1920 textbook, shows a boy pulling another child on a sled. His force has both a horizontal component and a vertical one, but only the horizontal one accelerates the sled. (The vertical component just partially cancels the force of gravity, causing a decrease in the normal force between the runners and the snow.) There are two triangles in the figure. One triangle's hypotenuse is the rope, and the other's is the magnitude of the force. These triangles are similar, so their internal angles are all the same, but they are not the same triangle. One is a distance triangle, with sides measured in meters, the other a force triangle, with sides in newtons. In both cases, the horizontal leg is 93% as long as the hypotenuse. It does not make sense, however, to compare the sizes of the triangles — the force triangle is not smaller in any meaningful sense.



h / Example 5.



i / The applied force $\mathbf{F}_A$ pushes the block up the frictionless ramp.

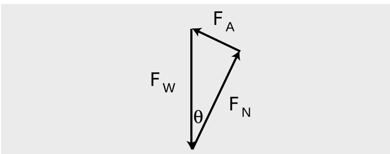
Pushing a block up a ramp — *example 6*

▷ Figure i shows a block being pushed up a frictionless ramp at constant speed by an externally applied force $\mathbf{F}_A$. How much force is required, in terms of the block's mass, m, and the angle of the ramp, θ?

▷ We analyze the forces on the block and introduce notation for the other forces besides $\mathbf{F}_A$:

force acting on block	*3rd-law partner*
ramp's normal force on block, $\mathbf{F}_N$, ↗	block's normal force on ramp, ↙
external object's force on block (type irrelevant), $\mathbf{F}_A$ ↖	block's force on external object (type irrelevant), ↘
planet earth's gravitational force on block, $\mathbf{F}_W$ ↓	block's gravitational force on earth, ↑

Because the block is being pushed up at constant speed, it has zero acceleration, and the total force on it must be zero. From figure j, we find

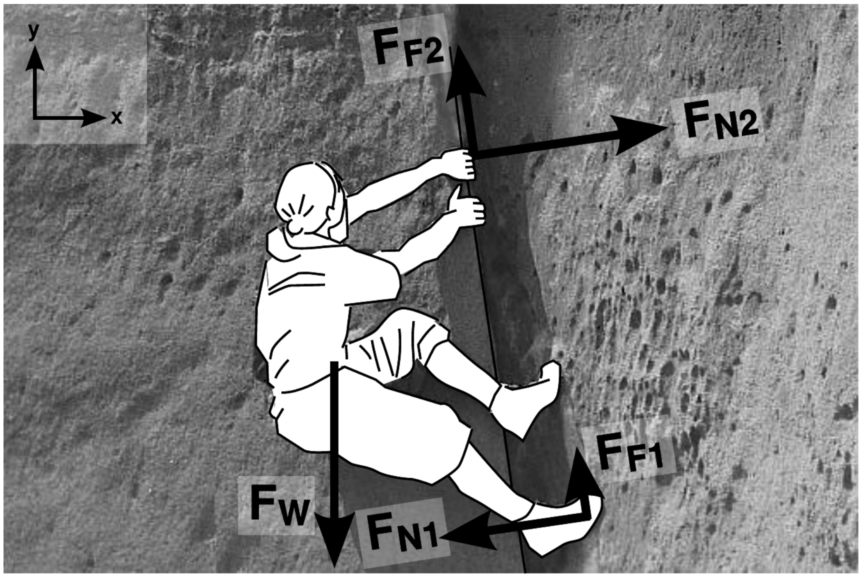


j / If the block is to move at constant velocity, Newton's first law says that the three force vectors acting on it must add up to zero. To perform vector addition, we put the vectors tip to tail, and in this case we are adding three vectors, so each one's tail goes against the tip of the previous one. Since they are supposed to add up to zero, the third vector's tip must come back to touch the tail of the first vector. They form a triangle, and since the applied force is perpendicular to the normal force, it is a right triangle.

$$\begin{aligned} |\mathbf{F}_A| &= |\mathbf{F}_W| \sin\theta \\ &= mg\sin\theta. \end{aligned}$$

Since the sine is always less than one, the applied force is always less than mg, i.e., pushing the block up the ramp is easier than lifting it straight up. This is presumably the principle on which the pyramids were constructed: the ancient Egyptians would have had a hard time applying the forces of enough slaves to equal the full weight of the huge blocks of stone.

Essentially the same analysis applies to several other simple machines, such as the wedge and the screw.



k / Example 7 and problem 22 on p. 251.

A layback *example 7*

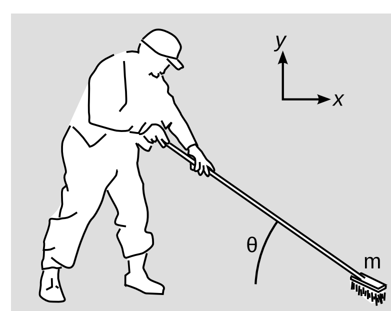
The figure shows a rock climber using a technique called a layback. He can make the normal forces $\mathbf{F}_{N1}$ and $\mathbf{F}_{N2}$ large, which has the side-effect of increasing the frictional forces $\mathbf{F}_{F1}$ and $\mathbf{F}_{F2}$, so that he doesn't slip down due to the gravitational (weight) force $\mathbf{F}_W$. The purpose of the problem is not to analyze all of this in detail, but simply to practice finding the components of the forces based on their magnitudes. To keep the notation simple, let's write F_{N1} for $|\mathbf{F}_{N1}|$, etc. The crack overhangs by a small, positive angle $\theta \approx 9°$.

In this example, we determine the x component of $\mathbf{F}_{N1}$. The other nine components are left as an exercise to the reader (problem 22, p. 251).

The easiest method is the one demonstrated in example 5 on p. 223. Casting vector $\mathbf{F}_{N1}$'s shadow on the ground, we can tell that it would point to the left, so its x component is negative. The only two possibilities for its x component are therefore $-F_{N1}\cos\theta$ or $-F_{N1}\sin\theta$. We expect this force to have a large x component and a much smaller y. Since θ is small, $\cos\theta \approx 1$, while $\sin\theta$ is small. Therefore the x component must be $-F_{N1}\cos\theta$.

Pushing a broom *example 8*

▷ Figure l shows a man pushing a broom at an angle θ relative to the horizontal. The mass m of the broom is concentrated at the brush. If the magnitude of the broom's acceleration is a, find the force F_H that the man must make on the handle.



l / Example 8.

▷ First we analyze all the forces on the brush.

force acting on brush	*3rd-law partner*
handle's normal force on brush, F_H, ↘	brush's normal force on handle, ↖
earth's gravitational force on brush, mg, ↓	brush's gravitational force on earth, ↑
floor's normal force on brush, F_N, ↑	brush's normal force on floor, ↓
floor's kinetic friction force on brush, F_k, ←	brush's kinetic friction force on floor, →

Newton's second law is:

$$\mathbf{a} = \frac{\mathbf{F}_H + m\mathbf{g} + \mathbf{F}_N + \mathbf{F}_k}{m},$$

where the addition is vector addition. If we actually want to carry out the vector addition of the forces, we have to do either graphical addition (as in example 6) or analytic addition. Let's do analytic addition, which means finding all the components of the forces, adding the x's, and adding the y's.

Most of the forces have components that are trivial to express in terms of their magnitudes, the exception being $\mathbf{F}_H$, whose components we can determine using the technique demonstrated in example 5 on p. 223 and example 7 on p. 241. Using the coordinate system shown in the figure, the results are:

$\mathrm{F}_{Hx} = F_H \cos\theta$	$\mathrm{F}_{Hy} = -F_H \sin\theta$
$\mathrm{mg}_x = 0$	$\mathrm{mg}_y = -mg$
$\mathrm{F}_{Nx} = 0$	$\mathrm{F}_{Ny} = F_N$
$\mathrm{F}_{kx} = -F_k$	$\mathrm{F}_{ky} = 0$

Note that we don't yet know the magnitudes F_H, F_N, and F_k. That's all right. First we need to set up Newton's laws, and *then* we can worry about solving the equations.

Newton's second law in the x direction gives:

[1] $$a = \frac{F_H \cos\theta - F_k}{m}$$

The acceleration in the vertical direction is zero, so Newton's second law in the y direction tells us that

[2] $$0 = -F_H \sin\theta - mg + F_N.$$

Finally, we have the relationship between kinetic friction and the normal force,

[3] $$F_k = \mu_k F_N.$$

Equations [1]-[3] are three equations, which we can use to determine the three unknowns, F_H, F_N, and F_k. Straightforward algebra gives

$$F_H = m\left(\frac{a + \mu_k g}{\cos\theta - \mu_k \sin\theta}\right)$$

▷ *Solved problem: A cargo plane* *page 249, problem 9*

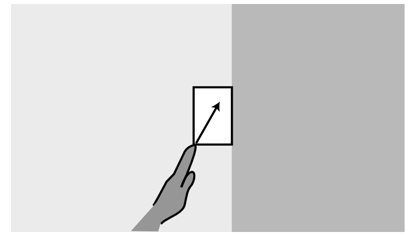
▷ *Solved problem: The angle of repose* *page 250, problem 13*

▷ *Solved problem: A wagon* *page 250, problem 14*

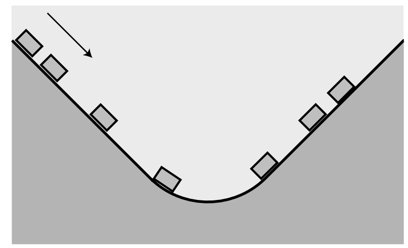
Discussion questions

A The figure shows a block being pressed diagonally upward against a wall, causing it to slide up the wall. Analyze the forces involved, including their directions.

B The figure shows a roller coaster car rolling down and then up under the influence of gravity. Sketch the car's velocity vectors and acceleration vectors. Pick an interesting point in the motion and sketch a set of force vectors acting on the car whose vector sum could have resulted in the right acceleration vector.



Discussion question A.



Discussion question B.

8.4 More about calculus with vectors

Our definition of the derivative of a vector implies the familiar properties

$$\frac{\mathrm{d}(c\mathbf{f})}{\mathrm{d}t} = c\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} \qquad [c \text{ is a constant}]$$

and

$$\frac{\mathrm{d}(\mathbf{f}+\mathbf{g})}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}.$$

The integral of a vector is likewise defined as integrating component by component.

The second derivative of a vector *example 9*

▷ Two objects have positions as functions of time given by the equations

$$\mathbf{r}_1 = 3t^2\hat{\mathbf{x}} + t\hat{\mathbf{y}}$$

and

$$\mathbf{r}_2 = 3t^4\hat{\mathbf{x}} + t\hat{\mathbf{y}}.$$

Find both objects' accelerations using calculus. Could either answer have been found without calculus?

▷ Taking the first derivative of each component, we find

$$\mathbf{v}_1 = 6t\hat{\mathbf{x}} + \hat{\mathbf{y}}$$
$$\mathbf{v}_2 = 12t^3\hat{\mathbf{x}} + \hat{\mathbf{y}},$$

and taking the derivatives again gives acceleration,

$$\mathbf{a}_1 = 6\hat{\mathbf{x}}$$
$$\mathbf{a}_2 = 36t^2\hat{\mathbf{x}}.$$

The first object's acceleration could have been found without calculus, simply by comparing the x and y coordinates with the constant-acceleration equation $\Delta x = v_o \Delta t + \frac{1}{2} a \Delta t^2$. The second equation, however, isn't just a second-order polynomial in t, so the acceleration isn't constant, and we really did need calculus to find the corresponding acceleration.

The integral of a vector — *example 10*

▷ Starting from rest, a flying saucer of mass m is observed to vary its propulsion with mathematical precision according to the equation

$$\mathbf{F} = bt^{42}\hat{\mathbf{x}} + ct^{137}\hat{\mathbf{y}}.$$

(The aliens inform us that the numbers 42 and 137 have a special religious significance for them.) Find the saucer's velocity as a function of time.

▷ From the given force, we can easily find the acceleration

$$\begin{aligned} \mathbf{a} &= \frac{\mathbf{F}}{m} \\ &= \frac{b}{m} t^{42}\hat{\mathbf{x}} + \frac{c}{m} t^{137}\hat{\mathbf{y}}. \end{aligned}$$

The velocity vector $\mathbf{v}$ is the integral with respect to time of the acceleration,

$$\begin{aligned} \mathbf{v} &= \int \mathbf{a}\, \mathrm{d}t \\ &= \int \left(\frac{b}{m} t^{42}\hat{\mathbf{x}} + \frac{c}{m} t^{137}\hat{\mathbf{y}} \right) \mathrm{d}t, \end{aligned}$$

and integrating component by component gives

$$\begin{aligned} &= \left(\int \frac{b}{m} t^{42}\, \mathrm{d}t \right) \hat{\mathbf{x}} + \left(\int \frac{c}{m} t^{137}\, \mathrm{d}t \right) \hat{\mathbf{y}} \\ &= \frac{b}{43m} t^{43}\hat{\mathbf{x}} + \frac{c}{138m} t^{138}\hat{\mathbf{y}}, \end{aligned}$$

where we have omitted the constants of integration, since the saucer was starting from rest.

A fire-extinguisher stunt on ice — *example 11*

▷ Prof. Puerile smuggles a fire extinguisher into a skating rink. Climbing out onto the ice without any skates on, he sits down and pushes off from the wall with his feet, acquiring an initial velocity $v_o\hat{\mathbf{y}}$. At $t = 0$, he then discharges the fire extinguisher at a 45-degree angle so that it applies a force to him that is backward and to the left, i.e., along the negative y axis and the positive x axis. The fire extinguisher's force is strong at first, but then dies down according to the equation $|\mathbf{F}| = b - ct$, where b and c are constants. Find the professor's velocity as a function of time.

▷ Measured counterclockwise from the x axis, the angle of the force vector becomes 315°. Breaking the force down into x and y components, we have

$$\begin{aligned} F_x &= |\mathbf{F}|\cos 315^\circ \\ &= (b - ct) \\ F_y &= |\mathbf{F}|\sin 315^\circ \\ &= (-b + ct). \end{aligned}$$

In unit vector notation, this is

$$F = (b - ct)\hat{\mathbf{x}} + (-b + ct)\hat{\mathbf{y}}.$$

Newton's second law gives

$$\begin{aligned} \mathbf{a} &= \mathbf{F}/m \\ &= \frac{b - ct}{\sqrt{2}m}\hat{\mathbf{x}} + \frac{-b + ct}{\sqrt{2}m}\hat{\mathbf{y}}. \end{aligned}$$

To find the velocity vector as a function of time, we need to integrate the acceleration vector with respect to time,

$$\begin{aligned} \mathbf{v} &= \int \mathbf{a}\,\mathrm{d}t \\ &= \int \left(\frac{b - ct}{\sqrt{2}m}\,\hat{\mathbf{x}} + \frac{-b + ct}{\sqrt{2}m}\,\hat{\mathbf{y}} \right) \mathrm{d}t \\ &= \frac{1}{\sqrt{2}m} \int \left[(b - ct)\,\hat{\mathbf{x}} \;+\; (-b + ct)\,\hat{\mathbf{y}} \right] \mathrm{d}t \end{aligned}$$

A vector function can be integrated component by component, so this can be broken down into two integrals,

$$\begin{aligned} \mathbf{v} &= \frac{\hat{\mathbf{x}}}{\sqrt{2}m} \int (b - ct)\,\mathrm{d}t \;+\; \frac{\hat{\mathbf{y}}}{\sqrt{2}m} \int (-b + ct)\,\mathrm{d}t \\ &= \left(\frac{bt - \frac{1}{2}ct^2}{\sqrt{2}m} + \text{constant \#1} \right) \hat{\mathbf{x}} + \left(\frac{-bt + \frac{1}{2}ct^2}{\sqrt{2}m} + \text{constant \#2} \right) \hat{\mathbf{y}} \end{aligned}$$

Here the physical significance of the two constants of integration is that they give the initial velocity. Constant #1 is therefore zero, and constant #2 must equal v_o. The final result is

$$\mathbf{v} = \left(\frac{bt - \frac{1}{2}ct^2}{\sqrt{2}m} \right) \hat{\mathbf{x}} + \left(\frac{-bt + \frac{1}{2}ct^2}{\sqrt{2}m} + v_\text{o} \right) \hat{\mathbf{y}}.$$

Summary

The velocity vector points in the direction of the object's motion. Relative motion can be described by vector addition of velocities.

The acceleration vector need not point in the same direction as the object's motion. We use the word "acceleration" to describe any change in an object's velocity vector, which can be either a change in its magnitude or a change in its direction.

An important application of the vector addition of forces is the use of Newton's first law to analyze mechanical systems.

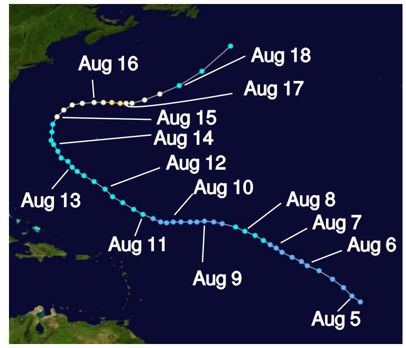
Problems

Key

√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

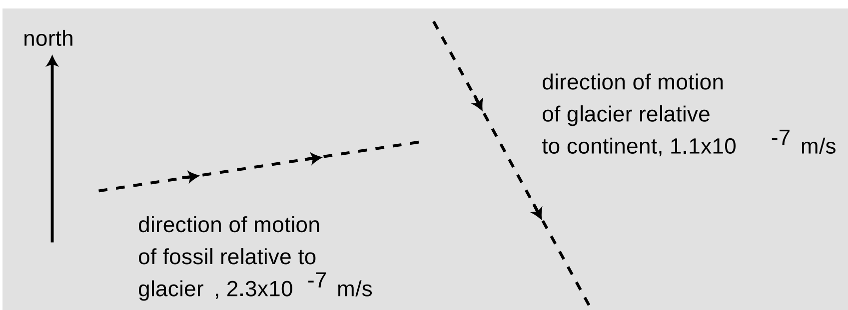
1 Is it possible for a helicopter to have an acceleration due east and a velocity due west? If so, what would be going on? If not, why not?

2 The figure shows the path followed by Hurricane Irene in 2005 as it moved north. The dots show the location of the center of the storm at six-hour intervals, with lighter dots at the time when the storm reached its greatest intensity. Find the time when the storm's center had a velocity vector to the northeast and an acceleration vector to the southeast. Explain.



Problem 2.

3 Annie Oakley, riding north on horseback at 30 mi/hr, shoots her rifle, aiming horizontally and to the northeast. The muzzle speed of the rifle is 140 mi/hr. When the bullet hits a defenseless fuzzy animal, what is its speed of impact? Neglect air resistance, and ignore the vertical motion of the bullet. ▷ Solution, p. 550

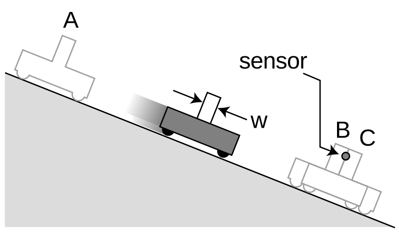


Problem 4.

4 As shown in the diagram, a dinosaur fossil is slowly moving down the slope of a glacier under the influence of wind, rain and gravity. At the same time, the glacier is moving relative to the continent underneath. The dashed lines represent the directions but not the magnitudes of the velocities. Pick a scale, and use graphical addition of vectors to find the magnitude and the direction of the fossil's velocity relative to the continent. You will need a ruler and protractor. √

5 A bird is initially flying horizontally east at 21.1 m/s, but one second later it has changed direction so that it is flying horizontally and 7° north of east, at the same speed. What are the magnitude and direction of its acceleration vector during that one second time interval? (Assume its acceleration was roughly constant.) √

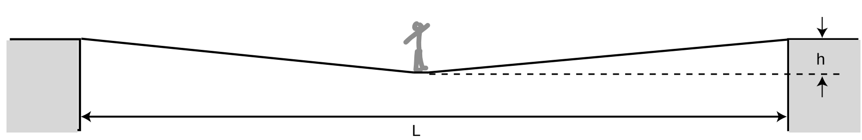
6 A gun is aimed horizontally to the west. The gun is fired, and the bullet leaves the muzzle at $t = 0$. The bullet's position vector as a function of time is $\mathbf{r} = b\hat{\mathbf{x}} + ct\hat{\mathbf{y}} + dt^2\hat{\mathbf{z}}$, where b, c, and d are positive constants.
(a) What units would b, c, and d need to have for the equation to make sense?
(b) Find the bullet's velocity and acceleration as functions of time.
(c) Give physical interpretations of b, c, d, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.



Problem 7.

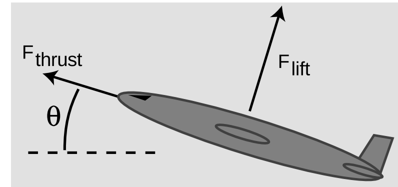
7 The figure shows an experiment in which a cart is released from rest at A, and accelerates down the slope through a distance x until it passes through a sensor's light beam. The point of the experiment is to determine the cart's acceleration. At B, a cardboard vane mounted on the cart enters the light beam, blocking the light beam, and starts an electronic timer running. At C, the vane emerges from the beam, and the timer stops.
(a) Find the final velocity of the cart in terms of the width w of the vane and the time t_b for which the sensor's light beam was blocked. ✓
(b) Find the magnitude of the cart's acceleration in terms of the measurable quantities x, t_b, and w. ✓
(c) Analyze the forces in which the cart participates, using a table in the format introduced in section 5.3. Assume friction is negligible.
(d) Find a theoretical value for the acceleration of the cart, which could be compared with the experimentally observed value extracted in part b. Express the theoretical value in terms of the angle θ of the slope, and the strength g of the gravitational field. ✓

8 A person of mass M stands in the middle of a tightrope, which is fixed at the ends to two buildings separated by a horizontal distance L. The rope sags in the middle, stretching and lengthening the rope slightly.
(a) If the tightrope walker wants the rope to sag vertically by no more than a height h, find the minimum tension, T, that the rope must be able to withstand without breaking, in terms of h, g, M, and L. ✓
(b) Based on your equation, explain why it is not possible to get $h = 0$, and give a physical interpretation.



Problem 8.

9 A cargo plane has taken off from a tiny airstrip in the Andes, and is climbing at constant speed, at an angle of $\theta = 17°$ with respect to horizontal. Its engines supply a thrust of $F_{\text{thrust}} = 200$ kN, and the lift from its wings is $F_{\text{lift}} = 654$ kN. Assume that air resistance (drag) is negligible, so the only forces acting are thrust, lift, and weight. What is its mass, in kg? ▷ Solution, p. 550



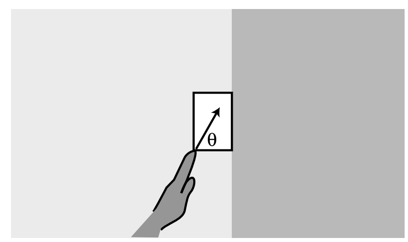
Problem 9.

10 A skier of mass m is coasting down a slope inclined at an angle θ compared to horizontal. Assume for simplicity that the treatment of kinetic friction given in chapter 5 is appropriate here, although a soft and wet surface actually behaves a little differently. The coefficient of kinetic friction acting between the skis and the snow is μ_k, and in addition the skier experiences an air friction force of magnitude bv^2, where b is a constant.

(a) Find the maximum speed that the skier will attain, in terms of the variables m, g, θ, μ_k, and b. √

(b) For angles below a certain minimum angle θ_{min}, the equation gives a result that is not mathematically meaningful. Find an equation for θ_{min}, and give a physical explanation of what is happening for $\theta < \theta_{min}$. √

11 Your hand presses a block of mass m against a wall with a force $\mathbf{F}_H$ acting at an angle θ, as shown in the figure. Find the minimum and maximum possible values of $|\mathbf{F}_H|$ that can keep the block stationary, in terms of m, g, θ, and μ_s, the coefficient of static friction between the block and the wall. Check both your answers in the case of $\theta = 90°$, and interpret the case where the maximum force is infinite. √



Problem 11.

12 Driving down a hill inclined at an angle θ with respect to horizontal, you slam on the brakes to keep from hitting a deer. Your antilock brakes kick in, and you don't skid.

(a) Analyze the forces. (Ignore rolling resistance and air friction.)

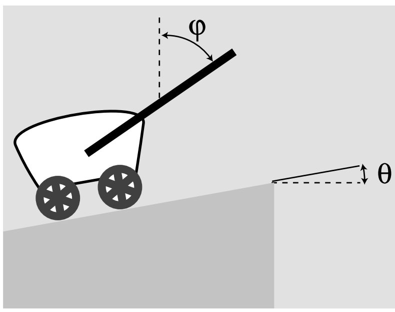
(b) Find the car's maximum possible deceleration, a (expressed as a positive number), in terms of g, θ, and the relevant coefficient of friction. √

(c) Explain physically why the car's mass has no effect on your answer.

(d) Discuss the mathematical behavior and physical interpretation of your result for negative values of θ.

(e) Do the same for very large positive values of θ.

13 The angle of repose is the maximum slope on which an object will not slide. On airless, geologically inert bodies like the moon or an asteroid, the only thing that determines whether dust or rubble will stay on a slope is whether the slope is less steep than the angle of repose. (See figure n, p. 286.)
(a) Find an equation for the angle of repose, deciding for yourself what are the relevant variables.
(b) On an asteroid, where g can be thousands of times lower than on Earth, would rubble be able to lie at a steeper angle of repose?
▷ Solution, p. 550

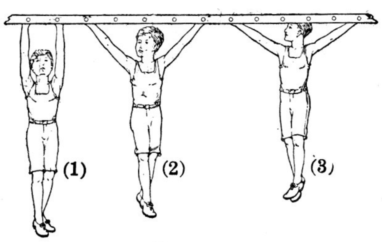


Problem 14.

14 A wagon is being pulled at constant speed up a slope θ by a rope that makes an angle ϕ with the vertical.
(a) Assuming negligible friction, show that the tension in the rope is given by the equation

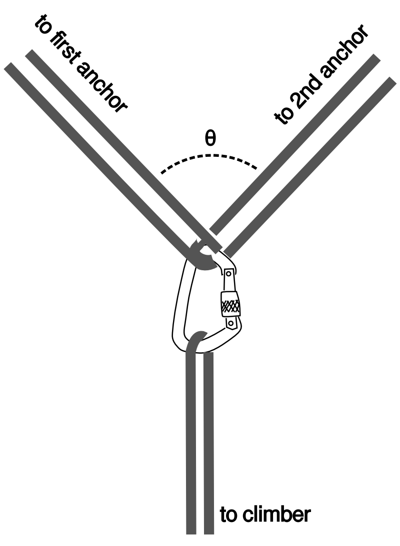
$$F_T = \frac{\sin\theta}{\sin(\theta+\phi)} F_W,$$

where F_W is the weight force acting on the wagon.
(b) Interpret this equation in the special cases of $\phi = 0$ and $\phi = 180° - \theta$.
▷ Solution, p. 551



Problem 15 (Millikan and Gale, 1920).

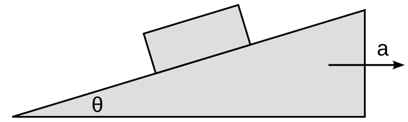
15 The figure shows a boy hanging in three positions: (1) with his arms straight up, (2) with his arms at 45 degrees, and (3) with his arms at 60 degrees with respect to the vertical. Compare the tension in his arms in the three cases.



Problem 16.

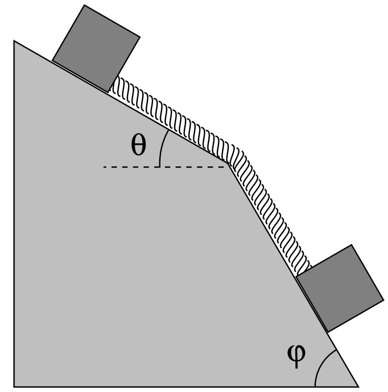
16 For safety, mountain climbers often wear a climbing harness and tie in to other climbers on a rope team or to anchors such as pitons or snow anchors. When using anchors, the climber usually wants to tie in to more than one, both for extra strength and for redundancy in case one fails. The figure shows such an arrangement, with the climber hanging from a pair of anchors forming a "Y" at an angle θ. The usual advice is to make $\theta < 90°$; for large values of θ, the stress placed on the anchors can be many times greater than the actual load L, so that two anchors are actually *less* safe than one.
(a) Find the force S at each anchor in terms of L and θ. ✓
(b) Verify that your answer makes sense in the case of $\theta = 0$.
(c) Interpret your answer in the case of $\theta = 180°$.
(d) What is the smallest value of θ for which S equals or exceeds L, so that for larger angles a failure of at least one anchor is *more* likely than it would have been with a single anchor? ✓

17 (a) A block is sitting on a wedge inclined at an angle θ with respect to horizontal. Someone grabs the wedge and moves it horizontally with acceleration a. The motion is in the direction shown by the arrow in the figure. Find the maximum acceleration that can be applied without causing the block to slide downhill. √
(b) Show that your answer to part a has the right units.
(c) Show that it also has the right dependence on θ, by comparing its mathematical behavior to its physically expected behavior.



Problem 17.

18 The two blocks shown in the figure have equal mass, m, and the surface is frictionless. (a) What is the tension in the massless rope? ▷ Hint, p. 542 √
(b) Show that the units of your answer make sense.
(c) Check the physical behavior of your answer in the special cases of $\phi \leq \theta$ and $\theta = 0$, $\phi = 90°$.

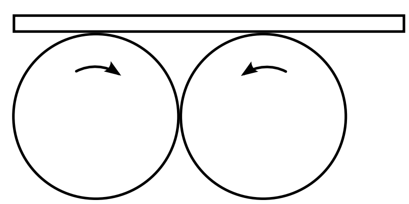


Problem 18.

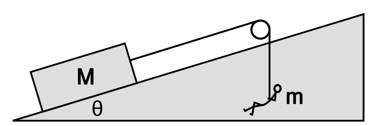
19 The photo shows a coil of rope wound around a smooth metal post. A large amount of tension is applied at the bottom of the coil, but only a tiny force, supplied by a piece of sticky tape, is needed at the top to keep the rope from slipping. Show that the ratio of these two forces increases exponentially with the number of turns of rope, and find an expression for that ratio. ▷ Hint, p. 542 √ ⋆


Problem 19.

20 Two wheels of radius r rotate in the same vertical plane with angular velocities $+\Omega$ and $-\Omega$ (rates of rotation in radians per second) about axes that are parallel and at the same height. The wheels touch one another at a point on their circumferences, so that their rotations mesh like gears in a gear train. A board is laid on top of the wheels, so that two friction forces act upon it, one from each wheel. Characterize the three qualitatively different types of motion that the board can exhibit, depending on the initial conditions. ⋆

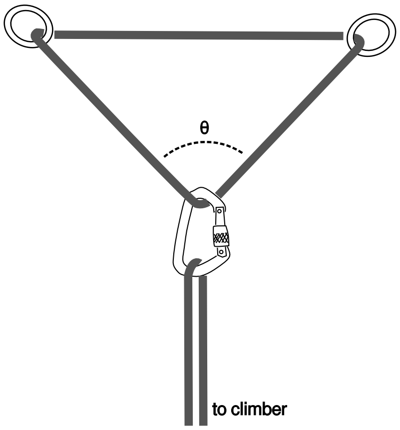

Problem 20.

21 (a) The person with mass m hangs from the rope, hauling the box of mass M up a slope inclined at an angle θ. There is friction between the box and the slope, described by the usual coefficients of friction. The pulley, however, is frictionless. Find the magnitude of the box's acceleration. √
(b) Show that the units of your answer make sense.
(c) Check the physical behavior of your answer in the special cases of $M = 0$ and $\theta = -90°$.



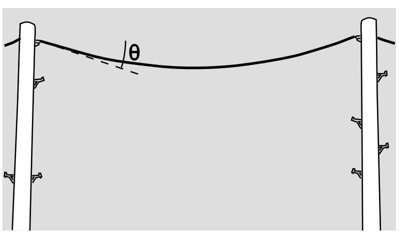
Problem 21.

22 Complete example 7 on p. 241 by expressing the remaining nine x and y components of the forces in terms of the five magnitudes and the small, positive angle $\theta \approx 9°$ by which the crack overhangs. √



Problem 23.

23 Problem 16 discussed a possible correct way of setting up a redundant anchor for mountaineering. The figure for this problem shows an incorrect way of doing it, by arranging the rope in a triangle (which we'll take to be isoceles). One of the bad things about the triangular arrangement is that it requires more force from the anchors, making them more likely to fail. (a) Using the same notation as in problem 16, find S in terms of L and θ. ✓
(b) Verify that your answer makes sense in the case of $\theta = 0$, and compare with the correct setup.

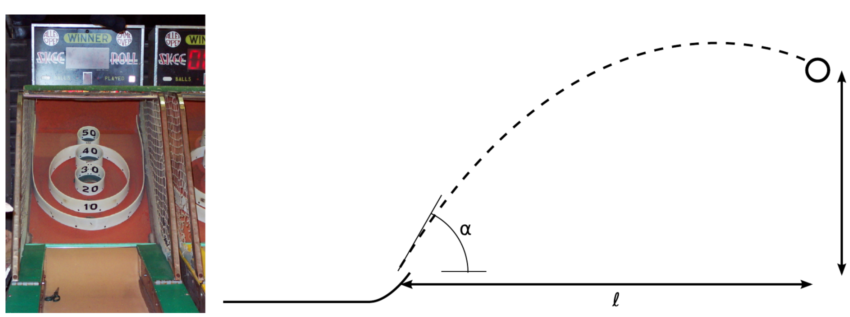


Problem 24.

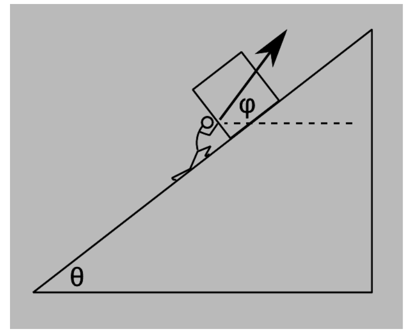
24 A telephone wire of mass m is strung between two poles, making an angle θ with the horizontal at each end. (a) Find the tension at the center. ✓
(b) Which is greater, the tension at the center or at the ends?

25 The figure shows an arcade game called skee ball that is similar to bowling. The player rolls the ball down a horizontal alley. The ball then rides up a curved lip and is launched at an initial speed u, at an angle α above horizontal. Suppose we want the ball to go into a hole that is at horizontal distance ℓ and height h, as shown in the figure.
(a) Find the initial speed u that is required, in terms of the other variables and g. ✓
(b) Check that your answer to part a has units that make sense.
(c) Check that your answer to part a depends on g in a way that makes sense. This means that you should first determine on physical grounds whether increasing g should increase u, or decrease it. Then see whether your answer to part a has this mathematical behavior.
(d) Do the same for the dependence on h.
(e) Interpret your equation in the case where $\alpha = 90°$.
(f) Interpret your equation in the case where $\tan \alpha = h/\ell$.
(g) Find u numerically if $h = 70$ cm, $\ell = 60$ cm, and $\alpha = 65°$. ✓

Problem 25.



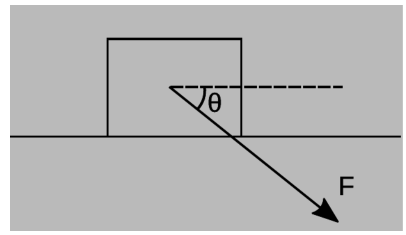
26 You are pushing a box up a ramp that is at an angle θ with respect to the horizontal. Friction acts between the box and the ramp, with coefficient μ. Suppose that your force is fixed in magnitude, but can be applied at any desired angle φ above the horizontal. Find the optimal value of φ. $\surd$ $\star$



Problem 26.

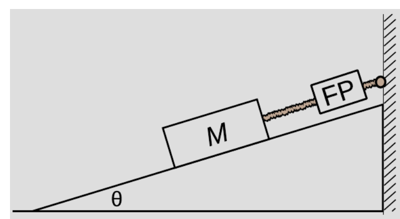
27 A plane flies toward a city directly north and a distance D away. The wind speed is u, and the plane's speed with respect to the wind is v.
(a) If the wind is blowing from the west (towards the east), what direction should the plane head (what angle west of north)? $\surd$
(b) How long does it take the plane to get to the city? $\surd$
(c) Check that your answer to part b has units that make sense.
(d) Comment on the behavior of your answer in the case where $u = v$. *[problem by B. Shotwell]*

28 A force F is applied to a box of mass M at an angle θ below the horizontal (see figure). The coefficient of static friction between the box and the floor is μ_s, and the coefficient of kinetic friction between the two surfaces is μ_k.
(a) What is the magnitude of the normal force on the box from the floor? $\surd$
(b) What is the minimum value of F to get the box to start moving from rest? $\surd$
(c) What is the value of F so that the box will move with constant velocity (assuming it is already moving)? $\surd$
(d) If θ is greater than some critical angle θ_{crit}, it is impossible to have the scenario described in part c. What is θ_{crit}? $\surd$ *[problem by B. Shotwell]*



Problem 28.

29 (a) A mass M is at rest on a fixed, frictionless ramp inclined at angle θ with respect to the horizontal. The mass is connected to the force probe, as shown. What is the reading on the force probe? $\surd$
(b) Check that your answer to part a makes sense in the special cases $\theta = 0$ and $\theta = 90°$. *[problem by B. Shotwell]*



Problem 29.

Exercise 8: Vectors and motion

Each diagram on page 255 shows the motion of an object in an $x - y$ plane. Each dot is one location of the object at one moment in time. The time interval from one dot to the next is always the same, so you can think of the vector that connects one dot to the next as a $\mathbf{v}$ vector, and subtract to find $\Delta\mathbf{v}$ vectors.

1. Suppose the object in diagram 1 is moving from the top left to the bottom right. Deduce whatever you can about the force acting on it. Does the force always have the same magnitude? The same direction?

Invent a physical situation that this diagram could represent.

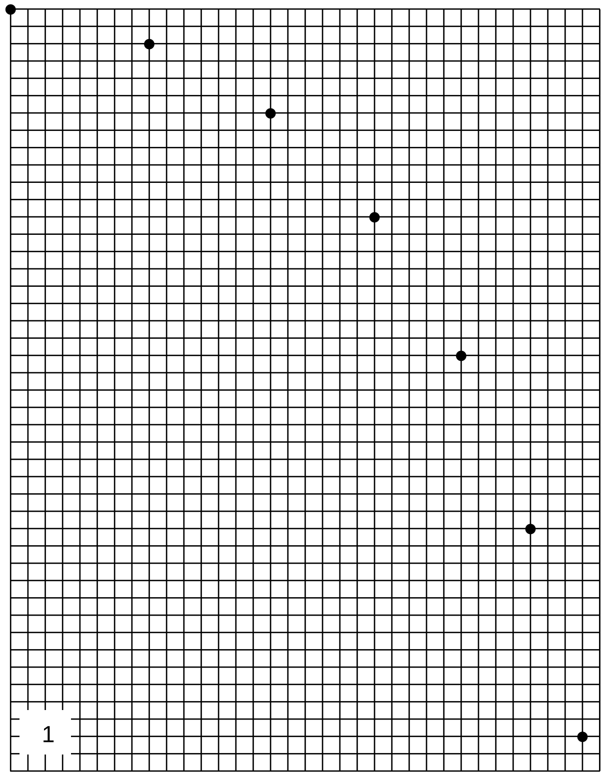
What if you reinterpret the diagram by reversing the object's direction of motion? Redo the construction of one of the $\Delta\mathbf{v}$ vectors and see what happens.

2. What can you deduce about the force that is acting in diagram 2?

Invent a physical situation that diagram 2 could represent.

3. What can you deduce about the force that is acting in diagram 3?

Invent a physical situation.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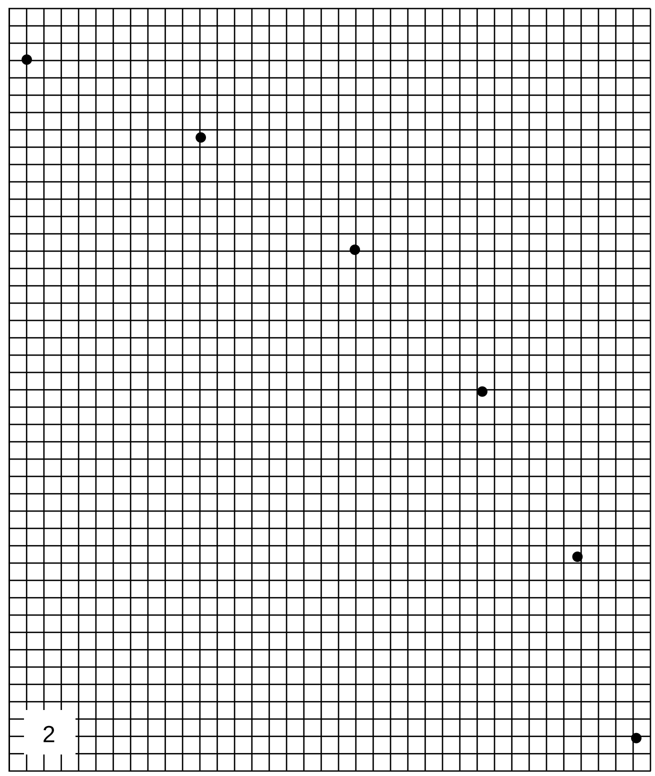
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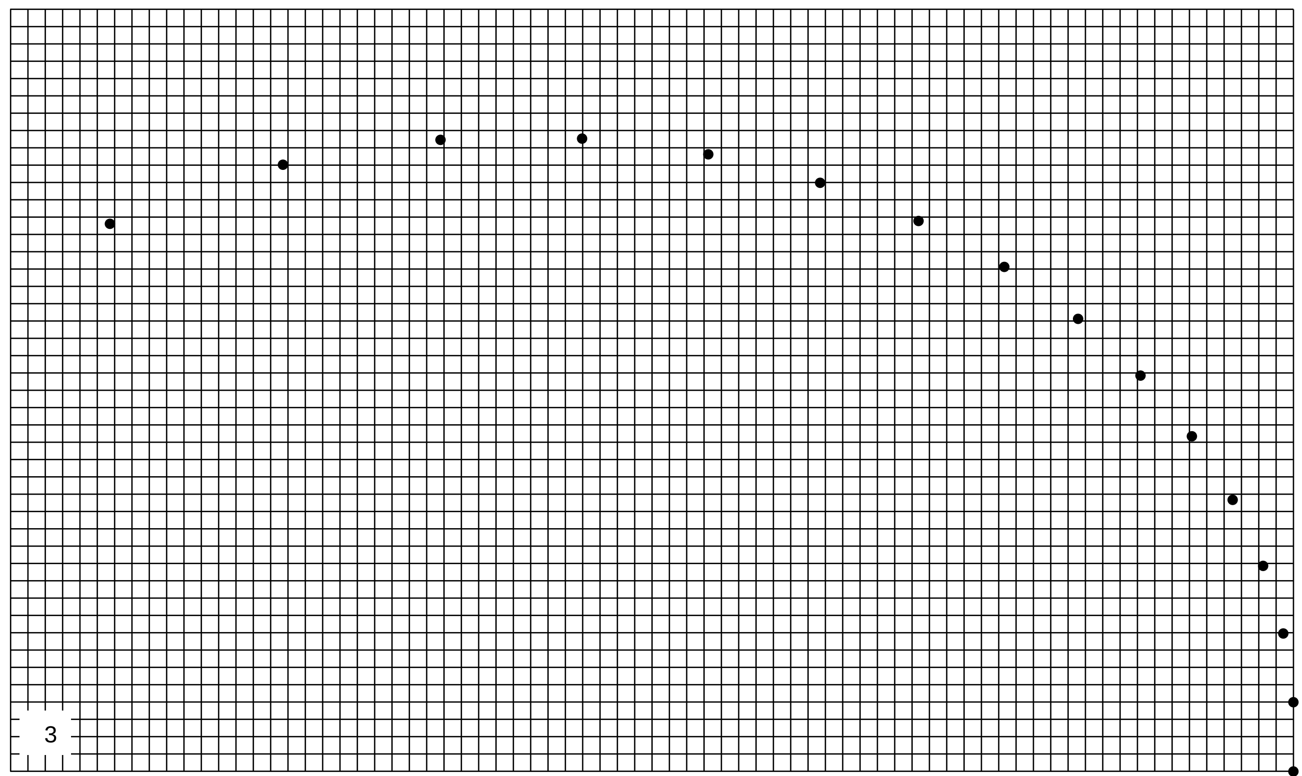
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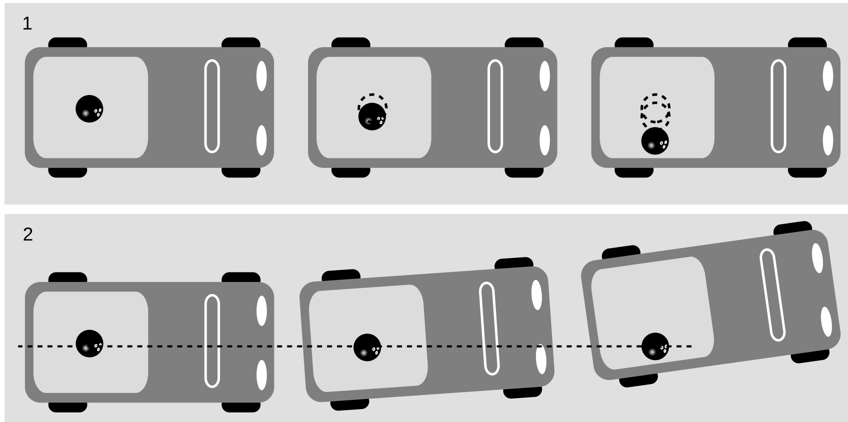
Chapter 9
Circular motion

9.1 Conceptual framework

I now live fifteen minutes from Disneyland, so my friends and family in my native Northern California think it's a little strange that I've never visited the Magic Kingdom again since a childhood trip to the south. The truth is that for me as a preschooler, Disneyland was not the Happiest Place on Earth. My mother took me on a ride in which little cars shaped like rocket ships circled rapidly around a central pillar. I knew I was going to die. There was a force trying to throw me outward, and the safety features of the ride would surely have been inadequate if I hadn't screamed the whole time to make sure Mom would hold on to me. Afterward, she seemed surprisingly indifferent to the extreme danger we had experienced.

Circular motion does not produce an outward force

My younger self's understanding of circular motion was partly right and partly wrong. I was wrong in believing that there was a force pulling me outward, away from the center of the circle. The easiest way to understand this is to bring back the parable of the bowling ball in the pickup truck from chapter 4. As the truck makes a left turn, the driver looks in the rearview mirror and thinks that some mysterious force is pulling the ball outward, but the truck is accelerating, so the driver's frame of reference is not an inertial frame. Newton's laws are violated in a noninertial frame, so the ball appears to accelerate without any actual force acting on it. Because we are used to inertial frames, in which accelerations are caused by forces, the ball's acceleration creates a vivid illusion that there must be an outward force.

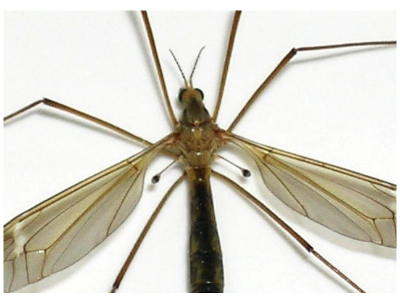


a / 1. In the turning truck's frame of reference, the ball appears to violate Newton's laws, displaying a sideways acceleration that is not the result of a force-interaction with any other object. 2. In an inertial frame of reference, such as the frame fixed to the earth's surface, the ball obeys Newton's first law. No forces are acting on it, and it continues moving in a straight line. It is the truck that is participating in an interaction with the asphalt, the truck that accelerates as it should according to Newton's second law.

In an inertial frame everything makes more sense. The ball has no force on it, and goes straight as required by Newton's first law. The truck has a force on it from the asphalt, and responds to it by accelerating (changing the direction of its velocity vector) as Newton's second law says it should.

The halteres *example 1*

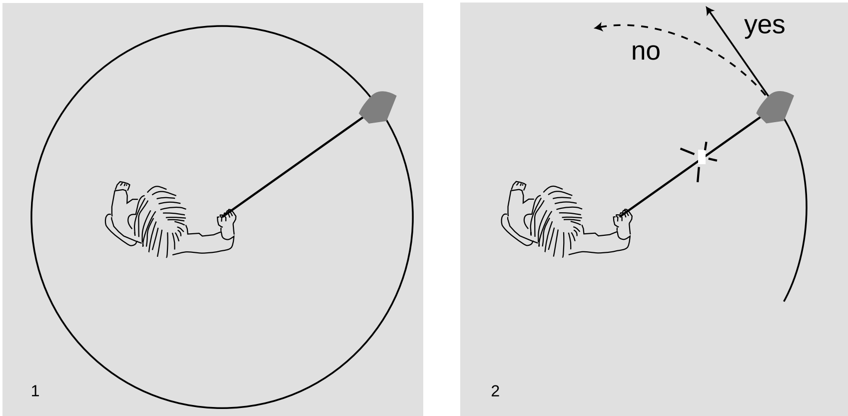
Another interesting example is an insect organ called the halteres, a pair of small knobbed limbs behind the wings, which vibrate up and down and help the insect to maintain its orientation in flight. The halteres evolved from a second pair of wings possessed by earlier insects. Suppose, for example, that the halteres are on their upward stroke, and at that moment an air current causes the fly to pitch its nose down. The halteres follow Newton's first law, continuing to rise vertically, but in the fly's rotating frame of reference, it seems as though they have been subjected to a backward force. The fly has special sensory organs that perceive this twist, and help it to correct itself by raising its nose.



b / This crane fly's halteres help it to maintain its orientation in flight.

Circular motion does not persist without a force

I was correct, however, on a different point about the Disneyland ride. To make me curve around with the car, I really did need some force such as a force from my mother, friction from the seat, or a normal force from the side of the car. (In fact, all three forces were probably adding together.) One of the reasons why Galileo failed to



c / 1. An overhead view of a person swinging a rock on a rope. A force from the string is required to make the rock's velocity vector keep changing direction. 2. If the string breaks, the rock will follow Newton's first law and go straight instead of continuing around the circle.

refine the principle of inertia into a quantitative statement like Newton's first law is that he was not sure whether motion without a force would naturally be circular or linear. In fact, the most impressive examples he knew of the persistence of motion were mostly circular: the spinning of a top or the rotation of the earth, for example. Newton realized that in examples such as these, there really were forces at work. Atoms on the surface of the top are prevented from flying off straight by the ordinary force that keeps atoms stuck together in solid matter. The earth is nearly all liquid, but gravitational forces pull all its parts inward.

d / Sparks fly away along tangents to a grinding wheel.

Uniform and nonuniform circular motion

Circular motion always involves a change in the direction of the velocity vector, but it is also possible for the magnitude of the velocity to change at the same time. Circular motion is referred to as *uniform* if $|\mathbf{v}|$ is constant, and *nonuniform* if it is changing.

Your speedometer tells you the magnitude of your car's velocity vector, so when you go around a curve while keeping your speedometer needle steady, you are executing uniform circular motion. If your speedometer reading is changing as you turn, your circular motion is nonuniform. Uniform circular motion is simpler to analyze mathematically, so we will attack it first and then pass to the nonuniform case.

self-check A

Which of these are examples of uniform circular motion and which are nonuniform?

(1) the clothes in a clothes dryer (assuming they remain against the inside of the drum, even at the top)

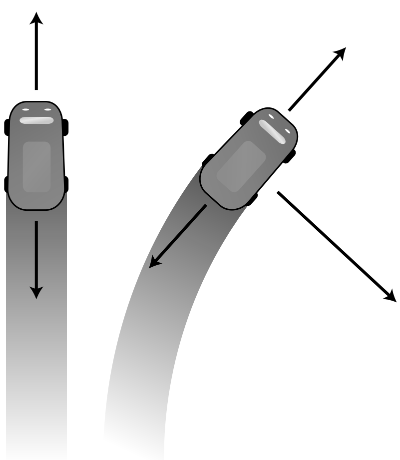
(2) a rock on the end of a string being whirled in a vertical circle ▷ Answer, p. 560

e / To make the brick go in a circle, I had to exert an inward force on the rope.

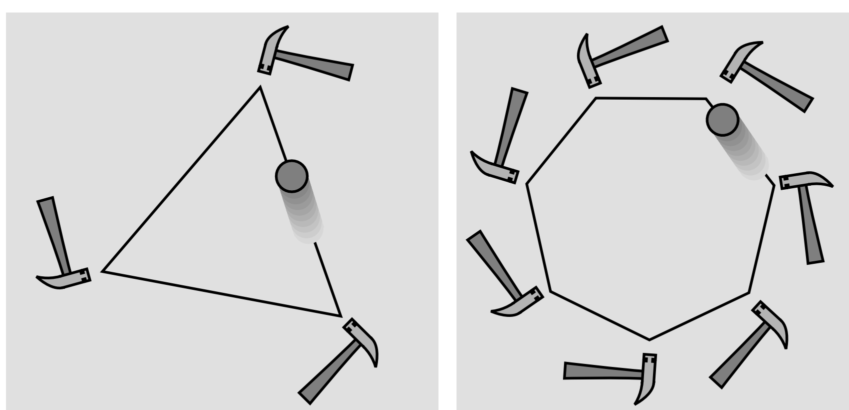
Only an inward force is required for uniform circular motion.

Figure c showed the string pulling in straight along a radius of the circle, but many people believe that when they are doing this they must be "leading" the rock a little to keep it moving along. That is, they believe that the force required to produce uniform circular motion is not directly inward but at a slight angle to the radius of the circle. This intuition is incorrect, which you can easily verify for yourself now if you have some string handy. It is only while you are getting the object going that your force needs to be at an angle to the radius. During this initial period of speeding up, the motion is not uniform. Once you settle down into uniform circular motion, you only apply an inward force.

If you have not done the experiment for yourself, here is a theoretical argument to convince you of this fact. We have discussed in chapter 6 the principle that forces have no perpendicular effects. To keep the rock from speeding up or slowing down, we only need to make sure that our force is perpendicular to its direction of motion. We are then guaranteed that its forward motion will remain unaffected: our force can have no perpendicular effect, and there is no other force acting on the rock which could slow it down. The rock requires no forward force to maintain its forward motion, any more than a projectile needs a horizontal force to "help it over the top" of its arc.



g / When a car is going straight at constant speed, the forward and backward forces on it are canceling out, producing a total force of zero. When it moves in a circle at constant speed, there are three forces on it, but the forward and backward forces cancel out, so the vector sum is an inward force.



f / A series of three hammer taps makes the rolling ball trace a triangle, seven hammers a heptagon. If the number of hammers was large enough, the ball would essentially be experiencing a steady inward force, and it would go in a circle. In no case is any forward force necessary.

Why, then, does a car driving in circles in a parking lot stop executing uniform circular motion if you take your foot off the gas? The source of confusion here is that Newton's laws predict an object's motion based on the *total* force acting on it. A car driving in circles has three forces on it

(1) an inward force from the asphalt, controlled with the steering wheel;

(2) a forward force from the asphalt, controlled with the gas pedal; and

(3) backward forces from air resistance and rolling resistance.

You need to make sure there is a forward force on the car so that the backward forces will be exactly canceled out, creating a vector sum that points directly inward.

h / Example 2.

A motorcycle making a turn *example 2*

The motorcyclist in figure h is moving along an arc of a circle. It looks like he's chosen to ride the slanted surface of the dirt at a place where it makes just the angle he wants, allowing him to get the force he needs on the tires as a normal force, without needing any frictional force. The dirt's normal force on the tires points up and to our left. The vertical component of that force is canceled by gravity, while its horizontal component causes him to curve.

In uniform circular motion, the acceleration vector is inward.

Since experiments show that the force vector points directly inward, Newton's second law implies that the acceleration vector points inward as well. This fact can also be proven on purely kinematical grounds, and we will do so in the next section.

Clock-comparison tests of Newton's first law *example 3*

Immediately after his original statement of the first law in the *Principia Mathematica*, Newton offers the supporting example of a spinning top, which only slows down because of friction. He describes the different parts of the top as being held together by "cohesion," i.e., internal forces. Because these forces act toward the center, they don't speed up or slow down the motion. The applicability of the first law, which only describes linear motion, may be more clear if we simply take figure f as a model of rotation. Between hammer taps, the ball experiences no force, so by the first law it doesn't speed up or slow down.

Suppose that we want to subject the first law to a stringent experimental test.[1] The law predicts that if we use a clock to measure the rate of rotation of an object spinning frictionlessly, it won't "naturally" slow down as Aristotle would have expected. But what is a clock but something with hands that rotate at a fixed rate? In

[1] Page 73 lists places in this book where we describe experimental tests of Newton's first law.

other words, we are comparing one clock with another. This is called a clock-comparison experiment. Suppose that the laws of physics weren't purely Newtonian, and there really was a very slight Aristotelian tendency for motion to slow down in the absence of friction. If we compare two clocks, they should both slow down, but if they aren't the same type of clock, then it seems unlikely that they would slow down at exactly the same rate, and over time they should drift further and further apart.

High-precision clock-comparison experiments have been done using a variety of clocks. In atomic clocks, the thing spinning is an atom. Astronomers can observe the rotation of collapsed stars called pulars, which, unlike the earth, can rotate with almost no disturbance due to geological activity or friction induced by the tides. In these experiments, the pulsars are observed to match the rates of the atomic clocks with a drift of less than about 10^{-6} seconds over a period of 10 years.[2] Atomic clocks using atoms of different elements drift relative to one another by no more than about 10^{-16} per year.[3]

It is not presently possible to do experiments with a similar level of precision using human-scale rotating objects. However, a set of gyroscopes aboard the Gravity Probe B satellite were allowed to spin weightlessly in a vacuum, without any physical contact that would have caused kinetic friction. Their rotation was extremely accurately monitored for the purposes of another experiment (a test of Einstein's theory of general relativity, which was the purpose of the mission), and they were found to be spinning down so gradually that they would have taken about 10,000 years to slow down by a factor of two. This rate was consistent with estimates of the amount of friction to be expected from the small amount of residual gas present in the vacuum chambers.

A subtle point in the interpretation of these experiments is that if there was a slight tendency for motion to slow down, we would have to decide what it was supposed to slow down relative to. A straight-line motion that is slowing down in some frame of reference can always be described as *speeding up* in some other appropriately chosen frame (problem 12, p. 98). If the laws of physics did have this slight Aristotelianism mixed in, we could wait for the anomalous acceleration or deceleration to stop. The object we were observing would then define a special or "preferred" frame of reference. Standard theories of physics do not have such a preferred frame, and clock-comparison experiments can be viewed as tests of the existence of such a frame. Another test for the existence of a preferred frame was described on p. 155.

[2] Matsakis *et al.*, Astronomy and Astrophysics 326 (1997) 924. Freely available online at adsabs.harvard.edu.

[3] Guéna *et al.*, arxiv.org/abs/1205.4235

Discussion questions

A In the game of crack the whip, a line of people stand holding hands, and then they start sweeping out a circle. One person is at the center, and rotates without changing location. At the opposite end is the person who is running the fastest, in a wide circle. In this game, someone always ends up losing their grip and flying off. Suppose the person on the end loses her grip. What path does she follow as she goes flying off? Draw an overhead view. (Assume she is going so fast that she is really just trying to put one foot in front of the other fast enough to keep from falling; she is not able to get any significant horizontal force between her feet and the ground.)

B Suppose the person on the outside is still holding on, but feels that she may loose her grip at any moment. What force or forces are acting on her, and in what directions are they? (We are not interested in the vertical forces, which are the earth's gravitational force pulling down, and the ground's normal force pushing up.) Make a table in the format shown in section 5.3.

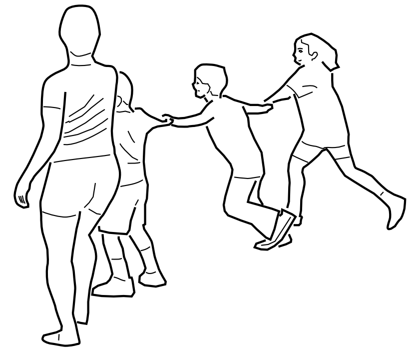
C Suppose the person on the outside is still holding on, but feels that she may loose her grip at any moment. What is wrong with the following analysis of the situation? "The person whose hand she's holding exerts an inward force on her, and because of Newton's third law, there's an equal and opposite force acting outward. That outward force is the one she feels throwing her outward, and the outward force is what might make her go flying off, if it's strong enough."

D If the only force felt by the person on the outside is an inward force, why doesn't she go straight in?

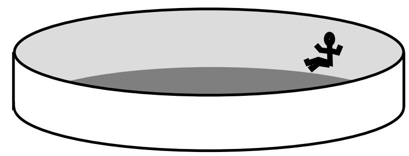
E In the amusement park ride shown in the figure, the cylinder spins faster and faster until the customer can pick her feet up off the floor without falling. In the old Coney Island version of the ride, the floor actually dropped out like a trap door, showing the ocean below. (There is also a version in which the whole thing tilts up diagonally, but we're discussing the version that stays flat.) If there is no outward force acting on her, why does she stick to the wall? Analyze all the forces on her.

F What is an example of circular motion where the inward force is a normal force? What is an example of circular motion where the inward force is friction? What is an example of circular motion where the inward force is the sum of more than one force?

G Does the acceleration vector always change continuously in circular motion? The velocity vector?



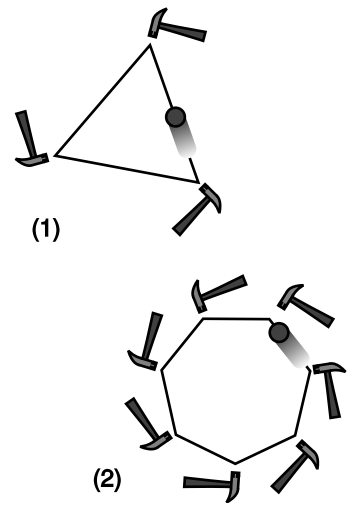
Discussion questions A-D



Discussion question E.

9.2 Uniform circular motion

In this section I derive some convenient results, which you will use frequently, for the acceleration of an object performing uniform circular motion.



i / This figure shows an intuitive justification for the fact proved mathematically in this section, that the direction of the force and acceleration in circular motion is inward. The heptagon, 2, is a better approximation to a circle than the triangle, 1. To make an infinitely good approximation to circular motion, we would need to use an infinitely large number of infinitesimal taps, which would amount to a steady inward force.

An object moving in a circle of radius r in the x-y plane has

$$\begin{aligned} x &= r \cos \omega t \qquad \text{and} \\ y &= r \sin \omega t, \end{aligned}$$

where ω is the number of radians traveled per second, and the positive or negative sign indicates whether the motion is clockwise or counterclockwise.

Differentiating, we find that the components of the velocity are

$$\begin{aligned} v_x &= -\omega r \sin \omega t \qquad \text{and} \\ v_y &= \quad \omega r \cos \omega t, \end{aligned}$$

and for the acceleration we have

$$\begin{aligned} a_x &= -\omega^2 r \cos \omega t \qquad \text{and} \\ a_y &= -\omega^2 r \sin \omega t. \end{aligned}$$

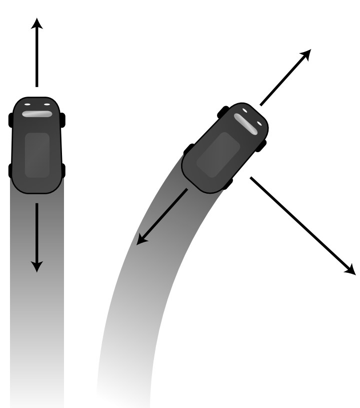
The acceleration vector has cosines and sines in the same places as the $\mathbf{r}$ vector, but with minus signs in front, so it points in the opposite direction, i.e., toward the center of the circle. By Newton's second law, $\mathbf{a}=\mathbf{F}/m$, this shows that the force must be inward as well; without this force, the object would fly off straight.

The magnitude of the acceleration is

$$\begin{aligned} |\mathbf{a}| &= \sqrt{a_x^2 + a_y^2} \\ &= \omega^2 r. \end{aligned}$$

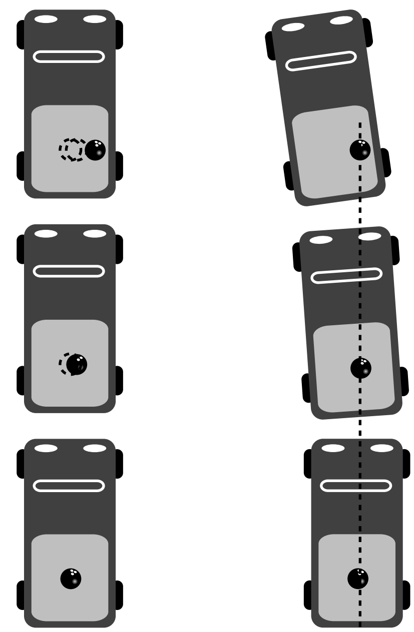
It makes sense that ω is squared, since reversing the sign of ω corresponds to reversing the direction of motion, but the acceleration is toward the center of the circle, regardless of whether the motion is clockwise or counterclockwise. This result can also be rewritten in the form

$$|\mathbf{a}| = \frac{|\mathbf{v}|^2}{r}.$$



j / The total force in the forward-backward direction is zero in both cases.

These results are counterintuitive. Until Newton, physicists and laypeople alike had assumed that the planets would need a force to push them *forward* in their orbits. Figure i may help to make it more plausible that only an inward force is required. A forward force might be needed in order to cancel out a backward force such as friction, j, but the total force in the forward-backward direction needs to be exactly zero for constant-speed motion. When you are in a car undergoing circular motion, there is also a strong illusion of an *outward* force. But what object could be making such a force? The car's seat makes an inward force on you, not an outward one. There is no object that could be exerting an outward force on your body. In reality, this force is an illusion that comes from our brain's intuitive efforts to interpret the situation within a noninertial frame of reference. As shown in figure k, we can describe everything perfectly well in an inertial frame of reference, such as the frame attached to the sidewalk. In such a frame, the bowling ball goes straight because there is *no* force on it. The wall of the truck's bed hits the ball, not the other way around.



k / There is no outward force on the bowling ball, but in the noninertial frame it seems like one exists.

Force required to turn on a bike *example 4*

▷ A bicyclist is making a turn along an arc of a circle with radius 20 m, at a speed of 5 m/s. If the combined mass of the cyclist plus the bike is 60 kg, how great a static friction force must the road be able to exert on the tires?

▷ Taking the magnitudes of both sides of Newton's second law gives

$$\begin{aligned}|\mathbf{F}| &= |m\mathbf{a}| \\ &= m|\mathbf{a}|.\end{aligned}$$

Substituting $|\mathbf{a}| = |\mathbf{v}|^2/r$ gives

$$\begin{aligned}|\mathbf{F}| &= m|\mathbf{v}|^2/r \\ &\approx 80\text{ N}\end{aligned}$$

(rounded off to one sig fig).

Don't hug the center line on a curve! *example 5*

▷ You're driving on a mountain road with a steep drop on your right. When making a left turn, is it safer to hug the center line or to stay closer to the outside of the road?

▷ You want whichever choice involves the least acceleration, because that will require the least force and entail the least risk of exceeding the maximum force of static friction. Assuming the curve is an arc of a circle and your speed is constant, your car is performing uniform circular motion, with $|\mathbf{a}| = |\mathbf{v}|^2/r$. The dependence on the square of the speed shows that driving slowly is the main safety measure you can take, but for any given speed you also want to have the largest possible value of r. Even though your instinct is to keep away from that scary precipice, you are actually less likely to skid if you keep toward the outside, because then you are describing a larger circle.

Acceleration related to radius and period of rotation *example 6*

▷ How can the equation for the acceleration in uniform circular motion be rewritten in terms of the radius of the circle and the period, T, of the motion, i.e., the time required to go around once?

▷ The period can be related to the speed as follows:

$$|\mathbf{v}| = \frac{\text{circumference}}{T} = 2\pi r/T.$$

Substituting into the equation $|\mathbf{a}| = |\mathbf{v}|^2/r$ gives

$$|\mathbf{a}| = \frac{4\pi^2 r}{T^2}.$$

l / Example 7.

A clothes dryer *example 7*

▷ My clothes dryer has a drum with an inside radius of 35 cm, and it spins at 48 revolutions per minute. What is the acceleration of the clothes inside?

▷ We can solve this by finding the period and plugging in to the result of the previous example. If it makes 48 revolutions in one minute, then the period is 1/48 of a minute, or 1.25 s. To get an acceleration in mks units, we must convert the radius to 0.35 m. Plugging in, the result is 8.8 m/s^2.

More about clothes dryers! *example 8*

▷ In a discussion question in the previous section, we made the assumption that the clothes remain against the inside of the drum as they go over the top. In light of the previous example, is this a correct assumption?

▷ No. We know that there must be some minimum speed at which the motor can run that will result in the clothes just barely staying against the inside of the drum as they go over the top. If the clothes dryer ran at just this minimum speed, then there would be no normal force on the clothes at the top: they would be on the verge of losing contact. The only force acting on them at the top would be the force of gravity, which would give them an acceleration of $g = 9.8\ \text{m/s}^2$. The actual dryer must be running slower than this minimum speed, because it produces an acceleration of only 8.8 m/s^2. My theory is that this is done intentionally, to make the clothes mix and tumble.

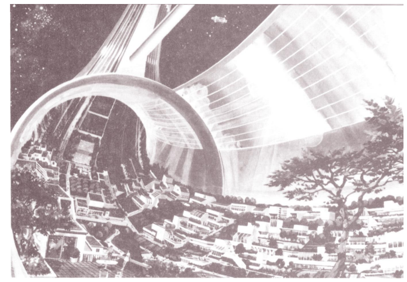
▷ *Solved problem: The tilt-a-whirl* *page 270, problem 3*

▷ *Solved problem: An off-ramp* *page 271, problem 5*

Discussion questions

A A certain amount of force is needed to provide the acceleration of circular motion. What if we are exerting a force perpendicular to the direction of motion in an attempt to make an object trace a circle of radius r, but the force isn't as big as $m|\mathbf{v}|^2/r$?

B Suppose a rotating space station, as in figure m on page 267, is built. It gives its occupants the illusion of ordinary gravity. What happens when a person in the station lets go of a ball? What happens when she throws a ball straight "up" in the air (i.e., towards the center)?



m / Discussion question B. An artist's conception of a rotating space colony in the form of a giant wheel. A person living in this noninertial frame of reference has an illusion of a force pulling her outward, toward the deck, for the same reason that a person in the pickup truck has the illusion of a force pulling the bowling ball. By adjusting the speed of rotation, the designers can make an acceleration $|\mathbf{v}|^2/r$ equal to the usual acceleration of gravity on earth. On earth, your acceleration standing on the ground is zero, and a falling rock heads for your feet with an acceleration of 9.8 m/s^2. A person standing on the deck of the space colony has an *upward* acceleration of 9.8 m/s^2, and when she lets go of a rock, her feet head *up* at the nonaccelerating rock. To her, it seems the same as true gravity.

9.3 Nonuniform circular motion

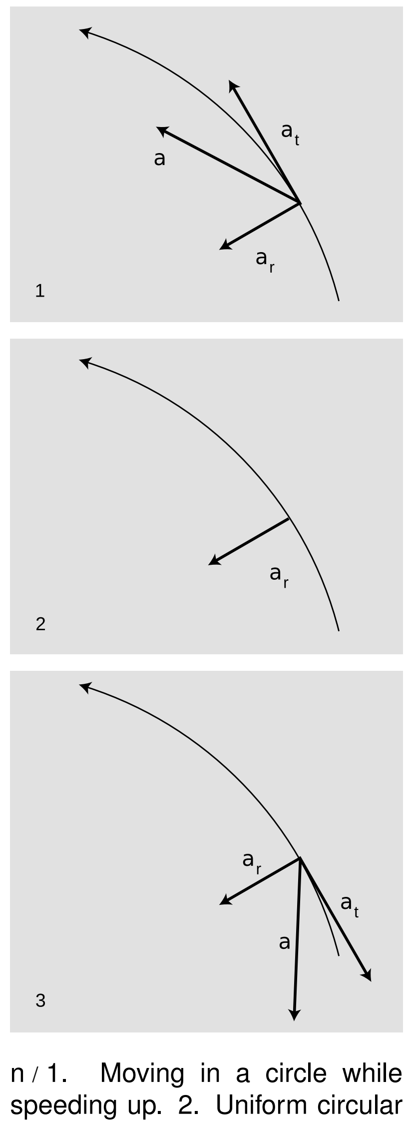
What about nonuniform circular motion? Although so far we have been discussing components of vectors along fixed x and y axes, it now becomes convenient to discuss components of the acceleration vector along the radial line (in-out) and the tangential line (along the direction of motion). For nonuniform circular motion, the radial component of the acceleration obeys the same equation as for uniform circular motion,

$$a_r = v^2/r,$$

where $v = |\mathbf{v}|$, but the acceleration vector also has a tangential component,

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t}.$$

The latter quantity has a simple interpretation. If you are going around a curve in your car, and the speedometer needle is moving, the tangential component of the acceleration vector is simply what you would have thought the acceleration was if you saw the speedometer and didn't know you were going around a curve.



n / 1. Moving in a circle while speeding up. 2. Uniform circular motion. 3. Slowing down.

Slow down before a turn, not during it. *example 9*

▷ When you're making a turn in your car and you're afraid you may skid, isn't it a good idea to slow down?

▷ If the turn is an arc of a circle, and you've already completed part of the turn at constant speed without skidding, then the road and tires are apparently capable of enough static friction to supply an acceleration of $|\mathbf{v}|^2/r$. There is no reason why you would skid out now if you haven't already. If you get nervous and brake, however, then you need to have a tangential acceleration component in addition to the radial one you were already able to produce successfully. This would require an acceleration vector with a greater magnitude, which in turn would require a larger force. Static friction might not be able to supply that much force, and you might skid out. The safer thing to do is to approach the turn at a comfortably low speed.

▹ *Solved problem: A bike race* *page 272, problem 10*

Summary

Selected vocabulary

uniform circular motion	circular motion in which the magnitude of the velocity vector remains constant
nonuniform circular motion	circular motion in which the magnitude of the velocity vector changes
radial	parallel to the radius of a circle; the in-out direction
tangential	tangent to the circle, perpendicular to the radial direction

Notation

a_r	radial acceleration; the component of the acceleration vector along the in-out direction
a_t	tangential acceleration; the component of the acceleration vector tangent to the circle

Summary

If an object is to have circular motion, a force must be exerted on it toward the center of the circle. There is no outward force on the object; the illusion of an outward force comes from our experiences in which our point of view was rotating, so that we were viewing things in a noninertial frame.

An object undergoing uniform circular motion has an inward acceleration vector of magnitude

$$|\mathbf{a}| = v^2/r,$$

where $v = |\mathbf{v}|$. In nonuniform circular motion, the radial and tangential components of the acceleration vector are

$$a_r = v^2/r$$
$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t}.$$

Problems

Key

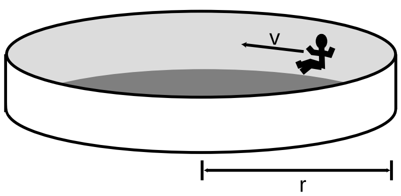
√ A computerized answer check is available online.

∫ A problem that requires calculus.

⋆ A difficult problem.

1 Show that the expression $|\mathbf{v}|^2/r$ has the units of acceleration.

2 A plane is flown in a loop-the-loop of radius 1.00 km. The plane starts out flying upside-down, straight and level, then begins curving up along the circular loop, and is right-side up when it reaches the top. (The plane may slow down somewhat on the way up.) How fast must the plane be going at the top if the pilot is to experience no force from the seat or the seatbelt while at the top of the loop? √



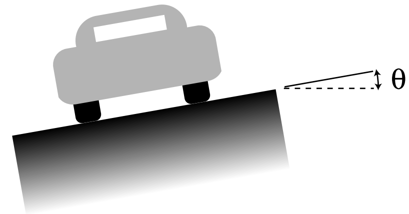
Problem 3.

3 The amusement park ride shown in the figure consists of a cylindrical room that rotates about its vertical axis. When the rotation is fast enough, a person against the wall can pick his or her feet up off the floor and remain "stuck" to the wall without falling.
(a) Suppose the rotation results in the person having a speed v. The radius of the cylinder is r, the person's mass is m, the downward acceleration of gravity is g, and the coefficient of static friction between the person and the wall is μ_s. Find an equation for the speed, v, required, in terms of the other variables. (You will find that one of the variables cancels out.)
(b) Now suppose two people are riding the ride. Huy is wearing denim, and Gina is wearing polyester, so Huy's coefficient of static friction is three times greater. The ride starts from rest, and as it begins rotating faster and faster, Gina must wait longer before being able to lift her feet without sliding to the floor. Based on your equation from part a, how many times greater must the speed be before Gina can lift her feet without sliding down? ▷ Solution, p. 551 ⋆

4 The bright star Sirius has a mass of 4.02×10^{30} kg and lies at a distance of 8.1×10^{16} m from our solar system. Suppose you're standing on a merry-go-round carousel rotating with a period of 10 seconds, and Sirius is on the horizon. You adopt a rotating, noninertial frame of reference, in which the carousel is at rest, and the universe is spinning around it. If you drop a corndog, you see it accelerate horizontally away from the axis, and you interpret this as the result of some horizontal force. This force does not actually exist; it only seems to exist because you're insisting on using a noninertial frame. Similarly, calculate the force that seems to act on Sirius in this frame of reference. Comment on the physical plausibility of this force, and on what object could be exerting it. √

5 An engineer is designing a curved off-ramp for a freeway. Since the off-ramp is curved, she wants to bank it to make it less likely that motorists going too fast will wipe out. If the radius of the curve is r, how great should the banking angle, θ, be so that for a car going at a speed v, no static friction force whatsoever is required to allow the car to make the curve? State your answer in terms of v, r, and g, and show that the mass of the car is irrelevant.

▷ Solution, p. 552



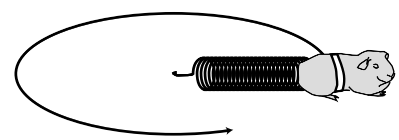
Problem 5.

6 Lionel brand toy trains come with sections of track in standard lengths and shapes. For circular arcs, the most commonly used sections have diameters of 662 and 1067 mm at the inside of the outer rail. The maximum speed at which a train can take the broader curve without flying off the tracks is 0.95 m/s. At what speed must the train be operated to avoid derailing on the tighter curve? √

7 Psychology professor R.O. Dent requests funding for an experiment on compulsive thrill-seeking behavior in guinea pigs, in which the subject is to be attached to the end of a spring and whirled around in a horizontal circle. The spring has relaxed length b, and obeys Hooke's law with spring constant k. It is stiff enough to keep from bending significantly under the guinea pig's weight.

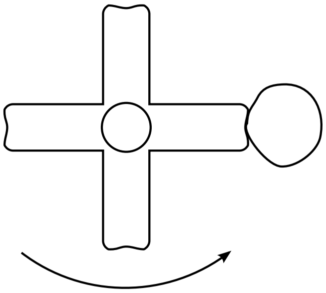
(a) Calculate the length of the spring when it is undergoing steady circular motion in which one rotation takes a time T. Express your result in terms of k, b, T, and the guinea pig's mass m. √

(b) The ethics committee somehow fails to veto the experiment, but the safety committee expresses concern. Why? Does your equation do anything unusual, or even spectacular, for any particular value of T? What do you think is the physical significance of this mathematical behavior?



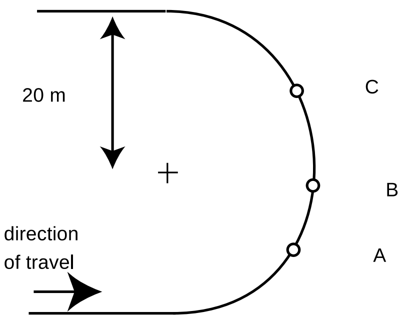
Problem 7.

8 The acceleration of an object in uniform circular motion can be given either by $|\mathbf{a}| = |\mathbf{v}|^2/r$ or, equivalently, by $|\mathbf{a}| = 4\pi^2 r/T^2$, where T is the time required for one cycle(example 6 on page 265). Person A says based on the first equation that the acceleration in circular motion is greater when the circle is smaller. Person B, arguing from the second equation, says that the acceleration is smaller when the circle is smaller. Rewrite the two statements so that they are less misleading, eliminating the supposed paradox. [Based on a problem by Arnold Arons.]



Problem 9.

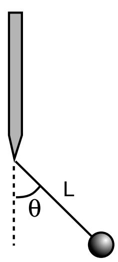
9 When you're done using an electric mixer, you can get most of the batter off of the beaters by lifting them out of the batter with the motor running at a high enough speed. Let's imagine, to make things easier to visualize, that we instead have a piece of tape stuck to one of the beaters.
(a) Explain why static friction has no effect on whether or not the tape flies off.
(b) Analyze the forces in which the tape participates, using a table in the format shown in section 5.3.
(c) Suppose you find that the tape doesn't fly off when the motor is on a low speed, but at a greater speed, the tape won't stay on. Why would the greater speed change things? [Hint: If you don't invoke any law of physics, you haven't explained it.]



Problem 10.

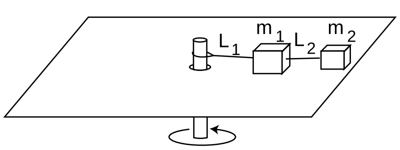
10 Three cyclists in a race are rounding a semicircular curve. At the moment depicted, cyclist A is using her brakes to apply a force of 375 N to her bike. Cyclist B is coasting. Cyclist C is pedaling, resulting in a force of 375 N on her bike. Each cyclist, with her bike, has a mass of 75 kg. At the instant shown, the instantaneous speed of all three cyclists is 10 m/s. On the diagram, draw each cyclist's acceleration vector with its tail on top of her present position, indicating the directions and lengths reasonably accurately. Indicate approximately the consistent scale you are using for all three acceleration vectors. Extreme precision is not necessary as long as the directions are approximately right, and lengths of vectors that should be equal appear roughly equal, etc. Assume all three cyclists are traveling along the road all the time, not wandering across their lane or wiping out and going off the road.

▷ Solution, p. 552



Problem 11.

11 The figure shows a ball on the end of a string of length L attached to a vertical rod which is spun about its vertical axis by a motor. The period (time for one rotation) is P.
(a) Analyze the forces in which the ball participates.
(b) Find how the angle θ depends on P, g, and L. [Hints: (1) Write down Newton's second law for the vertical and horizontal components of force and acceleration. This gives two equations, which can be solved for the two unknowns, θ and the tension in the string. (2) If you introduce variables like v and r, relate them to the variables your solution is supposed to contain, and eliminate them.] √
(c) What happens mathematically to your solution if the motor is run very slowly (very large values of P)? Physically, what do you think would actually happen in this case?



Problem 12.

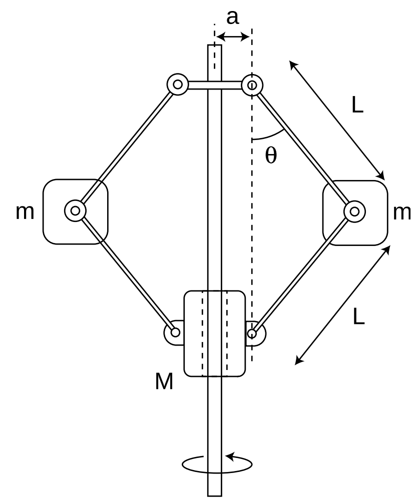
12 The figure shows two blocks of masses m_1 and m_2 sliding in circles on a frictionless table. Find the tension in the strings if the period of rotation (time required for one rotation) is P. √

13 This problem is now number 19 in ch. 12, p. 356.

14 The figure shows an old-fashioned device called a flyball governor, used for keeping an engine running at the correct speed. The whole thing rotates about the vertical shaft, and the mass M is free to slide up and down. This mass would have a connection (not shown) to a valve that controlled the engine. If, for instance, the engine ran too fast, the mass would rise, causing the engine to slow back down.
(a) Show that in the special case of $a = 0$, the angle θ is given by

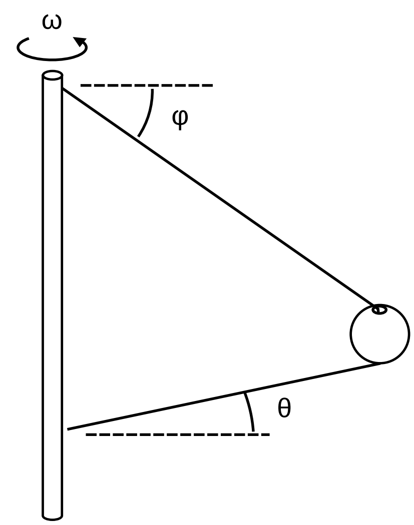
$$\theta = \cos^{-1}\left(\frac{g(m+M)P^2}{4\pi^2 mL}\right),$$

where P is the period of rotation (time required for one complete rotation).
(b) There is no closed-form solution for θ in the general case where a is not zero. However, explain how the undesirable low-speed behavior of the $a = 0$ device would be improved by making a nonzero. ⋆



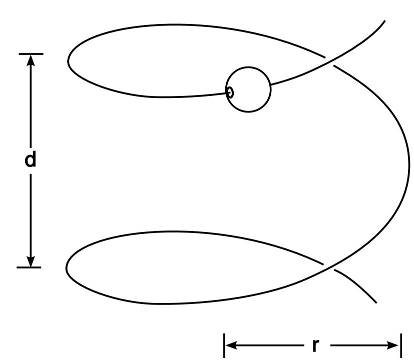
Problem 14.

15 The vertical post rotates at frequency ω. The bead slides freely along the string, reaching an equilibrium in which its distance from the axis is r and the angles θ and ϕ have some particular values. Find ϕ in terms of θ, g, ω, and r. √ ⋆

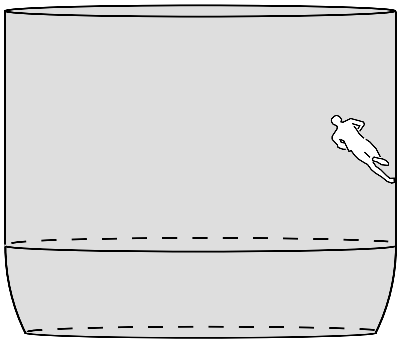


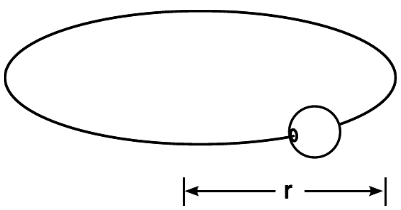
Problem 15.

16 A bead slides down along a piece of wire that is in the shape of a helix. The helix lies on the surface of a vertical cylinder of radius r, and the vertical distance between turns is d.
(a) Ordinarily when an object slides downhill under the influence of kinetic friction, the velocity-independence of kinetic friction implies that the acceleration is constant, and therefore there is no limit to the object's velocity. Explain the physical reason why this argument fails here, so that the bead will in fact have some limiting velocity.
(b) Find the limiting velocity.
(c) Show that your result has the correct behavior in the limit of $r \to \infty$. [Problem by B. Korsunsky.] √ ⋆

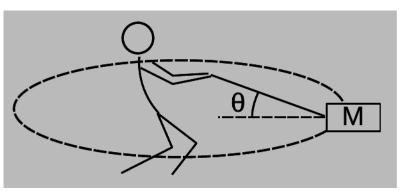


Problem 16.


Problem 17.



Problem 18.



Problem 20.

17 In a well known stunt from circuses and carnivals, a motorcyclist rides around inside a big bowl, gradually speeding up and rising higher. Eventually the cyclist can get up to where the walls of the bowl are vertical. Let's estimate the conditions under which a running human could do the same thing.
(a) If the runner can run at speed v, and her shoes have a coefficient of static friction μ_s, what is the maximum radius of the circle? ✓
(b) Show that the units of your answer make sense.
(c) Check that its dependence on the variables makes sense.
(d) Evaluate your result numerically for $v = 10$ m/s (the speed of an olympic sprinter) and $\mu_s = 5$. (This is roughly the highest coefficient of static friction ever achieved for surfaces that are not sticky. The surface has an array of microscopic fibers like a hair brush, and is inspired by the hairs on the feet of a gecko. These assumptions are not necessarily realistic, since the person would have to run at an angle, which would be physically awkward.) ✓

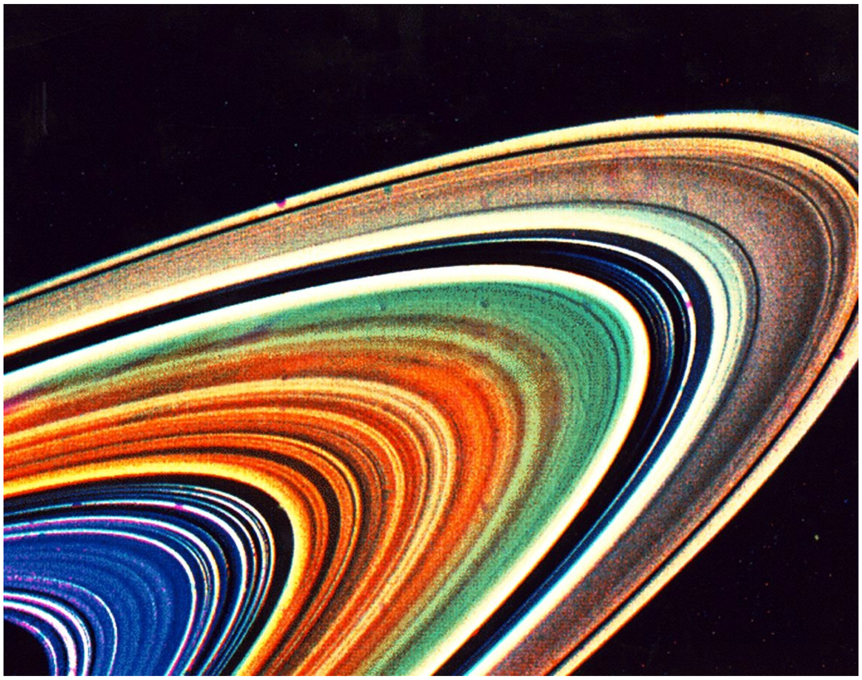
18 Find the motion of a bead that slides with coefficient of kinetic friction μ on a circular wire of radius r. Neglect gravity. [This requires a couple of standard techniques for solving a differential equation, but not obscure or tricky ones.] ⋆

19 A car is approaching the top of a hill of radius of curvature R.
(a) If the normal force that the driver feels at the top of the hill is 1/3 of their weight, how fast is the car going? ✓
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence of your answer on the variables makes sense.

[problem by B. Shotwell]

20 Tommy the playground bully is whirling a brick tied to the end of a rope. The rope makes an angle θ with respect to the horizontal, and the brick undergoes circular motion with radius R.
(a) What is the speed of the brick? ✓
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence of your answer on the variables makes sense, and comment on the limit $\theta \to 0$.

[problem by B. Shotwell]



Gravity is the only really important force on the cosmic scale. This false-color representation of Saturn's rings was made from an image sent back by the Voyager 2 space probe. The rings are composed of innumerable tiny ice particles orbiting in circles under the influence of saturn's gravity.

Chapter 10
Gravity

a / Johannes Kepler found a mathematical description of the motion of the planets, which led to Newton's theory of gravity.

Cruise your radio dial today and try to find any popular song that would have been imaginable without Louis Armstrong. By introducing solo improvisation into jazz, Armstrong took apart the jigsaw puzzle of popular music and fit the pieces back together in a different way. In the same way, Newton reassembled our view of the universe. Consider the titles of some recent physics books written for the general reader: The God Particle, Dreams of a Final Theory. Without Newton, such attempts at universal understanding would not merely have seemed a little pretentious, they simply would not have occurred to anyone.

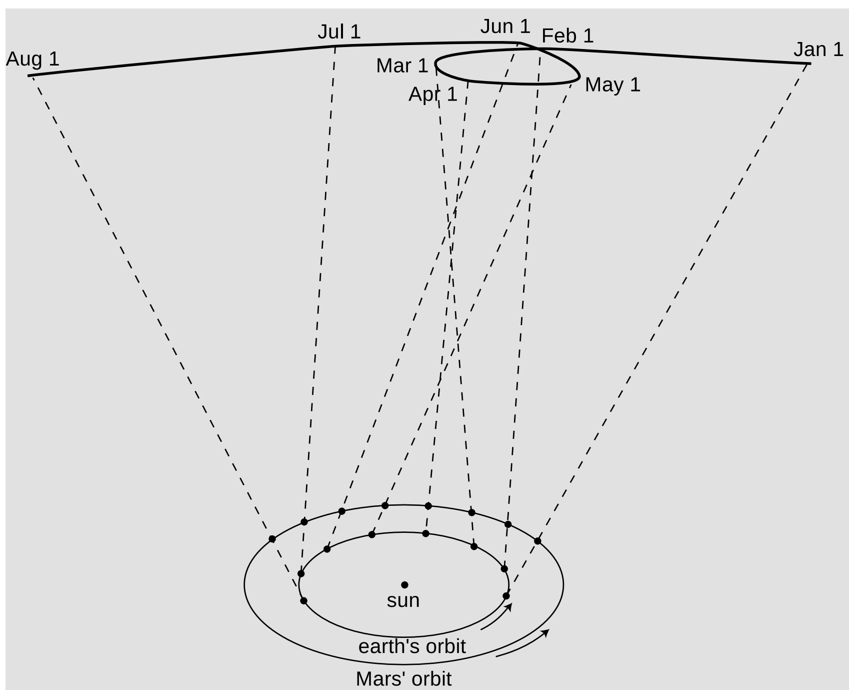
This chapter is about Newton's theory of gravity, which he used to explain the motion of the planets as they orbited the sun. Whereas this book has concentrated on Newton's laws of motion, leaving gravity as a dessert, Newton tosses off the laws of motion in the first 20 pages of the Principia Mathematica and then spends the next 130 discussing the motion of the planets. Clearly he saw this as the crucial scientific focus of his work. Why? Because in it he

b / Tycho Brahe made his name as an astronomer by showing that the bright new star, today called a supernova, that appeared in the skies in 1572 was far beyond the Earth's atmosphere. This, along with Galileo's discovery of sunspots, showed that contrary to Aristotle, the heavens were not perfect and unchanging. Brahe's fame as an astronomer brought him patronage from King Frederick II, allowing him to carry out his historic high-precision measurements of the planets' motions. A contradictory character, Brahe enjoyed lecturing other nobles about the evils of dueling, but had lost his own nose in a youthful duel and had it replaced with a prosthesis made of an alloy of gold and silver. Willing to endure scandal in order to marry a peasant, he nevertheless used the feudal powers given to him by the king to impose harsh forced labor on the inhabitants of his parishes. The result of their work, an Italian-style palace with an observatory on top, surely ranks as one of the most luxurious science labs ever built. Kepler described Brahe as dying of a ruptured bladder after falling from a wagon on the way home from a party, but other contemporary accounts and modern medical analysis suggest mercury poisoning, possibly as a result of court intrigue.

showed that the same laws of motion applied to the heavens as to the earth, and that the gravitational force that made an apple fall was the same as the force that kept the earth's motion from carrying it away from the sun. What was radical about Newton was not his laws of motion but his concept of a universal science of physics.

10.1 Kepler's laws

Newton wouldn't have been able to figure out *why* the planets move the way they do if it hadn't been for the astronomer Tycho Brahe (1546-1601) and his protege Johannes Kepler (1571-1630), who together came up with the first simple and accurate description of *how* the planets actually do move. The difficulty of their task is suggested by figure c, which shows how the relatively simple orbital motions of the earth and Mars combine so that as seen from earth Mars appears to be staggering in loops like a drunken sailor.



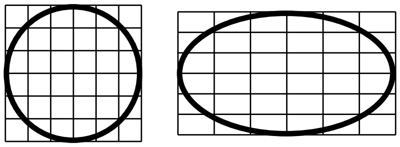
c / As the Earth and Mars revolve around the sun at different rates, the combined effect of their motions makes Mars appear to trace a strange, looped path across the background of the distant stars.

Brahe, the last of the great naked-eye astronomers, collected extensive data on the motions of the planets over a period of many years, taking the giant step from the previous observations' accuracy of about 10 minutes of arc (10/60 of a degree) to an unprecedented 1 minute. The quality of his work is all the more remarkable considering that his observatory consisted of four giant brass protractors mounted upright in his castle in Denmark. Four different observers would simultaneously measure the position of a planet in order to check for mistakes and reduce random errors.

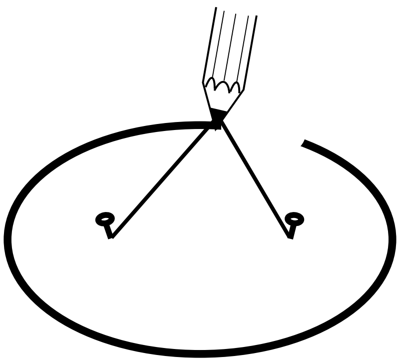
With Brahe's death, it fell to his former assistant Kepler to try to make some sense out of the volumes of data. Kepler, in contradiction to his late boss, had formed a prejudice, a correct one as it turned out, in favor of the theory that the earth and planets revolved around the sun, rather than the earth staying fixed and everything rotating about it. Although motion is relative, it is not just a matter of opinion what circles what. The earth's rotation and revolution about the sun make it a noninertial reference frame, which causes detectable violations of Newton's laws when one attempts to describe sufficiently precise experiments in the earth-fixed frame. Although such direct experiments were not carried out until the 19th century, what convinced everyone of the sun-centered system in the 17th century was that Kepler was able to come up with a surprisingly simple set of mathematical and geometrical rules for describing the planets' motion using the sun-centered assumption. After 900 pages of calculations and many false starts and dead-end ideas, Kepler finally synthesized the data into the following three laws:

> **Kepler's elliptical orbit law**
> The planets orbit the sun in elliptical orbits with the sun at one focus.
>
> **Kepler's equal-area law**
> The line connecting a planet to the sun sweeps out equal areas in equal amounts of time.
>
> **Kepler's law of periods**
> The time required for a planet to orbit the sun, called its period, is proportional to the long axis of the ellipse raised to the 3/2 power. The constant of proportionality is the same for all the planets.

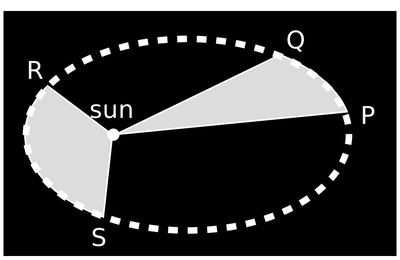
Although the planets' orbits are ellipses rather than circles, most are very close to being circular. The earth's orbit, for instance, is only flattened by 1.7% relative to a circle. In the special case of a planet in a circular orbit, the two foci (plural of "focus") coincide at the center of the circle, and Kepler's elliptical orbit law thus says that the circle is centered on the sun. The equal-area law implies that a planet in a circular orbit moves around the sun with constant speed. For a circular orbit, the law of periods then amounts to a statement that the time for one orbit is proportional to $r^{3/2}$, where r is the radius. If all the planets were moving in their orbits at the same speed, then the time for one orbit would simply depend on the circumference of the circle, so it would only be proportional to r to the first power. The more drastic dependence on $r^{3/2}$ means



d / An ellipse is a circle that has been distorted by shrinking and stretching along perpendicular axes.



e / An ellipse can be constructed by tying a string to two pins and drawing like this with the pencil stretching the string taut. Each pin constitutes one focus of the ellipse.



f / If the time interval taken by the planet to move from P to Q is equal to the time interval from R to S, then according to Kepler's equal-area law, the two shaded areas are equal. The planet is moving faster during interval RS than it did during PQ, which Newton later determined was due to the sun's gravitational force accelerating it. The equal-area law predicts exactly how much it will speed up.

that the outer planets must be moving more slowly than the inner planets.

10.2 Newton's law of gravity

The sun's force on the planets obeys an inverse square law.

Kepler's laws were a beautifully simple explanation of what the planets did, but they didn't address why they moved as they did. Did the sun exert a force that pulled a planet toward the center of its orbit, or, as suggested by Descartes, were the planets circulating in a whirlpool of some unknown liquid? Kepler, working in the Aristotelian tradition, hypothesized not just an inward force exerted by the sun on the planet, but also a second force in the direction of motion to keep the planet from slowing down. Some speculated that the sun attracted the planets magnetically.

Once Newton had formulated his laws of motion and taught them to some of his friends, they began trying to connect them to Kepler's laws. It was clear now that an inward force would be needed to bend the planets' paths. This force was presumably an attraction between the sun and each planet. (Although the sun does accelerate in response to the attractions of the planets, its mass is so great that the effect had never been detected by the prenewtonian astronomers.) Since the outer planets were moving slowly along more gently curving paths than the inner planets, their accelerations were apparently less. This could be explained if the sun's force was determined by distance, becoming weaker for the farther planets. Physicists were also familiar with the noncontact forces of electricity and magnetism, and knew that they fell off rapidly with distance, so this made sense.

In the approximation of a circular orbit, the magnitude of the sun's force on the planet would have to be

[1] $$F = ma = mv^2/r.$$

Now although this equation has the magnitude, v, of the velocity vector in it, what Newton expected was that there would be a more fundamental underlying equation for the force of the sun on a planet, and that that equation would involve the distance, r, from the sun to the object, but not the object's speed, v — motion doesn't make objects lighter or heavier.

self-check A

If eq. [1] really was generally applicable, what would happen to an object released at rest in some empty region of the solar system? ▷ Answer, p. 560

Equation [1] was thus a useful piece of information which could be related to the data on the planets simply because the planets happened to be going in nearly circular orbits, but Newton wanted to combine it with other equations and eliminate v algebraically in order to find a deeper truth.

To eliminate v, Newton used the equation

$$v = \frac{\text{circumference}}{T} = \frac{2\pi r}{T}. \qquad [2]$$

Of course this equation would also only be valid for planets in nearly circular orbits. Plugging this into eq. [1] to eliminate v gives

$$F = \frac{4\pi^2 mr}{T^2}. \qquad [3]$$

This unfortunately has the side-effect of bringing in the period, T, which we expect on similar physical grounds will not occur in the final answer. That's where the circular-orbit case, $T \propto r^{3/2}$, of Kepler's law of periods comes in. Using it to eliminate T gives a result that depends only on the mass of the planet and its distance from the sun:

$$F \propto m/r^2. \qquad \text{[force of the sun on a planet of mass } m \text{ at a distance } r \text{ from the sun; same proportionality constant for all the planets]}$$

(Since Kepler's law of periods is only a proportionality, the final result is a proportionality rather than an equation, so there is no point in hanging on to the factor of $4\pi^2$.)

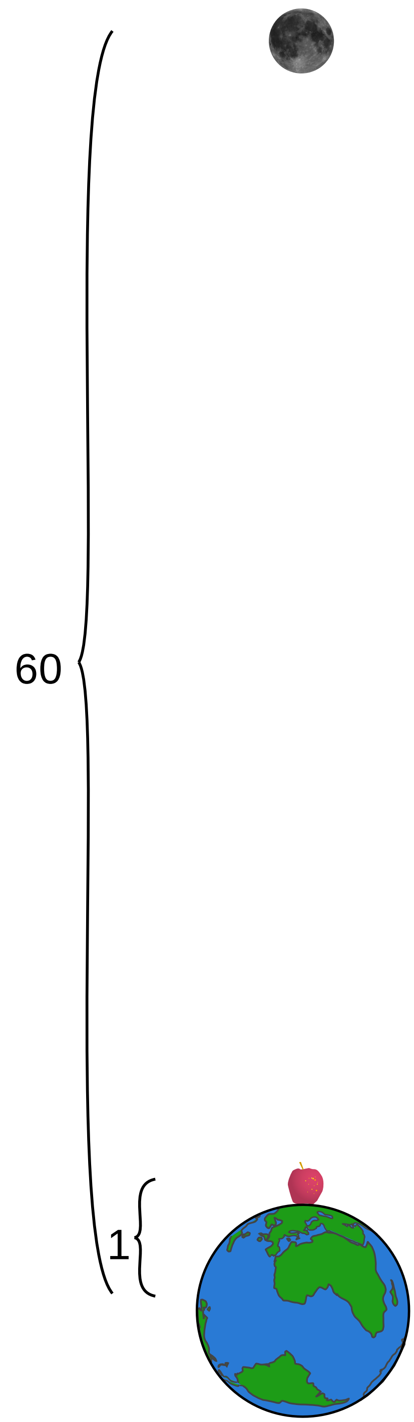
As an example, the "twin planets" Uranus and Neptune have nearly the same mass, but Neptune is about twice as far from the sun as Uranus, so the sun's gravitational force on Neptune is about four times smaller.

self-check B
Fill in the steps leading from equation [3] to $F \propto m/r^2$. ▷ Answer, p. 560

The forces between heavenly bodies are the same type of force as terrestrial gravity.

OK, but what kind of force was it? It probably wasn't magnetic, since magnetic forces have nothing to do with mass. Then came Newton's great insight. Lying under an apple tree and looking up at the moon in the sky, he saw an apple fall. Might not the earth also attract the moon with the same kind of gravitational force? The moon orbits the earth in the same way that the planets orbit the sun, so maybe the earth's force on the falling apple, the earth's force on the moon, and the sun's force on a planet were all the same type of force.

There was an easy way to test this hypothesis numerically. If it was true, then we would expect the gravitational forces exerted by



g / The moon's acceleration is 60^2 = 3600 times smaller than the apple's.

the earth to follow the same $F \propto m/r^2$ rule as the forces exerted by the sun, but with a different constant of proportionality appropriate to the earth's gravitational strength. The issue arises now of how to define the distance, r, between the earth and the apple. An apple in England is closer to some parts of the earth than to others, but suppose we take r to be the distance from the center of the earth to the apple, i.e., the radius of the earth. (The issue of how to measure r did not arise in the analysis of the planets' motions because the sun and planets are so small compared to the distances separating them.) Calling the proportionality constant k, we have

$$F_{\text{earth on apple}} = k\, m_{\text{apple}}/r^2_{\text{earth}}$$
$$F_{\text{earth on moon}} = k\, m_{\text{moon}}/d^2_{\text{earth-moon}}.$$

Newton's second law says $a = F/m$, so

$$a_{\text{apple}} = k \;/\; r^2_{earth}$$
$$a_{\text{moon}} = k \;/\; d^2_{\text{earth-moon}}.$$

The Greek astronomer Hipparchus had already found 2000 years before that the distance from the earth to the moon was about 60 times the radius of the earth, so if Newton's hypothesis was right, the acceleration of the moon would have to be $60^2 = 3600$ times less than the acceleration of the falling apple.

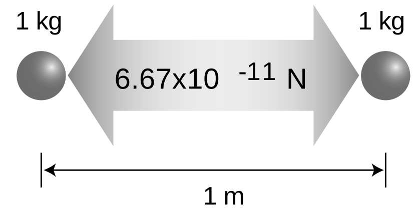
Applying $a = v^2/r$ to the acceleration of the moon yielded an acceleration that was indeed 3600 times smaller than 9.8 m/s^2, and Newton was convinced he had unlocked the secret of the mysterious force that kept the moon and planets in their orbits.

Newton's law of gravity

The proportionality $F \propto m/r^2$ for the gravitational force on an object of mass m only has a consistent proportionality constant for various objects if they are being acted on by the gravity of the same object. Clearly the sun's gravitational strength is far greater than the earth's, since the planets all orbit the sun and do not exhibit any very large accelerations caused by the earth (or by one another). What property of the sun gives it its great gravitational strength? Its great volume? Its great mass? Its great temperature? Newton reasoned that if the force was proportional to the mass of the object being acted on, then it would also make sense if the determining factor in the gravitational strength of the object exerting the force was its own mass. Assuming there were no other factors affecting the gravitational force, then the only other thing needed to make quantitative predictions of gravitational forces would be a proportionality constant. Newton called that proportionality constant G, so here is the complete form of the law of gravity he hypothesized.

Newton's law of gravity

$$F = \frac{Gm_1m_2}{r^2}$$ [gravitational force between objects of mass m_1 and m_2, separated by a distance r; r is not the radius of anything]



h / Students often have a hard time understanding the physical meaning of *G*. It's just a proportionality constant that tells you how strong gravitational forces are. If you could change it, all the gravitational forces all over the universe would get stronger or weaker. Numerically, the gravitational attraction between two 1-kg masses separated by a distance of 1 m is 6.67×10^{-11} N, and this is what *G* is in SI units.

Newton conceived of gravity as an attraction between any two masses in the universe. The constant G tells us how many newtons the attractive force is for two 1-kg masses separated by a distance of 1 m. The experimental determination of G in ordinary units (as opposed to the special, nonmetric, units used in astronomy) is described in section 10.5. This difficult measurement was not accomplished until long after Newton's death.

The units of G *example 1*

▷ What are the units of *G*?

▷ Solving for *G* in Newton's law of gravity gives

$$G = \frac{Fr^2}{m_1 m_2},$$

so the units of *G* must be $\text{N}\cdot\text{m}^2/\text{kg}^2$. Fully adorned with units, the value of *G* is $6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.

Newton's third law *example 2*

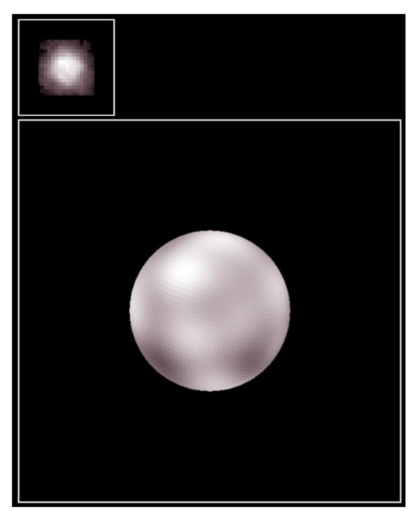
▷ Is Newton's law of gravity consistent with Newton's third law?

▷ The third law requires two things. First, m_1's force on m_2 should be the same as m_2's force on m_1. This works out, because the product $m_1 m_2$ gives the same result if we interchange the labels 1 and 2. Second, the forces should be in opposite directions. This condition is also satisfied, because Newton's law of gravity refers to an attraction: each mass pulls the other toward itself.

Pluto and Charon *example 3*

▷ Pluto's moon Charon is unusually large considering Pluto's size, giving them the character of a double planet. Their masses are 1.25×10^{22} and $1.9x10^{21}$ kg, and their average distance from one another is 1.96×10^4 km. What is the gravitational force between them?

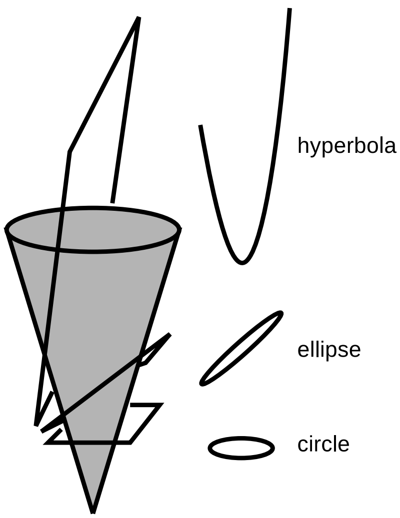
▷ If we want to use the value of *G* expressed in SI (meter-kilogram-second) units, we first have to convert the distance to $1.96 \times$



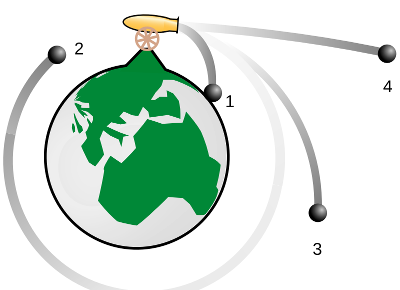
i / Example 3. Computer-enhanced images of Pluto and Charon, taken by the Hubble Space Telescope.

10^7 m. The force is

$$\frac{\left(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(1.25\times10^{22}\ \text{kg}\right)\left(1.9\times10^{21}\ \text{kg}\right)}{\left(1.96\times10^{7}\ \text{m}\right)^2}$$
$$= 4.1\times10^{18}\ \text{N}$$



j / The conic sections are the curves made by cutting the surface of an infinite cone with a plane.



k / An imaginary cannon able to shoot cannonballs at very high speeds is placed on top of an imaginary, very tall mountain that reaches up above the atmosphere. Depending on the speed at which the ball is fired, it may end up in a tightly curved elliptical orbit, 1, a circular orbit, 2, a bigger elliptical orbit, 3, or a nearly straight hyperbolic orbit, 4.

The proportionality to $1/r^2$ in Newton's law of gravity was not entirely unexpected. Proportionalities to $1/r^2$ are found in many other phenomena in which some effect spreads out from a point. For instance, the intensity of the light from a candle is proportional to $1/r^2$, because at a distance r from the candle, the light has to be spread out over the surface of an imaginary sphere of area $4\pi r^2$. The same is true for the intensity of sound from a firecracker, or the intensity of gamma radiation emitted by the Chernobyl reactor. It's important, however, to realize that this is only an analogy. Force does not travel through space as sound or light does, and force is not a substance that can be spread thicker or thinner like butter on toast.

Although several of Newton's contemporaries had speculated that the force of gravity might be proportional to $1/r^2$, none of them, even the ones who had learned Newton's laws of motion, had had any luck proving that the resulting orbits would be ellipses, as Kepler had found empirically. Newton did succeed in proving that elliptical orbits would result from a $1/r^2$ force, but we postpone the proof until the chapter 15 because it can be accomplished much more easily using the concepts of energy and angular momentum.

Newton also predicted that orbits in the shape of hyperbolas should be possible, and he was right. Some comets, for instance, orbit the sun in very elongated ellipses, but others pass through the solar system on hyperbolic paths, never to return. Just as the trajectory of a faster baseball pitch is flatter than that of a more slowly thrown ball, so the curvature of a planet's orbit depends on its speed. A spacecraft can be launched at relatively low speed, resulting in a circular orbit about the earth, or it can be launched at a higher speed, giving a more gently curved ellipse that reaches farther from the earth, or it can be launched at a very high speed which puts it in an even less curved hyperbolic orbit. As you go very far out on a hyperbola, it approaches a straight line, i.e., its curvature eventually becomes nearly zero.

Newton also was able to prove that Kepler's second law (sweeping out equal areas in equal time intervals) was a logical consequence of his law of gravity. Newton's version of the proof is moderately complicated, but the proof becomes trivial once you understand the concept of angular momentum, which will be covered later in the course. The proof will therefore be deferred until section 15.9.

self-check C

Which of Kepler's laws would it make sense to apply to hyperbolic orbits? ▷ Answer, p. 560

▷ *Solved problem: Visiting Ceres* *page 296, problem 1*

▷ *Solved problem: Why a equals g* *page 300, problem 20*

▷ *Solved problem: Ida and Dactyl* *page 300, problem 21*

▷ *Solved problem: Another solar system* *page 297, problem 6*

▷ *Solved problem: Weight loss* *page 296, problem 3*

▷ *Solved problem: The receding moon* *page 300, problem 22*

Discussion questions

A How could Newton find the speed of the moon to plug in to $a = v^2/r$?

B Two projectiles of different mass shot out of guns on the surface of the earth at the same speed and angle will follow the same trajectories, assuming that air friction is negligible. (You can verify this by throwing two objects together from your hand and seeing if they separate or stay side by side.) What corresponding fact would be true for satellites of the earth having different masses?

C What is wrong with the following statement? "A comet in an elliptical orbit speeds up as it approaches the sun, because the sun's force on it is increasing."

D Why would it not make sense to expect the earth's gravitational force on a bowling ball to be inversely proportional to the square of the distance between their surfaces rather than their centers?

E Does the earth accelerate as a result of the moon's gravitational force on it? Suppose two planets were bound to each other gravitationally the way the earth and moon are, but the two planets had equal masses. What would their motion be like?

F Spacecraft normally operate by firing their engines only for a few minutes at a time, and an interplanetary probe will spend months or years on its way to its destination without thrust. Suppose a spacecraft is in a circular orbit around Mars, and it then briefly fires its engines in reverse, causing a sudden decrease in speed. What will this do to its orbit? What about a forward thrust?

10.3 Apparent weightlessness

If you ask somebody at the bus stop why astronauts are weightless, you'll probably get one of the following two incorrect answers:

(1) They're weightless because they're so far from the earth.

(2) They're weightless because they're moving so fast.

The first answer is wrong, because the vast majority of astronauts never get more than a thousand miles from the earth's surface. The reduction in gravity caused by their altitude is significant, but not 100%. The second answer is wrong because Newton's law of gravity only depends on distance, not speed.

The correct answer is that astronauts in orbit around the earth are not really weightless at all. Their weightlessness is only apparent. If there was no gravitational force on the spaceship, it would obey Newton's first law and move off on a straight line, rather than orbiting the earth. Likewise, the astronauts inside the spaceship are in orbit just like the spaceship itself, with the earth's gravitational force continually twisting their velocity vectors around. The reason they appear to be weightless is that they are in the same orbit as the spaceship, so although the earth's gravity curves their trajectory down toward the deck, the deck drops out from under them at the same rate.

Apparent weightlessness can also be experienced on earth. Any time you jump up in the air, you experience the same kind of apparent weightlessness that the astronauts do. While in the air, you can lift your arms more easily than normal, because gravity does not make them fall any faster than the rest of your body, which is falling out from under them. The Russian air force now takes rich foreign tourists up in a big cargo plane and gives them the feeling of weightlessness for a short period of time while the plane is nose-down and dropping like a rock.

10.4 Vector addition of gravitational forces

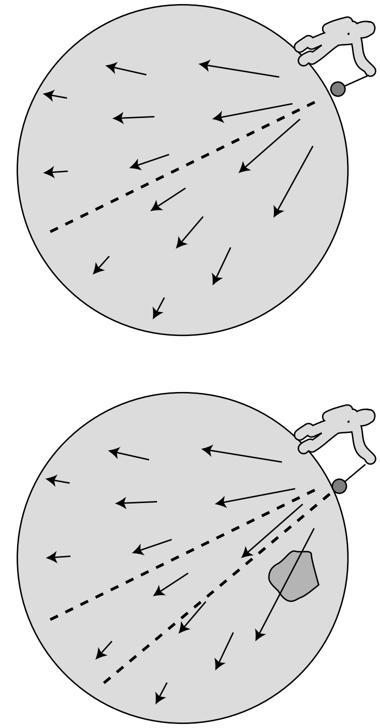
> Pick a flower on earth and you move the farthest star.
>
> *Paul Dirac*

When you stand on the ground, which part of the earth is pulling down on you with its gravitational force? Most people are tempted to say that the effect only comes from the part directly under you, since gravity always pulls straight down. Here are three observations that might help to change your mind:

- If you jump up in the air, gravity does not stop affecting you just because you are not touching the earth: gravity is a noncontact force. That means you are not immune from the gravity of distant parts of our planet just because you are not touching them.

- Gravitational effects are not blocked by intervening matter. For instance, in an eclipse of the moon, the earth is lined up directly between the sun and the moon, but only the sun's light is blocked from reaching the moon, not its gravitational force — if the sun's gravitational force on the moon was blocked in this situation, astronomers would be able to tell because the moon's acceleration would change suddenly. A more subtle but more easily observable example is that the tides are caused by the moon's gravity, and tidal effects can occur on the side of the earth facing away from the moon. Thus, far-off parts of the earth are not prevented from attracting you with their gravity just because there is other stuff between you and them.

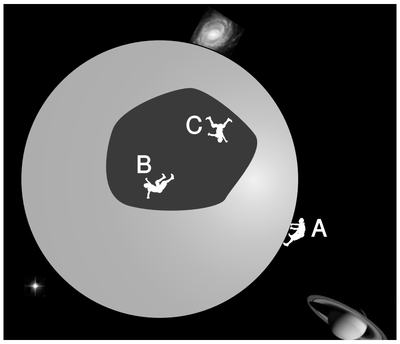
- Prospectors sometimes search for underground deposits of dense minerals by measuring the direction of the local gravitational forces, i.e., the direction things fall or the direction a plumb bob hangs. For instance, the gravitational forces in the region to the west of such a deposit would point along a line slightly to the east of the earth's center. Just because the total gravitational force on you points down, that doesn't mean that only the parts of the earth directly below you are attracting you. It's just that the sideways components of all the force vectors acting on you come very close to canceling out.



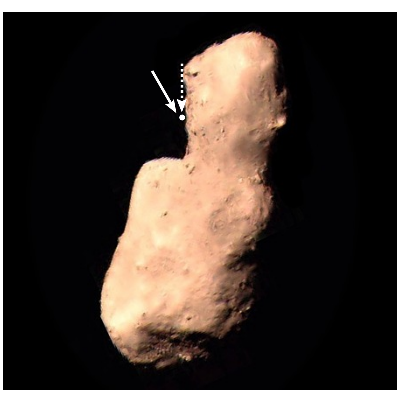
l / Gravity only appears to pull straight down because the near perfect symmetry of the earth makes the sideways components of the total force on an object cancel almost exactly. If the symmetry is broken, e.g., by a dense mineral deposit, the total force is a little off to the side.

A cubic centimeter of lava in the earth's mantle, a grain of silica inside Mt. Kilimanjaro, and a flea on a cat in Paris are all attracting you with their gravity. What you feel is the vector sum of all the gravitational forces exerted by all the atoms of our planet, and for that matter by all the atoms in the universe.

When Newton tested his theory of gravity by comparing the orbital acceleration of the moon to the acceleration of a falling apple on earth, he assumed he could compute the earth's force on the apple using the distance from the apple to the earth's center. Was he wrong? After all, it isn't just the earth's center attracting the apple, it's the whole earth. A kilogram of dirt a few feet under his backyard in England would have a much greater force on the apple than a kilogram of molten rock deep under Australia, thousands of miles away. There's really no obvious reason why the force should come out right if you just pretend that the earth's whole mass is concentrated at its center. Also, we know that the earth has some parts that are more dense, and some parts that are less dense. The solid crust, on which we live, is considerably less dense than the molten rock on which it floats. By all rights, the computation of the vector sum of all the forces exerted by all the earth's parts should be a horrendous mess.



m / Cut-away view of a spherical shell of mass. A, who is outside the shell, feels gravitational forces from every part of the shell — stronger forces from the closer parts, and weaker ones from the parts farther away. The shell theorem states that the vector sum of all the forces is the same as if all the mass had been concentrated at the center of the shell. B, at the center, is clearly weightless, because the shell's gravitational forces cancel out. Surprisingly, C also feels exactly zero gravitational force.



n / The asteroid Toutatis, imaged by the space probe Chang'e-2 in 2012, is shaped like a bowling pin.

Actually, Newton had sound reasons for treating the earth's mass as if it was concentrated at its center. First, although Newton no doubt suspected the earth's density was nonuniform, he knew that the direction of its total gravitational force was very nearly toward the earth's center. That was strong evidence that the distribution of mass was very symmetric, so that we can think of the earth as being made of layers, like an onion, with each layer having constant density throughout. (Today there is further evidence for symmetry based on measurements of how the vibrations from earthquakes and nuclear explosions travel through the earth.) He then considered the gravitational forces exerted by a single such thin shell, and proved the following theorem, known as the shell theorem:

> If an object lies outside a thin, spherical shell of mass, then the vector sum of all the gravitational forces exerted by all the parts of the shell is the same as if the shell's mass had been concentrated at its center. If the object lies inside the shell, then all the gravitational forces cancel out exactly.

For terrestrial gravity, each shell acts as though its mass was at the center, so the result is the same as if the whole mass was there. The shell theorem is proved on p. 292.

The second part of the shell theorem, about the gravitational forces canceling inside the shell, is a little surprising. Obviously the forces would all cancel out if you were at the exact center of a shell, but it's not at all obvious that they should still cancel out perfectly if you are inside the shell but off-center. The whole idea might seem academic, since we don't know of any hollow planets in our solar system that astronauts could hope to visit, but actually it's a useful result for understanding gravity within the earth, which is an important issue in geology. It doesn't matter that the earth is not actually hollow. In a mine shaft at a depth of, say, 2 km, we can use the shell theorem to tell us that the outermost 2 km of the earth has no net gravitational effect, and the gravitational force is the same as what would be produced if the remaining, deeper, parts of the earth were all concentrated at its center.

The shell theorem doesn't apply to things that aren't spherical. At the point marked with a dot in figure n, we might imagine that gravity was in the direction shown by the dashed arrow, pointing toward the asteroid's center of mass, so that the surface would be a vertical cliff almost a kilometer tall. In reality, calculations based on the assumption of uniform density show that the direction of the gravitational field is approximately as shown by the solid arrow, making the slope only about 60°.[1] This happens because gravity at this location is more strongly affected by the nearby "neck" than by the more distant "belly." This slope is still believed to be too steep to keep dirt and rocks from sliding off (see problem 13, p. 250).

[1] Hudson *et al.*, Icarus 161 (2003) 346

self-check D

Suppose you're at the bottom of a deep mineshaft, which means you're still quite far from the center of the earth. The shell theorem says that the shell of mass you've gone inside exerts zero total force on you. Discuss which parts of the shell are attracting you in which directions, and how strong these forces are. Explain why it's at least plausible that they cancel. ▷ Answer, p. 560

Discussion questions

A If you hold an apple, does the apple exert a gravitational force on the earth? Is it much weaker than the earth's gravitational force on the apple? Why doesn't the earth seem to accelerate upward when you drop the apple?

B When astronauts travel from the earth to the moon, how does the gravitational force on them change as they progress?

C How would the gravity in the first-floor lobby of a massive skyscraper compare with the gravity in an open field outside of the city?

D In a few billion years, the sun will start undergoing changes that will eventually result in its puffing up into a red giant star. (Near the beginning of this process, the earth's oceans will boil off, and by the end, the sun will probably swallow the earth completely.) As the sun's surface starts to get closer and closer to the earth, how will the earth's orbit be affected?

10.5 Weighing the earth

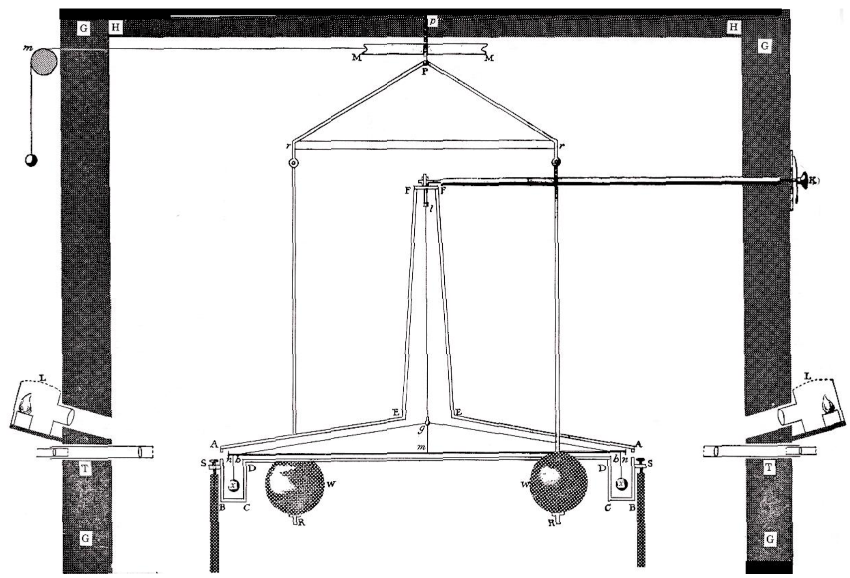
Let's look more closely at the application of Newton's law of gravity to objects on the earth's surface. Since the earth's gravitational force is the same as if its mass was all concentrated at its center, the force on a falling object of mass m is given by

$$F = G\, M_{\text{earth}}\, m \,/\, r_{\text{earth}}^2 .$$

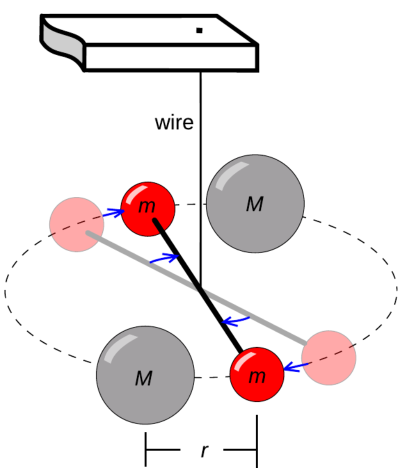
The object's acceleration equals F/m, so the object's mass cancels out and we get the same acceleration for all falling objects, as we knew we should:

$$g = G\, M_{\text{earth}} \,/\, r_{\text{earth}}^2 .$$

Newton knew neither the mass of the earth nor a numerical value for the constant G. But if someone could measure G, then it would be possible for the first time in history to determine the mass of the earth! The only way to measure G is to measure the gravitational force between two objects of known mass, but that's an exceedingly difficult task, because the force between any two objects of ordinary size is extremely small. The English physicist Henry Cavendish was



o / Cavendish's apparatus. The two large balls are fixed in place, but the rod from which the two small balls hang is free to twist under the influence of the gravitational forces.



p / A simplified version of Cavendish's apparatus.

the first to succeed, using the apparatus shown in figures o and p. The two larger balls were lead spheres 8 inches in diameter, and each one attracted the small ball near it. The two small balls hung from the ends of a horizontal rod, which itself hung by a thin thread. The frame from which the larger balls hung could be rotated by hand about a vertical axis, so that for instance the large ball on the right would pull its neighboring small ball toward us and while the small ball on the left would be pulled away from us. The thread from which the small balls hung would thus be twisted through a small angle, and by calibrating the twist of the thread with known forces, the actual gravitational force could be determined. Cavendish set up the whole apparatus in a room of his house, nailing all the doors shut to keep air currents from disturbing the delicate apparatus. The results had to be observed through telescopes stuck through holes drilled in the walls. Cavendish's experiment provided the first numerical values for G and for the mass of the earth. The presently accepted value of G is $6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.

Knowing G not only allowed the determination of the earth's mass but also those of the sun and the other planets. For instance, by observing the acceleration of one of Jupiter's moons, we can infer the mass of Jupiter. The following table gives the distances of the planets from the sun and the masses of the sun and planets. (Other data are given in the back of the book.)

	average distance from the sun, in units of the earth's average distance from the sun	mass, in units of the earth's mass
sun	—	330,000
Mercury	0.38	0.056
Venus	0.72	0.82
earth	1	1
Mars	1.5	0.11
Jupiter	5.2	320
Saturn	9.5	95
Uranus	19	14
Neptune	30	17
Pluto	39	0.002

The following example applies the numerical techniques of section 4.6.

From the earth to the moon *example 4*

The Apollo 11 mission landed the first humans on the moon in 1969. In this example, we'll estimate the time it took to get to the moon, and compare our estimate with the actual time, which was 73.0708 hours from the engine burn that took the ship out of earth orbit to the engine burn that inserted it into lunar orbit. During this time, the ship was coasting with the engines off, except for a small course-correction burn, which we neglect. More importantly, we do the calculation for a straight-line trajectory rather than the real S-shaped one, so the result can only be expected to agree roughly with what really happened. The following data come from the original press kit, which NASA has scanned and posted on the Web:

initial altitude	3.363×10^5 m
initial velocity	1.083×10^4 m/s

The endpoint of the the straight-line trajectory is a free-fall impact on the lunar surface, which is also unrealistic (luckily for the astronauts).

The force acting on the ship is

$$F = -\frac{GM_e m}{r^2} + \frac{GM_m m}{(r_m - r)^2},$$

but since everything is proportional to the mass of the ship, m, we can divide it out

$$\frac{F}{m} = -\frac{GM_e}{r^2} + \frac{GM_m}{(r_m - r)^2},$$

and the variables `F` in the program is actually the force per unit mass F/m. The program is a straightforward modification of the function `meteor` on page 152.

```
import math
def apollo(vi,n):
  bigg=6.67e-11     # gravitational constant, SI
  me=5.97e24        # mass of earth, kg
  mm=7.35e22        # mass of moon, kg
  em=3.84e8         # earth-moon distance, m
  re=6.378e6        # radius of earth, m
  rm=1.74e6         # radius of moon, m
  v=vi
  x=re+3.363e5      # re+initial altitude
  xf=em-rm          # surface of moon
  dt = 360000./n    # split 100 hours into n parts
  t = 0.
  for i in range(n):
    dx = v*dt
    x = x+dx                # Change x.
    if x>xf:
      return t/3600.
    a = -bigg*me/x**2+bigg*mm/(em-x)**2
    t = t + dt
    dv = a*dt
    v = v+dv
```

```
>>> print apollo(1.083e4,1000000)
59.7488999991
>>> vi=1.083e4
```

This is pretty decent agreement with the real-world time of 73 hours, considering the wildly inaccurate trajectory assumed. It's interesting to see how much the duration of the trip changes if we increase the initial velocity by only ten percent:

```
>>> print apollo(1.2e4,1000000)
18.3682
```

The most important reason for using the lower speed was that if something had gone wrong, the ship would have been able to whip around the moon and take a "free return" trajectory back to the earth, without having to do any further burns. At a higher speed, the ship would have had so much kinetic energy that in the absence of any further engine burns, it would have escaped from the earth-moon system. The Apollo 13 mission had to take a free return trajectory after an explosion crippled the spacecraft.

Discussion questions

A It would have been difficult for Cavendish to start designing an experiment without at least some idea of the order of magnitude of *G*. How could he estimate it in advance to within a factor of 10?

B Fill in the details of how one would determine Jupiter's mass by observing the acceleration of one of its moons. Why is it only necessary to know the acceleration of the moon, not the actual force acting on it? Why don't we need to know the mass of the moon? What about a planet that has no moons, such as Venus — how could its mass be found?

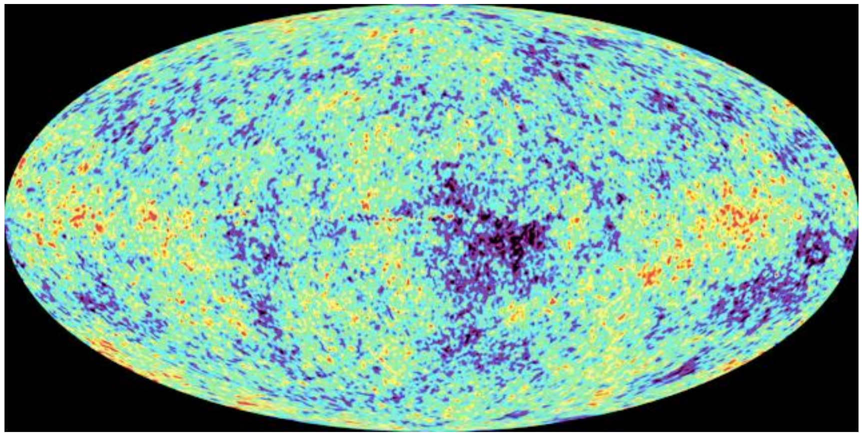
10.6 ⋆ Dark energy

Until recently, physicists thought they understood gravity fairly well. Einstein had modified Newton's theory, but certain characteristics of gravitational forces were firmly established. For one thing, they were always attractive. If gravity always attracts, then it is logical to ask why the universe doesn't collapse. Newton had answered this question by saying that if the universe was infinite in all directions, then it would have no geometric center toward which it would collapse; the forces on any particular star or planet exerted by distant parts of the universe would tend to cancel out by symmetry. More careful calculations, however, show that Newton's universe would have a tendency to collapse on smaller scales: any part of the universe that happened to be slightly more dense than average would contract further, and this contraction would result in stronger gravitational forces, which would cause even more rapid contraction, and so on.

When Einstein overhauled gravity, the same problem reared its ugly head. Like Newton, Einstein was predisposed to believe in a universe that was static, so he added a special repulsive term to his equations, intended to prevent a collapse. This term was not associated with any interaction of mass with mass, but represented merely an overall tendency for space itself to expand unless restrained by the matter that inhabited it. It turns out that Einstein's solution, like Newton's, is unstable. Furthermore, it was soon discovered observationally that the universe was expanding, and this was interpreted by creating the Big Bang model, in which the universe's current expansion is the aftermath of a fantastically hot explosion.

An expanding universe, unlike a static one, was capable of being explained with Einstein's equations, without any repulsion term. The universe's expansion would simply slow down over time due to the attractive gravitational forces. After these developments, Einstein said woefully that adding the repulsive term, known as the cosmological constant, had been the greatest blunder of his life.

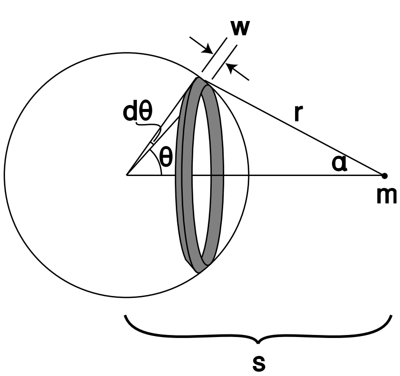
This was the state of things until 1999, when evidence began to turn up that the universe's expansion has been speeding up rather than slowing down! The first evidence came from using a telescope as a sort of time machine: light from a distant galaxy may have taken billions of years to reach us, so we are seeing it as it was far in the past. Looking back in time, astronomers saw the universe expanding at speeds that were lower, rather than higher. At first



q / The WMAP probe's map of the cosmic microwave background is like a "baby picture" of the universe.

they were mortified, since this was exactly the opposite of what had been expected. The statistical quality of the data was also not good enough to constitute ironclad proof, and there were worries about systematic errors. The case for an accelerating expansion has however been supported by high-precision mapping of the dim, sky-wide afterglow of the Big Bang, known as the cosmic microwave background.

So now Einstein's "greatest blunder" has been resurrected. Since we don't actually know whether or not this self-repulsion of space has a constant strength, the term "cosmological *constant*" has lost currency. Nowadays physicists usually refer to the phenomenon as "dark energy." Picking an impressive-sounding name for it should not obscure the fact that we know absolutely nothing about the nature of the effect or why it exists.



r / A spherical shell of mass M interacts with a pointlike mass m.

10.7 ⋆ Proof of the shell theorem

Referring to figure r, let b be the radius of the shell, h its thickness, and ρ its density. Its volume is then $V=(\text{area})(\text{thickness})=4\pi b^2 h$, and its mass is $M = \rho V = 4\pi\rho b^2 h$. The strategy is to divide the shell up into rings as shown, with each ring extending from θ to $\theta + \mathrm{d}\theta$. Since the ring is infinitesimally skinny, its entire mass lies at the same distance, r, from mass m. The width of such a ring is found by the definition of radian measure to be $w = b\,\mathrm{d}\theta$, and its mass is $\mathrm{d}M = (\rho)(\text{circumference})(\text{thickness})(\text{width})= (\rho)(2\pi b\sin\theta)(h)(b\,\mathrm{d}\theta)=2\pi\rho b^2 h\sin\theta\,\mathrm{d}\theta$. To save writing, we define $A = GMm/s^2$. For the case where m is outside the shell, our goal is to prove that the force F acting on m equals A. Let the axis of symmetry be x, and let the contribution of this ring to the total force be $\mathrm{d}F_x$.

$$\begin{aligned}
F &= \int \mathrm{d}F_x \\
&= \int \frac{Gm\,\mathrm{d}M}{r^2}\cos\alpha \\
&= \int \frac{Gm \cdot 2\pi\rho b^2 h \sin\theta\,\mathrm{d}\theta}{r^2}\cos\alpha \\
&= \left(\frac{s^2}{2}\right) A \int \frac{\sin\theta\,\mathrm{d}\theta}{r^2}\cos\alpha
\end{aligned}$$

From the law of cosines we find

$$\begin{aligned}
r^2 &= b^2 + s^2 - 2bs\cos\theta, \\
b^2 &= r^2 + s^2 - 2rs\cos\alpha,
\end{aligned}$$

and differentiation of the former gives

$$2r\,\mathrm{d}r = 2bs\sin\theta\,\mathrm{d}\theta.$$

We can now write the integrand entirely in terms of the single variable of integration r.

$$\begin{aligned}
F &= \left(\frac{s}{2b}\right) A \int_{s-b}^{s+b} \frac{r\,\mathrm{d}r}{r^2}\cos\alpha \\
&= \left(\frac{1}{4b}\right) A \int_{s-b}^{s+b} \frac{\mathrm{d}r}{r}\left(r + \frac{s^2-b^2}{r}\right) \\
&= \left(\frac{1}{4b}\right) A(2b+2b) \\
&= A
\end{aligned}$$

This is what we wanted to prove for the case where m is on the outside. The inside case is problem 27. A more elegant method of proof is to use Gauss's theorem, which is usually introduced in a class on electricity and magnetism or vector calculus; the concept is that the gravitational field can be visualized in terms of lines of gravitational force spreading out from a mass, and the number of lines coming out through a surface is independent of the exact geometry of the surface and the mass distribution. It is interesting to note that the result depends on both the fact that the exponent of r in Newton's law of gravity is -2 (problem 28) and on the fact that space has three dimensions.

Summary

Selected vocabulary

ellipse . . . . a flattened circle; one of the conic sections

conic section . . . a curve formed by the intersection of a plane and an infinite cone

hyperbola . . . . another conic section; it does not close back on itself

period . . . . the time required for a planet to complete one orbit; more generally, the time for one repetition of some repeating motion

focus . . . . one of two special points inside an ellipse: the ellipse consists of all points such that the sum of the distances to the two foci equals a certain number; a hyperbola also has a focus

Notation

G . . . . the constant of proportionality in Newton's law of gravity; the gravitational force of attraction between two 1-kg spheres at a center-to-center distance of 1 m

Summary

Kepler deduced three empirical laws from data on the motion of the planets:

Kepler's elliptical orbit law: The planets orbit the sun in elliptical orbits with the sun at one focus.

Kepler's equal-area law: The line connecting a planet to the sun sweeps out equal areas in equal amounts of time.

Kepler's law of periods: The time required for a planet to orbit the sun is proportional to the long axis of the ellipse raised to the 3/2 power. The constant of proportionality is the same for all the planets.

Newton was able to find a more fundamental explanation for these laws. Newton's law of gravity states that the magnitude of the attractive force between any two objects in the universe is given by

$$F = Gm_1m_2/r^2.$$

Weightlessness of objects in orbit around the earth is only apparent. An astronaut inside a spaceship is simply falling along with the spaceship. Since the spaceship is falling out from under the astronaut, it appears as though there was no gravity accelerating the astronaut down toward the deck.

Gravitational forces, like all other forces, add like vectors. A gravitational force such as we ordinarily feel is the vector sum of all the forces exerted by all the parts of the earth. As a consequence of this, Newton proved the *shell theorem* for gravitational forces:

If an object lies outside a thin, uniform shell of mass, then the vector sum of all the gravitational forces exerted by all the parts of the shell is the same as if all the shell's mass was concentrated at its center. If the object lies inside the shell, then all the gravitational forces cancel out exactly.

Problems

Key

√ A computerized answer check is available online.

∫ A problem that requires calculus.

⋆ A difficult problem.

1 Ceres, the largest asteroid in our solar system, is a spherical body with a mass 6000 times less than the earth's, and a radius which is 13 times smaller. If an astronaut who weighs 400 N on earth is visiting the surface of Ceres, what is her weight?

▷ Solution, p. 552

2 Roy has a mass of 60 kg. Laurie has a mass of 65 kg. They are 1.5 m apart.
(a) What is the magnitude of the gravitational force of the earth on Roy?
(b) What is the magnitude of Roy's gravitational force on the earth?
(c) What is the magnitude of the gravitational force between Roy and Laurie?
(d) What is the magnitude of the gravitational force between Laurie and the sun? √

3 (a) A certain vile alien gangster lives on the surface of an asteroid, where his weight is 0.20 N. He decides he needs to lose weight without reducing his consumption of princesses, so he's going to move to a different asteroid where his weight will be 0.10 N. The real estate agent's database has asteroids listed by mass, however, not by surface gravity. Assuming that all asteroids are spherical and have the same density, how should the mass of his new asteroid compare with that of his old one?
(b) Jupiter's mass is 318 times the Earth's, and its gravity is about twice Earth's. Is this consistent with the results of part a? If not, how do you explain the discrepancy? ▷ Solution, p. 552

4 The planet Uranus has a mass of 8.68×10^{25} kg and a radius of 2.56×10^{4} km. The figure shows the relative sizes of Uranus and Earth.
(a) Compute the ratio g_U/g_E, where g_U is the strength of the gravitational field at the surface of Uranus and g_E is the corresponding quantity at the surface of the Earth. √
(b) What is surprising about this result? How do you explain it?

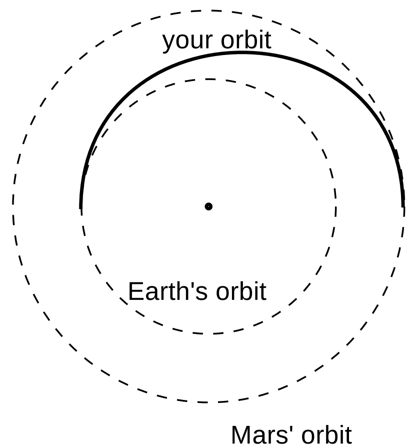
Problem 4.

5 How high above the Earth's surface must a rocket be in order to have 1/100 the weight it would have at the surface? Express your answer in units of the radius of the Earth. √

6 Astronomers have detected a solar system consisting of three planets orbiting the star Upsilon Andromedae. The planets have been named b, c, and d. Planet b's average distance from the star is 0.059 A.U., and planet c's average distance is 0.83 A.U., where an astronomical unit or A.U. is defined as the distance from the Earth to the sun. For technical reasons, it is possible to determine the ratios of the planets' masses, but their masses cannot presently be determined in absolute units. Planet c's mass is 3.0 times that of planet b. Compare the star's average gravitational force on planet c with its average force on planet b. [Based on a problem by Arnold Arons.] ▷ Solution, p. 553

7 The star Lalande 21185 was found in 1996 to have two planets in roughly circular orbits, with periods of 6 and 30 years. What is the ratio of the two planets' orbital radii? √

8 You are considering going on a space voyage to Mars, in which your route would be half an ellipse, tangent to the Earth's orbit at one end and tangent to Mars' orbit at the other. Your spacecraft's engines will only be used at the beginning and end, not during the voyage. How long would the outward leg of your trip last? (Assume the orbits of Earth and Mars are circular.) √



Problem 8.

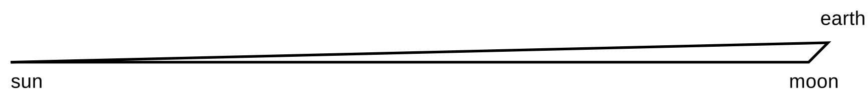
9 Where would an object have to be located so that it would experience zero total gravitational force from the earth and moon? √

10 In a Star Trek episode, the Enterprise is in a circular orbit around a planet when something happens to the engines. Spock then tells Kirk that the ship will spiral into the planet's surface unless they can fix the engines. Is this scientifically correct? Why?

11 Astronomers have recently observed stars orbiting at very high speeds around an unknown object near the center of our galaxy. For stars orbiting at distances of about 10^{14} m from the object, the orbital velocities are about 10^{6} m/s. Assuming the orbits are circular, estimate the mass of the object, in units of the mass of the sun, 2×10^{30} kg. If the object was a tightly packed cluster of normal stars, it should be a very bright source of light. Since no visible light is detected coming from it, it is instead believed to be a supermassive black hole. √

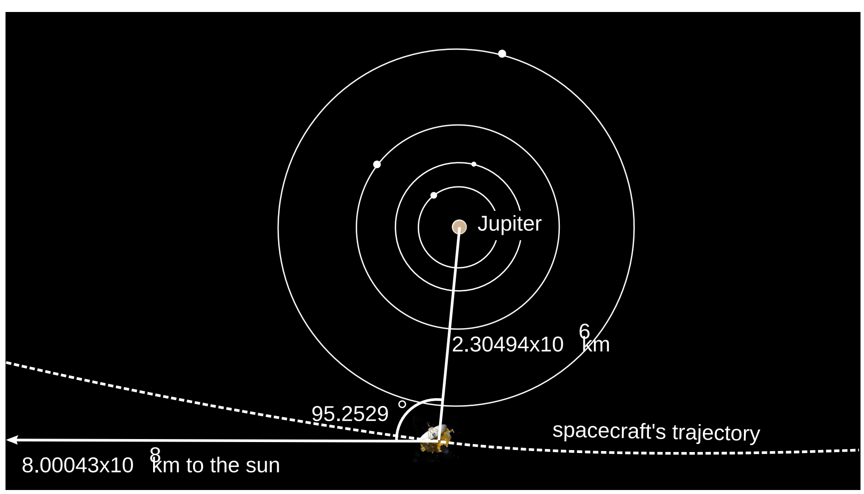
12 During a solar eclipse, the moon, earth and sun all lie on the same line, with the moon between the earth and sun. Define your coordinates so that the earth and moon lie at greater x values than the sun. For each force, give the correct sign as well as the magnitude. (a) What force is exerted on the moon by the sun? (b) On the moon by the earth? (c) On the earth by the sun? (d) What total force is exerted on the sun? (e) On the moon? (f) On the earth? √

13 Suppose that on a certain day there is a crescent moon, and you can tell by the shape of the crescent that the earth, sun and moon form a triangle with a 135° interior angle at the moon's corner. What is the magnitude of the total gravitational force of the earth and the sun on the moon? (If you haven't done problem 12 already, you might want to try it first, since it's easier, and some of its results can be recycled in this problem.) √



Problem 13.

14 On Feb. 28, 2007, the New Horizons space probe, on its way to a 2015 flyby of Pluto, passed by the planet Jupiter for a gravity-assisted maneuver that increased its speed and changed its course. The dashed line in the figure shows the spacecraft's trajectory, which is curved because of three forces: the force of the exhaust gases from the probe's own engines, the sun's gravitational force, and Jupiter's gravitational force. Find the magnitude of the total gravitational force acting on the probe. You will find that the sun's force is much smaller than Jupiter's, so that the magnitude of the total force is determined almost entirely by Jupiter's force. However, this is a high-precision problem, and you will find that the total force is slightly different from Jupiter's force. √



Problem 14: New Horizons at its closest approach to Jupiter. (Jupiter's four largest moons are shown for illustrative purposes.) The masses are:
sun: 1.9891×10^{30} kg
Jupiter: 1.8986×10^{27} kg
New Horizons: 465.0 kg

15 The International Space Station orbits at an average altitude of about 370 km above sea level. Compute the value of g at that altitude. √

16 (a) If the earth was of uniform density, would your weight be increased or decreased at the bottom of a mine shaft? Explain.
(b) In real life, objects weigh slightly more at the bottom of a mine shaft. What does that allow us to infer about the Earth? ⋆

17 (a) A geosynchronous orbit is one in which the satellite orbits above the equator, and has an orbital period of 24 hours, so that it is always above the same point on the spinning earth. Calculate the altitude of such a satellite. √
(b) What is the gravitational field experienced by the satellite? Give your answer as a percentage in relation to the gravitational field at the earth's surface. √

18 If a bullet is shot straight up at a high enough velocity, it will never return to the earth. This is known as the escape velocity. We will discuss escape velocity using the concept of energy later in the course, but it can also be gotten at using straightforward calculus. In this problem, you will analyze the motion of an object of mass m whose initial velocity is *exactly* equal to escape velocity. We assume that it is starting from the surface of a spherically symmetric planet of mass M and radius b. The trick is to guess at the general form of the solution, and then determine the solution in more detail. Assume (as is true) that the solution is of the form $r = kt^p$, where r is the object's distance from the center of the planet at time t, and k and p are constants.
(a) Find the acceleration, and use Newton's second law and Newton's law of gravity to determine k and p. You should find that the result is independent of m. √
(b) What happens to the velocity as t approaches infinity?
(c) Determine escape velocity from the Earth's surface. √

19 (a) Suppose a rotating spherical body such as a planet has a radius r and a uniform density ρ, and the time required for one rotation is T. At the surface of the planet, the apparent acceleration of a falling object is reduced by the acceleration of the ground out from under it. Derive an equation for the apparent acceleration of gravity, g, at the equator in terms of r, ρ, T, and G. √
(b) Applying your equation from a, by what fraction is your apparent weight reduced at the equator compared to the poles, due to the Earth's rotation? √
(c) Using your equation from a, derive an equation giving the value of T for which the apparent acceleration of gravity becomes zero, i.e., objects can spontaneously drift off the surface of the planet. Show that T only depends on ρ, and not on r. √
(d) Applying your equation from c, how long would a day have to be in order to reduce the apparent weight of objects at the equator of the Earth to zero? [Answer: 1.4 hours]
(e) Astronomers have discovered objects they called pulsars, which emit bursts of radiation at regular intervals of less than a second.

If a pulsar is to be interpreted as a rotating sphere beaming out a natural "searchlight" that sweeps past the earth with each rotation, use your equation from c to show that its density would have to be much greater than that of ordinary matter.
(f) Astrophysicists predicted decades ago that certain stars that used up their sources of energy could collapse, forming a ball of neutrons with the fantastic density of $\sim 10^{17}$ kg/m^3. If this is what pulsars really are, use your equation from c to explain why no pulsar has ever been observed that flashes with a period of less than 1 ms or so.

20 Prove, based on Newton's laws of motion and Newton's law of gravity, that all falling objects have the same acceleration if they are dropped at the same location on the earth and if other forces such as friction are unimportant. Do not just say, "$g = 9.8$ m/s^2 – it's constant." You are supposed to be *proving* that g should be the same number for all objects. ▷ Solution, p. 553

Problem 21.

21 The figure shows an image from the Galileo space probe taken during its August 1993 flyby of the asteroid Ida. Astronomers were surprised when Galileo detected a smaller object orbiting Ida. This smaller object, the only known satellite of an asteroid in our solar system, was christened Dactyl, after the mythical creatures who lived on Mount Ida, and who protected the infant Zeus. For scale, Ida is about the size and shape of Orange County, and Dactyl the size of a college campus. Galileo was unfortunately unable to measure the time, T, required for Dactyl to orbit Ida. If it had, astronomers would have been able to make the first accurate determination of the mass and density of an asteroid. Find an equation for the density, ρ, of Ida in terms of Ida's known volume, V, the known radius, r, of Dactyl's orbit, and the lamentably unknown variable T. (This is the same technique that was used successfully for determining the masses and densities of the planets that have moons.) ▷ Solution, p. 553

22 As discussed in more detail in example 3 on p. 430, tidal interactions with the earth are causing the moon's orbit to grow gradually larger. Laser beams bounced off of a mirror left on the moon by astronauts have allowed a measurement of the moon's rate of recession, which is about 4 cm per year. This means that the gravitational force acting between earth and moon is decreasing. By what fraction does the force decrease with each 27-day orbit? [Based on a problem by Arnold Arons.]
▷ Hint, p. 542 ▷ Solution, p. 553

23 Astronomers calculating orbits of planets often work in a nonmetric system of units, in which the unit of time is the year, the unit of mass is the sun's mass, and the unit of distance is the astronomical unit (A.U.), defined as half the long axis of the earth's orbit. In these units, find an exact expression for the gravitational constant, G. √

24 Suppose that we inhabited a universe in which, instead of Newton's law of gravity, we had $F = k\sqrt{m_1 m_2}/r^2$, where k is some constant with different units than G. (The force is still attractive.) However, we assume that $a = F/m$ and the rest of Newtonian physics remains true, and we use $a = F/m$ to define our mass scale, so that, e.g., a mass of 2 kg is one which exhibits half the acceleration when the same force is applied to it as to a 1 kg mass.
(a) Is this new law of gravity consistent with Newton's third law?
(b) Suppose you lived in such a universe, and you dropped two unequal masses side by side. What would happen?
(c) Numerically, suppose a 1.0-kg object falls with an acceleration of 10 m/s^2. What would be the acceleration of a rain drop with a mass of 0.1 g? Would you want to go out in the rain?
(d) If a falling object broke into two unequal pieces while it fell, what would happen?
(e) Invent a law of gravity that results in behavior that is the opposite of what you found in part b. [Based on a problem by Arnold Arons.]

25 The structures that we see in the universe, such as solar systems, galaxies, and clusters of galaxies, are believed to have condensed from clumps that formed, due to gravitational attraction, in preexisting clouds of gas and dust. Observations of the cosmic microwave background radiation (p. 292) suggest that the mixture of hot hydrogen and helium that existed soon after the Big Bang was extremely uniform, but not perfectly so. We can imagine that any region that started out a little more dense would form a natural center for the collapse of a clump. Suppose that we have a spherical region with density ρ and radius r, and for simplicity let's just assume that it's surrounded by vacuum. (a) Find the acceleration of the material at the edge of the cloud. To what power of r is it proportional? √
(b) The cloud will take a time t to collapse to some fraction of its original size. Show that t is independent of r.

Remark: This result suggests that structures would get a chance to form at all scales in the universe. That is, solar systems would not form before galaxies got to, or vice versa. It is therefore physically natural that when we look at the universe at essentially all scales less than a billion light-years, we see structure. ⋆

26 You have a fixed amount of material with a fixed density. If the material is formed into some shape S, then there will be some point in space at which the resulting gravitational field attains its maximum value g_S. What shape maximizes g_S? ⋆

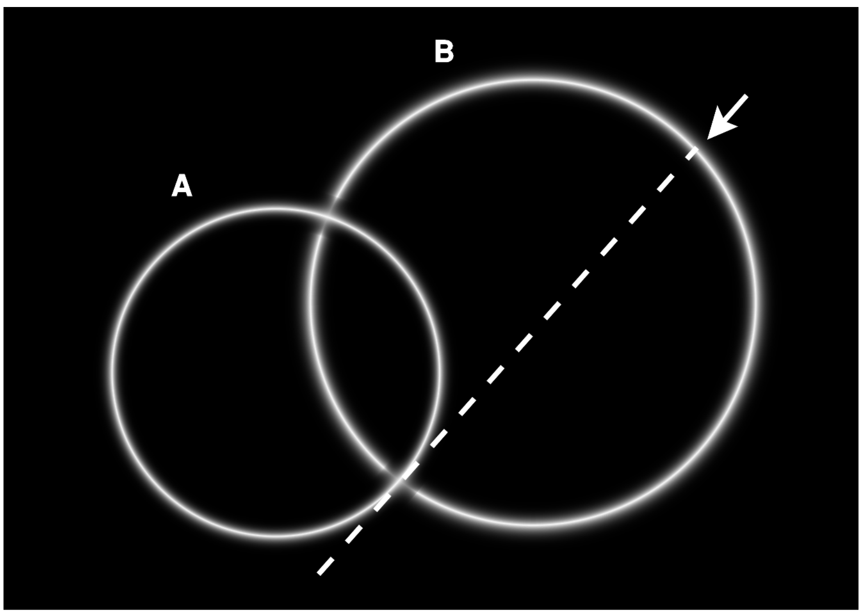
27 Complete the proof of the shell theorem in section 10.7 by filling in the case where m is inside the shell.

28 The shell theorem was proved in section 10.7.Prove that the theorem fails if the exponent of r in Newton's law of gravity differs from -2.

29 The shell theorem describes two cases, inside and outside. Show that for an alternative law of gravity $F = GMmr$ (with r^1 rather than r^{-2}), the outside case still holds.

30 On an airless body such as the moon, there is no atmospheric friction, so it should be possible for a satellite to orbit at a very low altitude, just high enough to keep from hitting the mountains. (a) Suppose that such a body is a smooth sphere of uniform density ρ and radius r. Find the velocity required for a ground-skimming orbit. ✓
(b) A typical asteroid has a density of about 2 g/cm^3, i.e., twice that of water. (This is a lot lower than the density of the earth's crust, probably indicating that the low gravity is not enough to compact the material very tightly, leaving lots of empty space inside.) Suppose that it is possible for an astronaut in a spacesuit to jump at 2 m/s. Find the radius of the largest asteroid on which it would be possible to jump into a ground-skimming orbit. ✓

31 The figure shows a region of outer space in which two stars have exploded, leaving behind two overlapping spherical shells of gas, which we assume to remain at rest. The figure is a cross-section in a plane containing the shells' centers. A space probe is released with a very small initial speed at the point indicated by the arrow, initially moving in the direction indicated by the dashed line. Without any further information, predict as much as possible about the path followed by the probe and its changes in speed along that path. ⋆



Problem 31.

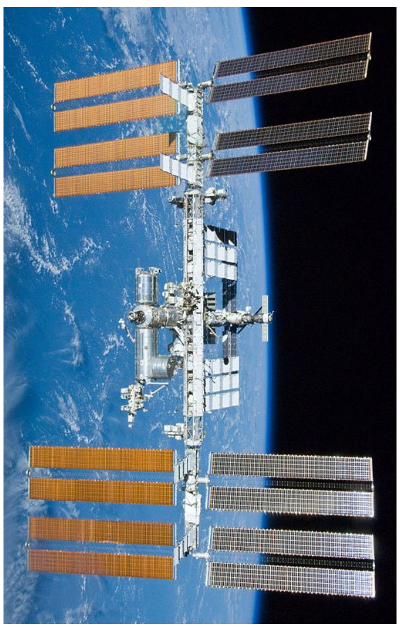
32 Approximate the earth's density as being constant. (a) Find the gravitational field at a point P inside the earth and half-way between the center and the surface. Express your result as a ratio g_P/g_S relative to the field we experience at the surface. (b) As a check on your answer, make sure that the same reasoning leads to a reasonable result when the fraction 1/2 is replaced by the value 0 (P being the earth's center) or the value 1 (P being a point on the surface).

33 The earth is divided into solid inner core, a liquid outer core, and a plastic mantle. Physical properties such as density change discontinuously at the boundaries between one layer and the next. Although the density is not completely constant within each region, we will approximate it as being so for the purposes of this problem. (We neglect the crust as well.) Let R be the radius of the earth as a whole and M its mass. The following table gives a model of some properties of the three layers, as determined by methods such as the observation of earthquake waves that have propagated from one side of the planet to the other.

region	*outer radius*$/R$	*mass*$/M$
mantle	1	0.69
outer core	0.55	0.29
inner core	0.19	0.017

The boundary between the mantle and the outer core is referred to as the Gutenberg discontinuity. Let g_s be the strength of the earth's gravitational field at its surface and g_G its value at the Gutenberg discontinuity. Find g_G/g_s. √

34 The figure shows the International Space Station (ISS). The ISS orbits the earth once every 92.6 minutes. It is desirable to keep the same side of the station always oriented toward the earth, which means that the station has to rotate with the same period. In the photo, the direction of orbital motion is left or right on the page, so the rotation is about the axis shown as up and down on the page. The greatest distance of any pressurized compartment from the axis of rotation is 36.5 meters. Find the acceleration, and the apparent weight of a 60 kg astronaut at that location. √



Problem 34.

Exercise 10: The shell theorem

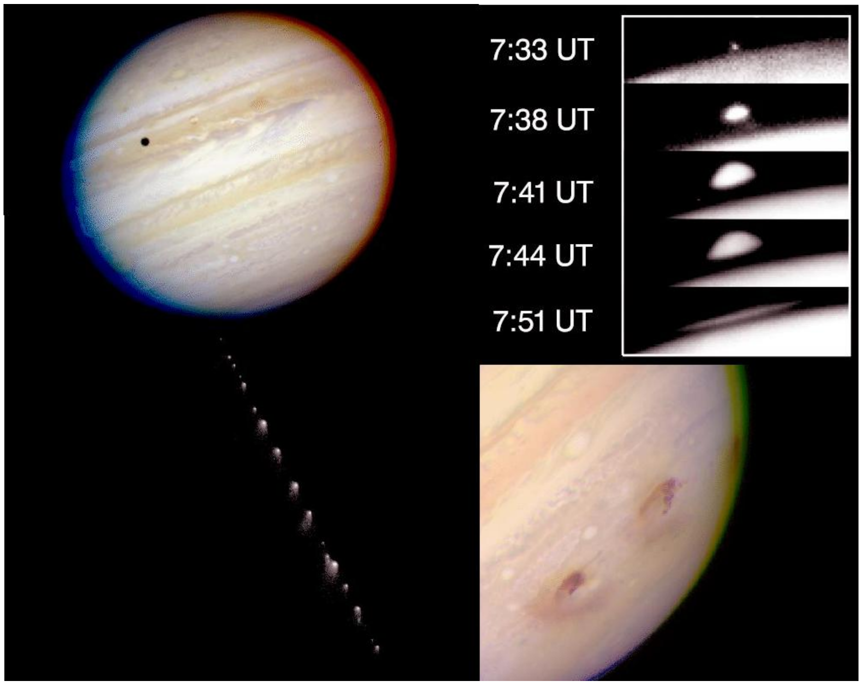
This exercise is an approximate numerical test of the shell theorem. There are seven masses A-G, each being one kilogram. Masses A-F, each one meter from the center, form a shape like two Egyptian pyramids joined at their bases; this is a rough approximation to a six-kilogram spherical shell of mass. Mass G is five meters from the center of the main group. The class will divide into six groups and split up the work required in order to calculate the vector sum of the six gravitational forces exerted on mass G. Depending on the size of the class, more than one group may be assigned to deal with the contribution of the same mass to the total force, and the redundant groups can check each other's results.

1. Discuss as a class what can be done to simplify the task of calculating the vector sum, and how to organize things so that each group can work in parallel with the others.

2. Each group should write its results on the board in units of piconewtons, retaining five significant figures of precision. Everyone will need to use the same value for the gravitational constant, $G = 6.6743 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.

3. The class will determine the vector sum and compare with the result that would be obtained with the shell theorem.

Conservation laws



In July of 1994, Comet Shoemaker-Levy struck the planet Jupiter, depositing 7×10^{22} joules of energy, and incidentally giving rise to a series of Hollywood movies in which our own planet is threatened by an impact by a comet or asteroid. There is evidence that such an impact caused the extinction of the dinosaurs. Left: Jupiter's gravitational force on the near side of the comet was greater than on the far side, and this difference in force tore up the comet into a string of fragments. Two separate telescope images have been combined to create the illusion of a point of view just behind the comet. (The colored fringes at the edges of Jupiter are artifacts of the imaging system.) Top: A series of images of the plume of superheated gas kicked up by the impact of one of the fragments. The plume is about the size of North America. Bottom: An image after all the impacts were over, showing the damage done.

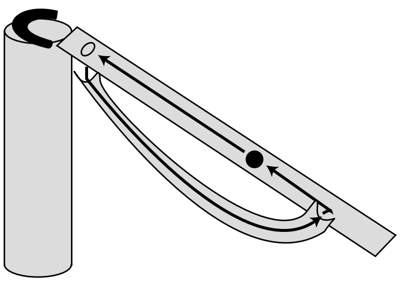
Chapter 11

Conservation of energy

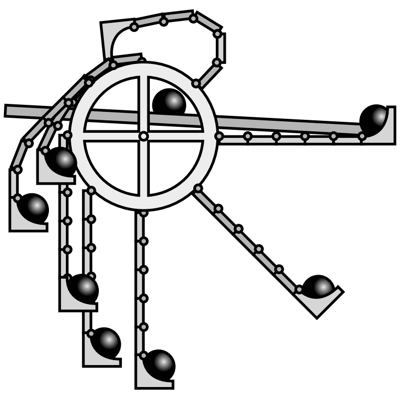
11.1 The search for a perpetual motion machine

Don't underestimate greed and laziness as forces for progress. Modern chemistry was born from the collision of lust for gold with distaste for the hard work of finding it and digging it up. Failed efforts by generations of alchemists to turn lead into gold led finally to the conclusion that it could not be done: certain substances, the chemical elements, are fundamental, and chemical reactions can neither increase nor decrease the amount of an element such as gold.

Now flash forward to the early industrial age. Greed and laziness have created the factory, the train, and the ocean liner, but in each of these is a boiler room where someone gets sweaty shoveling the coal to fuel the steam engine. Generations of inventors have tried to create a machine, called a perpetual motion machine, that would run forever without fuel. Such a machine is not forbidden by Newton's laws of motion, which are built around the concepts of force and inertia. Force is free, and can be multiplied indefinitely with pulleys, gears, or levers. The principle of inertia seems even to encourage the belief that a cleverly constructed machine might not ever run down.



a / The magnet draws the ball to the top of the ramp, where it falls through the hole and rolls back to the bottom.



b / As the wheel spins clockwise, the flexible arms sweep around and bend and unbend. By dropping off its ball on the ramp, the arm is supposed to make itself lighter and easier to lift over the top. Picking its own ball back up again on the right, it helps to pull the right side down.

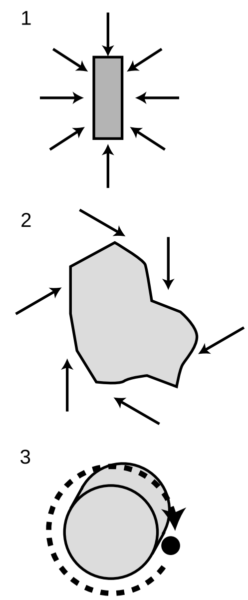
Figures a and b show two of the innumerable perpetual motion machines that have been proposed. The reason these two examples don't work is not much different from the reason all the others have failed. Consider machine a. Even if we assume that a properly shaped ramp would keep the ball rolling smoothly through each cycle, friction would always be at work. The designer imagined that the machine would repeat the same motion over and over again, so that every time it reached a given point its speed would be exactly the same as the last time. But because of friction, the speed would actually be reduced a little with each cycle, until finally the ball would no longer be able to make it over the top.

Friction has a way of creeping into all moving systems. The rotating earth might seem like a perfect perpetual motion machine, since it is isolated in the vacuum of outer space with nothing to exert frictional forces on it. But in fact our planet's rotation has slowed drastically since it first formed, and the earth continues to slow its rotation, making today just a little longer than yesterday. The very subtle source of friction is the tides. The moon's gravity raises bulges in the earth's oceans, and as the earth rotates the bulges progress around the planet. Where the bulges encounter land, there is friction, which slows the earth's rotation very gradually.

11.2 Energy

The analysis based on friction is somewhat superficial, however. One could understand friction perfectly well and yet imagine the following situation. Astronauts bring back a piece of magnetic ore from the moon which does not behave like ordinary magnets. A normal bar magnet, c/1, attracts a piece of iron essentially directly toward it, and has no left- or right-handedness. The moon rock, however, exerts forces that form a whirlpool pattern around it, 2. NASA goes to a machine shop and has the moon rock put in a lathe and machined down to a smooth cylinder, 3. If we now release a ball bearing on the surface of the cylinder, the magnetic force whips it around and around at ever higher speeds. Of course there is some friction, but there is a net gain in speed with each revolution.

Physicists would lay long odds against the discovery of such a moon rock, not just because it breaks the rules that magnets normally obey but because, like the alchemists, they have discovered a very deep and fundamental principle of nature which forbids certain things from happening. The first alchemist who deserved to be called a chemist was the one who realized one day, "In all these attempts to create gold where there was none before, all I've been doing is shuffling the same atoms back and forth among different test tubes. The only way to increase the amount of gold in my laboratory is to bring some in through the door." It was like having some of your money in a checking account and some in a savings account. Transferring money from one account into the other doesn't change the total amount.



c / A mysterious moon rock makes a perpetual motion machine.

We say that the number of grams of gold is a *conserved* quantity. In this context, the word "conserve" does not have its usual meaning of trying not to waste something. In physics, a conserved quantity is something that you wouldn't be able to get rid of even if you wanted to. Conservation laws in physics always refer to a *closed system*, meaning a region of space with boundaries through which the quantity in question is not passing. In our example, the alchemist's laboratory is a closed system because no gold is coming in or out through the doors.

Conservation of mass *example 1*

In figure d, the stream of water is fatter near the mouth of the faucet, and skinnier lower down. This is because the water speeds up as it falls. If the cross-sectional area of the stream was equal all along its length, then the rate of flow through a lower cross-section would be greater than the rate of flow through a cross-section higher up. Since the flow is steady, the amount of water between the two cross-sections stays constant. The cross-sectional area of the stream must therefore shrink in inverse proportion to the increasing speed of the falling water. This is an example of conservation of mass.


d / Example 1.

In general, the amount of any particular substance is not conserved. Chemical reactions can change one substance into another, and nuclear reactions can even change one element into another. The total mass of all substances is however conserved:

the law of conservation of mass

The total mass of a closed system always remains constant. Mass cannot be created or destroyed, but only transferred from one system to another.

A similar lightbulb eventually lit up in the heads of the people who had been frustrated trying to build a perpetual motion machine. In perpetual motion machine a, consider the motion of one of the balls. It performs a cycle of rising and falling. On the way down it gains speed, and coming up it slows back down. Having a greater speed is like having more money in your checking account, and being high up is like having more in your savings account. The device is simply shuffling funds back and forth between the two. Having more balls doesn't change anything fundamentally. Not only that, but friction is always draining off money into a third "bank account:" heat. The reason we rub our hands together when we're cold is that kinetic friction heats things up. The continual buildup in the "heat account" leaves less and less for the "motion account" and "height account," causing the machine eventually to run down.

These insights can be distilled into the following basic principle of physics:

the law of conservation of energy

It is possible to give a numerical rating, called energy, to the state of a physical system. The total energy is found by adding up contributions from characteristics of the system such as motion of objects in it, heating of the objects, and the relative positions of objects that interact via forces. The total energy of a closed system always remains constant. Energy cannot be created or destroyed, but only transferred from one system to another.

The moon rock story violates conservation of energy because the rock-cylinder and the ball together constitute a closed system. Once the ball has made one revolution around the cylinder, its position relative to the cylinder is exactly the same as before, so the numerical energy rating associated with its position is the same as before. Since the total amount of energy must remain constant, it is impossible for the ball to have a greater speed after one revolution. If it had picked up speed, it would have more energy associated with motion, the same amount of energy associated with position, and a little more energy associated with heating through friction. There cannot be a net increase in energy.

Converting one form of energy to another — *example 2*

Dropping a rock: The rock loses energy because of its changing position with respect to the earth. Nearly all that energy is transformed into energy of motion, except for a small amount lost to heat created by air friction.

Sliding in to home base: The runner's energy of motion is nearly all converted into heat via friction with the ground.

Accelerating a car: The gasoline has energy stored in it, which is released as heat by burning it inside the engine. Perhaps 10%

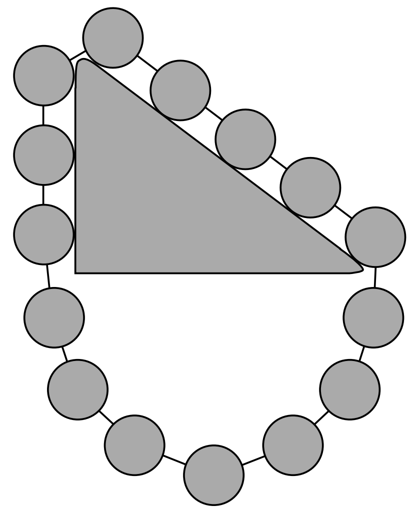
of this heat energy is converted into the car's energy of motion. The rest remains in the form of heat, which is carried away by the exhaust.

Cruising in a car: As you cruise at constant speed in your car, all the energy of the burning gas is being converted into heat. The tires and engine get hot, and heat is also dissipated into the air through the radiator and the exhaust.

Stepping on the brakes: All the energy of the car's motion is converted into heat in the brake shoes.

Stevin's machine *example 3*

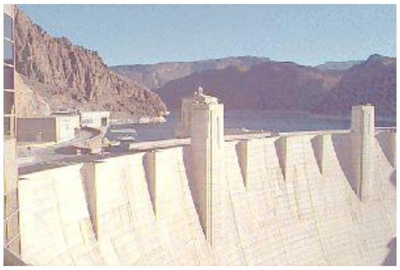
The Dutch mathematician and engineer Simon Stevin proposed the imaginary machine shown in figure e, which he had inscribed on his tombstone. This is an interesting example, because it shows a link between the force concept used earlier in this course, and the energy concept being developed now.



e / Example 3.

The point of the imaginary machine is to show the mechanical advantage of an inclined plane. In this example, the triangle has the proportions 3-4-5, but the argument works for any right triangle. We imagine that the chain of balls slides without friction, so that no energy is ever converted into heat. If we were to slide the chain clockwise by one step, then each ball would take the place of the one in front of it, and the over all configuration would be exactly the same. Since energy is something that only depends on the state of the system, the energy would have to be the same. Similarly for a counterclockwise rotation, no energy of position would be released by gravity. This means that if we place the chain on the triangle, and release it at rest, it can't start moving, because there would be no way for it to convert energy of position into energy of motion. Thus the chain must be perfectly balanced. Now by symmetry, the arc of the chain hanging underneath the triangle has equal tension at both ends, so removing this arc wouldn't affect the balance of the rest of the chain. This means that a weight of three units hanging vertically balances a weight of five units hanging diagonally along the hypotenuse.

The mechanical advantage of the inclined plane is therefore $5/3$, which is exactly the same as the result, $1/\sin\theta$, that we got on p. 240 by analyzing force vectors. What this shows is that Newton's laws and conservation laws are not logically separate, but rather are very closely related descriptions of nature. In the cases where Newton's laws are true, they give the same answers as the conservation laws. This is an example of a more general idea, called the correspondence principle, about how science progresses over time. When a newer, more general theory is proposed to replace an older theory, the new theory must agree with the old one in the realm of applicability of the old theory, since the old theory only became accepted as a valid theory by being verified experimentally in a variety of experiments. In other words, the new theory must be backward-compatible with the old one. Even though conservation laws can prove things that Newton's laws can't (that perpetual motion is impossible, for example), they aren't going to *disprove* Newton's laws when applied to mechanical systems where we already knew Newton's laws were valid.



Discussion question A. The water behind the Hoover Dam has energy because of its position relative to the planet earth, which is attracting it with a gravitational force. Letting water down to the bottom of the dam converts that energy into energy of motion. When the water reaches the bottom of the dam, it hits turbine blades that drive generators, and its energy of motion is converted into electrical energy.

Discussion question

A Hydroelectric power (water flowing over a dam to spin turbines) appears to be completely free. Does this violate conservation of energy? If not, then what is the ultimate source of the electrical energy produced by a hydroelectric plant?

B How does the proof in example 3 fail if the assumption of a frictionless surface doesn't hold?

11.3 A numerical scale of energy

Energy comes in a variety of forms, and physicists didn't discover all of them right away. They had to start somewhere, so they picked one form of energy to use as a standard for creating a numerical energy scale. (In fact the history is complicated, and several different energy units were defined before it was realized that there was a single general energy concept that deserved a single consistent unit of measurement.) One practical approach is to define an energy unit based on heating water. The SI unit of energy is the joule, J, (rhymes with "cool"), named after the British physicist James Joule. One Joule is the amount of energy required in order to heat 0.24 g of water by 1°C. The number 0.24 is not worth memorizing. A convenient way of restating this definition is that when heating water, heat $= cm\Delta T$, where ΔT is the change in temperature in °C, m is the mass, and we have defined the joule by defining the constant c, called the specific heat capacity of water, to have the value 4.2×10^3 J/kg·°C.

Note that heat, which is a form of energy, is completely different from temperature, which is not. Twice as much heat energy is required to prepare two cups of coffee as to make one, but two cups of coffee mixed together don't have double the temperature. In other words, the temperature of an object tells us how hot it is, but the heat energy contained in an object also takes into account the object's mass and what it is made of.[1]

Later we will encounter other quantities that are conserved in physics, such as momentum and angular momentum, and the method for defining them will be similar to the one we have used for energy:

[1] In standard, formal terminology, there is another, finer distinction. The word "heat" is used only to indicate an amount of energy that is transferred, whereas "thermal energy" indicates an amount of energy contained in an object. I'm informal on this point, and refer to both as heat, but you should be aware of the distinction.

pick some standard form of it, and then measure other forms by comparison with this standard. The flexible and adaptable nature of this procedure is part of what has made conservation laws such a durable basis for the evolution of physics.

Heating a swimming pool — *example 4*

▷ If electricity costs 3.9 cents per MJ (1 MJ = 1 megajoule = 10^6 J), how much does it cost to heat a 26000-gallon swimming pool from 10°C to 18°C?

▷ Converting gallons to cm^3 gives

$$26000\text{ gallons} \times \frac{3780\text{ cm}^3}{1\text{ gallon}} = 9.8 \times 10^7\text{ cm}^3.$$

Water has a density of 1 gram per cubic centimeter, so the mass of the water is 9.8×10^4 kg. The energy needed to heat the swimming pool is

$$mc\Delta T = 3.3 \times 10^3\text{ MJ}.$$

The cost of the electricity is $(3.3 \times 10^3$ MJ)($0.039/MJ)=$130.

Irish coffee — *example 5*

▷ You make a cup of Irish coffee out of 300 g of coffee at 100°C and 30 g of pure ethyl alcohol at 20°C. The specific heat capacity of ethanol is 2.4×10^3 J/kg·°C (i.e., alcohol is easier to heat than water). What temperature is the final mixture?

▷ Adding up all the energy after mixing has to give the same result as the total before mixing. We let the subscript *i* stand for the initial situation, before mixing, and *f* for the final situation, and use subscripts *c* for the coffee and *a* for the alcohol. In this notation, we have

$$\text{total initial energy} = \text{total final energy}$$
$$E_{ci} + E_{ai} = E_{cf} + E_{af}.$$

We assume coffee has the same heat-carrying properties as water. Our information about the heat-carrying properties of the two substances is stated in terms of the *change* in energy required for a certain *change* in temperature, so we rearrange the equation to express everything in terms of energy differences:

$$E_{af} - E_{ai} = E_{ci} - E_{cf}.$$

Using the heat capacities c_c for coffee (water) and c_a for alcohol, we have

$$E_{ci} - E_{cf} = (T_{ci} - T_{cf})m_c c_c \qquad \text{and}$$
$$E_{af} - E_{ai} = (T_{af} - T_{ai})m_a c_a.$$

Setting these two quantities to be equal, we have

$$(T_{af} - T_{ai})m_a c_a = (T_{ci} - T_{cf})m_c c_c.$$

In the final mixture the two substances must be at the same temperature, so we can use a single symbol $T_f = T_{cf} = T_{af}$ for the two quantities previously represented by two different symbols,

$$(T_f - T_{ai})m_a c_a = (T_{ci} - T_f)m_c c_c.$$

Solving for T_f gives

$$\begin{aligned} T_f &= \frac{T_{ci} m_c c_c + T_{ai} m_a c_a}{m_c c_c + m_a c_a} \\ &= 96^\circ\text{C}. \end{aligned}$$

Once a numerical scale of energy has been established for some form of energy such as heat, it can easily be extended to other types of energy. For instance, the energy stored in one gallon of gasoline can be determined by putting some gasoline and some water in an insulated chamber, igniting the gas, and measuring the rise in the water's temperature. (The fact that the apparatus is known as a "bomb calorimeter" will give you some idea of how dangerous these experiments are if you don't take the right safety precautions.) Here are some examples of other types of energy that can be measured using the same units of joules:

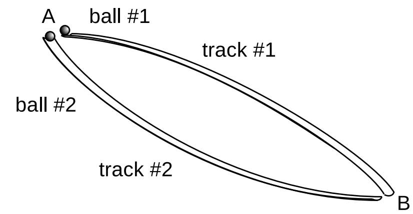
type of energy	**example**
chemical energy released by burning	About 50 MJ are released by burning a kg of gasoline.
energy required to break an object	When a person suffers a spiral fracture of the thighbone (a common type in skiing accidents), about 2 J of energy go into breaking the bone.
energy required to melt a solid substance	7 MJ are required to melt 1 kg of tin.
chemical energy released by digesting food	A bowl of Cheeries with milk provides us with about 800 kJ of usable energy.
raising a mass against the force of gravity	Lifting 1.0 kg through a height of 1.0 m requires 9.8 J.
nuclear energy released in fission	1 kg of uranium oxide fuel consumed by a reactor releases 2×10^{12} J of stored nuclear energy.

It is interesting to note the disproportion between the megajoule energies we consume as food and the joule-sized energies we expend in physical activities. If we could perceive the flow of energy around us the way we perceive the flow of water, eating a bowl of cereal would be like swallowing a bathtub's worth of energy, the continual loss of body heat to one's environment would be like an energy-hose left on all day, and lifting a bag of cement would be like flicking it with a few tiny energy-drops. The human body is tremendously inefficient. The calories we "burn" in heavy exercise are almost all dissipated directly as body heat.

You take the high road and I'll take the low road. *example 6*

▷ Figure f shows two ramps which two balls will roll down. Compare their final speeds, when they reach point B. Assume friction is negligible.

▷ Each ball loses some energy because of its decreasing height above the earth, and conservation of energy says that it must gain an equal amount of energy of motion (minus a little heat created by friction). The balls lose the same amount of height, so their final speeds must be equal.



f / Example 6.

It's impressive to note the complete impossibility of solving this problem using only Newton's laws. Even if the shape of the track had been given mathematically, it would have been a formidable task to compute the balls' final speed based on vector addition of the normal force and gravitational force at each point along the way.

How new forms of energy are discovered

Textbooks often give the impression that a sophisticated physics concept was created by one person who had an inspiration one day, but in reality it is more in the nature of science to rough out an idea and then gradually refine it over many years. The idea of energy was tinkered with from the early 1800's on, and new types of energy kept getting added to the list.

To establish the existence of a new form of energy, a physicist has to

(1) show that it could be converted to and from other forms of energy; and

(2) show that it related to some definite measurable property of the object, for example its temperature, motion, position relative to another object, or being in a solid or liquid state.

For example, energy is released when a piece of iron is soaked in water, so apparently there is some form of energy already stored in the iron. The release of this energy can also be related to a definite measurable property of the chunk of metal: it turns reddish-orange. There has been a chemical change in its physical state, which we call rusting.

Although the list of types of energy kept getting longer and longer, it was clear that many of the types were just variations on a theme. There is an obvious similarity between the energy needed to melt ice and to melt butter, or between the rusting of iron and many other chemical reactions. The topic of the next chapter is how this process of simplification reduced all the types of energy to a very small number (four, according to the way I've chosen to count them).

It might seem that if the principle of conservation of energy ever appeared to be violated, we could fix it up simply by inventing some new type of energy to compensate for the discrepancy. This would be like balancing your checkbook by adding in an imaginary deposit or withdrawal to make your figures agree with the bank's statements. Step (2) above guards against this kind of chicanery. In the 1920s there were experiments that suggested energy was not conserved in radioactive processes. Precise measurements of the energy released in the radioactive decay of a given type of atom showed inconsistent results. One atom might decay and release, say, 1.1×10^{-10} J of energy, which had presumably been stored in some mysterious form in the nucleus. But in a later measurement, an atom of exactly the same type might release 1.2×10^{-10} J. Atoms of the same type are supposed to be identical, so both atoms were thought to have started out with the same energy. If the amount released was random, then apparently the total amount of energy was not the same after the decay as before, i.e., energy was not conserved.

Only later was it found that a previously unknown particle, which is very hard to detect, was being spewed out in the decay. The particle, now called a neutrino, was carrying off some energy, and if this previously unsuspected form of energy was added in, energy was found to be conserved after all. The discovery of the energy discrepancies is seen with hindsight as being step (1) in the establishment of a new form of energy, and the discovery of the neutrino was step (2). But during the decade or so between step (1) and step (2) (the accumulation of evidence was gradual), physicists had the admirable honesty to admit that the cherished principle of conservation of energy might have to be discarded.

self-check A

How would you carry out the two steps given above in order to establish that some form of energy was stored in a stretched or compressed spring? ▷ Answer, p. 560

Mass Into Energy

Einstein showed that mass itself could be converted to and from energy, according to his celebrated equation $E = mc^2$, in which c is the speed of light. We thus speak of mass as simply another form of energy, and it is valid to measure it in units of joules. The mass of a 15-gram pencil corresponds to about 1.3×10^{15} J. The issue is largely academic in the case of the pencil, because very violent processes such as nuclear reactions are required in order to convert any significant fraction of an object's mass into energy. Cosmic rays, however, are continually striking you and your surroundings and converting part of their energy of motion into the mass of newly created particles. A single high-energy cosmic ray can create a "shower" of millions of previously nonexistent particles when it strikes the atmosphere. Einstein's theories are discussed later in this book.

Even today, when the energy concept is relatively mature and stable, a new form of energy has been proposed based on observations of distant galaxies whose light began its voyage to us billions of years ago. Astronomers have found that the universe's continuing expansion, resulting from the Big Bang, has not been decelerating as rapidly in the last few billion years as would have been expected from gravitational forces. They suggest that a new form of energy may be at work.

Discussion question

A I'm not making this up. XS Energy Drink has ads that read like this: *All the "Energy" ... Without the Sugar! Only 8 Calories!* Comment on this.

11.4 Kinetic energy

The technical term for the energy associated with motion is kinetic energy, from the Greek word for motion. (The root is the same as the root of the word "cinema" for a motion picture, and in French the term for kinetic energy is "énergie cinétique.") To find how much kinetic energy is possessed by a given moving object, we must convert all its kinetic energy into heat energy, which we have chosen as the standard reference type of energy. We could do this, for example, by firing projectiles into a tank of water and measuring the increase in temperature of the water as a function of the projectile's mass and velocity. Consider the following data from a series of three such experiments:

m (kg)	**v** (m/s)	**energy** (J)
1.00	1.00	0.50
1.00	2.00	2.00
2.00	1.00	1.00

Comparing the first experiment with the second, we see that doubling the object's velocity doesn't just double its energy, it quadruples it. If we compare the first and third lines, however, we find that doubling the mass only doubles the energy. This suggests that kinetic energy is proportional to mass and to the square of velocity, $KE \propto mv^2$, and further experiments of this type would indeed establish such a general rule. The proportionality factor equals 0.5 because of the design of the metric system, so the kinetic energy of a moving object is given by

$$KE = \frac{1}{2}mv^2.$$

The metric system is based on the meter, kilogram, and second, with other units being derived from those. Comparing the units on the left and right sides of the equation shows that the joule can be reexpressed in terms of the basic units as $\text{kg}\cdot\text{m}^2/\text{s}^2$.

Energy released by a comet impact *example 7*

▷ Comet Shoemaker-Levy, which struck the planet Jupiter in 1994, had a mass of roughly 4×10^{13} kg, and was moving at a speed of 60 km/s. Compare the kinetic energy released in the impact to the total energy in the world's nuclear arsenals, which is 2×10^{19} J. Assume for the sake of simplicity that Jupiter was at rest.

▷ Since we assume Jupiter was at rest, we can imagine that the comet stopped completely on impact, and 100% of its kinetic energy was converted to heat and sound. We first convert the speed to mks units, $v = 6 \times 10^4$ m/s, and then plug in to the equation to find that the comet's kinetic energy was roughly 7×10^{22} J, or about 3000 times the energy in the world's nuclear arsenals.

Energy and relative motion

Galileo's Aristotelian enemies (and it is no exaggeration to call them enemies!) would probably have objected to conservation of energy. Galilean got in trouble by claiming that an object in motion would continue in motion indefinitely in the absence of a force. This is not so different from the idea that an object's kinetic energy stays the same unless there is a mechanism like frictional heating for converting that energy into some other form.

More subtly, however, it's not immediately obvious that what we've learned so far about energy is strictly mathematically consistent with Galileo's principle that motion is relative. Suppose we verify that a certain process, say the collision of two pool balls, conserves energy as measured in a certain frame of reference: the sum of the balls' kinetic energies before the collision is equal to their sum after the collision. But what if we were to measure everything in a frame of reference that was in a different state of motion? It's not immediately obvious that the total energy before the collision will still equal the total energy after the collision. It *does* still work out. Homework problem 13, p. 333, gives a simple numerical example, and the general proof is taken up in problem 15 on p. 422 (with the solution given in the back of the book).

Why kinetic energy obeys the equation it does

I've presented the magic expression for kinetic energy, $(1/2)mv^2$, as a purely empirical fact. Does it have any deeper reason that might be knowable to us mere mortals? Yes and no. It contains three factors, and we need to consider each separately.

The reason for the factor of 1/2 is understandable, but only as an arbitrary historical choice. The metric system was designed so that some of the equations relating to energy would come out looking simple, at the expense of some others, which had to have inconvenient conversion factors in front. If we were using the old British Engineering System of units in this course, then we'd have the British Thermal Unit (BTU) as our unit of energy. In that system, the equation you'd learn for kinetic energy would have an inconvenient proportionality constant, $KE = \left(1.29 \times 10^{-3}\right) mv^2$, with KE measured in units of BTUs, v measured in feet per second, and so on. At the expense of this inconvenient equation for kinetic energy, the designers of the British Engineering System got a simple rule for calculating the energy required to heat water: one BTU per degree Fahrenheit per pound. The inventor of kinetic energy, Thomas Young, actually defined it as $KE = mv^2$, which meant that all his other equations had to be different from ours by a factor of two. All these systems of units work just fine as long as they are not combined with one another in an inconsistent way.

The proportionality to m is inevitable because the energy concept is based on the idea that we add up energy contributions from all the objects within a system. Therefore it is logically necessary that a 2 kg object moving at 1 m/s have the same kinetic energy as two 1 kg objects moving side-by-side at the same speed.

What about the proportionality to v^2? Consider:

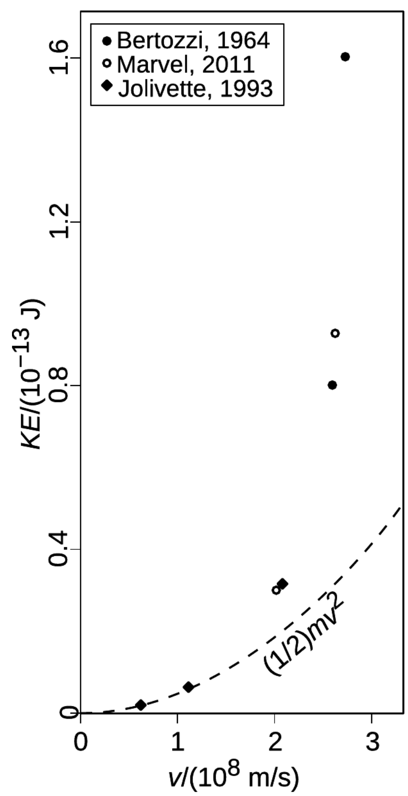
1. It's surprisingly hard to tamper with this factor without breaking things: see discussion questions A and B on p. 322.

2. The proportionality to v^2 is not even correct, except as a low-velocity approximation. Experiments show deviations from the v^2 rule at high speeds (figure g), an effect that is related to Einstein's theory of relativity.

3. As described on p. 320, we want conservation of energy to keep working when we switch frames of reference. The fact that this does work for $KE \propto v^2$ is intimately connected with the assumption that when we change frames, velocities add as described in section 2.5. This assumption turns out to be an approximation, which only works well at low velocities.

4. Conservation laws are of more general validity than Newton's laws, which apply to material objects moving at low speeds. Under the conditions where Newton's laws are accurate, they follow logically from the conservation laws. Therefore we need kinetic energy to have low-velocity behavior that ends up correctly reproducing Newton's laws.

So *under a certain set of low-velocity approximations*, $KE \propto v^2$ is what works. We verify in problem 15, p. 422, that it satisfies criterion 3, and we show in section 13.6, p. 374, that it is the *only* such relation that satisfies criterion 4.



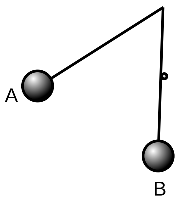
g / Kinetic energies of electrons measured in three experiments. At high velocities, the equation $KE = (1/2)mv^2$ becomes a poor approximation.

Discussion questions

A Suppose that, like Young or Einstein, you were trying out different equations for kinetic energy to see if they agreed with the experimental data. Based on the meaning of positive and negative signs of velocity, why would you suspect that a proportionality to mv would be less likely than mv^2?

B As in discussion question A, try to think of an argument showing that $m(v^2 + v^4)$ is not a possible formula for kinetic energy.

C The figure shows a pendulum that is released at A and caught by a peg as it passes through the vertical, B. To what height will the bob rise on the right?



Discussion question C

11.5 Power

A car may have plenty of energy in its gas tank, but still may not be able to increase its kinetic energy rapidly. A Porsche doesn't necessarily have more energy in its gas tank than a Hyundai, it is just able to transfer it more quickly. The rate of transferring energy from one form to another is called *power*. The definition can be written as an equation,

$$P = \frac{\Delta E}{\Delta t},$$

where the use of the delta notation in the symbol ΔE has the usual interpretation: the final amount of energy in a certain form minus the initial amount that was present in that form. Power has units of J/s, which are abbreviated as watts, W (rhymes with "lots").

If the rate of energy transfer is not constant, the power at any instant can be defined as the derivative $\mathrm{d}E/\mathrm{d}t$

Converting kilowatt-hours to joules *example 8*

▷ The electric company bills you for energy in units of kilowatt-hours (kilowatts multiplied by hours) rather than in SI units of joules. How many joules is a kilowatt-hour?

▷ 1 kilowatt-hour = (1 kW)(1 hour) = (1000 J/s)(3600 s) = 3.6 MJ.

Human wattage *example 9*

▷ A typical person consumes 2000 kcal of food in a day, and converts nearly all of that directly to heat. Compare the person's heat output to the rate of energy consumption of a 100-watt lightbulb.

▷ Looking up the conversion factor from calories to joules, we find

$$\Delta E = 2000\ \text{kcal} \times \frac{1000\ \text{cal}}{1\ \text{kcal}} \times \frac{4.18\ \text{J}}{1\ \text{cal}} = 8 \times 10^6\ \text{J}$$

for our daily energy consumption. Converting the time interval likewise into mks,

$$\Delta t = 1\ \text{day} \times \frac{24\ \text{hours}}{1\ \text{day}} \times \frac{60\ \text{min}}{1\ \text{hour}} \times \frac{60\ \text{s}}{1\ \text{min}} = 9 \times 10^4\ \text{s}.$$

Dividing, we find that our power dissipated as heat is 90 J/s = 90 W, about the same as a lightbulb.

Wind power density *example 10*

Wind power is a renewable energy resource, but it is most practical in areas where the wind is both strong and reliably strong. When a horizontal-axis wind turbine faces directly into a wind flowing at speed v, the air it intercepts in time Δt forms a cylinder whose length is $v\Delta t$, and whose mass is proportional to the same factor. The kinetic energy of this cylinder represents the maximum energy that can theoretically be extracted in this time. Since the mass is proportional to v, the kinetic energy is proportional to $v \times v^2 = v^3$. That is, the "wind power density" varies as the cube of the wind's speed.

It is easy to confuse the concepts of force, energy, and power, especially since they are synonyms in ordinary speech. The table on the following page may help to clear this up:

	force	energy	power
conceptual definition	A force is an interaction between two objects that causes a push or a pull. A force can be defined as anything that is capable of changing an object's state of motion.	Heating an object, making it move faster, or increasing its distance from another object that is attracting it are all examples of things that would require fuel or physical effort. All these things can be quantified using a single scale of measurement, and we describe them all as forms of energy.	Power is the rate at which energy is transformed from one form to another or transferred from one object to another.
operational definition	A spring scale can be used to measure force.	If we define a unit of energy as the amount required to heat a certain amount of water by a 1°C, then we can measure any other quantity of energy by transferring it into heat in water and measuring the temperature increase.	Measure the change in the amount of some form of energy possessed by an object, and divide by the amount of time required for the change to occur.
scalar or vector?	vector — has a direction in space which is the direction in which it pulls or pushes	scalar — has no direction in space	scalar — has no direction in space
unit	newtons (N)	joules (J)	watts (W) = joules/s
Can it run out? Does it cost money?	No. I don't have to pay a monthly bill for the meganewtons of force required to hold up my house.	Yes. We pay money for gasoline, electrical energy, batteries, etc., because they contain energy.	More power means you are paying money at a higher rate. A 100-W lightbulb costs a certain number of cents per hour.
Can it be a property of an object?	No. A force is a relationship between two interacting objects. A home-run baseball doesn't "have" force.	Yes. What a home-run baseball has is kinetic energy, not force.	Not really. A 100-W lightbulb doesn't "have" 100 W. 100 J/s is the rate at which it converts electrical energy into light.

11.6 ⋆ Massless particles

Failure of Newton's laws

One of the main reasons for preferring conservation laws to Newton's laws as a foundation for physics is that conservation laws are more general. For example, Newton's laws apply only to matter, whereas conservation laws can handle light as well. No experiment in Newton's day had ever shown anything but zero for the mass or weight of a ray of light, and substituting $m = 0$ into $a = F/m$ results in an infinite acceleration, which doesn't make sense. With hindsight, this is to be expected because of relativity (section 2.6). Newton's laws are only a good approximation for velocities that are small compared to c, the maximum speed of cause and effect. But light travels at c, so Newton's laws are not a good approximation to the behavior of light.

For insight into the behavior of things that go at exactly c, let's consider a case where something goes very close to c. A typical 22-caliber rifle shoots a bullet with a mass of about 3 g at a speed of about 400 m/s. Now consider the firing of such a rifle as seen through an ultra-powerful telescope by an alien in a distant galaxy. We happen to be firing in the direction away from the alien, who gets a view from over our shoulder. Since the universe is expanding, our two galaxies are receding from each other. In the alien's frame, our own galaxy is the one that is moving — let's say at $c-(200\text{ m/s})$. If the two velocities simply added, the bullet would be moving at $c+(200\text{ m/s})$. But velocities don't simply add and subtract relativistically (p. 90), and applying the correct equation for relativistic combination of velocities, we find that in the alien's frame, the bullet flies at only $c-(199.9995\text{ m/s})$. That is, according to the alien, the energy in the gunpowder only succeeded in accelerating the bullet by 0.0005 m/s! If we insisted on believing in $KE = (1/2)mv^2$, this would clearly violate conservation of energy in the alien's frame of reference. KE must not only get bigger faster than $(1/2)mv^2$ as v approaches c, it must blow up to infinity. This gives a mechanical explanation for why no material object can ever reach or exceed c, which is reassuring because speeds greater than c lead to violation of causality.

Ultrarelativistic motion

The bullet as seen in the alien's frame of reference is an example of an ultrarelativistic particle, meaning one moving very close to c. We can fairly easily infer quite a bit about how kinetic energy must behave at ultrarelativistic speeds. We know that it must get larger and larger, and the question is how large it is when the speed differs from c by some small amount.

A good way of thinking about an ultrarelativistic particle is that it's a particle with a very small mass. For example, the subatomic particle called the neutrino has a very small mass, thousands of times smaller than that of the electron. Neutrinos are emitted in radioactive decay, and because the neutrino's mass is so small, the amount of energy available in these decays is always enough to accelerate it to very close to the speed of light. Nobody has ever succeeded in observing a neutrino that was *not* ultrarelativistic. When a particle's mass is very small, the mass becomes difficult to measure. For almost 70 years after the neutrino was discovered, its mass was thought to be zero. Similarly, we currently believe that a ray of light has no mass, but it is always possible that its mass will be found to be nonzero at some point in the future. A ray of light can be modeled as an ultrarelativistic particle.

Let's compare ultrarelativistic particles with train cars. A single car with kinetic energy E has different properties than a train of two cars each with kinetic energy $E/2$. The single car has half the mass and a speed that is greater by a factor of $\sqrt{2}$. But the same is not true for ultrarelativistic particles. Since an idealized ultrarelativistic particle has a mass too small to be detectable in any experiment, we can't detect the difference between m and $2m$. Furthermore, ultrarelativistic particles move at close to c, so there is no observable difference in speed. Thus we expect that a single ultrarelativistic particle with energy E compared with two such particles, each with energy $E/2$, should have all the same properties as measured by a mechanical detector.

An idealized zero-mass particle also has no frame in which it can be at rest. It always travels at c, and no matter how fast we chase after it, we can never catch up. We can, however, observe it in different frames of reference, and we will find that its energy is different. For example, distant galaxies are receding from us at substantial fractions of c, and when we observe them through a telescope, they appear very dim not just because they are very far away but also because their light has less energy in our frame than in a frame at rest relative to the source. This effect must be such that changing frames of reference according to a specific Lorentz transformation always changes the energy of the particle by a fixed factor, regardless of the particle's original energy; for if not, then the effect of a Lorentz transformation on a single particle of energy E would be different from its effect on two particles of energy $E/2$.

How does this energy-shift factor depend on the velocity v of the Lorentz transformation? Actually, it is more convenient to express this in terms of a different variable rather than v. In nonrelativistic physics, we change frames of reference simply by adding a constant onto all our velocities, but this is only a low-velocity approximation. For this reason, it will be more convenient to work with a variable s, defined as the factor by which the long diagonal of a parallelogram like the ones in section 2.6 stretches under a Lorentz transformation. For example, we found in problem 21 on p. 100 that a velocity of $0.6c$ corresponds to a stretch factor $s = 2$. The convenient thing about stretch factors is that when we change to a new frame of reference, they simply multiply. For example, in problem 21 you found the result of combining a velocity of $0.6c$ with another velocity of $0.6c$ by drawing a parallelogram with its long axis stretched by a factor of $2 \times 2 = 4$. The relation between s and v is given by $s = \sqrt{(1+v)/(1-v)}$ (in units with $c = 1$; see problems 18 on p. 99 and 22 on p. 101).

A low-speed approximation *example 11*

What happens when the the velocity is small compared to c? In units where $c = 1$, this means that v is small compared to 1. The stretch factor $s = \sqrt{(1+v)/(1-v)}$ can then be approximated by taking $1/(1-v) \approx 1+v$ and $\sqrt{1+\epsilon} \approx 1+\epsilon/2$, so that $s \approx 1+v$.

Let's write $f(s)$ for the energy-shift factor that results from a given Lorentz transformation. Since a Lorentz transformation s_1 followed by a second transformation s_2 is equivalent to a single transformation by $s_1 s_2$, we must have $f(s_1 s_2) = f(s_1)f(s_2)$. This tightly constrains the form of the function f; it must be something like $f(s) = s^n$, where n is a constant. The interpretation of n is that under a Lorentz transformation corresponding to 1% of c, energies of ultrarelativistic particles change by about n% (making the approximation that $v = .01$ gives $s \approx 1.01$). We postpone until p. 415 the proof that $n = 1$, which is also in agreement with experiments with rays of light.

Our final result is that the energy of an ultrarelativistic particle is simply proportional to its Lorentz "stretch factor" s. Even in the case where the particle is truly massless, so that s doesn't have any finite value, we can still find how the energy differs according to different observers by finding the s of the Lorentz transformation between the two observers' frames of reference.

An astronomical energy shift *example 12*

▷ For quantum-mechanical reasons, a hydrogen atom can only exist in states with certain specific energies. By conservation of energy, the atom can therefore only absorb or emit light that has an energy equal to the difference between two such atomic energies. The outer atmosphere of a star is mostly made of monoatomic hydrogen, and one of the energies that a hydrogen atom can absorb or emit is 3.0276×10^{-19} J. When we observe light from stars in the Andromeda Galaxy, it has an energy of 3.0306×10^{-19} J. If this is assumed to be due entirely to the motion of the Milky Way and Andromeda Galaxy relative to one another, along the line connecting them, find the direction and magnitude of this velocity.

▷ The energy is shifted upward, which means that the Andromeda Galaxy is moving toward us. (Galaxies at cosmological distances are always observed to be receding from one another, but this doesn't necessarily hold for galaxies as close as these.) Relating the energy shift to the velocity, we have

$$\frac{E'}{E} = s = \sqrt{(1+v)/(1-v)}.$$

Since the shift is only about one part per thousand, the velocity is small compared to c — or small compared to 1 in units where $c = 1$. Therefore we can approximate as in example 11, $s \approx 1+v$, and we find

$$v \approx s - 1 = \frac{E'}{E} - 1 = 1.0 \times 10^{-3}.$$

This is in units where $c = 1$. Converting to SI units, where $c \neq 1$, we have $v = (1.0 \times 10^{-3})c = 300\ \text{km}/s$. Although the Andromeda Galaxy's tangential motion is not accurately known, it is considered likely that it will collide with the Milky Way in a few billion years.

A symmetry property of the energy shift — *example 13*

Suppose that A and B are at rest relative to one another, but C is moving along the line between A and B. A sends a pulse of laser light to C, who then measures its energy and transmits another pulse to B having the same energy. The pulse accumulates two energy shifts, and the result is their product $s(v)s(-v)$. But C didn't actually need to absorb the original pulse and retransmit it; the results would have been the same if C had just stayed out of the way. Therefore this product must equal 1, so we must have $s(-v)s(v) = 1$, which can be verified directly from the equation.

The Ives-Stilwell experiment — *example 14*

The result of example 13 was the basis of one of the earliest laboratory tests of special relativity, by Ives and Stilwell in 1938. They observed the light emitted by a beam of excited H_2^+ and H_3^+ ions with speeds of a few tenths of a percent of c. Measuring the light from both ahead of and behind the beams, they found that the product $s(v)s(-v)$ was equal to 1, as predicted by relativity. If relativity had been false, then one would have expected the product to differ from 1 by an amount that would have been detectable in their experiment. In 2003, Saathoff et al. carried out an extremely precise version of the Ives-Stilwell technique with Li^+ ions moving at 6.4% of c. The energies observed, in units of 10^{-28} J, were:

$$\begin{aligned}
E_o &= 3620927488 \pm 3 \\
&\qquad \text{(unshifted energy)} \\
E_o s(v) &= 3859620256 \pm 0.6 \\
&\qquad \text{(shifted energy, forward)} \\
E_o s(-v) &= 3396996334 \pm 3 \\
&\qquad \text{(shifted energy, backward)} \\
\sqrt{E_o s(v) \cdot E_o s(-v)} &= 3620927487 \pm 2
\end{aligned}$$

The results show incredibly precise agreement between E_o and $\sqrt{E_o s(v) \cdot E_o s(-v)}$, as expected relativistically because $s(v)s(-v)$ is supposed to equal 1. The agreement extends to 9 significant figures, whereas if relativity had been false there should have been a relative disagreement of about $v^2 = .004$, i.e., a discrepancy in the third significant figure. The spectacular agreement with theory has made this experiment a lightning rod for anti-relativity kooks.

Summary

Selected vocabulary

energy . . . . . . A numerical scale used to measure the heat, motion, or other properties that would require fuel or physical effort to put into an object; a scalar quantity with units of joules (J).

power . . . . The rate of transferring energy; a scalar quantity with units of watts (W).

kinetic energy . . The energy an object possesses because of its motion.

heat . . . . A form of energy that relates to temperature. Heat is different from temperature because an object with twice as much mass requires twice as much heat to increase its temperature by the same amount. Heat is measured in joules, temperature in degrees. (In standard terminology, there is another, finer distinction between heat and thermal energy, which is discussed below. In this book, I informally refer to both as heat.)

temperature . . . What a thermometer measures. Objects left in contact with each other tend to reach the same temperature. Cf. heat. As discussed in more detail in chapter 2, temperature is essentially a measure of the average kinetic energy per molecule.

Notation

E . . . . energy

J . . . . joules, the SI unit of energy

KE . . . . kinetic energy

P . . . . power

W . . . . watts, the SI unit of power; equivalent to J/s

Other terminology and notation

Q or ΔQ . . . . . the amount of heat transferred into or out of an object

K or T . . . . . . alternative symbols for kinetic energy, used in the scientific literature and in most advanced textbooks

thermal energy . Careful writers make a distinction between heat and thermal energy, but the distinction is often ignored in casual speech, even among physicists. Properly, thermal energy is used to mean the total amount of energy possessed by an object, while heat indicates the amount of thermal energy transferred in or out. The term heat is used in this book to include both meanings.

Summary

Heating an object, making it move faster, or increasing its distance from another object that is attracting it are all examples of things that would require fuel or physical effort. All these things can be quantified using a single scale of measurement, and we describe them all as forms of *energy*. The SI unit of energy is the Joule. The reason why energy is a useful and important quantity is that it is always conserved. That is, it cannot be created or destroyed but only transferred between objects or changed from one form to another. Conservation of energy is the most important and broadly applicable of all the laws of physics, more fundamental and general even than Newton's laws of motion.

Heating an object requires a certain amount of energy per degree of temperature and per unit mass, which depends on the substance of which the object consists. Heat and temperature are completely different things. Heat is a form of energy, and its SI unit is the joule (J). Temperature is not a measure of energy. Heating twice as much of something requires twice as much heat, but double the amount of a substance does not have double the temperature.

The energy that an object possesses because of its motion is called kinetic energy. Kinetic energy is related to the mass of the object and the magnitude of its velocity vector by the equation

$$KE = \frac{1}{2}mv^2.$$

Power is the rate at which energy is transformed from one form to another or transferred from one object to another,

$$P = \frac{\mathrm{d}E}{\mathrm{d}t}$$

The SI unit of power is the watt (W).

The equation $KE = (1/2)mv^2$ is a nonrelativistic approximation, valid at speeds that are small compared to c. In the opposite limit, of a particle with a speed very close to c, the energy is proportional to the "stretch factor" of the Lorentz transformation, $s = \sqrt{(1+v)/(1-v)}$ (in units with $c = 1$), for $v \to +c$ and $1/s$ for $v \to -c$. This gives a mechanical explanation for why no material object can ever reach or exceed c, which is reassuring because speeds greater than c lead to violation of causality.

Problems

Key

$\surd$ A computerized answer check is available online.
$\int$ A problem that requires calculus.
$\star$ A difficult problem.

1 Can kinetic energy ever be less than zero? Explain. [Based on a problem by Serway and Faughn.]

2 Estimate the kinetic energy of an Olympic sprinter.

3 You are driving your car, and you hit a brick wall head on, at full speed. The car has a mass of 1500 kg. The kinetic energy released is a measure of how much destruction will be done to the car and to your body. Calculate the energy released if you are traveling at (a) 40 mi/hr, and again (b) if you're going 80 mi/hr. What is counterintuitive about this, and what implication does this have for driving at high speeds? $\surd$

4 The following table gives the amount of energy required in order to heat, melt, or boil a gram of water.

heat 1 g of ice by 1°C	2.05 J
melt 1 g of ice	333 J
heat 1 g of water by 1°C	4.19 J
boil 1 g of water	2500 J
heat 1 g of steam by 1°C	2.01 J

(a) How much energy is required in order to convert 1.00 g of ice at -20 °C into steam at 137 °C? $\surd$
(b) What is the minimum amount of hot water that could melt 1.00 g of ice? $\surd$

5 A closed system can be a bad thing — for an astronaut sealed inside a space suit, getting rid of body heat can be difficult. Suppose a 60-kg astronaut is performing vigorous physical activity, expending 200 W of power. If none of the heat can escape from her space suit, how long will it take before her body temperature rises by 6°C(11°F), an amount sufficient to kill her? Assume that the amount of heat required to raise her body temperature by 1°C is the same as it would be for an equal mass of water. Express your answer in units of minutes. $\surd$

6 A bullet flies through the air, passes through a paperback book, and then continues to fly through the air beyond the book. When is there a force? When is there energy? ▷ Solution, p. 553

7 Experiments show that the power consumed by a boat's engine is approximately proportional to the third power of its speed. (We assume that it is moving at constant speed.) (a) When a boat is crusing at constant speed, what type of energy transformation do you think is being performed? (b) If you upgrade to a motor with double the power, by what factor is your boat's crusing speed increased? [Based on a problem by Arnold Arons.]
▷ Solution, p. 553

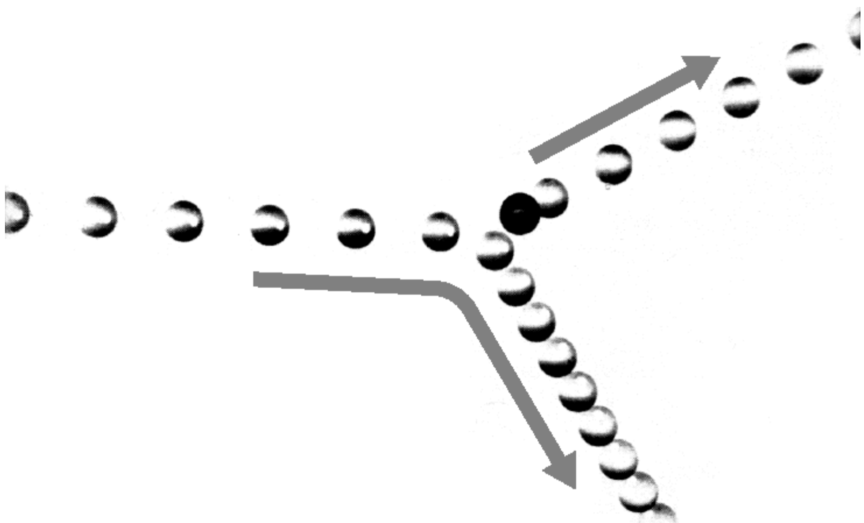
8 Object A has a kinetic energy of 13.4 J. Object B has a mass that is greater by a factor of 3.77, but is moving more slowly by a factor of 2.34. What is object B's kinetic energy? [Based on a problem by Arnold Arons.] ▷ Solution, p. 554

9 Example 10 on page 323 showed that the power produced by a wind turbine is proportional to the cube of the wind speed v. Von Kármán found empirically that when a fluid flows turbulently over a surface, the speed of the fluid is often well approximated by $v \propto z^{1/7}$, where z is the distance from the surface. Wind turbine towers are often constructed at heights of 50 m, but surveys of wind speeds are usually conducted at heights of about 3 m. By what factor should the predicted wind power density be scaled up relative to the survey data? ✓

10 The moon doesn't really just orbit the Earth. By Newton's third law, the moon's gravitational force on the earth is the same as the earth's force on the moon, and the earth must respond to the moon's force by accelerating. If we consider the earth and moon in isolation and ignore outside forces, then Newton's first law says their common center of mass doesn't accelerate, i.e., the earth wobbles around the center of mass of the earth-moon system once per month, and the moon also orbits around this point. The moon's mass is 81 times smaller than the earth's. Compare the kinetic energies of the earth and moon. (We know that the center of mass is a kind of balance point, so it must be closer to the earth than to the moon. In fact, the distance from the earth to the center of mass is 1/81 of the distance from the moon to the center of mass, which makes sense intuitively, and can be proved rigorously using the equation on page 404.)

11 My 1.25 kW microwave oven takes 126 seconds to bring 250 g of water from room temperature to a boil. What percentage of the power is being wasted? Where might the rest of the energy be going? ▷ Solution, p. 554

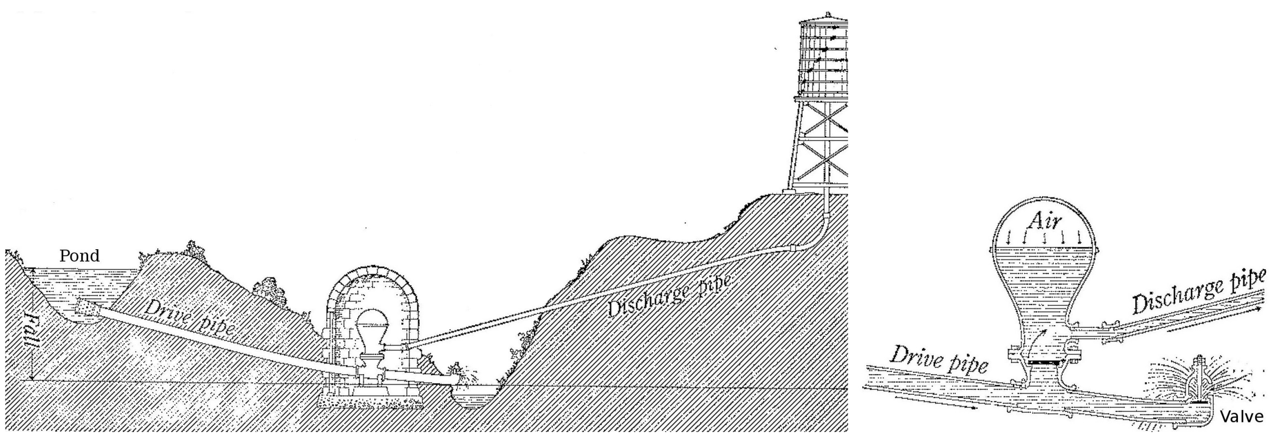
12 The multiflash photograph shows a collision between two pool balls. The ball that was initially at rest shows up as a dark image in its initial position, because its image was exposed several times before it was struck and began moving. By making *measurements* on the figure, determine *numerically* whether or not energy appears to have been conserved in the collision. What systematic effects would limit the accuracy of your test? [From an example in PSSC Physics.]



Problem 12.

13 This problem is a numerical example of the imaginary experiment discussed on p. 320 regarding the relationship between energy and relative motion. Let's say that the pool balls both have masses of 1.00 kg. Suppose that in the frame of reference of the pool table, the cue ball moves at a speed of 1.00 m/s toward the eight ball, which is initially at rest. The collision is head-on, and as you can verify for yourself the next time you're playing pool, the result of such a collision is that the incoming ball stops dead and the ball that was struck takes off with the same speed originally possessed by the incoming ball. (This is actually a bit of an idealization. To keep things simple, we're ignoring the spin of the balls, and we assume that no energy is liberated by the collision as heat or sound.) (a) Calculate the total initial kinetic energy and the total final kinetic energy, and verify that they are equal. (b) Now carry out the whole calculation again in the frame of reference that is moving in the same direction that the cue ball was initially moving, but at a speed of 0.50 m/s. In this frame of reference, both balls have nonzero initial and final velocities, which are different from what they were in the table's frame. [See also problem 15 on p. 422.]

14 One theory about the destruction of the space shuttle Columbia in 2003 is that one of its wings had been damaged on liftoff by a chunk of foam insulation that fell off of one of its external fuel tanks. The New York Times reported on June 5, 2003, that NASA engineers had recreated the impact to see if it would damage a mock-up of the shuttle's wing. "Before last week's test, many engineers at NASA said they thought lightweight foam could not harm the seemingly tough composite panels, and privately predicted that the foam would bounce off harmlessly, like a Nerf ball." In fact, the 1.7-pound piece of foam, moving at 531 miles per hour, did serious damage. A member of the board investigating the disaster said this demonstrated that "people's intuitive sense of physics is sometimes way off." (a) Compute the kinetic energy of the foam, and (b) compare with the energy of a 170-pound boulder moving at 5.3 miles per hour (the speed it would have if you dropped it from about knee-level). ✓
(c) The boulder is a hundred times more massive, but its speed is a hundred times smaller, so what's counterintuitive about your results?



15 The figure above is from a classic 1920 physics textbook by Millikan and Gale. It represents a method for raising the water from the pond up to the water tower, at a higher level, without using a pump. Water is allowed into the drive pipe, and once it is flowing fast enough, it forces the valve at the bottom closed. Explain how this works in terms of conservation of mass and energy. (Cf. example 1 on page 311.)

16 All stars, including our sun, show variations in their light output to some degree. Some stars vary their brightness by a factor of two or even more, but our sun has remained relatively steady during the hundred years or so that accurate data have been collected. Nevertheless, it is possible that climate variations such as ice ages are related to long-term irregularities in the sun's light output. If the sun was to increase its light output even slightly, it could melt enough Antarctic ice to flood all the world's coastal cities. The total sunlight that falls on Antarctica amounts to about 1×10^{16} watts. Presently, this heat input to the poles is balanced by the loss of heat via winds, ocean currents, and emission of infrared light, so that there is no net melting or freezing of ice at the poles from year to year. Suppose that the sun changes its light output by some small percentage, but there is no change in the rate of heat loss by the polar caps. Estimate the percentage by which the sun's light output would have to increase in order to melt enough ice to raise the level of the oceans by 10 meters over a period of 10 years. (This would be enough to flood New York, London, and many other cities.) Melting 1 kg of ice requires 3×10^3 J.

17 Estimate the kinetic energy of a buzzing fly's wing. (You may wish to review section 1.3 on order-of-magnitude estimates.)

18 A blade of grass moves upward as it grows. Estimate its kinetic energy. (You may wish to review section 1.3 on order-of-magnitude estimates.)

Do these forms of energy have anything in common?

Chapter 12
Simplifying the energy zoo

Variety is the spice of life, not of science. The figure shows a few examples from the bewildering array of forms of energy that surrounds us. The physicist's psyche rebels against the prospect of a long laundry list of types of energy, each of which would require its own equations, concepts, notation, and terminology. The point at which we've arrived in the study of energy is analogous to the period in the 1960's when a half a dozen new subatomic particles were being discovered every year in particle accelerators. It was an embarrassment. Physicists began to speak of the "particle zoo," and it seemed that the subatomic world was distressingly complex. The particle zoo was simplified by the realization that most of the new particles being whipped up were simply clusters of a previously unsuspected set of more fundamental particles (which were whimsically dubbed quarks, a made-up word from a line of poetry by James Joyce, “Three quarks for Master Mark.”) The energy zoo can also be simplified, and it is the purpose of this chapter to demonstrate the hidden similarities between forms of energy as seemingly different as heat and motion.

a / A vivid demonstration that heat is a form of motion. A small amount of boiling water is poured into the empty can, which rapidly fills up with hot steam. The can is then sealed tightly, and soon crumples. This can be explained as follows. The high temperature of the steam is interpreted as a high average speed of random motions of its molecules. Before the lid was put on the can, the rapidly moving steam molecules pushed their way out of the can, forcing the slower air molecules out of the way. As the steam inside the can thinned out, a stable situation was soon achieved, in which the force from the less dense steam molecules moving at high speed balanced against the force from the more dense but slower air molecules outside. The cap was put on, and after a while the steam inside the can reached the same temperature as the air outside. The force from the cool, thin steam no longer matched the force from the cool, dense air outside, and the imbalance of forces crushed the can.

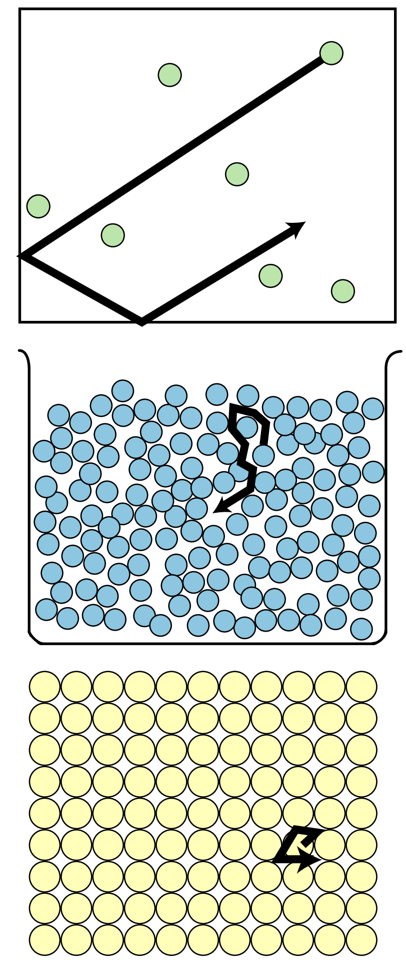
12.1 Heat is kinetic energy

What is heat really? Is it an invisible fluid that your bare feet soak up from a hot sidewalk? Can one ever remove all the heat from an object? Is there a maximum to the temperature scale?

The theory of heat as a fluid seemed to explain why colder objects absorbed heat from hotter ones, but once it became clear that heat was a form of energy, it began to seem unlikely that a material substance could transform itself into and out of all those other forms of energy like motion or light. For instance, a compost pile gets hot, and we describe this as a case where, through the action of bacteria, chemical energy stored in the plant cuttings is transformed into heat energy. The heating occurs even if there is no nearby warmer object that could have been leaking “heat fluid” into the pile.

An alternative interpretation of heat was suggested by the theory that matter is made of atoms. Since gases are thousands of times less dense than solids or liquids, the atoms (or clusters of atoms called molecules) in a gas must be far apart. In that case, what is keeping all the air molecules from settling into a thin film on the floor of the room in which you are reading this book? The simplest explanation is that they are moving very rapidly, continually ricocheting off of the floor, walls, and ceiling. Though bizarre, the cloud-of-bullets image of a gas did give a natural explanation for the surprising ability of something as tenuous as a gas to exert huge forces. Your car's tires can hold it up because you have pumped extra molecules into them. The inside of the tire gets hit by molecules more often than the outside, forcing it to stretch and stiffen.

The outward forces of the air in your car's tires increase even further when you drive on the freeway for a while, heating up the rubber and the air inside. This type of observation leads naturally to the conclusion that hotter matter differs from colder in that its atoms' random motion is more rapid. In a liquid, the motion could be visualized as people in a milling crowd shoving past each other more quickly. In a solid, where the atoms are packed together, the motion is a random vibration of each atom as it knocks against its neighbors.


b / Random motion of atoms in a gas, a liquid, and a solid.

We thus achieve a great simplification in the theory of heat. Heat is simply a form of kinetic energy, the total kinetic energy of random motion of all the atoms in an object. With this new understanding, it becomes possible to answer at one stroke the questions posed at the beginning of the section. Yes, it is at least theoretically possible to remove all the heat from an object. The coldest possible temperature, known as absolute zero, is that at which all the atoms have zero velocity, so that their kinetic energies, $(1/2)mv^2$, are all zero. No, there is no maximum amount of heat that a certain quantity of matter can have, and no maximum to the temperature scale, since arbitrarily large values of v can create arbitrarily large amounts of kinetic energy per atom.

The kinetic theory of heat also provides a simple explanation of the true nature of temperature. Temperature is a measure of the amount of energy per molecule, whereas heat is the total amount of energy possessed by all the molecules in an object.

There is an entire branch of physics, called thermodynamics, that deals with heat and temperature and forms the basis for technologies such as refrigeration.

Thermodynamics is not covered in this book, and I have provided here only a brief overview of the thermodynamic concepts that relate directly to energy, glossing over at least one point that would be dealt with more carefully in a thermodynamics course: it is really only true for a gas that all the heat is in the form of kinetic energy. In solids and liquids, the atoms are close enough to each other to exert intense electrical forces on each other, and there is therefore another type of energy involved, the energy associated with the atoms' distances from each other. Strictly speaking, heat energy is defined not as energy associated with random motion of molecules but as any form of energy that can be conducted between objects in contact, without any force.

12.2 Potential energy: energy of distance or closeness

We have already seen many examples of energy related to the distance between interacting objects. When two objects participate in an attractive noncontact force, energy is required to bring them farther apart. In both of the perpetual motion machines that started off the previous chapter, one of the types of energy involved was the energy associated with the distance between the balls and the earth, which attract each other gravitationally. In the perpetual motion machine with the magnet on the pedestal, there was also energy associated with the distance between the magnet and the iron ball, which were attracting each other.

The opposite happens with repulsive forces: two socks with the same type of static electric charge will repel each other, and cannot be pushed closer together without supplying energy.

In general, the term *potential energy,* with algebra symbol PE, is used for the energy associated with the distance between two objects that attract or repel each other via a force that depends on the distance between them. Forces that are not determined by distance do not have potential energy associated with them. For instance, the normal force acts only between objects that have zero distance between them, and depends on other factors besides the fact that the distance is zero. There is no potential energy associated with the normal force.

c / The skater has converted all his kinetic energy into potential energy on the way up the side of the pool.

The following are some commonplace examples of potential energy:

gravitational potential energy: The skateboarder in the photo has risen from the bottom of the pool, converting kinetic energy into gravitational potential energy. After being at rest for an instant, he will go back down, converting PE back into KE.

magnetic potential energy: When a magnetic compass needle is allowed to rotate, the poles of the compass change their distances from the earth's north and south magnetic poles, converting magnetic potential energy into kinetic energy. (Eventually the kinetic energy is all changed into heat by friction, and the needle settles down in the position that minimizes its potential energy.)

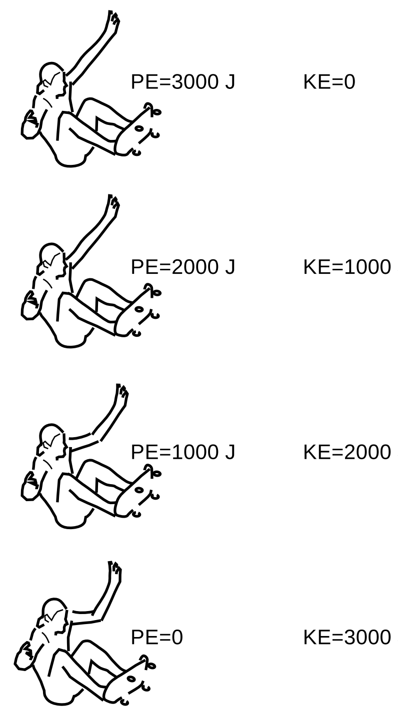
electrical potential energy: Socks coming out of the dryer cling together because of attractive electrical forces. Energy is required in order to separate them.

potential energy of bending or stretching: The force between the two ends of a spring depends on the distance between them, i.e., on the length of the spring. If a car is pressed down on its shock absorbers and then released, the potential energy stored in the spring is transformed into kinetic and gravitational potential energy as the car bounces back up.

I have deliberately avoided introducing the term potential energy up until this point, because it tends to produce unfortunate connotations in the minds of students who have not yet been inoculated with a careful description of the construction of a numerical energy scale. Specifically, there is a tendency to generalize the term inappropriately to apply to any situation where there is the "potential" for something to happen: "I took a break from digging, but I had potential energy because I knew I'd be ready to work hard again in a few minutes."

An equation for gravitational potential energy

All the vital points about potential energy can be made by focusing on the example of gravitational potential energy. For simplicity, we treat only vertical motion, and motion close to the surface of the earth, where the gravitational force is nearly constant. (The generalization to the three dimensions and varying forces is more easily accomplished using the concept of work, which is the subject of the next chapter.)



d / As the skater free-falls, his PE is converted into KE. (The numbers would be equally valid as a description of his motion on the way up.)

To find an equation for gravitational PE, we examine the case of free fall, in which energy is transformed between kinetic energy and gravitational PE. Whatever energy is lost in one form is gained in an equal amount in the other form, so using the notation ΔKE to stand for $KE_f - KE_i$ and a similar notation for PE, we have

[1] $$\Delta KE = -\Delta PE_{grav}.$$

It will be convenient to refer to the object as falling, so that PE is being changed into KE, but the math applies equally well to an object slowing down on its way up. We know an equation for kinetic energy,

[2] $$KE = \frac{1}{2}mv^2,$$

so if we can relate v to height, y, we will be able to relate ΔPE to y, which would tell us what we want to know about potential energy. The y component of the velocity can be connected to the height via the constant acceleration equation

[3] $$v_f^2 = v_i^2 + 2a\Delta y,$$

and Newton's second law provides the acceleration,

[4] $$a = F/m,$$

in terms of the gravitational force.

The algebra is simple because both equation [2] and equation [3] have velocity to the second power. Equation [2] can be solved for v^2 to give $v^2 = 2KE/m$, and substituting this into equation [3], we find

$$2\frac{KE_f}{m} = 2\frac{KE_i}{m} + 2a\Delta y.$$

Making use of equations [1] and [4] gives the simple result

$$\Delta PE_{grav} = -F\Delta y.$$

[change in gravitational PE resulting from a change in height Δy; F is the gravitational force on the object, i.e., its weight; valid only near the surface of the earth, where F is constant]

Dropping a rock *example 1*

▷ If you drop a 1-kg rock from a height of 1 m, how many joules of KE does it have on impact with the ground? (Assume that any energy transformed into heat by air friction is negligible.)

▷ If we choose the y axis to point up, then F_y is negative, and equals $-(1\text{ kg})(g) = -9.8$ N. A decrease in y is represented by a negative value of Δy, $\Delta y = -1$ m, so the change in potential energy is $-(-9.8\text{ N})(-1\text{ m}) \approx -10$ J. (The proof that newtons multiplied by meters give units of joules is left as a homework problem.) Conservation of energy says that the loss of this amount of PE must be accompanied by a corresponding increase in KE of 10 J.

It may be dismaying to note how many minus signs had to be handled correctly even in this relatively simple example: a total of four. Rather than depending on yourself to avoid any mistakes with signs, it is better to check whether the final result make sense physically. If it doesn't, just reverse the sign.

Although the equation for gravitational potential energy was derived by imagining a situation where it was transformed into kinetic energy, the equation can be used in any context, because all the types of energy are freely convertible into each other.

Gravitational PE converted directly into heat *example 2*

▷ A 50-kg firefighter slides down a 5-m pole at constant velocity. How much heat is produced?

▷ Since she slides down at constant velocity, there is no change in KE. Heat and gravitational PE are the only forms of energy that change. Ignoring plus and minus signs, the gravitational force on her body equals mg, and the amount of energy transformed is

$$(mg)(5\text{ m}) = 2500\text{ J}.$$

On physical grounds, we know that there must have been an increase (positive change) in the heat energy in her hands and in the flagpole.

Here are some questions and answers about the interpretation of the equation $\Delta PE_{grav} = -F\Delta y$ for gravitational potential energy.

Question: In a nutshell, why is there a minus sign in the equation?
Answer: It is because we increase the PE by moving the object in the *opposite* direction compared to the gravitational force.

Question: Why do we only get an equation for the *change* in potential energy? Don't I really want an equation for the potential energy itself?
Answer: No, you really don't. This relates to a basic fact about potential energy, which is that it is not a well defined quantity in the absolute sense. Only changes in potential energy are unambiguously defined. If you and I both observe a rock falling, and agree that it deposits 10 J of energy in the dirt when it hits, then we will be forced to agree that the 10 J of KE must have come from a loss of 10 joules of PE. But I might claim that it started with 37 J of PE and ended with 27, while you might swear just as truthfully that it had 109 J initially and 99 at the end. It is possible to pick some specific height as a reference level and say that the PE is zero there, but it's easier and safer just to work with changes in PE and avoid absolute PE altogether.

Question: You referred to potential energy as the energy that *two* objects have because of their distance from each other. If a rock falls, the object is the rock. Where's the other object?
Answer: Newton's third law guarantees that there will always be two objects. The other object is the planet earth.

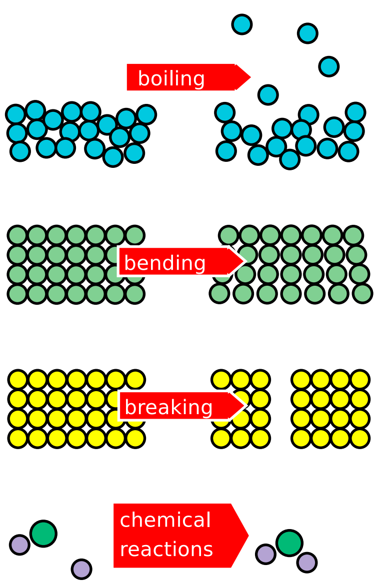
Question: If the other object is the earth, are we talking about the distance from the rock to the center of the earth or the distance from the rock to the surface of the earth?
Answer: It doesn't matter. All that matters is the change in distance, Δy, not y. Measuring from the earth's center or its surface are just two equally valid choices of a reference point for defining absolute PE.

Question: Which object contains the PE, the rock or the earth?

Answer: We may refer casually to the PE of the rock, but technically the PE is a relationship between the earth and the rock, and we should refer to the earth and the rock together as possessing the PE.

Question: How would this be any different for a force other than gravity?
Answer: It wouldn't. The result was derived under the assumption of constant force, but the result would be valid for any other situation where two objects interacted through a constant force. Gravity is unusual, however, in that the gravitational force on an object is so nearly constant under ordinary conditions. The magnetic force between a magnet and a refrigerator, on the other hand, changes drastically with distance. The math is a little more complex for a varying force, but the concepts are the same.

Question: Suppose a pencil is balanced on its tip and then falls over. The pencil is simultaneously changing its height and rotating, so the height change is different for different parts of the object. The bottom of the pencil doesn't lose any height at all. What do you do in this situation?
Answer: The general philosophy of energy is that an object's energy is found by adding up the energy of every little part of it. You could thus add up the changes in potential energy of all the little parts of the pencil to find the total change in potential energy. Luckily there's an easier way! The derivation of the equation for gravitational potential energy used Newton's second law, which deals with the acceleration of the object's center of mass (i.e., its balance point). If you just define Δy as the height change of the center of mass, everything works out. A huge Ferris wheel can be rotated without putting in or taking out any PE, because its center of mass is staying at the same height.



e / All these energy transformations turn out at the atomic level to be changes in potential energy resulting from changes in the distances between atoms.

self-check A

A ball thrown straight up will have the same speed on impact with the ground as a ball thrown straight down at the same speed. How can this be explained using potential energy? ▷ Answer, p. 560

Discussion question

A You throw a steel ball up in the air. How can you prove based on conservation of energy that it has the same speed when it falls back into your hand? What if you throw a feather up — is energy not conserved in this case?

12.3 All energy is potential or kinetic

In the same way that we found that a change in temperature is really only a change in kinetic energy at the atomic level, we now find that every other form of energy turns out to be a form of potential energy. Boiling, for instance, means knocking some of the atoms (or molecules) out of the liquid and into the space above, where they constitute a gas. There is a net attractive force between essentially any two atoms that are next to each other, which is why matter always prefers to be packed tightly in the solid or liquid state unless we supply enough potential energy to pull it apart into a gas. This explains why water stops getting hotter when it reaches the boiling point: the power being pumped into the water by your stove begins going into potential energy rather than kinetic energy.

As shown in figure e, every stored form of energy that we encounter in everyday life turns out to be a form of potential energy at the atomic level. The forces between atoms are electrical and magnetic in nature, so these are actually electrical and magnetic potential energies.

Even if we wish to include nuclear reactions in the picture, there still turn out to be only four fundamental types of energy:

kinetic energy (including heat)

gravitational PE

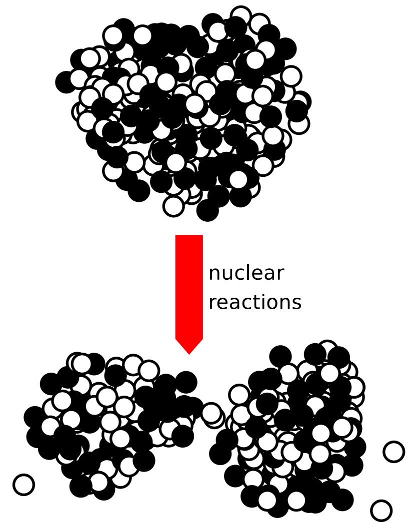
electrical and magnetic PE (including light)

nuclear PE

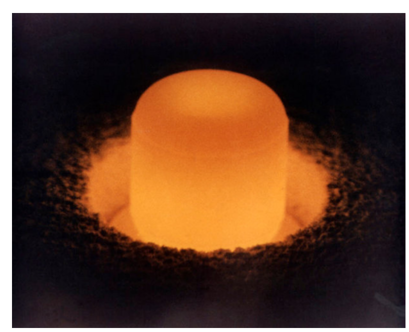
How does light fit into this picture? Optional section 11.6 discussed the idea of modeling a ray of light as a stream of massless particles. But the way in which we described the energy of such particles was completely different from the use of $KE = (1/2)mv^2$ for objects made of atoms. Since the purpose of this chapter has been to bring every form of energy under the same roof, this inconsistency feels unsatisfying. Section 12.5 eliminates this inconsistency.

Discussion question

A Referring back to the pictures at the beginning of the chapter, how do all these forms of energy fit into the shortened list of categories given above?



f / This figure looks similar to the previous ones, but the scale is a million times smaller. The little balls are the neutrons and protons that make up the tiny nucleus at the center of the uranium atom. When the nucleus splits (fissions), the potential energy change is partly electrical and partly a change in the potential energy derived from the force that holds atomic nuclei together (known as the strong nuclear force).



g / A pellet of plutonium-238 glows with its own heat. Its nuclear potential energy is being converted into heat, a form of kinetic energy. Pellets of this type are used as power supplies on some space probes.

12.4 Applications

Heat transfer

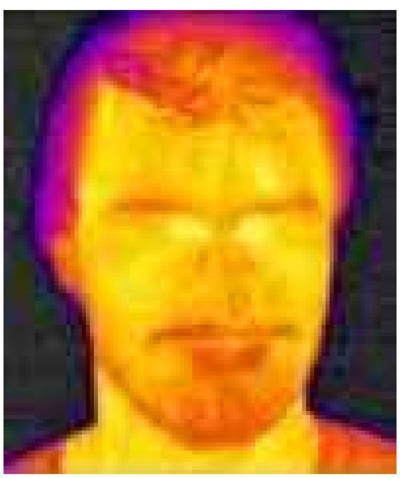
Conduction

When you hold a hot potato in your hand, energy is transferred from the hot object to the cooler one. Our microscopic picture of this process (figure b, p. 339) tells us that the heat transfer can only occur at the surface of contact, where one layer of atoms in the potato skin make contact with one such layer in the hand. This type of heat transfer is called *conduction*, and its rate is proportional to both the surface area and the temperature difference.

Convection

In a gas or a liquid, a faster method of heat transfer can occur, because hotter or colder parts of the fluid can flow, physically transporting their heat energy from one place to another. This mechanism of heat transfer, *convection*, is at work in Los Angeles when hot Santa Ana winds blow in from the Mojave Desert. On a cold day, the reason you feel warmer when there is no wind is that your skin warms a thin layer of air near it by conduction. If a gust of wind comes along, convection robs you of this layer. A thermos bottle has inner and outer walls separated by a layer of vacuum, which prevents heat transport by conduction or convection, except for a tiny amount of conduction through the thin connection between the walls, near the neck, which has a small cross-sectional area.

Radiation



h / A portrait of a man's face made with infrared light, a color of light that lies beyond the red end of the visible rainbow. His warm skin emits quite a bit of infrared light energy, while his hair, at a lower temperature, emits less.

The glow of the sun or a candle flame is an example of heat transfer by *radiation*. In this context, "radiation" just means anything that radiates outward from a source, including, in these examples, ordinary visible light. The power is proportional to the surface area. It also depends very dramatically on the absolute temperature, $P \propto T^4$.

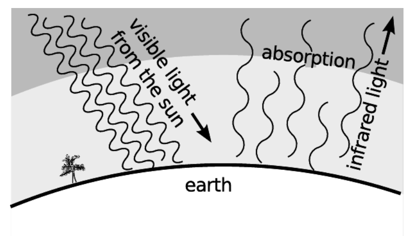
We can easily understand the reason for radiation based on the picture of heat as random kinetic energy at the atomic scale. Atoms are made out of subatomic particles, such as electrons and nuclei, that carry electric charge. When a charged particle vibrates, it creates wave disturbances in the electric and magnetic fields, and the waves have a frequency (number of vibrations per second) that matches the frequency of the particle's motion. If this frequency is in the right range, they constitute visible light . In figure g, the nuclear and electrical potential energy in the plutonium pellet cause the pellet to heat up, and an equilibrium is reached, in which the heat is radiated away just as quickly as it is produced. When an object is closer to room temperature, it glows in the invisible infrared part of the spectrum (figure h).

Earth's energy equilibrium

Our planet receives a nearly constant amount of energy from the sun (about 1.8×10^{17} W). If it hadn't had any mechanism for getting rid of that energy, the result would have been some kind of catastrophic explosion soon after its formation. Even a 10% imbalance between energy input and output, if maintained steadily from the time of the Roman Empire until the present, would have been enough to raise the oceans to a boil. So evidently the earth does dump this energy somehow. How does it do it? Our planet is surrounded by the vacuum of outer space, like the ultimate thermos bottle. Therefore it can't expel heat by conduction or convection, but it does radiate in the infrared, and this is the *only* available mechanism for cooling.

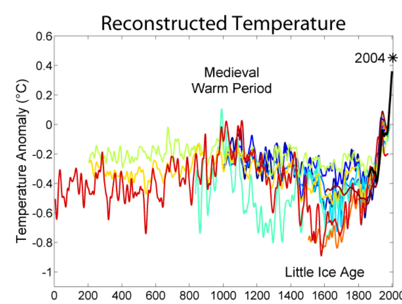
Global warming

It was realized starting around 1930 that this created a dangerous vulnerability in our biosphere. Our atmosphere is only about 0.04% carbon dioxide, but carbon dioxide is an extraordinarily efficient absorber of infrared light. It is, however, transparent to visible light. Therefore any increase in the concentration of carbon dioxide would decrease the efficiency of cooling by radiation, while allowing in just as much heat input from visible light. When we burn fossil fuels such as gasoline or coal, we release into the atmosphere carbon that had previously been locked away underground. This results in a shift to a new energy balance. The average temperature T of the land increases until the T^4 dependence of radiation compensates for the additional absorption of infrared light.



i / The "greenhouse effect." Carbon dioxide in the atmosphere allows visible light in, but partially blocks the reemitted infrared light.

By about 1980, a clear scientific consensus had emerged that this effect was real, that it was caused by human activity, and that it had resulted in an abrupt increase in the earth's average temperature. We know, for example, from radioisotope studies that the effect has not been caused by the release of carbon dioxide in volcanic eruptions. The temperature increase has been verified by multiple independent methods, including studies of tree rings and coral reefs. Detailed computer models have correctly predicted a number of effects that were later verified empirically, including a rise in sea levels, and day-night and pole-equator variations. There is no longer any controversy among climate scientists about the existence or cause of the effect.



j / Global average temperatures over the last 2000 years. The black line is from thermometer measurements. The colored lines are from various indirect indicators such as tree rings, ice cores, buried pollen, and corals.

One solution to the problem is to replace fossil fuels with renewable sources of energy such as solar power and wind. However, these cannot be brought online fast enough to prevent severe warming all by themselves, so nuclear power is also a critical piece of the puzzle.

12.5 ⋆ E=mc^2

In section 11.6 we found the relativistic expression for kinetic energy in the limiting case of an ultrarelativistic particle, i.e., one with a speed very close to c: its energy is proportional to the "stretch factor" of the Lorentz transformation, $s = \sqrt{(1+v)/(1-v)}$ (in units with $c = 1$), for $v \to +c$ and $1/s$ for $v \to -c$. What about intermediate cases, like $v = c/2$?

k / The match is lit inside the bell jar. It burns, and energy escapes from the jar in the form of light. After it stops burning, all the same atoms are still in the jar: none have entered or escaped. The figure shows the outcome expected before relativity, which was that the mass measured on the balance would remain exactly the same. This is not what happens in reality.

When we are forced to tinker with a time-honored theory, our first instinct should always be to tinker as conservatively as possible. Although we've been forced to admit that kinetic energy doesn't vary as $v^2/2$ at relativistic speeds, the next most conservative thing we could do would be to assume that the *only* change necessary is to replace the factor of $v^2/2$ in the nonrelativistic expression for kinetic energy with some other function, which would have to act like s or $1/s$ for $v \to \pm c$. I suspect that this is what Einstein thought when he completed his original paper on relativity in 1905, because it wasn't until later that year that he published a second paper showing that this still wasn't enough of a change to produce a working theory. We now know that there is something more that needs to be changed about prerelativistic physics, and this is the assumption that mass is only a property of material particles such as atoms (figure k). Call this the "atoms-only hypothesis."

Now that we know the correct relativistic way of finding the energy of a ray of light, it turns out that we can use that to find what we were originally seeking, which was the energy of a material object. The following discussion closely follows Einstein's.

Suppose that a material object O of mass m_o, initially at rest in a certain frame A, emits two rays of light, each with energy $E/2$. By conservation of energy, the object must have lost an amount of energy equal to E. By symmetry, O remains at rest.

We now switch to a different frame of reference B moving at some arbitrary speed corresponding to a stretch factor S. The change of frames means that we're chasing one ray, so that its energy is scaled down to $(E/2)S^{-1}$, while running away from the other, whose energy gets boosted to $(E/2)S$. In frame B, as in A, O retains the same speed after emission of the light. But observers in frames A and B disagree on how much energy O has lost, the discrepancy being

$$E\left[\frac{1}{2}(S+S^{-1})-1\right].$$

Let's consider the case where B's velocity relative to A is small. Expanding the above expression in a Taylor series in v, the discrepancy in O's energy loss is approximately

$$\frac{1}{2}Ev^2/c^2.$$

The interpretation is that when O reduced its energy by E in order to make the light rays, it reduced its *mass* from m_o to $m_o - m$, where $m = E/c^2$. Rearranging factors, we have Einstein's famous

$$E = mc^2.$$

This derivation entailed an approximation, and redoing it without the approximation entails some complexity.[1] It turns out, however, to be valid in general.

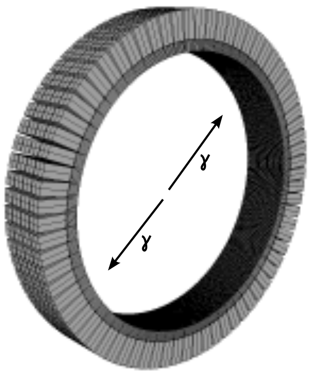
We find that mass is not simply a built-in property of the particles that make up an object, with the object's mass being the sum of the masses of its particles. Rather, mass and energy are equivalent, so that if the experiment of figure k is carried out with a sufficiently precise balance, the reading will drop because of the mass equivalent of the energy emitted as light.

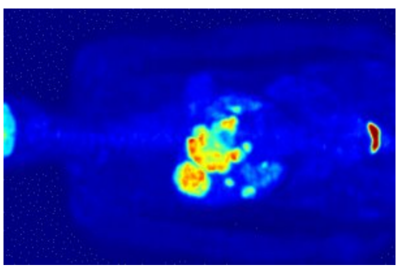
The equation $E = mc^2$ tells us how much energy is equivalent to how much mass: the conversion factor is the square of the speed of light, c. Since c a big number, you get a really really big number when you multiply it by itself to get c^2. This means that even a small amount of mass is equivalent to a very large amount of energy. Conversely, an ordinary amount of energy corresponds to an extremely small mass, and this is why nobody detected the non-null result of experiments like the one in figure k hundreds of years ago.

The big event here is mass-energy equivalence, but we can also harvest a result for the energy of a material particle moving at a certain speed. Plugging in $S = \sqrt{(1+v)/(1-v)}$ to the equation above for the energy discrepancy of object O between frames A and B, we find $m(\gamma - 1)c^2$. This is the difference between O's energy in frame B and its energy when it is at rest, but since mass and energy are equivalent, we assign it energy mc^2 when it is at rest. The result is that the energy is

$$E = m\gamma c^2.$$

[1]See Ohanian, "Einstein's $E = mc^2$ mistakes," `arxiv.org/abs/0805.1400`, and Jammer, Concepts of Mass in Contemporary Physics and Philosophy.





l / Top: A PET scanner. Middle: Each positron annihilates with an electron, producing two gamma-rays that fly off back-to-back. When two gamma rays are observed simultaneously in the ring of detectors, they are assumed to come from the same annihilation event, and the point at which they were emitted must lie on the line connecting the two detectors. Bottom: A scan of a person's torso. The body has concentrated the radioactive tracer around the stomach, indicating an abnormal medical condition.

Electron-positron annihilation — *example 3*

Natural radioactivity in the earth produces positrons, which are like electrons but have the opposite charge. A form of antimatter, positrons annihilate with electrons to produce gamma rays, a form of high-frequency light. Such a process would have been considered impossible before Einstein, because conservation of mass and energy were believed to be separate principles, and this process eliminates 100% of the original mass. The amount of energy produced by annihilating 1 kg of matter with 1 kg of antimatter is

$$\begin{aligned} E &= mc^2 \\ &= (2\ \text{kg})\left(3.0\times 10^8\ \text{m/s}\right)^2 \\ &= 2\times 10^{17}\ \text{J}, \end{aligned}$$

which is on the same order of magnitude as a day's energy consumption for the entire world's population!

Positron annihilation forms the basis for the medical imaging technique called a PET (positron emission tomography) scan, in which a positron-emitting chemical is injected into the patient and mapped by the emission of gamma rays from the parts of the body where it accumulates.

A rusting nail — *example 4*

▷ An iron nail is left in a cup of water until it turns entirely to rust. The energy released is about 0.5 MJ. In theory, would a sufficiently precise scale register a change in mass? If so, how much?

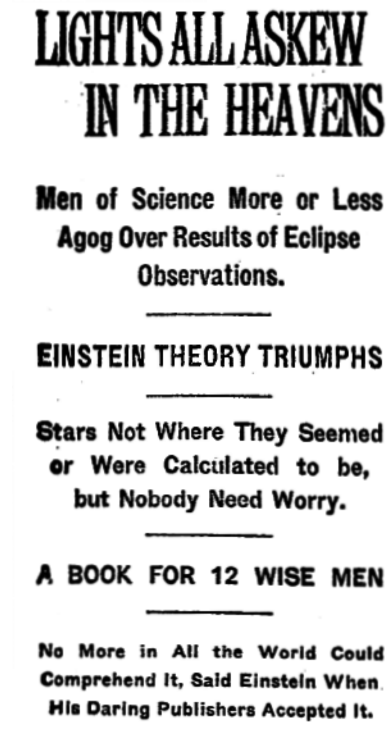
▷ The energy will appear as heat, which will be lost to the environment. The total mass-energy of the cup, water, and iron will indeed be lessened by 0.5 MJ. (If it had been perfectly insulated, there would have been no change, since the heat energy would have been trapped in the cup.) The speed of light is $c = 3\times 10^8$ meters per second, so converting to mass units, we have

$$\begin{aligned} m &= \frac{E}{c^2} \\ &= \frac{0.5\times 10^6\ \text{J}}{\left(3\times 10^8\ \text{m/s}\right)^2} \\ &= 6\times 10^{-12}\ \text{kilograms}. \end{aligned}$$

The change in mass is too small to measure with any practical technique. This is because the square of the speed of light is such a large number.

Gravity bending light — *example 5*

Gravity is a universal attraction between things that have mass, and since the energy in a beam of light is equivalent to some very small amount of mass, we expect that light will be affected by gravity, although the effect should be very small. The first important experimental confirmation of relativity came in 1919 when stars next to the sun during a solar eclipse were observed to have shifted a little from their ordinary position. (If there was no eclipse, the glare of the sun would prevent the stars from being observed.) Starlight had been deflected by the sun's gravity. Figure n is a photographic negative, so the circle that appears bright is actually the dark face of the moon, and the dark area is really the bright corona of the sun. The stars, marked by lines above and below them, appeared at positions slightly different than their normal ones.


LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

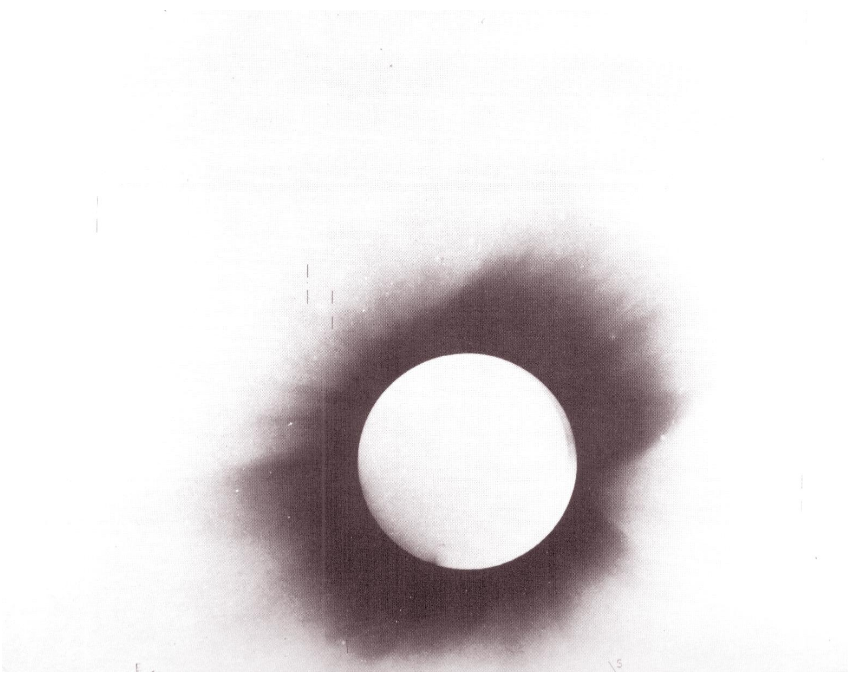
EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

m / A New York Times headline from November 10, 1919, describing the observations discussed in example 5.



n / Example 5.

Black holes *example 6*

A star with sufficiently strong gravity can prevent light from leaving. Quite a few black holes have been detected via their gravitational forces on neighboring stars or clouds of gas and dust.

Summary

Selected vocabulary

potential energy — the energy having to do with the distance between two objects that interact via a noncontact force

Notation

PE . . . . potential energy

Other terminology and notation

U or V . . . . . . symbols used for potential energy in the scientific literature and in most advanced textbooks

Summary

Historically, the energy concept was only invented to include a few phenomena, but it was later generalized more and more to apply to new situations, for example nuclear reactions. This generalizing process resulted in an undesirably long list of types of energy, each of which apparently behaved according to its own rules.

The first step in simplifying the picture came with the realization that heat was a form of random motion on the atomic level, i.e., heat was nothing more than the kinetic energy of atoms.

A second and even greater simplification was achieved with the realization that all the other apparently mysterious forms of energy actually had to do with changing the distances between atoms (or similar processes in nuclei). This type of energy, which relates to the distance between objects that interact via a force, is therefore of great importance. We call it potential energy.

Most of the important ideas about potential energy can be understood by studying the example of gravitational potential energy. The change in an object's gravitational potential energy is given by

$$\Delta PE_{grav} = -F_{grav}\Delta y, \qquad \text{[if } F_{grav} \text{ is constant, i.e., the the motion is all near the Earth's surface]}$$

The most important thing to understand about potential energy is that there is no unambiguous way to define it in an absolute sense. The only thing that everyone can agree on is how much the potential energy has changed from one moment in time to some later moment in time.

An implication of Einstein's theory of special relativity is that mass and energy are equivalent, as expressed by the famous $E = mc^2$. The energy of a material object is given by $E = m\gamma c^2$.

Problems

Key

$\surd$ A computerized answer check is available online.
$\int$ A problem that requires calculus.
$\star$ A difficult problem.

1 A ball rolls up a ramp, turns around, and comes back down. When does it have the greatest gravitational potential energy? The greatest kinetic energy? [Based on a problem by Serway and Faughn.]

2 Anya and Ivan lean over a balcony side by side. Anya throws a penny downward with an initial speed of 5 m/s. Ivan throws a penny upward with the same speed. Both pennies end up on the ground below. Compare their kinetic energies and velocities on impact.

3 Can gravitational potential energy ever be negative? Note that the question refers to PE, not ΔPE, so that you must think about how the choice of a reference level comes into play. [Based on a problem by Serway and Faughn.]

4 (a) You release a magnet on a tabletop near a big piece of iron, and the magnet slides across the table to the iron. Does the magnetic potential energy increase, or decrease? Explain.
(b) Suppose instead that you have two repelling magnets. You give them an initial push towards each other, so they decelerate while approaching each other. Does the magnetic potential energy increase or decrease? Explain.

5 Let E_b be the energy required to boil one kg of water. (a) Find an equation for the minimum height from which a bucket of water must be dropped if the energy released on impact is to vaporize it. Assume that all the heat goes into the water, not into the dirt it strikes, and ignore the relatively small amount of energy required to heat the water from room temperature to 100°C. [Numerical check, not for credit: Plugging in $E_b = 2.3$ MJ/kg should give a result of 230 km.] $\surd$
(b) Show that the units of your answer in part a come out right based on the units given for E_b.

6 A grasshopper with a mass of 110 mg falls from rest from a height of 310 cm. On the way down, it dissipates 1.1 mJ of heat due to air resistance. At what speed, in m/s, does it hit the ground? ▹ Solution, p. 554

7 At a given temperature, the average kinetic energy per molecule is a fixed value, so for instance in air, the more massive oxygen molecules are moving more slowly on the average than the nitrogen molecules. The ratio of the masses of oxygen and nitrogen molecules is 16.00 to 14.01. Now suppose a vessel containing some air is surrounded by a vacuum, and the vessel has a tiny hole in it, which allows the air to slowly leak out. The molecules are bouncing around randomly, so a given molecule will have to "try" many times before it gets lucky enough to head out through the hole. Find the rate at which oxygen leaks divided by the rate at which nitrogen leaks. (Define this rate according to the fraction of the gas that leaks out in a given time, not the mass or number of molecules leaked per unit time.) √

8 A person on a bicycle is to coast down a ramp of height h and then pass through a circular loop of radius r. What is the smallest value of h for which the cyclist will complete the loop without falling? (Ignore the kinetic energy of the spinning wheels.) √

9 Problem 9 has been deleted. ⋆

10 Students are often tempted to think of potential energy and kinetic energy as if they were always related to each other, like yin and yang. To show this is incorrect, give examples of physical situations in which (a) PE is converted to another form of PE, and (b) KE is converted to another form of KE. ▷ Solution, p. 554

11 Lord Kelvin, a physicist, told the story of how he encountered James Joule when Joule was on his honeymoon. As he traveled, Joule would stop with his wife at various waterfalls, and measure the difference in temperature between the top of the waterfall and the still water at the bottom. (a) It would surprise most people to learn that the temperature increased. Why should there be any such effect, and why would Joule care? How would this relate to the energy concept, of which he was the principal inventor? (b) How much of a gain in temperature should there be between the top and bottom of a 50-meter waterfall? (c) What assumptions did you have to make in order to calculate your answer to part b? In reality, would the temperature change be more than or less than what you calculated? [Based on a problem by Arnold Arons.] √

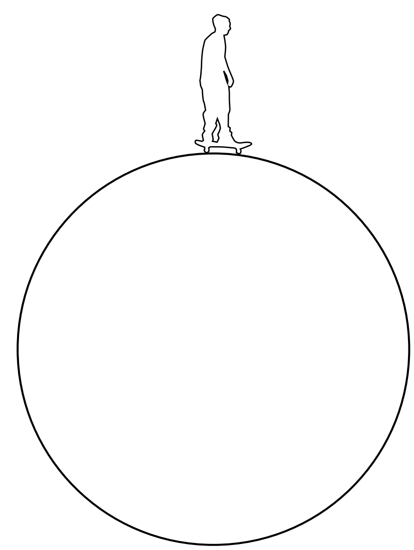
12 Make an order-of-magnitude estimate of the power represented by the loss of gravitational energy of the water going over Niagara Falls. If the hydroelectric plant at the bottom of the falls could convert 100% of this to electrical power, roughly how many households could be powered? ▷ Solution, p. 554

13 When you buy a helium-filled balloon, the seller has to inflate it from a large metal cylinder of the compressed gas. The helium inside the cylinder has energy, as can be demonstrated for example by releasing a little of it into the air: you hear a hissing sound, and that sound energy must have come from somewhere. The total amount of energy in the cylinder is very large, and if the valve is inadvertently damaged or broken off, the cylinder can behave like a bomb or a rocket.

Suppose the company that puts the gas in the cylinders prepares cylinder A with half the normal amount of pure helium, and cylinder B with the normal amount. Cylinder B has twice as much energy, and yet the temperatures of both cylinders are the same. Explain, at the atomic level, what form of energy is involved, and why cylinder B has twice as much.

14 Explain in terms of conservation of energy why sweating cools your body, even though the sweat is at the same temperature as your body. Describe the forms of energy involved in this energy transformation. Why don't you get the same cooling effect if you wipe the sweat off with a towel? Hint: The sweat is evaporating.

15 (a) A circular hoop of mass m and radius r spins like a wheel while its center remains at rest. Let ω (Greek letter omega) be the number of radians it covers per unit time, i.e., $\omega = 2\pi/T$, where the period, T, is the time for one revolution. Show that its kinetic energy equals $(1/2)m\omega^2 r^2$.
(b) Show that the answer to part a has the right units. (Note that radians aren't really units, since the definition of a radian is a unitless ratio of two lengths.)
(c) If such a hoop rolls with its center moving at velocity v, its kinetic energy equals $(1/2)mv^2$, plus the amount of kinetic energy found in part a. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have. ⋆



Problem 16.

16 A skateboarder starts at rest nearly at the top of a giant cylinder, and begins rolling down its side. (If he started exactly at rest and exactly at the top, he would never get going!) Show that his board loses contact with the pipe after he has dropped by a height equal to one third the radius of the pipe. ▷ Solution, p. 554 ⋆

17 In example 7 on page 87, I remarked that accelerating a macroscopic (i.e., not microscopic) object to close to the speed of light would require an unreasonable amount of energy. Suppose that the starship Enterprise from Star Trek has a mass of 8.0×10^7 kg, about the same as the Queen Elizabeth 2. Compute the kinetic energy it would have to have if it was moving at half the speed of light. Compare with the total energy content of the world's nuclear arsenals, which is about 10^{21} J. √

18 (a) A free neutron (as opposed to a neutron bound into an atomic nucleus) is unstable, and undergoes spontaneous radioactive decay into a proton, an electron, and an antineutrino. The masses of the particles involved are as follows:

neutron	1.67495×10^{-27} kg
proton	1.67265×10^{-27} kg
electron	0.00091×10^{-27} kg
antineutrino	$< 10^{-35}$ kg

Find the energy released in the decay of a free neutron. √

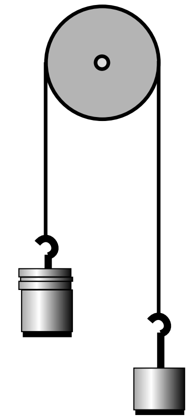
(b) Neutrons and protons make up essentially all of the mass of the ordinary matter around us. We observe that the universe around us has no free neutrons, but lots of free protons (the nuclei of hydrogen, which is the element that 90% of the universe is made of). We find neutrons only inside nuclei along with other neutrons and protons, not on their own.

If there are processes that can convert neutrons into protons, we might imagine that there could also be proton-to-neutron conversions, and indeed such a process does occur sometimes in nuclei that contain both neutrons and protons: a proton can decay into a neutron, a positron, and a neutrino. A positron is a particle with the same properties as an electron, except that its electrical charge is positive. A neutrino, like an antineutrino, has negligible mass.

Although such a process can occur within a nucleus, explain why it cannot happen to a free proton. (If it could, hydrogen would be radioactive, and you wouldn't exist!)

19 A little kid in my neighborhood came home from shopping with his mother. They live on a hill, with their driveway oriented perpendicular to the slope. Their minivan was parked in the driveway, and while she was bringing groceries inside, he unlocked the parking brake and put the car in neutral. The steering wheel was locked with the wheels banked. The car rolled downhill in a circular arc with the driveway at its top, eventually crashing through the wall of a neighbor's living room. (Nobody was hurt.) Suppose the neighbor's house hadn't intervened. The car just rolls freely, and we want to know whether it will ever skid. Static friction acts between the asphalt and the tires with coefficient μ_s, the radius of the circle is r, the slope of the hill is θ, and the gravitational field has strength g. Find the maximum value of θ such that the car will never skid. √ ⋆

20 The figure shows two unequal masses, M and m, connected by a string running over a pulley. This system was analyzed previously in problem 20 on p. 198, using Newton's laws.
(a) Analyze the system using conservation of energy instead. Find the speed the weights gain after being released from rest and traveling a distance h. ✓
(b) Use your result from part a to find the acceleration, reproducing the result of the earlier problem. ✓



Problem 20.

21 In 2003, physicist and philosopher John Norton came up with the following apparent paradox, in which Newton's laws, which appear deterministic, can produce nondeterministic results. Suppose that a bead moves frictionlessly on a curved wire under the influence of gravity. The shape of the wire is defined by the function $y(x)$, which passes through the origin, and the bead is released from rest at the origin. For convenience of notation, choose units such that $g = 1$, and define $\dot{y} = \mathrm{d}y/\,\mathrm{d}t$ and $y' = \mathrm{d}y/\,\mathrm{d}x$.
(a) Show that the equation of motion is

$$y = -\frac{1}{2}\dot{y}^2\left(1 + y'^{-2}\right).$$

(b) To simplify the calculations, assume from now on that $y' \ll 1$. Find a shape for the wire such that $x = t^4$ is a solution. (Ignore units.) ✓
(c) Show that not just the motion assumed in part b, but any motion of the following form is a solution:

$$x = \begin{cases} 0 & \text{if } t \le t_0 \\ (t - t_0)^4 & \text{if } t \ge t_0 \end{cases}$$

This is remarkable because there is no physical principle that determines t_0, so if we place the bead at rest at the origin, there is no way to predict when it will start moving. ⋆

Chapter 13

Work: the transfer of mechanical energy

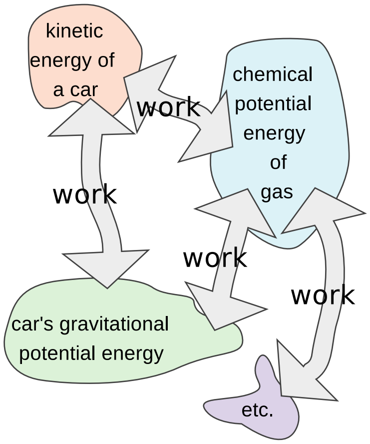
13.1 Work: the transfer of mechanical energy

The concept of work

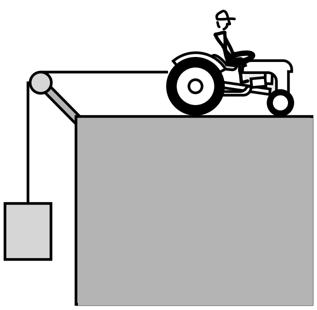
The mass contained in a closed system is a conserved quantity, but if the system is not closed, we also have ways of measuring the amount of mass that goes in or out. The water company does this with a meter that records your water use.

Likewise, we often have a system that is not closed, and would like to know how much energy comes in or out. Energy, however, is not a physical substance like water, so energy transfer cannot be measured with the same kind of meter. How can we tell, for instance, how much useful energy a tractor can "put out" on one tank of gas?

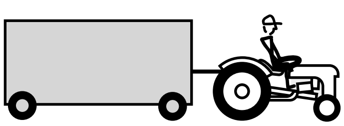
The law of conservation of energy guarantees that all the chem-



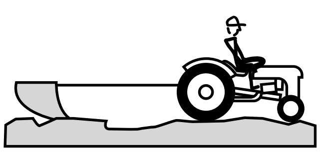
a / Work is a transfer of energy.



b / The tractor raises the weight over the pulley, increasing its gravitational potential energy.



c / The tractor accelerates the trailer, increasing its kinetic energy.



d / The tractor pulls a plow. Energy is expended in frictional heating of the plow and the dirt, and in breaking dirt clods and lifting dirt up to the sides of the furrow.

ical energy in the gasoline will reappear in some form, but not necessarily in a form that is useful for doing farm work. Tractors, like cars, are extremely inefficient, and typically 90% of the energy they consume is converted directly into heat, which is carried away by the exhaust and the air flowing over the radiator. We wish to distinguish the energy that comes out directly as heat from the energy that serves to accelerate a trailer or to plow a field, so we define a technical meaning of the ordinary word "work" to express the distinction:

> **definition of work**
> Work is the amount of energy transferred into or out of a system, not counting energy transferred by heat conduction.

self-check A

Based on this definition, is work a vector, or a scalar? What are its units? ▷ Answer, p. 560

The conduction of heat is to be distinguished from heating by friction. When a hot potato heats up your hands by conduction, the energy transfer occurs without any force, but when friction heats your car's brake shoes, there is a force involved. The transfer of energy with and without a force are measured by completely different methods, so we wish to include heat transfer by frictional heating under the definition of work, but not heat transfer by conduction. The definition of work could thus be restated as the amount of energy transferred by forces.

Calculating work as force multiplied by distance

The examples in figures b-d show that there are many different ways in which energy can be transferred. Even so, all these examples have two things in common:

1. A force is involved.
2. The tractor travels some distance as it does the work.

In b, the increase in the height of the weight, Δy, is the same as the distance the tractor travels, which we'll call d. For simplicity, we discuss the case where the tractor raises the weight at constant speed, so that there is no change in the kinetic energy of the weight, and we assume that there is negligible friction in the pulley, so that the force the tractor applies to the rope is the same as the rope's upward force on the weight. By Newton's first law, these forces are also of the same magnitude as the earth's gravitational force on the weight. The increase in the weight's potential energy is given by $F\Delta y$, so the work done by the tractor on the weight equals Fd, the product of the force and the distance moved:

$$W = Fd.$$

In example c, the tractor's force on the trailer accelerates it, increasing its kinetic energy. If frictional forces on the trailer are negligible, then the increase in the trailer's kinetic energy can be found using the same algebra that was used on page 341 to find the potential energy due to gravity. Just as in example b, we have

$$W = Fd.$$

Does this equation always give the right answer? Well, sort of. In example d, there are two quantities of work you might want to calculate: the work done by the tractor on the plow and the work done by the plow on the dirt. These two quantities can't both equal Fd. Most of the energy transmitted through the cable goes into frictional heating of the plow and the dirt. The work done by the plow on the dirt is less than the work done by the tractor on the plow, by an amount equal to the heat absorbed by the plow. It turns out that the equation $W = Fd$ gives the work done by the tractor, not the work done by the plow. How are you supposed to know when the equation will work and when it won't? The somewhat complex answer is postponed until section 13.6. Until then, we will restrict ourselves to examples in which $W = Fd$ gives the right answer; essentially the reason the ambiguities come up is that when one surface is slipping past another, d may be hard to define, because the two surfaces move different distances.

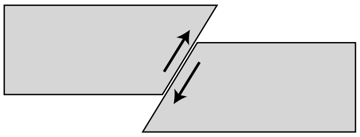
e / The baseball pitcher put kinetic energy into the ball, so he did work on it. To do the greatest possible amount of work, he applied the greatest possible force over the greatest possible distance.

We have also been using examples in which the force is in the same direction as the motion, and the force is constant. (If the force was not constant, we would have to represent it with a function, not a symbol that stands for a number.) To summarize, we have:

rule for calculating work (simplest version)
The work done by a force can be calculated as

$$W = Fd,$$

if the force is constant and in the same direction as the motion. Some ambiguities are encountered in cases such as kinetic friction.

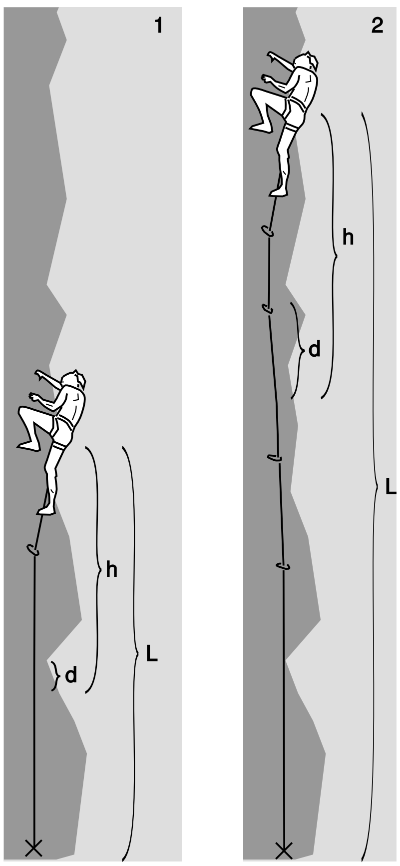


f / Example 1.

Mechanical work done in an earthquake — *example 1*

▷ In 1998, geologists discovered evidence for a big prehistoric earthquake in Pasadena, between 10,000 and 15,000 years ago. They found that the two sides of the fault moved 6.7 m relative to one another, and estimated that the force between them was 1.3×10^{17} N. How much energy was released?

▷ Multiplying the force by the distance gives 9×10^{17} J. For comparison, the Northridge earthquake of 1994, which killed 57 people and did 40 billion dollars of damage, released 22 times less energy.



g / Example 2. Surprisingly, the climber is in more danger at 1 than at 2. The distance d is the amount by which the rope will stretch while work is done to transfer the kinetic energy of a fall out of her body.

The fall factor — *example 2*

Counterintuitively, the rock climber may be in more danger in figure g/1 than later when she gets up to position g/2.

Along her route, the climber has placed removable rock anchors (not shown) and carabiners attached to the anchors. She clips the rope into each carabiner so that it can travel but can't pop out. In both 1 and 2, she has ascended a certain distance above her last anchor, so that if she falls, she will drop through a height h that is about twice this distance, and this fall height is about the same in both cases. In fact, h is somewhat larger than twice her height above the last anchor, because the rope is intentionally designed to stretch under the big force of a falling climber who suddenly brings it taut.

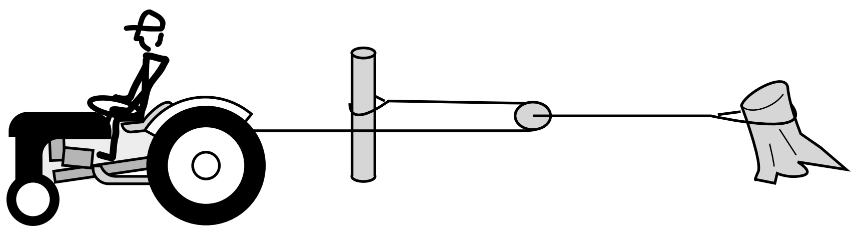
To see why we want a stretchy rope, consider the equation $F = W/d$ in the case where d is zero; F would theoretically become infinite. In a fall, the climber loses a fixed amount of gravitational energy mgh. This is transformed into an equal amount of kinetic energy as she falls, and eventually this kinetic energy has to be transferred out of her body when the rope comes up taut. If the rope was not stretchy, then the distance traveled at the point where the rope attaches to her harness would be zero, and the force exerted would theoretically be infinite. Before the rope reached the theoretically infinite tension F it would break (or her back would break, or her anchors would be pulled out of the rock). We want the rope to be stretchy enough to make d fairly big, so that dividing W by d gives a small force.[1]

In g/1 and g/2, the fall h is about the same. What is different is the length L of rope that has been paid out. A longer rope can stretch more, so the distance d traveled after the "catch" is proportional to L. Combining $F = W/d$, $W \propto h$, and $d \propto L$, we have $F \propto h/L$. For these reasons, rock climbers define a *fall factor* $f = h/L$. The larger fall factor in g/1 is more dangerous.

[1]Actually F isn't constant, because the tension in the rope increases steadily as it stretches, but this is irrelevant to the present analysis.

Machines can increase force, but not work.

Figure h shows a pulley arrangement for doubling the force supplied by the tractor (book 1, section 5.6). The tension in the left-hand rope is equal throughout, assuming negligible friction, so there are two forces pulling the pulley to the left, each equal to the original force exerted by the tractor on the rope. This doubled force is transmitted through the right-hand rope to the stump.



h / The pulley doubles the force the tractor can exert on the stump.

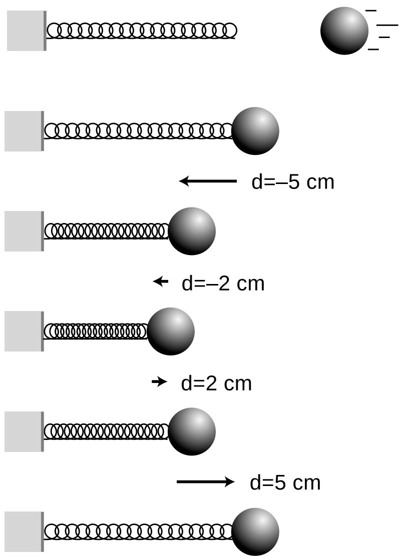
It might seem as though this arrangement would also double the work done by the tractor, but look again. As the tractor moves forward 2 meters, 1 meter of rope comes around the pulley, and the pulley moves 1 m to the left. Although the pulley exerts double the force on the stump, the pulley and stump only move half as far, so the work done on the stump is no greater that it would have been without the pulley.

The same is true for any mechanical arrangement that increases or decreases force, such as the gears on a ten-speed bike. You can't get out more work than you put in, because that would violate conservation of energy. If you shift gears so that your force on the pedals is amplified, the result is that you just have to spin the pedals more times.

No work is done without motion.

It strikes most students as nonsensical when they are told that if they stand still and hold a heavy bag of cement, they are doing no work on the bag. Even if it makes sense mathematically that $W = Fd$ gives zero when d is zero, it seems to violate common sense. You would certainly become tired! The solution is simple. Physicists have taken over the common word "work" and given it a new technical meaning, which is the transfer of energy. The energy of the bag of cement is not changing, and that is what the physicist means by saying no work is done on the bag.

There is a transformation of energy, but it is taking place entirely within your own muscles, which are converting chemical energy into heat. Physiologically, a human muscle is not like a tree limb, which can support a weight indefinitely without the expenditure of energy. Each muscle cell's contraction is generated by zillions of little molecular machines, which take turns supporting the tension. When a particular molecule goes on or off duty, it moves, and since it moves while exerting a force, it is doing work. There is work, but it is work done by one molecule in a muscle cell on another.



i / Whenever energy is transferred out of the spring, the same amount has to be transferred into the ball, and vice versa. As the spring compresses, the ball is doing positive work on the spring (giving up its KE and transferring energy into the spring as PE), and as it decompresses the ball is doing negative work (extracting energy).

Positive and negative work

When object A transfers energy to object B, we say that A does positive work on B. B is said to do negative work on A. In other words, a machine like a tractor is defined as doing positive work. This use of the plus and minus signs relates in a logical and consistent way to their use in indicating the directions of force and motion in one dimension. In figure i, suppose we choose a coordinate system with the x axis pointing to the right. Then the force the spring exerts on the ball is always a positive number. The ball's motion, however, changes directions. The symbol d is really just a shorter way of writing the familiar quantity Δx, whose positive and negative signs indicate direction.

While the ball is moving to the left, we use $d < 0$ to represent its direction of motion, and the work done by the spring, Fd, comes out negative. This indicates that the spring is taking kinetic energy out of the ball, and accepting it in the form of its own potential energy.

As the ball is reaccelerated to the right, it has $d > 0$, Fd is positive, and the spring does positive work on the ball. Potential energy is transferred out of the spring and deposited in the ball as kinetic energy.

In summary:

rule for calculating work (including cases of negative work)
The work done by a force can be calculated as

$$W = Fd,$$

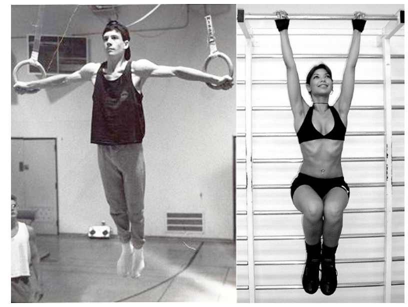
if the force is constant and along the same line as the motion. The quantity d is to be interpreted as a synonym for Δx, i.e., positive and negative signs are used to indicate the direction of motion. Some ambiguities are encountered in cases such as kinetic friction.

self-check B
In figure i, what about the work done by the ball on the spring? ▷ Answer, p. 560

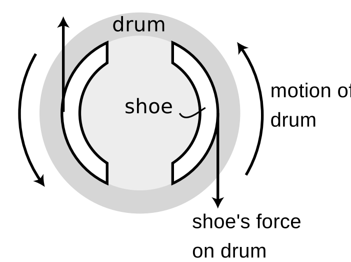
There are many examples where the transfer of energy out of an object cancels out the transfer of energy in. When the tractor pulls the plow with a rope, the rope does negative work on the tractor and positive work on the plow. The total work done by the rope is zero, which makes sense, since it is not changing its energy.

It may seem that when your arms do negative work by lowering a bag of cement, the cement is not really transferring energy into your body. If your body was storing potential energy like a compressed spring, you would be able to raise and lower a weight all day, recycling the same energy. The bag of cement does transfer energy into your body, but your body accepts it as heat, not as potential energy. The tension in the muscles that control the speed of the motion also results in the conversion of chemical energy to heat, for the same physiological reasons discussed previously in the case where you just hold the bag still.

One of the advantages of electric cars over gasoline-powered cars is that it is just as easy to put energy back in a battery as it is to take energy out. When you step on the brakes in a gas car, the brake shoes do negative work on the rest of the car. The kinetic energy of the car is transmitted through the brakes and accepted by the brake shoes in the form of heat. The energy cannot be recovered. Electric cars, however, are designed to use regenerative braking. The brakes don't use friction at all. They are electrical, and when you step on the brake, the negative work done by the brakes means they accept the energy and put it in the battery for later use. This is one of the reasons why an electric car is far better for the environment than a gas car, even if the ultimate source of the electrical energy happens to be the burning of oil in the electric company's plant. The electric car recycles the same energy over and over, and only dissipates heat due to air friction and rolling resistance, not braking. (The electric company's power plant can also be fitted with expensive pollution-reduction equipment that would be prohibitively expensive or bulky for a passenger car.)



j / *Left:* No mechanical work occurs in the man's body while he holds himself motionless. There is a transformation of chemical energy into heat, but this happens at the microscopic level inside the tensed muscles. *Right:* When the woman lifts herself, her arms do positive work on her body, transforming chemical energy into gravitational potential energy and heat. On the way back down, the arms' work is negative; gravitational potential energy is transformed into heat. (In exercise physiology, the man is said to be doing isometric exercise, while the woman's is concentric and then eccentric.)



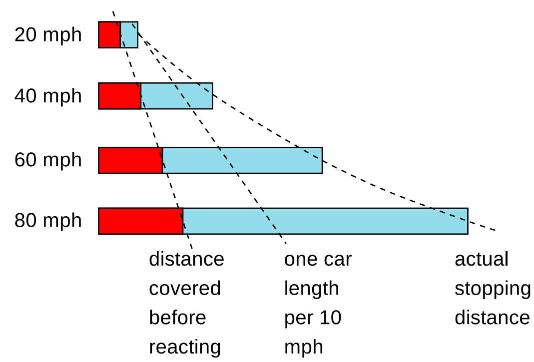
k / Because the force is in the opposite direction compared to the motion, the brake shoe does negative work on the drum, i.e., accepts energy from it in the form of heat.

Discussion questions

A Besides the presence of a force, what other things differentiate the processes of frictional heating and heat conduction?

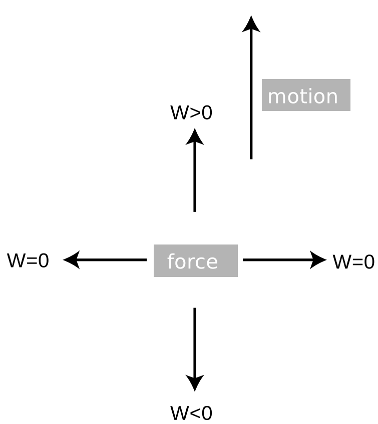
B Criticize the following incorrect statement: "A force doesn't do any work unless it's causing the object to move."

C To stop your car, you must first have time to react, and then it takes some time for the car to slow down. Both of these times contribute to the distance you will travel before you can stop. The figure shows how the average stopping distance increases with speed. Because the stopping distance increases more and more rapidly as you go faster, the rule of one car length per 10 m.p.h. of speed is not conservative enough at high speeds. In terms of work and kinetic energy, what is the reason for the more rapid increase at high speeds?



Discussion question C.

13.2 Work in three dimensions



m / A force can do positive, negative, or zero work, depending on its direction relative to the direction of the motion.

A force perpendicular to the motion does no work.

Suppose work is being done to change an object's kinetic energy. A force in the same direction as its motion will speed it up, and a force in the opposite direction will slow it down. As we have already seen, this is described as doing positive work or doing negative work on the object. All the examples discussed up until now have been of motion in one dimension, but in three dimensions the force can be at any angle θ with respect to the direction of motion.

What if the force is perpendicular to the direction of motion? We have already seen that a force perpendicular to the motion results in circular motion at constant speed. The kinetic energy does not change, and we conclude that no work is done when the force is perpendicular to the motion.

So far we have been reasoning about the case of a single force acting on an object, and changing only its kinetic energy. The result is more generally true, however. For instance, imagine a hockey puck sliding across the ice. The ice makes an upward normal force, but does not transfer energy to or from the puck.

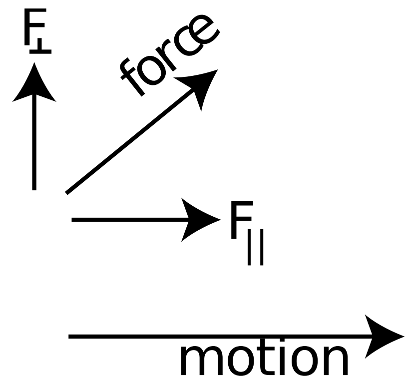
Forces at other angles

Suppose the force is at some other angle with respect to the motion, say $\theta = 45°$. Such a force could be broken down into two components, one along the direction of the motion and the other perpendicular to it. The force vector equals the vector sum of its two components, and the principle of vector addition of forces thus tells us that the work done by the total force cannot be any different than the sum of the works that would be done by the two forces by themselves. Since the component perpendicular to the motion does no work, the work done by the force must be

$$W = F_{\|}|\mathbf{d}|, \qquad \text{[work done by a constant force]}$$

where the vector $\mathbf{d}$ is simply a less cumbersome version of the notation $\Delta\mathbf{r}$. This result can be rewritten via trigonometry as

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta. \qquad \text{[work done by a constant force]}$$



n / Work is only done by the component of the force parallel to the motion.

Even though this equation has vectors in it, it depends only on their magnitudes, and the magnitude of a vector is a scalar. Work is therefore still a scalar quantity, which only makes sense if it is defined as the transfer of energy. Ten gallons of gasoline have the ability to do a certain amount of mechanical work, and when you pull in to a full-service gas station you don't have to say "Fill 'er up with 10 gallons of south-going gas."

Students often wonder why this equation involves a cosine rather than a sine, or ask if it would ever be a sine. In vector addition, the treatment of sines and cosines seemed more equal and democratic, so why is the cosine so special now? The answer is that if we are going to describe, say, a velocity vector, we must give both the component *parallel* to the x axis and the component *perpendicular* to the x axis (i.e., the y component). In calculating work, however, the force component perpendicular to the motion is irrelevant — it changes the direction of motion without increasing or decreasing the energy of the object on which it acts. In this context, it is *only* the parallel force component that matters, so only the cosine occurs.

self-check C

(a) Work is the transfer of energy. According to this definition, is the horse in the picture doing work on the pack? (b) If you calculate work by the method described in this section, is the horse in figure o doing work on the pack? ▷ Answer, p. 561

o / Self-check. (Breaking Trail, by Walter E. Bohl.)

Pushing a broom *example 3*

▷ If you exert a force of 21 N on a push broom, at an angle 35 degrees below horizontal, and walk for 5.0 m, how much work do you do? What is the physical significance of this quantity of work?

▷ Using the second equation above, the work done equals

$$(21\ \text{N})(5.0\ \text{m})(\cos 35°) = 86\ \text{J}.$$

The form of energy being transferred is heat in the floor and the broom's bristles. This comes from the chemical energy stored in your body. (The majority of the calories you burn are dissipated directly as heat inside your body rather than doing any work on the broom. The 86 J is only the amount of energy transferred through the broom's handle.)

A violin *example 4*

As a violinist draws the bow across a string, the bow hairs exert both a normal force and a kinetic frictional force on the string. The normal force is perpendicular to the direction of motion, and does no work. However, the frictional force is in the same direction as the motion of the bow, so it does work: energy is transferred to the string, causing it to vibrate.

One way of playing a violin more loudly is to use longer strokes. Since $W = Fd$, the greater distance results in more work.

A second way of getting a louder sound is to press the bow more firmly against the strings. This increases the normal force, and although the normal force itself does no work, an increase in the normal force has the side effect of increasing the frictional force, thereby increasing $W = Fd$.

The violinist moves the bow back and forth, and sound is produced on both the "up-bow" (the stroke toward the player's left) and the "down-bow" (to the right). One may, for example, play a series of notes in alternation between up-bows and down-bows. However, if the notes are of unequal length, the up and down motions tend to be unequal, and if the player is not careful, she can run out of bow in the middle of a note! To keep this from happening, one can move the bow more quickly on the shorter notes, but the resulting increase in d will make the shorter notes louder than they should be. A skilled player compensates by reducing the force.

13.3 The dot product

Up until now, we have not found any physically useful way to define the multiplication of two vectors. It would be possible, for instance, to multiply two vectors component by component to form a third vector, but there are no physical situations where such a multiplication would be useful.

The equation $W = |\mathbf{F}||\mathbf{d}|\cos\theta$ is an example of a sort of multiplication of vectors that is useful. The result is a scalar, not a vector, and this is therefore often referred to as the *scalar product* of the vectors $\mathbf{F}$ and $\mathbf{d}$. There is a standard shorthand notation for this operation,

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta, \qquad \text{[definition of the notation } \mathbf{A}\cdot\mathbf{B}\text{; } \theta \text{ is the angle between vectors } \mathbf{A} \text{ and } \mathbf{B}]$$

and because of this notation, a more common term for this operation is the *dot product*. In dot product notation, the equation for work is simply

$$W = \mathbf{F}\cdot\mathbf{d}.$$

The dot product has the following geometric interpretation:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= |\mathbf{A}|(\text{component of } \mathbf{B} \text{ parallel to } \mathbf{A})\\ &= |\mathbf{B}|(\text{component of } \mathbf{A} \text{ parallel to } \mathbf{B})\end{aligned}$$

The dot product has some of the properties possessed by ordinary multiplication of numbers,

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= \mathbf{B}\cdot\mathbf{A}\\ \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\\ (c\mathbf{A})\cdot\mathbf{B} &= c\left(\mathbf{A}\cdot\mathbf{B}\right),\end{aligned}$$

but it lacks one other: the ability to undo multiplication by dividing.

If you know the components of two vectors, you can easily calculate their dot product as follows:

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

(This can be proved by first analyzing the special case where each vector has only an x component, and the similar cases for y and z. We can then use the rule $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$ to make a generalization by writing each vector as the sum of its x, y, and z components. See homework problem 23.)

Magnitude expressed with a dot product — *example 5*

If we take the dot product of any vector **b** with itself, we find

$$\begin{aligned}\mathbf{b}\cdot\mathbf{b} &= \left(b_x\hat{\mathbf{x}}+b_y\hat{\mathbf{y}}+b_z\hat{\mathbf{z}}\right)\cdot\left(b_x\hat{\mathbf{x}}+b_y\hat{\mathbf{y}}+b_z\hat{\mathbf{z}}\right)\\ &= b_x^2+b_y^2+b_z^2,\end{aligned}$$

so its magnitude can be expressed as

$$|\mathbf{b}| = \sqrt{\mathbf{b}\cdot\mathbf{b}}.$$

We will often write b^2 to mean $\mathbf{b}\cdot\mathbf{b}$, when the context makes it clear what is intended. For example, we could express kinetic energy as $(1/2)m|\mathbf{v}|^2$, $(1/2)m\mathbf{v}\cdot\mathbf{v}$, or $(1/2)mv^2$. In the third version, nothing but context tells us that v really stands for the magnitude of some vector **v**.

Towing a barge — *example 6*

▷ A mule pulls a barge with a force $\mathbf{F}$=(1100 N)$\hat{\mathbf{x}}$ + (400 N)$\hat{\mathbf{y}}$, and the total distance it travels is (1000 m)$\hat{\mathbf{x}}$. How much work does it do?

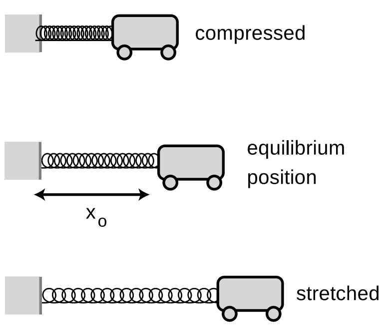
▷ The dot product is 1.1×10^6 N·m = 1.1×10^6 J.

13.4 Varying force

Up until now we have done no actual calculations of work in cases where the force was not constant. The question of how to treat such cases is mathematically analogous to the issue of how to generalize the equation (distance) = (velocity)(time) to cases where the velocity was not constant. We have to make the equation into an integral:

$$W = \int F \, dx$$

The examples in this section are ones in which the force is varying, but is always along the same line as the motion.



p / The spring does work on the cart. (Unlike the ball in section 13.1, the cart is attached to the spring.)

self-check D

In which of the following examples would it be OK to calculate work using Fd, and in which ones would you have to integrate?

(a) A fishing boat cruises with a net dragging behind it.

(b) A magnet leaps onto a refrigerator from a distance.

(c) Earth's gravity does work on an outward-bound space probe. ▷ Answer, p. 561

Work done by a spring — *example 7*

An important and straightforward example is the calculation of the work done by a spring that obeys Hooke's law,

$$F \approx -k(x - x_o),$$

where x_o is the equilibrium position and the minus sign is because this is the force being exerted by the spring, not the force that would have to act on the spring to keep it at this position. That is, if the position of the cart in figure p is to the right of equilibrium, the spring pulls back to the left, and vice-versa. Integrating, we find that the work done between x_1 and x_2 is

$$W = -\frac{1}{2}k(x - x_o)^2\Big|_{x_1}^{x_2} .$$

Work done by gravity — *example 8*

Another important example is the work done by gravity when the change in height is not small enough to assume a constant force. Newton's law of gravity is

$$F = \frac{GMm}{r^2},$$

which can be integrated to give

$$W = \int_{r_1}^{r_2} \frac{GMm}{r^2}\,\mathrm{d}r$$
$$= -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

13.5 Work and potential energy

The techniques for calculating work can also be applied to the calculation of potential energy. If a certain force depends only on the distance between the two participating objects, then the energy released by changing the distance between them is defined as the potential energy, and the amount of potential energy lost equals minus the work done by the force,

$$\Delta PE = -W.$$

The minus sign occurs because positive work indicates that the potential energy is being expended and converted to some other form.

It is sometimes convenient to pick some arbitrary position as a reference position, and derive an equation for once and for all that gives the potential energy relative to this position

$$PE_x = -W_{\text{ref}\rightarrow x}. \qquad \text{[potential energy at a point } x\text{]}$$

To find the energy transferred into or out of potential energy, one then subtracts two different values of this equation.

These equations might almost make it look as though work and energy were the same thing, but they are not. First, potential energy measures the energy that a system *has* stored in it, while work measures how much energy is *transferred* in or out. Second, the techniques for calculating work can be used to find the amount of energy transferred in many situations where there is no potential energy involved, as when we calculate the amount of kinetic energy transformed into heat by a car's brake shoes.

A toy gun *example 9*

▷ A toy gun uses a spring with a spring constant of 10 N/m to shoot a ping-pong ball of mass 5 g. The spring is compressed to 10 cm shorter than its equilibrium length when the gun is loaded. At what speed is the ball released?

▷ The equilibrium point is the natural choice for a reference point. Using the equation found previously for the work, we have

$$PE_x = \frac{1}{2}k\left(x - x_o\right)^2.$$

The spring loses contact with the ball at the equilibrium point, so the final potential energy is

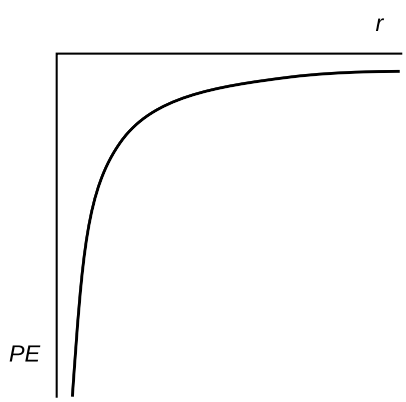
$$PE_f = 0.$$

The initial potential energy is

$$PE_i = \frac{1}{2}(10\ \text{N/m})(0.10\ \text{m})^2. \\ = 0.05\ \text{J}.$$

The loss in potential energy of 0.05 J means an increase in kinetic energy of the same amount. The velocity of the ball is found by solving the equation $KE = (1/2)mv^2$ for v,

$$v = \sqrt{\frac{2KE}{m}} \\ = \sqrt{\frac{(2)(0.05\ \text{J})}{0.005\ \text{kg}}} \\ = 4\ \text{m/s}.$$



q / Example 10, gravitational potential energy as a function of distance.

Gravitational potential energy *example 10*

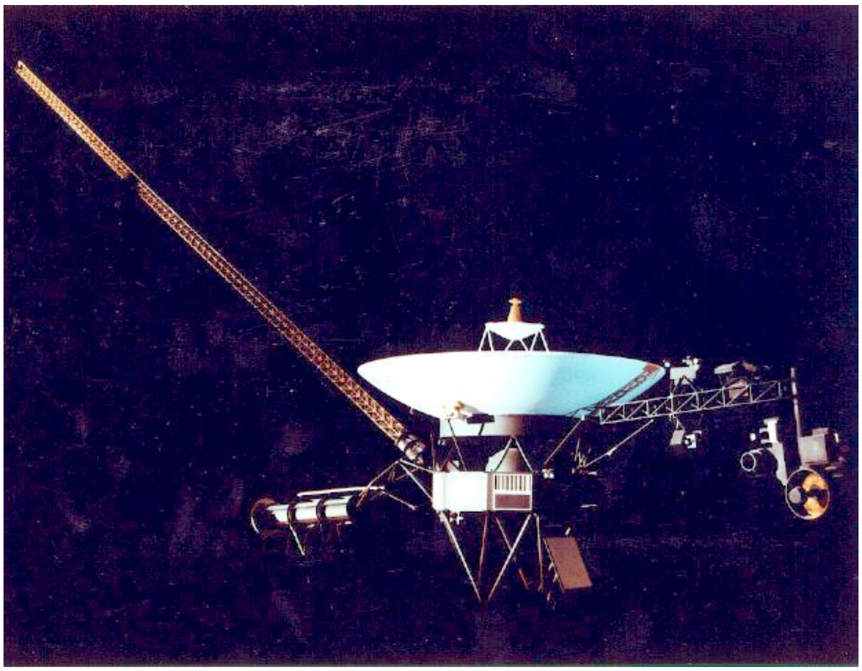
▷ We have already found the equation $\Delta PE = -F\Delta y$ for the gravitational potential energy when the change in height is not enough to cause a significant change in the gravitational force F. What if the change in height is enough so that this assumption is no longer valid? Use the equation $W = GMm(1/r_2 - 1/r_1)$ derived in example 8 to find the potential energy, using $r = \infty$ as a reference point.

▷ The potential energy equals minus the work that would have to be done to bring the object from $r_1 = \infty$ to $r = r_2$, which is

$$PE = -\frac{GMm}{r}.$$

This is simpler than the equation for the work, which is an example of why it is advantageous to record an equation for potential energy relative to some reference point, rather than an equation for work.

Although the equations derived in the previous two examples may seem arcane and not particularly useful except for toy designers and rocket scientists, their usefulness is actually greater than it appears. The equation for the potential energy of a spring can be adapted to any other case in which an object is compressed, stretched, twisted, or bent. While you are not likely to use the equation for gravitational potential energy for anything practical, it is directly analogous to an equation that is extremely useful in chemistry, which is the equation for the potential energy of an electron at a distance r from the nucleus of its atom. As discussed in more detail later in the course, the electrical force between the electron and the nucleus is proportional to $1/r^2$, just like the gravitational force between two masses. Since the equation for the force is of the same form, so is the equation for the potential energy.



r / The twin Voyager space probes were perhaps the greatest scientific successes of the space program. Over a period of decades, they flew by all the planets of the outer solar system, probably accomplishing more of scientific interest than the entire space shuttle program at a tiny fraction of the cost. Both Voyager probes completed their final planetary flybys with speeds greater than the escape velocity at that distance from the sun, and so headed on out of the solar system on hyperbolic orbits, never to return. Radio contact has been lost, and they are now likely to travel interstellar space for billions of years without colliding with anything or being detected by any intelligent species.

Discussion questions

A What does the graph of $PE = (1/2)k\,(x - x_o)^2$ look like as a function of x? Discuss the physical significance of its features.

B What does the graph of $PE = -GMm/r$ look like as a function of r? Discuss the physical significance of its features. How would the equation and graph change if some other reference point was chosen rather than $r = \infty$?

C Starting at a distance r from a planet of mass M, how fast must an object be moving in order to have a hyperbolic orbit, i.e., one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it?

D Does a spring have an "escape velocity?"

E If the form of energy being transferred is potential energy, then the equations $F = \mathrm{d}W/\mathrm{d}x$ and $W = \int F\,\mathrm{d}x$ become $F = -\mathrm{d}PE/\mathrm{d}x$ and $PE = -\int F\,\mathrm{d}x$. How would you then apply the following calculus concepts: zero derivative at minima and maxima, and the second derivative test for concavity up or down.

13.6 ⋆ When does work equal force times distance?

In the example of the tractor pulling the plow discussed on page 361, the work did not equal Fd. The purpose of this section is to explain more fully how the quantity Fd can and cannot be used. To simplify things, I write Fd throughout this section, but more generally everything said here would be true for the area under the graph of $F_{\|}$ versus d.

The following two theorems allow most of the ambiguity to be cleared up.

> **the work-kinetic-energy theorem**
> The change in kinetic energy associated with the motion of an object's center of mass is related to the total force acting on it and to the distance traveled by its center of mass according to the equation $\Delta KE_{cm} = F_{\text{total}} d_{cm}$.

A proof is given in the sidebar, along with some interpretation of how this result relates to the logical structure of our presentation. Note that despite the traditional name, it does not necessarily tell the amount of work done, since the forces acting on the object could be changing other types of energy besides the KE associated with its center of mass motion.

The second theorem does relate directly to work:

> When a contact force acts between two objects and the two surfaces do not slip past each other, the work done equals Fd, where d is the distance traveled by the point of contact.

This one has no generally accepted name, so we refer to it simply as the second theorem.

A great number of physical situations can be analyzed with these two theorems, and often it is advantageous to apply both of them to the same situation.

An ice skater pushing off from a wall — *example 11*

The work-kinetic energy theorem tells us how to calculate the skater's kinetic energy if we know the amount of force and the distance her center of mass travels while she is pushing off.

The second theorem tells us that the wall does no work on the skater. This makes sense, since the wall does not have any source of energy.

Absorbing an impact without recoiling? — *example 12*

▷ Is it possible to absorb an impact without recoiling? For instance, would a brick wall "give" at all if hit by a ping-pong ball?

▷ There will always be a recoil. In the example proposed, the wall will surely have some energy transferred to it in the form of heat

The work-KE theorem

Proof

For simplicity, we have assumed F_{total} to be constant, and therefore $a_{\text{cm}} = F_{\text{total}}/m$ is also constant, and the constant-acceleration equation

$$v_{\text{cm,f}}^2 = v_{\text{cm,i}}^2 + 2a_{\text{cm}}\Delta x_{\text{cm}}$$

applies. Multiplying by $m/2$ on both sides and applying Newton's second law gives

$$KE_{\text{cm,f}}^2 = KE_{\text{cm,i}}^2 + F_{\text{total}}\Delta x_{\text{cm}},$$

which is the result that was to be proved.

Further interpretation

The logical structure of this book is that although Newton's laws are discussed before conservation laws, the conservation laws are taken to be fundamental, since they are true even in cases where Newton's laws fail. Many treatments of this subject present the work-KE theorem as a proof that kinetic energy behaves as $(1/2)mv^2$. This is a matter of taste, but one can just as well rearrange the equations in the proof above to solve for the unknown a_{cm} and prove Newton's second law as a consequence of conservation of energy. Ultimately we have a great deal of freedom in choosing which equations to take as definitions, which to take as empirically verified laws of nature, and which to take as theorems.

Regardless of how we slice things, we require both mathematical consistency and consistency with experiment. As described on p. 321, the work-KE theorem is an important part of this interlocking system of relationships.

and vibration. The second theorem tells us that we can only have nonzero work if the distance traveled by the point of contact is nonzero.

Dragging a refrigerator at constant velocity *example 13*

Newton's first law tells us that the total force on the refrigerator must be zero: your force is canceling the floor's kinetic frictional force. The work-kinetic energy theorem is therefore true but useless. It tells us that there is zero total force on the refrigerator, and that the refrigerator's kinetic energy doesn't change.

The second theorem tells us that the work you do equals your hand's force on the refrigerator multiplied by the distance traveled. Since we know the floor has no source of energy, the only way for the floor and refrigerator to gain energy is from the work you do. We can thus calculate the total heat dissipated by friction in the refrigerator and the floor.

Note that there is no way to find how much of the heat is dissipated in the floor and how much in the refrigerator.

Accelerating a cart *example 14*

If you push on a cart and accelerate it, there are two forces acting on the cart: your hand's force, and the static frictional force of the ground pushing on the wheels in the opposite direction.

Applying the second theorem to your force tells us how to calculate the work you do.

Applying the second theorem to the floor's force tells us that the floor does no work on the cart. There is no motion at the point of contact, because the atoms in the floor are not moving. (The atoms in the surface of the wheel are also momentarily at rest when they touch the floor.) This makes sense, since the floor does not have any source of energy.

The work-kinetic energy theorem refers to the total force, and because the floor's backward force cancels part of your force, the total force is less than your force. This tells us that only part of your work goes into the kinetic energy associated with the forward motion of the cart's center of mass. The rest goes into rotation of the wheels.

13.7 ⋆ Uniqueness of the dot product

In this section I prove that the vector dot product is unique, in the sense that there is no other possible way to define it that is consistent with rotational invariance and that reduces appropriately to ordinary multiplication in one dimension.

Suppose we want to find some way to multiply two vectors to get a scalar, and we don't know how this operation should be defined.

Let's consider what we would get by performing this operation on various combinations of the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. Rotational invariance requires that we handle the three coordinate axes in the same way, without giving special treatment to any of them, so we must have $\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}}$. This is supposed to be a way of generalizing ordinary multiplication, so for consistency with the property $1 \times 1 = 1$ of ordinary numbers, the result of multiplying a magnitude-one vector by itself had better be the scalar 1, so $\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$. Furthermore, there is no way to satisfy rotational invariance unless we define the mixed products to be zero, $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$; for example, a 90-degree rotation of our frame of reference about the z axis reverses the sign of $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}$, but rotational invariance requires that $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}$ produce the same result either way, and zero is the only number that stays the same when we reverse its sign. Establishing these six products of unit vectors suffices to define the operation in general, since any two vectors that we want to multiply can be broken down into components, e.g., $(2\hat{\mathbf{x}}+3\hat{\mathbf{z}})\cdot\hat{\mathbf{z}} = 2\hat{\mathbf{x}}\cdot\hat{\mathbf{z}}+3\hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 0+3 = 3$. Thus by requiring rotational invariance and consistency with multiplication of ordinary numbers, we find that there is only one possible way to define a multiplication operation on two vectors that gives a scalar as the result. (There is, however, a different operation, discussed in chapter 15, which multiplies two vectors to give a vector.)

13.8 ⋆ A dot product for relativity?

In section 13.7 I showed that the dot product is the only physically sensible way to multiply two vectors to get a scalar. This is essentially because the outcome of experiments shouldn't depend on which way we rotate the laboratory. Dot products relate to the lengths of vectors and the angles between them, and rotations don't change lengths or angles.

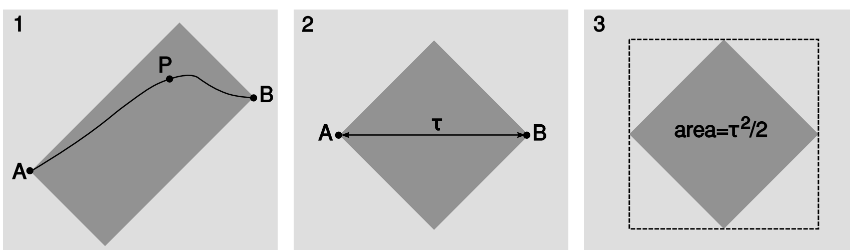
Let's consider how this would apply to relativity. Relativity tells us that the length of a measuring rod is *not* absolute. Rotating the lab won't change its length, but changing the lab's state of motion will. The rod's length is greatest in the frame that is at rest relative to the rod. This suggests that relativity requires some new variation on the dot product: some slightly different way of multiplying two vectors to find a number that doesn't depend on the frame of reference.

Clock time

We do know of a number that stays the same in all frames of reference. In figure am on p. 85, we proved that the Lorentz transformation doesn't change the area of a shape in the x-t plane. We used this only as a stepping stone toward the Lorentz transformation, but it is natural to wonder whether this kind of area has any physical interest of its own.

The equal-area result is not relativistic, since the proof never appeals to property 5 on page 81. Cases I and II on page 84 also have the equal-area property. We can see this clearly in a Galilean transformation like figure ag on p. 82, where the distortion of the rectangle could be accomplished by cutting it into vertical slices and then displacing the slices upward without changing their areas.

But the area does have a nice interpretation in the relativistic case. Suppose that we have events A (Charles VII is restored to the throne) and B (Joan of Arc is executed). Now imagine that technologically advanced aliens want to be present at both A and B, but in the interim they wish to fly away in their spaceship, be present at some other event P (perhaps a news conference at which they give an update on the events taking place on earth), but get back in time for B. Since nothing can go faster than c (which we take to equal 1 in appropriate units), P cannot be too far away. The set of all possible events P forms a rectangle, figure s/1, in the $x-t$ plane that has A and B at opposite corners and whose edges have slopes equal to ± 1. We call this type of rectangle a light-rectangle, because its sides could represent the motion of rays of light.



s / 1. The gray light-rectangle represents the set of all events such as P that could be visited after A and before B.
2. The rectangle becomes a square in the frame in which A and B occur at the same location in space.
3. The area of the dashed square is τ^2, so the area of the gray square is $\tau^2/2$.

The area of this rectangle will be the same regardless of one's frame of reference. In particular, we could choose a special frame of reference, panel 2 of the figure, such that A and B occur in the same place. (They do not occur at the same place, for example, in the sun's frame, because the earth is spinning and going around the sun.) Since the speed c, which equals 1 in our units, is the same in all frames of reference, and the sides of the rectangle had slopes ± 1 in frame 1, they must still have slopes ± 1 in frame 2. The rectangle becomes a square with its diagonals parallel to the x and t axes, and the length of these diagonals equals the time τ elapsed on a clock that is at rest in frame 2, i.e., a clock that glides through space at constant velocity from A to B, meeting up with the planet earth at the appointed time. As shown in panel 3 of the figure, the area of the gray regions can be interpreted as half the square of this gliding-clock time.

If events A and B are separated by a distance x and a time t, then in general $t^2 - x^2$ gives the square of the gliding-clock time. Proof: Based on units, the expression must have the form $(\ldots)t^2 + (\ldots)tx + (\ldots)x^2$, where each $(\ldots)$ represents a unitless constant. The tx coefficient must be zero by property 2 on p. 81. For consistency with figure s/3, the t^2 coefficient must equal 1. Since the area vanishes for $x = t$, the x^2 coefficient must equal -1.

When $|x|$ is greater than $|t|$, events A and B are so far apart in space and so close together in time that it would be impossible to have a cause and effect relationship between them, since $c = 1$ is the maximum speed of cause and effect. In this situation $t^2 - x^2$ is negative and cannot be interpreted as a clock time, but it can be interpreted as minus the square of the distance between A and B as measured by rulers at rest in a frame in which A and B are simultaneous.

Four-vectors

No matter what, $t^2 - x^2$ is the same as measured in all frames of reference. Geometrically, it plays the same role in the x-t plane that ruler measurements play in the Euclidean plane. In Euclidean geometry, the ruler-distance between any two points stays the same regardless of rotation, i.e., regardless of the angle from which we view the scene; according to the Pythagorean theorem, the square of this distance is $x^2 + y^2$. In the x-t plane, $t^2 - x^2$ stays the same regardless of the frame of reference. This suggests that by analogy with the dot product

$$x_1x_2 + y_1y_2$$

in the Euclidean x-y plane, we define a similar operation in the x-t plane,

$$t_1t_2 - x_1x_2.$$

Putting in the other two spatial dimensions, we have

$$t_1t_2 - x_1x_2 - y_1y_2 - z_1z_2.$$

A mathematical object like (t, x, y, z) is referred to as a four-vector, as opposed to a three-vector like (x, y, z). The term "dot product" has connotations of referring only to three-vectors, so the operation of taking the scalar product of two four-vectors is usually referred to instead as the "inner product." There are various ways of notating the inner product of vectors $\mathbf{a}$ and $\mathbf{b}$, such as $\mathbf{a} \cdot \mathbf{b}$ or $< \mathbf{a}, \mathbf{b} >$.

The magnitude of a three-vector is defined by taking the square root of its dot product with itself, and this square root is always a real number, because a vector's dot product with itself is always positive. But the inner product of a four-vector with itself can be positive, zero, or negative, and in these cases the vector is referred to as timelike, lightlike, spacelike, respectively. Since material objects can never go as fast as c, the vector $(\Delta t, \Delta x, \Delta y, \Delta z)$ describing an object's motion from one event to another is always timelike.

The twin paradox *example 15*

One of the classic paradoxes of relativity, known as the twin paradox, is usually stated something like this. Alice and Betty are identical twins. Betty goes on a space voyage at relativistic speeds, traveling away from the earth and then turning around and coming back. Meanwhile, Alice stays on earth. When Betty returns, she is younger than Alice because of relativistic time dilation. But isn't it valid to say that Betty's spaceship is standing still and the earth moving? In that description, wouldn't Alice end up younger and Betty older?

The most common way of explaining the non-paradoxical nature of this paradox is that although special relativity says that inertial motion is relative, it doesn't say that noninertial motion is relative. In this respect it is the same as Newtonian mechanics. Betty experiences accelerations on her voyage, but Alice doesn't. Therefore there is no doubt about who actually went on the trip and who didn't.

This resolution, however, may not be entirely satisfying because it makes it sound as if relativistic time dilation is not occurring while Betty's ship cruises at constant velocity, but only while the ship is speeding up or slowing down. This would appear to contradict our earlier interpretation of relativistic time dilation, which was that a clock runs fastest according to an observer at rest relative to the clock. Furthermore, if it's acceleration that causes the effect, should we be looking for some new formula that computes time dilation based on acceleration?

The first thing to realize is that there is no unambiguous way to decide during which part of Betty's journey the time dilation is occurring. To do this, we could need to be able to compare Alice and Betty's clocks many times over the course of the trip. But neither twin has any way of finding out what her sister's clock reads "now," except by exchanging radio signals, which travel at the speed of light. The speed-of-light lag vanishes only at the beginning and end of the trip, when the twins are in the same place.

Furthermore, we can use the inner product to show that the accumulated difference in clock time doesn't depend on the details of how Betty carries out her accelerations and decelerations. In fact, we can get the right answer simply by assuming that these changes in velocities occur instantaneously.

In Euclidean geometry, the triangle inequality $|\mathbf{b}+\mathbf{c}| < |\mathbf{b}| + |\mathbf{c}|$ follows from

$$(|\mathbf{b}| + |\mathbf{c}|)^2 - (\mathbf{b}+\mathbf{c})\cdot(\mathbf{b}+\mathbf{c}) = 2(|\mathbf{b}||\mathbf{c}| - \mathbf{b}\cdot\mathbf{c}) \geq 0.$$

The reason this quantity always comes out positive is that for two vectors of fixed magnitude, the greatest dot product is always achieved in the case where they lie along the same direction.

In the geometry of the x-t plane, the situation is different. Suppose that **b** and **c** are timelike vectors, so that they represent possible $(\Delta t, \Delta x, \ldots)$ vectors for Betty on the outward and return legs of her trip. Then $\mathbf{a} = \mathbf{b} + \mathbf{c}$ describes the vector for Alice's motion. Alice goes by a direct route through the x-t plane while Betty takes a detour. The magnitude of each timelike vector represents the time elapsed on a clock carried by that twin. The triangle equality is now reversed, becoming $|\mathbf{b} + \mathbf{c}| > |\mathbf{b}| + |\mathbf{c}|$. The difference from the Euclidean case arises because inner products are no longer necessarily maximized if vectors are in the same direction. E.g., for two lightlike vectors, $\mathbf{b} \cdot \mathbf{c}$ vanishes entirely if **b** and **c** are parallel. For timelike vectors, parallelism actually minimizes the inner product rather than maximizing it.[2]

[2]Proof: Let **b** and **c** be parallel and timelike, and directed forward in time. Adopt a frame of reference in which every spatial component of each vector vanishes. This entails no loss of generality, since inner products are invariant under such a transformation. Now let **b** and **c** be pulled away from parallelism, like opening a pair of scissors in the $x - t$ plane. This reduces $b_t c_t$, while causing $b_x c_x$ to become negative. Both effects increase the inner product.

Summary

Selected vocabulary

work . . . . the amount of energy transferred into or out of a system, excluding energy transferred by heat conduction

Notation

W . . . . work

Summary

Work is a measure of the transfer of mechanical energy, i.e., the transfer of energy by a force rather than by heat conduction. When the force is constant, work can usually be calculated as

$$W = F_{\|}|\mathbf{d}|, \qquad \text{[only if the force is constant]}$$

where $\mathbf{d}$ is simply a less cumbersome notation for $\Delta\mathbf{r}$, the vector from the initial position to the final position. Thus,

- A force in the same direction as the motion does positive work, i.e., transfers energy into the object on which it acts.
- A force in the opposite direction compared to the motion does negative work, i.e., transfers energy out of the object on which it acts.
- When there is no motion, no mechanical work is done. The human body burns calories when it exerts a force without moving, but this is an internal energy transfer of energy within the body, and thus does not fall within the scientific definition of work.
- A force perpendicular to the motion does no work.

When the force is not constant, the above equation should be generalized as an integral, $\int F_{\|}\,dx$.

There is only one meaningful (rotationally invariant) way of defining a multiplication of vectors whose result is a scalar, and it is known as the vector *dot product:*

$$\begin{aligned}\mathbf{b}\cdot\mathbf{c} &= b_x c_x + b_y c_y + b_z c_z \\ &= |\mathbf{b}|\,|\mathbf{c}|\,\cos\theta_{bc}.\end{aligned}$$

The dot product has most of the usual properties associated with multiplication, except that there is no "dot division." The dot product can be used to compute mechanical work as $W = \mathbf{F}\cdot\mathbf{d}$.

Machines such as pulleys, levers, and gears may increase or decrease a force, but they can never increase or decrease the amount of work done. That would violate conservation of energy unless the

machine had some source of stored energy or some way to accept and store up energy.

There are some situations in which the equation $W = F_{\|} \, |\mathbf{d}|$ is ambiguous or not true, and these issues are discussed rigorously in section 13.6. However, problems can usually be avoided by analyzing the types of energy being transferred before plunging into the math. In any case there is no substitute for a physical understanding of the processes involved.

The techniques developed for calculating work can also be applied to the calculation of potential energy. We fix some position as a reference position, and calculate the potential energy for some other position, x, as

$$PE_x = -W_{\text{ref}\to x}.$$

The following two equations for potential energy have broader significance than might be suspected based on the limited situations in which they were derived:

$$PE = \frac{1}{2} k \left(x - x_\text{o}\right)^2 .$$

[potential energy of a spring having spring constant k, when stretched or compressed from the equilibrium position x_o; analogous equations apply for the twisting, bending, compression, or stretching of any object.]

$$PE = -\frac{GMm}{r}$$

[gravitational potential energy of objects of masses M and m, separated by a distance r; an analogous equation applies to the electrical potential energy of an electron in an atom.]

Problems

Key

√ A computerized answer check is available online.

∫ A problem that requires calculus.

⋆ A difficult problem.

1 Two speedboats are identical, but one has more people aboard than the other. Although the total masses of the two boats are unequal, suppose that they happen to have the same kinetic energy. In a boat, as in a car, it's important to be able to stop in time to avoid hitting things. (a) If the frictional force from the water is the same in both cases, how will the boats' stopping distances compare? Explain. (b) Compare the times required for the boats to stop.

2 In each of the following situations, is the work being done positive, negative, or zero? (a) a bull paws the ground; (b) a fishing boat pulls a net through the water behind it; (c) the water resists the motion of the net through it; (d) you stand behind a pickup truck and lower a bale of hay from the truck's bed to the ground. Explain. [Based on a problem by Serway and Faughn.]

A bull paws the ground, as in problem 2.

3 (a) Suppose work is done in one-dimensional motion. What happens to the work if you reverse the direction of the positive coordinate axis? Base your answer directly on the definition of work. (b) Now answer the question based on the $W = Fd$ rule.

4 Does it make sense to say that work is conserved?

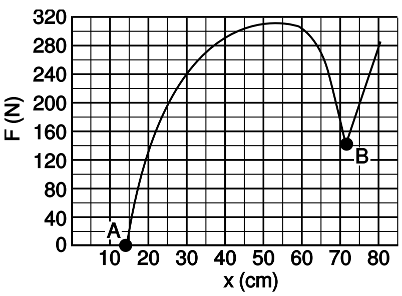
▷ Solution, p. 555

5 A microwave oven works by twisting molecules one way and then the other, counterclockwise and then clockwise about their own centers, millions of times a second. If you put an ice cube or a stick of butter in a microwave, you'll observe that the solid doesn't heat very quickly, although eventually melting begins in one small spot. Once this spot forms, it grows rapidly, while the rest of the solid remains solid; it appears that a microwave oven heats a liquid much more rapidly than a solid. Explain why this should happen, based on the atomic-level description of heat, solids, and liquids. (See, e.g., figure b on page 339.)

Don't repeat the following common mistakes:

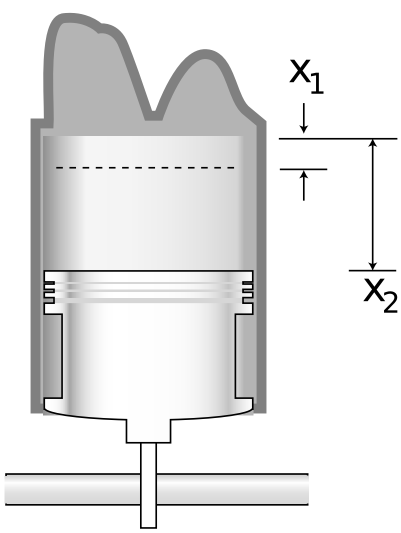
In a solid, the atoms are packed more tightly and have less space between them. Not true. Ice floats because it's *less* dense than water.

In a liquid, the atoms are moving much faster. No, the difference in average speed between ice at −1°C and water at 1°C is only 0.4%.



Problem 6.

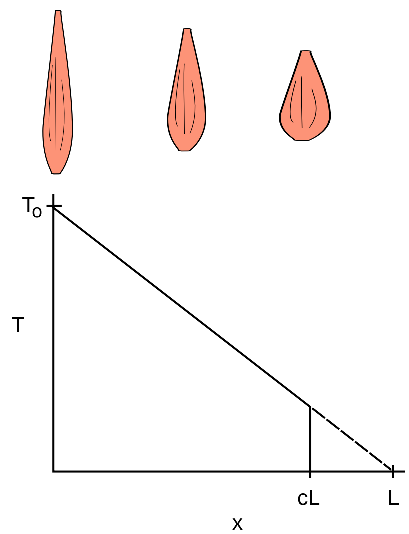
6 Most modern bow hunters in the U.S. use a fancy mechanical bow called a compound bow, which looks nothing like what most people imagine when they think of a bow and arrow. It has a system of pulleys designed to produce the force curve shown in the figure, where F is the force required to pull the string back, and x is the distance between the string and the center of the bow's body. It is not a linear Hooke's-law graph, as it would be for an old-fashioned bow. The big advantage of the design is that relatively little force is required to hold the bow stretched to point B on the graph. This is the force required from the hunter in order to hold the bow ready while waiting for a shot. Since it may be necessary to wait a long time, this force can't be too big. An old-fashioned bow, designed to require the same amount of force when fully drawn, would shoot arrows at much lower speeds, since its graph would be a straight line from A to B. For the graph shown in the figure (taken from realistic data), find the speed at which a 26 g arrow is released, assuming that 70% of the mechanical work done by the hand is actually transmitted to the arrow. (The other 30% is lost to frictional heating inside the bow and kinetic energy of the recoiling and vibrating bow.) ✓



Problem 7: A cylinder from the 1965 Rambler's engine. The piston is shown in its pushed out position. The two bulges at the top are for the valves that let fresh air-gas mixture in. Based on a figure from Motor Service's Automotive Encyclopedia, Toboldt and Purvis.

7 In the power stroke of a car's gasoline engine, the fuel-air mixture is ignited by the spark plug, explodes, and pushes the piston out. The exploding mixture's force on the piston head is greatest at the beginning of the explosion, and decreases as the mixture expands. It can be approximated by $F = a/x$, where x is the distance from the cylinder to the piston head, and a is a constant with units of N$\cdot$m. (Actually $a/x^{1.4}$ would be more accurate, but the problem works out more nicely with a/x!) The piston begins its stroke at $x = x_1$, and ends at $x = x_2$.
(a) Find the amount of work done in one stroke by one cylinder. ✓
(b) The 1965 Rambler had six cylinders, each with $a = 220$ N$\cdot$m, $x_1 = 1.2$ cm, and $x_2 = 10.2$ cm. Assume the engine is running at 4800 r.p.m., so that during one minute, each of the six cylinders performs 2400 power strokes. (Power strokes only happen every other revolution.) Find the engine's power, in units of horsepower (1 hp=746 W). ✓
(c) The compression ratio of an engine is defined as x_2/x_1. Explain in words why the car's power would be exactly the same if x_1 and x_2 were, say, halved or tripled, maintaining the same compression ratio of 8.5. Explain why this would *not* quite be true with the more realistic force equation $F = a/x^{1.4}$.

8 The figure, redrawn from *Gray's Anatomy*, shows the tension of which a muscle is capable. The variable x is defined as the contraction of the muscle from its maximum length L, so that at $x = 0$ the muscle has length L, and at $x = L$ the muscle would theoretically have zero length. In reality, the muscle can only contract to $x = cL$, where c is less than 1. When the muscle is extended to its maximum length, at $x = 0$, it is capable of the greatest tension, T_o. As the muscle contracts, however, it becomes weaker. Gray suggests approximating this function as a linear decrease, which would theoretically extrapolate to zero at $x = L$. (a) Find the maximum work the muscle can do in one contraction, in terms of c, L, and T_o. √
(b) Show that your answer to part a has the right units.
(c) Show that your answer to part a has the right behavior when $c = 0$ and when $c = 1$.
(d) Gray also states that the absolute maximum tension T_o has been found to be approximately proportional to the muscle's cross-sectional area A (which is presumably measured at $x = 0$), with proportionality constant k. Approximating the muscle as a cylinder, show that your answer from part a can be reexpressed in terms of the volume, V, eliminating L and A. √
(e) Evaluate your result numerically for a biceps muscle with a volume of 200 cm^3, with $c = 0.8$ and $k = 100\ N/cm^2$ as estimated by Gray. √



Problem 8.

9 In the earth's atmosphere, the molecules are constantly moving around. Because temperature is a measure of kinetic energy per molecule, the average kinetic energy of each type of molecule is the same, e.g., the average KE of the O_2 molecules is the same as the average KE of the N_2 molecules. (a) If the mass of an O_2 molecule is eight times greater than that of a He atom, what is the ratio of their average speeds? Which way is the ratio, i.e., which is typically moving faster? (b) Use your result from part a to explain why any helium occurring naturally in the atmosphere has long since escaped into outer space, never to return. (Helium is obtained commercially by extracting it from rocks.) You may want to do problem 12 first, for insight. √

10 Weiping lifts a rock with a weight of 1.0 N through a height of 1.0 m, and then lowers it back down to the starting point. Bubba pushes a table 1.0 m across the floor at constant speed, requiring a force of 1.0 N, and then pushes it back to where it started. (a) Compare the total work done by Weiping and Bubba. (b) Check that your answers to part a make sense, using the definition of work: work is the transfer of energy. In your answer, you'll need to discuss what specific type of energy is involved in each case.

11 In one of his more flamboyant moments, Galileo wrote "Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon." Find the speed of an ant that falls to earth from the distance of the moon at the moment when it is about to enter the atmosphere. Assume it is released from a point that is not actually near the moon, so the moon's gravity is negligible. You will need the result of example 10 on p. 372. √

12 Starting at a distance r from a planet of mass M, how fast must an object be moving in order to have a hyperbolic orbit, i.e., one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it? √

13 A projectile is moving directly away from a planet of mass M at exactly escape velocity. (a) Find r, the distance from the projectile to the center of the planet, as a function of time, t, and also find $v(t)$. √
(b) Check the units of your answer.
(c) Does v show the correct behavior as t approaches infinity?
▷ Hint, p. 542

14 A car starts from rest at $t = 0$, and starts speeding up with constant acceleration. (a) Find the car's kinetic energy in terms of its mass, m, acceleration, a, and the time, t. (b) Your answer in the previous part also equals the amount of work, W, done from $t = 0$ until time t. Take the derivative of the previous expression to find the power expended by the car at time t. (c) Suppose two cars with the same mass both start from rest at the same time, but one has twice as much acceleration as the other. At any moment, how many times more power is being dissipated by the more quickly accelerating car? (The answer is not 2.) √

15 A car accelerates from rest. At low speeds, its acceleration is limited by static friction, so that if we press too hard on the gas, we will "burn rubber" (or, for many newer cars, a computerized traction-control system will override the gas pedal). At higher speeds, the limit on acceleration comes from the power of the engine, which puts a limit on how fast kinetic energy can be developed.
(a) Show that if a force F is applied to an object moving at speed v, the power required is given by $P = vF$.
(b) Find the speed v at which we cross over from the first regime described above to the second. At speeds higher than this, the engine does not have enough power to burn rubber. Express your result in terms of the car's power P, its mass m, the coefficient of static friction μ_s, and g. √
(c) Show that your answer to part b has units that make sense.
(d) Show that the dependence of your answer on each of the four variables makes sense physically.
(e) The 2010 Maserati Gran Turismo Convertible has a maximum power of 3.23×10^5 W (433 horsepower) and a mass (including a 50-kg driver) of 2.03×10^3 kg. (This power is the maximum the engine can supply at its optimum frequency of 7600 r.p.m. Presumably the automatic transmission is designed so a gear is available in which the engine will be running at very nearly this frequency when the car is moving at v.) Rubber on asphalt has $\mu_s \approx 0.9$. Find v for this car. Answer: 18 m/s, or about 40 miles per hour.
(f) Our analysis has neglected air friction, which can probably be approximated as a force proportional to v^2. The existence of this force is the reason that the car has a maximum speed, which is 176 miles per hour. To get a feeling for how good an approximation it is to ignore air friction, find what fraction of the engine's maximum power is being used to overcome air resistance when the car is moving at the speed v found in part e. Answer: 1%

16 In 1935, Yukawa proposed an early theory of the force that held the neutrons and protons together in the nucleus. His equation for the potential energy of two such particles, at a center-to-center distance r, was $PE(r) = gr^{-1}e^{-r/a}$, where g parametrizes the strength of the interaction, e is the base of natural logarithms, and a is about 10^{-15} m. Find the force between two nucleons that would be consistent with this equation for the potential energy. √

17 The magnitude of the force between two magnets separated by a distance r can be approximated as kr^{-3} for large values of r. The constant k depends on the strengths of the magnets and the relative orientations of their north and south poles. Two magnets are released on a slippery surface at an initial distance r_i, and begin sliding towards each other. What will be the total kinetic energy of the two magnets when they reach a final distance r_f? (Ignore friction.) √

18 A rail gun is a device like a train on a track, with the train propelled by a powerful electrical pulse. Very high speeds have been demonstrated in test models, and rail guns have been proposed as an alternative to rockets for sending into outer space any object that would be strong enough to survive the extreme accelerations. Suppose that the rail gun capsule is launched straight up, and that the force of air friction acting on it is given by $F = be^{-cx}$, where x is the altitude, b and c are constants, and e is the base of natural logarithms. The exponential decay occurs because the atmosphere gets thinner with increasing altitude. (In reality, the force would probably drop off even faster than an exponential, because the capsule would be slowing down somewhat.) Find the amount of kinetic energy lost by the capsule due to air friction between when it is launched and when it is completely beyond the atmosphere. (Gravity is negligible, since the air friction force is much greater than the gravitational force.) √

19 A certain binary star system consists of two stars with masses m_1 and m_2, separated by a distance b. A comet, originally nearly at rest in deep space, drops into the system and at a certain point in time arrives at the midpoint between the two stars. For that moment in time, find its velocity, v, symbolically in terms of b, m_1, m_2, and fundamental constants. √

20 Find the angle between the following two vectors:

$$\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$$
$$4\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}$$

▷ Hint, p. 542 √

21 An airplane flies in the positive direction along the x axis, through crosswinds that exert a force $\mathbf{F} = (a + bx)\hat{\mathbf{x}} + (c + dx)\hat{\mathbf{y}}$. Find the work done by the wind on the plane, and by the plane on the wind, in traveling from the origin to position x. √

22 Prove that the dot product defined in section 13.3 is rotationally invariant in the sense of section 7.5.

23 Fill in the details of the proof of $\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$ on page 369.

24 A space probe of mass m is dropped into a previously unexplored spherical cloud of gas and dust, and accelerates toward the center of the cloud under the influence of the cloud's gravity. Measurements of its velocity allow its potential energy, PE, to be determined as a function of the distance r from the cloud's center. The mass in the cloud is distributed in a spherically symmetric way, so its density, $\rho(r)$, depends only on r and not on the angular coordinates. Show that by finding PE, one can infer $\rho(r)$ as follows:

$$\rho(r) = \frac{1}{4\pi G m r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}PE}{\mathrm{d}r}\right).$$

⋆

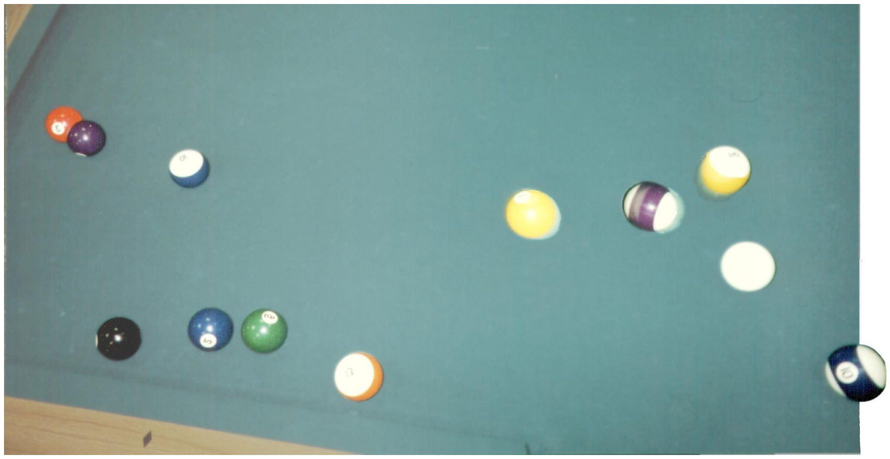
25 The purpose of this problem is to estimate the height of the tides. The main reason for the tides is the moon's gravity, and we'll neglect the effect of the sun. Also, real tides are heavily influenced by landforms that channel the flow of water, but we'll think of the earth as if it was completely covered with oceans. Under these assumptions, the ocean surface should be a surface of constant U/m. That is, a thimbleful of water, m, should not be able to gain or lose any gravitational energy by moving from one point on the ocean surface to another. If only the spherical earth's gravity was present, then we'd have $U/m = -GM_e/r$, and a surface of constant U/m would be a surface of constant r, i.e., the ocean's surface would be spherical. Taking into account the moon's gravity, the main effect is to shift the center of the sphere, but the sphere also becomes slightly distorted into an approximately ellipsoidal shape. (The shift of the center is not physically related to the tides, since the solid part of the earth tends to be centered within the oceans; really, this effect has to do with the motion of the whole earth through space, and the way that it wobbles due to the moon's gravity.) Determine the amount by which the long axis of the ellipsoid exceeds the short axis. ▷ Hint, p. 542 ⋆

26 A mass moving in one dimension is attached to a horizontal spring. It slides on the surface below it, with equal coefficients of static and kinetic friction, $\mu_k = \mu_s$. The equilibrium position is $x = 0$. If the mass is pulled to some initial position and released from rest, it will complete some number of oscillations before friction brings it to a stop. When released from $x = a$ ($a > 0$), it completes exactly 1/4 of an oscillation, i.e., it stops precisely at $x = 0$. Similarly, define $b > 0$ as the greatest x from which it could be released and comlete 1/2 of an oscillation, stopping on the far side and not coming back toward equilibrium. Find b/a. Hint: To keep the algebra simple, set every fixed parameter of the system equal to 1. √

27 "Big wall" climbing is a specialized type of rock climbing that involves going up tall cliffs such as the ones in Yosemite, usually with the climbers spending at least one night sleeping on a natural ledge or an artificial "portaledge." In this style of climbing, each pitch of the climb involves strenuously hauling up several heavy bags of gear — a fact that has caused these climbs to be referred to as "vertical ditch digging." (a) If an 80 kg haul bag has to be pulled up the full length of a 60 m rope, how much work is done? (b) Since it can be difficult to lift 80 kg, a 2:1 pulley is often used. The hauler then lifts the equivalent of 40 kg, but has to pull in 120 m of rope. How much work is done in this case? √

28 Let a and b be any two numbers (not both zero), and let $\mathbf{u} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}}$. Find a (nonzero) second vector $\mathbf{v}$ in the x-y plane that is perpendicular to $\mathbf{u}$.

29 A soccer ball of mass m is moving at speed v when you kick it in the same direction it is moving. You kick it with constant force F, and you want to triple the ball's speed. Over what distance must your foot be in contact with the ball? √ *[problem by B. Shotwell]*



Pool balls exchange momentum.

Chapter 14

Conservation of momentum

In many subfields of physics these days, it is possible to read an entire issue of a journal without ever encountering an equation involving force or a reference to Newton's laws of motion. In the last hundred and fifty years, an entirely different framework has been developed for physics, based on conservation laws.

The new approach is not just preferred because it is in fashion. It applies inside an atom or near a black hole, where Newton's laws do not. Even in everyday situations the new approach can be superior. We have already seen how perpetual motion machines could be designed that were too complex to be easily debunked by Newton's laws. The beauty of conservation laws is that they tell us something must remain the same, regardless of the complexity of the process.

So far we have discussed only two conservation laws, the laws of conservation of mass and energy. Is there any reason to believe that further conservation laws are needed in order to replace Newton's laws as a complete description of nature? Yes. Conservation of mass and energy do not relate in any way to the three dimensions of space, because both are scalars. Conservation of energy, for instance, does not prevent the planet earth from abruptly making a 90-degree turn and heading straight into the sun, because kinetic energy does not depend on direction. In this chapter, we develop a new conserved quantity, called momentum, which is a vector.

14.1 Momentum

A conserved quantity of motion

Your first encounter with conservation of momentum may have come as a small child unjustly confined to a shopping cart. You spot something interesting to play with, like the display case of imported wine down at the end of the aisle, and decide to push the cart over there. But being imprisoned by Dad in the cart was not the only injustice that day. There was a far greater conspiracy to thwart your young id, one that originated in the laws of nature. Pushing forward did nudge the cart forward, but it pushed you backward. If the wheels of the cart were well lubricated, it wouldn't matter how you jerked, yanked, or kicked off from the back of the cart. You could not cause any overall forward motion of the entire system consisting of the cart with you inside.

In the Newtonian framework, we describe this as arising from Newton's third law. The cart made a force on you that was equal and opposite to your force on it. In the framework of conservation laws, we cannot attribute your frustration to conservation of energy. It would have been perfectly possible for you to transform some of the internal chemical energy stored in your body to kinetic energy of the cart and your body.

The following characteristics of the situation suggest that there may be a new conservation law involved:

A closed system is involved. All conservation laws deal with closed systems. You and the cart are a closed system, since the well-oiled wheels prevent the floor from making any forward force on you.

Something remains unchanged. The overall velocity of the system started out being zero, and you cannot change it. This vague reference to "overall velocity" can be made more precise: it is the velocity of the system's center of mass that cannot be changed.

Something can be transferred back and forth without changing the total amount. If we define forward as positive and backward as negative, then one part of the system can gain positive motion if another part acquires negative motion. If we don't want to worry about positive and negative signs, we can imagine that the whole cart was initially gliding forward on its well-oiled wheels. By kicking off from the back of the cart, you could increase your own velocity, but this inevitably causes the cart to slow down.

It thus appears that there is some numerical measure of an object's quantity of motion that is conserved when you add up all the objects within a system.

Momentum

Although velocity has been referred to, it is not the total velocity of a closed system that remains constant. If it was, then firing a gun would cause the gun to recoil at the same velocity as the bullet! The gun does recoil, but at a much lower velocity than the bullet. Newton's third law tells us

$$F_{gun\ on\ bullet} = -F_{bullet\ on\ gun},$$

and assuming a constant force for simplicity, Newton's second law allows us to change this to

$$m_{bullet}\frac{\Delta v_{bullet}}{\Delta t} = -m_{gun}\frac{\Delta v_{gun}}{\Delta t}.$$

Thus if the gun has 100 times more mass than the bullet, it will recoil at a velocity that is 100 times smaller and in the opposite direction, represented by the opposite sign. The quantity mv is therefore apparently a useful measure of motion, and we give it a name, *momentum*, and a symbol, p. (As far as I know, the letter "p" was just chosen at random, since "m" was already being used for mass.) The situations discussed so far have been one-dimensional, but in three-dimensional situations it is treated as a vector.

definition of momentum for material objects
The momentum of a material object, i.e., a piece of matter, is defined as

$$\mathbf{p} = m\mathbf{v},$$

the product of the object's mass and its velocity vector.

The units of momentum are kg·m/s, and there is unfortunately no abbreviation for this clumsy combination of units.

The reasoning leading up to the definition of momentum was all based on the search for a conservation law, and the only reason why we bother to define such a quantity is that experiments show it is conserved:

the law of conservation of momentum
In any closed system, the vector sum of all the momenta remains constant,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} + \ldots = \mathbf{p}_{1f} + \mathbf{p}_{2f} + \ldots,$$

where i labels the initial and f the final momenta. (A closed system is one on which no external forces act.)

This chapter first addresses the one-dimensional case, in which the direction of the momentum can be taken into account by using plus and minus signs. We then pass to three dimensions, necessitating the use of vector addition.

A subtle point about conservation laws is that they all refer to "closed systems," but "closed" means different things in different cases. When discussing conservation of mass, "closed" means a system that doesn't have matter moving in or out of it. With energy, we mean that there is no work or heat transfer occurring across the boundary of the system. For momentum conservation, "closed" means there are no external *forces* reaching into the system.

A cannon *example 1*

▷ A cannon of mass 1000 kg fires a 10-kg shell at a velocity of 200 m/s. At what speed does the cannon recoil?

▷ The law of conservation of momentum tells us that

$$p_{cannon,i} + p_{shell,i} = p_{cannon,f} + p_{shell,f}.$$

Choosing a coordinate system in which the cannon points in the positive direction, the given information is

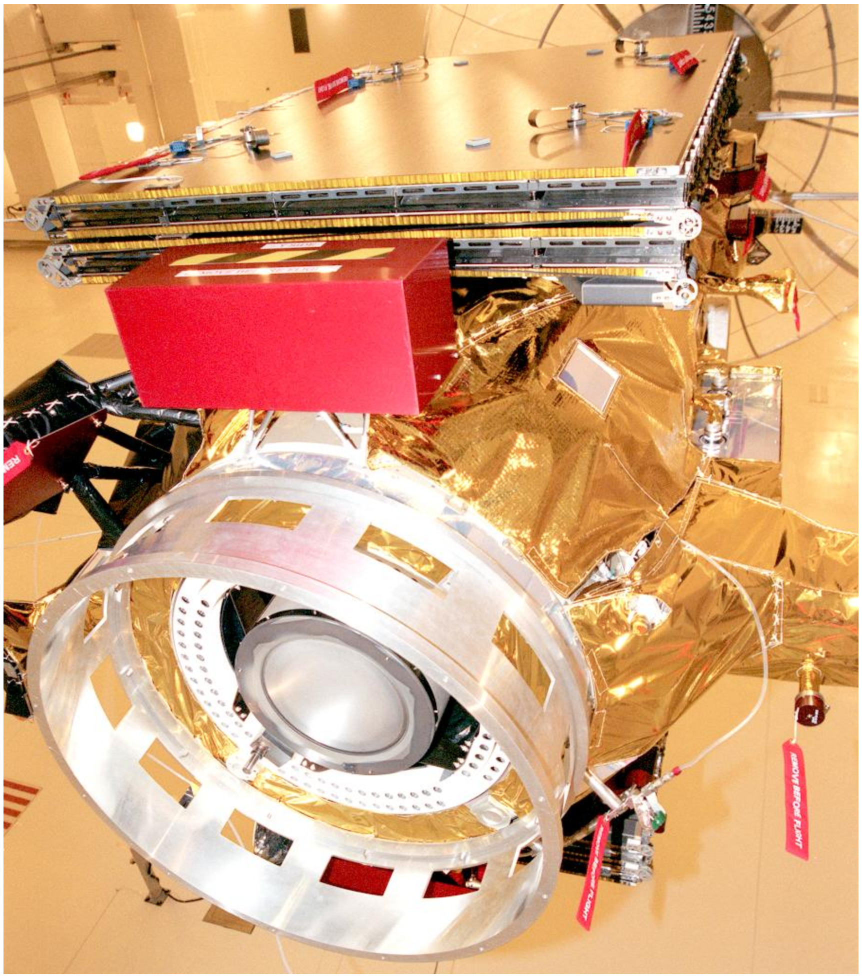
$$\begin{aligned} p_{cannon,i} &= 0 \\ p_{shell,i} &= 0 \\ p_{shell,f} &= 2000\ \text{kg}\cdot\text{m/s}. \end{aligned}$$

We must have $p_{cannon,f} = -2000\ \text{kg}\cdot\text{m/s}$, so the recoil velocity of the cannon is -2 m/s.

Ion drive for propelling spacecraft *example 2*

▷ The experimental solar-powered ion drive of the Deep Space 1 space probe expels its xenon gas exhaust at a speed of 30,000 m/s, ten times faster than the exhaust velocity for a typical chemical-fuel rocket engine. Roughly how many times greater is the maximum speed this spacecraft can reach, compared with a chemical-fueled probe with the same mass of fuel ("reaction mass") available for pushing out the back as exhaust?

▷ Momentum equals mass multiplied by velocity. Both spacecraft are assumed to have the same amount of reaction mass, and the ion drive's exhaust has a velocity ten times greater, so the momentum of its exhaust is ten times greater. Before the engine starts firing, neither the probe nor the exhaust has any momentum, so the total momentum of the system is zero. By conservation of momentum, the total momentum must also be zero after



a / The ion drive engine of the NASA Deep Space 1 probe, shown under construction (left) and being tested in a vacuum chamber (right) prior to its October 1998 launch. Intended mainly as a test vehicle for new technologies, the craft nevertheless carried out a successful scientific program that included a flyby of a comet.

all the exhaust has been expelled. If we define the positive direction as the direction the spacecraft is going, then the negative momentum of the exhaust is canceled by the positive momentum of the spacecraft. The ion drive allows a final speed that is ten times greater. (This simplified analysis ignores the fact that the reaction mass expelled later in the burn is not moving backward as fast, because of the forward speed of the already-moving spacecraft.)

Generalization of the momentum concept

As with all the conservation laws, the law of conservation of momentum has evolved over time. In the 1800's it was found that a beam of light striking an object would give it some momentum, even though light has no mass, and would therefore have no momentum according to the above definition. Rather than discarding the principle of conservation of momentum, the physicists of the time decided to see if the definition of momentum could be extended to include momentum carried by light. The process is analogous to the process outlined on page 317 for identifying new forms of energy. The first step was the discovery that light could impart momentum to matter, and the second step was to show that the momentum possessed by light could be related in a definite way to observable properties of the light. They found that conservation of momentum could be successfully generalized by attributing to a beam of light a momentum vector in the direction of the light's motion and having a magnitude proportional to the amount of energy the light possessed. The momentum of light is negligible under ordinary circumstances, e.g., a flashlight left on for an hour would only absorb about 10^{-5} kg·m/s of momentum as it recoiled.

b / Steam and other gases boiling off of the nucleus of Halley's comet. This close-up photo was taken by the European Giotto space probe, which passed within 596 km of the nucleus on March 13, 1986.

The tail of a comet — *example 3*

Momentum is not always equal to mv. Like many comets, Halley's comet has a very elongated elliptical orbit. About once per century, its orbit brings it close to the sun. The comet's head, or nucleus, is composed of dirty ice, so the energy deposited by the intense sunlight boils off steam and dust, b. The sunlight does not just carry energy, however — it also carries momentum. The momentum of the sunlight impacting on the smaller dust particles pushes them away from the sun, forming a tail, c. By analogy with matter, for which momentum equals mv, you would expect that massless light would have zero momentum, but the equation $p = mv$ is not the correct one for light, and light does have momentum. (The gases typically form a second, distinct tail whose motion is controlled by the sun's magnetic field.)

c / Halley's comet, in a much less magnified view from a ground-based telescope.

The reason for bringing this up is not so that you can plug numbers into a formulas in these exotic situations. The point is that the conservation laws have proven so sturdy exactly because they can easily be amended to fit new circumstances. Newton's laws are no longer at the center of the stage of physics because they did not have the same adaptability. More generally, the moral of this story is the provisional nature of scientific truth.

It should also be noted that conservation of momentum is not a consequence of Newton's laws, as is often asserted in textbooks. Newton's laws do not apply to light, and therefore could not possibly be used to prove anything about a concept as general as the conservation of momentum in its modern form.

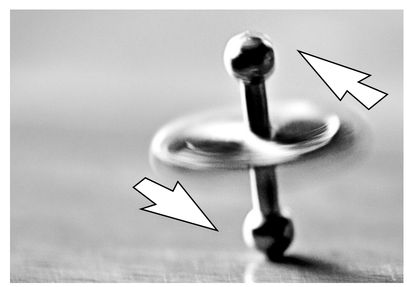
Momentum compared to kinetic energy

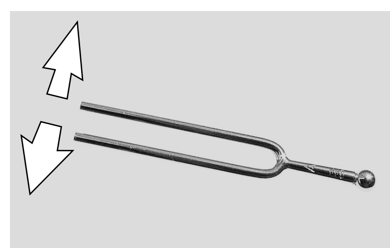
Momentum and kinetic energy are both measures of the quantity of motion, and a sideshow in the Newton-Leibnitz controversy over who invented calculus was an argument over whether mv (i.e., momentum) or mv^2 (i.e., kinetic energy without the 1/2 in front) was the "true" measure of motion. The modern student can certainly be excused for wondering why we need both quantities, when their complementary nature was not evident to the greatest minds of the 1700's. The following table highlights their differences.

kinetic energy ...	**momentum ...**
is a scalar.	is a vector.
is not changed by a force perpendicular to the motion, which changes only the direction of the velocity vector.	is changed by any force, since a change in either the magnitude or the direction of the velocity vector will result in a change in the momentum vector.
is always positive, and cannot cancel out.	cancels with momentum in the opposite direction.
can be traded for other forms of energy that do not involve motion. KE is not a conserved quantity by itself.	is always conserved in a closed system.
is quadrupled if the velocity is doubled.	is doubled if the velocity is doubled.

A spinning top *example 4*

A spinning top has zero total momentum, because for every moving point, there is another point on the opposite side that cancels its momentum. It does, however, have kinetic energy.





d / Examples 4 and 5. The momenta cancel, but the energies don't.

Why a tuning fork has two prongs *example 5*

A tuning fork is made with two prongs so that they can vibrate in opposite directions, canceling their momenta. In a hypothetical version with only one prong, the momentum would have to oscillate, and this momentum would have to come from somewhere, such as the hand holding the fork. The result would be that vibrations would be transmitted to the hand and rapidly die out. In a two-prong fork, the two momenta cancel, but the energies don't.

Momentum and kinetic energy in firing a rifle *example 6*

The rifle and bullet have zero momentum and zero kinetic energy to start with. When the trigger is pulled, the bullet gains some momentum in the forward direction, but this is canceled by the rifle's backward momentum, so the total momentum is still zero. The kinetic energies of the gun and bullet are both positive scalars, however, and do not cancel. The total kinetic energy is allowed to increase, because kinetic energy is being traded for other forms of energy. Initially there is chemical energy in the gunpowder. This chemical energy is converted into heat, sound, and kinetic energy. The gun's "backward" kinetic energy does not refrigerate the shooter's shoulder!

The wobbly earth *example 7*

As the moon completes half a circle around the earth, its motion reverses direction. This does not involve any change in kinetic energy, and the earth's gravitational force does not do any work on the moon. The reversed velocity vector does, however, imply a reversed momentum vector, so conservation of momentum in the closed earth-moon system tells us that the earth must also change its momentum. In fact, the earth wobbles in a little "orbit" about a point below its surface on the line connecting it and the moon. The two bodies' momentum vectors always point in opposite directions and cancel each other out.

The earth and moon get a divorce *example 8*

Why can't the moon suddenly decide to fly off one way and the earth the other way? It is not forbidden by conservation of momentum, because the moon's newly acquired momentum in one direction could be canceled out by the change in the momentum of the earth, supposing the earth headed the opposite direction at the appropriate, slower speed. The catastrophe is forbidden by conservation of energy, because both their energies would have to increase greatly.

Momentum and kinetic energy of a glacier *example 9*

A cubic-kilometer glacier would have a mass of about 10^{12} kg. If it moves at a speed of 10^{-5} m/s, then its momentum is 10^7 $\text{kg}\cdot\text{m}/\text{s}$. This is the kind of heroic-scale result we expect, perhaps the equivalent of the space shuttle taking off, or all the cars in LA driving in the same direction at freeway speed. Its kinetic energy, however, is only 50 J, the equivalent of the calories contained in a poppy seed or the energy in a drop of gasoline too small to be seen without a microscope. The surprisingly small kinetic energy is because kinetic energy is proportional to the square of the velocity, and the square of a small number is an even smaller number.

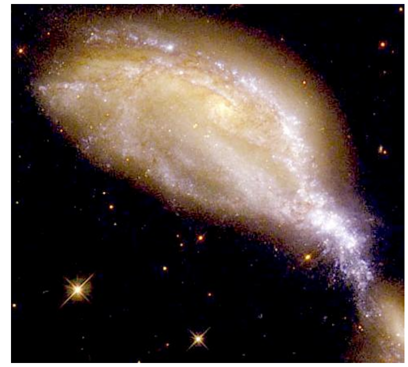
Discussion questions

A If all the air molecules in the room settled down in a thin film on the floor, would that violate conservation of momentum as well as conservation of energy?

B A refrigerator has coils in the back that get hot, and heat is molecular motion. These moving molecules have both energy and momentum. Why doesn't the refrigerator need to be tied to the wall to keep it from recoiling from the momentum it loses out the back?

14.2 Collisions in one dimension

Physicists employ the term "collision" in a broader sense than ordinary usage, applying it to any situation where objects interact for a certain period of time. A bat hitting a baseball, a radioactively emitted particle damaging DNA, and a gun and a bullet going their separate ways are all examples of collisions in this sense. Physical contact is not even required. A comet swinging past the sun on a hyperbolic orbit is considered to undergo a collision, even though it never touches the sun. All that matters is that the comet and the sun exerted gravitational forces on each other.



e / This Hubble Space Telescope photo shows a small galaxy (yellow blob in the lower right) that has collided with a larger galaxy (spiral near the center), producing a wave of star formation (blue track) due to the shock waves passing through the galaxies' clouds of gas. This is considered a collision in the physics sense, even though it is statistically certain that no star in either galaxy ever struck a star in the other. (This is because the stars are very small compared to the distances between them.)

The reason for broadening the term "collision" in this way is that all of these situations can be attacked mathematically using the same conservation laws in similar ways. In the first example, conservation of momentum is all that is required.

Getting rear-ended — *example 10*

▷ Ms. Chang is rear-ended at a stop light by Mr. Nelson, and sues to make him pay her medical bills. He testifies that he was only going 35 miles per hour when he hit Ms. Chang. She thinks he was going much faster than that. The cars skidded together after the impact, and measurements of the length of the skid marks and the coefficient of friction show that their joint velocity immediately after the impact was 19 miles per hour. Mr. Nelson's Nissan weighs 3100 pounds, and Ms. Chang 's Cadillac weighs 5200 pounds. Is Mr. Nelson telling the truth?

▷ Since the cars skidded together, we can write down the equation for conservation of momentum using only two velocities, v for Mr. Nelson's velocity before the crash, and v' for their joint velocity afterward:

$$m_N v = m_N v' + m_C v'.$$

Solving for the unknown, v, we find

$$v = \left(1 + \frac{m_C}{m_N}\right) v'.$$

Although we are given the weights in pounds, a unit of force, the ratio of the masses is the same as the ratio of the weights, and we find v = 51 miles per hour. He is lying.

The above example was simple because both cars had the same velocity afterward. In many one-dimensional collisions, however, the two objects do not stick. If we wish to predict the result of such a collision, conservation of momentum does not suffice, because both velocities after the collision are unknown, so we have one equation in two unknowns.

Conservation of energy can provide a second equation, but its application is not as straightforward, because kinetic energy is only the particular form of energy that has to do with motion. In many collisions, part of the kinetic energy that was present before the collision is used to create heat or sound, or to break the objects or permanently bend them. Cars, in fact, are carefully designed to crumple in a collision. Crumpling the car uses up energy, and that's good because the goal is to get rid of all that kinetic energy in a relatively safe and controlled way. At the opposite extreme, a superball is "super" because it emerges from a collision with almost all its original kinetic energy, having only stored it briefly as potential energy while it was being squashed by the impact.

Collisions of the superball type, in which almost no kinetic energy is converted to other forms of energy, can thus be analyzed more thoroughly, because they have $KE_f = KE_i$, as opposed to the less useful inequality $KE_f < KE_i$ for a case like a tennis ball bouncing on grass.

Pool balls colliding head-on *example 11*

▷ Two pool balls collide head-on, so that the collision is restricted to one dimension. Pool balls are constructed so as to lose as little kinetic energy as possible in a collision, so under the assumption that no kinetic energy is converted to any other form of energy, what can we predict about the results of such a collision?

▷ Pool balls have identical masses, so we use the same symbol m for both. Conservation of momentum and no loss of kinetic energy give us the two equations

$$mv_{1i} + mv_{2i} = mv_{1f} + mv_{2f}$$
$$\frac{1}{2}mv_{1i}^2 + \frac{1}{2}mv_{2i}^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2$$

The masses and the factors of 1/2 can be divided out, and we eliminate the cumbersome subscripts by replacing the symbols v_{1i},... with the symbols A, B, C, and D:

$$A + B = C + D$$
$$A^2 + B^2 = C^2 + D^2.$$

A little experimentation with numbers shows that given values of A and B, it is impossible to find C and D that satisfy these equations unless C and D equal A and B, or C and D are the same as A and B but swapped around. A formal proof of this fact is given in the sidebar. In the special case where ball 2 is initially at rest, this tells us that ball 1 is stopped dead by the collision, and ball 2 heads off at the velocity originally possessed by ball 1. This behavior will be familiar to players of pool.

Gory details of the proof in example 11

The equation $A + B = C + D$ says that the change in one ball's velocity is equal and opposite to the change in the other's. We invent a symbol $x = C - A$ for the change in ball 1's velocity. The second equation can then be rewritten as $A^2+B^2 = (A+x)^2+(B-x)^2$. Squaring out the quantities in parentheses and then simplifying, we get $0 = Ax - Bx + x^2$. The equation has the trivial solution $x = 0$, i.e., neither ball's velocity is changed, but this is physically impossible because the balls can't travel through each other like ghosts. Assuming $x \neq 0$, we can divide by x and solve for $x = B - A$. This means that ball 1 has gained an amount of velocity exactly right to match ball 2's initial velocity, and vice-versa. The balls must have swapped velocities.

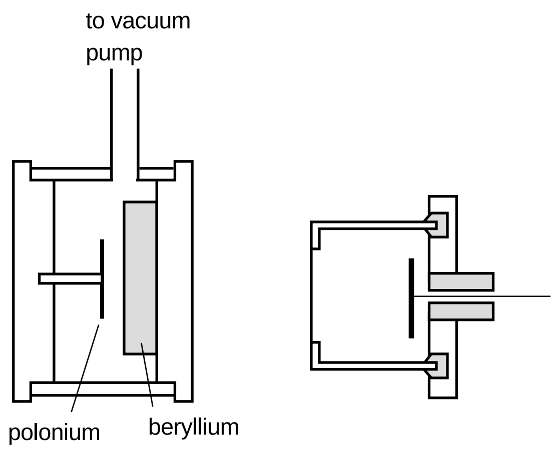
Often, as in the example above, the details of the algebra are the least interesting part of the problem, and considerable physical insight can be gained simply by counting the number of unknowns and comparing to the number of equations. Suppose a beginner at pool notices a case where her cue ball hits an initially stationary ball and stops dead. “Wow, what a good trick,” she thinks. “I bet I could never do that again in a million years.” But she tries again, and finds that she can’t help doing it even if she doesn’t want to. Luckily she has just learned about collisions in her physics course. Once she has written down the equations for conservation of energy and no loss of kinetic energy, she really doesn’t have to complete the algebra. She knows that she has two equations in two unknowns, so there must be a well-defined solution. Once she has seen the result of one such collision, she knows that the same thing must happen every time. The same thing would happen with colliding marbles or croquet balls. It doesn’t matter if the masses or velocities are different, because that just multiplies both equations by some constant factor.

The discovery of the neutron

This was the type of reasoning employed by James Chadwick in his 1932 discovery of the neutron. At the time, the atom was imagined to be made out of two types of fundamental particles, protons and electrons. The protons were far more massive, and clustered together in the atom’s core, or nucleus. Attractive electrical forces caused the electrons to orbit the nucleus in circles, in much the same way that gravitational forces kept the planets from cruising out of the solar system. Experiments showed that the helium nucleus, for instance, exerted exactly twice as much electrical force on an electron as a nucleus of hydrogen, the smallest atom, and this was explained by saying that helium had two protons to hydrogen’s one. The trouble was that according to this model, helium would have two electrons and two protons, giving it precisely twice the mass of a hydrogen atom with one of each. In fact, helium has about four times the mass of hydrogen.

Chadwick suspected that the helium nucleus possessed two additional particles of a new type, which did not participate in electrical forces at all, i.e., were electrically neutral. If these particles had very nearly the same mass as protons, then the four-to-one mass ratio of helium and hydrogen could be explained. In 1930, a new type of radiation was discovered that seemed to fit this description. It was electrically neutral, and seemed to be coming from the nuclei of light elements that had been exposed to other types of radiation. At this time, however, reports of new types of particles were a dime a dozen, and most of them turned out to be either clusters made of previously known particles or else previously known particles with higher energies. Many physicists believed that the “new” particle that had attracted Chadwick’s interest was really a previously known particle called a gamma ray, which was electrically neutral. Since gamma rays have no mass, Chadwick decided to try to determine the new particle’s mass and see if it was nonzero and approximately equal to the mass of a proton.

Unfortunately a subatomic particle is not something you can just put on a scale and weigh. Chadwick came up with an ingenious solution. The masses of the nuclei of the various chemical elements were already known, and techniques had already been developed for measuring the speed of a rapidly moving nucleus. He therefore set out to bombard samples of selected elements with the mysterious new particles. When a direct, head-on collision occurred between a mystery particle and the nucleus of one of the target atoms, the nucleus would be knocked out of the atom, and he would measure its velocity.



f / Chadwick's subatomic pool table. A disk of the naturally occurring metal polonium provides a source of radiation capable of kicking neutrons out of the beryllium nuclei. The type of radiation emitted by the polonium is easily absorbed by a few mm of air, so the air has to be pumped out of the left-hand chamber. The neutrons, Chadwick's mystery particles, penetrate matter far more readily, and fly out through the wall and into the chamber on the right, which is filled with nitrogen or hydrogen gas. When a neutron collides with a nitrogen or hydrogen nucleus, it kicks it out of its atom at high speed, and this recoiling nucleus then rips apart thousands of other atoms of the gas. The result is an electrical pulse that can be detected in the wire on the right. Physicists had already calibrated this type of apparatus so that they could translate the strength of the electrical pulse into the velocity of the recoiling nucleus. The whole apparatus shown in the figure would fit in the palm of your hand, in dramatic contrast to today's giant particle accelerators.

Suppose, for instance, that we bombard a sample of hydrogen atoms with the mystery particles. Since the participants in the collision are fundamental particles, there is no way for kinetic energy to be converted into heat or any other form of energy, and Chadwick thus had two equations in three unknowns:

equation #1: conservation of momentum

equation #2: no loss of kinetic energy

unknown #1: mass of the mystery particle

unknown #2: initial velocity of the mystery particle

unknown #3: final velocity of the mystery particle

The number of unknowns is greater than the number of equations, so there is no unique solution. But by creating collisions with nuclei of another element, nitrogen, he gained two more equations at the expense of only one more unknown:

equation #3: conservation of momentum in the new collision

equation #4: no loss of kinetic energy in the new collision

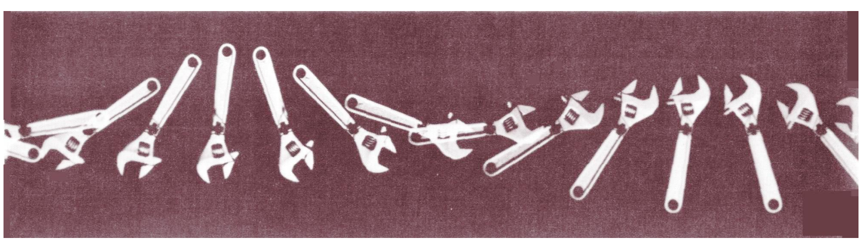
unknown #4: final velocity of the mystery particle in the new collision

He was thus able to solve for all the unknowns, including the mass of the mystery particle, which was indeed within 1% of the mass of a proton. He named the new particle the neutron, since it is electrically neutral.

Discussion question

A Good pool players learn to make the cue ball spin, which can cause it not to stop dead in a head-on collision with a stationary ball. If this does not violate the laws of physics, what hidden assumption was there in the example above?

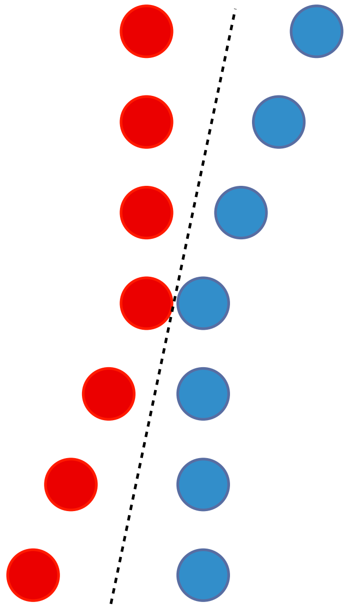
14.3 ⋆ Relationship of momentum to the center of mass



g / In this multiple-flash photograph, we see the wrench from above as it flies through the air, rotating as it goes. Its center of mass, marked with the black cross, travels along a straight line, unlike the other points on the wrench, which execute loops.

We have already discussed the idea of the center of mass on p. 61, but using the concept of momentum we can now find a mathematical method for defining the center of mass, explain why the motion of an object's center of mass usually exhibits simpler motion than any other point, and gain a very simple and powerful way of understanding collisions.

The first step is to realize that the center of mass concept can be applied to systems containing more than one object. Even something like a wrench, which we think of as one object, is really made of many atoms. The center of mass is particularly easy to visualize in the case shown on the left, where two identical hockey pucks col-


h / Two hockey pucks collide. Their mutual center of mass traces the straight path shown by the dashed line.

lide. It is clear on grounds of symmetry that their center of mass must be at the midpoint between them. After all, we previously defined the center of mass as the balance point, and if the two hockey pucks were joined with a very lightweight rod whose own mass was negligible, they would obviously balance at the midpoint. It doesn't matter that the hockey pucks are two separate objects. It is still true that the motion of their center of mass is exceptionally simple, just like that of the wrench's center of mass.

The x coordinate of the hockey pucks' center of mass is thus given by $x_{cm} = (x_1 + x_2)/2$, i.e., the arithmetic average of their x coordinates. Why is its motion so simple? It has to do with conservation of momentum. Since the hockey pucks are not being acted on by any net external force, they constitute a closed system, and their total momentum is conserved. Their total momentum is

$$\begin{aligned} mv_1 + mv_2 &= m(v_1 + v_2) \\ &= m\left(\frac{\Delta x_1}{\Delta t} + \frac{\Delta x_2}{\Delta t}\right) \\ &= \frac{m}{\Delta t}\Delta\left(x_1 + x_2\right) \\ &= m\frac{2\Delta x_{cm}}{\Delta t} \\ &= m_{total}v_{cm} \end{aligned}$$

In other words, the total momentum of the system is the same as if all its mass was concentrated at the center of mass point. Since the total momentum is conserved, the x component of the center of mass's velocity vector cannot change. The same is also true for the other components, so the center of mass must move along a straight line at constant speed.

The above relationship between the total momentum and the motion of the center of mass applies to any system, even if it is not closed.

total momentum related to center of mass motion
The total momentum of any system is related to its total mass and the velocity of its center of mass by the equation

$$\mathbf{p}_{total} = m_{total}\mathbf{v}_{cm}.$$

What about a system containing objects with unequal masses, or containing more than two objects? The reasoning above can be generalized to a weighted average

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \ldots}{m_1 + m_2 + \ldots},$$

with similar equations for the y and z coordinates.

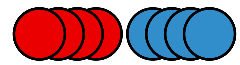
Momentum in different frames of reference

Absolute motion is supposed to be undetectable, i.e., the laws of physics are supposed to be equally valid in all inertial frames of reference. If we first calculate some momenta in one frame of reference and find that momentum is conserved, and then rework the whole problem in some other frame of reference that is moving with respect to the first, the numerical values of the momenta will all be different. Even so, momentum will still be conserved. All that matters is that we work a single problem in one consistent frame of reference.

One way of proving this is to apply the equation $\mathbf{p}_{total} = m_{total}\mathbf{v}_{cm}$. If the velocity of frame B relative to frame A is $\mathbf{v}_{BA}$, then the only effect of changing frames of reference is to change $\mathbf{v}_{cm}$ from its original value to $\mathbf{v}_{cm} + \mathbf{v}_{BA}$. This adds a constant onto the momentum vector, which has no effect on conservation of momentum.

The center of mass frame of reference

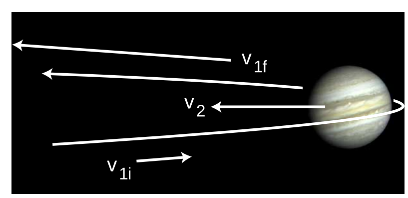
A particularly useful frame of reference in many cases is the frame that moves along with the center of mass, called the center of mass (c.m.) frame. In this frame, the total momentum is zero. The following examples show how the center of mass frame can be a powerful tool for simplifying our understanding of collisions.



i / Moving your head so that you are always looking down from right above the center of mass, you observe the collision of the two hockey pucks in the center of mass frame.

A collision of pool balls viewed in the c.m. frame — *example 12*

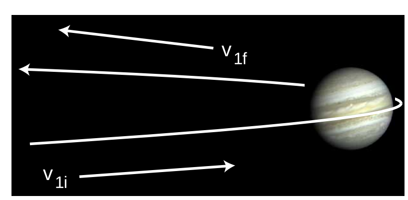
If you move your head so that your eye is always above the point halfway in between the two pool balls, you are viewing things in the center of mass frame. In this frame, the balls come toward the center of mass at equal speeds. By symmetry, they must therefore recoil at equal speeds along the lines on which they entered. Since the balls have essentially swapped paths in the center of mass frame, the same must also be true in any other frame. This is the same result that required laborious algebra to prove previously without the concept of the center of mass frame.



j / The slingshot effect viewed in the sun's frame of reference. Jupiter is moving to the left, and the collision is head-on.

The slingshot effect — *example 13*

It is a counterintuitive fact that a spacecraft can pick up speed by swinging around a planet, if it arrives in the opposite direction compared to the planet's motion. Although there is no physical contact, we treat the encounter as a one-dimensional collision, and analyze it in the center of mass frame. Figure j shows such a "collision," with a space probe whipping around Jupiter. In the sun's frame of reference, Jupiter is moving.



k / The slingshot viewed in the frame of the center of mass of the Jupiter-spacecraft system.

What about the center of mass frame? Since Jupiter is so much more massive than the spacecraft, the center of mass is essentially fixed at Jupiter's center, and Jupiter has zero velocity in the center of mass frame, as shown in figure k. The c.m. frame is moving to the left compared to the sun-fixed frame used in j, so the spacecraft's initial velocity is greater in this frame.

Things are simpler in the center of mass frame, because it is more symmetric. In the complicated sun-fixed frame, the incoming leg of the encounter is rapid, because the two bodies are rushing toward each other, while their separation on the outbound leg is more gradual, because Jupiter is trying to catch up. In the c.m. frame, Jupiter is sitting still, and there is perfect symmetry between the incoming and outgoing legs, so by symmetry we have $v_{1f} = -v_{1i}$. Going back to the sun-fixed frame, the spacecraft's final velocity is increased by the frames' motion relative to each other. In the sun-fixed frame, the spacecraft's velocity has increased greatly.

The result can also be understood in terms of work and energy. In Jupiter's frame, Jupiter is not doing any work on the spacecraft as it rounds the back of the planet, because the motion is perpendicular to the force. But in the sun's frame, the spacecraft's velocity vector at the same moment has a large component to the left, so Jupiter is doing work on it.

Discussion questions

A Make up a numerical example of two unequal masses moving in one dimension at constant velocity, and verify the equation $p_{total} = m_{total}v_{cm}$ over a time interval of one second.

B A more massive tennis racquet or baseball bat makes the ball fly off faster. Explain why this is true, using the center of mass frame. For simplicity, assume that the racquet or bat is simply sitting still before the collision, and that the hitter's hands do not make any force large enough to have a significant effect over the short duration of the impact.

14.4 Momentum transfer

The rate of change of momentum

As with conservation of energy, we need a way to measure and calculate the transfer of momentum into or out of a system when the system is not closed. In the case of energy, the answer was rather complicated, and entirely different techniques had to be used for measuring the transfer of mechanical energy (work) and the transfer of heat by conduction. For momentum, the situation is far simpler.

In the simplest case, the system consists of a single object acted on by a constant external force. Since it is only the object's velocity that can change, not its mass, the momentum transferred is

$$\Delta \mathbf{p} = m\Delta \mathbf{v},$$

which with the help of $\mathbf{a} = \mathbf{F}/m$ and the constant-acceleration equation $\mathbf{a} = \Delta\mathbf{v}/\Delta t$ becomes

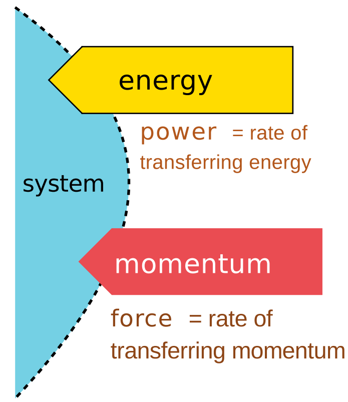
$$\begin{aligned}\Delta \mathbf{p} &= m\mathbf{a}\Delta t\\ &= \mathbf{F}\Delta t.\end{aligned}$$

Thus the rate of transfer of momentum, i.e., the number of kg·m/s absorbed per second, is simply the external force,

$$\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta t}.$$

[relationship between the force on an object and the rate of change of its momentum; valid only if the force is constant]

This is just a restatement of Newton's second law, and in fact Newton originally stated it this way. As shown in figure l, the relationship between force and momentum is directly analogous to that between power and energy.



l / Power and force are the rates at which energy and momentum are transferred.

The situation is not materially altered for a system composed of many objects. There may be forces between the objects, but the internal forces cannot change the system's momentum. (If they did, then removing the external forces would result in a closed system that could change its own momentum, like the mythical man who could pull himself up by his own bootstraps. That would violate conservation of momentum.) The equation above becomes

$$\mathbf{F}_{total} = \frac{\Delta \mathbf{p}_{total}}{\Delta t}.$$

[relationship between the total external force on a system and the rate of change of its total momentum; valid only if the force is constant]

Walking into a lamppost — *example 14*

▷ Starting from rest, you begin walking, bringing your momentum up to 100 kg·m/s. You walk straight into a lamppost. Why is the momentum change of −100 kg·m/s caused by the lamppost so much more painful than the change of +100 kg·m/s when you started walking?

▷ The situation is one-dimensional, so we can dispense with the vector notation. It probably takes you about 1 s to speed up initially, so the ground's force on you is $F = \Delta p/\Delta t \approx 100$ N. Your impact with the lamppost, however, is over in the blink of an eye, say 1/10 s or less. Dividing by this much smaller Δt gives a much larger force, perhaps thousands of newtons. (The negative sign simply indicates that the force is in the opposite direction.)

This is also the principle of airbags in cars. The time required for the airbag to decelerate your head is fairly long, the time required for your face to travel 20 or 30 cm. Without an airbag, your face would hit the dashboard, and the time interval would be the much shorter time taken by your skull to move a couple of centimeters while your face compressed. Note that either way, the same amount of mechanical work has to be done on your head: enough to eliminate all its kinetic energy.

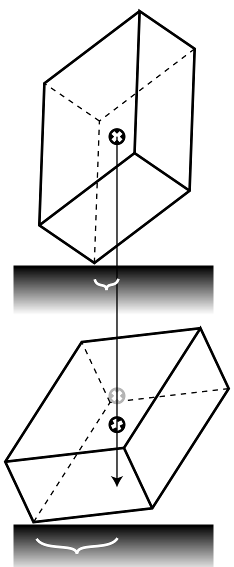
m / The airbag increases Δt so as to reduce $F = \Delta p/\Delta t$.

Ion drive for spacecraft *example 15*

▷ The ion drive of the Deep Space 1 spacecraft, pictured on page 395 and discussed in example 2, produces a thrust of 90 mN (millinewtons). It carries about 80 kg of reaction mass, which it ejects at a speed of 30,000 m/s. For how long can the engine continue supplying this amount of thrust before running out of reaction mass to shove out the back?

▷ Solving the equation $F = \Delta p/\Delta t$ for the unknown Δt, and treating force and momentum as scalars since the problem is one-dimensional, we find

$$\begin{aligned}\Delta t &= \frac{\Delta p}{F}\\ &= \frac{m_{exhaust}\Delta v_{exhaust}}{F}\\ &= \frac{(80\ \text{kg})(30,000\ \text{m/s})}{0.090\ \text{N}}\\ &= 2.7\times10^{7}\ \text{s}\\ &= 300\ \text{days}\end{aligned}$$



n / Example 16.

A toppling box *example 16*

If you place a box on a frictionless surface, it will fall over with a very complicated motion that is hard to predict in detail. We know, however, that its center of mass moves in the same direction as its momentum vector points. There are two forces, a normal force and a gravitational force, both of which are vertical. (The gravitational force is actually many gravitational forces acting on all the atoms in the box.) The total force must be vertical, so the momentum vector must be purely vertical too, and the center of mass travels vertically. This is true even if the box bounces and tumbles. [Based on an example by Kleppner and Kolenkow.]

Discussion question

A Many collisions, like the collision of a bat with a baseball, appear to be instantaneous. Most people also would not imagine the bat and ball as bending or being compressed during the collision. Consider the following possibilities:

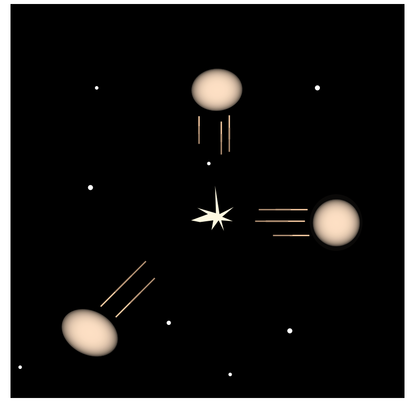
1. The collision is instantaneous.
2. The collision takes a finite amount of time, during which the ball and bat retain their shapes and remain in contact.
3. The collision takes a finite amount of time, during which the ball and bat are bending or being compressed.

How can two of these be ruled out based on energy or momentum considerations?

14.5 Momentum in three dimensions

In this section we discuss how the concepts applied previously to one-dimensional situations can be used as well in three dimensions. Often vector addition is all that is needed to solve a problem:

An explosion — *example 17*



o / Example 17.

▷ Astronomers observe the planet Mars as the Martians fight a nuclear war. The Martian bombs are so powerful that they rip the planet into three separate pieces of liquified rock, all having the same mass. If one fragment flies off with velocity components

$$v_{1x} = 0$$
$$v_{1y} = 1.0 \times 10^4 \text{ km/hr},$$

and the second with

$$v_{2x} = 1.0 \times 10^4 \text{ km/hr}$$
$$v_{2y} = 0,$$

(all in the center of mass frame) what is the magnitude of the third one's velocity?

▷ In the center of mass frame, the planet initially had zero momentum. After the explosion, the vector sum of the momenta must still be zero. Vector addition can be done by adding components, so

$$mv_{1x} + mv_{2x} + mv_{3x} = 0, \quad \text{and}$$
$$mv_{1y} + mv_{2y} + mv_{3y} = 0,$$

where we have used the same symbol m for all the terms, because the fragments all have the same mass. The masses can be eliminated by dividing each equation by m, and we find

$$v_{3x} = -1.0 \times 10^4 \text{ km/hr}$$
$$v_{3y} = -1.0 \times 10^4 \text{ km/hr}$$

which gives a magnitude of

$$|\mathbf{v}_3| = \sqrt{v_{3x}^2 + v_{3y}^2}$$
$$= 1.4 \times 10^4 \text{ km/hr}$$

The center of mass

In three dimensions, we have the vector equations

$$\mathbf{F}_{total} = \frac{\Delta \mathbf{p}_{total}}{\Delta t}$$

and

$$\mathbf{p}_{total} = m_{total}\mathbf{v}_{cm}.$$

The following is an example of their use.

The bola — *example 18*

The bola, similar to the North American lasso, is used by South American gauchos to catch small animals by tangling up their legs in the three leather thongs. The motion of the whirling bola through the air is extremely complicated, and would be a challenge to analyze mathematically. The motion of its center of mass, however, is much simpler. The only forces on it are gravitational, so

$$\mathbf{F}_{total} = m_{total}\mathbf{g}.$$

Using the equation $\mathbf{F}_{total} = \Delta\mathbf{p}_{total}/\Delta t$, we find

$$\Delta\mathbf{p}_{total}/\Delta t = m_{total}\mathbf{g},$$

and since the mass is constant, the equation $\mathbf{p}_{total} = m_{total}\mathbf{v}_{cm}$ allows us to change this to

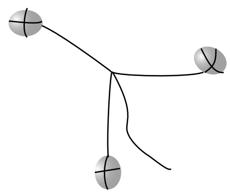
$$m_{total}\Delta\mathbf{v}_{cm}/\Delta t = m_{total}\mathbf{g}.$$

The mass cancels, and $\Delta\mathbf{v}_{cm}/\Delta t$ is simply the acceleration of the center of mass, so

$$\mathbf{a}_{cm} = \mathbf{g}.$$

In other words, the motion of the system is the same as if all its mass was concentrated at and moving with the center of mass. The bola has a constant downward acceleration equal to g, and flies along the same parabola as any other projectile thrown with the same initial center of mass velocity. Throwing a bola with the correct rotation is presumably a difficult skill, but making it hit its target is no harder than it is with a ball or a single rock.

[Based on an example by Kleppner and Kolenkow.]



p / Example 18.

Counting equations and unknowns

Counting equations and unknowns is just as useful as in one dimension, but every object's momentum vector has three components, so an unknown momentum vector counts as three unknowns. Conservation of momentum is a single vector equation, but it says that all three components of the total momentum vector stay constant, so we count it as three equations. Of course if the motion happens to be confined to two dimensions, then we need only count vectors as having two components.

A two-car crash with sticking — *example 19*

Suppose two cars collide, stick together, and skid off together. If we know the cars' initial momentum vectors, we can count equations and unknowns as follows:

unknown #1: x component of cars' final, total momentum

unknown #2: y component of cars' final, total momentum

equation #1: conservation of the total p_x

equation #2: conservation of the total p_y

Since the number of equations equals the number of unknowns, there must be one unique solution for their total momentum vector after the crash. In other words, the speed and direction at which their common center of mass moves off together is unaffected by factors such as whether the cars collide center-to-center or catch each other a little off-center.

Shooting pool *example 20*

Two pool balls collide, and as before we assume there is no decrease in the total kinetic energy, i.e., no energy converted from KE into other forms. As in the previous example, we assume we are given the initial velocities and want to find the final velocities. The equations and unknowns are:

unknown #1: x component of ball #1's final momentum

unknown #2: y component of ball #1's final momentum

unknown #3: x component of ball #2's final momentum

unknown #4: y component of ball #2's final momentum

equation #1: conservation of the total p_x

equation #2: conservation of the total p_y

equation #3: no decrease in total KE

Note that we do not count the balls' final kinetic energies as unknowns, because knowing the momentum vector, one can always find the velocity and thus the kinetic energy. The number of equations is less than the number of unknowns, so no unique result is guaranteed. This is what makes pool an interesting game. By aiming the cue ball to one side of the target ball you can have some control over the balls' speeds and directions of motion after the collision.

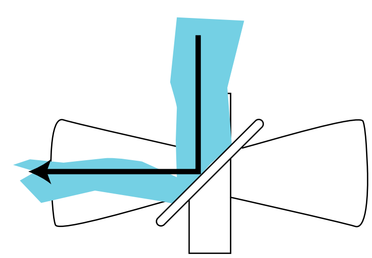
It is not possible, however, to choose any combination of final speeds and directions. For instance, a certain shot may give the correct direction of motion for the target ball, making it go into a pocket, but may also have the undesired side-effect of making the cue ball go in a pocket.

Calculations with the momentum vector

The following example illustrates how a force is required to change the direction of the momentum vector, just as one would be required to change its magnitude.

A turbine *example 21*

▷ In a hydroelectric plant, water flowing over a dam drives a turbine, which runs a generator to make electric power. The figure shows a simplified physical model of the water hitting the turbine, in which it is assumed that the stream of water comes in at a



q / Example 21.

45°angle with respect to the turbine blade, and bounces off at a 90°angle at nearly the same speed. The water flows at a rate R, in units of kg/s, and the speed of the water is v. What are the magnitude and direction of the water's force on the turbine?

▷ In a time interval Δt, the mass of water that strikes the blade is $R\Delta$t, and the magnitude of its initial momentum is $mv = vR\Delta t$. The water's final momentum vector is of the same magnitude, but in the perpendicular direction. By Newton's third law, the water's force on the blade is equal and opposite to the blade's force on the water. Since the force is constant, we can use the equation

$$F_{\text{blade on water}} = \frac{\Delta \mathbf{p}_{water}}{\Delta t}.$$

Choosing the x axis to be to the right and the y axis to be up, this can be broken down into components as

$$\begin{aligned} F_{\text{blade on water},x} &= \frac{\Delta p_{\text{water},x}}{\Delta t} \\ &= \frac{-vR\Delta t - 0}{\Delta t} \\ &= -vR \end{aligned}$$

and

$$\begin{aligned} F_{\text{blade on water},y} &= \frac{\Delta p_{\text{water},y}}{\Delta t} \\ &= \frac{0 - (-vR\Delta t)}{\Delta t} \\ &= vR. \end{aligned}$$

The water's force on the blade thus has components

$$\begin{aligned} F_{\text{water on blade},x} &= vR \\ F_{\text{water on blade},y} &= -vR. \end{aligned}$$

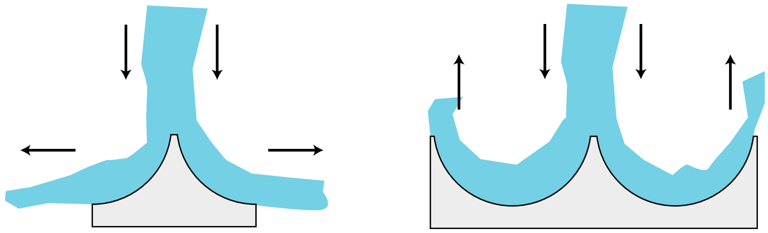
In situations like this, it is always a good idea to check that the result makes sense physically. The x component of the water's force on the blade is positive, which is correct since we know the blade will be pushed to the right. The y component is negative, which also makes sense because the water must push the blade down. The magnitude of the water's force on the blade is

$$|F_{\text{water on blade}}| = \sqrt{2}vR$$

and its direction is at a 45-degree angle down and to the right.

Discussion questions

A The figures show a jet of water striking two different objects. How does the total downward force compare in the two cases? How could this fact be used to create a better waterwheel? (Such a waterwheel is known as a Pelton wheel.)

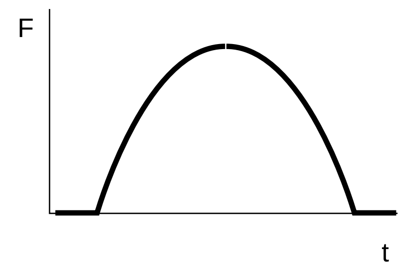


Discussion question A.

14.6 Applications of calculus

Few real collisions involve a constant force. For example, when a tennis ball hits a racquet, the strings stretch and the ball flattens dramatically. They are both acting like springs that obey Hooke's law, which says that the force is proportional to the amount of stretching or flattening. The force is therefore small at first, ramps up to a maximum when the ball is about to reverse directions, and ramps back down again as the ball is on its way back out. The equation $F = \Delta p/\Delta t$, derived under the assumption of constant acceleration, does not apply here, and the force does not even have a single well-defined numerical value that could be plugged in to the equation.

This is like every other situation where an equation of the form $foo = \Delta bar/\Delta baz$ has to be generalized to the case where the rate of change isn't constant. We have $F = \mathrm{d}p/\mathrm{d}t$ and, by the fundamental theorem of calculus, $\Delta p = \int F\,\mathrm{d}t$, which can be interpreted as the area under the $F-t$ graph, figure r.



r / The $F-t$ graph for a tennis racquet hitting a ball might look like this. The amount of momentum transferred equals the area under the curve.

Rain falling into a moving cart — *example 22*

▷ If 1 kg/s of rain falls vertically into a 10-kg cart that is rolling without friction at an initial speed of 1.0 m/s, what is the effect on the speed of the cart when the rain first starts falling?

▷ The rain and the cart make horizontal forces on each other, but there is no external horizontal force on the rain-plus-cart system, so the horizontal motion obeys

$$F = \frac{\mathrm{d}(mv)}{\mathrm{d}t} = 0$$

We use the product rule to find

$$0 = \frac{\mathrm{d}m}{\mathrm{d}t}v + m\frac{\mathrm{d}v}{\mathrm{d}t}.$$

We are trying to find how v changes, so we solve for dv/dt,

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{v}{m}\frac{\mathrm{d}m}{\mathrm{d}t}\\ &= -\left(\frac{1\ \mathrm{m/s}}{10\ \mathrm{kg}}\right)(1\ \mathrm{kg/s})\\ &= -0.1\ \mathrm{m/s^2}.\end{aligned}$$

(This is only at the moment when the rain starts to fall.)

Finally we note that there are cases where $F = ma$ is not just less convenient than $F = \mathrm{d}p/\,\mathrm{d}t$ but in fact $F = ma$ is wrong and $F = \mathrm{d}p/\,\mathrm{d}t$ is right. A good example is the formation of a comet's tail by sunlight. We cannot use $F = ma$ to describe this process, since we are dealing with a collision of light with matter, whereas Newton's laws only apply to matter. The equation $F = \mathrm{d}p/\,\mathrm{d}t$, on the other hand, allows us to find the force experienced by an atom of gas in the comet's tail if we know the rate at which the momentum vectors of light rays are being turned around by reflection from the atom.

14.7 ⋆ Relativistic momentum

How does momentum behave in relativity?

Newtonian mechanics has two different measures of motion, kinetic energy and momentum, and the relationship between them is nonlinear. Doubling your car's momentum quadruples its kinetic energy.

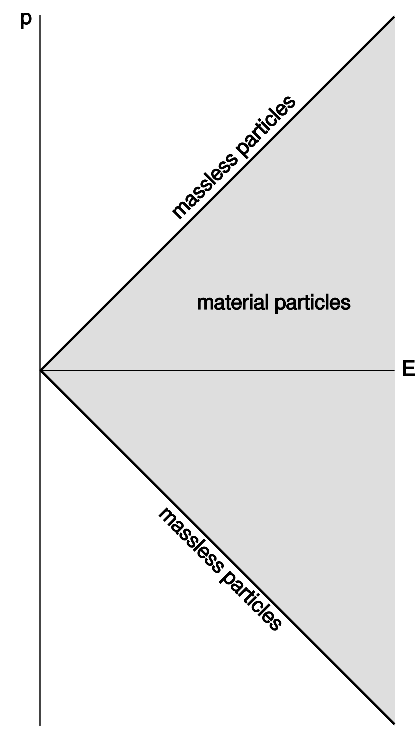
But nonrelativistic mechanics can't handle massless particles, which are always ultrarelativistic. We saw in section 11.6 that ultrarelativistic particles are "generic," in the sense that they have no individual mechanical properties other than an energy and a direction of motion. Therefore the relationship between kinetic energy and momentum must be *linear* for ultrarelativistic particles. Indeed, experiments verify that light has momentum, and doubling the energy of a ray of light doubles its momentum rather than quadrupling it.

How can we make sense of these energy-momentum relationships, which seem to take on two completely different forms in the limiting cases of very low and very high velocities?

The first step is realize that since mass and energy are equivalent (section 12.5), we will get more of an apples-to-apples comparison if we stop talking about a material object's *kinetic* energy and consider instead its *total* energy E, which includes a contribution from its mass.

On a graph of p versus E, massless particles, which have $E \propto |p|$, lie on two diagonal lines that connect at the origin. If we like, we can pick units such that the slopes of these lines are plus and minus one. Material particles lie to the right of these lines. For example, a car sitting in a parking lot has $p = 0$ and $E = mc^2$.

Now what happens to such a graph when we change to a different frame or reference that is in motion relative to the original frame? A massless particle still has to act like a massless particle, so the diagonals are simply stretched or contracted along their own lengths. A transformation that always takes a line to a line is a linear transformation (p. 82), and if the transformation between different frames of reference preserves the linearity of the lines $p = E$ and $p = -E$, then it's natural to suspect that it is actually some kind of linear transformation. In fact the transformation must be linear (p. 82), because conservation of energy and momentum involve addition, and we need these laws to be valid in all frames of reference. By the same reasoning as in figure am on p. 85, the transformation must be area-preserving. We then have the same three cases to consider as in figure aj on p. 84. Case I is ruled out because it would imply that particles keep the same energy when we change frames. (This is what would happen if c were infinite, so that the mass-equivalent E/c^2 of a given energy was zero, and therefore E would be interpreted purely as the mass.) Case II can't be right because it doesn't preserve the $E = |p|$ diagonals. We are left with case III, which establishes the following aesthetically appealing fact: *the p-E plane transforms according to exactly the same kind of Lorentz transformation as the x-t plane.* That is, (E, p_x, p_y, p_z) is a four-vector (p. 378) just like (t, x, y, z). This is a highly desirable result. If it were not true, it would be like having to learn different mathematical rules for different kinds of three-vectors in Newtonian mechanics.



s / In the p-E plane, massless particles lie on the two diagonals, while particles with mass lie to the right.

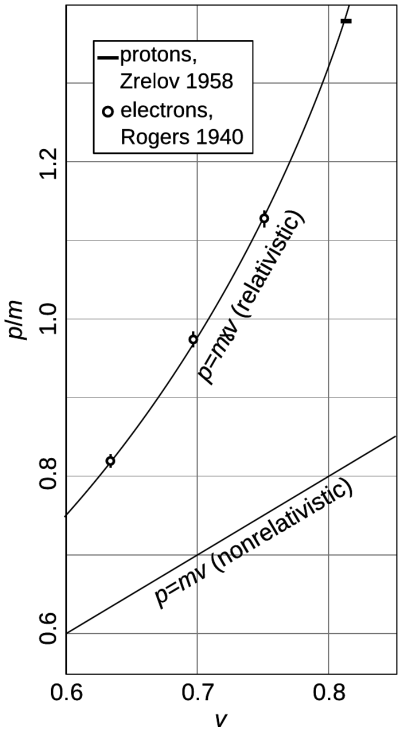
The only remaining issue to settle is whether the choice of units that gives invariant 45-degree diagonals in the x-t plane is the same as the choice of units that gives such diagonals in the p-E plane. That is, we need to establish that the c that applies to x and t is equal to the c' needed for p and E, i.e., that the velocity scales of the two graphs are matched up. This is true because in the Newtonian limit, the total mass-energy E is essentially just the particle's mass, and then $p/E \approx p/m \approx v$. This establishes that the velocity scales are matched at small velocities, which implies that they coincide for all velocities, since a large velocity, even one approaching c, can be built up from many small increments. (This also establishes that the exponent n defined on p. 327 equals 1 as claimed.)

Suppose that a particle is at rest. Then it has $p = 0$ and mass-energy E equal to its mass m. Therefore the inner product of its (E, p) four-vector with itself equals m^2. In other words, the "magnitude" of the energy-momentum four-vector is simply equal to the particle's mass. If we transform into a different frame of reference, the inner product stays the same. We can therefore always interpret the magnitude of an energy-momentum four-vector as the mass. In symbols,

$$m^2 = E^2 - p^2,$$

or, in units with $c \neq 1$,

$$(mc^2)^2 = E^2 - (pc)^2.$$



t / Two early high-precision tests of the relativistic equation $p = m\gamma v$ for momentum. Graphing p/m rather than p allows the data for electrons and protons to be placed on the same graph. Natural units are used, so that the horizontal axis is the velocity in units of c, and the vertical axis is the unitless quantity p/mc. The very small error bars for the data point from Zrelov are represented by the height of the black rectangle.

self-check A

Interpret this relationship in the case where $m = 0$. ▷ Answer, p. 561

Since we already have an equation $E = m\gamma$ for the energy of a material particle in terms of its velocity, we can find a similar equation for the momentum:

$$\begin{aligned} p &= \sqrt{E^2 - m^2} \\ &= m\sqrt{\gamma^2 - 1} \\ &= m\sqrt{\frac{1}{1-v^2} - 1} \\ &= m\gamma v. \end{aligned}$$

As a material particle gets closer and closer to c, its momentum approaches infinity, so that an infinite force would be required in order to reach c. Figure t shows experimental data confirming the relativistic equation for momentum.

Light rays don't interact *example 23*

We observe that when two rays of light cross paths, they continue through one another without bouncing like material objects. This behavior follows directly from conservation of energy-momentum.

Any two vectors can be contained in a single plane, so we can choose our coordinates so that both rays have vanishing p_z. By choosing the state of motion of our coordinate system appropriately, we can also make $p_y = 0$, so that the collision takes place along a single line parallel to the x axis. Since only p_x is nonzero, we write it simply as p. In the resulting p-E plane, there are two possibilities: either the rays both lie along the same diagonal, or they lie along different diagonals. If they lie along the same diagonal, then there can't be a collision, because the two rays are both moving in the same direction at the same speed c, and the trailing one will never catch up with the leading one.

Now suppose they lie along different diagonals. We add their energy-momentum vectors to get their total energy-momentum, which will lie in the gray area of figure s. That is, a pair of light rays taken as a single system act sort of like a material object with a nonzero mass.[1] By a Lorentz transformation, we can always find a frame in which this total energy-momentum vector lies along the E axis. This is a frame in which the momenta of the two rays cancel, and we have a symmetric head-on collision between two rays of equal energy. It is the "center-of-mass" frame, although neither object has any mass on an individual basis. For convenience, let's assume that the x-y-z coordinate system was chosen so that its origin was at rest in this frame.

[1] If you construct a box out of mirrors and put some light inside, it has weight, and theoretically even has a gravitational field! This is an example of the fact that mass is not additive in relativity. Two objects, each with zero mass, can have an aggregate mass that is nonzero.

Since the collision occurs along the x axis, by symmetry it is not possible for the rays after the collision to depart from the x axis; for if they did, then there would be nothing to determine the orientation of the plane in which they emerged.[2] Therefore we are justified in continuing to use the same p_x-E plane to analyze the four-vectors of the rays after the collision.

Let each ray have energy E in the frame described above. Given this total energy-momentum vector, how can we cook up two energy-momentum vectors for the final state such that energy and momentum will have been conserved? Since there is zero total momentum, our only choice is two light rays, one with energy-momentum vector (E, E) and one with $(E, -E)$. But this is exactly the same as our initial state, except that we can arbitrarily choose the roles of the two rays to have been interchanged. Such an interchanging is only a matter of labeling, so there is no observable sense in which the rays have collided.[3]

[2] In quantum mechanics, there is a loophole here. Quantum mechanics allows certain kinds of randomness, so that the symmetry can be broken by letting the outgoing rays be observed in a plane with some random orientation.

[3] There is a second loophole here, which is that a ray of light is actually a wave, and a wave has other properties besides energy and momentum. It has a wavelength, and some waves also have a property called polarization. As a mechanical analogy for polarization, consider a rope stretched taut. Side-to-side vibrations can propagate along the rope, and these vibrations can occur in any plane that coincides with the rope. The orientation of this plane is referred to as the polarization of the wave. Returning to the case of the colliding light rays, it is possible to have nontrivial collisions in the sense that the rays could affect one another's wavelengths and polarizations. Although this doesn't actually happen with non-quantum-mechanical light waves, it can happen with other types of waves; see, e.g., Hu et al., `arxiv.org/abs/hep-ph/9502276`, figure 2. The title of example 23 is only valid if a "ray" is taken to be something that lacks wave structure. The wave nature of light is not evident in everyday life from observations with apparatus such as flashlights, mirrors, and eyeglasses, so we expect the result to hold under those circumstances, and it does. E.g., flashlight beams do pass through one anther without interacting.

Summary

Selected vocabulary

momentum . . .	a measure of motion, equal to mv for material objects
collision	an interaction between moving objects that lasts for a certain time
center of mass . .	the balance point or average position of the mass in a system

Notation

$\mathbf{p}$	the momentum vector
cm	center of mass, as in x_{cm}, a_{cm}, etc.

Other terminology and notation

impulse, I, J . .	the amount of momentum transferred, Δp
elastic collision .	one in which no KE is converted into other forms of energy
inelastic collision	one in which some KE is converted to other forms of energy

Summary

If two objects interact via a force, Newton's third law guarantees that any change in one's velocity vector will be accompanied by a change in the other's which is in the opposite direction. Intuitively, this means that if the two objects are not acted on by any external force, they cannot cooperate to change their overall state of motion. This can be made quantitative by saying that the quantity $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ must remain constant as long as the only forces are the internal ones between the two objects. This is a conservation law, called the conservation of momentum, and like the conservation of energy, it has evolved over time to include more and more phenomena unknown at the time the concept was invented. The momentum of a material object is

$$\mathbf{p} = m\mathbf{v},$$

but this is more like a standard for comparison of momenta rather than a definition. For instance, light has momentum, but has no mass, and the above equation is not the right equation for light. The law of conservation of momentum says that the total momentum of any closed system, i.e., the vector sum of the momentum vectors of all the things in the system, is a constant.

An important application of the momentum concept is to collisions, i.e., interactions between moving objects that last for a certain amount of time while the objects are in contact or near each other. Conservation of momentum tells us that certain outcomes of a collision are impossible, and in some cases may even be sufficient to predict the motion after the collision. In other cases, conservation of momentum does not provide enough equations to find all the unknowns. In some collisions, such as the collision of a superball with the floor, very little kinetic energy is converted into other forms of energy, and this provides one more equation, which may suffice to predict the outcome.

The total momentum of a system can be related to its total mass and the velocity of its center of mass by the equation

$$\mathbf{p}_{total} = m_{total}\mathbf{v}_{cm}.$$

The center of mass, introduced on an intuitive basis in book 1 as the "balance point" of an object, can be generalized to any system containing any number of objects, and is defined mathematically as the weighted average of the positions of all the parts of all the objects,

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots},$$

with similar equations for the y and z coordinates.

The frame of reference moving with the center of mass of a closed system is always a valid inertial frame, and many problems can be greatly simplified by working them in the inertial frame. For example, any collision between two objects appears in the c.m. frame as a head-on one-dimensional collision.

When a system is not closed, the rate at which momentum is transferred in or out is simply the total force being exerted externally on the system,

$$\mathbf{F}_{total} = \frac{\mathrm{d}\mathbf{p}_{total}}{\mathrm{d}t}.$$

Problems

Key

√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

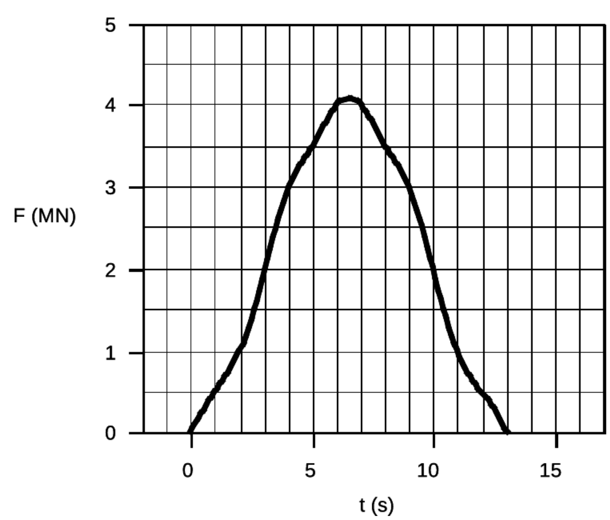
1 Derive a formula expressing the kinetic energy of an object in terms of its momentum and mass. √

2 Two people in a rowboat wish to move around without causing the boat to move. What should be true about their total momentum? Explain.

3 A firework shoots up into the air, and just before it explodes it has a certain momentum and kinetic energy. What can you say about the momenta and kinetic energies of the pieces immediately after the explosion? [Based on a problem from PSSC Physics.]
▷ Solution, p. 555

4 A bullet leaves the barrel of a gun with a kinetic energy of 90 J. The gun barrel is 50 cm long. The gun has a mass of 4 kg, the bullet 10 g.
(a) Find the bullet's final velocity. √
(b) Find the bullet's final momentum. √
(c) Find the momentum of the recoiling gun.
(d) Find the kinetic energy of the recoiling gun, and explain why the recoiling gun does not kill the shooter. √

Problem 5

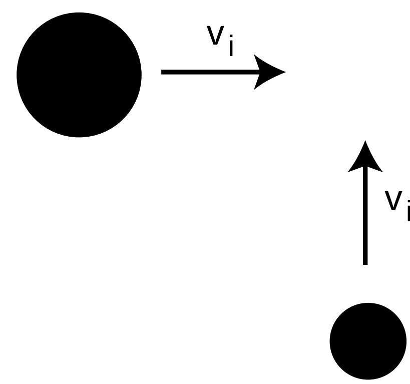


5 The graph shows the force, in meganewtons, exerted by a rocket engine on the rocket as a function of time. If the rocket's mass is 4000 kg, at what speed is the rocket moving when the engine stops firing? Assume it goes straight up, and neglect the force of gravity, which is much less than a meganewton. √

6 Cosmic rays are particles from outer space, mostly protons and atomic nuclei, that are continually bombarding the earth. Most of them, although they are moving extremely fast, have no discernible effect even if they hit your body, because their masses are so small. Their energies vary, however, and a very small minority of them have extremely large energies. In some cases the energy is as much as several Joules, which is comparable to the KE of a well thrown rock! If you are in a plane at a high altitude and are so incredibly unlucky as to be hit by one of these rare ultra-high-energy cosmic rays, what would you notice, the momentum imparted to your body, the energy dissipated in your body as heat, or both? Base your conclusions on numerical estimates, not just random speculation. (At these high speeds, one should really take into account the deviations from Newtonian physics described by Einstein's special theory of relativity. Don't worry about that, though.)

7 Show that for a body made up of many *equal* masses, the equation for the center of mass becomes a simple average of all the positions of the masses.

8 The figure shows a view from above of a collision about to happen between two air hockey pucks sliding without friction. They have the same speed, v_i, before the collision, but the big puck is 2.3 times more massive than the small one. Their sides have sticky stuff on them, so when they collide, they will stick together. At what angle will they emerge from the collision? In addition to giving a numerical answer, please indicate by drawing on the figure how your angle is defined. ▷ Solution, p. 555



Problem 8

9 A learjet traveling due east at 300 mi/hr collides with a jumbo jet which was heading southwest at 150 mi/hr. The jumbo jet's mass is 5.0 times greater than that of the learjet. When they collide, the learjet sticks into the fuselage of the jumbo jet, and they fall to earth together. Their engines stop functioning immediately after the collision. On a map, what will be the direction from the location of the collision to the place where the wreckage hits the ground? (Give an angle.) √

10 A very massive object with velocity v collides head-on with an object at rest whose mass is very small. No kinetic energy is converted into other forms. Prove that the low-mass object recoils with velocity $2v$. [Hint: Use the center-of-mass frame of reference.]

11 A mass m moving at velocity v collides with a stationary target having the same mass m. Find the maximum amount of energy that can be released as heat and sound. √

12 When the contents of a refrigerator cool down, the changed molecular speeds imply changes in both momentum and energy. Why, then, does a fridge transfer *power* through its radiator coils, but not *force*? ▷ Solution, p. 555

13 A 10-kg bowling ball moving at 2.0 m/s hits a 1.0-kg bowling pin, which is initially at rest. The other pins are all gone already, and the collision is head-on, so that the motion is one-dimensional. Assume that negligible amounts of heat and sound are produced. Find the velocity of the pin immediately after the collision.

14 A ball of mass $3m$ collides head-on with an initially stationary ball of mass m. No kinetic energy is transformed into heat or sound. In what direction is the mass-$3m$ ball moving after the collision, and how fast is it going compared to its original velocity? √

15 Suppose a system consisting of pointlike particles has a total kinetic energy K_{cm} measured in the center-of-mass frame of reference. Since they are pointlike, they cannot have any energy due to internal motion.
(a) Prove that in a different frame of reference, moving with velocity $\mathbf{u}$ relative to the center-of-mass frame, the total kinetic energy equals $K_{cm} + M|\mathbf{u}|^2/2$, where M is the total mass. [Hint: You can save yourself a lot of writing if you express the total kinetic energy using the dot product.] ▷ Solution, p. 556
(b) Use this to prove that if energy is conserved in one frame of reference, then it is conserved in every frame of reference. The total energy equals the total kinetic energy plus the sum of the potential energies due to the particles' interactions with each other, which we assume depends only on the distance between particles. [For a simpler numerical example, see problem 13 on p. 333.] ⋆

16 The big difference between the equations for momentum and kinetic energy is that one is proportional to v and one to v^2. Both, however, are proportional to m. Suppose someone tells you that there's a third quantity, funkosity, defined as $f = m^2v$, and that funkosity is conserved. How do you know your leg is being pulled?
▷ Solution, p. 556

17 A rocket ejects exhaust with an exhaust velocity u. The rate at which the exhaust mass is used (mass per unit time) is b. We assume that the rocket accelerates in a straight line starting from rest, and that no external forces act on it. Let the rocket's initial mass (fuel plus the body and payload) be m_i, and m_f be its final mass, after all the fuel is used up. (a) Find the rocket's final velocity, v, in terms of u, m_i, and m_f. Neglect the effects of special relativity. (b) A typical exhaust velocity for chemical rocket engines is 4000 m/s. Estimate the initial mass of a rocket that could accelerate a one-ton payload to 10% of the speed of light, and show that this design won't work. (For the sake of the estimate, ignore the mass of the fuel tanks. The speed is fairly small compared to c, so it's not an unreasonable approximation to ignore relativity.) ✓ ⋆

18 A flexible rope of mass m and length L slides without friction over the edge of a table. Let x be the length of the rope that is hanging over the edge at a given moment in time.
(a) Show that x satisfies the equation of motion $\mathrm{d}^2x/\mathrm{d}t^2 = gx/L$. [Hint: Use $F = \mathrm{d}p/\mathrm{d}t$, which allows you to handle the two parts of the rope separately even though mass is moving out of one part and into the other.]
(b) Give a physical explanation for the fact that a larger value of x on the right-hand side of the equation leads to a greater value of the acceleration on the left side.
(c) When we take the second derivative of the function $x(t)$ we are supposed to get essentially the same function back again, except for a constant out in front. The function e^x has the property that it is unchanged by differentiation, so it is reasonable to look for solutions to this problem that are of the form $x = be^{ct}$, where b and c are constants. Show that this does indeed provide a solution for two specific values of c (and for any value of b).
(d) Show that the sum of any two solutions to the equation of motion is also a solution.
(e) Find the solution for the case where the rope starts at rest at $t = 0$ with some nonzero value of x. ⋆

19 (a) Find a relativistic equation for the velocity of an object in terms of its mass and momentum (eliminating γ). ✓
(b) Show that your result is approximately the same as the nonrelativistic value, p/m, at low velocities.
(c) Show that very large momenta result in speeds close to the speed of light. ⋆

20 The force acting on an object is $F = At^2$. The object is at rest at time $t = 0$. What is its momentum at $t = T$?
$\checkmark$ *[problem by B. Shotwell]*

21 A bullet of mass m strikes a block of mass M which is hanging by a string of length L from the ceiling. It is observed that, after the sticky collision, the maximum angle that the string makes with the vertical is θ. This setup is called a ballistic pendulum, and it can be used to measure the speed of the bullet.
(a) What vertical height does the block reach? $\checkmark$
(b) What was the speed of the block just after the collision? $\checkmark$
(c) What was the speed of the bullet just before it struck the block?
$\checkmark$ *[problem by B. Shotwell]*

22 A car of mass M and a truck of mass $2M$ collide head-on with equal speeds v, and the collision is perfectly inelastic.
(a) What is the magnitude of the change in momentum of the car? $\checkmark$
(b) What is the magnitude of the change in momentum of the truck? $\checkmark$
(c) What is the final speed of the two vehicles? $\checkmark$
(d) What fraction of the initial kinetic energy was lost as a result of the collision? $\checkmark$ *[problem by B. Shotwell]*

A tornado touches down in Spring Hill, Kansas, May 20, 1957.

Chapter 15

Conservation of angular momentum

"Sure, and maybe the sun won't come up tomorrow." Of course, the sun only appears to go up and down because the earth spins, so the cliche should really refer to the unlikelihood of the earth's stopping its rotation abruptly during the night. Why can't it stop? It wouldn't violate conservation of momentum, because the earth's rotation doesn't add anything to its momentum. While California spins in one direction, some equally massive part of India goes the opposite way, canceling its momentum. A halt to Earth's rotation would entail a drop in kinetic energy, but that energy could simply be converted into some other form, such as heat.

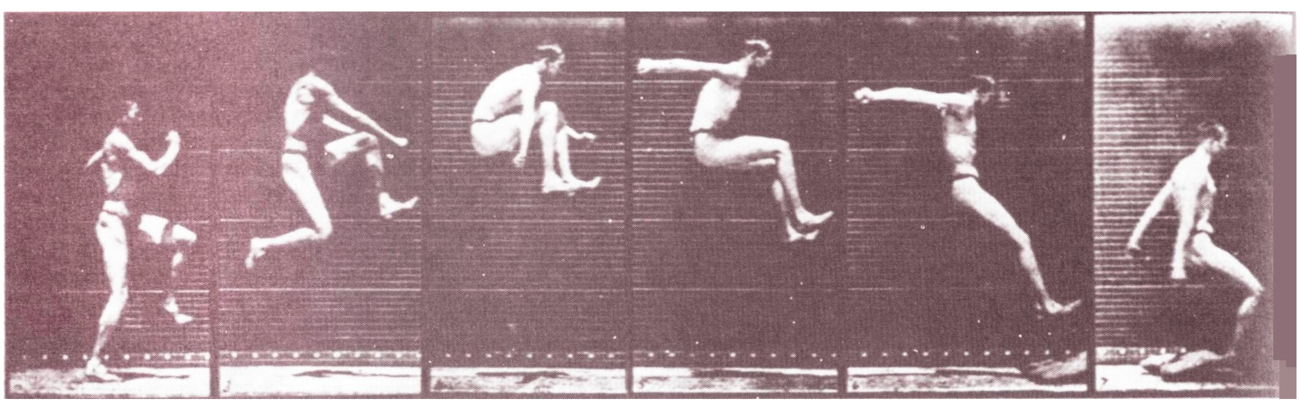
Other examples along these lines are not hard to find. A hydrogen atom spins at the same rate for billions of years. A high-diver who is rotating when he comes off the board does not need to make any physical effort to continue rotating, and indeed would be unable to stop rotating before he hit the water.

These observations have the hallmarks of a conservation law:

A closed system is involved. Nothing is making an effort to twist the earth, the hydrogen atom, or the high-diver. They are isolated from rotation-changing influences, i.e., they are closed systems.

Something remains unchanged. There appears to be a numerical quantity for measuring rotational motion such that the total amount of that quantity remains constant in a closed system.

Something can be transferred back and forth without changing the total amount. In figure a, the jumper wants to get his feet out in front of him so he can keep from doing a "face plant" when he lands. Bringing his feet forward would involve a certain quantity of counterclockwise rotation, but he didn't start out with any rotation when he left the ground. Suppose we consider counterclockwise as positive and clockwise as negative. The only way his legs can acquire some positive rotation is if some other part of his body picks up an equal amount of negative rotation. This is why he swings his arms up behind him, clockwise.



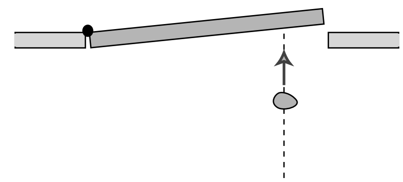
a / An early photograph of an old-fashioned long-jump.

What numerical measure of rotational motion is conserved? Car engines and old-fashioned LP records have speeds of rotation measured in rotations per minute (r.p.m.), but the number of rotations per minute (or per second) is not a conserved quantity. A twirling figure skater, for instance, can pull her arms in to increase her r.p.m.'s. The first section of this chapter deals with the numerical definition of the quantity of rotation that results in a valid conservation law.

15.1 Conservation of angular momentum

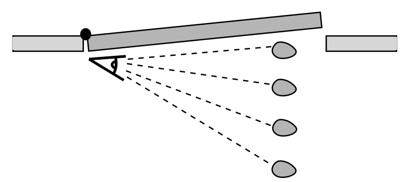
When most people think of rotation, they think of a solid object like a wheel rotating in a circle around a fixed point. Examples of this type of rotation, called rigid rotation or rigid-body rotation, include a spinning top, a seated child's swinging leg, and a helicopter's spinning propeller. Rotation, however, is a much more general phenomenon, and includes noncircular examples such as a comet in an elliptical orbit around the sun, or a cyclone, in which the core completes a circle more quickly than the outer parts.

If there is a numerical measure of rotational motion that is a conserved quantity, then it must include nonrigid cases like these, since nonrigid rotation can be traded back and forth with rigid rotation. For instance, there is a trick for finding out if an egg is raw or hardboiled. If you spin a hardboiled egg and then stop it briefly with your finger, it stops dead. But if you do the same with a raw egg, it springs back into rotation because the soft interior was still swirling around within the momentarily motionless shell. The pattern of flow of the liquid part is presumably very complex and nonuniform due to the asymmetric shape of the egg and the different consistencies of the yolk and the white, but there is apparently some way to describe the liquid's total amount of rotation with a single number, of which some percentage is given back to the shell when you release it.



b / An overhead view of a piece of putty being thrown at a door. Even though the putty is neither spinning nor traveling along a curve, we must define it as having some kind of "rotation" because it is able to make the door rotate.

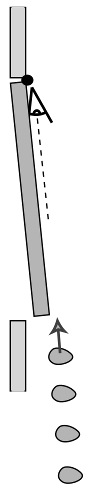
The best strategy is to devise a way of defining the amount of rotation of a single small part of a system. The amount of rotation of a system such as a cyclone will then be defined as the total of all the contributions from its many small parts.



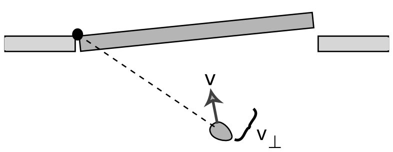
c / As seen by someone standing at the axis, the putty changes its angular position. We therefore define it as having angular momentum.

The quest for a conserved quantity of rotation even requires us to broaden the rotation concept to include cases where the motion doesn't repeat or even curve around. If you throw a piece of putty at a door, the door will recoil and start rotating. The putty was traveling straight, not in a circle, but if there is to be a general conservation law that can cover this situation, it appears that we must describe the putty as having had some "rotation," which it then gave up to the door. The best way of thinking about it is to attribute rotation to any moving object or part of an object that changes its angle in relation to the axis of rotation. In the putty-and-door example, the hinge of the door is the natural point to think of as an axis, and the putty changes its angle as seen by someone standing at the hinge. For this reason, the conserved quantity we are investigating is called *angular* momentum. The symbol for angular momentum can't be a or m, since those are used for acceleration and mass, so the symbol L is arbitrarily chosen instead.

Imagine a 1-kg blob of putty, thrown at the door at a speed of 1 m/s, which hits the door at a distance of 1 m from the hinge. We define this blob to have 1 unit of angular momentum. When it hits the door, the door will recoil and start rotating. We can use the speed at which the door recoils as a measure of the angular momentum the blob brought in.[1]



d / A putty blob thrown directly at the axis has no angular motion, and therefore no angular momentum. It will not cause the door to rotate.



e / Only the component of the velocity vector perpendicular to the dashed line should be counted into the definition of angular momentum.

Experiments show, not surprisingly, that a 2-kg blob thrown in the same way makes the door rotate twice as fast, so the angular momentum of the putty blob must be proportional to mass,

$$L \propto m.$$

Similarly, experiments show that doubling the velocity of the blob will have a doubling effect on the result, so its angular momentum must be proportional to its velocity as well,

$$L \propto mv.$$

You have undoubtedly had the experience of approaching a closed door with one of those bar-shaped handles on it and pushing on the wrong side, the side close to the hinges. You feel like an idiot, because you have so little leverage that you can hardly budge the door. The same would be true with the putty blob. Experiments would show that the amount of rotation the blob can give to the door is proportional to the distance, r, from the axis of rotation, so angular momentum must also be proportional to r,

$$L \propto mvr.$$

We are almost done, but there is one missing ingredient. We know on grounds of symmetry that a putty ball thrown directly inward toward the hinge will have no angular momentum to give to the door. After all, there would not even be any way to decide whether the ball's rotation was clockwise or counterclockwise in this situation. It is therefore only the component of the blob's velocity vector perpendicular to the door that should be counted in its angular momentum,

$$L = mv_\perp r.$$

More generally, $v_\perp$ should be thought of as the component of the object's velocity vector that is perpendicular to the line joining the object to the axis of rotation.

We find that this equation agrees with the definition of the original putty blob as having one unit of angular momentum, and we can now see that the units of angular momentum are (kg·m/s)·m, i.e., kg·m^2/s. This gives us a way of calculating the angular momentum of any material object or any system consisting of material objects:

[1] We assume that the door is much more massive than the blob. Under this assumption, the speed at which the door recoils is much less than the original speed of the blob, so the blob has lost essentially all its angular momentum, and given it to the door.

angular momentum of a material object

The angular momentum of a moving particle is

$$L = mv_\perp r,$$

where m is its mass, $v_\perp$ is the component of its velocity vector perpendicular to the line joining it to the axis of rotation, and r is its distance from the axis. Positive and negative signs are used to describe opposite directions of rotation.

The angular momentum of a finite-sized object or a system of many objects is found by dividing it up into many small parts, applying the equation to each part, and adding to find the total amount of angular momentum.

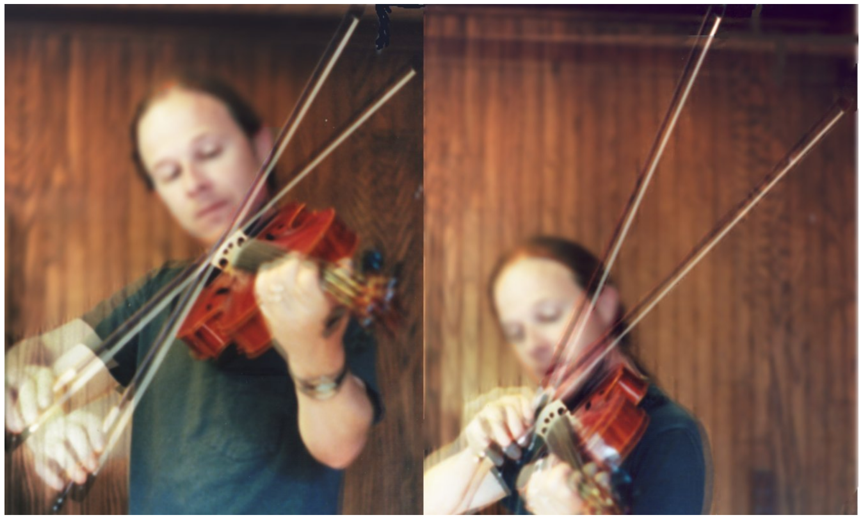
Note that r is not necessarily the radius of a circle. (As implied by the qualifiers, matter isn't the only thing that can have angular momentum. Light can also have angular momentum, and the above equation would not apply to light.)

Conservation of angular momentum has been verified over and over again by experiment, and is now believed to be one of the three most fundamental principles of physics, along with conservation of energy and momentum.

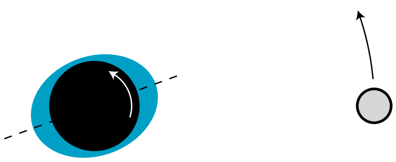
A figure skater pulls her arms in — *example 1*

When a figure skater is twirling, there is very little friction between her and the ice, so she is essentially a closed system, and her angular momentum is conserved. If she pulls her arms in, she is decreasing r for all the atoms in her arms. It would violate conservation of angular momentum if she then continued rotating at the same speed, i.e., taking the same amount of time for each revolution, because her arms' contributions to her angular momentum would have decreased, and no other part of her would have increased its angular momentum. This is impossible because it would violate conservation of angular momentum. If her total angular momentum is to remain constant, the decrease in r for her arms must be compensated for by an overall increase in her rate of rotation. That is, by pulling her arms in, she substantially reduces the time for each rotation.

f / A figure skater pulls in her arms so that she can execute a spin more rapidly.



g / Example 2.



h / Example 3. A view of the earth-moon system from above the north pole. All distances have been highly distorted for legibility. The earth's rotation is counterclockwise from this point of view (arrow). The moon's gravity creates a bulge on the side near it, because its gravitational pull is stronger there, and an "anti-bulge" on the far side, since its gravity there is weaker. For simplicity, let's focus on the tidal bulge closer to the moon. Its frictional force is trying to slow down the earth's rotation, so its force on the earth's solid crust is toward the bottom of the figure. By Newton's third law, the crust must thus make a force on the bulge which is toward the top of the figure. This causes the bulge to be pulled forward at a slight angle, and the bulge's gravity therefore pulls the moon forward, accelerating its orbital motion about the earth and flinging it outward.

Changing the axis *example 2*

An object's angular momentum can be different depending on the axis about which it rotates. Figure g shows two double-exposure photographs a viola player tipping the bow in order to cross from one string to another. Much more angular momentum is required when playing near the bow's handle, called the frog, as in the panel on the right; not only are most of the atoms in the bow at greater distances, r, from the axis of rotation, but the ones in the tip also have more momentum, p. It is difficult for the player to quickly transfer a large angular momentum into the bow, and then transfer it back out just as quickly. (In the language of section 15.4, large torques are required.) This is one of the reasons that string players tend to stay near the middle of the bow as much as possible.

Earth's slowing rotation and the receding moon *example 3*

As noted in chapter 1, the earth's rotation is actually slowing down very gradually, with the kinetic energy being dissipated as heat by friction between the land and the tidal bulges raised in the seas by the earth's gravity. Does this mean that angular momentum is not really perfectly conserved? No, it just means that the earth is not quite a closed system by itself. If we consider the earth and moon as a system, then the angular momentum lost by the earth must be gained by the moon somehow. In fact very precise measurements of the distance between the earth and the moon have been carried out by bouncing laser beams off of a mirror left there by astronauts, and these measurements show that the moon is receding from the earth at a rate of 4 centimeters per year! The moon's greater value of r means that it has a greater angular momentum, and the increase turns out to be exactly the amount lost by the earth. In the days of the dinosaurs, the days were significantly shorter, and the moon was closer and appeared bigger in the sky.

But what force is causing the moon to speed up, drawing it out into a larger orbit? It is the gravitational forces of the earth's tidal bulges. The effect is described qualitatively in the caption of the figure. The result would obviously be extremely difficult to calculate directly, and this is one of those situations where a conservation law allows us to make precise quantitative statements about the outcome of a process when the calculation of the process itself would be prohibitively complex.

Restriction to rotation in a plane

Is angular momentum a vector, or a scalar? It does have a direction in space, but it's a direction of rotation, not a straight-line direction like the directions of vectors such as velocity or force. It turns out (see problem 25) that there is a way of defining angular momentum as a vector, but until section 15.8 the examples will be confined to a single plane of rotation, i.e., effectively two-dimensional situations. In this special case, we can choose to visualize the plane of rotation from one side or the other, and to define clockwise and counterclockwise rotation as having opposite signs of angular momentum.

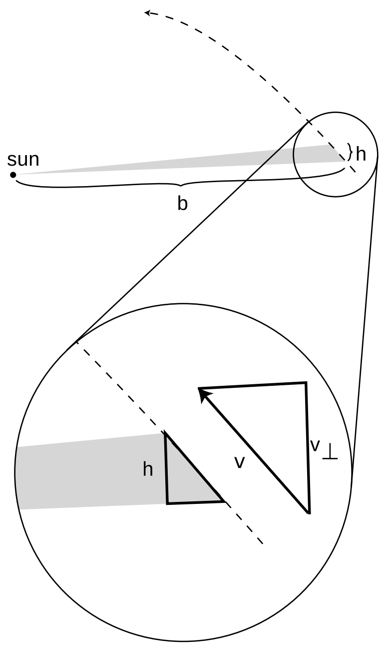
Discussion question

A Conservation of plain old momentum, p, can be thought of as the greatly expanded and modified descendant of Galileo's original principle of inertia, that no force is required to keep an object in motion. The principle of inertia is counterintuitive, and there are many situations in which it appears superficially that a force *is* needed to maintain motion, as maintained by Aristotle. Think of a situation in which conservation of angular momentum, L, also seems to be violated, making it seem incorrectly that something external must act on a closed system to keep its angular momentum from "running down."

15.2 Angular momentum in planetary motion

We now discuss the application of conservation of angular momentum to planetary motion, both because of its intrinsic importance and because it is a good way to develop a visual intuition for angular momentum.

Kepler's law of equal areas states that the area swept out by a planet in a certain length of time is always the same. Angular momentum had not been invented in Kepler's time, and he did not even know the most basic physical facts about the forces at work. He thought of this law as an entirely empirical and unexpectedly simple way of summarizing his data, a rule that succeeded in describing and predicting how the planets sped up and slowed down in their elliptical paths. It is now fairly simple, however, to show that the equal area law amounts to a statement that the planet's angular momentum stays constant.



i / The planet's angular momentum is related to the rate at which it sweeps out area.

There is no simple geometrical rule for the area of a pie wedge cut out of an ellipse, but if we consider a very short time interval, as shown in figure i, the shaded shape swept out by the planet is very nearly a triangle. We do know how to compute the area of a triangle. It is one half the product of the base and the height:

$$\text{area} = \frac{1}{2}bh.$$

We wish to relate this to angular momentum, which contains the variables r and $v_\perp$. If we consider the sun to be the axis of rotation, then the variable r is identical to the base of the triangle, $r=b$. Referring to the magnified portion of the figure, $v_\perp$ can be related to h, because the two right triangles are similar:

$$\frac{h}{\text{distance traveled}} = \frac{v_\perp}{|\mathbf{v}|}$$

The area can thus be rewritten as

$$\text{area} = \frac{1}{2}r\frac{v_\perp(\text{distance traveled})}{|\mathbf{v}|}.$$

The distance traveled equals $|\mathbf{v}|\Delta t$, so this simplifies to

$$\text{area} = \frac{1}{2}rv_\perp\Delta t.$$

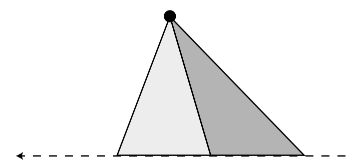
We have found the following relationship between angular momentum and the rate at which area is swept out:

$$L = 2m\frac{\text{area}}{\Delta t}.$$

The factor of 2 in front is simply a matter of convention, since any conserved quantity would be an equally valid conserved quantity if you multiplied it by a constant. The factor of m was not relevant to Kepler, who did not know the planets' masses, and who was only describing the motion of one planet at a time.

We thus find that Kepler's equal-area law is equivalent to a statement that the planet's angular momentum remains constant. But wait, why should it remain constant? — the planet is not a closed system, since it is being acted on by the sun's gravitational force. There are two valid answers. The first is that it is actually the total angular momentum of the sun plus the planet that is conserved. The sun, however, is millions of times more massive than the typical planet, so it accelerates very little in response to the planet's gravitational force. It is thus a good approximation to say that the sun doesn't move at all, so that no angular momentum is transferred between it and the planet.

The second answer is that to change the planet's angular momentum requires not just a force but a force applied in a certain way. In section 15.4 we discuss the transfer of angular momentum by a force, but the basic idea here is that a force directly in toward the axis does not change the angular momentum.

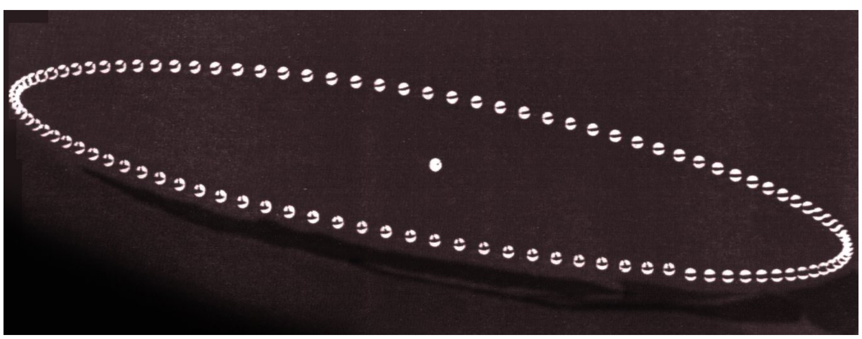


Discussion question A.

Discussion questions

A Suppose an object is simply traveling in a straight line at constant speed. If we pick some point not on the line and call it the axis of rotation, is area swept out by the object at a constant rate? Would it matter if we chose a different axis?

B The figure is a strobe photo of a pendulum bob, taken from underneath the pendulum looking straight up. The black string can't be seen in the photograph. The bob was given a slight sideways push when it was released, so it did not swing in a plane. The bright spot marks the center, i.e., the position the bob would have if it hung straight down at us. Does the bob's angular momentum appear to remain constant if we consider the center to be the axis of rotation? What if we choose a different axis?

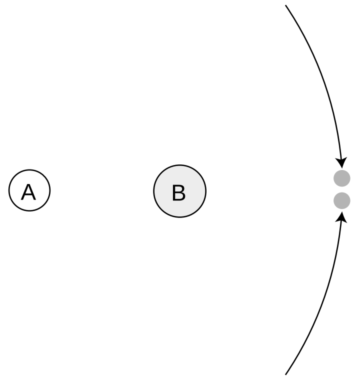


Discussion question B.

15.3 Two theorems about angular momentum

With plain old momentum, p, we had the freedom to work in any inertial frame of reference we liked. The same object could have different values of momentum in two different frames, if the frames were not at rest with respect to each other. Conservation of momentum, however, would be true in either frame. As long as we employed a single frame consistently throughout a calculation, everything would work.

The same is true for angular momentum, and in addition there is an ambiguity that arises from the definition of an axis of rotation. For a wheel, the natural choice of an axis of rotation is obviously the axle, but what about an egg rotating on its side? The egg has an asymmetric shape, and thus no clearly defined geometric center. A similar issue arises for a cyclone, which does not even have a sharply defined shape, or for a complicated machine with many gears. The following theorem, the first of two presented in this section without proof, explains how to deal with this issue. Although I have put descriptive titles above both theorems, they have no generally accepted names.



j / Example 4.

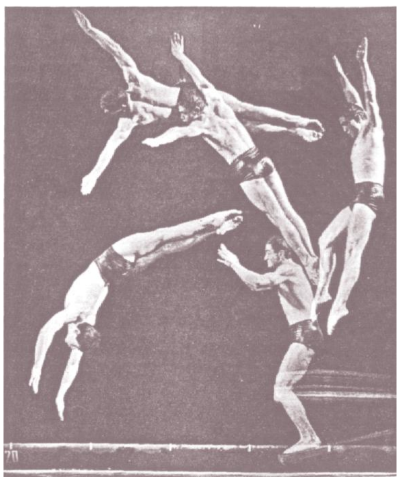
the choice of axis theorem
It is entirely arbitrary what point one defines as the axis for purposes of calculating angular momentum. If a closed system's angular momentum is conserved when calculated with one choice of axis, then it will also be conserved for any other choice. Likewise, any inertial frame of reference may be used.

Colliding asteroids described with different axes *example 4*

Observers on planets A and *B* both see the two asteroids colliding. The asteroids are of equal mass and their impact speeds are the same. Astronomers on each planet decide to define their own planet as the axis of rotation. Planet A is twice as far from the collision as planet *B*. The asteroids collide and stick. For simplicity, assume planets A and *B* are both at rest.

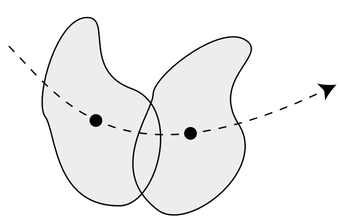
With planet A as the axis, the two asteroids have the same amount of angular momentum, but one has positive angular momentum and the other has negative. Before the collision, the total angular momentum is therefore zero. After the collision, the two asteroids will have stopped moving, and again the total angular momentum is zero. The total angular momentum both before and after the collision is zero, so angular momentum is conserved if you choose planet A as the axis.

The only difference with planet B as axis is that r is smaller by a factor of two, so all the angular momenta are halved. Even though the angular momenta are different than the ones calculated by planet A, angular momentum is still conserved.



k / Everyone has a strong tendency to think of the diver as rotating about his own center of mass. However, he is flying in an arc, and he also has angular momentum because of this motion.

The earth spins on its own axis once a day, but simultaneously travels in its circular one-year orbit around the sun, so any given part of it traces out a complicated loopy path. It would seem difficult to calculate the earth's angular momentum, but it turns out that there is an intuitively appealing shortcut: we can simply add up the angular momentum due to its spin plus that arising from its center of mass's circular motion around the sun. This is a special case of the following general theorem:



l / This rigid object has angular momentum both because it is spinning about its center of mass and because it is moving through space.

the spin theorem
An object's angular momentum with respect to some outside axis A can be found by adding up two parts:
(1) The first part is the object's angular momentum found by using its own center of mass as the axis, i.e., the angular

momentum the object has because it is spinning.
(2) The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

A system with its center of mass at rest *example 5*
In the special case of an object whose center of mass is at rest, the spin theorem implies that the object's angular momentum is the same regardless of what axis we choose. (This is an even stronger statement than the choice of axis theorem, which only guarantees that angular momentum is conserved for any given choice of axis, without specifying that it is the same for all such choices.)

Discussion question

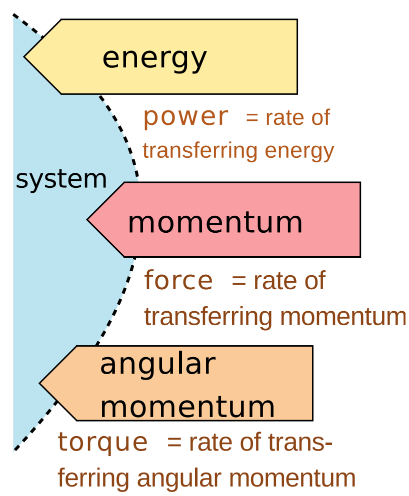
A In the example of the colliding asteroids, suppose planet A was moving toward the top of the page, at the same speed as the bottom asteroid. How would planet A's astronomers describe the angular momenta of the asteroids? Would angular momentum still be conserved?

15.4 Torque: the rate of transfer of angular momentum

Force can be interpreted as the rate of transfer of momentum. The equivalent in the case of angular momentum is called *torque* (rhymes with "fork"). Where force tells us how hard we are pushing or pulling on something, torque indicates how hard we are twisting on it. Torque is represented by the Greek letter tau, τ, and the rate of change of an object's angular momentum equals the total torque acting on it,

$$\tau_{total} = \frac{\mathrm{d}L}{\mathrm{d}t}.$$

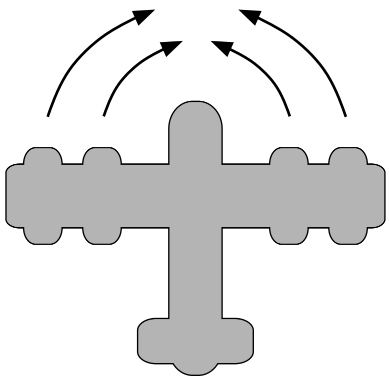
As with force and momentum, it often happens that angular momentum recedes into the background and we focus our interest on the torques. The torque-focused point of view is exemplified by the fact that many scientifically untrained but mechanically apt people know all about torque, but none of them have heard of angular momentum. Car enthusiasts eagerly compare engines' torques, and there is a tool called a torque wrench which allows one to apply a desired amount of torque to a screw and avoid overtightening it.



m / Energy, momentum, and angular momentum can be transferred. The rates of transfer are called power, force, and torque.

Torque distinguished from force

Of course a force is necessary in order to create a torque — you can't twist a screw without pushing on the wrench — but force and torque are two different things. One distinction between them is direction. We use positive and negative signs to represent forces in the two possible directions along a line. The direction of a torque, however, is clockwise or counterclockwise, not a linear direction.



n / The plane's four engines produce zero total torque but not zero total force.

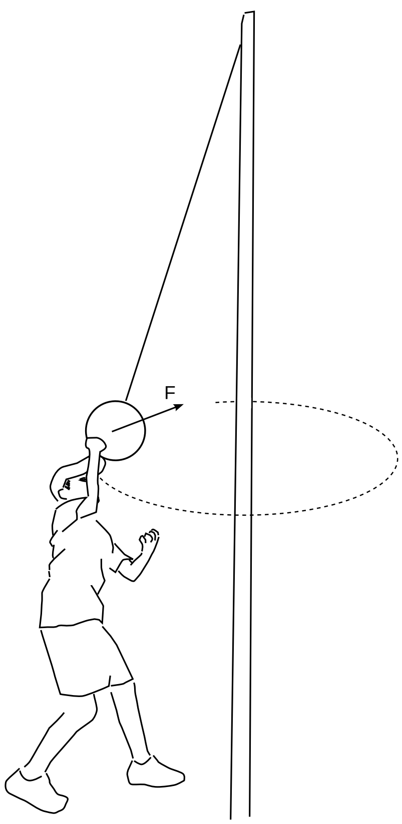
The other difference between torque and force is a matter of leverage. A given force applied at a door's knob will change the door's angular momentum twice as rapidly as the same force applied halfway between the knob and the hinge. The same amount of force produces different amounts of torque in these two cases.

It is possible to have a zero total torque with a nonzero total force. An airplane with four jet engines, n, would be designed so that their forces are balanced on the left and right. Their forces are all in the same direction, but the clockwise torques of two of the engines are canceled by the counterclockwise torques of the other two, giving zero total torque.

Conversely we can have zero total force and nonzero total torque. A merry-go-round's engine needs to supply a nonzero torque on it to bring it up to speed, but there is zero total force on it. If there was not zero total force on it, its center of mass would accelerate!

Relationship between force and torque

How do we calculate the amount of torque produced by a given force? Since it depends on leverage, we should expect it to depend on the distance between the axis and the point of application of the force. We'll derive an equation relating torque to force for a particular very simple situation, and state without proof that the equation actually applies to all situations.



o / The boy makes a torque on the tetherball.

To try to pin down this relationship more precisely, let's imagine hitting a tetherball, figure o. The boy applies a force F to the ball for a short time Δt, accelerating the ball from rest to a velocity v. Since force is the rate of transfer of momentum, we have

$$F = \frac{m\Delta v}{\Delta t}.$$

Since the initial velocity is zero, Δv is the same as the final velocity v. Multiplying both sides by r gives

$$Fr = \frac{mvr}{\Delta t}.$$

But mvr is simply the amount of angular momentum he's given the ball, so $mvr/\Delta t$ also equals the amount of torque he applied. The result of this example is

$$\tau = Fr.$$

Figure o was drawn so that the force F was in the direction tangent to the circle, i.e., perpendicular to the radius r. If the boy had applied a force *parallel* to the radius line, either directly inward or outward, then the ball would not have picked up any clockwise or counterclockwise angular momentum.

If a force acts at an angle other than 0 or 90°with respect to the line joining the object and the axis, it would be only the component of the force perpendicular to the line that would produce a torque,

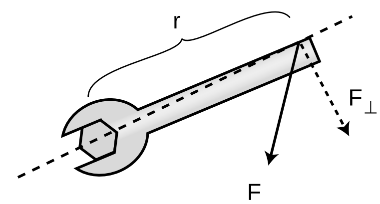
$$\tau = F_{\perp} r.$$

Although this result was proved under a simplified set of circumstances, it is more generally valid:

relationship between force and torque
The rate at which a force transfers angular momentum to an object, i.e., the torque produced by the force, is given by

$$|\boldsymbol{\tau}| = r|F_{\perp}|,$$

where r is the distance from the axis to the point of application of the force, and $F_{\perp}$ is the component of the force that is perpendicular to the line joining the axis to the point of application.

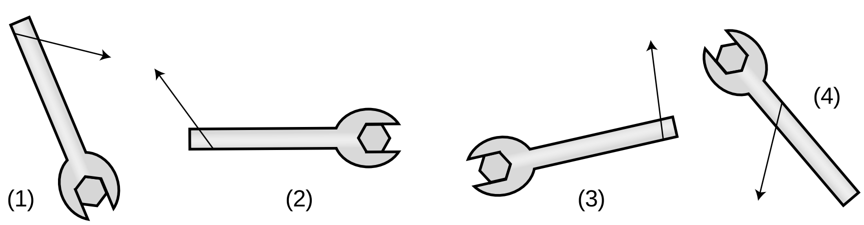


p / The geometric relationships referred to in the relationship between force and torque.

The equation is stated with absolute value signs because the positive and negative signs of force and torque indicate different things, so there is no useful relationship between them. The sign of the torque must be found by physical inspection of the case at hand.

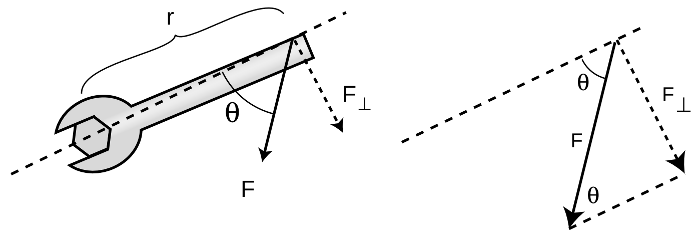
From the equation, we see that the units of torque can be written as newtons multiplied by meters. Metric torque wrenches are calibrated in N·m, but American ones use foot-pounds, which is also a unit of distance multiplied by a unit of force. We know from our study of mechanical work that newtons multiplied by meters equal joules, but torque is a completely different quantity from work, and nobody writes torques with units of joules, even though it would be technically correct.

self-check A
Compare the magnitudes and signs of the four torques shown in the figure. ▷ Answer, p. 561



How torque depends on the direction of the force *example 6*
▷ How can the torque applied to the wrench in the figure be expressed in terms of r, $|F|$, and the angle θ between these two vectors?

▷ The force vector and its $F_\perp$ component form the hypotenuse and one leg of a right triangle,

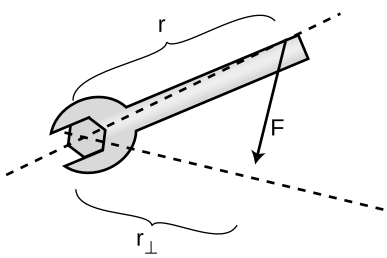


and the interior angle opposite to $F_\perp$ equals θ. The absolute value of $F_\perp$ can thus be expressed as

$$F_\perp = |\mathbf{F}| \sin\theta,$$

leading to

$$|\tau| = r|\mathbf{F}| \sin\theta.$$



q / The quantity $r_\perp$.

Sometimes torque can be more neatly visualized in terms of the quantity $r_\perp$ shown in figure q, which gives us a third way of expressing the relationship between torque and force:

$$|\boldsymbol{\tau}| = r_\perp |\mathbf{F}|.$$

Of course you would not want to go and memorize all three equations for torque. Starting from any one of them you could easily derive the other two using trigonometry. Familiarizing yourself with them can however clue you in to easier avenues of attack on certain problems.

The torque due to gravity

Up until now we've been thinking in terms of a force that acts at a single point on an object, such as the force of your hand on the wrench. This is of course an approximation, and for an extremely realistic calculation of your hand's torque on the wrench you might need to add up the torques exerted by each square millimeter where your skin touches the wrench. This is seldom necessary. But in the case of a gravitational force, there is never any single point at which the force is applied. Our planet is exerting a separate tug on every brick in the Leaning Tower of Pisa, and the total gravitational torque on the tower is the sum of the torques contributed by all the little forces. Luckily there is a trick that allows us to avoid such a massive calculation. It turns out that for purposes of computing the total gravitational torque on an object, you can get the right answer by just pretending that the whole gravitational force acts at the object's center of mass.

Gravitational torque on an outstretched arm *example 7*

▷ Your arm has a mass of 3.0 kg, and its center of mass is 30 cm from your shoulder. What is the gravitational torque on your arm when it is stretched out horizontally to one side, taking the shoulder to be the axis?

▷ The total gravitational force acting on your arm is

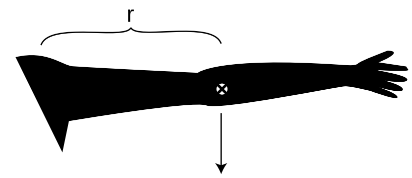
$$|F| = (3.0\text{ kg})(9.8\text{ m/s}^2) = 29\text{ N}.$$

For the purpose of calculating the gravitational torque, we can treat the force as if it acted at the arm's center of mass. The force is straight down, which is perpendicular to the line connecting the shoulder to the center of mass, so

$$F_\perp = |F| = 29\text{ N}.$$

Continuing to pretend that the force acts at the center of the arm, r equals 30 cm = 0.30 m, so the torque is

$$\tau = rF_\perp = 9\text{ N}\cdot\text{m}.$$



r / Example 7.

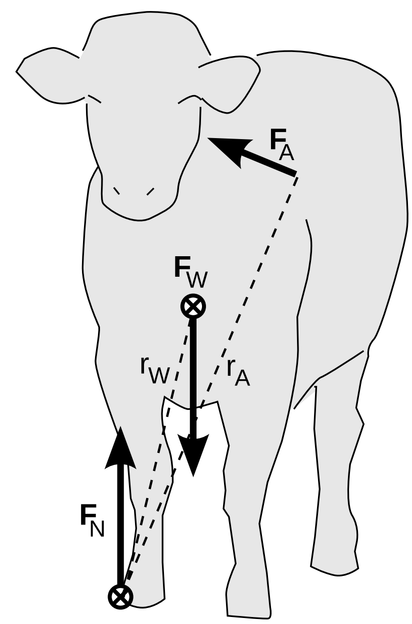
Cow tipping *example 8*

In 2005, Dr. Margo Lillie and her graduate student Tracy Boechler published a study claiming to debunk cow tipping. Their claim was based on an analysis of the torques that would be required to tip a cow, which showed that one person wouldn't be able to make enough torque to do it. A lively discussion ensued on the popular web site slashdot.org ("news for nerds, stuff that matters") concerning the validity of the study. Personally, I had always assumed that cow-tipping was a group sport anyway, but as a physicist, I also had some quibbles with their calculation. Here's my own analysis.

There are three forces on the cow: the force of gravity $\mathbf{F}_W$, the ground's normal force $\mathbf{F}_N$, and the tippers' force $\mathbf{F}_A$.

As soon as the cow's left hooves (on the right from our point of view) break contact with the ground, the ground's force is being applied only to hooves on the other side. We don't know the ground's force, and we don't want to find it. Therefore we take the axis to be at its point of application, so that its torque is zero.

For the purpose of computing torques, we can pretend that gravity acts at the cow's center of mass, which I've placed a little lower than the center of its torso, since its legs and head also have some mass, and the legs are more massive than the head and stick out farther, so they lower the c.m. more than the head raises it. The angle θ_W between the vertical gravitational force and the line r_W is about 14°. (An estimate by Matt Semke at the University of Nebraska-Lincoln gives 20°, which is in the same ballpark.)



s / Example 8.

To generate the maximum possible torque with the least possible force, the tippers want to push at a point as far as possible from the axis, which will be the shoulder on the other side, and they want to push at a 90 degree angle with respect to the radius line r_A.

When the tippers are just barely applying enough force to raise the cow's hooves on one side, the total torque has to be just slightly more than zero. (In reality, they want to push a lot harder than this — hard enough to impart a lot of angular momentum to the cow fair in a short time, before it gets mad and hurts them. We're just trying to calculate the bare minimum force they can possibly use, which is the question that science can answer.) Setting the total torque equal to zero,

$$\tau_N + \tau_W + \tau_A = 0,$$

and letting counterclockwise torques be positive, we have

$$0 - mgr_W \sin\theta_W + F_A r_A \sin 90^\circ = 0$$

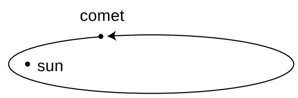
$$\begin{aligned} F_A &= \frac{r_W}{r_A} mg \sin\theta_W \\ &\approx \frac{1}{1.5}(680\ \text{kg})(9.8\ \text{m/s}^2) \sin 14^\circ \\ &= 1100\ \text{N}. \end{aligned}$$

The 680 kg figure for the typical mass of a cow is due to Lillie and Boechler, who are veterinarians, so I assume it's fairly accurate. My estimate of 1100 N comes out significantly lower than their 1400 N figure, mainly because their incorrect placement of the center of mass gives $\theta_W = 24^\circ$. I don't think 1100 N is an impossible amount of force to require of one big, strong person (it's equivalent to lifting about 110 kg, or 240 pounds), but given that the tippers need to impart a large angular momentum fairly quickly, it's probably true that several people would be required.

The main practical issue with cow tipping is that cows generally sleep lying down. Falling on its side can also seriously injure a cow.

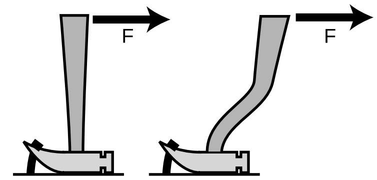
Discussion questions

A This series of discussion questions deals with past students' incorrect reasoning about the following problem.

> Suppose a comet is at the point in its orbit shown in the figure. The only force on the comet is the sun's gravitational force.
>
> 
>
>
> Throughout the question, define all torques and angular momenta using the sun as the axis.
>
> (1) Is the sun producing a nonzero torque on the comet? Explain.
> (2) Is the comet's angular momentum increasing, decreasing, or staying the same? Explain.

Explain what is wrong with the following answers. In some cases, the answer is correct, but the reasoning leading up to it is wrong. (a) Incorrect answer to part (1): "Yes, because the sun is exerting a force on the comet, and the comet is a certain distance from the sun."
(b) Incorrect answer to part (1): "No, because the torques cancel out."
(c) Incorrect answer to part (2): "Increasing, because the comet is speeding up."

B Which claw hammer would make it easier to get the nail out of the wood if the same force was applied in the same direction?



Discussion question B.

C You whirl a rock over your head on the end of a string, and gradually pull in the string, eventually cutting the radius in half. What happens to the rock's angular momentum? What changes occur in its speed, the time required for one revolution, and its acceleration? Why might the string break?

D A helicopter has, in addition to the huge fan blades on top, a smaller propeller mounted on the tail that rotates in a vertical plane. Why?

E The photo shows an amusement park ride whose two cars rotate in opposite directions. Why is this a good design?

Discussion question E.

15.5 Statics

Equilibrium

There are many cases where a system is not closed but maintains constant angular momentum. When a merry-go-round is running at constant angular momentum, the engine's torque is being canceled by the torque due to friction.

When an object has constant momentum and constant angular momentum, we say that it is in equilibrium. In symbols,

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0 \text{ and } \frac{\mathrm{d}L}{\mathrm{d}t} = 0 \qquad \text{[conditions for equilibrium]}$$

(or equivalently, zero total force and zero total torque). This is a scientific redefinition of the common English word, since in ordinary speech nobody would describe a car spinning out on an icy road as being in equilibrium.

t / The windmills are not closed systems, but angular momentum is being transferred out of them at the same rate it is transferred in, resulting in constant angular momentum. To get an idea of the huge scale of the modern windmill farm, note the sizes of the trucks and trailers.

Very commonly, however, we are interested in cases where an object is not only in equilibrium but also at rest, and this corresponds more closely to the usual meaning of the word. Trees and bridges have been designed by evolution and engineers to stay at rest, and to do so they must have not just zero total force acting on them but zero total torque. It is not enough that they don't fall down, they also must not tip over. Statics is the branch of physics concerned with problems such as these.

Solving statics problems is now simply a matter of applying and combining some things you already know:

- You know the behaviors of the various types of forces, for example that a frictional force is always parallel to the surface of contact.
- You know about vector addition of forces. It is the vector sum of the forces that must equal zero to produce equilibrium.
- You know about torque. The total torque acting on an object must be zero if it is to be in equilibrium.
- You know that the choice of axis is arbitrary, so you can make a choice of axis that makes the problem easy to solve.

In general, this type of problem could involve four equations in four unknowns: three equations that say the force components add up to zero, and one equation that says the total torque is zero. Most cases you'll encounter will not be this complicated. In the following example, only the equation for zero total torque is required in order to get an answer.

Art! *example 9*

▷ The abstract sculpture shown in figure u contains a cube of mass m and sides of length b. The cube rests on top of a cylinder, which is off-center by a distance a. Find the tension in the cable.

▷ There are four forces on the cube: a gravitational force mg, the force F_T from the cable, the upward normal force from the cylinder, F_N, and the horizontal static frictional force from the cylinder, F_s.

The total force on the cube in the vertical direction is zero:

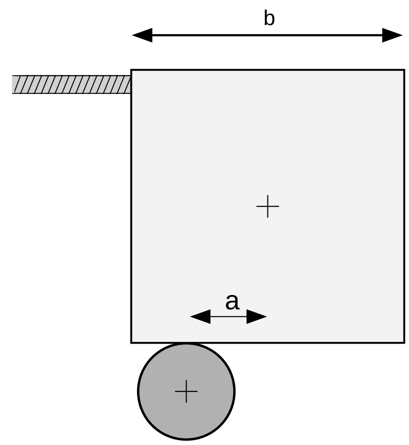
$$F_N - mg = 0.$$

As our axis for defining torques, it's convenient to choose the point of contact between the cube and the cylinder, because then neither F_s nor F_N makes any torque. The cable's torque is counterclockwise, the torque due to gravity is clockwise. Letting counterclockwise torques be positive, and using the convenient equation $\tau = r_\perp F$, we find the equation for the total torque:

$$bF_T - mga = 0.$$

We could also write down the equation saying that the total horizontal force is zero, but that would bring in the cylinder's frictional force on the cube, which we don't know and don't need to find. We already have two equations in the two unknowns F_T and F_N, so there's no need to make it into three equations in three unknowns. Solving the first equation for $F_N = mg$, we then substitute into the second equation to eliminate F_N, and solve for $F_T = (a/b)mg$.

As a check, our result makes sense when $a = 0$; the cube is balanced on the cylinder, so the cable goes slack.

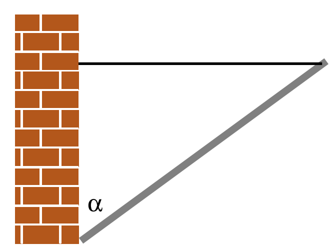


u / Example 9.

A flagpole *example 10*

▷ A 10-kg flagpole is being held up by a lightweight horizontal cable, and is propped against the foot of a wall as shown in the figure. If the cable is only capable of supporting a tension of 70 N, how great can the angle α be without breaking the cable?

▷ All three objects in the figure are supposed to be in equilibrium: the pole, the cable, and the wall. Whichever of the three objects we pick to investigate, all the forces and torques on it have to cancel out. It is not particularly helpful to analyze the forces and torques on the wall, since it has forces on it from the ground that are not given and that we don't want to find. We could study the forces and torques on the cable, but that doesn't let us use the given information about the pole. The object we need to analyze is the pole.



v / Example 10.

The pole has three forces on it, each of which may also result in a torque: (1) the gravitational force, (2) the cable's force, and (3) the wall's force.

We are free to define an axis of rotation at any point we wish, and it is helpful to define it to lie at the bottom end of the pole, since by that definition the wall's force on the pole is applied at $r = 0$ and thus makes no torque on the pole. This is good, because we don't know what the wall's force on the pole is, and we are not trying to find it.

With this choice of axis, there are two nonzero torques on the pole, a counterclockwise torque from the cable and a clockwise torque from gravity. Choosing to represent counterclockwise torques as positive numbers, and using the equation $|\tau| = r|F|\sin\theta$, we have

$$r_{cable}|F_{cable}|\sin\theta_{cable} - r_{grav}|F_{grav}|\sin\theta_{grav} = 0.$$

A little geometry gives $\theta_{cable} = 90° - \alpha$ and $\theta_{grav} = \alpha$, so

$$r_{cable}|F_{cable}|\sin(90° - \alpha) - r_{grav}|F_{grav}|\sin\alpha = 0.$$

The gravitational force can be considered as acting at the pole's center of mass, i.e., at its geometrical center, so r_{cable} is twice r_{grav}, and we can simplify the equation to read

$$2|F_{cable}|\sin(90° - \alpha) - |F_{grav}|\sin\alpha = 0.$$

These are all quantities we were given, except for α, which is the angle we want to find. To solve for α we need to use the trig identity $\sin(90° - x) = \cos x$,

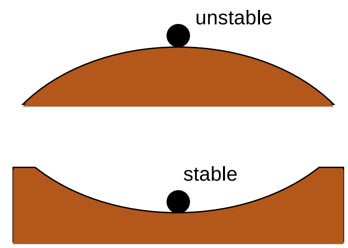
$$2|F_{cable}|\cos\alpha - |F_{grav}|\sin\alpha = 0,$$

which allows us to find

$$\begin{aligned}
\tan\alpha &= 2\frac{|\mathbf{F}_{cable}|}{|\mathbf{F}_{grav}|} \\
\alpha &= \tan^{-1}\left(2\frac{|\mathbf{F}_{cable}|}{|\mathbf{F}_{grav}|}\right) \\
&= \tan^{-1}\left(2\times\frac{70\ \text{N}}{98\ \text{N}}\right) \\
&= 55°.
\end{aligned}$$

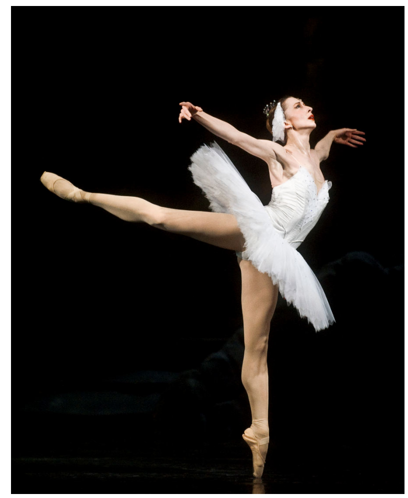
Stable and unstable equilibria

A pencil balanced upright on its tip could theoretically be in equilibrium, but even if it was initially perfectly balanced, it would topple in response to the first air current or vibration from a passing truck. The pencil can be put in equilibrium, but not in stable equilibrium. The things around us that we really do see staying still are all in stable equilibrium.



w / Stable and unstable equilibria.

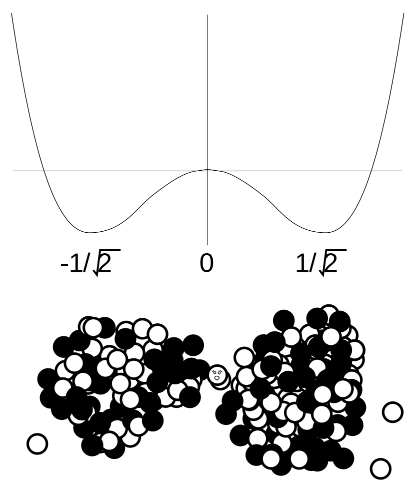
Why is one equilibrium stable and another unstable? Try pushing your own nose to the left or the right. If you push it a millimeter to the left, your head responds with a gentle force to the right, which keeps your nose from flying off of your face. If you push your nose a centimeter to the left, your face's force on your nose becomes much stronger. The defining characteristic of a stable equilibrium is that the farther the object is moved away from equilibrium, the stronger the force is that tries to bring it back.


x / The dancer's equilibrium is unstable. If she didn't constantly make tiny adjustments, she would tip over.

The opposite is true for an unstable equilibrium. In the top figure, the ball resting on the round hill theoretically has zero total force on it when it is exactly at the top. But in reality the total force will not be exactly zero, and the ball will begin to move off to one side. Once it has moved, the net force on the ball is greater than it was, and it accelerates more rapidly. In an unstable equilibrium, the farther the object gets from equilibrium, the stronger the force that pushes it farther from equilibrium.

This idea can be rephrased in terms of energy. The difference between the stable and unstable equilibria shown in figure w is that in the stable equilibrium, the potential energy is at a minimum, and moving to either side of equilibrium will increase it, whereas the unstable equilibrium represents a maximum.

Note that we are using the term "stable" in a weaker sense than in ordinary speech. A domino standing upright is stable in the sense we are using, since it will not spontaneously fall over in response to a sneeze from across the room or the vibration from a passing truck. We would only call it unstable in the technical sense if it could be toppled by *any* force, no matter how small. In everyday usage, of course, it would be considered unstable, since the force required to topple it is so small.



y / Example 11.

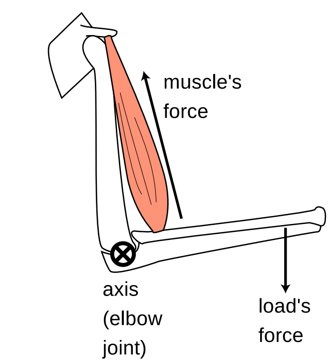
An application of calculus *example 11*

▷ Nancy Neutron is living in a uranium nucleus that is undergoing fission. Nancy's potential energy as a function of position can be approximated by $PE = x^4 - x^2$, where all the units and numerical constants have been suppressed for simplicity. Use calculus to locate the equilibrium points, and determine whether they are stable or unstable.

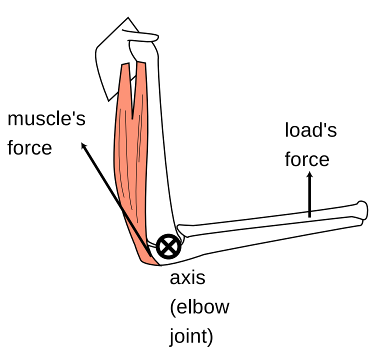
▷ The equilibrium points occur where the PE is at a minimum or maximum, and minima and maxima occur where the derivative (which equals minus the force on Nancy) is zero. This derivative is $dPE/dx = 4x^3 - 2x$, and setting it equal to zero, we have $x = 0, \pm 1/\sqrt{2}$. Minima occur where the second derivative is positive, and maxima where it is negative. The second derivative is $12x^2 - 2$, which is negative at $x = 0$ (unstable) and positive at $x = \pm 1/\sqrt{2}$ (stable). Interpretation: the graph of the PE is shaped like a rounded letter 'W,' with the two troughs representing the two halves of the splitting nucleus. Nancy is going to have to decide which half she wants to go with.

15.6 Simple Machines: the lever

Although we have discussed some simple machines such as the pulley, without the concept of torque we were not yet ready to address the lever, which is the machine nature used in designing living things, almost to the exclusion of all others. (We can speculate what life on our planet might have been like if living things had evolved wheels, gears, pulleys, and screws.) The figures show two examples of levers within your arm. Different muscles are used to flex and extend the arm, because muscles work only by contraction.



z / The biceps muscle flexes the arm.



aa / The triceps extends the arm.

Analyzing example z physically, there are two forces that do work. When we lift a load with our biceps muscle, the muscle does positive work, because it brings the bone in the forearm in the direction it is moving. The load's force on the arm does negative work, because the arm moves in the direction opposite to the load's force. This makes sense, because we expect our arm to do positive work on the load, so the load must do an equal amount of negative work on the arm. (If the biceps was lowering a load, the signs of the works would be reversed. Any muscle is capable of doing either positive or negative work.)

There is also a third force on the forearm: the force of the upper arm's bone exerted on the forearm at the elbow joint (not shown with an arrow in the figure). This force does no work, because the elbow joint is not moving.

Because the elbow joint is motionless, it is natural to define our torques using the joint as the axis. The situation now becomes quite simple, because the upper arm bone's force exerted at the elbow neither does work nor creates a torque. We can ignore it completely. In any lever there is such a point, called the fulcrum.

If we restrict ourselves to the case in which the forearm rotates with constant angular momentum, then we know that the total torque on the forearm is zero,

$$\tau_{\text{muscle}} + \tau_{\text{load}} = 0.$$

If we choose to represent counterclockwise torques as positive, then the muscle's torque is positive, and the load's is negative. In terms of their absolute values,

$$|\tau_{\text{muscle}}| = |\tau_{\text{load}}|.$$

Assuming for simplicity that both forces act at angles of 90°with respect to the lines connecting the axis to the points at which they act, the absolute values of the torques are

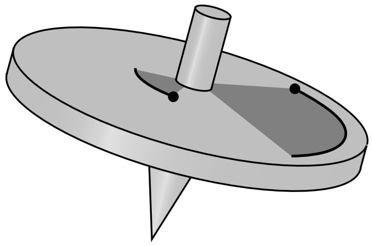
$$r_{\text{muscle}} F_{\text{muscle}} = r_{\text{load}} F_{\text{arm}},$$

where r_{muscle}, the distance from the elbow joint to the biceps' point of insertion on the forearm, is only a few cm, while r_{load} might be 30 cm or so. The force exerted by the muscle must therefore be about ten times the force exerted by the load. We thus see that this lever is a force reducer. In general, a lever may be used either to increase or to reduce a force.

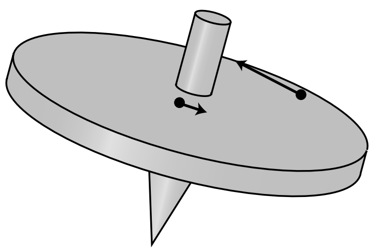
Why did our arms evolve so as to reduce force? In general, your body is built for compactness and maximum speed of motion rather than maximum force. This is the main anatomical difference between us and the Neanderthals (their brains covered the same range of sizes as those of modern humans), and it seems to have worked for us.

As with all machines, the lever is incapable of changing the amount of mechanical work we can do. A lever that increases force will always reduce motion, and vice versa, leaving the amount of work unchanged.

It is worth noting how simple and yet how powerful this analysis was. It was simple because we were well prepared with the concepts of torque and mechanical work. In anatomy textbooks, whose readers are assumed not to know physics, there is usually a long and complicated discussion of the different types of levers. For example, the biceps lever, z, would be classified as a class III lever, since it has the fulcrum and load on the ends and the muscle's force acting in the middle. The triceps, aa, is called a class I lever, because the load and muscle's force are on the ends and the fulcrum is in the middle. How tiresome! With a firm grasp of the concept of torque, we realize that all such examples can be analyzed in much the same way. Physics is at its best when it lets us understand many apparently complicated phenomena in terms of a few simple yet powerful concepts.



ab / The two atoms cover the same angle in a given time interval.



ac / Their velocity vectors, however, differ in both magnitude and direction.

15.7 Rigid-body rotation

Kinematics

When a rigid object rotates, every part of it (every atom) moves in a circle, covering the same angle in the same amount of time, ab. Every atom has a different velocity vector, ac. Since all the velocities are different, we can't measure the speed of rotation of the top by giving a single velocity. We can, however, specify its speed of rotation consistently in terms of angle per unit time. Let the position of some reference point on the top be denoted by its angle θ, measured in a circle around the axis. For reasons that will become more apparent shortly, we measure all our angles in radians. Then the change in the angular position of any point on the top can be written as $\mathrm{d}\theta$, and all parts of the top have the same value of $\mathrm{d}\theta$ over a certain time interval $\mathrm{d}t$. We define the angular velocity, ω (Greek omega),

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} \qquad .$$

[definition of angular velocity; θ in units of radians]

The relationship between ω and t is exactly analogous to that between x and t for the motion of a particle through space.

self-check B

If two different people chose two different reference points on the top in order to define θ=0, how would their θ-t graphs differ? What effect would this have on the angular velocities? ▷ Answer, p. 561

The angular velocity has units of radians per second, rad/s. However, radians are not really units at all. The radian measure of an angle is defined, as the length of the circular arc it makes, divided by the radius of the circle. Dividing one length by another gives a unitless quantity, so anything with units of radians is really unitless. We can therefore simplify the units of angular velocity, and call them inverse seconds, s^{-1}.

A 78-rpm record *example 12*

▷ In the early 20th century, the standard format for music recordings was a plastic disk that held a single song and rotated at 78 rpm (revolutions per minute). What was the angular velocity of such a disk?

▷ If we measure angles in units of revolutions and time in units of minutes, then 78 rpm is the angular velocity. Using standard physics units of radians/second, however, we have

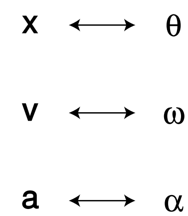
$$\frac{\text{78 revolutions}}{\text{1 minute}} \times \frac{2\pi\text{ radians}}{\text{1 revolution}} \times \frac{\text{1 minute}}{\text{60 seconds}} = 8.2\ \mathrm{s}^{-1}.$$

In the absence of any torque, a rigid body will rotate indefinitely with the same angular velocity. If the angular velocity is changing because of a torque, we define an angular acceleration,

$$\alpha = \frac{d\omega}{dt}, \qquad \text{[definition of angular acceleration]}$$

The symbol is the Greek letter alpha. The units of this quantity are rad/s^2, or simply s^{-2}.

The mathematical relationship between ω and θ is the same as the one between v and x, and similarly for α and a. We can thus make a system of analogies, ad, and recycle all the familiar kinematic equations for constant-acceleration motion.



ad / Analogies between rotational and linear quantities.

The synodic period *example 13*

Mars takes nearly twice as long as the Earth to complete an orbit. If the two planets are alongside one another on a certain day, then one year later, Earth will be back at the same place, but Mars will have moved on, and it will take more time for Earth to finish catching up. Angular velocities add and subtract, just as velocity vectors do. If the two planets' angular velocities are ω_1 and ω_2, then the angular velocity of one relative to the other is $\omega_1 - \omega_2$. The corresponding period, $1/(1/T_1 - 1/T_2)$ is known as the synodic period.

A neutron star *example 14*

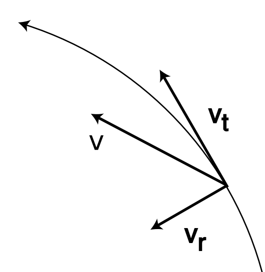
▷ A neutron star is initially observed to be rotating with an angular velocity of $2.0\ s^{-1}$, determined via the radio pulses it emits. If its angular acceleration is a constant $-1.0 \times 10^{-8}\ s^{-2}$, how many rotations will it complete before it stops? (In reality, the angular acceleration is not always constant; sudden changes often occur, and are referred to as "starquakes!")

▷ The equation $v_f^2 - v_i^2 = 2a\Delta x$ can be translated into $\omega_f^2 - \omega_i^2 = 2\alpha\Delta\theta$, giving

$$\begin{aligned}\Delta\theta &= (\omega_f^2 - \omega_i^2)/2\alpha \\ &= 2.0 \times 10^8 \text{ radians} \\ &= 3.2 \times 10^7 \text{ rotations.}\end{aligned}$$

Relations between angular quantities and motion of a point

It is often necessary to be able to relate the angular quantities to the motion of a particular point on the rotating object. As we develop these, we will encounter the first example where the advantages of radians over degrees become apparent.



ae / We construct a coordinate system that coincides with the location and motion of the moving point of interest at a certain moment.

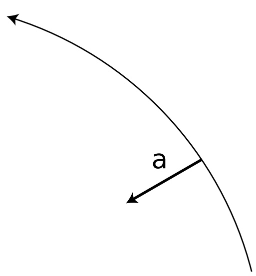
The speed at which a point on the object moves depends on both the object's angular velocity ω and the point's distance r from the axis. We adopt a coordinate system, ae, with an inward (radial) axis and a tangential axis. The length of the infinitesimal circular arc ds traveled by the point in a time interval dt is related to $d\theta$ by the definition of radian measure, $d\theta = ds/r$, where positive and negative values of ds represent the two possible directions of motion along the tangential axis. We then have $v_t = ds/\,dt = r\,d\theta/\,dt = \omega r$, or

$$v_t = \omega r. \qquad \text{[tangential velocity of a point at a distance } r \text{ from the axis of rotation]}$$

The radial component is zero, since the point is not moving inward or outward,

$$v_r = 0. \qquad \text{[radial velocity of a point at a distance } r \text{ from the axis of rotation]}$$

Note that we had to use the definition of radian measure in this derivation. Suppose instead we had used units of degrees for our angles and degrees per second for angular velocities. The relationship between $d\theta_{degrees}$ and ds is $d\theta_{degrees} = (360/2\pi)s/r$, where the extra conversion factor of $(360/2\pi)$ comes from that fact that there are 360 degrees in a full circle, which is equivalent to 2π radians. The equation for v_t would then have been $v_t = (2\pi/360)(\omega_{degrees\ per\ second})(r)$, which would have been much messier. Simplicity, then, is the reason for using radians rather than degrees; by using radians we avoid infecting all our equations with annoying conversion factors.



af / Even if the rotating object has zero angular acceleration, every point on it has an acceleration towards the center.

Since the velocity of a point on the object is directly proportional to the angular velocity, you might expect that its acceleration would be directly proportional to the angular acceleration. This is not true, however. Even if the angular acceleration is zero, i.e., if the object is rotating at constant angular velocity, every point on it will have an acceleration vector directed toward the axis, af. As derived on page 264, the magnitude of this acceleration is

$$a_r = \omega^2 r. \qquad \text{[radial acceleration of a point at a distance } r \text{ from the axis]}$$

For the tangential component, any change in the angular velocity $d\omega$ will lead to a change $d\omega \cdot r$ in the tangential velocity, so it is easily shown that

$$a_t = \alpha r. \qquad \text{[tangential acceleration of a point at a distance } r \text{ from the axis]}$$

self-check C

Positive and negative signs of ω represent rotation in opposite directions. Why does it therefore make sense physically that ω is raised to the first power in the equation for v_t and to the second power in the one for a_r? ▷ Answer, p. 561

Radial acceleration at the surface of the Earth *example 15*

▷ What is your radial acceleration due to the rotation of the earth if you are at the equator?

▷ At the equator, your distance from the Earth's rotation axis is the same as the radius of the spherical Earth, 6.4×10^6 m. Your angular velocity is

$$\begin{aligned}\omega &= \frac{2\pi \text{ radians}}{1 \text{ day}} \\ &= 7.3 \times 10^{-5}\ \text{s}^{-1},\end{aligned}$$

which gives an acceleration of

$$\begin{aligned}a_r &= \omega^2 r \\ &= 0.034\ \text{m/s}^2.\end{aligned}$$

The angular velocity was a very small number, but the radius was a very big number. Squaring a very small number, however, gives a very very small number, so the ω^2 factor "wins," and the final result is small.

If you're standing on a bathroom scale, this small acceleration is provided by the imbalance between the downward force of gravity and the slightly weaker upward normal force of the scale on your foot. The scale reading is therefore a little lower than it should be.

Dynamics

If we want to connect all this kinematics to anything dynamical, we need to see how it relates to torque and angular momentum. Our strategy will be to tackle angular momentum first, since angular momentum relates to motion, and to use the additive property of angular momentum: the angular momentum of a system of particles equals the sum of the angular momenta of all the individual particles. The angular momentum of one particle within our rigidly rotating object, $L = mv_\perp r$, can be rewritten as $L = r\,p \sin\theta$, where r and p are the magnitudes of the particle's $\mathbf{r}$ and momentum vectors, and θ is the angle between these two vectors. (The $\mathbf{r}$ vector points outward perpendicularly from the axis to the particle's position in space.) In rigid-body rotation the angle θ is $90°$, so we have simply $L = rp$. Relating this to angular velocity, we have $L = rp = (r)(mv) = (r)(m\omega r) = mr^2\omega$. The particle's contribution to the total angular momentum is proportional to ω, with a proportionality constant mr^2. We refer to mr^2 as the particle's contribution to the object's total *moment of inertia*, I, where "moment" is used in the sense of "important," as in "momentous" — a bigger value of I tells us the particle is more important for determining the total angular momentum. The total moment of inertia

x	$\longleftrightarrow$	θ
v	$\longleftrightarrow$	ω
a	$\longleftrightarrow$	α
m	$\longleftrightarrow$	I
p	$\longleftrightarrow$	L
F	$\longleftrightarrow$	τ

ag / Analogies between rotational and linear quantities.

is

$$I = \sum m_i r_i^2, \qquad \text{[definition of the moment of inertia; for rigid-body rotation in a plane; } r \text{ is the distance from the axis, measured perpendicular to the axis]}$$

The angular momentum of a rigidly rotating body is then

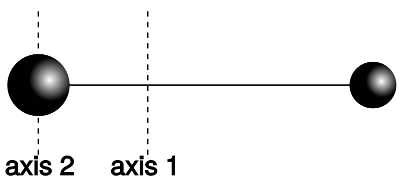
$$L = I\omega. \qquad \text{[angular momentum of rigid-body rotation in a plane]}$$

Since torque is defined as $\mathrm{d}L/\,\mathrm{d}t$, and a rigid body has a constant moment of inertia, we have $\tau = \mathrm{d}L/\,\mathrm{d}t = I\,\mathrm{d}\omega/\,\mathrm{d}t = I\alpha$,

$$\tau = I\alpha, \qquad \text{[relationship between torque and angular acceleration for rigid-body rotation in a plane]}$$

which is analogous to $F = ma$.

The complete system of analogies between linear motion and rigid-body rotation is given in figure ag.



ah / Example 16

A barbell *example 16*

▷ The barbell shown in figure ah consists of two small, dense, massive balls at the ends of a very light rod. The balls have masses of 2.0 kg and 1.0 kg, and the length of the rod is 3.0 m. Find the moment of inertia of the rod (1) for rotation about its center of mass, and (2) for rotation about the center of the more massive ball.

▷ (1) The ball's center of mass lies 1/3 of the way from the greater mass to the lesser mass, i.e., 1.0 m from one and 2.0 m from the other. Since the balls are small, we approximate them as if they were two pointlike particles. The moment of inertia is

$$\begin{aligned} I &= (2.0\ \text{kg})(1.0\ \text{m})^2 + (1.0\ \text{kg})(2.0\ \text{m})^2 \\ &= 2.0\ \text{kg}\cdot\text{m}^2 + 4.0\ \text{kg}\cdot\text{m}^2 \\ &= 6.0\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

Perhaps counterintuitively, the less massive ball contributes far more to the moment of inertia.

(2) The big ball theoretically contributes a little bit to the moment of inertia, since essentially none of its atoms are exactly at r=0. However, since the balls are said to be small and dense, we assume all the big ball's atoms are so close to the axis that we can ignore their small contributions to the total moment of inertia:

$$\begin{aligned} I &= (1.0\ \text{kg})(3.0\ \text{m})^2 \\ &= 9.0\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

This example shows that the moment of inertia depends on the choice of axis. For example, it is easier to wiggle a pen about its center than about one end.

The parallel axis theorem *example 17*

▷ Generalizing the previous example, suppose we pick any axis parallel to axis 1, but offset from it by a distance h. Part (2) of the previous example then corresponds to the special case of $h = -1.0$ m (negative being to the left). What is the moment of inertia about this new axis?

▷ The big ball's distance from the new axis is (1.0 m)+h, and the small one's is (2.0 m)-h. The new moment of inertia is

$$\begin{aligned} I &= (2.0\ \text{kg})[(1.0\ \text{m})+h]^2 + (1.0\ \text{kg})[(2.0\ \text{m}) - h]^2 \\ &= 6.0\ \text{kg}\cdot\text{m}^2 + (4.0\ \text{kg}\cdot\text{m})h - (4.0\ \text{kg}\cdot\text{m})h + (3.0\ \text{kg})h^2. \end{aligned}$$

The constant term is the same as the moment of inertia about the center-of-mass axis, the first-order terms cancel out, and the third term is just the total mass multiplied by h^2. The interested reader will have no difficulty in generalizing this to any set of particles (problem 27, p. 483), resulting in the parallel axis theorem: If an object of total mass M rotates about a line at a distance h from its center of mass, then its moment of inertia equals $I_{cm} + Mh^2$, where I_{cm} is the moment of inertia for rotation about a parallel line through the center of mass.

Scaling of the moment of inertia *example 18*

▷ (1) Suppose two objects have the same mass and the same shape, but one is less dense, and larger by a factor k. How do their moments of inertia compare?

(2) What if the densities are equal rather than the masses?

▷ (1) This is like increasing all the distances between atoms by a factor k. All the r's become greater by this factor, so the moment of inertia is increased by a factor of k^2.

(2) This introduces an increase in mass by a factor of k^3, so the moment of inertia of the bigger object is greater by a factor of k^5.

Iterated integrals

In various places in this book, starting with subsection 15.7.5, we'll come across integrals stuck inside other integrals. These are known as iterated integrals, or double integrals, triple integrals, etc. Similar concepts crop up all the time even when you're not doing calculus, so let's start by imagining such an example. Suppose you want to count how many squares there are on a chess board, and you don't know how to multiply eight times eight. You could start from the upper left, count eight squares across, then continue with the second row, and so on, until you how counted every square, giving the result of 64. In slightly more formal mathematical language, we could write the following recipe: for each row, r, from 1 to 8, consider the columns, c, from 1 to 8, and add one to the count for each one of them. Using the sigma notation, this becomes

$$\sum_{r=1}^{8}\sum_{c=1}^{8} 1.$$

If you're familiar with computer programming, then you can think of this as a sum that could be calculated using a loop nested inside another loop. To evaluate the result (again, assuming we don't know how to multiply, so we have to use brute force), we can first evaluate the inside sum, which equals 8, giving

$$\sum_{r=1}^{8} 8.$$

Notice how the "dummy" variable c has disappeared. Finally we do the outside sum, over r, and find the result of 64.

Now imagine doing the same thing with the pixels on a TV screen. The electron beam sweeps across the screen, painting the pixels in each row, one at a time. This is really no different than the example of the chess board, but because the pixels are so small, you normally think of the image on a TV screen as continuous rather than discrete. This is the idea of an integral in calculus. Suppose we want to find the area of a rectangle of width a and height b, and we don't know that we can just multiply to get the area ab. The brute force way to do this is to break up the rectangle into a grid of infinitesimally small squares, each having width $\mathrm{d}x$ and height $\mathrm{d}y$, and therefore the infinitesimal area $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$. For convenience, we'll imagine that the rectangle's lower left corner is at the origin. Then the area is given by this integral:

$$\begin{aligned}\text{area} &= \int_{y=0}^{b}\int_{x=0}^{a} \mathrm{d}A \\ &= \int_{y=0}^{b}\int_{x=0}^{a} \mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Notice how the leftmost integral sign, over y, and the rightmost differential, $\mathrm{d}y$, act like bookends, or the pieces of bread on a sandwich. Inside them, we have the integral sign that runs over x, and the differential $\mathrm{d}x$ that matches it on the right. Finally, on the innermost layer, we'd normally have the thing we're integrating, but here's it's 1, so I've omitted it. Writing the lower limits of the integrals with $x =$ and $y =$ helps to keep it straight which integral goes with which differential. The result is

$$\begin{aligned}
\text{area} &= \int_{y=0}^{b}\int_{x=0}^{a} \mathrm{d}A \\
&= \int_{y=0}^{b}\int_{x=0}^{a} \mathrm{d}x\,\mathrm{d}y \\
&= \int_{y=0}^{b}\left(\int_{x=0}^{a} \mathrm{d}x\right)\mathrm{d}y \\
&= \int_{y=0}^{b} a\,\mathrm{d}y \\
&= a\int_{y=0}^{b} \mathrm{d}y \\
&= ab.
\end{aligned}$$

Area of a triangle *example 19*

▷ Find the area of a 45-45-90 right triangle having legs a.

▷ Let the triangle's hypotenuse run from the origin to the point (a, a), and let its legs run from the origin to $(0, a)$, and then to (a, a). In other words, the triangle sits on top of its hypotenuse. Then the integral can be set up the same way as the one before, but for a particular value of y, values of x only run from 0 (on the y axis) to y (on the hypotenuse). We then have

$$\begin{aligned}
\text{area} &= \int_{y=0}^{a}\int_{x=0}^{y} \mathrm{d}A \\
&= \int_{y=0}^{a}\int_{x=0}^{y} \mathrm{d}x\,\mathrm{d}y \\
&= \int_{y=0}^{a}\left(\int_{x=0}^{y} \mathrm{d}x\right)\mathrm{d}y \\
&= \int_{y=0}^{a} y\,\mathrm{d}y \\
&= \frac{1}{2}a^2
\end{aligned}$$

Note that in this example, because the upper end of the x values depends on the value of y, it makes a difference which order we do the integrals in. The x integral has to be on the inside, and we have to do it first.

Volume of a cube *example 20*

▷ Find the volume of a cube with sides of length a.

▷ This is a three-dimensional example, so we'll have integrals nested three deep, and the thing we're integrating is the volume $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

$$\begin{aligned}
\text{volume} &= \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{z=0}^{a}\int_{y=0}^{a} a\,\mathrm{d}y\,\mathrm{d}z \\
&= a\int_{z=0}^{a}\int_{y=0}^{a} \mathrm{d}y\,\mathrm{d}z \\
&= a\int_{z=0}^{a} a\,\mathrm{d}z \\
&= a^3
\end{aligned}$$

Area of a circle *example 21*

▷ Find the area of a circle.

▷ To make it easy, let's find the area of a semicircle and then double it. Let the circle's radius be r, and let it be centered on the origin and bounded below by the x axis. Then the curved edge is given by the equation $r^2 = x^2 + y^2$, or $y = \sqrt{r^2 - x^2}$. Since the y integral's limit depends on x, the x integral has to be on the outside. The area is

$$\begin{aligned}
\text{area} &= \int_{x=-r}^{r}\int_{y=0}^{\sqrt{r^2-x^2}} \mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=-r}^{r} \sqrt{r^2-x^2}\,\mathrm{d}x \\
&= r\int_{x=-r}^{r} \sqrt{1-(x/r)^2}\,\mathrm{d}x.
\end{aligned}$$

Substituting $u = x/r$,

$$\text{area} = r^2\int_{u=-1}^{1} \sqrt{1-u^2}\,\mathrm{d}u$$

The definite integral equals π, as you can find using a trig substitution or simply by looking it up in a table, and the result is, as expected, $\pi r^2/2$ for the area of the semicircle.

Finding moments of inertia by integration

When calculating the moment of inertia of an ordinary-sized object with perhaps 10^{26} atoms, it would be impossible to do an actual sum over atoms, even with the world's fastest supercomputer. Calculus, however, offers a tool, the integral, for breaking a sum down to infinitely many small parts. If we don't worry about the existence of atoms, then we can use an integral to compute a moment of inertia as if the object was smooth and continuous throughout, rather than granular at the atomic level. Of course this granularity typically has a negligible effect on the result unless the object is itself an individual molecule. This subsection consists of three examples of how to do such a computation, at three distinct levels of mathematical complication.

Moment of inertia of a thin rod

What is the moment of inertia of a thin rod of mass M and length L about a line perpendicular to the rod and passing through its center? We generalize the discrete sum

$$I = \sum m_i r_i^2$$

to a continuous one,

$$\begin{aligned} I &= \int r^2 \, \mathrm{d}m \\ &= \int_{-L/2}^{L/2} x^2 \, \frac{M}{L} \, \mathrm{d}x \qquad [r = |x|, \text{ so } r^2 = x^2] \\ &= \frac{1}{12} M L^2 \end{aligned}$$

In this example the object was one-dimensional, which made the math simple. The next example shows a strategy that can be used to simplify the math for objects that are three-dimensional, but possess some kind of symmetry.

Moment of inertia of a disk

What is the moment of inertia of a disk of radius b, thickness t, and mass M, for rotation about its central axis?

We break the disk down into concentric circular rings of thickness $\mathrm{d}r$. Since all the mass in a given circular slice has essentially the same value of r (ranging only from r to $r + \mathrm{d}r$), the slice's contribution to the total moment of inertia is simply $r^2 \, \mathrm{d}m$. We then have

$$\begin{aligned} I &= \int r^2 \, \mathrm{d}m \\ &= \int r^2 \rho \, \mathrm{d}V, \end{aligned}$$

where $V = \pi b^2 t$ is the total volume, $\rho = M/V = M/\pi b^2 t$ is the density, and the volume of one slice can be calculated as the volume enclosed by its outer surface minus the volume enclosed by its inner surface, $\mathrm{d}V = \pi(r + \mathrm{d}r)^2 t - \pi r^2 t = 2\pi t r \, \mathrm{d}r$.

$$\begin{aligned} I &= \int_0^b r^2 \frac{M}{\pi b^2 t} \, 2\pi t \, r \, \mathrm{d}r \\ &= \frac{1}{2} M b^2. \end{aligned}$$

In the most general case where there is no symmetry about the rotation axis, we must use iterated integrals, as discussed in subsection 15.7.4. The example of the disk possessed two types of symmetry with respect to the rotation axis: (1) the disk is the same when rotated through any angle about the axis, and (2) all slices perpendicular to the axis are the same. These two symmetries reduced the number of layers of integrals from three to one. The following example possesses only one symmetry, of type (2), and we simply set it up as a triple integral. You may not have seen multiple integrals yet in a math course. If so, just skim this example.

Moment of inertia of a cube

What is the moment of inertia of a cube of side b, for rotation about an axis that passes through its center and is parallel to four of its faces? Let the origin be at the center of the cube, and let x be the rotation axis.

$$\begin{aligned}
I &= \int r^2 \, \mathrm{d}m \\
&= \rho \int r^2 \, \mathrm{d}V \\
&= \rho \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \left(y^2 + z^2\right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \left(y^2 + z^2\right) \mathrm{d}y \, \mathrm{d}z
\end{aligned}$$

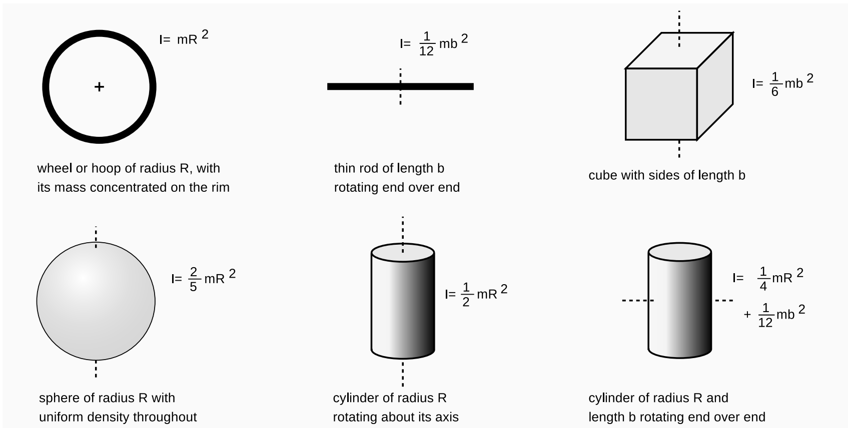
The fact that the last step is a trivial integral results from the symmetry of the problem. The integrand of the remaining double integral breaks down into two terms, each of which depends on only one of the variables, so we break it into two integrals,

$$I = \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} y^2 \, \mathrm{d}y \, \mathrm{d}z + \rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} z^2 \, \mathrm{d}y \, \mathrm{d}z$$

which we know have identical results. We therefore only need to evaluate one of them and double the result:

$$\begin{aligned}
I &= 2\rho b \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} z^2 \, \mathrm{d}y \, \mathrm{d}\, z \\
&= 2\rho b^2 \int_{-b/2}^{b/2} z^2 \, \mathrm{d}z \\
&= \frac{1}{6} \rho b^5 \\
&= \frac{1}{6} M b^2
\end{aligned}$$

Figure ai shows the moments of inertia of some shapes, which were evaluated with techniques like these.



ai / Moments of inertia of some geometric shapes.

The hammer throw *example 22*

▷ In the men's Olympic hammer throw, a steel ball of radius 6.1 cm is swung on the end of a wire of length 1.22 m. What fraction of the ball's angular momentum comes from its rotation, as opposed to its motion through space?

▷ It's always important to solve problems symbolically first, and plug in numbers only at the end, so let the radius of the ball be b, and the length of the wire ℓ. If the time the ball takes to go once around the circle is T, then this is also the time it takes to revolve once around its own axis. Its speed is $v = 2\pi\ell/T$, so its angular momentum due to its motion through space is $mv\ell = 2\pi m\ell^2/T$. Its angular momentum due to its rotation around its own center is $(4\pi/5)mb^2/T$. The ratio of these two angular momenta is $(2/5)(b/\ell)^2 = 1.0\times10^{-3}$. The angular momentum due to the ball's spin is extremely small.

Toppling a rod *example 23*

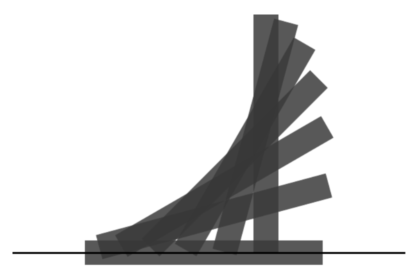
▷ A rod of length b and mass m stands upright. We want to strike the rod at the bottom, causing it to fall and land flat. Find the momentum, p, that should be delivered, in terms of m, b, and g. Can this really be done without having the rod scrape on the floor?

▷ This is a nice example of a question that can very nearly be answered based only on units. Since the three variables, m, b, and g, all have different units, they can't be added or subtracted. The only way to combine them mathematically is by multiplication or division. Multiplying one of them by itself is exponentiation, so in general we expect that the answer must be of the form

$$p = Am^j b^k g^l,$$

where A, j, k, and l are unitless constants. The result has to have units of kg·m/s. To get kilograms to the first power, we need

$$j = 1,$$



aj / Example 23.

meters to the first power requires

$$k + l = 1,$$

and seconds to the power -1 implies

$$l = 1/2.$$

We find $j = 1$, $k = 1/2$, and $l = 1/2$, so the solution must be of the form

$$p = Am\sqrt{bg}.$$

Note that no physics was required!

Consideration of units, however, won't help us to find the unitless constant A. Let t be the time the rod takes to fall, so that $(1/2)gt^2 = b/2$. If the rod is going to land exactly on its side, then the number of revolutions it completes while in the air must be 1/4, or 3/4, or 5/4, ..., but all the possibilities greater than 1/4 would cause the head of the rod to collide with the floor prematurely. The rod must therefore rotate at a rate that would cause it to complete a full rotation in a time $T = 4t$, and it has angular momentum $L = (\pi/6)mb^2/T$.

The momentum lost by the object striking the rod is p, and by conservation of momentum, this is the amount of momentum, in the horizontal direction, that the rod acquires. In other words, the rod will fly forward a little. However, this has no effect on the solution to the problem. More importantly, the object striking the rod loses angular momentum $bp/2$, which is also transferred to the rod. Equating this to the expression above for L, we find $p = (\pi/12)m\sqrt{bg}$.

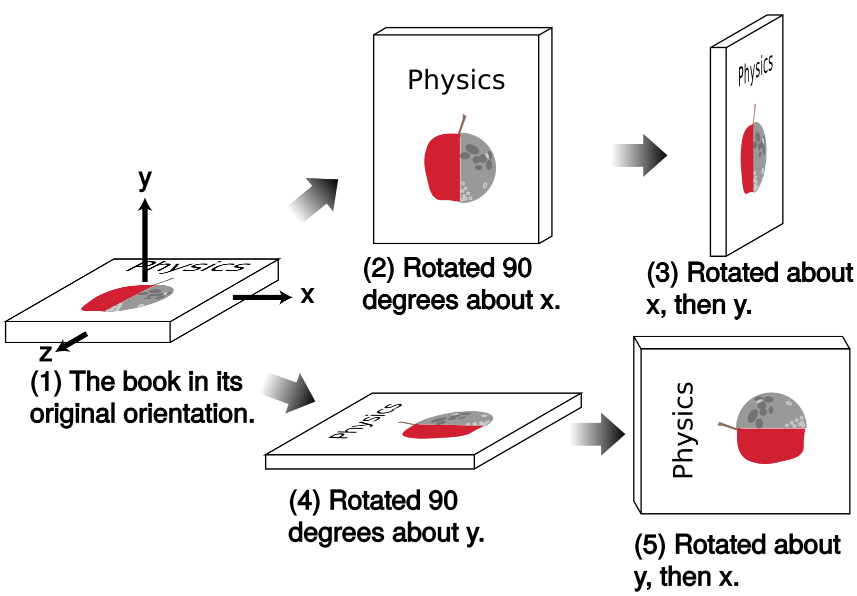
Finally, we need to know whether this can really be done without having the foot of the rod scrape on the floor. The figure shows that the answer is no for this rod of finite width, but it appears that the answer would be yes for a sufficiently thin rod. This is analyzed further in homework problem 46 on page 486.

15.8 Angular momentum in three dimensions

Conservation of angular momentum produces some surprising phenomena when extended to three dimensions. Try the following experiment, for example. Take off your shoe, and toss it in to the air, making it spin along its long (toe-to-heel) axis. You should observe a nice steady pattern of rotation. The same happens when you spin the shoe about its shortest (top-to-bottom) axis. But something unexpected happens when you spin it about its third (left-to-right) axis, which is intermediate in length between the other two. Instead of a steady pattern of rotation, you will observe something more complicated, with the shoe changing its orientation with respect to the rotation axis.

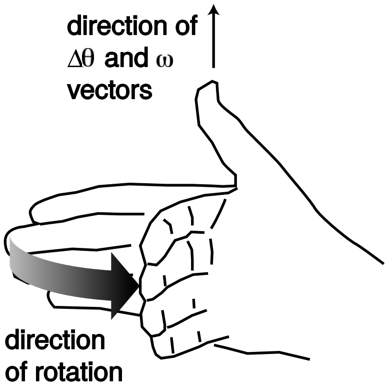
Rigid-body kinematics in three dimensions

How do we generalize rigid-body kinematics to three dimensions? When we wanted to generalize the kinematics of a moving particle to three dimensions, we made the numbers r, v, and a into vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$. This worked because these quantities all obeyed the same laws of vector addition. For instance, one of the laws of vector addition is that, just like addition of numbers, vector addition gives the same result regardless of the order of the two quantities being added. Thus you can step sideways 1 m to the right and then step forward 1 m, and the end result is the same as if you stepped forward first and then to the side. In order words, it didn't matter whether you took $\Delta\mathbf{r}_1 + \Delta\mathbf{r}_2$ or $\Delta\mathbf{r}_2 + \Delta\mathbf{r}_1$. In math this is called the commutative property of addition.



ak / Performing the rotations in one order gives one result, 3, while reversing the order gives a different result, 5.

Angular motion, unfortunately doesn't have this property, as shown in figure ak. Doing a rotation about the x axis and then about y gives one result, while doing them in the opposite order gives a different result. These operations don't "commute," i.e., it makes a difference what order you do them in.



al / The right-hand rule for associating a vector with a direction of rotation.

This means that there is in general no possible way to construct a $\Delta\boldsymbol{\theta}$ vector. However, if you try doing the operations shown in figure ak using small rotation, say about 10 degrees instead of 90, you'll find that the result is nearly the same regardless of what order you use; small rotations are very nearly commutative. Not only that, but the result of the two 10-degree rotations is about the same as a single, somewhat larger, rotation about an axis that lies symmetrically at between the x and y axes at 45 degree angles to each one. This is exactly what we would expect if the two small rotations did act like vectors whose directions were along the axis of rotation. We therefore define a $\mathrm{d}\boldsymbol{\theta}$ vector whose magnitude is the amount of rotation in units of radians, and whose direction is along the axis of rotation. Actually this definition is ambiguous, because there it could point in either direction along the axis. We therefore use a right-hand rule as shown in figure al to define the direction of the $\mathrm{d}\boldsymbol{\theta}$ vector, and the $\boldsymbol{\omega}$ vector, $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\theta}/\,\mathrm{d}t$, based on it. Aliens on planet Tammyfaye may decide to define it using their left hands rather than their right, but as long as they keep their scientific literature separate from ours, there is no problem. When entering a physics exam, always be sure to write a large warning note on your left hand in magic marker so that you won't be tempted to use it for the right-hand rule while keeping your pen in your right.

self-check D

Use the right-hand rule to determine the directions of the ω vectors in each rotation shown in figures ak/1 through ak/5. ▷ Answer, p. 561

Because the vector relationships among $\mathrm{d}\boldsymbol{\theta}$, $\boldsymbol{\omega}$, and $\boldsymbol{\alpha}$ are strictly analogous to the ones involving $\mathrm{d}\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ (with the proviso that we avoid describing large rotations using $\Delta\boldsymbol{\theta}$ vectors), any operation in $\mathbf{r}$-$\mathbf{v}$-$\mathbf{a}$ vector kinematics has an exact analog in $\boldsymbol{\theta}$-$\boldsymbol{\omega}$-$\boldsymbol{\alpha}$ kinematics.

Result of successive 10-degree rotations — *example 24*

▷ What is the result of two successive (positive) 10-degree rotations about the x and y axes? That is, what single rotation about a single axis would be equivalent to executing these in succession?

▷ The result is only going to be approximate, since 10 degrees is not an infinitesimally small angle, and we are not told in what order the rotations occur. To some approximation, however, we can add the $\Delta\theta$ vectors in exactly the same way we would add $\Delta\mathbf{r}$ vectors, so we have

$$\begin{aligned}\Delta\theta &\approx \Delta\theta_1 + \Delta\theta_2\\ &\approx (10\text{ degrees})\hat{\mathbf{x}} + (10\text{ degrees})\hat{\mathbf{y}}.\end{aligned}$$

This is a vector with a magnitude of $\sqrt{(10\text{ deg})^2 + (10\text{ deg})^2}$ =

14 deg, and it points along an axis midway between the *x* and *y* axes.

Angular momentum in three dimensions

The vector cross product

In order to expand our system of three-dimensional kinematics to include dynamics, we will have to generalize equations like $v_t = \omega r$, $\tau = rF \sin\theta_{rF}$, and $L = rp \sin\theta_{rp}$, each of which involves three quantities that we have either already defined as vectors or that we want to redefine as vectors. Although the first one appears to differ from the others in its form, it could just as well be rewritten as $v_t = \omega r \sin\theta_{\omega r}$, since $\theta_{\omega r} = 90°$, and $\sin\theta_{\omega r} = 1$.

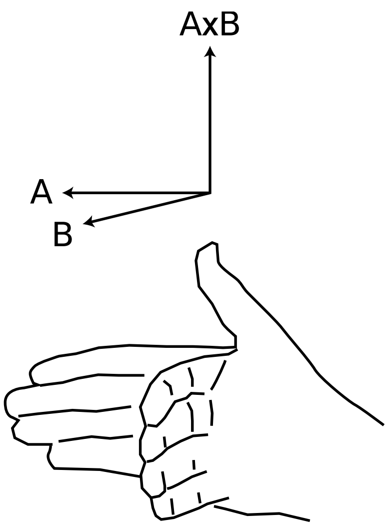
It thus appears that we have discovered something general about the physically useful way to relate three vectors in a multiplicative way: the magnitude of the result always seems to be proportional to the product of the magnitudes of the two vectors being "multiplied," and also to the sine of the angle between them.

Is this pattern just an accident? Actually the sine factor has a very important physical property: it goes to zero when the two vectors are parallel. This is a Good Thing. The generalization of angular momentum into a three-dimensional vector, for example, is presumably going to describe not just the clockwise or counterclockwise nature of the motion but also from which direction we would have to view the motion so that it was clockwise or counterclockwise. (A clock's hands go counterclockwise as seen from behind the clock, and don't rotate at all as seen from above or to the side.) Now suppose a particle is moving directly away from the origin, so that its $\mathbf{r}$ and $\mathbf{p}$ vectors are parallel. It is not going around the origin from any point of view, so its angular momentum vector had better be zero.

Thinking in a slightly more abstract way, we would expect the angular momentum vector to point perpendicular to the plane of motion, just as the angular velocity vector points perpendicular to the plane of motion. The plane of motion is the plane containing both $\mathbf{r}$ and $\mathbf{p}$, if we place the two vectors tail-to-tail. But if $\mathbf{r}$ and $\mathbf{p}$ are parallel and are placed tail-to-tail, then there are infinitely many planes containing them both. To pick one of these planes in preference to the others would violate the symmetry of space, since they should all be equally good. Thus the zero-if-parallel property is a necessary consequence of the underlying symmetry of the laws of physics.

The following definition of a kind of vector multiplication is consistent with everything we've seen so far, and on p. 473 we'll prove that the definition is unique, i.e., if we believe in the symmetry of space, it is essentially the only way of defining the multiplication of two vectors to produce a third vector:

Definition of the vector cross product:
The cross product $\mathbf{A}\times\mathbf{B}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as follows:
(1) Its magnitude is defined by $|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta_{AB}$, where θ_{AB} is the angle between $\mathbf{A}$ and $\mathbf{B}$ when they are placed tail-to-tail.
(2) Its direction is along the line perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. Of the two such directions, it is the one that obeys the right-hand rule shown in figure am.



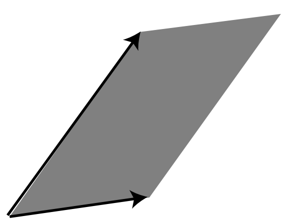
am / The right-hand rule for the direction of the vector cross product.

The name "cross product" refers to the symbol, and distinguishes it from the dot product, which acts on two vectors but produces a scalar.

Although the vector cross-product has nearly all the properties of numerical multiplication, e.g., $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$, it lacks the usual property of commutativity. Try applying the right-hand rule to find the direction of the vector cross product $\mathbf{B}\times\mathbf{A}$ using the two vectors shown in the figure. This requires starting with a flattened hand with the four fingers pointing along $\mathbf{B}$, and then curling the hand so that the fingers point along $\mathbf{A}$. The only possible way to do this is to point your thumb toward the floor, in the opposite direction. Thus for the vector cross product we have

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A},$$

a property known as anticommutativity. The vector cross product is also not associative, i.e., $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is usually not the same as $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$.

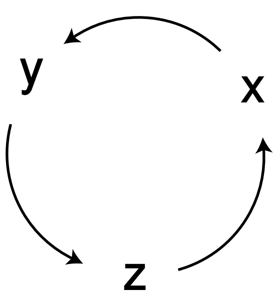


an / The magnitude of the cross product is the area of the shaded parallelogram.

A geometric interpretation of the cross product, an, is that if both $\mathbf{A}$ and $\mathbf{B}$ are vectors with units of distance, then the magnitude of their cross product can be interpreted as the area of the parallelogram they form when placed tail-to-tail.

A useful expression for the components of the vector cross product in terms of the components of the two vectors being multiplied is as follows:

$$(\mathbf{A}\times\mathbf{B})_x = A_yB_z - B_yA_z$$
$$(\mathbf{A}\times\mathbf{B})_y = A_zB_x - B_zA_x$$
$$(\mathbf{A}\times\mathbf{B})_z = A_xB_y - B_xA_y$$



ao / A cyclic change in the *x*, *y*, and *z* subscripts.

I'll prove later that these expressions are equivalent to the previous definition of the cross product. Although they may appear formidable, they have a simple structure: the subscripts on the right are the other two besides the one on the left, and each equation is related to the preceding one by a cyclic change in the subscripts, ao. If the subscripts were not treated in some completely symmetric manner like this, then the definition would provide some way to distinguish one axis from another, which would violate the symmetry of space.

self-check E

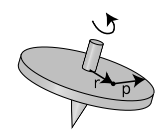
Show that the component equations are consistent with the rule $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. ▷ Answer, p. 561

Angular momentum in three dimensions

In terms of the vector cross product, we have:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$
$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

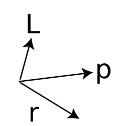
But wait, how do we know these equations are even correct? For instance, how do we know that the quantity defined by $\mathbf{r} \times \mathbf{p}$ is in fact conserved? Well, just as we saw on page 375 that the dot product is unique (i.e., can only be defined in one way while observing rotational invariance), the cross product is also unique, as proved on page 473. If $\mathbf{r} \times \mathbf{p}$ was not conserved, then there could not be any generally conserved quantity that would reduce to our old definition of angular momentum in the special case of plane rotation. This doesn't prove conservation of angular momentum — only experiments can prove that — but it does prove that if angular momentum is conserved in three dimensions, there is only one possible way to generalize from two dimensions to three.



ap / The position and momentum vectors of an atom in the spinning top.

Angular momentum of a spinning top *example 25*

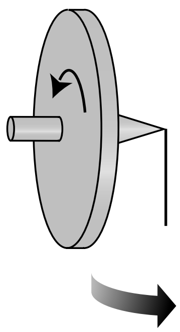
As an illustration, we consider the angular momentum of a spinning top. Figures ap and aq show the use of the vector cross product to determine the contribution of a representative atom to the total angular momentum. Since every other atom's angular momentum vector will be in the same direction, this will also be the direction of the total angular momentum of the top. This happens to be rigid-body rotation, and perhaps not surprisingly, the angular momentum vector is along the same direction as the angular velocity vector.



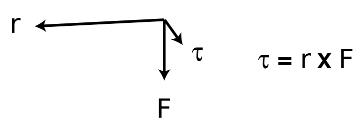
aq / The right-hand rule for the atom's contribution to the angular momentum.

Three important points are illustrated by this example: (1) When we do the full three-dimensional treatment of angular momentum, the "axis" from which we measure the position vectors is just an arbitrarily chosen point. If this had not been rigid-body rotation, we would not even have been able to identify a single line about which every atom circled. (2) Starting from figure ap, we had to rearrange the vectors to get them tail-to-tail before applying the right-hand rule. If we had attempted to apply the right-hand rule to figure ap, the direction of the result would have been exactly the opposite of the correct answer. (3) The equation $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ cannot be applied all at once to an entire system of particles. The total momentum of the top is zero, which would give an erroneous result of zero angular momentum (never mind the fact that **r** is not well defined for the top as a whole).

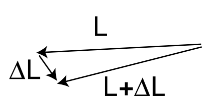
Doing the right-hand rule like this requires some practice. I urge you to make models like aq out of rolled up pieces of paper and to practice with the model in various orientations until it becomes natural.



ar / A top is supported at its tip by a pinhead. (More practical devices to demonstrate this would use a double bearing.)



as / The torque made by gravity is in the horizontal plane.



at / The $\Delta\mathbf{L}$ vector is in the same direction as the torque, out of the page.

Precession — *example 26*

Figure ar shows a counterintuitive example of the concepts we've been discussing. One expects the torque due to gravity to cause the top to flop down. Instead, the top remains spinning in the horizontal plane, but its axis of rotation starts moving in the direction shown by the shaded arrow. This phenomenon is called precession. Figure as shows that the torque due to gravity is out of the page. (Actually we should add up all the torques on all the atoms in the top, but the qualitative result is the same.) Since torque is the rate of change of angular momentum, $\tau = d\mathbf{L}/dt$, the $\Delta\mathbf{L}$ vector must be in the same direction as the torque (division by a positive scalar doesn't change the direction of the vector). As shown in at, this causes the angular momentum vector to twist in space without changing its magnitude.

For similar reasons, the Earth's axis precesses once every 26,000 years (although not through a great circle, since the angle between the axis and the force isn't 90 degrees as in figure ar). This precession is due to a torque exerted by the moon. If the Earth was a perfect sphere, there could be no precession effect due to symmetry. However, the Earth's own rotation causes it to be slightly flattened (oblate) relative to a perfect sphere, giving it "love handles" on which the moon's gravity can act. The moon's gravity on the nearer side of the equatorial bulge is stronger, so the torques do not cancel out perfectly. Presently the earth's axis very nearly lines up with the star Polaris, but in 12,000 years, the pole star will be Vega instead.

The frisbee — *example 27*

The flow of the air over a flying frisbee generates lift, and the lift at the front and back of the frisbee isn't necessarily balanced. If you throw a frisbee without rotating it, as if you were shooting a basketball with two hands, you'll find that it pitches, i.e., its nose goes either up or down. When I do this with my frisbee, it goes nose down, which apparently means that the lift at the back of the disc is greater than the lift at the front. The two torques are unbalanced, resulting in a total torque that points to the left.

The way you actually throw a frisbee is with one hand, putting a lot of spin on it. If you throw backhand, which is how most people first learn to do it, the angular momentum vector points down (assuming you're right-handed). On my frisbee, the aerodynamic torque to the left would therefore tend to make the angular momentum vector precess in the clockwise direction as seen by the thrower. This would cause the disc to roll to the right, and therefore follow a curved trajectory. Some specialized discs, used in the sport of disc golf, are actually designed intentionally to show this behavior; they're known as "understable" discs. However, the typical frisbee that most people play with is designed to be stable: as the disc rolls to one side, the airflow around it is altered in way that tends to bring the disc back into level flight. Such a disc will therefore tend to fly in a straight line, provided that it is thrown with enough angular momentum.

Finding a cross product by components — *example 28*

▷ What is the torque produced by a force given by $\hat{\mathbf{x}}+2\hat{\mathbf{y}}+3\hat{\mathbf{z}}$ (in units of Newtons) acting on a point whose radius vector is $4\hat{\mathbf{x}}+2\hat{\mathbf{y}}$ (in meters)?

r	**4**	**5**	**0**
F	**1**	**2**	**3**

au / Example 28.

▷ It's helpful to make a table of the components as shown in the figure. The results are

$$\begin{aligned}\tau_x &= r_y F_z - F_y r_z = 15\ \text{N}\cdot\text{m}\\ \tau_y &= r_z F_x - F_z r_x = -12\ \text{N}\cdot\text{m}\\ \tau_z &= r_x F_y - F_x r_y = 3\ \text{N}\cdot\text{m}\end{aligned}$$

Torque and angular momentum — *example 29*

In this example, we prove explicitly the consistency of the equations involving torque and angular momentum that we proved above based purely on symmetry. Starting from the definition of torque, we have

$$\begin{aligned}\tau &= \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \mathbf{r}_i\times\mathbf{p}_i\\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i\times\mathbf{p}_i).\end{aligned}$$

The derivative of a cross product can be evaluated in the same way as the derivative of an ordinary scalar product:

$$\tau = \sum_i \left[\left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\times\mathbf{p}_i\right)+\left(\mathbf{r}_i\times\frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t}\right)\right]$$

The first term is zero for each particle, since the velocity vector is parallel to the momentum vector. The derivative appearing in the second term is the force acting on the particle, so

$$\tau = \sum_i \mathbf{r}_i\times\mathbf{F}_i,$$

which is the relationship we set out to prove.

Rigid-body dynamics in three dimensions

The student who is not madly in love with mathematics may wish to skip the rest of this section after absorbing the statement that, for a typical, asymmetric object, the angular momentum vector and the angular velocity vector need not be parallel. That is, only for a body that possesses symmetry about the rotation axis is it true that $\mathbf{L} = I\boldsymbol{\omega}$ (the rotational equivalent of $\mathbf{p} = m\mathbf{v}$) for some scalar I.

Let's evaluate the angular momentum of a rigidly rotating system of particles:

$$\begin{aligned}
\mathbf{L} &= \sum_i \mathbf{r}_i \times \mathbf{p}_i \\
&= \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i \\
&= \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)
\end{aligned}$$

An important mathematical skill is to know when to give up and back off. This is a complicated expression, and there is no reason to expect it to simplify and, for example, take the form of a scalar multiplied by ω. Instead we examine its general characteristics. If we expanded it using the equation that gives the components of a vector cross product, every term would have one of the ω components raised to the first power, multiplied by a bunch of other stuff. The most general possible form for the result is

$$\begin{aligned}
L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\
L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\
L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z,
\end{aligned}$$

which you may recognize as a case of matrix multiplication. The moment of inertia is not a scalar, and not a three-component vector. It is a matrix specified by nine numbers, called its matrix elements.

The elements of the moment of inertia matrix will depend on our choice of a coordinate system. In general, there will be some special coordinate system, in which the matrix has a simple diagonal form:

$$\begin{aligned}
L_x &= I_{xx}\omega_x \\
L_y &= \qquad I_{yy}\omega_y \\
L_z &= \qquad\qquad I_{zz}\omega_z.
\end{aligned}$$

The three special axes that cause this simplification are called the principal axes of the object, and the corresponding coordinate system is the principal axis system. For symmetric shapes such as a rectangular box or an ellipsoid, the principal axes lie along the intersections of the three symmetry planes, but even an asymmetric body has principal axes.

We can also generalize the plane-rotation equation $K = (1/2)I\omega^2$ to three dimensions as follows:

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i v_i^2 \\ &= \frac{1}{2} \sum_i m_i (\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (\boldsymbol{\omega} \times \mathbf{r}_i) \end{aligned}$$

We want an equation involving the moment of inertia, and this has some evident similarities to the sum we originally wrote down for the moment of inertia. To massage it into the right shape, we need the vector identity $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}$, which we state without proof. We then write

$$\begin{aligned} K &= \frac{1}{2} \sum_i m_i \left[\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)\right] \cdot \boldsymbol{\omega} \\ &= \frac{1}{2} \boldsymbol{\omega} \cdot \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &= \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega} \end{aligned}$$

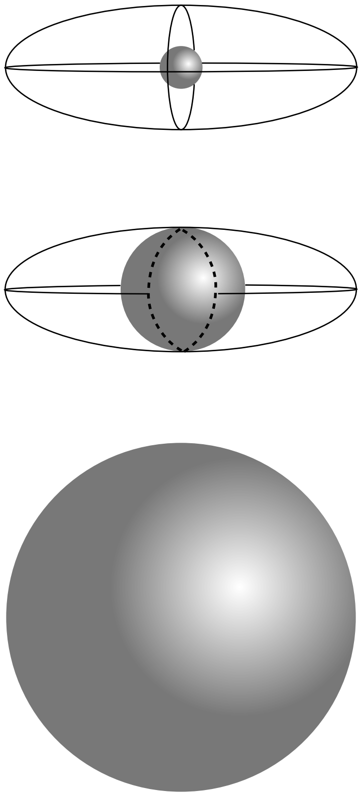
As a reward for all this hard work, let's analyze the problem of the spinning shoe that I posed at the beginning of the chapter. The three rotation axes referred to there are approximately the principal axes of the shoe. While the shoe is in the air, no external torques are acting on it, so its angular momentum vector must be constant in magnitude and direction. Its kinetic energy is also constant. That's in the room's frame of reference, however. The principal axis frame is attached to the shoe, and tumbles madly along with it. In the principal axis frame, the kinetic energy and the magnitude of the angular momentum stay constant, but the actual direction of the angular momentum need not stay fixed (as you saw in the case of rotation that was initially about the intermediate-length axis). Constant $|\mathbf{L}|$ gives

$$L_x^2 + L_y^2 + L_z^2 = \text{constant}.$$

In the principal axis frame, it's easy to solve for the components of $\boldsymbol{\omega}$ in terms of the components of $\mathbf{L}$, so we eliminate ω from the expression $2K = \mathbf{L} \cdot \boldsymbol{\omega}$, giving

$$\frac{1}{I_{xx}} L_x^2 + \frac{1}{I_{yy}} L_y^2 + \frac{1}{I_{zz}} L_z^2 = \text{constant \#2}.$$

The first equation is the equation of a sphere in the three dimensional space occupied by the angular momentum vector, while the second one is the equation of an ellipsoid. The top figure corresponds to the case of rotation about the shortest axis, which has the greatest moment of inertia element. The intersection of the two



av / Visualizing surfaces of constant energy and angular momentum in L_x-L_y-L_z space.

aw / The Explorer I satellite.

surfaces consists only of the two points at the front and back of the sphere. The angular momentum is confined to one of these points, and can't change its direction, i.e., its orientation with respect to the principal axis system, which is another way of saying that the shoe can't change its orientation with respect to the angular momentum vector. In the bottom figure, the shoe is rotating about the longest axis. Now the angular momentum vector is trapped at one of the two points on the right or left. In the case of rotation about the axis with the intermediate moment of inertia element, however, the intersection of the sphere and the ellipsoid is not just a pair of isolated points but the curve shown with the dashed line. The relative orientation of the shoe and the angular momentum vector can and will change.

One application of the moment of inertia tensor is to video games that simulate car racing or flying airplanes.

One more exotic example has to do with nuclear physics. Although you have probably visualized atomic nuclei as nothing more than featureless points, or perhaps tiny spheres, they are often ellipsoids with one long axis and two shorter, equal ones. Although a spinning nucleus normally gets rid of its angular momentum via gamma ray emission within a period of time on the order of picoseconds, it may happen that a deformed nucleus gets into a state in which has a large angular momentum is along its long axis, which is a very stable mode of rotation. Such states can live for seconds or even years! (There is more to the story — this is the topic on which I wrote my Ph.D. thesis — but the basic insight applies even though the full treatment requires fancy quantum mechanics.)

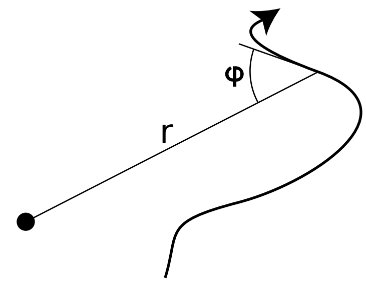
Our analysis has so far assumed that the kinetic energy of rotation energy can't be converted into other forms of energy such as heat, sound, or vibration. When this assumption fails, then rotation about the axis of least moment of inertia becomes unstable, and will eventually convert itself into rotation about the axis whose moment of inertia is greatest. This happened to the U.S.'s first artificial satellite, Explorer I, launched in 1958. Note the long, floppy antennas, which tended to dissipate kinetic energy into vibration. It had been designed to spin about its minimimum-moment-of-inertia axis, but almost immediately, as soon as it was in space, it began spinning end over end. It was nevertheless able to carry out its science mission, which didn't depend on being able to maintain a stable orientation, and it discovered the Van Allen radiation belts.

15.9 ⋆ Proof of Kepler's elliptical orbit law

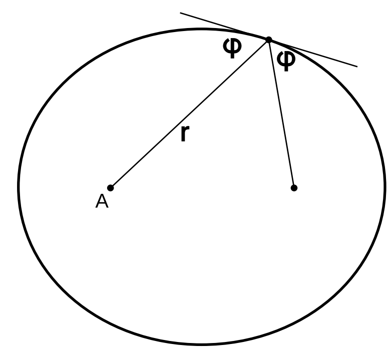
Kepler determined purely empirically that the planets' orbits were ellipses, without understanding the underlying reason in terms of physical law. Newton's proof of this fact based on his laws of motion and law of gravity was considered his crowning achievement both by him and by his contemporaries, because it showed that the same physical laws could be used to analyze both the heavens and the earth. Newton's proof was very lengthy, but by applying the more recent concepts of conservation of energy and angular momentum we can carry out the proof quite simply and succinctly, and without calculus.

The basic idea of the proof is that we want to describe the shape of the planet's orbit with an equation, and then show that this equation is exactly the one that represents an ellipse. Newton's original proof had to be very complicated because it was based directly on his laws of motion, which include time as a variable. To make any statement about the shape of the orbit, he had to eliminate time from his equations, leaving only space variables. But conservation laws tell us that certain things don't change over time, so they have already had time eliminated from them.

There are many ways of representing a curve by an equation, of which the most familiar is $y = ax + b$ for a line in two dimensions. It would be perfectly possible to describe a planet's orbit using an $x - y$ equation like this, but remember that we are applying conservation of angular momentum, and the space variables that occur in the equation for angular momentum are the distance from the axis, r, and the angle between the velocity vector and the r vector, which we will call φ. The planet will have φ=90°when it is moving perpendicular to the r vector, i.e., at the moments when it is at its smallest or greatest distances from the sun. When φ is less than 90°the planet is approaching the sun, and when it is greater than 90°it is receding from it. Describing a curve with an $r - \varphi$ equation is like telling a driver in a parking lot a certain rule for what direction to steer based on the distance from a certain streetlight in the middle of the lot.



ax / The $r - \varphi$ representation of a curve.



ay / Proof that the two angles labeled φ are in fact equal: The definition of an ellipse is that the sum of the distances from the two foci stays constant. If we move a small distance ℓ along the ellipse, then one distance shrinks by an amount $\ell \cos \varphi_1$, while the other grows by $\ell \cos \varphi_2$. These are equal, so $\varphi_1 = \varphi_2$.

The proof is broken into the three parts for easier digestion. The first part is a simple and intuitively reasonable geometrical fact about ellipses, whose proof we relegate to the caption of figure ay; you will not be missing much if you merely absorb the result without reading the proof.

(1) If we use one of the two foci of an ellipse as an axis for defining the variables r and φ, then the angle between the tangent line and the line drawn to the other focus is the same as φ, i.e., the two angles labeled φ in figure ay are in fact equal.

The other two parts form the meat of our proof. We state the results first and then prove them.

(2) A planet, moving under the influence of the sun's gravity with less than the energy required to escape, obeys an equation of

the form

$$\sin\varphi = \frac{1}{\sqrt{-pr^2+qr}},$$

where p and q are positive constants that depend on the planet's energy and angular momentum.

(3) A curve is an ellipse if and only if its $r-\varphi$ equation is of the form

$$\sin\varphi = \frac{1}{\sqrt{-pr^2+qr}},$$

where p and q are positive constants that depend on the size and shape of the ellipse.

Proof of part (2)

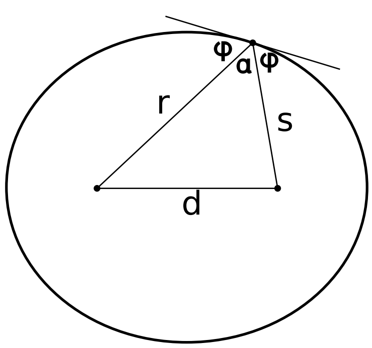
The component of the planet's velocity vector that is perpendicular to the $\mathbf{r}$ vector is $v_\perp = v\sin\varphi$, so conservation of angular momentum tells us that $L = mrv\sin\varphi$ is a constant. Since the planet's mass is a constant, this is the same as the condition

$$rv\sin\varphi = \text{constant}.$$

Conservation of energy gives

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}.$$

We solve the first equation for v and plug into the second equation to eliminate v. Straightforward algebra then leads to the equation claimed above, with the constant p being positive because of our assumption that the planet's energy is insufficient to escape from the sun, i.e., its total energy is negative.



az / Proof of part (3).

Proof of part (3)

We define the quantities α, d, and s as shown in the figure. The law of cosines gives

$$d^2 = r^2 + s^2 - 2rs\cos\alpha.$$

Using $\alpha = 180°-2\varphi$ and the trigonometric identities $\cos(180°-x) = -\cos x$ and $\cos 2x = 1-2\sin^2 x$, we can rewrite this as

$$d^2 = r^2 + s^2 - 2rs\left(2\sin^2\varphi - 1\right).$$

Straightforward algebra transforms this into

$$\sin\,\varphi = \sqrt{\frac{(r+s)^2-d^2}{4rs}}.$$

Since $r+s$ is constant, the top of the fraction is constant, and the denominator can be rewritten as $4rs = 4r(\text{constant} - r)$, which is equivalent to the desired form.

15.10 Some theorems and proofs

In this section I prove three theorems stated earlier, and state a fourth theorem whose proof is left as an exercise.

Uniqueness of the cross product

The vector cross product as we have defined it has the following properties:
(1) It does not violate rotational invariance.
(2) It has the property $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.
(3) It has the property $\mathbf{A} \times (k\mathbf{B}) = k(\mathbf{A} \times \mathbf{B})$, where k is a scalar.

Theorem: The definition we have given is the only possible method of multiplying two vectors to make a third vector which has these properties, with the exception of trivial redefinitions which just involve multiplying all the results by the same constant or swapping the names of the axes. (Specifically, using a left-hand rule rather than a right-hand rule corresponds to multiplying all the results by -1.)

Proof: We prove only the uniqueness of the definition, without explicitly proving that it has properties (1) through (3).

Using properties (2) and (3), we can break down any vector multiplication $(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$ into terms involving cross products of unit vectors.

A "self-term" like $\hat{\mathbf{x}} \times \hat{\mathbf{x}}$ must either be zero or lie along the x axis, since any other direction would violate property (1). If it was not zero, then $(-\hat{\mathbf{x}}) \times (-\hat{\mathbf{x}})$ would have to lie in the opposite direction to avoid breaking rotational invariance, but property (3) says that $(-\hat{\mathbf{x}}) \times (-\hat{\mathbf{x}})$ is the same as $\hat{\mathbf{x}} \times \hat{\mathbf{x}}$, which is a contradiction. Therefore the self-terms must be zero.

An "other-term" like $\hat{\mathbf{x}} \times \hat{\mathbf{y}}$ could conceivably have components in the x-y plane and along the z axis. If there was a nonzero component in the x-y plane, symmetry would require that it lie along the diagonal between the x and y axes, and similarly the in-the-plane component of $(-\hat{\mathbf{x}}) \times \hat{\mathbf{y}}$ would have to be along the other diagonal in the x-y plane. Property (3), however, requires that $(-\hat{\mathbf{x}}) \times \hat{\mathbf{y}}$ equal $-(\hat{\mathbf{x}} \times \hat{\mathbf{y}})$, which would be along the original diagonal. The only way it can lie along both diagonals is if it is zero.

We now know that $\hat{\mathbf{x}} \times \hat{\mathbf{y}}$ must lie along the z axis. Since we are not interested in trivial differences in definitions, we can fix $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, ignoring peurile possibilities such as $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = 7\hat{\mathbf{z}}$ or the left-handed definition $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{z}}$. Given $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, the symmetry of space requires that similar relations hold for $\hat{\mathbf{y}} \times \hat{\mathbf{z}}$ and $\hat{\mathbf{z}} \times \hat{\mathbf{x}}$, with at most a difference in sign. A difference in sign could always be eliminated by swapping the names of some of the axes, so ignoring possible trivial differences in definitions we can assume that the cyclically related set of relations $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$, and $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$

holds. Since the arbitrary cross-product with which we started can be broken down into these simpler ones, the cross product is uniquely defined.

Choice of axis theorem

Theorem: Suppose a closed system of material particles conserves angular momentum in one frame of reference, with the axis taken to be at the origin. Then conservation of angular momentum is unaffected if the origin is relocated or if we change to a frame of reference that is in constant-velocity motion with respect to the first one. The theorem also holds in the case where the system is not closed, but the total external force is zero.

Proof: In the original frame of reference, angular momentum is conserved, so we have $\mathrm{d}\mathbf{L}/\mathrm{d}t=0$. From example 29 on page 467, this derivative can be rewritten as

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \sum_i \mathbf{r}_i \times \mathbf{F}_i,$$

where $\mathbf{F}_i$ is the total force acting on particle i. In other words, we're adding up all the torques on all the particles.

By changing to the new frame of reference, we have changed the position vector of each particle according to $\mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{k} - \mathbf{u}t$, where $\mathbf{k}$ is a constant vector that indicates the relative position of the new origin at $t = 0$, and $\mathbf{u}$ is the velocity of the new frame with respect to the old one. The forces are all the same in the new frame of reference, however. In the new frame, the rate of change of the angular momentum is

$$\begin{aligned}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= \sum_i (\mathbf{r}_i + \mathbf{k} - \mathbf{u}t) \times \mathbf{F}_i \\ &= \sum_i \mathbf{r}_i \times \mathbf{F}_i + (\mathbf{k} - \mathbf{u}t) \times \sum_i \mathbf{F}_i.\end{aligned}$$

The first term is the expression for the rate of change of the angular momentum in the original frame of reference, which is zero by assumption. The second term vanishes by Newton's third law; since the system is closed, every force $\mathbf{F}_i$ cancels with some force $\mathbf{F}_j$. (If external forces act, but they add up to zero, then the sum can be broken up into a sum of internal forces and a sum of external forces, each of which is zero.) The rate of change of the angular momentum is therefore zero in the new frame of reference.

Spin theorem

Theorem: An object's angular momentum with respect to some outside axis A can be found by adding up two parts:
(1) The first part is the object's angular momentum found by using its own center of mass as the axis, i.e., the angular momentum the

object has because it is spinning.
(2) The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

Proof: Let the system's center of mass be at $\mathbf{r}_{cm}$, and let particle i lie at position $\mathbf{r}_{cm} + \mathbf{d}_i$. Then the total angular momentum is

$$\begin{aligned}\mathbf{L} &= \sum_i (\mathbf{r}_{cm} + \mathbf{d}_i) \times \mathbf{p}_i \\ &= \mathbf{r}_{cm} \times \sum_i \mathbf{p}_i + \sum_i \mathbf{d}_i \times \mathbf{p}_i,\end{aligned}$$

which establishes the result claimed, since we can identify the first term with (2) and the second with (1).

Parallel axis theorem

Suppose an object has mass m, and moment of inertia I_o for rotation about some axis A passing through its center of mass. Given a new axis B, parallel to A and lying at a distance h from it, the object's moment of inertia is given by $I_o + mh^2$.

The proof of this theorem is left as an exercise (problem 27, p. 483).

Summary

Selected vocabulary

angular momentum	a measure of rotational motion; a conserved quantity for a closed system
axis	An arbitrarily chosen point used in the definition of angular momentum. Any object whose direction changes relative to the axis is considered to have angular momentum. No matter what axis is chosen, the angular momentum of a closed system is conserved.
torque	the rate of change of angular momentum; a numerical measure of a force's ability to twist on an object
equilibrium	a state in which an object's momentum and angular momentum are constant
stable equilibrium	one in which a force always acts to bring the object back to a certain point
unstable equilibrium	one in which any deviation of the object from its equilibrium position results in a force pushing it even farther away

Notation

L	angular momentum
t	torque
T	the period the time required for a rigidly rotating body to complete one rotation
ω	the angular velocity, $\mathrm{d}\theta/\mathrm{d}t$
moment of inertia, I	the proportionality constant in the equation $L = I\omega$

Summary

Angular momentum is a measure of rotational motion which is conserved for a closed system. This book only discusses angular momentum for rotation of material objects in two dimensions. Not all rotation is rigid like that of a wheel or a spinning top. An example of nonrigid rotation is a cyclone, in which the inner parts take less time to complete a revolution than the outer parts. In order to define a measure of rotational motion general enough to include nonrigid rotation, we define the angular momentum of a system by dividing it up into small parts, and adding up all the angular momenta of the small parts, which we think of as tiny particles. We arbitrarily choose some point in space, the *axis*, and we say that anything that changes its direction relative to that point possesses angular momentum. The angular momentum of a single particle is

$$L = mv_{\perp}r,$$

where $v_{\perp}$ is the component of its velocity perpendicular to the line joining it to the axis, and r is its distance from the axis. Positive and negative signs of angular momentum are used to indicate clockwise and counterclockwise rotation.

The *choice of axis theorem* states that any axis may be used for defining angular momentum. If a system's angular momentum is constant for one choice of axis, then it is also constant for any other choice of axis.

The *spin theorem* states that an object's angular momentum with respect to some outside axis A can be found by adding up two parts:

(1) The first part is the object's angular momentum found by using its own center of mass as the axis, i.e., the angular momentum the object has because it is spinning.

(2) The other part equals the angular momentum that the object would have with respect to the axis A if it had all its mass concentrated at and moving with its center of mass.

Torque is the rate of change of angular momentum. The torque a force can produce is a measure of its ability to twist on an object. The relationship between force and torque is

$$|\boldsymbol{\tau}| = r|F_\perp|,$$

where r is the distance from the axis to the point where the force is applied, and $F_\perp$ is the component of the force perpendicular to the line connecting the axis to the point of application. Statics problems can be solved by setting the total force and total torque on an object equal to zero and solving for the unknowns.

In the special case of a *rigid body* rotating in a single plane, we define

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} \qquad \text{[angular velocity]}$$

and

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t}, \qquad \text{[angular acceleration]}$$

in terms of which we have

$$L = I\omega$$

and

$$\tau = I\alpha,$$

where the *moment of inertia*, I, is defined as

$$I = \sum m_i r_i^2,$$

summing over all the atoms in the object (or using calculus to perform a continuous sum, i.e. an integral). The relationship between the angular quantities and the linear ones is

$$v_t = \omega r \qquad \text{[tangential velocity of a point]}$$

$$v_r = 0 \qquad \text{[radial velocity of a point]}$$

$$a_t = \alpha r. \qquad \text{[radial acceleration of a point at a distance } r \text{ from the axis]}$$

$$a_r = \omega^2 r \qquad \text{[radial acceleration of a point at a distance } r \text{ from the axis]}$$

In three dimensions, torque and angular momentum are vectors, and are expressed in terms of the vector *cross product*, which is the only rotationally invariant way of defining a multiplication of two vectors that produces a third vector:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

In general, the cross product of vectors **b** and **c** has magnitude

$$|\mathbf{b} \times \mathbf{c}| = |\mathbf{b}|\,|\mathbf{c}| \sin\theta_{bc},$$

which can be interpreted geometrically as the area of the parallelogram formed by the two vectors when they are placed tail-to-tail. The direction of the cross product lies along the line which is perpendicular to both vectors; of the two such directions, we choose the one that is right-handed, in the sense that if we point the fingers of the flattened right hand along **b**, then bend the knuckles to point the fingers along **c**, the thumb gives the direction of $\mathbf{b} \times \mathbf{c}$. In terms of components, the cross product is

$$(\mathbf{b} \times \mathbf{c})_x = b_y c_z - c_y b_z$$

$$(\mathbf{b} \times \mathbf{c})_y = b_z c_x - c_z b_x$$

$$(\mathbf{b} \times \mathbf{c})_z = b_x c_y - c_x b_y$$

The cross product has the disconcerting properties

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \qquad \text{[noncommutative]}$$

and

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \qquad \text{[nonassociative]},$$

and there is no "cross-division."

For rigid-body rotation in three dimensions, we define an angular velocity vector $\boldsymbol{\omega}$, which lies along the axis of rotation and bears a right-hand relationship to it. Except in special cases, there is no scalar moment of inertia for which $\mathbf{L} = I\boldsymbol{\omega}$; the moment of inertia must be expressed as a matrix.

Problems

Key

√ A computerized answer check is available online.

∫ A problem that requires calculus.

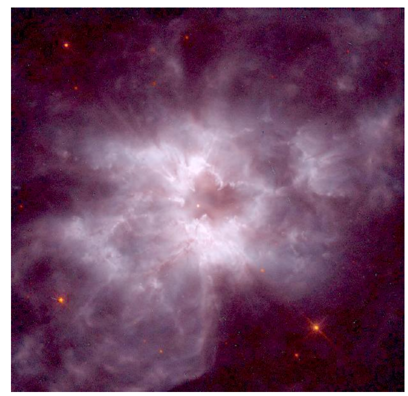
⋆ A difficult problem.

1 A skilled motorcyclist can ride up a ramp, fly through the air, and land on another ramp. Why would it be useful for the rider to speed up or slow down the back wheel while in the air?

2 An object thrown straight up in the air is momentarily at rest when it reaches the top of its motion. Does that mean that it is in equilibrium at that point? Explain.

3 An object is observed to have constant angular momentum. Can you conclude that no torques are acting on it? Explain. [Based on a problem by Serway and Faughn.]

4 The sun turns on its axis once every 26.0 days. Its mass is 2.0×10^{30} kg and its radius is 7.0×10^{8} m. Assume it is a rigid sphere of uniform density.
(a) What is the sun's angular momentum? √
In a few billion years, astrophysicists predict that the sun will use up all its sources of nuclear energy, and will collapse into a ball of exotic, dense matter known as a white dwarf. Assume that its radius becomes 5.8×10^{6} m (similar to the size of the Earth.) Assume it does not lose any mass between now and then. (Don't be fooled by the photo, which makes it look like nearly all of the star was thrown off by the explosion. The visually prominent gas cloud is actually thinner than the best laboratory vacuum ever produced on earth. Certainly a little bit of mass is actually lost, but it is not at all unreasonable to make an approximation of zero loss of mass as we are doing.)
(b) What will its angular momentum be?
(c) How long will it take to turn once on its axis? √

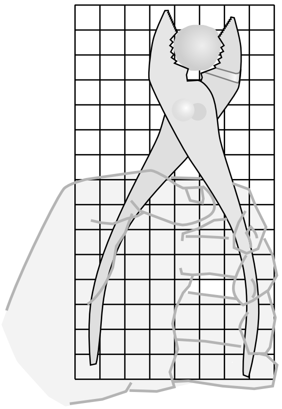


Problem 4.

5 (a) Alice says Cathy's body has zero momentum, but Bob says Cathy's momentum is nonzero. Nobody is lying or making a mistake. How is this possible? Give a concrete example.
(b) Alice and Bob agree that Dong's body has nonzero momentum, but disagree about Dong's angular momentum, which Alice says is zero, and Bob says is nonzero. Explain.

6 Two objects have the same momentum vector. Assume that they are not spinning; they only have angular momentum due to their motion through space. Can you conclude that their angular momenta are the same? Explain. [Based on a problem by Serway and Faughn.]

7 You are trying to loosen a stuck bolt on your RV using a big wrench that is 50 cm long. If you hang from the wrench, and your mass is 55 kg, what is the maximum torque you can exert on the bolt? √



Problem 8.

8 The figure shows scale drawing of a pair of pliers being used to crack a nut, with an appropriately reduced centimeter grid. Warning: do not attempt this at home; it is bad manners. If the force required to crack the nut is 300 N, estimate the force required of the person's hand. ▷ Solution, p. 556

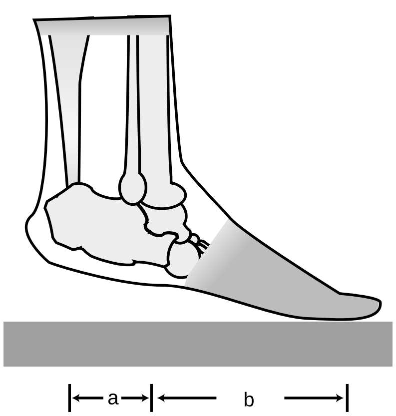
Problem 9.

9 Make a rough estimate of the mechanical advantage of the lever shown in the figure. In other words, for a given amount of force applied on the handle, how many times greater is the resulting force on the cork?

10 A physical therapist wants her patient to rehabilitate his injured elbow by laying his arm flat on a table, and then lifting a 2.1 kg mass by bending his elbow. In this situation, the weight is 33 cm from his elbow. He calls her back, complaining that it hurts him to grasp the weight. He asks if he can strap a bigger weight onto his arm, only 17 cm from his elbow. How much mass should she tell him to use so that he will be exerting the same torque? (He is raising his forearm itself, as well as the weight.) √

11 Two horizontal tree branches on the same tree have equal diameters, but one branch is twice as long as the other. Give a quantitative comparison of the torques where the branches join the trunk. [Thanks to Bong Kang.]

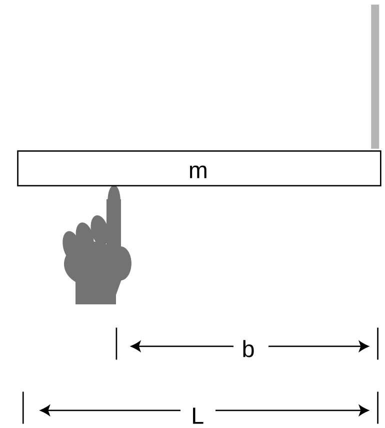
12 A ball is connected by a string to a vertical post. The ball is set in horizontal motion so that it starts winding the string around the post. Assume that the motion is confined to a horizontal plane, i.e., ignore gravity. Michelle and Astrid are trying to predict the final velocity of the ball when it reaches the post. Michelle says that according to conservation of angular momentum, the ball has to speed up as it approaches the post. Astrid says that according to conservation of energy, the ball has to keep a constant speed. Who is right? [Hint: How is this different from the case where you whirl a rock in a circle on a string and gradually reel in the string?]



Problem 13.

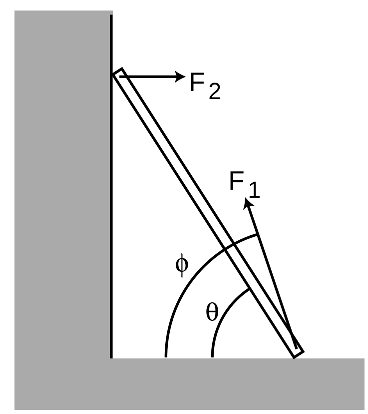
13 A person of weight W stands on the ball of one foot. Find the tension in the calf muscle and the force exerted by the shinbones on the bones of the foot, in terms of W, a, and b. For simplicity, assume that all the forces are at 90-degree angles to the foot, i.e., neglect the angle between the foot and the floor. √

14 The rod in the figure is supported by the finger and the string.
(a) Find the tension, T, in the string, and the force, F, from the finger, in terms of m, b, L, and g. ✓
(b) Comment on the cases $b = L$ and $b = L/2$.
(c) Are any values of b unphysical?



Problem 14.

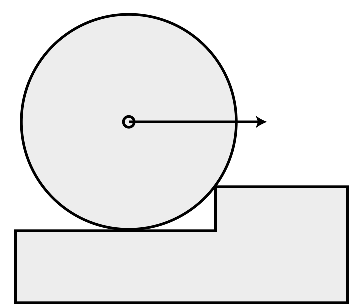
15 A uniform ladder of mass m and length L leans against a smooth wall, making an angle θ with respect to the ground. The dirt exerts a normal force and a frictional force on the ladder, producing a force vector with magnitude F_1 at an angle ϕ with respect to the ground. Since the wall is smooth, it exerts only a normal force on the ladder; let its magnitude be F_2.
(a) Explain why ϕ must be greater than θ. No math is needed.
(b) Choose any numerical values you like for m and L, and show that the ladder can be in equilibrium (zero torque and zero total force vector) for $\theta = 45.00°$ and $\phi = 63.43°$.



Problems 15 and 16.

16 Continuing problem 15, find an equation for ϕ in terms of θ, and show that m and L do not enter into the equation. Do not assume any numerical values for any of the variables. You will need the trig identity $\sin(a - b) = \sin a \cos b - \sin b \cos a$. (As a numerical check on your result, you may wish to check that the angles given in part b of the previous problem satisfy your equation.) ✓ ⋆

17 (a) Find the minimum horizontal force which, applied at the axle, will pull a wheel over a step. Invent algebra symbols for whatever quantities you find to be relevant, and give your answer in symbolic form. [Hints: There are four forces on the wheel at first, but only three when it lifts off. Normal forces are always perpendicular to the surface of contact. Note that the corner of the step cannot be perfectly sharp, so the surface of contact for this force really coincides with the surface of the wheel.]
(b) Under what circumstances does your result become infinite? Give a physical interpretation.

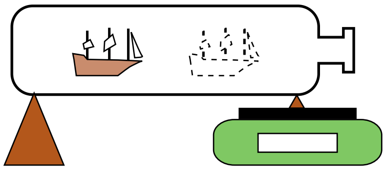


Problem 17.

18 In the 1950's, serious articles began appearing in magazines like *Life* predicting that world domination would be achieved by the nation that could put nuclear bombs in orbiting space stations, from which they could be dropped at will. In fact it can be quite difficult to get an orbiting object to come down. Let the object have energy $E = KE + PE$ and angular momentum L. Assume that the energy is negative, i.e., the object is moving at less than escape velocity. Show that it can never reach a radius less than

$$r_{min} = \frac{GMm}{2E}\left(-1 + \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}\right).$$

[Note that both factors are negative, giving a positive result.]



Problem 19.

19 You wish to determine the mass of a ship in a bottle without taking it out. Show that this can be done with the setup shown in the figure, with a scale supporting the bottle at one end, provided that it is possible to take readings with the ship slid to several different locations. Note that you can't determine the position of the ship's center of mass just by looking at it, and likewise for the bottle. In particular, you can't just say, "position the ship right on top of the fulcrum" or "position it right on top of the balance." ⋆

20 Two atoms will interact via electrical forces between their protons and electrons. One fairly good approximation to the potential energy is the Lennard-Jones potential,

$$PE(r) = k\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right],$$

where r is the center-to-center distance between the atoms.

Show that (a) there is an equilibrium point at $r = a$, (b) the equilibrium is stable, and (c) the energy required to bring the atoms from their equilibrium separation to infinity is k. [Hints: The first two parts can be done more easily by setting $a = 1$, since the value of a only changes the distance scale. One way to do part b is by graphing.]

21 Suppose that we lived in a universe in which Newton's law of gravity gave forces proportional to r^{-7} rather than r^{-2}. Which, if any, of Kepler's laws would still be true? Which would be completely false? Which would be different, but in a way that could be calculated with straightforward algebra?

22 Show that a sphere of radius R that is rolling without slipping has angular momentum and momentum in the ratio $L/p = (2/5)R$.

23 Suppose a bowling ball is initially thrown so that it has no angular momentum at all, i.e., it is initially just sliding down the lane. Eventually kinetic friction will get it spinning fast enough so that it is rolling without slipping. Show that the final velocity of the ball equals 5/7 of its initial velocity. [Hint: You'll need the result of problem 22.]

24 Penguins are playful animals. Tux the Penguin invents a new game using a natural circular depression in the ice. He waddles at top speed toward the crater, aiming off to the side, and then hops into the air and lands on his belly just inside its lip. He then belly-surfs, moving in a circle around the rim. The ice is frictionless, so his speed is constant. Is Tux's angular momentum zero, or nonzero? What about the total torque acting on him? Take the center of the crater to be the axis. Explain your answers.

25 A massless rod of length ℓ has weights, each of mass m, attached to its ends. The rod is initially put in a horizontal position, and laid on an off-center fulcrum located at a distance b from the rod's center. The rod will topple. (a) Calculate the total gravitational torque on the rod directly, by adding the two torques. (b) Verify that this gives the same result as would have been obtained by taking the entire gravitational force as acting at the center of mass.

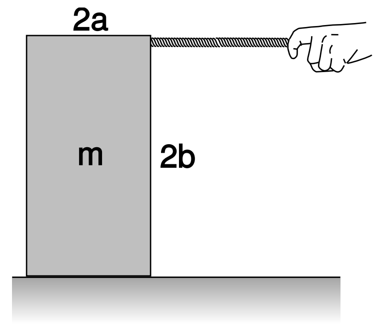
26 Use analogies to find the equivalents of the following equations for rotation in a plane:

$$KE = p^2/2m$$
$$\Delta x = v_o \Delta t + (1/2)a\Delta t^2$$
$$W = F\Delta x$$

Example: $v = \Delta x/\Delta t \rightarrow \omega = \Delta\theta/\Delta t$

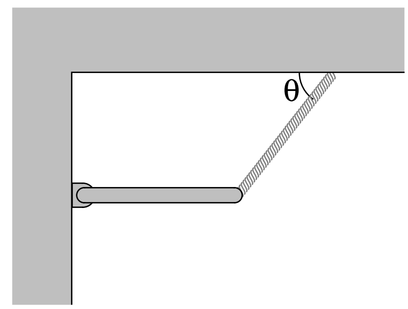
27 Prove the parallel axis theorem stated on page 475.

28 The box shown in the figure is being accelerated by pulling on it with the rope.
(a) Assume the floor is frictionless. What is the maximum force that can be applied without causing the box to tip over?
▷ Hint, p. 542 ✓
(b) Repeat part a, but now let the coefficient of friction be μ. ✓
(c) What happens to your answer to part b when the box is sufficiently tall? How do you interpret this?



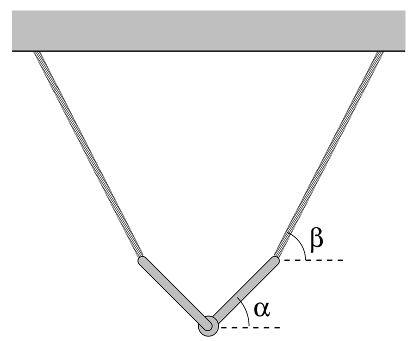
Problem 28.

29 (a) The bar of mass m is attached at the wall with a hinge, and is supported on the right by a massless cable. Find the tension, T, in the cable in terms of the angle θ. ✓
(b) Interpreting your answer to part a, what would be the best angle to use if we wanted to minimize the strain on the cable?
(c) Again interpreting your answer to part a, for what angles does the result misbehave mathematically? Interpet this physically.



Problem 29.

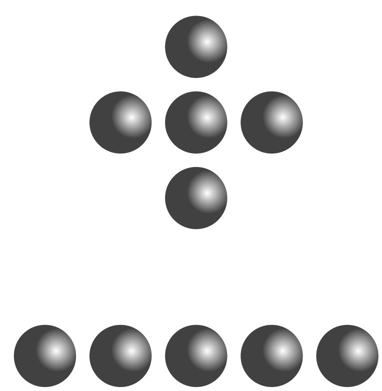
30 (a) The two identical rods are attached to one another with a hinge, and are supported by the two massless cables. Find the angle α in terms of the angle β, and show that the result is a purely geometric one, independent of the other variables involved. ✓
(b) Using your answer to part a, sketch the configurations for $\beta \rightarrow 0$, $\beta = 45°$, and $\beta = 90°$. Do your results make sense intuitively?



Problem 30.

31 (a) Find the angular velocities of the earth's rotation and of the earth's motion around the sun. ✓
(b) Which motion involves the greater acceleration?

32 Give a numerical comparison of the two molecules' moments of inertia for rotation in the plane of the page about their centers of mass. ✓


Problem 32

33 Find the angular momentum of a particle whose position is $\mathbf{r} = 3\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}$ (in meters) and whose momentum is $\mathbf{p} = -2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$ (in kg·m/s). ✓

34 Find a vector that is perpendicular to both of the following two vectors:

$$\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$$
$$4\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}$$

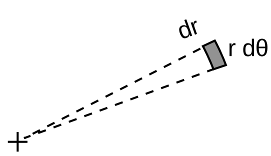
✓

35 Prove property (3) of the vector cross product from the theorem on page 473.

36 Prove the anticommutative property of the vector cross product, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, using the expressions for the components of the cross product.

37 Find three vectors with which you can demonstrate that the vector cross product need not be associative, i.e., that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ need not be the same as $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

38 Which of the following expressions make sense, and which are nonsense? For those that make sense, indicate whether the result is a vector or a scalar.
(a) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$
(b) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
(c) $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$



Problem 39

39 (a) As suggested in the figure, find the area of the infinitesimal region expressed in polar coordinates as lying between r and $r + \mathrm{d}r$ and between θ and $\theta + \mathrm{d}\theta$. ✓
(b) Generalize this to find the infinitesimal element of volume in cylindrical coordinates (r, θ, z), where the Cartesian z axis is perpendicular to the directions measured by r and θ. ✓
(c) Find the moment of inertia for rotation about its axis of a cone whose mass is M, whose height is h, and whose base has a radius b. ✓

40 Find the moment of inertia of a solid rectangular box of mass M and uniform density, whose sides are of length a, b, and c, for rotation about an axis through its center parallel to the edges of length a. ✓

41 The nucleus ^{168}Er (erbium-168) contains 68 protons (which is what makes it a nucleus of the element erbium) and 100 neutrons. It has an ellipsoidal shape like an American football, with one long axis and two short axes that are of equal diameter. Because this is a subatomic system, consisting of only 168 particles, its behavior shows some clear quantum-mechanical properties. It can only have certain energy levels, and it makes quantum leaps between these levels. Also, its angular momentum can only have certain values, which are all multiples of 2.109×10^{-34} kg · m^2/s. The table shows some of the observed angular momenta and energies of ^{168}Er, in SI units (kg · m^2/s and joules).

$L \times 10^{34}$	$E \times 10^{14}$
0	0
2.109	1.2786
4.218	4.2311
6.327	8.7919
8.437	14.8731
10.546	22.3798
12.655	31.135
14.764	41.206
16.873	52.223

(a) These data can be described to a good approximation as a rigid end-over-end rotation. Estimate a single best-fit value for the moment of inertia from the data, and check how well the data agree with the assumption of rigid-body rotation. ▷ Hint, p. 542 √

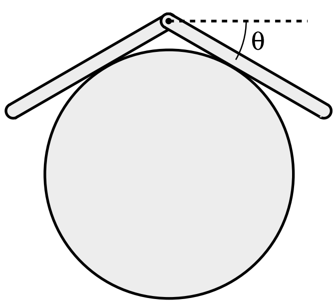
(b) Check whether this moment of inertia is on the right order of magnitude. The moment of inertia depends on both the size and the shape of the nucleus. For the sake of this rough check, ignore the fact that the nucleus is not quite spherical. To estimate its size, use the fact that a neutron or proton has a volume of about 1 fm^3 (one cubic femtometer, where 1 fm $= 10^{-15}$ m), and assume they are closely packed in the nucleus.

42 (a) Prove the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ by expanding the product in terms of its components. Note that because the x, y, and z components are treated symmetrically in the definitions of the vector cross product, it is only necessary to carry out the proof for the x component of the result.

(b) Applying this to the angular momentum of a rigidly rotating body, $L = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm$, show that the diagonal elements of the moment of inertia tensor can be expressed as, e.g., $I_{xx} = \int (y^2 + z^2)\, dm$.

(c) Find the diagonal elements of the moment of inertia matrix of an ellipsoid with axes of lengths a, b, and c, in the principal-axis frame, and with the axis at the center. √ ⋆

43 When we talk about rigid-body rotation, the concept of a perfectly rigid body can only be an idealization. In reality, any object will compress, expand, or deform to some extent when subjected to the strain of rotation. However, if we let it settle down for a while, perhaps it will reach a new equilibrium. As an example, suppose we fill a centrifuge tube with some compressible substance like shaving cream or Wonder Bread. We can model the contents of the tube as a one-dimensional line of mass, extending from $r = 0$ to $r = \ell$. Once the rotation starts, we expect that the contents will be most compressed near the "floor" of the tube at $r = \ell$; this is both because the inward force required for circular motion increases with r for a fixed ω, and because the part at the floor has the greatest amount of material pressing "down" (actually outward) on it. The linear density $\mathrm{d}m/\,\mathrm{d}r$, in units of kg/m, should therefore increase as a function of r. Suppose that we have $\mathrm{d}m/\,\mathrm{d}r = \mu e^{r/\ell}$, where μ is a constant. Find the moment of inertia. ✓



Problem 44.

44 Two bars of length L are connected with a hinge and placed on a frictionless cylinder of radius r. (a) Show that the angle θ shown in the figure is related to the unitless ratio r/L by the equation

$$\frac{r}{L} = \frac{\cos^2\theta}{2\tan\theta}.$$

(b) Discuss the physical behavior of this equation for very large and very small values of r/L. ⋆

45 Let two sides of a triangle be given by the vectors $\mathbf{A}$ and $\mathbf{B}$, with their tails at the origin, and let mass m be uniformly distributed on the interior of the triangle. (a) Show that the distance of the triangle's center of mass from the intersection of sides $\mathbf{A}$ and $\mathbf{B}$ is given by $\frac{1}{3}|\mathbf{A}+\mathbf{B}|$.
(b) Consider the quadrilateral with mass $2m$, and vertices at the origin, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}+\mathbf{B}$. Show that its moment of inertia, for rotation about an axis perpendicular to it and passing through its center of mass, is $\frac{m}{6}(A^2+B^2)$.
(c) Show that the moment of inertia for rotation about an axis perpendicular to the plane of the original triangle, and passing through its center of mass, is $\frac{m}{18}(A^2+B^2-\mathbf{A}\cdot\mathbf{B})$. Hint: Combine the results of parts a and b with the result of problem 27. ⋆

46 In example 23 on page 459, prove that if the rod is sufficiently thin, it can be toppled without scraping on the floor.
▷ Solution, p. 556 ⋆

47 A yo-yo of total mass m consists of two solid cylinders of radius R, connected by a small spindle of negligible mass and radius r. The top of the string is held motionless while the string unrolls from the spindle. Show that the acceleration of the yo-yo is $g/(1 + R^2/2r^2)$. [Hint: The acceleration and the tension in the string are unknown. Use $\tau = \Delta L/\Delta t$ and $F = ma$ to determine these two unknowns.] ⋆

48 We have n identical books of width w, and we wish to stack them at the edge of a table so that they extend the maximum possible distance L_n beyond the edge. Surprisingly, it is possible to have values of L_n that are greater than w, even with fairly small n. For large n, however, L_n begins to grow very slowly. Our goal is to find L_n for a given n. We adopt the restriction that only one book is ever used at a given height.[2] (a) Use proof by induction to find L_n, expressing your result as a sum. (b) Find a sufficiently tight lower bound on this sum, as a closed-form expression, to prove that 1,202,604 books suffice for $L > 7w$. ⋆

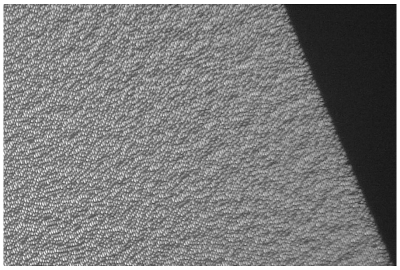
49 A certain function f takes two vectors as inputs and gives an output that is also a vector. The function can be defined in such a way that it is rotationally invariant, and it is also well defined regardless of the units of the vectors. It takes on the following values for the following inputs:

$$\begin{aligned} f(\hat{\mathbf{x}}, \hat{\mathbf{y}}) &= -\hat{\mathbf{z}} \\ f(2\hat{\mathbf{x}}, \hat{\mathbf{y}}) &= -8\hat{\mathbf{z}} \\ f(\hat{\mathbf{x}}, 2\hat{\mathbf{y}}) &= -2\hat{\mathbf{z}} \end{aligned}$$

Prove that the given information uniquely determines f, and give an explicit expression for it. ⋆

50 (a) Find the moment of inertia of a uniform square of mass m and with sides of length b, for rotation in its own plane, about one of its corners. √
(b) The square is balanced on one corner on a frictionless surface. An infinitesimal perturbation causes it to topple. Find its angular velocity at the moment when its side slaps the surface. √ ⋆

[2] When this restriction is lifted, the calculation of L_n becomes a much more difficult problem, which was partially solved in 2009 by Paterson, Peres, Thorup, Winkler, and Zwick.



Problem 51.

51 The figure shows a microscopic view of the innermost tracks of a music CD. The pits represent the pattern of ones and zeroes that encode the musical waveform. Because the laser that reads the data has to sweep over a fixed amount of data per unit time, the disc spins at a decreasing angular velocity as the music is played from the inside out. The linear velocity v, not the angular velocity, is constant. Each track is separated from its neighbors on either side by a fixed distance p, called the pitch. Although the tracks are actually concentric circles, we will idealize them in this problem as a type of spiral, called an Archimedean spiral, whose turns have constant spacing, p, along any radial line. Our goal is to find the angular acceleration of this idealized CD, in terms of the constants v and p, and the radius r at which the laser is positioned.
(a) Use geometrical reasoning to constrain the dependence of the result on p.
(b) Use units to further constrain the result up to a unitless multiplicative constant.
(c) Find the full result. [Hint: Find a differential equation involving r and its time derivative, and then solve this equation by separating variables.] ✓
(d) Consider the signs of the variables in your answer to part c, and show that your equation still makes sense when the direction of rotation is reversed.
(e) Similarly, check that your result makes sense regardless of whether we view the CD player from the front or the back. (Clockwise seen from one side is counterclockwise from the other.) ⋆

52 Neutron stars are the collapsed remnants of dead stars. They rotate quickly, and their rotation can be measured extremely accurately by radio astronomers. Some of them rotate at such a predictable rate that they can be used to count time about as accurately as the best atomic clocks. They do decelerate slowly, but this deceleration can be taken into account. One of the best-studied stars of this type[3] was observed continuously over a 10-year period. As of the benchmark date April 5, 2001, it was found to have

$$\omega = 1.091313551502333 \times 10^3 \text{ s}^{-1}$$

and

$$\alpha = -1.085991 \times 10^{-14} \text{ s}^{-2},$$

where the error bars in the final digit of each number are about ± 1. Astronomers often use the Julian year as their unit of time, where one Julian year is defined to be exactly 3.15576×10^7 s. Find the number of revolutions that this pulsar made over a period of 10 Julian years, starting from the benchmark date.

[3] Verbiest *et al.*, Astrophysical Journal 679 (675) 2008

√

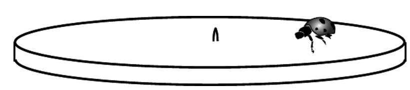
53 A disk, initially rotating at 120 radians per second, is slowed down with a constant angular acceleration of magnitude 4.0 s^{-2}. How many revolutions does the disk make before it comes to rest? [Problem by B. Shotwell.] √

54 A bell rings at the Tilden Park merry go round in Berkeley, California, and the carousel begins to move with an angular acceleration of 1.0×10^{-2} s^{-2}. How much time does it take to perform its first revolution?

√

55 A gasoline-powered car has a heavy wheel called a flywheel, whose main function is to add inertia to the motion of the engine so that it keeps spinning smoothly between power strokes of the cylinders. Suppose that a certain car's flywheel is spinning with angular velocity ω_0, but the car is then turned off, so that the engine and flywheel start to slow down as a result of friction. Assume that the angular acceleration is constant. After the flywheel has made N revolutions, it comes to rest. What is the magnitude of the angular acceleration? [Problem by B. Shotwell.] √

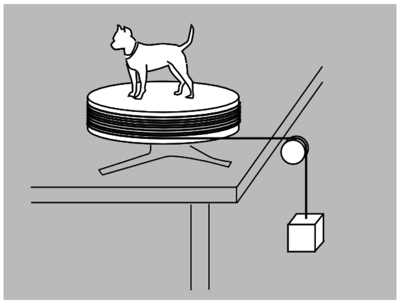
56 A rigid body rotates about a line according to $\theta = At^3 - Bt$ (valid for both negative and positive t).
(a) What is the angular velocity as a function of time? √
(b) What is the angular acceleration as a function of time? √
(c) There are two times when the angular velocity is zero. What is the positive time for which this is true? Call this t_+. √
(d) What is the average angular velocity over the time interval from 0 to t_+? [Problem by B. Shotwell.] √



Problems 57 and 58.

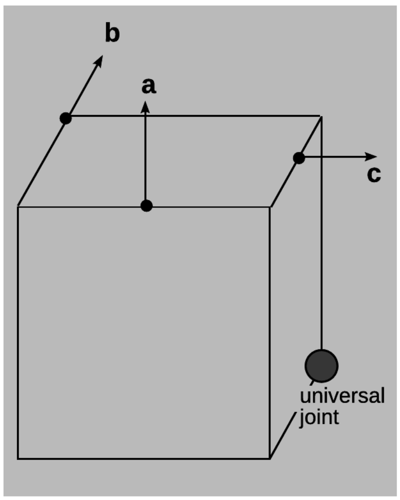
57 A bug stands on a horizontal turntable at distance r from the center. The coefficient of static friction between the bug and the turntable is μ_s. The turntable spins at constant angular frequency ω.
(a) Is the bug more likely to slip at small values of r, or large values?
(b) If the bug walks along a radius, what is the value of r at which it looses its footing? [Problem by B. Shotwell.] √

58 A bug stands on a horizontal turntable at distance r from the center. The coefficient of static friction between the bug and the turntable is μ_s. Starting from rest, the turntable begins rotating with angular acceleration α. What is the magnitude of the angular frequency at which the bug starts to slide? [Problem by B. Shotwell.]
√ ⋆



Problem 59.

59 The figure shows a tabletop experiment that can be used to determine an unknown moment of inertia. A rotating platform of radius R has a string wrapped around it. The string is threaded over a pulley and down to a hanging weight of mass m. The mass is released from rest, and its acceleration a is measured. Find the total moment of inertia I of the platform plus the object sitting on top of it. (The moment of inertia of the object itself can then be found by subtracting the value for the empty platform.) ✓



Problem 60.

60 The uniform cube has unit weight and sides of unit length. One corner is attached to a universal joint, i.e., a frictionless bearing that allows any type of rotation. If the cube is in equilibrium, find the magnitudes of the forces **a**, **b**, and **c**. ✓

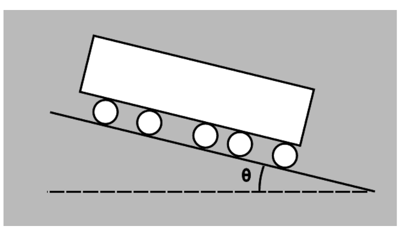
61 In this problem we investigate the notion of division by a vector.

(a) Given a nonzero vector **a** and a scalar b, suppose we wish to find a vector **u** that is the solution of $\mathbf{a}\cdot\mathbf{u}=b$. Show that the solution is not unique, and give a geometrical description of the solution set.

(b) Do the same thing for the equation $\mathbf{a}\times\mathbf{u}=\mathbf{c}$.

(c) Show that the *simultaneous* solution of these two equations exists and is unique.

Remark: This is one motivation for constructing the number system called the quaternions. For a certain period around 1900, quaternions were more popular than the system of vectors and scalars more commonly used today. They still have some important advantages over the scalar-vector system for certain applications, such as avoiding a phenomenon known as gimbal lock in controlling the orientation of bodies such as spacecraft. ⋆



Problem 62.

62 The figure shows a slab of mass M rolling freely down an inclined plane inclined at an angle θ to the horizontal. The slab is on top of a set of rollers, each of radius r, that roll without slipping at their top and bottom surfaces. The rollers may for example be cylinders, or spheres such as ball bearings. Each roller's center of mass coincides with its geometrical center. The sum of the masses of the rollers is m, and the sum of their moments of inertia (each about its own center) is I. Find the acceleration of the slab, and verify that your expression has the correct behavior in interesting limiting cases. ✓ ⋆

63 Vector $\mathbf{A}=(3.0\hat{\mathbf{x}}-4.0\hat{\mathbf{y}})$ meters, and vector $\mathbf{B}=(5.0\hat{\mathbf{x}}+12.0\hat{\mathbf{y}})$ meters. Find the following: (a) The magnitude of vector $\mathbf{A}-2\mathbf{B}$. ✓

(b) The dot product $\mathbf{A}\cdot\mathbf{B}$. ✓

(c) The cross product $\mathbf{A}\times\mathbf{B}$ (expressing the result in terms of its components). ✓

(d) The value of $(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})$. ✓

(e) The angle between the two vectors.

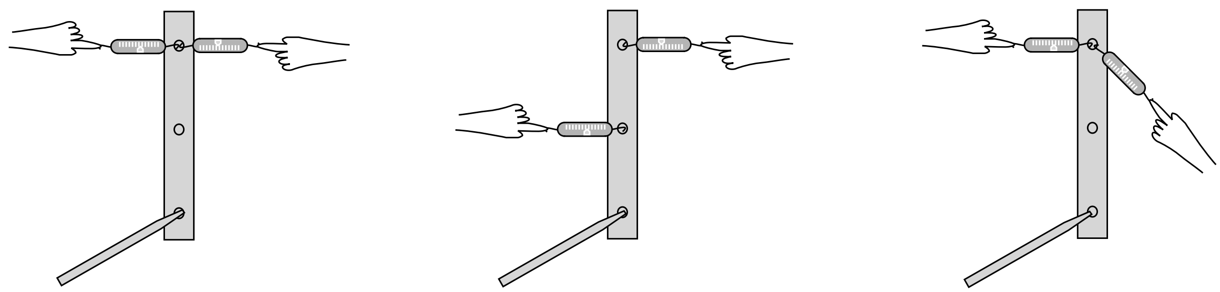
✓ *[problem by B. Shotwell]*

64 A disk starts from rest and rotates about a fixed axis, subject to a constant torque. The work done by the torque during the first revolution is W. What is the work done by the torque during the second revolution? $\surd$ *[problem by B. Shotwell]*

Exercise 15: Torque

Equipment:

- rulers with holes in them
- spring scales (two per group)



While one person holds the pencil which forms the axle for the ruler, the other members of the group pull on the scale and take readings. In each case, calculate the total torque on the ruler, and find out whether it equals zero to roughly within the accuracy of the experiment. Finish the calculations for each part before moving on to the next one.

Vibrations and resonance

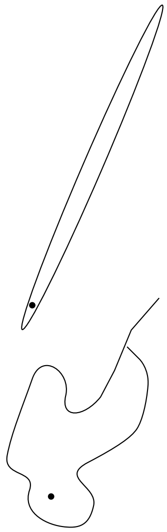
The vibrations of this electric bass string are converted to electrical vibrations, then to sound vibrations, and finally to vibrations of our eardrums.

Chapter 16
Vibrations

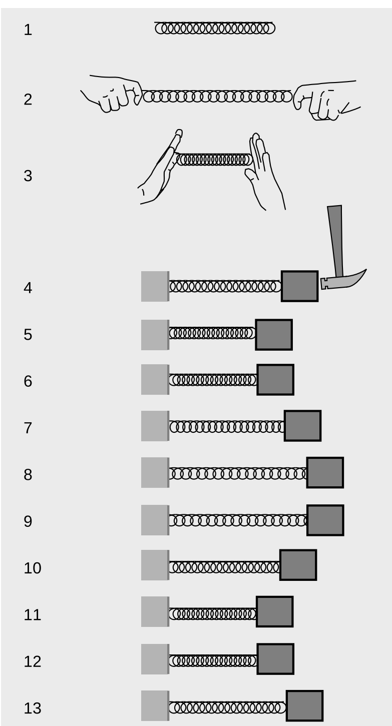
Dandelion. Cello. Read those two words, and your brain instantly conjures a stream of associations, the most prominent of which have to do with vibrations. Our mental category of "dandelion-ness" is strongly linked to the color of light waves that vibrate about half a million billion times a second: yellow. The velvety throb of a cello has as its most obvious characteristic a relatively low musical pitch — the note you are spontaneously imagining right now might be one whose sound vibrations repeat at a rate of a hundred times a second.

Evolution has designed our two most important senses around the assumption that not only will our environment be drenched with information-bearing vibrations, but in addition those vibrations will often be repetitive, so that we can judge colors and pitches by the rate of repetition. Granting that we do sometimes encounter non-repeating waves such as the consonant "sh," which has no recognizable pitch, why was Nature's assumption of repetition nevertheless so right in general?

Repeating phenomena occur throughout nature, from the orbits of electrons in atoms to the reappearance of Halley's Comet every 75 years. Ancient cultures tended to attribute repetitious phenomena



a / If we try to draw a non-repeating orbit for Halley's Comet, it will inevitably end up crossing itself.



b / A spring has an equilibrium length, 1, and can be stretched, 2, or compressed, 3. A mass attached to the spring can be set into motion initially, 4, and will then vibrate, 4-13.

like the seasons to the cyclical nature of time itself, but we now have a less mystical explanation. Suppose that instead of Halley's Comet's true, repeating elliptical orbit that closes seamlessly upon itself with each revolution, we decide to take a pen and draw a whimsical alternative path that never repeats. We will not be able to draw for very long without having the path cross itself. But at such a crossing point, the comet has returned to a place it visited once before, and since its potential energy is the same as it was on the last visit, conservation of energy proves that it must again have the same kinetic energy and therefore the same speed. Not only that, but the comet's direction of motion cannot be randomly chosen, because angular momentum must be conserved as well. Although this falls short of being an ironclad proof that the comet's orbit must repeat, it no longer seems surprising that it does.

Conservation laws, then, provide us with a good reason why repetitive motion is so prevalent in the universe. But it goes deeper than that. Up to this point in your study of physics, I have been indoctrinating you with a mechanistic vision of the universe as a giant piece of clockwork. Breaking the clockwork down into smaller and smaller bits, we end up at the atomic level, where the electrons circling the nucleus resemble — well, little clocks! From this point of view, particles of matter are the fundamental building blocks of everything, and vibrations and waves are just a couple of the tricks that groups of particles can do. But at the beginning of the 20th century, the tables were turned. A chain of discoveries initiated by Albert Einstein led to the realization that the so-called subatomic "particles" were in fact waves. In this new world-view, it is vibrations and waves that are fundamental, and the formation of matter is just one of the tricks that waves can do.

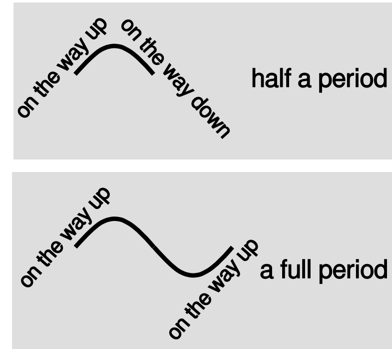
16.1 Period, frequency, and amplitude

Figure b shows our most basic example of a vibration. With no forces on it, the spring assumes its equilibrium length, b/1. It can be stretched, 2, or compressed, 3. We attach the spring to a wall on the left and to a mass on the right. If we now hit the mass with a hammer, 4, it oscillates as shown in the series of snapshots, 4-13. If we assume that the mass slides back and forth without friction and that the motion is one-dimensional, then conservation of energy proves that the motion must be repetitive. When the block comes back to its initial position again, 7, its potential energy is the same again, so it must have the same kinetic energy again. The motion is in the opposite direction, however. Finally, at 10, it returns to its initial position with the same kinetic energy and the same direction of motion. The motion has gone through one complete cycle, and will now repeat forever in the absence of friction.

The usual physics terminology for motion that repeats itself over and over is periodic motion, and the time required for one repetition is called the period, T. (The symbol P is not used because of the possible confusion with momentum.) One complete repetition of the motion is called a cycle.

We are used to referring to short-period sound vibrations as "high" in pitch, and it sounds odd to have to say that high pitches have low periods. It is therefore more common to discuss the rapidity of a vibration in terms of the number of vibrations per second, a quantity called the frequency, f. Since the period is the number of seconds per cycle and the frequency is the number of cycles per second, they are reciprocals of each other,

$$f = 1/T.$$



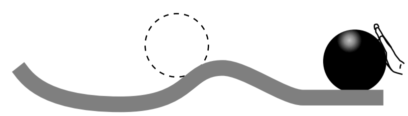
c / Position-versus-time graphs for half a period and a full period.

The forms of various equations turn out to be simpler when they are expressed not in terms of f but in terms of $\omega = 2\pi f$. It's not a coincidence that this relationship looks the same as the one relating angular velocity and frequency in circular motion. In machines, mechanical linkages are used to convert back and forth between vibrational motion and circular motion. For example, a car engine's pistons oscillate in their cylinders at a frequency f, driving the crankshaft at the same frequency f. Either of these motions can be described using ω instead of f, even though only in the case of the crankshaft's rotational motion does it make sense to interpret ω as the number of radians per second. When the motion is not rotational, we usually refer to ω as the angular frequency, and we often use the word "frequency" to mean either f or ω, relying on context to make the meaning clear.

d / The locomotive's wheels spin at a frequency of f cycles per second, which can also be described as ω radians per second. The mechanical linkages allow the linear vibration of the steam engine's pistons, at frequency f, to drive the wheels.

A carnival game — *example 1*

In the carnival game shown in figure e, the rube is supposed to push the bowling ball on the track just hard enough so that it goes over the hump and into the valley, but does not come back out again. If the only types of energy involved are kinetic and potential, this is impossible. Suppose you expect the ball to come back to a point such as the one shown with the dashed outline, then stop and turn around. It would already have passed through this point once before, going to the left on its way into the valley. It was moving then, so conservation of energy tells us that it cannot be at rest when it comes back to the same point. The motion that the customer hopes for is physically impossible. There is a physically possible periodic motion in which the ball rolls back and forth, staying confined within the valley, but there is no way to get the ball into that motion beginning from the place where we start. There is a way to beat the game, though. If you put enough spin on the ball, you can create enough kinetic friction so that a significant amount of heat is generated. Conservation of energy then allows the ball to be at rest when it comes back to a point



e / Example 1.

like the outlined one, because kinetic energy has been converted into heat.

Period and frequency of a fly's wing-beats *example 2*

A Victorian parlor trick was to listen to the pitch of a fly's buzz, reproduce the musical note on the piano, and announce how many times the fly's wings had flapped in one second. If the fly's wings flap, say, 200 times in one second, then the frequency of their motion is $f = 200/1\ \text{s} = 200\ \text{s}^{-1}$. The period is one 200th of a second, $T = 1/f = (1/200)\ \text{s} = 0.005\ \text{s}$.

Units of inverse second, s^{-1}, are awkward in speech, so an abbreviation has been created. One Hertz, named in honor of a pioneer of radio technology, is one cycle per second. In abbreviated form, $1\ \text{Hz} = 1\ \text{s}^{-1}$. This is the familiar unit used for the frequencies on the radio dial.

Frequency of a radio station *example 3*

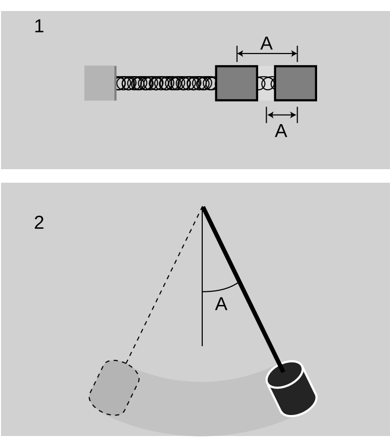
▷ KKJZ's frequency is 88.1 MHz. What does this mean, and what period does this correspond to?

▷ The metric prefix M- is mega-, i.e., millions. The radio waves emitted by KKJZ's transmitting antenna vibrate 88.1 million times per second. This corresponds to a period of

$$T = 1/f = 1.14 \times 10^{-8}\ \text{s}.$$

This example shows a second reason why we normally speak in terms of frequency rather than period: it would be painful to have to refer to such small time intervals routinely. I could abbreviate by telling people that KKJZ's period was 11.4 nanoseconds, but most people are more familiar with the big metric prefixes than with the small ones.

Units of frequency are also commonly used to specify the speeds of computers. The idea is that all the little circuits on a computer chip are synchronized by the very fast ticks of an electronic clock, so that the circuits can all cooperate on a task without getting ahead or behind. Adding two numbers might require, say, 30 clock cycles. Microcomputers these days operate at clock frequencies of about a gigahertz.



f / 1. The amplitude of the vibrations of the mass on a spring could be defined in two different ways. It would have units of distance. 2. The amplitude of a swinging pendulum would more naturally be defined as an angle.

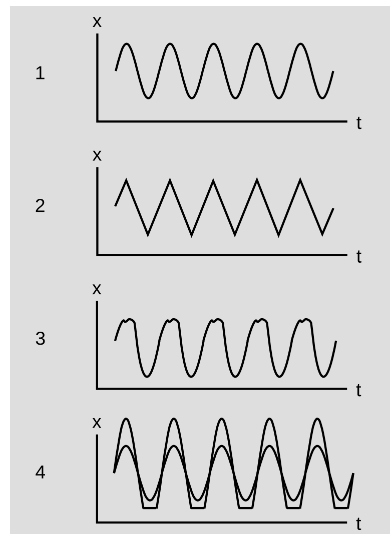
We have discussed how to measure how fast something vibrates, but not how big the vibrations are. The general term for this is amplitude, A. The definition of amplitude depends on the system being discussed, and two people discussing the same system may not even use the same definition. In the example of the block on the end of the spring, f/1, the amplitude will be measured in distance units such as cm. One could work in terms of the distance traveled by the block from the extreme left to the extreme right, but it would be somewhat more common in physics to use the distance from the center to one extreme. The former is usually referred to as the peak-to-peak amplitude, since the extremes of the motion look like mountain peaks or upside-down mountain peaks on a graph of position versus time.

In other situations we would not even use the same units for amplitude. The amplitude of a child on a swing, or a pendulum, f/2, would most conveniently be measured as an angle, not a distance, since her feet will move a greater distance than her head. The electrical vibrations in a radio receiver would be measured in electrical units such as volts or amperes.

16.2 Simple harmonic motion

Why are sine-wave vibrations so common?

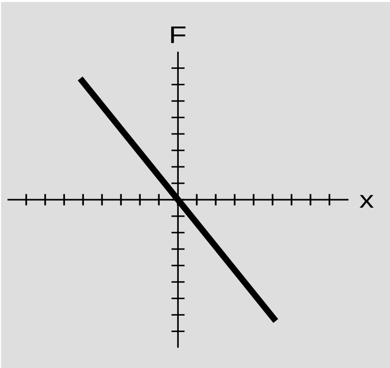
If we actually construct the mass-on-a-spring system discussed in the previous section and measure its motion accurately, we will find that its $x-t$ graph is nearly a perfect sine-wave shape, as shown in figure g/1. (We call it a "sine wave" or "sinusoidal" even if it is a cosine, or a sine or cosine shifted by some arbitrary horizontal amount.) It may not be surprising that it is a wiggle of this general sort, but why is it a specific mathematically perfect shape? Why is it not a sawtooth shape like 2 or some other shape like 3? The mystery deepens as we find that a vast number of apparently unrelated vibrating systems show the same mathematical feature. A tuning fork, a sapling pulled to one side and released, a car bouncing on its shock absorbers, all these systems will exhibit sine-wave motion under one condition: the amplitude of the motion must be small.



g / Sinusoidal and non-sinusoidal vibrations.

It is not hard to see intuitively why extremes of amplitude would act differently. For example, a car that is bouncing lightly on its shock absorbers may behave smoothly, but if we try to double the amplitude of the vibrations the bottom of the car may begin hitting the ground, g/4. (Although we are assuming for simplicity in this chapter that energy is never dissipated, this is clearly not a very realistic assumption in this example. Each time the car hits the ground it will convert quite a bit of its potential and kinetic energy into heat and sound, so the vibrations would actually die out quite quickly, rather than repeating for many cycles as shown in the figure.)

The key to understanding how an object vibrates is to know how the force on the object depends on the object's position. If an object is vibrating to the right and left, then it must have a leftward force on it when it is on the right side, and a rightward force when it is on the left side. In one dimension, we can represent the direction of the force using a positive or negative sign, and since the force changes from positive to negative there must be a point in the middle where the force is zero. This is the equilibrium point, where the object would stay at rest if it was released at rest. For convenience of notation throughout this chapter, we will define the origin of our coordinate system so that x equals zero at equilibrium.



h / The force exerted by an ideal spring, which behaves exactly according to Hooke's law.

The simplest example is the mass on a spring, for which the force on the mass is given by Hooke's law,

$$F = -kx.$$

We can visualize the behavior of this force using a graph of F versus x, as shown in figure h. The graph is a line, and the spring constant, k, is equal to minus its slope. A stiffer spring has a larger value of k and a steeper slope. Hooke's law is only an approximation, but it works very well for most springs in real life, as long as the spring isn't compressed or stretched so much that it is permanently bent or damaged.

The following important theorem relates the motion graph to the force graph.

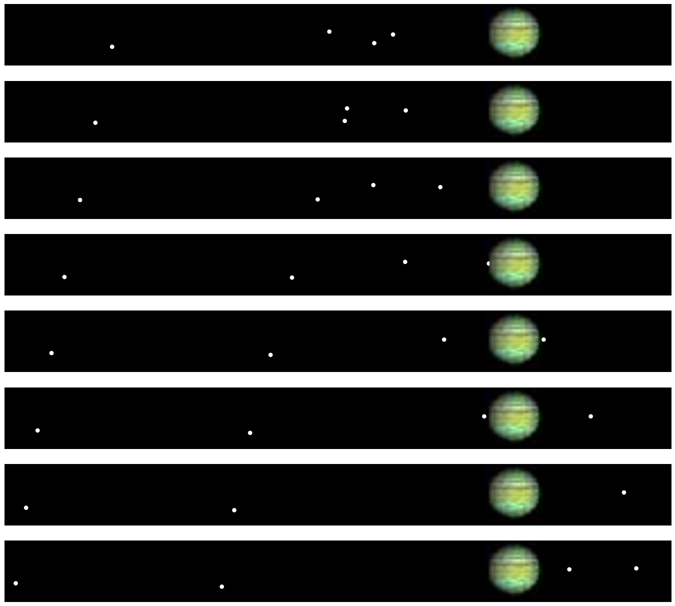
> Theorem: A linear force graph makes a sinusoidal motion graph.
>
> If the total force on a vibrating object depends only on the object's position, and is related to the objects displacement from equilibrium by an equation of the form $F = -kx$, then the object's motion displays a sinusoidal graph with frequency $\omega = \sqrt{k/m}$.

Proof: By Newton's second law, $-kx = ma$, so we need a function $x(t)$ that satisfies the equation $\mathrm{d}^2 x/\mathrm{d}t^2 = -cx$, where for convenience we write c for k/m. This type of equation is called a differential equation, because it relates a function to its own derivative (in this case the second derivative).

Just to make things easier to think about, suppose that we happen to have an oscillator with $c = 1$. Then our goal is to find a function whose second derivative is equal to minus the original function. We know of two such functions, the sine and the cosine. These two solutions can be combined to make anything of the form $P\sin t+Q\cos t$, where P and Q are constants, and the result will still be a solution. Using trig identities, such an expression can always be rewritten as $A\cos(t + \delta)$.

Now what about the more general case where c need not equal 1? The role of c in $\mathrm{d}^2 x/\mathrm{d}t^2 = -cx$ is to set the time scale. For example, suppose we produce a fake video of an object oscillating according to $A\cos(t + \delta)$, which violates Newton's second law because c doesn't equal 1, so the acceleration is too small. We can always make the video physically accurate by speeding it up. This suggests generalizing the solution to $A\cos(\omega t + \delta)$. Plugging in to the differential equation, we find that $\omega = \sqrt{k/m}$, and $T = 2\pi/\omega$ brings us to the claimed result.

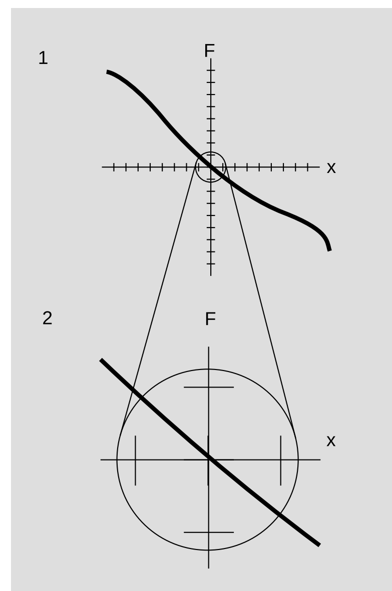
We've proved that anything of this form is a solution, but we



i / Because simple harmonic motion involves sinusoidal functions, it is equivalent to circular motion that has been projected into one dimension. This figure shows a simulated view of Jupiter and its four largest moons at intervals of three hours. Seen from the side from within the plane of the solar system, the circular orbits appear linear. In coordinates with the origin at Jupiter, a moon has coordinates $x = r\cos\theta$ and $y = r\sin\theta$, where $\theta = \omega t$. If the view is along the y axis, then we see only the x motion, which is of the form $A\cos(\omega t)$.

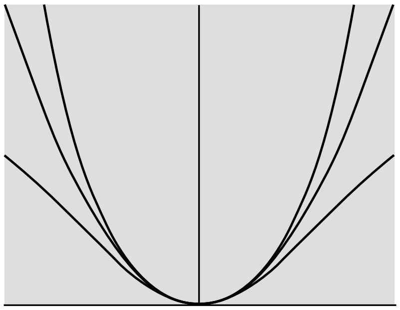
haven't shown that any solution must be of this form. Physically, this must be true because the motion is fully determined by the oscillator's initial position and initial velocity, which can always be matched by choosing A and δ appropriately. Mathematically, the uniqueness result is a standard one about second-order differential equations.

This may seem like only an obscure theorem about the mass-on-a-spring system, but figure j shows it to be far more general than that. Figure j/1 depicts a force curve that is not a straight line. A system with this $F-x$ curve would have large-amplitude vibrations that were complex and not sinusoidal. But the same system would exhibit sinusoidal small-amplitude vibrations. This is because any curve looks linear from very close up. If we magnify the $F-x$ graph as shown in figure j/2, it becomes very difficult to tell that the graph is not a straight line. If the vibrations were confined to the region shown in j/2, they would be very nearly sinusoidal. This is the reason why sinusoidal vibrations are a universal feature of all vibrating systems, if we restrict ourselves to small amplitudes. The theorem is therefore of great general significance. It applies throughout the universe, to objects ranging from vibrating stars to vibrating nuclei. A sinusoidal vibration is known as simple harmonic motion.



j / Seen from close up, any $F-x$ curve looks like a line.

This relates to the fundamental idea behind differential calculus, which is that up close, any smooth function looks linear. To characterize small oscillations about the equilibrium at $x = 0$ in figure h, all we need to know is the derivative $\mathrm{d}F/\,\mathrm{d}x|_0$, which equals $-k$. That is, a force function $F(x)$ has no "individuality" except as defined by k.


k / Three functions with the same curvature at $x = 0$.

Spring constant related to potential energy *example 4*

The same idea about lack of individuality can be expressed in terms of energy.

On a graph of PE versus x, an equilibrium is a local minimum. We can imagine an oscillation about this equilibrium point as if a marble was rolling back and forth in the depression of the graph. Let's choose a coordinate system in which $x = 0$ is the equilibrium, and since the potential energy is only well defined up to an additive constant, we'll simply define it to be zero at equilibrium:

$$PE(0) = 0$$

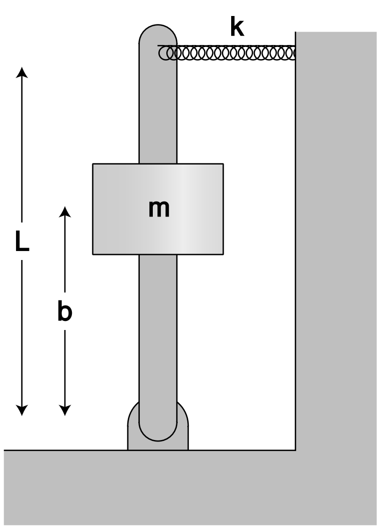
Since $x = 0$ is a local minimum,

$$\frac{\mathrm{d}PE}{\mathrm{d}x}(0) = 0.$$

There are still infinitely many functions that could satisfy these criteria, including the three shown in figure k, which are $x^2/2$, $x^2/2(1+x^2)$, and $(e^{3x}+e^{-3x}-2)/18$. Note, however, how all three functions are virtually identical right near the minimum. That's because they all have the same curvature. More specifically, each function has its second derivative equal to 1 at $x = 0$, and the second derivative is a measure of curvature. Since the $F = -\mathrm{d}PE/\mathrm{d}x$ and $k = -\mathrm{d}PE/\mathrm{d}x$, k equals the second derivative of the PE,

$$\frac{\mathrm{d}^2PE}{\mathrm{d}x^2}(0) = k.$$

As shown in figure k, any two functions that have $PE(0) = 0$, $\mathrm{d}PE/\mathrm{d}x = 0$, and $\mathrm{d}^2PE/\mathrm{d}x^2 = k$, with the same value of k, are virtually indistinguishable for small values of x, so if we want to analyze small oscillations, it doesn't even matter which function we assume. For simplicity, we can always use $PE(x) = (1/2)kx^2$, which is the form that gives a constant second derivative.



l / Example 5. The rod pivots on the hinge at the bottom.

A spring and a lever *example 5*

▷ What is the period of small oscillations of the system shown in the figure? Neglect the mass of the lever and the spring. Assume that the spring is so stiff that gravity is not an important effect. The spring is relaxed when the lever is vertical.

▷ This is a little tricky, because the spring constant k, although it is relevant, is *not* the k we should be putting into the equation $\omega = \sqrt{k/m}$. I find this easier to understand by working with energy rather than force. (Another method would be to use torque, as in problem 15.) The k that goes into $\sqrt{k/m}$ has to be the second derivative of PE with respect to the position, x, of the mass that's moving. The energy PE stored in the spring depends on how far the *tip* of the lever is from the center. This distance equals $(L/b)x$, so the energy in the spring is

$$PE = \frac{1}{2}k\left(\frac{L}{b}x\right)^2 = \frac{kL^2}{2b^2}x^2,$$

and the k we have to put in $T = 2\pi\sqrt{m/k}$ is

$$\frac{d^2PE}{dx^2} = \frac{kL^2}{b^2}.$$

The result is

$$\omega = \sqrt{\frac{kL^2}{mb^2}} = \frac{L}{b}\sqrt{\frac{k}{m}}$$

The leverage of the lever makes it as if the spring was stronger, decreasing the period of the oscillations by a factor of b/L.

Water in a U-shaped tube — *example 6*

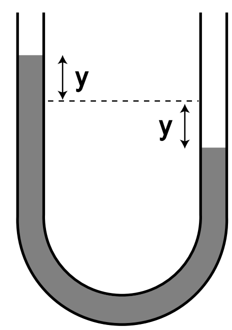
▷ The U-shaped tube in figure m has cross-sectional area A, and the density of the water inside is ρ. Find the gravitational potential energy as a function of the quantity y shown in the figure, show that there is an equilibrium at y=0, and find the frequency of oscillation of the water.

▷ Potential energy is only well defined up to an additive constant. To fix this constant, let's define PE to be zero when y=0. The difference between $PE(y)$ and $PE(0)$ is the energy that would be required to lift a water column of height y out of the right side, and place it above the dashed line, on the left side, raising it through a height y. This water column has height y and cross-sectional area A, so its volume is Ay, its mass is ρAy, and the energy required is $mgy=(\rho Ay)gy=\rho gAy^2$. We then have $PE(y) = PE(0)+\rho gAy^2 = \rho gAy^2$.

The "spring constant" is

$$k = \frac{d^2PE}{dy^2} = 2\rho gA.$$

This is an interesting example, because k can be calculated without any approximations, but the kinetic energy requires an approximation, because we don't know the details of the pattern of



m / Example 6.

flow of the water. It could be very complicated. There will be a tendency for the water near the walls to flow more slowly due to friction, and there may also be swirling, turbulent motion. However, if we make the approximation that all the water moves with the same velocity as the surface, dy/dt, then the mass-on-a-spring analysis applies. Letting L be the total length of the filled part of the tube, the mass is ρLA, and we have

$$\begin{aligned}\omega &= \sqrt{k/m}\\ &= \sqrt{\frac{2\rho gA}{\rho LA}}\\ &= \sqrt{\frac{2g}{L}}.\end{aligned}$$

Period is approximately independent of amplitude, if the amplitude is small.

Until now we have not even mentioned the most counterintuitive aspect of the equation $\omega = \sqrt{k/m}$: it does not depend on amplitude at all. Intuitively, most people would expect the mass-on-a-spring system to take longer to complete a cycle if the amplitude was larger. (We are comparing amplitudes that are different from each other, but both small enough that the theorem applies.) In fact the larger-amplitude vibrations take the same amount of time as the small-amplitude ones. This is because at large amplitudes, the force is greater, and therefore accelerates the object to higher speeds.

Legend has it that this fact was first noticed by Galileo during what was apparently a less than enthralling church service. A gust of wind would now and then start one of the chandeliers in the cathedral swaying back and forth, and he noticed that regardless of the amplitude of the vibrations, the period of oscillation seemed to be the same. Up until that time, he had been carrying out his physics experiments with such crude time-measuring techniques as feeling his own pulse or singing a tune to keep a musical beat. But after going home and testing a pendulum, he convinced himself that he had found a superior method of measuring time. Even without a fancy system of pulleys to keep the pendulum's vibrations from dying down, he could get very accurate time measurements, because the gradual decrease in amplitude due to friction would have no effect on the pendulum's period. (Galileo never produced a modern-style pendulum clock with pulleys, a minute hand, and a second hand, but within a generation the device had taken on the form that persisted for hundreds of years after.)

The pendulum *example 7*

▷ Compare the frequencies of pendula having bobs with different masses.

▷ From the equation $\omega = \sqrt{k/m}$, we might expect that a larger mass would lead to a lower frequency. However, increasing the mass also increases the forces that act on the pendulum: gravity and the tension in the string. This increases k as well as m, so the frequency of a pendulum is independent of m.

Discussion questions

A Suppose that a pendulum has a rigid arm mounted on a bearing, rather than a string tied at its top with a knot. The bob can then oscillate with center-to-side amplitudes greater than 90°. For the maximum amplitude of 180°, what can you say about the period?

B In the language of calculus, Newton's second law for a simple harmonic oscillator can be written in the form $d^2x/dt^2 = -(\ldots)x$, where "…" refers to a constant, and the minus sign says that if we pull the object away from equilibrium, a restoring force tries to bring it back to equlibrium, which is the opposite direction. This is why we get motion that looks like a sine or cosine function: these are functions that, when differentiated twice, give back the original function but with an opposite sign. Now consider the example described in discussion question A, where a pendulum is upright or nearly upright. How does the analysis play out differently?

Summary

Selected vocabulary

periodic motion .	motion that repeats itself over and over
period	the time required for one cycle of a periodic motion
frequency	the number of cycles per second, the inverse of the period
amplitude	the amount of vibration, often measured from the center to one side; may have different units depending on the nature of the vibration
simple harmonic motion	motion whose $x - t$ graph is a sine wave

Notation

T	period
f	frequency
A	amplitude
k	the slope of the graph of F versus x, where F is the total force acting on an object and x is the object's position; for a spring, this is known as the spring constant.
ω (Greek letter "omega")	$2\pi f$

Other terminology and notation

ν	The Greek letter ν, nu, is used in many books for frequency.

Summary

Periodic motion is common in the world around us because of conservation laws. An important example is one-dimensional motion in which the only two forms of energy involved are potential and kinetic; in such a situation, conservation of energy requires that an object repeat its motion, because otherwise when it came back to the same point, it would have to have a different kinetic energy and therefore a different total energy.

Not only are periodic vibrations very common, but small-amplitude vibrations are always sinusoidal as well. That is, the $x-t$ graph is a sine wave. This is because the graph of force versus position will always look like a straight line on a sufficiently small scale. This type of vibration is called simple harmonic motion. In simple harmonic motion, the frequency is independent of the amplitude, and is given by

$$\omega = \sqrt{k/m}.$$

Problems

Key

√ A computerized answer check is available online.

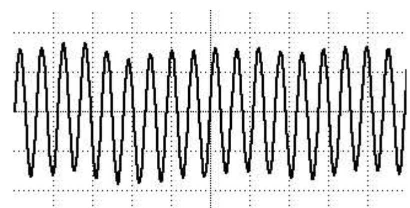
∫ A problem that requires calculus.

⋆ A difficult problem.

1 This problem has been deleted.

2 Many single-celled organisms propel themselves through water with long tails, which they wiggle back and forth. (The most obvious example is the sperm cell.) The frequency of the tail's vibration is typically about 10-15 Hz. To what range of periods does this range of frequencies correspond? √

3 The figure shows the oscillation of a microphone in response to the author whistling the musical note "A." The horizontal axis, representing time, has a scale of 1.0 ms per square. Find the period T, the frequency f, and the angular frequency ω. √

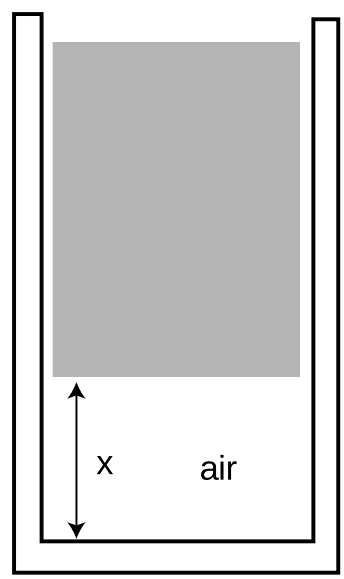


Problem 3.

4 (a) Pendulum 2 has a string twice as long as pendulum 1. If we define x as the distance traveled by the bob along a circle away from the bottom, how does the k of pendulum 2 compare with the k of pendulum 1? Give a numerical ratio. [Hint: the total force on the bob is the same if the angles away from the bottom are the same, but equal angles do not correspond to equal values of x.]

(b) Based on your answer from part (a), how does the period of pendulum 2 compare with the period of pendulum 1? Give a numerical ratio.

5 A pneumatic spring consists of a piston riding on top of the air in a cylinder. The upward force of the air on the piston is given by $F_{air} = ax^{-\beta}$, where $\beta = 1.4$ and a is a constant with funny units of $\text{N}\cdot\text{m}^{1.4}$. For simplicity, assume the air only supports the weight mg of the piston itself, although in practice this device is used to support some other object. The equilibrium position, x_0, is where mg equals $-F_{air}$. (Note that in the main text I have assumed the equilibrium position to be at $x = 0$, but that is not the natural choice here.) Assume friction is negligible, and consider a case where the amplitude of the vibrations is very small. Find the angular frequency of oscillation. √

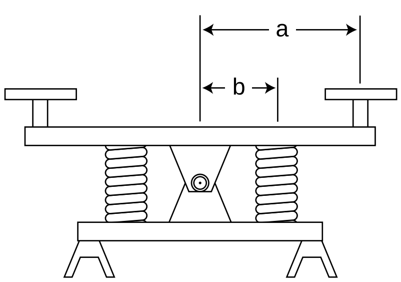


Problem 5.

6 Verify that energy is conserved in simple harmonic motion.

7 Consider the same pneumatic piston described in problem 5, but now imagine that the oscillations are not small. Sketch a graph of the total force on the piston as it would appear over this wider range of motion. For a wider range of motion, explain why the vibration of the piston about equilibrium is not simple harmonic motion, and sketch a graph of x vs t, showing roughly how the curve is different from a sine wave. [Hint: Acceleration corresponds to the curvature of the $x-t$ graph, so if the force is greater, the graph should curve around more quickly.]

8 Archimedes' principle states that an object partly or wholly immersed in fluid experiences a buoyant force equal to the weight of the fluid it displaces. For instance, if a boat is floating in water, the upward pressure of the water (vector sum of all the forces of the water pressing inward and upward on every square inch of its hull) must be equal to the weight of the water displaced, because if the boat was instantly removed and the hole in the water filled back in, the force of the surrounding water would be just the right amount to hold up this new "chunk" of water. (a) Show that a cube of mass m with edges of length b floating upright (not tilted) in a fluid of density ρ will have a draft (depth to which it sinks below the waterline) h given at equilibrium by $h_0 = m/b^2\rho$. (b) Find the total force on the cube when its draft is h, and verify that plugging in $h - h_0$ gives a total force of zero. (c) Find the cube's period of oscillation as it bobs up and down in the water, and show that can be expressed in terms of and g only. √



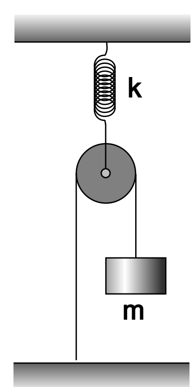
Problem 9.

9 The figure shows a see-saw with two springs at Codornices Park in Berkeley, California. Each spring has spring constant k, and a kid of mass m sits on each seat. (a) Find the period of vibration in terms of the variables k, m, a, and b. (b) Discuss the special case where $a = b$, rather than $a > b$ as in the real see-saw. (c) Show that your answer to part a also makes sense in the case of $b = 0$. √ ⋆

10 Show that the equation $\omega = \sqrt{k/m}$ has units that make sense.

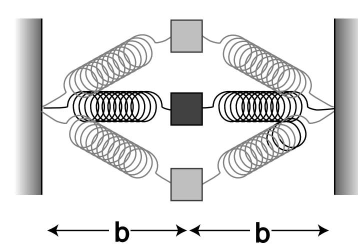
11 A hot scientific question of the 18th century was the shape of the earth: whether its radius was greater at the equator than at the poles, or the other way around. One method used to attack this question was to measure gravity accurately in different locations on the earth using pendula. If the highest and lowest latitudes accessible to explorers were 0 and 70 degrees, then the the strength of gravity would in reality be observed to vary over a range from about 9.780 to 9.826 $\mathrm{m/s^2}$. This change, amounting to 0.046 $\mathrm{m/s^2}$, is greater than the 0.022 $\mathrm{m/s^2}$ effect to be expected if the earth had been spherical. The greater effect occurs because the equator feels a reduction due not just to the acceleration of the spinning earth out from under it, but also to the greater radius of the earth at the equator. What is the accuracy with which the period of a one-second pendulum would have to be measured in order to prove that the earth was not a sphere, and that it bulged at the equator? $\checkmark$

12 A certain mass, when hung from a certain spring, causes the spring to stretch by an amount h compared to its equilibrium length. If the mass is displaced vertically from this equilibrium, it will oscillate up and down with a period T_{osc}. Give a numerical comparison between T_{osc} and T_{fall}, the time required for the mass to fall from rest through a height h, when it isn't attached to the spring. . $\checkmark$



Problem 13.

13 Find the period of vertical oscillations of the mass m. The spring, pulley, and ropes have negligible mass.
▷ Hint, p. 543 $\checkmark$



Problem 14.

14 The equilibrium length of each spring in the figure is b, so when the mass m is at the center, neither spring exerts any force on it. When the mass is displaced to the side, the springs stretch; their spring constants are both k.
(a) Find the energy, U, stored in the springs, as a function of y, the distance of the mass up or down from the center. $\checkmark$
(b) Show that the period of small up-down oscillations is infinite.

15 For a one-dimensional harmonic oscillator, the solution to the energy conservation equation,

$$U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{constant},$$

is an oscillation with frequency $\omega = \sqrt{k/m}$.

Now consider an analogous system consisting of a bar magnet hung from a thread, which acts like a magnetic compass. A normal compass is full of water, so its oscillations are strongly damped, but the magnet-on-a-thread compass has very little friction, and will oscillate repeatedly around its equilibrium direction. The magnetic energy of the bar magnet is

$$U = -Bm\cos\theta,$$

where B is a constant that measures the strength of the earth's magnetic field, m is a constant that parametrizes the strength of the magnet, and θ is the angle, measured in radians, between the bar magnet and magnetic north. The equilibrium occurs at $\theta = 0$, which is the minimum of U.

(a) Problem 26 on p. 483 gave some examples of how to construct analogies between rotational and linear motion. Using the same technique, translate the equation defining the linear quantity k to one that defines an analogous angular one κ (Greek letter kappa). Applying this to the present example, find an expression for κ. (Assume the thread is so thin that its stiffness does not have any significant effect compared to earth's magnetic field.) ✓
(b) Find the frequency of the compass's vibrations. ✓

16 A mass m on a spring oscillates around an equilibrium at $x = 0$. Any function $F(x)$ with an equilibrium at $x = 0$, $F(0) = 0$, can be approximated as $F(x) = -kx$, and if the spring's behavior is symmetric with respect to positive and negative values of x, so that $F(-x) = -F(x)$, then the next level of improvement in such an approximation would be $F(x) = -kx - bx^3$. The general idea here is that any smooth function can be approximated locally by a polynomial, and if you want a better approximation, you can use a polynomial with more terms in it. When you ask your calculator to calculate a function like sin or e^x, it's using a polynomial approximation with 10 or 12 terms. Physically, a spring with a positive value of b gets stiffer when stretched strongly than an "ideal" spring with $b = 0$. A spring with a negative b is like a person who cracks under stress — when you stretch it too much, it becomes more elastic than an ideal spring would. We should not expect any spring to give totally ideal behavior no matter no matter how much it is stretched. For example, there has to be some point at which it breaks.

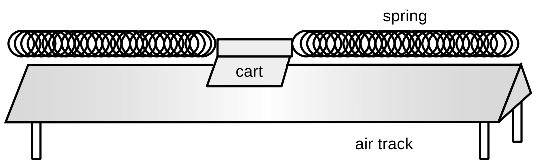
Do a numerical simulation of the oscillation of a mass on a spring whose force has a nonvanishing b. Is the period still independent of amplitude? Is the amplitude-independent equation for the period still approximately valid for small enough amplitudes? Does the addition of an x^3 term with $b > 0$ tend to increase the period, or decrease it? Include a printout of your program and its output with your homework paper.

17 An idealized pendulum consists of a pointlike mass m on the end of a massless, rigid rod of length L. Its amplitude, θ, is the angle the rod makes with the vertical when the pendulum is at the end of its swing. Write a numerical simulation to determine the period of the pendulum for any combination of m, L, and θ. Examine the effect of changing each variable while manipulating the others. $\star$

Exercise 16: Vibrations

Equipment:

- air track and carts of two different masses
- springs
- spring scales



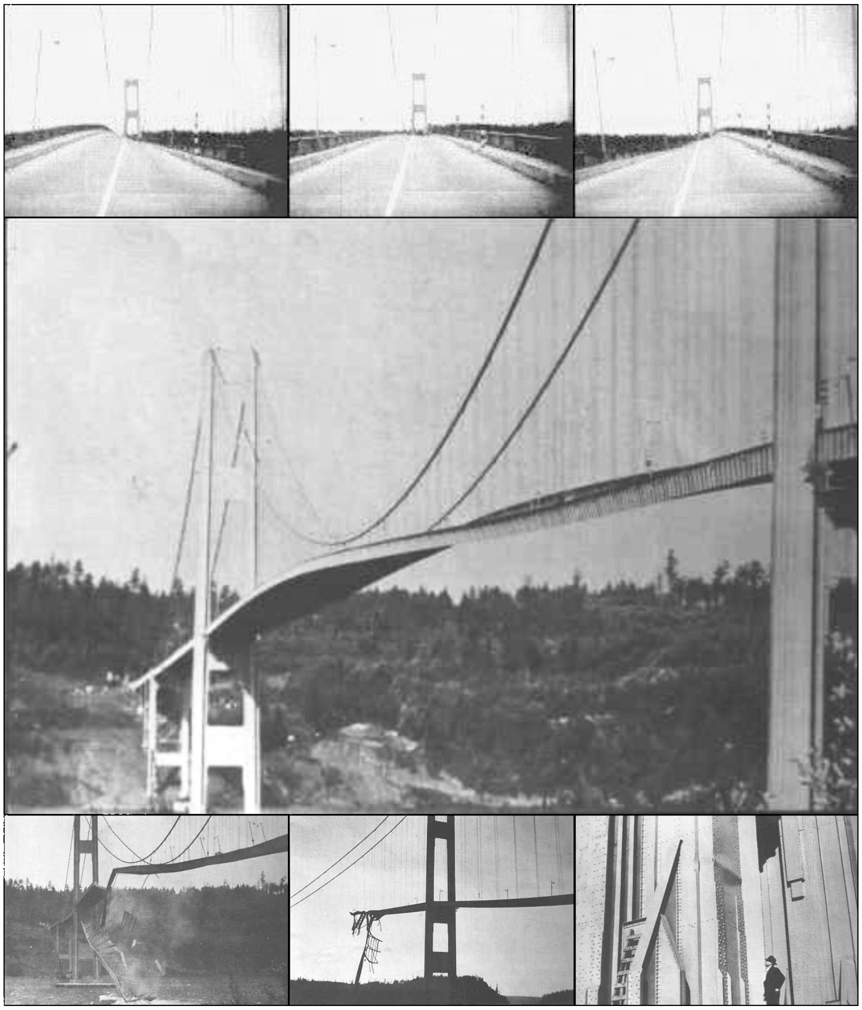
Place the cart on the air track and attach springs so that it can vibrate.

1. Test whether the period of vibration depends on amplitude. Try at least one moderate amplitude, for which the springs do not go slack, at least one amplitude that is large enough so that they do go slack, and one amplitude that's the very smallest you can possibly observe.

2. Try a cart with a different mass. Does the period change by the expected factor, based on the equation $\omega = \sqrt{k/m}$?

3. Use a spring scale to pull the cart away from equilibrium, and make a graph of force versus position. Is it linear? If so, what is its slope?

4. Test the equation $\omega = \sqrt{k/m}$ numerically.



Top: A series of images from a film of the Tacoma Narrows Bridge vibrating on the day it was to collapse. *Middle:* The bridge immediately before the collapse, with the sides vibrating 8.5 meters (28 feet) up and down. Note that the bridge is over a mile long. *Bottom:* During and after the final collapse. The right-hand picture gives a sense of the massive scale of the construction.

Chapter 17
Resonance

Soon after the mile-long Tacoma Narrows Bridge opened in July 1940, motorists began to notice its tendency to vibrate frighteningly in even a moderate wind. Nicknamed "Galloping Gertie," the bridge collapsed in a steady 42-mile-per-hour wind on November 7 of the same year. The following is an eyewitness report from a newspaper editor who found himself on the bridge as the vibrations approached the breaking point.

"Just as I drove past the towers, the bridge began to sway violently from side to side. Before I realized it, the tilt became so violent that I lost control of the car... I jammed on the brakes and got out, only to be thrown onto my face against the curb.

"Around me I could hear concrete cracking. I started to get my dog Tubby, but was thrown again before I could reach the car. The car itself began to slide from side to side of the roadway.

"On hands and knees most of the time, I crawled 500 yards or more to the towers... My breath was coming in gasps; my knees were raw and bleeding, my hands bruised and swollen from gripping the concrete curb... Toward the last, I risked rising to my feet and running a few yards at a time... Safely back at the toll plaza, I saw the bridge in its final collapse and saw my car plunge into the Narrows."

The ruins of the bridge formed an artificial reef, one of the world's largest. It was not replaced for ten years. The reason for its collapse was not substandard materials or construction, nor was the bridge under-designed: the piers were hundred-foot blocks of concrete, the girders massive and made of carbon steel. The bridge was destroyed because the bridge absorbed energy efficiently from the wind, but didn't dissipate it efficiently into heat. The replacement bridge, which has lasted half a century so far, was built smarter, not stronger. The engineers learned their lesson and simply included some slight modifications to avoid the phenomenon that spelled the doom of the first one.

17.1 Energy in vibrations

One way of describing the collapse of the bridge is that the bridge kept taking energy from the steadily blowing wind and building up more and more energetic vibrations. In this section, we discuss the energy contained in a vibration, and in the subsequent sections we will move on to the loss of energy and the adding of energy to a vibrating system, all with the goal of understanding the important phenomenon of resonance.

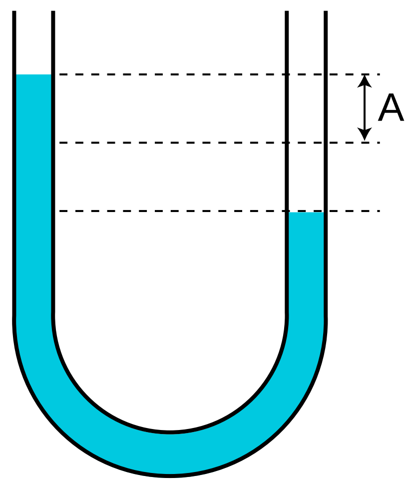
Going back to our standard example of a mass on a spring, we find that there are two forms of energy involved: the potential energy stored in the spring and the kinetic energy of the moving mass. We may start the system in motion either by hitting the mass to put in kinetic energy or by pulling it to one side to put in potential energy. Either way, the subsequent behavior of the system is identical. It trades energy back and forth between kinetic and potential energy. (We are still assuming there is no friction, so that no energy is converted to heat, and the system never runs down.)

The most important thing to understand about the energy content of vibrations is that the total energy is proportional to the square of the amplitude. Although the total energy is constant, it is instructive to consider two specific moments in the motion of the mass on a spring as examples. When the mass is all the way to one side, at rest and ready to reverse directions, all its energy is potential. We have already seen that the potential energy stored in a spring equals $(1/2)kx^2$, so the energy is proportional to the square of the amplitude. Now consider the moment when the mass is passing through the equilibrium point at $x = 0$. At this point it has no potential energy, but it does have kinetic energy. The velocity is proportional to the amplitude of the motion, and the kinetic energy, $(1/2)mv^2$, is proportional to the square of the velocity, so again we find that the energy is proportional to the square of the amplitude. The reason for singling out these two points is merely instructive; proving that energy is proportional to A^2 at any point would suffice to prove that energy is proportional to A^2 in general, since the energy is constant.

Are these conclusions restricted to the mass-on-a-spring example? No. We have already seen that $F = -kx$ is a valid approximation for any vibrating object, as long as the amplitude is small. We are thus left with a very general conclusion: the energy of any vibration is approximately proportional to the square of the amplitude, provided that the amplitude is small.

Water in a U-tube — *example 1*

If water is poured into a U-shaped tube as shown in the figure, it can undergo vibrations about equilibrium. The energy of such a vibration is most easily calculated by considering the "turnaround point" when the water has stopped and is about to reverse directions. At this point, it has only potential energy and no kinetic energy, so by calculating its potential energy we can find the energy of the vibration. This potential energy is the same as the work that would have to be done to take the water out of the right-hand side down to a depth A below the equilibrium level, raise it through a height A, and place it in the left-hand side. The weight of this chunk of water is proportional to A, and so is the height through which it must be lifted, so the energy is proportional to A^2.



a / Example 1.

The range of energies of sound waves — *example 2*

▷ The amplitude of vibration of your eardrum at the threshold of pain is about 10^6 times greater than the amplitude with which it vibrates in response to the softest sound you can hear. How many times greater is the energy with which your ear has to cope for the painfully loud sound, compared to the soft sound?

▷ The amplitude is 10^6 times greater, and energy is proportional to the square of the amplitude, so the energy is greater by a factor of 10^{12} . This is a phenomenally large factor!

We are only studying vibrations right now, not waves, so we are not yet concerned with how a sound wave works, or how the energy gets to us through the air. Note that because of the huge range of energies that our ear can sense, it would not be reasonable to have a sense of loudness that was additive. Consider, for instance, the following three levels of sound:

barely audible wind	
quiet conversation	10^5 times more energy than the wind
heavy metal concert ..	10^{12} times more energy than the wind

In terms of addition and subtraction, the difference between the wind and the quiet conversation is nothing compared to the difference between the quiet conversation and the heavy metal concert. Evolution wanted our sense of hearing to be able to encompass all these sounds without collapsing the bottom of the scale so that anything softer than the crack of doom would sound the same. So rather than making our sense of loudness additive, mother nature made it multiplicative. We sense the difference between the wind and the quiet conversation as spanning a range of about 5/12 as much as the whole range from the wind to the heavy metal concert. Although a detailed discussion of the decibel scale is not relevant here, the basic point to note about the decibel scale is that it is logarithmic. The zero of the decibel scale is close to the lower limit of human hearing, and adding 1 unit to the decibel measurement corresponds to *multiplying* the energy level (or actually the power per unit area) by a certain factor.

17.2 Energy lost from vibrations

Numerical treatment

An oscillator that has friction is referred to as damped. Let's use numerical techniques to find the motion of a damped oscillator that is released away from equilibrium, but experiences no driving force after that. We can expect that the motion will consist of oscillations that gradually die out.

Friction is in general a very complicated phenomenon, and for example video games with racing cars in them include extremely sophisticated models of the friction between the tires and the road. On p. 174 I presented a very simple model of friction that dates back to the French physicist Coulomb, who worked in the era of the French revolution. There is nothing sacred about this model. For example, it doesn't work well for lubricated surfaces. Most of the ideas we're going to learn about damped vibrations are at least qualitatively correct regardless of what technical assumptions are made about friction, but the math turns out to be much simpler if we choose, instead of the Coulomb model, one in which the force of friction on an object is given by $F=-bv$, where v is the object's speed and b is a constant. This is in contrast to the Coulomb model, in which the force is independent of speed.

Newton's second law, $a=F/m$, gives $a=(-kx-bv)/m$. This becomes a little prettier if we rewrite it in the form

$$ma+bv+kx=0,$$

which gives symmetric treatment to three terms involving x and its first and second derivatives, v and a.

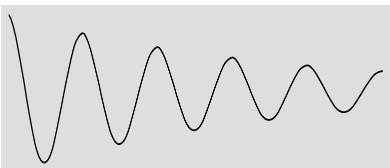
```
import math
k=39.4784  # chosen to give a period of 1 second
m=1.
b=0.211    # chosen to make the results simple
x=1.
v=0.
t=0.
dt=.01
n=1000
for j in range(n):
  x=x+v*dt
  a=(-k*x-b*v)/m
  if (v>0) and (v+a*dt<0) :
    print("turnaround at t=",t,", x=",x)
  v=v+a*dt
  t=t+dt
```

```
turnaround at t= 0.99 , x= 0.899919262445
turnaround at t= 1.99 , x= 0.809844934046
turnaround at t= 2.99 , x= 0.728777519477
turnaround at t= 3.99 , x= 0.655817260033
turnaround at t= 4.99 , x= 0.590154191135
turnaround at t= 5.99 , x= 0.531059189965
turnaround at t= 6.99 , x= 0.477875914756
turnaround at t= 7.99 , x= 0.430013546991
turnaround at t= 8.99 , x= 0.386940256644
turnaround at t= 9.99 , x= 0.348177318484
```

The spring constant, $k=4\pi=39.4784$ N/m, is designed so that if the undamped equation $f=(1/2\pi)\sqrt{k/m}$ was still true, the frequency would be 1 Hz. We start by noting that the addition of a small amount of damping doesn't seem to have changed the period at all, or at least not to within the accuracy of the calculation. You can check for yourself, however, that a large value of b, say 5 N·s/m, does change the period significantly.

We release the mass from $x = 1$ m, and after one cycle, it only comes back to about $x = 0.9$ m. I chose $b = 0.211$ N·s/m by fiddling around until I got this result, since a decrease of exactly 10% is easy to discuss. Notice how the amplitude after two cycles is about 0.81 m, i.e., 1 m times 0.9^2: the amplitude has again dropped by exactly 10%. This pattern continues for as long as the simulation runs, e.g., for the last two cycles, we have 0.34818/0.38694=0.89982, or almost exactly 0.9 again. It might have seemed capricious when I chose to use the unrealistic equation $F = -bv$, but this is the payoff. Only with $-bv$ friction do we get this kind of mathematically simple exponential decay.

Because the decay is exponential, it never dies out completely; this is different from the behavior we would have had with Coulomb friction, which does make objects grind completely to a stop at some point. With friction that acts like $F = -bv$, v gets smaller as the oscillations get smaller. The smaller and smaller force then causes them to die out at a rate that is slower and slower.



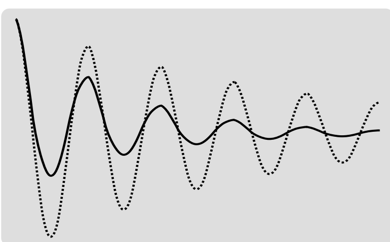
b / A damped sine wave, of the form $x = Ae^{-ct}\sin(\omega_f t + \delta)$.

Analytic treatment

Taking advantage of this unexpectedly simple result, let's find an analytic solution for the motion. The numerical output suggests that we assume a solution of the form

$$x = Ae^{-ct}\sin(\omega_f t + \delta),$$

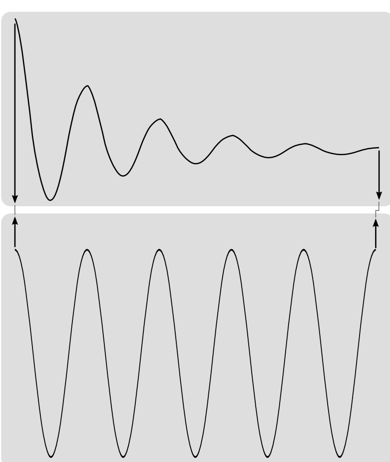
where the unknown constants ω_f and c will presumably be related to m, b, and k. The constant c indicates how quickly the oscillations die out. The constant ω_f is, as before, defined as 2π times the frequency, with the subscript f to indicate a free (undriven) solution. All our equations will come out much simpler if we use ωs everywhere instead of fs from now on, and, as physicists often do, I'll generally use the word "frequency" to refer to ω when the context makes it clear what I'm talking about. The phase angle δ has no real physical significance, since we can define $t = 0$ to be any moment in time we like.



c / Self-check A.

self-check A

In figure c, which graph has the greater value of c? ▷ Answer, p. 561



d / A damped sine wave is compared with an undamped one, with m and k kept the same and only b changed.

The factor A for the initial amplitude can also be omitted without loss of generality, since the equation we're trying to solve, $ma + bv + kx = 0$, is linear. That is, v and a are the first and second derivatives of x, and the derivative of Ax is simply A times the derivative of x. Thus, if $x(t)$ is a solution of the equation, then multiplying it by a constant gives an equally valid solution. This is another place where we see that a damping force proportional to v is the easiest to handle mathematically. For a damping force proportional to v^2, for example, we would have had to solve the equation $ma + bv^2 + kx = 0$, which is nonlinear.

For the purpose of determining ω_f and c, the most general form we need to consider is therefore $x = e^{-ct}\sin\omega_f t$, whose first and second derivatives are $v = e^{-ct}\left(-c\sin\omega_f t + \omega\cos\omega_f t\right)$ and $a = e^{-ct}\left(c^2\sin\omega_f t - 2\omega_f c\cos\omega_f t - \omega_f^2\sin\omega_f t\right)$. Plugging these into the equation $ma + bv + kx = 0$ and setting the sine and cosine parts equal to zero gives, after some tedious algebra,

$$c = \frac{b}{2m}$$

and

$$\omega_f = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

Intuitively, we expect friction to "slow down" the motion, as when we ride a bike into a big patch of mud. "Slow down," however, could have more than one meaning here. It could mean that the oscillator would take more time to complete each cycle, or it could mean that as time went on, the oscillations would die out, thus giving smaller velocities.

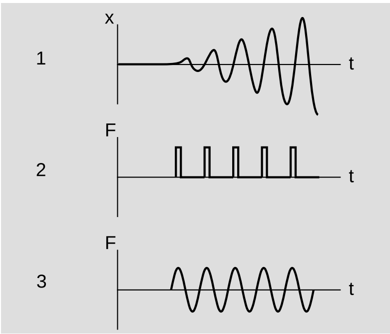
Our mathematical results show that both of these things happen. The first equation says that c, which indicates how quickly the oscillations damp out, is directly related to b, the strength of the damping.

The second equation, for the frequency, can be compared with the result from page 499 of $\sqrt{k/m}$ for the undamped system. Let's refer to this now as ω_o, to distinguish it from the actual frequency ω_f of the free oscillations when damping is present. The result for ω_f will be less than ω_o, due to the presence of the $b^2/4m^2$ term. This tells us that the addition of friction to the system does increase the time required for each cycle. However, it is very common for the $b^2/4m^2$ term to be negligible, so that $\omega_f \approx \omega_o$.

Figure d shows an example. The damping here is quite strong: after only one cycle of oscillation, the amplitude has already been reduced by a factor of 2, corresponding to a factor of 4 in energy. However, the frequency of the damped oscillator is only about 1% lower than that of the undamped one; after five periods, the accumulated lag is just barely visible in the offsetting of the arrows. We can see that extremely strong damping — even stronger than this — would have been necessary in order to make $\omega_f \approx \omega_o$ a poor approximation.

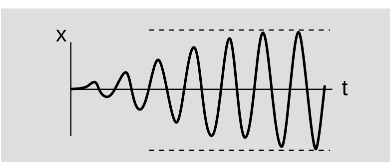
It is customary to describe the amount of damping with a quantity called the quality factor, Q, defined as the number of cycles required for the energy to fall off by a factor of 535. (The origin of this obscure numerical factor is $e^{2\pi}$, where $e = 2.71828\ldots$ is the base of natural logarithms. Choosing this particular number causes some of our later equations to come out nice and simple.) The terminology arises from the fact that friction is often considered a bad thing, so a mechanical device that can vibrate for many oscillations before it loses a significant fraction of its energy would be considered a high-quality device.

17.3 Putting energy into vibrations



e / 1. Pushing a child on a swing gradually puts more and more energy into her vibrations. 2. A fairly realistic graph of the driving force acting on the child. 3. A less realistic, but more mathematically simple, driving force.

When pushing a child on a swing, you cannot just apply a constant force. A constant force will move the swing out to a certain angle, but will not allow the swing to start swinging. Nor can you give short pushes at randomly chosen times. That type of random pushing would increase the child's kinetic energy whenever you happened to be pushing in the same direction as her motion, but it would reduce her energy when your pushing happened to be in the opposite direction compared to her motion. To make her build up her energy, you need to make your pushes rhythmic, pushing at the same point in each cycle. In other words, your force needs to form a repeating pattern with the same frequency as the normal frequency of vibration of the swing. Graph e/1 shows what the child's $x-t$ graph would look like as you gradually put more and more energy into her vibrations. A graph of your *force* versus time would probably look something like graph 2. It turns out, however, that it is much simpler mathematically to consider a vibration with energy being pumped into it by a driving force that is itself a sine-wave, 3. A good example of this is your eardrum being driven by the force of a sound wave.



f / The amplitude approaches a maximum.

Now we know realistically that the child on the swing will not keep increasing her energy forever, nor does your eardrum end up exploding because a continuing sound wave keeps pumping more and more energy into it. In any realistic system, there is energy going out as well as in. As the vibrations increase in amplitude, there is an increase in the amount of energy taken away by damping with each cycle. This occurs for two reasons. Work equals force times distance (or, more accurately, the area under the force-distance curve). As the amplitude of the vibrations increases, the damping force is being applied over a longer distance. Furthermore, the damping force usually increases with velocity (we usually assume for simplicity that it is proportional to velocity), and this also serves to increase the rate at which damping forces remove energy as the amplitude increases. Eventually (and small children and our eardrums are thankful for this!), the amplitude approaches a maximum value, f, at which energy is removed by the damping force just as quickly as it is being put in by the driving force.

This process of approaching a maximum amplitude happens extremely quickly in many cases, e.g., the ear or a radio receiver, and we don't even notice that it took a millisecond or a microsecond for the vibrations to "build up steam." We are therefore mainly interested in predicting the behavior of the system once it has had enough time to reach essentially its maximum amplitude. This is known as the steady-state behavior of a vibrating system.

Now comes the interesting part: what happens if the frequency of the driving force is mismatched to the frequency at which the system would naturally vibrate on its own? We all know that a radio station doesn't have to be tuned in exactly, although there is only a small range over which a given station can be received. The designers of the radio had to make the range fairly small to make it possible to eliminate unwanted stations that happened to be nearby in frequency, but it couldn't be too small or you wouldn't be able to adjust the knob accurately enough. (Even a digital radio can be tuned to 88.0 MHz and still bring in a station at 88.1 MHz.) The ear also has some natural frequency of vibration, but in this case the range of frequencies to which it can respond is quite broad. Evolution has made the ear's frequency response as broad as possible because it was to our ancestors' advantage to be able to hear everything from a low roar to a high-pitched shriek.

The remainder of this section develops four important facts about the response of a system to a driving force whose frequency is not necessarily the same as the system's natural frequency of vibration. The style is approximate and intuitive, but proofs are given in section 17.4.

First, although we know the ear has a frequency — about 4000 Hz — at which it would vibrate naturally, it does not vibrate at 4000 Hz in response to a low-pitched 200 Hz tone. It always responds at the frequency at which it is driven. Otherwise all pitches would sound like 4000 Hz to us. This is a general fact about driven vibrations:

(1) The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system's own natural frequency of vibration.

Now let's think about the amplitude of the steady-state response. Imagine that a child on a swing has a natural frequency of vibration of 1 Hz, but we are going to try to make her swing back and forth at 3 Hz. We intuitively realize that quite a large force would be needed to achieve an amplitude of even 30 cm, i.e., the amplitude is less in proportion to the force. When we push at the natural frequency of 1 Hz, we are essentially just pumping energy back into the system to compensate for the loss of energy due to the damping (friction) force. At 3 Hz, however, we are not just counteracting friction. We are also providing an extra force to make the child's momentum reverse itself more rapidly than it would if gravity and the tension in the chain were the only forces acting. It is as if we are artificially increasing the k of the swing, but this is wasted effort because we spend just as much time decelerating the child (taking energy out of the system) as accelerating her (putting energy in).

Now imagine the case in which we drive the child at a very low frequency, say 0.02 Hz or about one vibration per minute. We are essentially just holding the child in position while very slowly walking back and forth. Again we intuitively recognize that the amplitude will be very small in proportion to our driving force. Imagine how hard it would be to hold the child at our own head-level when she is at the end of her swing! As in the too-fast 3 Hz case, we are spending most of our effort in artificially changing the k of the swing, but now rather than reinforcing the gravity and tension forces we are working against them, effectively reducing k. Only a very small part of our force goes into counteracting friction, and the rest is used in repetitively putting potential energy in on the upswing and taking it back out on the downswing, without any long-term gain.

We can now generalize to make the following statement, which is true for all driven vibrations:

(2) A vibrating system resonates at its own natural frequency.[1] That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

An opera singer breaking a wine glass *example 3*

In order to break a wineglass by singing, an opera singer must first tap the glass to find its natural frequency of vibration, and then sing the same note back.

Collapse of the Nimitz Freeway in an earthquake *example 4*

g / The collapsed section of the Nimitz Freeway.

I led off the chapter with the dramatic collapse of the Tacoma Narrows Bridge, mainly because it was well documented by a local physics professor, and an unknown person made a movie of the collapse. The collapse of a section of the Nimitz Freeway in Oakland, CA, during a 1989 earthquake is however a simpler example to analyze.

An earthquake consists of many low-frequency vibrations that occur simultaneously, which is why it sounds like a rumble of indeterminate pitch rather than a low hum. The frequencies that we can hear are not even the strongest ones; most of the energy is in the form of vibrations in the range of frequencies from about 1 Hz to 10 Hz.

Now all the structures we build are resting on geological layers of dirt, mud, sand, or rock. When an earthquake wave comes along, the topmost layer acts like a system with a certain natural frequency of vibration, sort of like a cube of jello on a plate being shaken from side to side. The resonant frequency of the layer depends on how stiff it is and also on how deep it is. The ill-fated section of the Nimitz freeway was built on a layer of mud, and analysis by geologist Susan E. Hough of the U.S. Geological Survey shows that the mud layer's resonance was centered on about 2.5 Hz, and had a width covering a range from about 1 Hz to 4 Hz.

When the earthquake wave came along with its mixture of frequencies, the mud responded strongly to those that were close to its own natural 2.5 Hz frequency. Unfortunately, an engineering analysis after the quake showed that the overpass itself had a resonant frequency of 2.5 Hz as well! The mud responded strongly to the earthquake waves with frequencies close to 2.5 Hz, and the bridge responded strongly to the 2.5 Hz vibrations of the mud, causing sections of it to collapse.

Collapse of the Tacoma Narrows Bridge *example 5*

Let's now examine the more conceptually difficult case of the Tacoma Narrows Bridge. The surprise here is that the wind was steady. If the wind was blowing at constant velocity, why did it shake the bridge back and forth? The answer is a little complicated. Based on film footage and after-the-fact wind tunnel experiments, it appears that two different mechanisms were involved.

The first mechanism was the one responsible for the initial, relatively weak vibrations, and it involved resonance. As the wind moved over the bridge, it began acting like a kite or an airplane wing. As shown in the figure, it established swirling patterns of air flow around itself, of the kind that you can see in a moving cloud of smoke. As one of these swirls moved off of the bridge, there was an abrupt change in air pressure, which resulted in an up or down force on the bridge. We see something similar when a flag flaps in the wind, except that the flag's surface is usually vertical. This back-and-forth sequence of forces is exactly the kind of periodic driving force that would excite a resonance. The faster the wind, the more quickly the swirls would get across the bridge, and the higher the frequency of the driving force would be. At just the right velocity, the frequency would be the right one to excite the resonance. The wind-tunnel models, however, show that the pattern of vibration of the bridge excited by this mechanism would have been a different one than the one that finally destroyed the bridge.

The bridge was probably destroyed by a different mechanism, in which its vibrations at its own natural frequency of 0.2 Hz set up an alternating pattern of wind gusts in the air immediately around it, which then increased the amplitude of the bridge's vibrations. This vicious cycle fed upon itself, increasing the amplitude of the vibrations until the bridge finally collapsed.

As long as we're on the subject of collapsing bridges, it is worth bringing up the reports of bridges falling down when soldiers marching over them happened to step in rhythm with the bridge's natural frequency of oscillation. This is supposed to have happened in 1831 in Manchester, England, and again in 1849 in Anjou, France. Many modern engineers and scientists, however, are suspicious of the analysis of these reports. It is possible that the collapses had more to do with poor construction and overloading than with resonance. The Nimitz Freeway and Tacoma Narrows Bridge are far better documented, and occurred in an era when engineers' abilities to analyze the vibrations of a complex structure were much more advanced.

Emission and absorption of light waves by atoms *example 6*

In a very thin gas, the atoms are sufficiently far apart that they can act as individual vibrating systems. Although the vibrations are of a very strange and abstract type described by the theory of quantum mechanics, they nevertheless obey the same basic rules as ordinary mechanical vibrations. When a thin gas made of a certain element is heated, it emits light waves with certain specific frequencies, which are like a fingerprint of that element. As with all other vibrations, these atomic vibrations respond most strongly to a driving force that matches their own natural frequency. Thus if we have a relatively cold gas with light waves of various frequencies passing through it, the gas will absorb light at precisely those frequencies at which it would emit light if heated.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to Q.

This is fairly intuitive. The steady-state behavior is an equilibrium between energy input from the driving force and energy loss due to damping. A low-Q oscillator, i.e., one with strong damping, dumps its energy faster, resulting in lower-amplitude steady-state motion.

self-check B

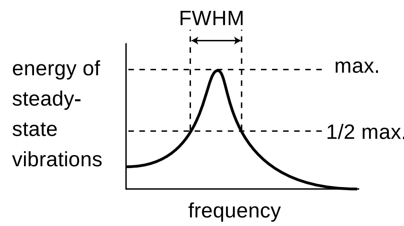
If an opera singer is shopping for a wine glass that she can impress her friends by breaking, what should she look for? ▷ Answer, p. 562

Piano strings ringing in sympathy with a sung note *example 7*

▷ A sufficiently loud musical note sung near a piano with the lid raised can cause the corresponding strings in the piano to vibrate. (A piano has a set of three strings for each note, all struck by the same hammer.) Why would this trick be unlikely to work with a violin?

▷ If you have heard the sound of a violin being plucked (the pizzicato effect), you know that the note dies away very quickly. In other words, a violin's Q is much lower than a piano's. This means that its resonances are much weaker in amplitude.

Our fourth and final fact about resonance is perhaps the most surprising. It gives us a way to determine numerically how wide a range of driving frequencies will produce a strong response. As shown in the graph, resonances do not suddenly fall off to zero outside a certain frequency range. It is usual to describe the width of a resonance by its full width at half-maximum (FWHM) as illustrated in figure h.



h / The definition of the full width at half maximum.

(4) The FWHM of a resonance is related to its Q and its resonant frequency f_{res} by the equation

$$\text{FWHM} = \frac{f_{res}}{Q}.$$

(This equation is only a good approximation when Q is large.)

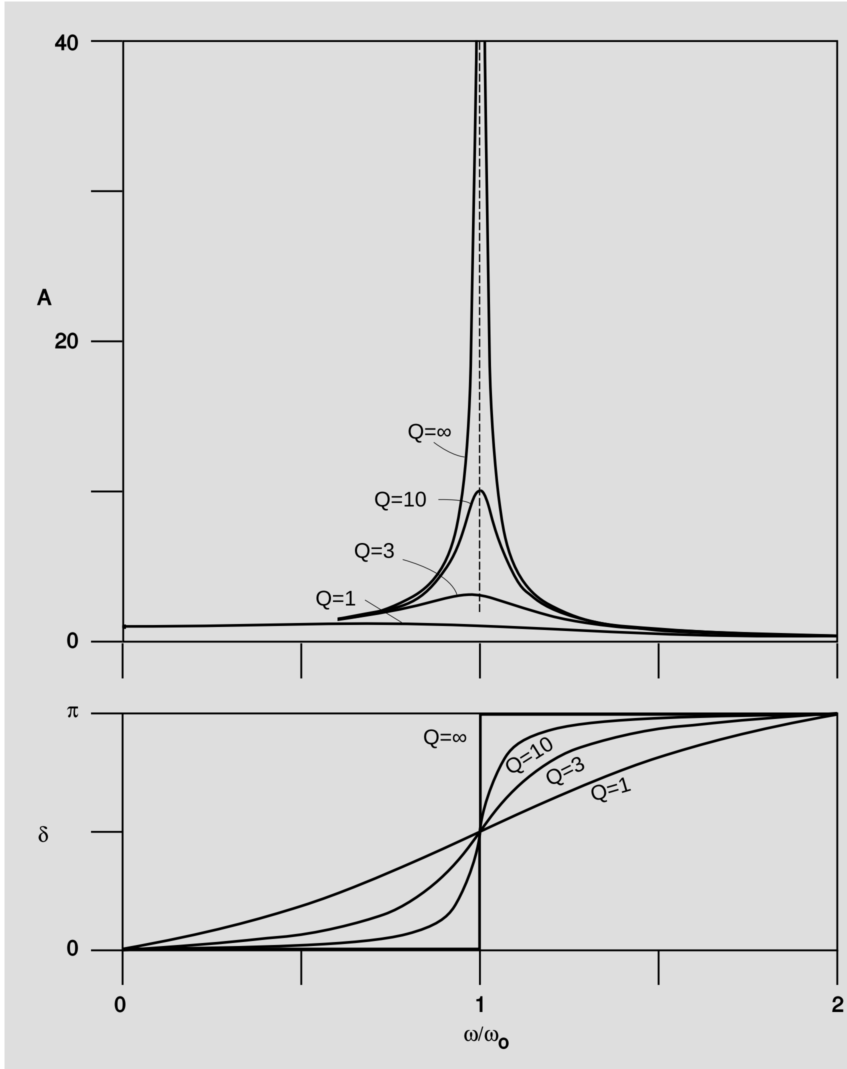
Why? It is not immediately obvious that there should be any logical relationship between Q and the FWHM. Here's the idea. As we have seen already, the reason why the response of an oscillator is smaller away from resonance is that much of the driving force is being used to make the system act as if it had a different k. Roughly speaking, the half-maximum points on the graph correspond to the places where the amount of the driving force being wasted in this way is the same as the amount of driving force being used productively to replace the energy being dumped out by the damping force. If the damping force is strong, then a large amount of force is needed to counteract it, and we can waste quite a bit of driving force on changing k before it becomes comparable to the damping force. If, on the other hand, the damping force is weak, then even a small amount of force being wasted on changing k will become significant in proportion, and we cannot get very far from the resonant frequency before the two are comparable.

The response is in general out of phase with the driving force by an angle δ.

Changing the pitch of a wind instrument *example 8*

▷ A saxophone player normally selects which note to play by choosing a certain fingering, which gives the saxophone a certain resonant frequency. The musician can also, however, change the pitch significantly by altering the tightness of her lips. This corresponds to driving the horn slightly off of resonance. If the pitch can be altered by about 5% up or down (about one musical half-step) without too much effort, roughly what is the Q of a saxophone?

▷ Five percent is the width on one side of the resonance, so the



i / Dependence of the amplitude and phase angle on the driving frequency. The undamped case is $Q = \infty$, and the other curves represent Q=1, 3, and 10. F_m, m, and ω_o are all set to 1.

full width is about 10%, FWHM / f_{res} = 0.1. This implies a Q of about 10, i.e., once the musician stops blowing, the horn will continue sounding for about 10 cycles before its energy falls off by a factor of 535. (Blues and jazz saxophone players will typically choose a mouthpiece that has a low Q, so that they can produce the bluesy pitch-slides typical of their style. "Legit," i.e., classically oriented players, use a higher-Q setup because their style only calls for enough pitch variation to produce a vibrato.)

Decay of a saxophone tone — *example 9*

▷ If a typical saxophone setup has a Q of about 10, how long will it take for a 100-Hz tone played on a baritone saxophone to die down by a factor of 535 in energy, after the player suddenly stops blowing?

▷ A Q of 10 means that it takes 10 cycles for the vibrations to die down in energy by a factor of 535. Ten cycles at a frequency of 100 Hz would correspond to a time of 0.1 seconds, which is not very long. This is why a saxophone note doesn't "ring" like a note played on a piano or an electric guitar.

Q of a radio receiver *example 10*

▷ A radio receiver used in the FM band needs to be tuned in to within about 0.1 MHz for signals at about 100 MHz. What is its Q?

▷ $Q = f_{res}/\text{FWHM} = 1000$. This is an extremely high Q compared to most mechanical systems.

Q of a stereo speaker *example 11*

We have already given one reason why a stereo speaker should have a low Q: otherwise it would continue ringing after the end of the musical note on the recording. The second reason is that we want it to be able to respond to a large range of frequencies.

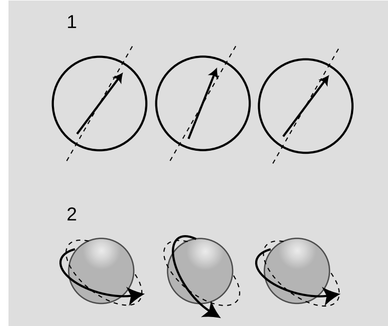
Nuclear magnetic resonance *example 12*

If you have ever played with a magnetic compass, you have undoubtedly noticed that if you shake it, it takes some time to settle down, j/1. As it settles down, it acts like a damped oscillator of the type we have been discussing. The compass needle is simply a small magnet, and the planet earth is a big magnet. The magnetic forces between them tend to bring the needle to an equilibrium position in which it lines up with the planet-earth-magnet.

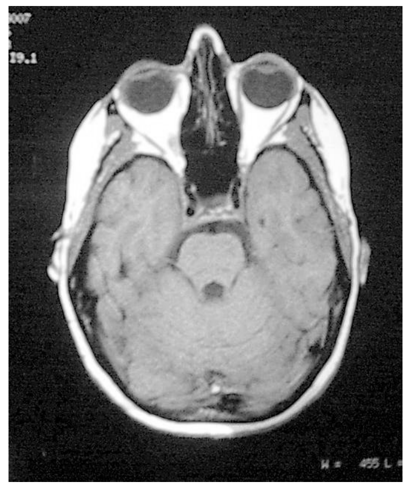
Essentially the same physics lies behind the technique called Nuclear Magnetic Resonance (NMR). NMR is a technique used to deduce the molecular structure of unknown chemical substances, and it is also used for making medical images of the inside of people's bodies. If you ever have an NMR scan, they will actually tell you you are undergoing "magnetic resonance imaging" or "MRI," because people are scared of the word "nuclear." In fact, the nuclei being referred to are simply the non-radioactive nuclei of atoms found naturally in your body.

Here's how NMR works. Your body contains large numbers of hydrogen atoms, each consisting of a small, lightweight electron orbiting around a large, heavy proton. That is, the nucleus of a hydrogen atom is just one proton. A proton is always spinning on its own axis, and the combination of its spin and its electrical charge causes it to behave like a tiny magnet. The principle is identical to that of an electromagnet, which consists of a coil of wire through which electrical charges pass; the circling motion of the charges in the coil of wire makes it magnetic, and in the same way, the circling motion of the proton's charge makes it magnetic.

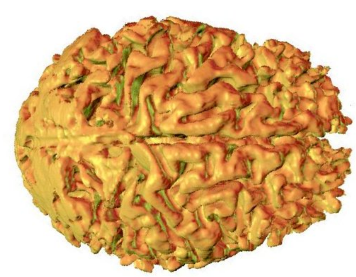
Now a proton in one of your body's hydrogen atoms finds itself surrounded by many other whirling, electrically charged particles: its own electron, plus the electrons and nuclei of the other nearby atoms. These neighbors act like magnets, and exert magnetic forces on the proton, j/2. The k of the vibrating proton is simply a



j / Example 12. 1. A compass needle vibrates about the equilibrium position under the influence of the earth's magnetic forces. 2. The orientation of a proton's spin vibrates around its equilibrium direction under the influence of the magnetic forces coming from the surrounding electrons and nuclei.



k / A member of the author's family, who turned out to be healthy.



l / A three-dimensional computer reconstruction of the shape of a human brain, based on magnetic resonance data.

measure of the total strength of these magnetic forces. Depending on the structure of the molecule in which the hydrogen atom finds itself, there will be a particular set of magnetic forces acting on the proton and a particular value of k. The NMR apparatus bombards the sample with radio waves, and if the frequency of the radio waves matches the resonant frequency of the proton, the proton will absorb radio-wave energy strongly and oscillate wildly. Its vibrations are damped not by friction, because there is no friction inside an atom, but by the reemission of radio waves.

By working backward through this chain of reasoning, one can determine the geometric arrangement of the hydrogen atom's neighboring atoms. It is also possible to locate atoms in space, allowing medical images to be made.

Finally, it should be noted that the behavior of the proton cannot be described entirely correctly by Newtonian physics. Its vibrations are of the strange and spooky kind described by the laws of quantum mechanics. It is impressive, however, that the few simple ideas we have learned about resonance can still be applied successfully to describe many aspects of this exotic system.

Discussion question

A Nikola Tesla, one of the inventors of radio and an archetypical mad scientist, told a credulous reporter in 1912 the following story about an application of resonance. He built an electric vibrator that fit in his pocket, and attached it to one of the steel beams of a building that was under construction in New York. Although the article in which he was quoted didn't say so, he presumably claimed to have tuned it to the resonant frequency of the building. "In a few minutes, I could feel the beam trembling. Gradually the trembling increased in intensity and extended throughout the whole great mass of steel. Finally, the structure began to creak and weave, and the steelworkers came to the ground panic-stricken, believing that there had been an earthquake. ... [If] I had kept on ten minutes more, I could have laid that building flat in the street." Is this physically plausible?

17.4 ⋆ Proofs

Our first goal is to predict the amplitude of the steady-state vibrations as a function of the frequency of the driving force and the amplitude of the driving force. With that equation in hand, we will then prove statements 2, 3, and 4 from section 17.3.

We have an external driving force $F = F_m \sin \omega t$, where the constant F_m indicates the maximum strength of the force in either direction. The equation of motion is

$$ma + bv + kx = F_m \sin \omega t. \quad [1]$$

For the steady-state motion, we're going to look for a solution of the form

$$x = A \sin(\omega t + \delta).$$

The left-hand side of the equation of motion will clearly be a sinusoidal function with frequency ω, so it can only equal the right-hand side if, as we have already implicitly assumed, the frequency of the motion matches the frequency of the driving force. This proves statement (1).

In contrast to the undriven case, here it's not possible to sweep A and δ under the rug. The amplitude of the steady-state motion, A, is actually the most interesting thing to know about the steady-state motion, and it's not true that we still have a solution no matter how we fiddle with A; if we have a solution for a certain value of A, then multiplying A by some constant would break the equality between the two sides of the equation of motion. It's also no longer true that we can get rid of δ simply be redefining when we start the clock; here δ represents a *difference* in time between the start of one cycle of the driving force and the start of the corresponding cycle of the motion.

The velocity and acceleration are $v = \omega A\cos(\omega t + \delta)$ and $a = -\omega^2 A\sin(\omega t + \delta)$, and if we plug these into the equation of motion, [1], and simplify a little, we find

[2] $$(k - m\omega^2)\sin(\omega t + \delta) + \omega b\cos(\omega t + \delta) = \frac{F_m}{A}\sin\omega t.$$

The sum of any two sinusoidal functions with the same frequency is also a sinusoidal, so the whole left side adds up to a sinusoidal. By fiddling with A and δ we can make the amplitudes and phases of the two sides of the equation match up.

Using the trig identities for the sine of a sum and cosine of a sum, we can change equation [2] into the form

$$\begin{aligned}&\left[(-m\omega^2 + k)\cos\delta - b\omega\sin\delta - F_m/A\right]\sin\omega t\\ &+\left[(-m\omega^2 + k)\sin\delta + b\omega\cos\delta\right]\cos\omega t = 0.\end{aligned}$$

Both the quantities in square brackets must equal zero, which gives us two equations we can use to determine the unknowns A and δ. The results are

[3] $$\delta = \tan^{-1}\frac{\omega\omega_o}{Q(\omega_o^2 - \omega^2)}$$

and

[4] $$A = \frac{F_m}{m\sqrt{(\omega^2 - \omega_o^2)^2 + \omega_o^2\omega^2 Q^{-2}}}.$$

Statement 2: maximum amplitude at resonance

Equation [4] makes it plausible that the amplitude is maximized when the system is driven at close to its resonant frequency. At

$f = f_o$, the first term inside the square root vanishes, and this makes the denominator as small as possible, causing the amplitude to be as big as possible. (Actually this is only approximately true, because it is possible to make A a little bigger by decreasing f a little below f_o, which makes the second term smaller. This technical issue is addressed in homework problem 3 on page 533.)

Statement 3: amplitude at resonance proportional to Q

Equation [4] shows that the amplitude at resonance is proportional to $1/b$, and the Q of the system is inversely proportional to b, so the amplitude at resonance is proportional to Q.

Statement 4: FWHM related to Q

We will satisfy ourselves by proving only the proportionality $FWHM \propto f_o/Q$, not the actual equation $FWHM = f_o/Q$. The energy is proportional to A^2, i.e., to the inverse of the quantity inside the square root in equation [4]. At resonance, the first term inside the square root vanishes, and the half-maximum points occur at frequencies for which the whole quantity inside the square root is double its value at resonance, i.e., when the two terms are equal. At the half-maximum points, we have

$$\begin{aligned} f^2 - f_o^2 &= \left(f_o \pm \frac{\text{FWHM}}{2}\right)^2 - f_o^2 \\ &= \pm f_o \cdot \text{FWHM} + \frac{1}{4}\text{FWHM}^2 \end{aligned}$$

If we assume that the width of the resonance is small compared to the resonant frequency, then the FWHM^2 term is negligible compared to the $f_o \cdot \text{FWHM}$ term, and setting the terms in equation 4 equal to each other gives

$$4\pi^2 m^2 \left(f_o \text{FWHM}\right)^2 = b^2 f^2.$$

We are assuming that the width of the resonance is small compared to the resonant frequency, so f and f_o can be taken as synonyms. Thus,

$$\text{FWHM} = \frac{b}{2\pi m}.$$

We wish to connect this to Q, which can be interpreted as the energy of the free (undriven) vibrations divided by the work done by damping in one cycle. The former equals $kA^2/2$, and the latter is proportional to the force, $bv \propto bAf_o$, multiplied by the distance traveled, A. (This is only a proportionality, not an equation, since the force is not constant.) We therefore find that Q is proportional to k/bf_o. The equation for the FWHM can then be restated as a proportionality $\text{FWHM} \propto k/Qf_o m \propto f_o/Q$.

Summary

Selected vocabulary

damping . . . . . the dissipation of a vibration's energy into heat energy, or the frictional force that causes the loss of energy

quality factor . . the number of oscillations required for a system's energy to fall off by a factor of 535 due to damping

driving force . . . an external force that pumps energy into a vibrating system

resonance . . . . the tendency of a vibrating system to respond most strongly to a driving force whose frequency is close to its own natural frequency of vibration

steady state . . . the behavior of a vibrating system after it has had plenty of time to settle into a steady response to a driving force

Notation

Q . . . . the quality factor

f_o . . . . the natural (resonant) frequency of a vibrating system, i.e., the frequency at which it would vibrate if it was simply kicked and left alone

f . . . . the frequency at which the system actually vibrates, which in the case of a driven system is equal to the frequency of the driving force, not the natural frequency

Summary

The energy of a vibration is always proportional to the square of the amplitude, assuming the amplitude is small. Energy is lost from a vibrating system for various reasons such as the conversion to heat via friction or the emission of sound. This effect, called damping, will cause the vibrations to decay exponentially unless energy is pumped into the system to replace the loss. A driving force that pumps energy into the system may drive the system at its own natural frequency or at some other frequency. When a vibrating system is driven by an external force, we are usually interested in its steady-state behavior, i.e., its behavior after it has had time to settle into a steady response to a driving force. In the steady state, the same amount of energy is pumped into the system during each cycle as is lost to damping during the same period.

The following are four important facts about a vibrating system being driven by an external force:

(1) The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system's own natural frequency of vibration.

(2) A vibrating system resonates at its own natural frequency. That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to Q.

(4) The FWHM of a resonance is related to its Q and its resonant frequency f_o by the equation

$$\text{FWHM} = \frac{f_o}{Q}.$$

(This equation is only a good approximation when Q is large.)

Problems

Key

- $\checkmark$ A computerized answer check is available online.
- $\int$ A problem that requires calculus.
- $\star$ A difficult problem.

1 If one stereo system is capable of producing 20 watts of sound power and another can put out 50 watts, how many times greater is the amplitude of the sound wave that can be created by the more powerful system? (Assume they are playing the same music.)

2 Many fish have an organ known as a swim bladder, an air-filled cavity whose main purpose is to control the fish's buoyancy and allow it to keep from rising or sinking without having to use its muscles. In some fish, however, the swim bladder (or a small extension of it) is linked to the ear and serves the additional purpose of amplifying sound waves. For a typical fish having such an anatomy, the bladder has a resonant frequency of 300 Hz, the bladder's Q is 3, and the maximum amplification is about a factor of 100 in energy. Over what range of frequencies would the amplification be at least a factor of 50?

3 As noted in section 17.4, it is only approximately true that the amplitude has its maximum at the natural frequency $(1/2\pi)\sqrt{k/m}$. Being more careful, we should actually define two different symbols, $f_\text{o} = (1/2\pi)\sqrt{k/m}$ and f_res for the slightly different frequency at which the amplitude is a maximum, i.e., the actual resonant frequency. In this notation, the amplitude as a function of frequency is

$$A = \frac{F}{2\pi\sqrt{4\pi^2 m^2\left(f^2 - f_0^2\right)^2 + b^2 f^2}}.$$

Show that the maximum occurs not at f_o but rather at

$$f_\text{res} = \sqrt{f_0^2 - \frac{b^2}{8\pi^2 m^2}} = \sqrt{f_0^2 - \frac{1}{2}\text{FWHM}^2}$$

Hint: Finding the frequency that minimizes the quantity inside the square root is equivalent to, but much easier than, finding the frequency that maximizes the amplitude.

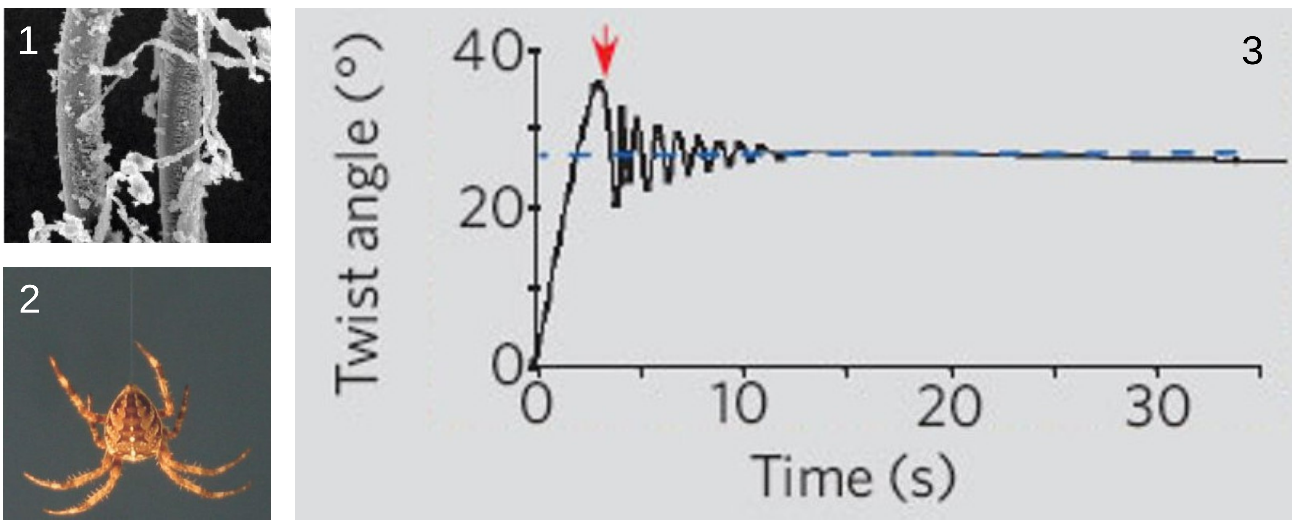
4 (a) Let W be the amount of work done by friction in the first cycle of oscillation, i.e., the amount of energy lost to heat. Find the fraction of the original energy E that remains in the oscillations after n cycles of motion.

(b) From this, prove the equation

$$\left(1 - \frac{W}{E}\right)^Q = e^{-2\pi}$$

(recalling that the number 535 in the definition of Q is $e^{2\pi}$).

(c) Use this to prove the approximation $1/Q \approx (1/2\pi)W/E$. (Hint: Use the approximation $\ln(1+x) \approx x$, which is valid for small values of x.)



Problem 5.

5 (a) We observe that the amplitude of a certain free oscillation decreases from A_o to A_o/Z after n oscillations. Find its Q. ✓
(b) The figure is from *Shape memory in Spider draglines*, Emile, Le Floch, and Vollrath, *Nature* 440:621 (2006). Panel 1 shows an electron microscope's image of a thread of spider silk. In 2, a spider is hanging from such a thread. From an evolutionary point of view, it's probably a bad thing for the spider if it twists back and forth while hanging like this. (We're referring to a back-and-forth rotation about the axis of the thread, not a swinging motion like a pendulum.) The authors speculate that such a vibration could make the spider easier for predators to see, and it also seems to me that it would be a bad thing just because the spider wouldn't be able to control its orientation and do what it was trying to do. Panel 3 shows a graph of such an oscillation, which the authors measured using a video camera and a computer, with a 0.1 g mass hung from it

in place of a spider. Compared to human-made fibers such as kevlar or copper wire, the spider thread has an unusual set of properties:

1. It has a low Q, so the vibrations damp out quickly.

2. It doesn't become brittle with repeated twisting as a copper wire would.

3. When twisted, it tends to settle in to a new equilibrium angle, rather than insisting on returning to its original angle. You can see this in panel 2, because although the experimenters initially twisted the wire by 35 degrees, the thread only performed oscillations with an amplitude much smaller than ± 35 degrees, settling down to a new equilibrium at 27 degrees.

4. Over much longer time scales (hours), the thread eventually resets itself to its original equilbrium angle (shown as zero degrees on the graph). (The graph reproduced here only shows the motion over a much shorter time scale.) Some human-made materials have this "memory" property as well, but they typically need to be heated in order to make them go back to their original shapes.

Focusing on property number 1, estimate the Q of spider silk from the graph. $\checkmark$

6 An oscillator with sufficiently strong damping has its maximum response at $\omega = 0$. Using equation [4] on p. 529 , find the value of Q at which this behavior sets in.

▷ Hint, p. 543 ▷ Answer, p. 562

7 The goal of this problem is to refine the proportionality FWHM $\propto f_{res}/Q$ into the equation FWHM $= f_{res}/Q$, i.e., to prove that the constant of proportionality equals 1.

(a) Show that the work done by a damping force $F = -bv$ over one cycle of steady-state motion equals $W_{damp} = -2\pi^2 bfA^2$. Hint: It is less confusing to calculate the work done over half a cycle, from $x = -A$ to $x = +A$, and then double it.

(b) Show that the fraction of the undriven oscillator's energy lost to damping over one cycle is $|W_{damp}|/E = 4\pi^2 bf/k$.

(c) Use the previous result, combined with the result of problem 4, to prove that Q equals $k/2\pi bf$.

(d) Combine the preceding result for Q with the equation FWHM $= b/2\pi m$ from section 17.4 to prove the equation FWHM $= f_{res}/Q$.

⋆

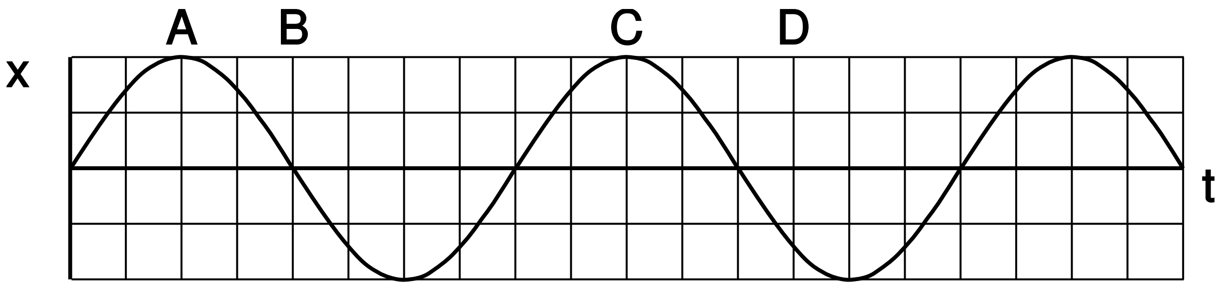
8 An oscillator has Q=6.00, and, for convenience, let's assume $F_m = 1.00$, $\omega_o = 1.00$, and $m = 1.00$. The usual approximations would give

$$\omega_{res} = \omega_o,$$
$$A_{res} = 6.00, \qquad \text{and}$$
$$\Delta\omega = 1/6.00.$$

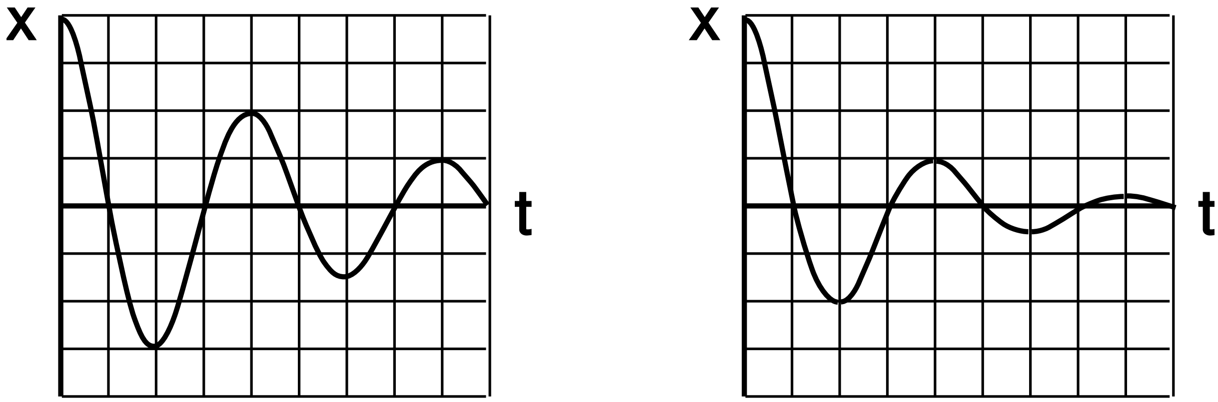
Determine these three quantities numerically using equation [4] on p. 529 , and compare with the approximations.

Exercise 17: Resonance

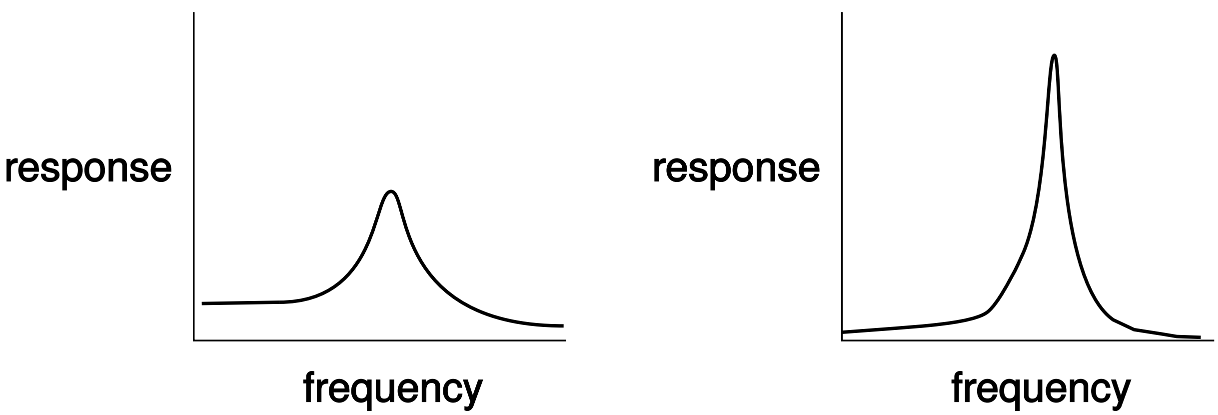
1. Compare the oscillator's energies at A, B, C, and D.



2. Compare the Q values of the two oscillators.



3. Match the x-t graphs in #2 with the amplitude-frequency graphs below.



Three essential mathematical skills

More often than not when a search-and-rescue team finds a hiker dead in the wilderness, it turns out that the person was not carrying some item from a short list of essentials, such as water and a map. There are three mathematical essentials in this course.

1. Converting units

basic technique: section 0.9, p. 28; conversion of area, volume, etc.: section 1.1, p. 39

Examples:

$$0.7\ \cancel{\text{kg}} \times \frac{10^3\ \text{g}}{1\ \cancel{\text{kg}}} = 700\ \text{g}.$$

To check that we have the conversion factor the right way up (10^3 rather then $1/10^3$), we note that the smaller unit of grams has been *compensated* for by making the number larger.

For units like m^2, kg/m^3, etc., we have to raise the conversion factor to the appropriate power:

$$4\ \text{m}^3 \times \left(\frac{10^3\ \text{mm}}{1\ \text{m}}\right)^3 = 4 \times 10^9\ \cancel{\text{m}^3} \times \frac{\text{mm}^3}{\cancel{\text{m}^3}} = 4 \times 10^9\ \text{mm}^3$$

Examples with solutions — p. 35, #1; p. 54, #2

Problems you can check at `lightandmatter.com/area1checker.html` — p. 35, #3; p. 35, #2; p. 35, #4; p. 54, #5; p. 54, #1

2. Reasoning about ratios and proportionalities

The technique is introduced in section 1.2, p. 41, in the context of area and volume, but it applies more generally to any relationship in which one variable depends on another raised to some power.

Example: When a car or truck travels over a road, there is wear and tear on the road surface, which incurs a cost. Studies show that the cost per kilometer of travel C is given by

$$C = kw^4,$$

where w is the weight per axle and k is a constant. The weight per axle is about 13 times higher for a semi-trailer than for my Honda Fit. How many times greater is the cost imposed on the federal government when the semi travels a given distance on an interstate freeway?

▷ First we convert the equation into a proportionality by throwing out k, which is the same for both vehicles:

$$C \propto w^4$$

Next we convert this proportionality to a statement about ratios:

$$\frac{C_1}{C_2} = \left(\frac{w_1}{w_2}\right)^4 \approx 29,000$$

Since the gas taxes paid by the trucker are nowhere near 29,000 times more than those I pay to drive my Fit the same distance, the federal government is effectively awarding a massive subsidy to the trucking company. Plus my Fit is cuter.

Examples with solutions — p. 54, #3; p. 54, #8; p. 54, #4; p. 122, #13; p. 122, #15; p. 270, #3; p. 296, #1; p. 297, #6; p. 296, #3; p. 332, #7; p. 332, #8

Problems you can check at `lightandmatter.com/area1checker.html` — p. 56, #13; p. 55, #9; p. 55, #10; p. 55, #11; p. 55, #12; p. 214, #4; p. 271, #6; p. 297, #7; p. 297, #8; p. 296, #4; p. 331, #3; p. 480, #10

3. Vector addition

section 7.3, p. 224

Example: The $\Delta\mathbf{r}$ vector from San Diego to Los Angeles has magnitude 190 km and direction 129°counterclockwise from east. The one from LA to Las Vegas is 370 km at 38°counterclockwise from east. Find the distance and direction from San Diego to Las Vegas.

▹ Graphical addition is discussed on p. 224. Here we concentrate on analytic addition, which involves adding the x components to find the total x component, and similarly for y. The trig needed in order to find the components of the second leg (LA to Vegas) is laid out in figure e on p. 221 and explained in detail in example 3 on p. 221:

$$\Delta x_2 = (370 \text{ km}) \cos 38^\circ = 292 \text{ km}$$
$$\Delta y_2 = (370 \text{ km}) \sin 38^\circ = 228 \text{ km}$$

(Since these are intermediate results, we keep an extra sig fig to avoid accumulating too much rounding error.) Once we understand the trig for one example, we don't need to reinvent the wheel every time. The pattern is completely universal, provided that we first make sure to get the angle expressed according to the usual trig convention, counterclockwise from the x axis. Applying the pattern to the first leg, we have:

$$\Delta x_1 = (190 \text{ km}) \cos 129^\circ = -120 \text{ km}$$
$$\Delta y_1 = (190 \text{ km}) \sin 129^\circ = 148 \text{ km}$$

For the vector directly from San Diego to Las Vegas, we have

$$\Delta x = \Delta x_1 + \Delta x_2 = 172 \text{ km}$$
$$\Delta y = \Delta y_1 + \Delta y_2 = 376 \text{ km}.$$

The distance from San Diego to Las Vegas is found using the Pythagorean theorem,

$$\sqrt{(172 \text{ km})^2 + (376 \text{ km})^2} = 410 \text{ km}$$

(rounded to two sig figs because it's one of our final results). The direction is one of the two possible values of the inverse tangent

$$\tan^{-1}(\Delta y/\Delta x) = \{65^\circ, 245^\circ\}.$$

Consulting a sketch shows that the first of these values is the correct one.

Examples with solutions — p. 247, #3; p. 249, #9; p. 421, #8

Problems you can check at `lightandmatter.com/area1checker.html` — p. 231, #3; p. 231, #4; p. 247, #4; p. 247, #5; p. 250, #16; p. 298, #13; p. 298, #14; p. 421, #9

Programming with python

The purpose of this tutorial is to help you get familiar with a computer programming language called Python, which I've chosen because (a) it's free, and (b) it's easy to use interactively. I won't assume you have any previous experience with computer programming; you won't need to learn very much Python, and what little you do need to learn I'll explain explicitly. If you really want to learn Python more thoroughly, there are a couple of excellent books that you can download for free on the Web:

> **How to Think Like a Computer Scientist (Python Version)**, Allen B. Downey, Jeffrey Elkner, Moshe Zadka, http://www.ibiblio.org/obp/
>
> **Dive Into Python**, Mark Pilgrim, http://diveintopython.net/

The first book is meant for people who have never programmed before, while the second is a more complete introduction aimed at veteran programmers who know a different language already.

Using Python as a calculator

The easiest way to get Python going is to go to the web site `ideone.com`. Under "choose a language," select Python. Inside the window where it says "paste your source code or insert template or sample," type `print(2+2)`. Click on the "submit" button. The result, 4, is shown under "output." In other words, you can use Python just like a calculator.

For compactness, I'll show examples in the following style:

```
>>> print(2+2)
4
```

Here the `>>>` is not something you would type yourself; it's just a marker to distinguish your input from the program's output. (In some other versions of Python, the computer will actually print out `>>>` as a prompt to tell you it's ready to type something.)

There are only a couple of things to watch out for. First, Python distinguishes between integers and real numbers, so the following gives an unexpected result:

```
>>>  print(2/3)
0
```

To get it to treat these values as real numbers, you have to use decimal points:

```
>>> print(2./3.)
0.66666666666666666663
```

Multiplication is represented by "*":

```
>>> print(2.*3.)
6.0
```

Also, Python doesn't know about its own library of math functions unless you tell it explicitly to load them in:

```
>>> print (sqrt(2.))
Traceback (most recent call last):
File ``<stdin>'', line 1, in ?
NameError: There is no variable named `sqrt'
```

Here are the steps you have to go through to calculate the square root of 2 successfully:

```
>>> import math
>>> print(math.sqrt(2.))
1.4142135623730951
```

The first line is just something you can make a habit of doing every time you start up Python. In the second line, the name of the square root function had to be prefixed with "`math.`" to tell Python where you wanted to get this "`sqrt`" function from. (All of this may seem like a nuisance if you're just using Python as a calculator, but it's a good way to design a programming language so that names of functions never conflict.)

Try it. Experiment and figure out whether Python's trig functions assume radians or degrees.

Variables

Python lets you define variables and assign values to them using an equals sign:

```
>>> dwarfs=7
>>> print(dwarfs)
>>> print(dwarfs+3)
7
10
```

Note that a variable in computer programming isn't quite like a variable in algebra. In algebra, if a=7 then a=7 always, throughout a particular calculation. But in a programming language, the variable name really represents a place in memory where a number can be stored, so you can change its value:

```
>>> dwarfs=7
>>> dwarfs=37
>>> print(dwarfs)
37
```

You can even do stuff like this,

```
>>> dwarfs=37
>>> dwarfs=dwarfs+1
>>> print(dwarfs)
38
```

In algebra it would be nonsense to have a variable equal to itself plus one, but in a computer program, it's not an assertion that the two things are equal, its a command to calculate the value of the expression on the right side of the equals, and then put that number into the memory location referred to by the variable name on the left.

Try it. What happens if you do dwarfs+1 = dwarfs? Do you understand why?

Functions

Somebody had to teach Python how to do functions like `sqrt`, and it's handy to be able to define your own functions in the same way. Here's how to do it:

```
>>> def double(x):
>>>   return 2.*x
>>> print(double(5.))
10.0
```

Note that the indentation is mandatory. The first and second lines define a function called `double`. The final line evaluates that function with an input of 5.

Loops

Suppose we want to add up all the numbers from 0 to 99.

Automating this kind of thing is exactly what computers are best at, and Python provides a mechanism for this called a loop:

```
>>> sum=0
>>> for j in range(100):
>>>   sum=sum+j
>>> print(sum)
4950
```

The stuff that gets repeated — the inside of the loop — has to be indented, just like in a function definition. Python always counts loops starting from 0, so for `j in range(100)` actually causes `j` to range from 0 to 99, not from 1 to 100.

Hints

Hints for chapter 8

Page 251, problem 18:
The easiest way to do this problem is to use two different coordinate systems: one that's tilted to coincide with the upper slope, and one that's tilted to coincide with the lower one.

Page 251, problem 19:
Consider a section of the rope subtending a very small angle, and find an approximate equation relating the normal force to the tension. Apply small-angle approximations to any trig functions occurring in your result. Eliminate all variables except for the tension and the angle, and separate these variables.

Hints for chapter 10

Page 300, problem 22:
If you try to calculate the two forces and subtract, your calculator will probably give a result of zero due to rounding. Instead, reason about the fractional amount by which the quantity $1/r^2$ will change. As a warm-up, you may wish to observe the percentage change in $1/r^2$ that results from changing r from 1 to 1.01.

Hints for chapter 13

Page 386, problem 13:
What does the total energy have to be if the projectile's velocity is exactly escape velocity? Write down conservation of energy, change v to $\mathrm{d}r/\,\mathrm{d}t$, separate the variables, and integrate.

Page 388, problem 20:
You can use the geometric interpretation of the dot product.

Page 389, problem 25:
The analytic approach is a little cumbersome, although it can be done by using approximations like $1/\sqrt{1+\epsilon} \approx 1-(1/2)\epsilon$. A more straightforward, brute-force method is simply to write a computer program that calculates U/m for a given point in spherical coordinates. By trial and error, you can fairly rapidly find the r that gives a desired value of U/m.

Hints for chapter 15

Page 483, problem 28:
The choice of axis theorem only applies to a closed system, or to a system acted on by a total force of zero. Even if the box is not going to rotate, its center of mass is going to accelerate, and this can still cause a change in its angular momentum, unless the right axis is chosen. For example, if the axis is chosen at the bottom right corner, then the box will start accumulating clockwise angular momentum, even if it is just accelerating to the right without rotating. Only by choosing the axis at the center of mass (or at some other point on the same horizontal line) do we get a constant, zero angular momentum.

Page 485, problem 41:
You'll need the result of problem 26 in order to relate the energy and angular momentum of a rigidly rotating body. Since this relationship involves a variable raised to a power, you can't just graph the data and get the moment of inertia directly. One way to get around this is to manipulate one of the variables to make the graph linear. Here is an example of this technique from another context. Suppose you were given a table of the masses, m, of cubical pieces of

wood, whose sides had various lengths, b. You want to find a best-fit value for the density of the wood. The relationship is $m = \rho b^3$. The graph of m versus b would be a curve, and you would not have any easy way to get the density from such a graph. But by graphing m versus b^3, you can produce a graph that is linear, and whose slope equals the density.

Hints for chapter 16

Page 509, problem 13:

The spring constant of this spring, k, is *not* the quantity you need in the equation for the period. What you need in that equation is the second derivative of the spring's energy with respect to the position of the thing that's oscillating. You need to start by finding the energy stored in the spring as a function of the vertical position, y, of the mass. This is similar to example 5 on page 502.

Hints for chapter 17

Page 535, problem 6:

The whole expression for the amplitude has maxima where the stuff inside the square root is at a minimum, and vice versa, so you can save yourself a lot of work by just working on the stuff inside the square root. For normal, large values of Q, the there are two extrema, one at $\omega = 0$ and one at resonance; one of these is a maximum and one is a minimum. You want to find out at what value of Q the zero-frequency extremum switches over from being a maximum to being a minimum.

Solutions to selected problems

Solutions for chapter 0

Page 35, problem 1:

$$134 \text{ mg} \times \frac{10^{-3} \text{ g}}{1 \text{ mg}} \times \frac{10^{-3} \text{ kg}}{1 \text{ g}} = 1.34 \times 10^{-4} \text{ kg}$$

Page 35, problem 7:

(a) Let's do 10.0 g and 1000 g. The arithmetic mean is 505 grams. It comes out to be 0.505 kg, which is consistent. (b) The geometric mean comes out to be 100 g or 0.1 kg, which is consistent. (c) If we multiply meters by meters, we get square meters. Multiplying grams by grams should give square grams! This sounds strange, but it makes sense. Taking the square root of square grams (g^2) gives grams again. (d) No. The superduper mean of two quantities with units of grams wouldn't even be something with units of grams! Related to this shortcoming is the fact that the superduper mean would fail the kind of consistency test carried out in the first two parts of the problem.

Page 36, problem 10:

(a) They're all defined in terms of the ratio of side of a triangle to another. For instance, the tangent is the length of the opposite side over the length of the adjacent side. Dividing meters by meters gives a unitless result, so the tangent, as well as the other trig functions, is unitless. (b) The tangent function gives a unitless result, so the units on the right-hand side had better cancel out. They do, because the top of the fraction has units of meters squared, and so does the bottom.

Solutions for chapter 1

Page 54, problem 1:

The proportionality $V \propto L^3$ can be restated in terms of ratios as $V_1/V_2 = (L_1/L_2)^3 = (1/10)^3 = 1/1000$, so scaling down the linear dimensions by a factor of 1/10 reduces the volume by 1/1000, to a milliliter.

Page 54, problem 2:

$$1\ \text{mm}^2 \times \left(\frac{1\ \text{cm}}{10\ \text{mm}}\right)^2 = 10^{-2}\ \text{cm}^2$$

Page 54, problem 3:

The bigger scope has a diameter that's ten times greater. Area scales as the square of the linear dimensions, so $A \propto d^2$, or in the language of ratios $A_1/A_2 = (d_1/d_2)^2 = 100$. Its light-gathering power is a hundred times greater.

Page 54, problem 4:

The cone of mixed gin and vermouth is the same shape as the cone of vermouth, but its linear dimensions are doubled. Translating the proportionality $V \propto L^3$ into an equation about ratios, we have $V_1/V_2 = (L_1/L_2)^3 = 8$. Since the ratio of the whole thing to the vermouth is 8, the ratio of gin to vermouth is 7.

Page 54, problem 8:

Since they differ by two steps on the Richter scale, the energy of the bigger quake is 10^4 times greater. The wave forms a hemisphere, and the surface area of the hemisphere over which the energy is spread is proportional to the square of its radius, $A \propto r^2$, or $r \propto \sqrt{A}$, which means $r_1/r_2 = \sqrt{A_1/A_2}$. If the amount of vibration was the same, then the surface areas must be in the ratio $A_1/A_2 = 10^4$, which means that the ratio of the radii is 10^2.

Page 57, problem 22:

Let's estimate the Great Wall's volume, and then figure out how many bricks that would represent. The wall is famous because it covers pretty much all of China's northern border, so let's say it's 1000 km long. From pictures, it looks like it's about 10 m high and 10 m wide, so the total volume would be $10^6\ \text{m} \times 10\ \text{m} \times 10\ \text{m} = 10^8\ \text{m}^3$. If a single brick has a volume of 1 liter, or $10^{-3}\ \text{m}^3$, then this represents about 10^{11} bricks. If one person can lay 10 bricks in an hour (taking into account all the preparation, etc.), then this would be 10^{10} man-hours.

Page 57, problem 24:

Directly guessing the number of jelly beans would be like guessing volume directly. That would be a mistake. Instead, we start by estimating the linear dimensions, in units of beans. The contents of the jar look like they're about 10 beans deep. Although the jar is a cylinder, its exact geometrical shape doesn't really matter for the purposes of our order-of-magnitude estimate. Let's pretend it's a rectangular jar. The horizontal dimensions are also something like 10 beans, so it looks like the jar has about $10 \times 10 \times 10$ or $\sim 10^3$ beans inside.

Solutions for chapter 2

Page 97, problem 1:

Since the lines are at intervals of one m/s and one second, each box represents one meter. From $t = 0$ to $t = 2$ s, the area under the curve represents a positive Δx of 6 m. (The triangle has half the area of the 2×6 rectangle it fits inside.) After $t = 2$ s, the area above the curve represents negative Δx. To get -6 m worth of area, we need to go out to $t = 6$ s, at which point the triangle under the axis has a width of 4 s and a height of 3 m/s, for an area of 6 m (half of 3×4).

Page 98, problem 8:
(a) Let f and g be functions. Then the chain rule states that if we construct the function $f(g(x))$, its derivative is

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

On the right-hand side, the units of dg on the top cancel with the units of dg on the bottom, so the units do match up with those of df/dx on the left.
(b) The cosine function requires a unitless input and produces a unitless output. Therefore A must have units of meters, and b must have units of s^{-1} (inverse seconds, or "per second"). A is the distance the object moves on either side of the origin, and b is a measure of how fast it vibrates back and forth (how many radians it passes through per second).
(b) The derivative is $v = dx/dt = -Ab\sin(bt)$, where the factor of b in front comes from the chain rule. The product Ab does have units of m/s. If we hadn't put in the factor of b as required by the chain rule, the units would have been wrong. Physically, it also makes sense that a larger b, indicating a more rapid vibration, produces a greater v.

Page 98, problem 10:
In one second, the ship moves v meters to the east, and the person moves v meters north relative to the deck. Relative to the water, he traces the diagonal of a triangle whose length is given by the Pythagorean theorem, $(v^2 + v^2)^{1/2} = \sqrt{2}v$. Relative to the water, he is moving at a 45-degree angle between north and east.

Page 98, problem 11:
Velocity is relative, so having to lean tells you nothing about the current value of the train's velocity. Fullerton is moving at a huge speed relative to Beijing, but that doesn't produce any noticeable effect in either city. The fact that you have to lean tells you that the train is *changing* its speed, but it doesn't tell you what the train's current speed is.

Page 98, problem 13:
To the person riding the moving bike, bug A is simply going in circles. The only difference between the motions of the two wheels is that one is traveling through space, but motion is relative, so this doesn't have any effect on the bugs. It's equally hard for each of them.

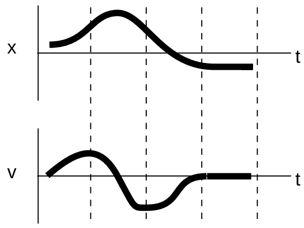
Solutions for chapter 3

Page 121, problem 1:
Taking g to be 10 m/s^2, the bullet loses 10 m/s of speed every second, so it will take 10 s to come to a stop, and then another 10 s to come back down, for a total of 20 s.

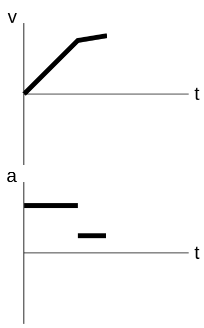
Page 121, problem 4:

$$\begin{aligned} v &= \frac{dx}{dt} \\ &= 10 - 3t^2 \\ a &= \frac{dv}{dt} \\ &= -6t \\ &= -18\ \text{m/s}^2 \end{aligned}$$

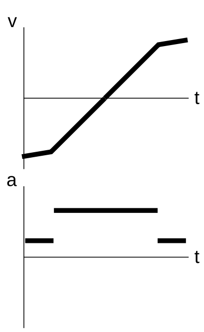
Page 121, problem 6:



Page 121, problem 7:
(a) We choose a coordinate system with positive pointing to the right. Some people might expect that the ball would slow down once it was on the more gentle ramp. This may be true if there is significant friction, but Galileo's experiments with inclined planes showed that when friction is negligible, a ball rolling on a ramp has constant acceleration, not constant speed. The speed stops increasing as quickly once the ball is on the more gentle slope, but it still keeps on increasing. The a-t graph can be drawn by inspecting the slope of the v-t graph.

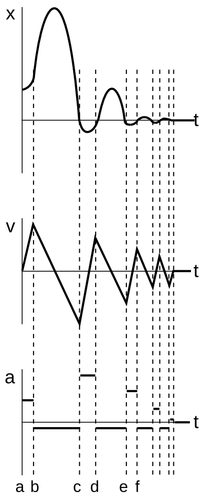


(b) The ball will roll back down, so the second half of the motion is the same as in part a. In the first (rising) half of the motion, the velocity is negative, since the motion is in the opposite direction compared to the positive x axis. The acceleration is again found by inspecting the slope of the v-t graph.



Page 121, problem 8:
This is a case where it's probably easiest to draw the acceleration graph first. While the ball is in the air (bc, de, etc.), the only force acting on it is gravity, so it must have the same, constant acceleration during each hop. Choosing a coordinate system where the positive x axis points up, this becomes a negative acceleration (force in the opposite direction compared to the axis). During the short times between hops when the ball is in contact with the ground (cd, ef, etc.), it experiences a large acceleration, which turns around its velocity very rapidly. These short positive accelerations probably aren't constant, but it's hard to know how they'd really look. We just idealize them as constant accelerations. Similarly, the hand's force on the ball

during the time ab is probably not constant, but we can draw it that way, since we don't know how to draw it more realistically. Since our acceleration graph consists of constant-acceleration segments, the velocity graph must consist of line segments, and the position graph must consist of parabolas. On the x graph, I chose zero to be the height of the center of the ball above the floor when the ball is just lying on the floor. When the ball is touching the floor and compressed, as in interval cd, its center is below this level, so its x is negative.



Page 122, problem 11:
(a) Solving for $\Delta x = \frac{1}{2}at^2$ for a, we find $a = 2\Delta x/t^2 = 5.51$ m/s^2. (b) $v = \sqrt{2a\Delta x} = 66.6$ m/s. (c) The actual car's final velocity is less than that of the idealized constant-acceleration car. If the real car and the idealized car covered the quarter mile in the same time but the real car was moving more slowly at the end than the idealized one, the real car must have been going faster than the idealized car at the beginning of the race. The real car apparently has a greater acceleration at the beginning, and less acceleration at the end. This make sense, because every car has some maximum speed, which is the speed beyond which it cannot accelerate.

Page 122, problem 13:
$\Delta x = \frac{1}{2}at^2$, so for a fixed value of Δx, we have $t \propto 1/\sqrt{a}$. Translating this into the language of ratios gives $t_M/t_E = \sqrt{a_E/a_M} = \sqrt{3} = 1.7$.

Page 122, problem 15:
We have $v_f^2 = 2a\Delta x$, so the distance is proportional to the square of the velocity. To get up to half the speed, the ball needs 1/4 the distance, i.e., $L/4$.

Page 127, problem 32:
(a) Other than w, the only thing with units that can occur in our answer is g. If we want to combine a distance and an acceleration to produce a time, the only way to do so is like $\sqrt{w/g}$, possibly multiplied by a unitless constant.
(b) It is convenient to introduce the notations L for the length of one side of the vee and h for the height, so that $L^2 = w^2 + h^2$. The acceleration is $a = g\sin\theta = gh/L$. To travel a distance L with this acceleration takes time

$$t = \sqrt{2L/a} = \sqrt{\left(\frac{2w}{g}\right)\left(\frac{h}{w}+\frac{w}{h}\right)}.$$

Let $x = h/w$. For a fixed value of w, this time is an increasing function of $x + 1/x$, so we want the value of x that minimizes this expression. Taking the derivative and setting it equal to zero

gives $x = 1$, or $h = w$. In other words, the time is minimized if the angle is 45°.
(c) Plugging $x = 1$ back in, we have $t^* = 2t = 4\sqrt{w/g}$, so the unitless factor was 4.

Solutions for chapter 4

Page 159, problem 1:

$a = \Delta v/\Delta t$, and also $a = F/m$, so

$$\begin{aligned}\Delta t &= \frac{\Delta v}{a} \\ &= \frac{m\Delta v}{F} \\ &= \frac{(1000\ \text{kg})(50\ \text{m/s} - 20\ \text{m/s})}{3000\ \text{N}} \\ &= 10\ \text{s}\end{aligned}$$

Page 159, problem 4:

(a) This is a measure of the box's resistance to a change in its state of motion, so it measures the box's mass. The experiment would come out the same in lunar gravity.
(b) This is a measure of how much gravitational force it feels, so it's a measure of weight. In lunar gravity, the box would make a softer sound when it hit.
(c) As in part a, this is a measure of its resistance to a change in its state of motion: its mass. Gravity isn't involved at all.

Solutions for chapter 5

Page 195, problem 1:

(a) The swimmer's acceleration is caused by the water's force on the swimmer, and the swimmer makes a backward force on the water, which accelerates the water backward. (b) The club's normal force on the ball accelerates the ball, and the ball makes a backward normal force on the club, which decelerates the club. (c) The bowstring's normal force accelerates the arrow, and the arrow also makes a backward normal force on the string. This force on the string causes the string to accelerate less rapidly than it would if the bow's force was the only one acting on it.
(d) The tracks' backward frictional force slows the locomotive down. The locomotive's forward frictional force causes the whole planet earth to accelerate by a tiny amount, which is too small to measure because the earth's mass is so great.

Page 195, problem 2:

The person's normal force on the box is paired with the box's normal force on the person. The dirt's frictional force on the box pairs with the box's frictional force on the dirt. The earth's gravitational force on the box matches the box's gravitational force on the earth.

Page 195, problem 3:

(a) A liter of water has a mass of 1.0 kg. The mass is the same in all three locations. Mass indicates how much an object resists a change in its motion. It has nothing to do with gravity.
(b) The term "weight" refers to the force of gravity on an object. The bottle's weight on earth is $F_W = mg = 9.8$ N. Its weight on the moon is about one sixth that value, and its weight in interstellar space is zero.

Page 200, problem 26:

(a)

top spring's rightward force on connector
...connector's leftward force on top spring
bottom spring's rightward force on connector
...connector's leftward force on bottom spring
hand's leftward force on connector
...connector's rightward force on hand

Looking at the three forces on the connector, we see that the hand's force must be double the force of either spring. The value of $x - x_o$ is the same for both springs and for the arrangement as a whole, so the spring constant must be $2k$. This corresponds to a stiffer spring (more force to produce the same extension).

(b) Forces in which the left spring participates:

hand's leftward force on left spring
...left spring's rightward force on hand
right spring's rightward force on left spring
...left spring's leftward force on right spring

Forces in which the right spring participates:

left spring's leftward force on right spring
...right spring's rightward force on left spring
wall's rightward force on right spring
...right spring's leftward force on wall

Since the left spring isn't accelerating, the total force on it must be zero, so the two forces acting on it must be equal in magnitude. The same applies to the two forces acting on the right spring. The forces between the two springs are connected by Newton's third law, so all eight of these forces must be equal in magnitude. Since the value of $x - x_o$ for the whole setup is double what it is for either spring individually, the spring constant of the whole setup must be $k/2$, which corresponds to a less stiff spring.

Page 200, problem 28:
(a) Spring constants in parallel add, so the spring constant has to be proportional to the cross-sectional area. Two springs in series give half the spring constant, three springs in series give 1/3, and so on, so the spring constant has to be inversely proportional to the length. Summarizing, we have $k \propto A/L$. (b) With the Young's modulus, we have $k = (A/L)E$.The spring constant has units of N/m, so the units of E would have to be N/m^2.

Solutions for chapter 7

Page 232, problem 7:
We'll use the same approach as in the example in section 7.5, which is to find an example such that when the calculation is carried out in a rotated frame of reference, the result is clearly not the same vector expressed in the new frame. Let $\mathbf{A} = \pi\hat{\mathbf{x}}$ in the original coordinate system. Then in this coordinate system $\mathbf{B} = 0$.

But now suppose we choose a new coordinate system, rotated by 10 degrees relative to the first one. In this new coordinate system, A_x is a little less than π. Since A_x is no longer a multiple of π, B_x is no longer zero, and $\mathbf{B}$ is no longer zero. The nonzero $\mathbf{B}$ computed in the new coordinate system is clearly not the same as the old $\mathbf{B}$ expressed in a new way, since rotating our coordinate system should not change the magnitudes of vectors.

Solutions for chapter 8

Page 247, problem 3:
We want to find out about the velocity vector v_{BG} of the bullet relative to the ground, so we need to add Annie's velocity relative to the ground v_{AG} to the bullet's velocity vector v_{BA} relative to her. Letting the positive x axis be east and y north, we have

$$\begin{aligned} v_{BA,x} &= (140 \text{ mi/hr}) \cos 45° \\ &= 100 \text{ mi/hr} \\ v_{BA,y} &= (140 \text{ mi/hr}) \sin 45° \\ &= 100 \text{ mi/hr} \end{aligned}$$

and

$$\begin{aligned} v_{AG,x} &= 0 \\ v_{AG,y} &= 30 \text{ mi/hr}. \end{aligned}$$

The bullet's velocity relative to the ground therefore has components

$$\begin{aligned} v_{BG,x} &= 100 \text{ mi/hr} \qquad \text{and} \\ v_{BG,y} &= 130 \text{ mi/hr}. \end{aligned}$$

Its speed on impact with the animal is the magnitude of this vector

$$\begin{aligned} |v_{BG}| &= \sqrt{(100 \text{ mi/hr})^2 + (130 \text{ mi/hr})^2} \\ &= 160 \text{ mi/hr} \end{aligned}$$

(rounded off to 2 significant figures).

Page 249, problem 9:
Since its velocity vector is constant, it has zero acceleration, and the sum of the force vectors acting on it must be zero. There are three forces acting on the plane: thrust, lift, and gravity. We are given the first two, and if we can find the third we can infer its mass. The sum of the y components of the forces is zero, so

$$\begin{aligned} 0 &= F_{thrust,y} + F_{lift,y} + F_{W,y} \\ &= |\mathbf{F}_{thrust}| \sin\theta + |\mathbf{F}_{lift}| \cos\theta - mg. \end{aligned}$$

The mass is

$$\begin{aligned} m &= (|\mathbf{F}_{thrust}| \sin\theta + |\mathbf{F}_{lift}| \cos\theta)/g \\ &= 7.0 \times 10^4 \text{ kg} \end{aligned}$$

Page 250, problem 13:
(a) If there was no friction, the angle of repose would be zero, so the coefficient of static friction, μ_s, will definitely matter. We also make up symbols θ, m and g for the angle of the slope, the mass of the object, and the acceleration of gravity. The forces form a triangle just like the one in example 6 on p. 240, but instead of a force applied by an external object, we have static friction, which is less than $\mu_s|\mathbf{F}_N|$. As in that example, $|\mathbf{F}_s| = mg\sin\theta$, and $|\mathbf{F}_s| < \mu_s|\mathbf{F}_N|$, so

$$mg\sin\theta < \mu_s|\mathbf{F}_N|.$$

From the same triangle, we have $|\mathbf{F}_N| = mg\cos\theta$, so

$$mg\sin\theta < \mu_s mg\cos\theta.$$

Rearranging,

$$\theta < \tan^{-1}\mu_s.$$

(b) Both m and g canceled out, so the angle of repose would be the same on an asteroid.

Page 250, problem 14:

(a) Since the wagon has no acceleration, the total forces in both the x and y directions must be zero. There are three forces acting on the wagon: $\mathbf{F}_T$, $\mathbf{F}_W$, and the normal force from the ground, $\mathbf{F}_N$. If we pick a coordinate system with x being horizontal and y vertical, then the angles of these forces measured counterclockwise from the x axis are $90° - \phi$, $270°$, and $90° + \theta$, respectively. We have

$$\begin{aligned} F_{x,total} &= |\mathbf{F}_T|\cos(90° - \phi) + |\mathbf{F}_W|\cos(270°) + |\mathbf{F}_N|\cos(90° + \theta) \\ F_{y,total} &= |\mathbf{F}_T|\sin(90° - \phi) + |\mathbf{F}_W|\sin(270°) + |\mathbf{F}_N|\sin(90° + \theta), \end{aligned}$$

which simplifies to

$$\begin{aligned} 0 &= |\mathbf{F}_T|\sin\phi - |\mathbf{F}_N|\sin\theta \\ 0 &= |\mathbf{F}_T|\cos\phi - |\mathbf{F}_W| + |\mathbf{F}_N|\cos\theta. \end{aligned}$$

The normal force is a quantity that we are not given and do not wish to find, so we should choose it to eliminate. Solving the first equation for $|\mathbf{F}_N| = (\sin\phi/\sin\theta)|\mathbf{F}_T|$, we eliminate $|\mathbf{F}_N|$ from the second equation,

$$0 = |\mathbf{F}_T|\cos\phi - |\mathbf{F}_W| + |\mathbf{F}_T|\sin\phi\cos\theta/\sin\theta$$

and solve for $|\mathbf{F}_T|$, finding

$$|\mathbf{F}_T| = \frac{|\mathbf{F}_W|}{\cos\phi + \sin\phi\cos\theta/\sin\theta}.$$

Multiplying both the top and the bottom of the fraction by $\sin\theta$, and using the trig identity for $\sin(\theta + \phi)$ gives the desired result,

$$|\mathbf{F}_T| = \frac{\sin\theta}{\sin(\theta + \phi)}|\mathbf{F}_W|.$$

(b) The case of $\phi = 0$, i.e., pulling straight up on the wagon, results in $|\mathbf{F}_T| = |\mathbf{F}_W|$: we simply support the wagon and it glides up the slope like a chair-lift on a ski slope. In the case of $\phi = 180° - \theta$, $|\mathbf{F}_T|$ becomes infinite. Physically this is because we are pulling directly into the ground, so no amount of force will suffice.

Solutions for chapter 9

Page 270, problem 3:

(a) The inward normal force must be sufficient to produce circular motion, so

$$|\mathbf{F}_N| = mv^2/r.$$

We are searching for the minimum speed, which is the speed at which the static friction force is just barely able to cancel out the downward gravitational force. The maximum force of static friction is

$$|\mathbf{F}_s| = \mu_s|\mathbf{F}_N|,$$

and this cancels the gravitational force, so

$$|\mathbf{F}_s| = mg.$$

Solving these three equations for v gives

$$v = \sqrt{\frac{gr}{\mu_s}}.$$

(b) Greater by a factor of $\sqrt{3}$.

Page 271, problem 5:
The inward force must be supplied by the inward component of the normal force,

$$|\mathbf{F}_N| \sin\theta = mv^2/r.$$

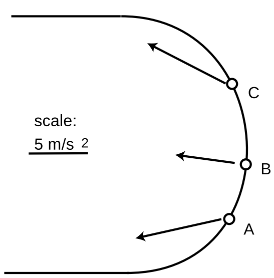
The upward component of the normal force must cancel the downward force of gravity,

$$|\mathbf{F}_N| \cos\theta = mg.$$

Eliminating $|\mathbf{F}_N|$ and solving for θ, we find

$$\theta = \tan^{-1}\left(\frac{v^2}{gr}\right).$$

Page 272, problem 10:
Each cyclist has a radial acceleration of $v^2/r = 5$ m/s^2. The tangential accelerations of cyclists A and B are 375 N/75 kg = 5 m/s^2.



Solutions for chapter 10

Page 296, problem 1:
Newton's law of gravity is $F = GMm/r^2$. Both G and the astronaut's mass m are the same in the two situations, so $F \propto Mr^{-2}$. In terms of ratios, this is

$$\frac{F_c}{F_e} = \frac{M_c}{M_e}\left(\frac{r_c}{r_e}\right)^{-2}.$$

The result is 11 N.

Page 296, problem 3:
(a) The asteroid's mass depends on the cube of its radius, and for a given mass the surface gravity depends on r^{-2}. The result is that surface gravity is directly proportional to radius. Half the gravity means half the radius, or one eighth the mass. (b) To agree with a, Earth's mass would have to be 1/8 Jupiter's. We assumed spherical shapes and equal density. Both

planets are at least roughly spherical, so the only way out of the contradiction is if Jupiter's density is significantly less than Earth's.

Page 297, problem 6:
Newton's law of gravity depends on the inverse square of the distance, so if the two planets' masses had been equal, then the factor of $0.83/0.059 = 14$ in distance would have caused the force on planet c to be $14^2 = 2.0 \times 10^2$ times weaker. However, planet c's mass is 3.0 times greater, so the force on it is only smaller by a factor of $2.0 \times 10^2/3.0 = 65$.

Page 300, problem 20:
Newton's law of gravity says $F = Gm_1m_2/r^2$, and Newton's second law says $F = m_2a$, so $Gm_1m_2/r^2 = m_2a$. Since m_2 cancels, a is independent of m_2.

Page 300, problem 21:
Newton's second law gives

$$F = m_Da_D,$$

where F is Ida's force on Dactyl. Using Newton's universal law of gravity, F= Gm_Im_D/r^2,and the equation $a = v^2/r$ for circular motion, we find

$$Gm_Im_D/r^2 = m_Dv^2/r.$$

Dactyl's mass cancels out, giving

$$Gm_I/r^2 = v^2/r.$$

Dactyl's velocity equals the circumference of its orbit divided by the time for one orbit: $v = 2\pi r/T$. Inserting this in the above equation and solving for m_I, we find

$$m_I = \frac{4\pi^2r^3}{GT^2},$$

so Ida's density is

$$\begin{aligned}\rho &= m_I/V \\ &= \frac{4\pi^2r^3}{GVT^2}.\end{aligned}$$

Page 300, problem 22:
Any fractional change in r results in double that amount of fractional change in $1/r^2$. For example, raising r by 1% causes $1/r^2$ to go down by very nearly 2%. A 27-day orbit is 1/13.5 of a year, so the fractional change in $1/r^2$ is

$$2 \times \frac{(4/13.5)\text{ cm}}{3.84 \times 10^5\text{ km}} \times \frac{1\text{ km}}{10^5\text{ cm}} = 1.5 \times 10^{-11}$$

Solutions for chapter 11

Page 332, problem 6:
A force is an interaction between two objects, so while the bullet is in the air, there is no force. There is only a force while the bullet is in contact with the book. There is energy the whole time, and the total amount doesn't change. The bullet has some kinetic energy, and transfers some of it to the book as heat, sound, and the energy required to tear a hole through the book.

Page 332, problem 7:
(a) The energy stored in the gasoline is being changed into heat via frictional heating, and also

probably into sound and into energy of water waves. Note that the kinetic energy of the propeller and the boat are not changing, so they are not involved in the energy transformation. (b) The crusing speed would be greater by a factor of the cube root of 2, or about a 26% increase.

Page 332, problem 8:
We don't have actual masses and velocities to plug in to the equation, but that's OK. We just have to reason in terms of ratios and proportionalities. Kinetic energy is proportional to mass and to the square of velocity, so B's kinetic energy equals

$$(13.4\ \text{J})(3.77)/(2.34)^2 = 9.23\ \text{J}$$

Page 332, problem 11:
Room temperature is about 20°C. The fraction of the energy that actually goes into heating the water is

$$\frac{(250\ \text{g})/(0.24\ \text{g}\cdot{}^\circ\text{C/J}) \times (100^\circ\text{C} - 20^\circ\text{C})}{(1.25 \times 10^3\ \text{J/s})\,(126\ \text{s})} = 0.53$$

So roughly half of the energy is wasted. The wasted energy might be in several forms: heating of the cup, heating of the oven itself, or leakage of microwaves from the oven.

Solutions for chapter 12

Page 353, problem 6:

$$\begin{aligned}
E_{total,i} &= E_{total,f} \\
PE_i + \text{heat}_i &= PE_f + KE_f + \text{heat}_f \\
\frac{1}{2}mv^2 &= PE_i - PE_f + \text{heat}_i - \text{heat}_f \\
&= -\Delta PE - \Delta\text{heat} \\
v &= \sqrt{2\left(\frac{-\Delta PE - \Delta\text{heat}}{m}\right)} \\
&= 6.4\ \text{m/s}
\end{aligned}$$

Page 354, problem 10:
(a) Example: As one child goes up on one side of a see-saw, another child on the other side comes down. (b) Example: A pool ball hits another pool ball, and transfers some KE.

Page 354, problem 12:
Suppose the river is 1 m deep, 100 m wide, and flows at a speed of 10 m/s, and that the falls are 100 m tall. In 1 second, the volume of water flowing over the falls is 10^3 m^3, with a mass of 10^6 kg. The potential energy released in one second is $(10^6\ \text{kg})(g)(100\ \text{m}) = 10^9$ J, so the power is 10^9 W. A typical household might have 10 hundred-watt applicances turned on at any given time, so it consumes about 10^3 watts on the average. The plant could supply a about million households with electricity.

Page 355, problem 16:

Let θ be the angle by which he has progressed around the pipe. Conservation of energy gives

$$\begin{aligned} E_{total,i} &= E_{total,f} \\ PE_i &= PE_f + KE_f \\ 0 &= \Delta PE + KE_f \\ 0 &= mgr(\cos\theta - 1) + \frac{1}{2}mv^2. \end{aligned}$$

While he is still in contact with the pipe, the radial component of his acceleration is

$$a_r = \frac{v^2}{r},$$

and making use of the previous equation we find

$$a_r = 2g(1 - \cos\theta).$$

There are two forces on him, a normal force from the pipe and a downward gravitational force from the earth. At the moment when he loses contact with the pipe, the normal force is zero, so the radial component, $mg\cos\theta$, of the gravitational force must equal ma_r,

$$mg\cos\theta = 2mg(1 - \cos\theta),$$

which gives

$$\cos\theta = \frac{2}{3}.$$

The amount by which he has dropped is $r(1 - \cos\theta)$, which equals $r/3$ at this moment.

Solutions for chapter 13

Page 383, problem 4:

No. Work describes how energy was transferred by some process. It isn't a measurable property of a system.

Solutions for chapter 14

Page 420, problem 3:

By conservation of momentum, the total momenta of the pieces after the explosion is the same as the momentum of the firework before the explosion. However, there is no law of conservation of kinetic energy, only a law of conservation of energy. The chemical energy in the gunpowder is converted into heat and kinetic energy when it explodes. All we can say about the kinetic energy of the pieces is that their total is greater than the kinetic energy before the explosion.

Page 421, problem 8:

Let m be the mass of the little puck and $M = 2.3m$ be the mass of the big one. All we need to do is find the direction of the total momentum vector before the collision, because the total momentum vector is the same after the collision. Given the two components of the momentum vector $p_x = Mv$ and $p_y = mv$, the direction of the vector is $\tan^{-1}(p_y/p_x) = 23°$ counterclockwise from the big puck's original direction of motion.

Page 422, problem 12:

Momentum is a vector. The total momentum of the molecules is always zero, since the momenta in different directions cancal out on the average. Cooling changes individual molecular momenta, but not the total.

Page 422, problem 15:
(a) Particle i had velocity v_i in the center-of-mass frame, and has velocity $v_i + u$ in the new frame. The total kinetic energy is

$$\frac{1}{2} m_1 \left(\mathbf{v}_1 + \mathbf{u}\right)^2 + \ldots,$$

where "…" indicates that the sum continues for all the particles. Rewriting this in terms of the vector dot product, we have

$$\frac{1}{2} m_1 \left(\mathbf{v}_1 + \mathbf{u}\right) \cdot \left(\mathbf{v}_1 + \mathbf{u}\right) + \ldots = \frac{1}{2} m_1 \left(\mathbf{v}_1 \cdot \mathbf{v}_1 + 2\mathbf{u} \cdot \mathbf{v}_1 + \mathbf{u} \cdot \mathbf{u}\right) + \ldots.$$

When we add up all the terms like the first one, we get K_{cm}. Adding up all the terms like the third one, we get $M|\mathbf{u}|^2/2$. The terms like the second term cancel out:

$$m_1 \mathbf{u} \cdot \mathbf{v}_1 + \ldots = \mathbf{u} \cdot \left(m_1 \mathbf{v}_1 + \ldots\right),$$

where the sum in brackets equals the total momentum in the center-of-mass frame, which is zero by definition.
(b) Changing frames of reference doesn't change the distances between the particles, so the potential energies are all unaffected by the change of frames of reference. Suppose that in a given frame of reference, frame 1, energy is conserved in some process: the initial and final energies add up to be the same. First let's transform to the center-of-mass frame. The potential energies are unaffected by the transformation, and the total kinetic energy is simply reduced by the quantity $M|\mathbf{u}_1|^2/2$, where $\mathbf{u}_1$ is the velocity of frame 1 relative to the center of mass. Subtracting the same constant from the initial and final energies still leaves them equal. Now we transform to frame 2. Again, the effect is simply to change the initial and final energies by adding the same constant.

Page 422, problem 16:
A conservation law is about addition: it says that when you add up a certain thing, the total always stays the same. Funkosity would violate the additive nature of conservation laws, because a two-kilogram mass would have twice as much funkosity as a pair of one-kilogram masses moving at the same speed.

Solutions for chapter 15

Page 480, problem 8:
The pliers are not moving, so their angular momentum remains constant at zero, and the total torque on them must be zero. Not only that, but each half of the pliers must have zero total torque on it. This tells us that the magnitude of the torque at one end must be the same as that at the other end. The distance from the axis to the nut is about 2.5 cm, and the distance from the axis to the centers of the palm and fingers are about 8 cm. The angles are close enough to 90° that we can pretend they're 90 degrees, considering the rough nature of the other assumptions and measurements. The result is (300 N)(2.5 cm) = (F)(8 cm), or $F = 90$ N.

Page 486, problem 46:
The foot of the rod is moving in a circle relative to the center of the rod, with speed $v = \pi b/T$, and acceleration $v^2/(b/2) = (\pi^2/8)g$. This acceleration is initially upward, and is greater in magnitude than g, so the foot of the rod will lift off without dragging. We could also worry about whether the foot of the rod would make contact with the floor again before the rod finishes up flat on its back. This is a question that can be settled by graphing, or simply by

inspection of figure aj on page 459. The key here is that the two parts of the acceleration are both independent of m and b, so the result is univeral, and it does suffice to check a graph in a single example. In practical terms, this tells us something about how difficult the trick is to do. Because $\pi^2/8 = 1.23$ isn't much greater than unity, a hit that is just a little too weak (by a factor of $1.23^{1/2} = 1.11$) will cause a fairly obvious qualitative change in the results. This is easily observed if you try it a few times with a pencil.

Answers to self-checks

Answers to self-checks for chapter 0

Page 15, self-check A:
If only he has the special powers, then his results can never be reproduced.

Page 17, self-check B:
They would have had to weigh the rays, or check for a loss of weight in the object from which they were have emitted. (For technical reasons, this was not a measurement they could actually do, hence the opportunity for disagreement.)

Page 23, self-check C:
A dictionary might define "strong" as "possessing powerful muscles," but that's not an operational definition, because it doesn't say how to measure strength numerically. One possible operational definition would be the number of pounds a person can bench press.

Page 27, self-check D:
A microsecond is 1000 times longer than a nanosecond, so it would seem like 1000 seconds, or about 20 minutes.

Page 28, self-check E:
Exponents have to do with multiplication, not addition. The first line should be 100 times longer than the second, not just twice as long.

Page 31, self-check F:
The various estimates differ by 5 to 10 million. The CIA's estimate includes a ridiculous number of gratuitous significant figures. Does the CIA understand that every day, people in are born in, die in, immigrate to, and emigrate from Nigeria?

Page 31, self-check G:
(1) 4; (2) 2; (3) 2

Answers to self-checks for chapter 1

Page 40, self-check A:
$1\ \text{yd}^2 \times (3\ \text{ft}/1\ \text{yd})^2 = 9\ \text{ft}^2$
$1\ \text{yd}^3 \times (3\ \text{ft}/1\ \text{yd})^3 = 27\ \text{ft}^3$

Page 46, self-check B:
$C_1/C_2 = (w_1/w_2)^4$

Answers to self-checks for chapter 2

Page 65, self-check A:
Coasting on a bike and coasting on skates give one-dimensional center-of-mass motion, but running and pedaling require moving body parts up and down, which makes the center of mass move up and down. The only example of rigid-body motion is coasting on skates. (Coasting on

a bike is not rigid-body motion, because the wheels twist.)

Page 65, self-check B:
By shifting his weight around, he can cause the center of mass not to coincide with the geometric center of the wheel.

Page 66, self-check C:
(1) a point in time; (2) time in the abstract sense; (3) a time interval

Page 68, self-check D:
Zero, because the "after" and "before" values of x are the same.

Page 73, self-check E:
(1) The effect only occurs during blastoff, when their velocity is changing. Once the rocket engines stop firing, their velocity stops changing, and they no longer feel any effect. (2) It is only an observable effect of your motion relative to the air.

Page 86, self-check F:
At $v = 0$, we get $\gamma = 1$, so $t = T$. There is no time distortion unless the two frames of reference are in relative motion.

Answers to self-checks for chapter 3

Page 107, self-check A:
Its speed increases at a steady rate, so in the next second it will travel 19 cm.

Answers to self-checks for chapter 4

Page 142, self-check A:
(1) The case of $\rho = 0$ represents an object falling in a vacuum, i.e., there is no density of air. The terminal velocity would be infinite. Physically, we know that an object falling in a vacuum would never stop speeding up, since there would be no force of air friction to cancel the force of gravity. (2) The 4-cm ball would have a mass that was greater by a factor of $4 \times 4 \times 4$, but its cross-sectional area would be greater by a factor of 4×4. Its terminal velocity would be greater by a factor of $\sqrt{4^3/4^2} = 2$. (3) It isn't of any general importance. It's just an example of one physical situation. You should not memorize it.

Page 145, self-check B:
(1) This is motion, not force. (2) This is a description of how the sub is able to get the water to produce a forward force on it. (3) The sub runs out of energy, not force.

Answers to self-checks for chapter 5

Page 167, self-check A:
The sprinter pushes backward against the ground, and by Newton's third law, the ground pushes forward on her. (Later in the race, she is no longer accelerating, but the ground's forward force is needed in order to cancel out the backward forces, such as air friction.)

Page 175, self-check B:
(1) It's kinetic friction, because her uniform is sliding over the dirt. (2) It's static friction, because even though the two surfaces are moving relative to the landscape, they're not slipping over each other. (3) Only kinetic friction creates heat, as when you rub your hands together. If you move your hands up and down together without sliding them across each other, no heat is produced by the static friction.

Page 175, self-check C:

By the POFOSTITO mnemonic, we know that each of the bird's forces on the trunk will be of the same type as the corresponding force of the tree on the bird, but in the opposite direction. The bird's feet make a normal force on the tree that is to the right and a static frictional force that is downward.

Page 176, self-check D:
Frictionless ice can certainly make a normal force, since otherwise a hockey puck would sink into the ice. Friction is not possible without a normal force, however: we can see this from the equation, or from common sense, e.g., while sliding down a rope you do not get any friction unless you grip the rope.

Page 177, self-check E:
(1) Normal forces are always perpendicular to the surface of contact, which means right or left in this figure. Normal forces are repulsive, so the cliff's force on the feet is to the right, i.e., away from the cliff. (2) Frictional forces are always parallel to the surface of contact, which means right or left in this figure. Static frictional forces are in the direction that would tend to keep the surfaces from slipping over each other. If the wheel was going to slip, its surface would be moving to the left, so the static frictional force on the wheel must be in the direction that would prevent this, i.e., to the right. This makes sense, because it is the static frictional force that accelerates the dragster. (3) Normal forces are always perpendicular to the surface of contact. In this diagram, that means either up and to the left or down and to the right. Normal forces are repulsive, so the ball is pushing the bat away from itself. Therefore the ball's force is down and to the right on this diagram.

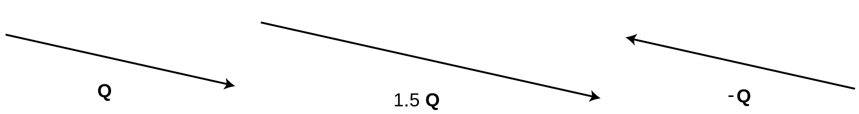
Answers to self-checks for chapter 6

Page 207, self-check A:
The wind increases the ball's overall speed. If you think about it in terms of overall speed, it's not so obvious that the increased speed is exactly sufficient to compensate for the greater distance. However, it becomes much simpler if you think about the forward motion and the sideways motion as two separate things. Suppose the ball is initially moving at one meter per second. Even if it picks up some sideways motion from the wind, it's still getting closer to the wall by one meter every second.

Answers to self-checks for chapter 7

Page 219, self-check A:
$\mathbf{v} = \Delta\mathbf{r}/\Delta t$

Page 220, self-check B:



Page 225, self-check C:
$\mathbf{A} - \mathbf{B}$ is equivalent to $\mathbf{A} + (-\mathbf{B})$, which can be calculated graphically by reversing $\mathbf{B}$ to form $-\mathbf{B}$, and then adding it to $\mathbf{A}$.

Answers to self-checks for chapter 8

Page 236, self-check A:
(1) It is speeding up, because the final velocity vector has the greater magnitude. (2) The result would be zero, which would make sense. (3) Speeding up produced a $\Delta\mathbf{v}$ vector in the same

direction as the motion. Slowing down would have given a $\Delta\mathbf{v}$ that pointed backward.

Page 237, self-check B:
As we have already seen, the projectile has $a_x = 0$ and $a_y = -g$, so the acceleration vector is pointing straight down.

Answers to self-checks for chapter 9

Page 259, self-check A:
(1) Uniform. They have the same motion as the drum itself, which is rotating as one solid piece. No part of the drum can be rotating at a different speed from any other part. (2) Nonuniform. Gravity speeds it up on the way down and slows it down on the way up.

Answers to self-checks for chapter 10

Page 278, self-check A:
It would just stay where it was. Plugging $v = 0$ into eq. [1] would give $F = 0$, so it would not accelerate from rest, and would never fall into the sun. No astronomer had ever observed an object that did that!

Page 279, self-check B:

$$F \propto mr/T^2 \propto mr/(r^{3/2})^2 \propto mr/r^3 = m/r^2$$

Page 282, self-check C:
The equal-area law makes equally good sense in the case of a hyperbolic orbit (and observations verify it). The elliptical orbit law had to be generalized by Newton to include hyperbolas. The law of periods doesn't make sense in the case of a hyperbolic orbit, because a hyperbola never closes back on itself, so the motion never repeats.

Page 287, self-check D:
Above you there is a small part of the shell, comprising only a tiny fraction of the earth's mass. This part pulls you up, while the whole remainder of the shell pulls you down. However, the part above you is extremely close, so it makes sense that its force on you would be far out of proportion to its small mass.

Answers to self-checks for chapter 11

Page 318, self-check A:
(1) A spring-loaded toy gun can cause a bullet to move, so the spring is capable of storing energy and then converting it into kinetic energy. (2) The amount of energy stored in the spring relates to the amount of compression, which can be measured with a ruler.

Answers to self-checks for chapter 12

Page 344, self-check A:
Both balls start from the same height and end at the same height, so they have the same Δy. This implies that their losses in potential energy are the same, so they must both have gained the same amount of kinetic energy.

Answers to self-checks for chapter 13

Page 360, self-check A:
Work is defined as the transfer of energy, so like energy it is a scalar with units of joules.

Page 364, self-check B:

Whenever energy is transferred out of the spring, the same amount has to be transferred into the ball, and vice versa. As the spring compresses, the ball is doing positive work on the spring (giving up its KE and transferring energy into the spring as PE), and as it decompresses the ball is doing negative work (extracting energy).

Page 367, self-check C:
(a) No. The pack is moving at constant velocity, so its kinetic energy is staying the same. It is only moving horizontally, so its gravitational potential energy is also staying the same. No energy transfer is occurring. (b) No. The horse's upward force on the pack forms a 90-degree angle with the direction of motion, so $\cos\theta = 0$, and no work is done.

Page 370, self-check D:
Only in (a) can we use Fd to calculate work. In (b) and (c), the force is changing as the distance changes.

Answers to self-checks for chapter 14

Page 415, self-check A:
When $m = 0$, we have $E = p$ (or $E = pc$, in units with $c \neq 1$), which is what we expect.

Answers to self-checks for chapter 15

Page 437, self-check A:
1, 2, and 4 all have the same sign, because they are trying to twist the wrench clockwise. The sign of torque 3 is opposite to the signs of the others. The magnitude of torque 3 is the greatest, since it has a large r, and the force is nearly all perpendicular to the wrench. Torques 1 and 2 are the same because they have the same values of r and $F_\perp$. Torque 4 is the smallest, due to its small r.

Page 448, self-check B:
One person's θ-t graph would simply be shifted up or down relative to the others. The derivative equals the slope of the tangent line, and this slope isn't changed when you shift the graph, so both people would agree on the angular velocity.

Page 450, self-check C:
Reversing the direction of ω also reverses the direction of motion, and this is reflected by the relationship between the plus and minus signs. In the equation for the radial acceleration, ω is squared, so even if ω is negative, the result is positive. This makes sense because the acceleration is always inward in circular motion, never outward.

Page 462, self-check D:
All the rotations around the x axis give ω vectors along the positive x axis (thumb pointing along positive x), and all the rotations about the y axis have ω vectors with positive y components.

Page 465, self-check E:
For example, if we take $(\mathbf{A}\times\mathbf{B})_x = A_yB_z - B_yA_z$ and reverse the A's and B's, we get $(\mathbf{B}\times\mathbf{A})_x = B_yA_z - A_yB_z$, which is just the negative of the original expression.

Answers to self-checks for chapter 17

Page 518, self-check A:
The two graphs start off with the same amplitude, but the solid curve loses amplitude more rapidly. For a given time, t, the quantity e^{-ct} is apparently smaller for the solid curve, meaning that ct is greater. The solid curve has the higher value of c.

Page 524, self-check B:
She should tap the wine glasses she finds in the store and look for one with a high Q, i.e., one whose vibrations die out very slowly. The one with the highest Q will have the highest-amplitude response to her driving force, making it more likely to break.

Answers

Answers for chapter 1

Page 55, problem 10:
Check: The actual number of species of lupine occurring in the San Gabriels is 22. You should find that your answer comes out in the same ballpark as this figure, but not exactly the same, of course, because the scaling rule is only a generalization.

Answers for chapter 6

Page 215, problem 5:
(a) $R = (2v^2/g)\sin\theta\cos\theta$ (c) $45\ ^\circ$

Page 215, problem 5:
(a) $R = (2v^2/g)\sin\theta\cos\theta$ (c) $45\ ^\circ$

Answers for chapter 7

Page 232, problem 6:
(a) The optimal angle is about 40°, and the resulting range is about 124 meters, which is about the length of a home run. (b) It goes about 9 meters farther. For comparison with reality, the stadium's web site claims a home run goes about 11 meters farther there than in a sea-level stadium.

Page 232, problem 6:
(a) The optimal angle is about 40°, and the resulting range is about 124 meters, which is about the length of a home run. (b) It goes about 9 meters farther. For comparison with reality, the stadium's web site claims a home run goes about 11 meters farther there than in a sea-level stadium.

Answers for chapter 17

Page 535, problem 6:
$Q = 1/\sqrt{2}$

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Trig Table

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0.000	1.000	0.000	30°	0.500	0.866	0.577	60°	0.866	0.500	1.732
1°	0.017	1.000	0.017	31°	0.515	0.857	0.601	61°	0.875	0.485	1.804
2°	0.035	0.999	0.035	32°	0.530	0.848	0.625	62°	0.883	0.469	1.881
3°	0.052	0.999	0.052	33°	0.545	0.839	0.649	63°	0.891	0.454	1.963
4°	0.070	0.998	0.070	34°	0.559	0.829	0.675	64°	0.899	0.438	2.050
5°	0.087	0.996	0.087	35°	0.574	0.819	0.700	65°	0.906	0.423	2.145
6°	0.105	0.995	0.105	36°	0.588	0.809	0.727	66°	0.914	0.407	2.246
7°	0.122	0.993	0.123	37°	0.602	0.799	0.754	67°	0.921	0.391	2.356
8°	0.139	0.990	0.141	38°	0.616	0.788	0.781	68°	0.927	0.375	2.475
9°	0.156	0.988	0.158	39°	0.629	0.777	0.810	69°	0.934	0.358	2.605
10°	0.174	0.985	0.176	40°	0.643	0.766	0.839	70°	0.940	0.342	2.747
11°	0.191	0.982	0.194	41°	0.656	0.755	0.869	71°	0.946	0.326	2.904
12°	0.208	0.978	0.213	42°	0.669	0.743	0.900	72°	0.951	0.309	3.078
13°	0.225	0.974	0.231	43°	0.682	0.731	0.933	73°	0.956	0.292	3.271
14°	0.242	0.970	0.249	44°	0.695	0.719	0.966	74°	0.961	0.276	3.487
15°	0.259	0.966	0.268	45°	0.707	0.707	1.000	75°	0.966	0.259	3.732
16°	0.276	0.961	0.287	46°	0.719	0.695	1.036	76°	0.970	0.242	4.011
17°	0.292	0.956	0.306	47°	0.731	0.682	1.072	77°	0.974	0.225	4.331
18°	0.309	0.951	0.325	48°	0.743	0.669	1.111	78°	0.978	0.208	4.705
19°	0.326	0.946	0.344	49°	0.755	0.656	1.150	79°	0.982	0.191	5.145
20°	0.342	0.940	0.364	50°	0.766	0.643	1.192	80°	0.985	0.174	5.671
21°	0.358	0.934	0.384	51°	0.777	0.629	1.235	81°	0.988	0.156	6.314
22°	0.375	0.927	0.404	52°	0.788	0.616	1.280	82°	0.990	0.139	7.115
23°	0.391	0.921	0.424	53°	0.799	0.602	1.327	83°	0.993	0.122	8.144
24°	0.407	0.914	0.445	54°	0.809	0.588	1.376	84°	0.995	0.105	9.514
25°	0.423	0.906	0.466	55°	0.819	0.574	1.428	85°	0.996	0.087	11.430
26°	0.438	0.899	0.488	56°	0.829	0.559	1.483	86°	0.998	0.070	14.301
27°	0.454	0.891	0.510	57°	0.839	0.545	1.540	87°	0.999	0.052	19.081
28°	0.469	0.883	0.532	58°	0.848	0.530	1.600	88°	0.999	0.035	28.636
29°	0.485	0.875	0.554	59°	0.857	0.515	1.664	89°	1.000	0.017	57.290
								90°	1.000	0.000	∞

Mathematical Review

Algebra

Quadratic equation:

The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Logarithms and exponentials:

$$\ln(ab) = \ln a + \ln b$$

$$e^{a+b} = e^a e^b$$

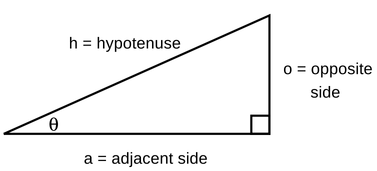
$$\ln e^x = e^{\ln x} = x$$

$$\ln(a^b) = b \ln a$$

Geometry, area, and volume

area of a triangle of base b and height h	$= \frac{1}{2}bh$
circumference of a circle of radius r	$= 2\pi r$
area of a circle of radius r	$= \pi r^2$
surface area of a sphere of radius r	$= 4\pi r^2$
volume of a sphere of radius r	$= \frac{4}{3}\pi r^3$

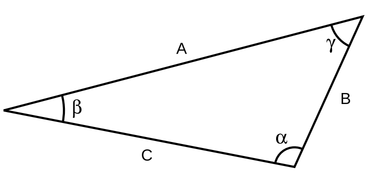
Trigonometry with a right triangle



$$\sin\theta = o/h \quad \cos\theta = a/h \quad \tan\theta = o/a$$

Pythagorean theorem: $h^2 = a^2 + o^2$

Trigonometry with any triangle



Law of Sines:

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}$$

Law of Cosines:

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$

Properties of the derivative and integral (for students in calculus-based courses)

Let f and g be functions of x, and let c be a constant.

Linearity of the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}(cf) = c\frac{\mathrm{d}f}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}$$

The chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'(g(x))g'(x)$$

Derivatives of products and quotients:

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = \frac{\mathrm{d}f}{\mathrm{d}x}g + \frac{\mathrm{d}g}{\mathrm{d}x}f$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{g}\right) = \frac{f'}{g} - \frac{fg'}{g^2}$$

Some derivatives:

$\frac{\mathrm{d}}{\mathrm{d}x}x^m = mx^{m-1}$, except for $m = 0$

$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$

$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$

The fundamental theorem of calculus:

$$\int \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = f$$

Linearity of the integral:

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x$$

$$\int [f(x) + g(x)] = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

Integration by parts:

$$\int f\,\mathrm{d}g = fg - \int g\,\mathrm{d}f$$

Useful Data

Metric Prefixes

M-	mega-	10^6
k-	kilo-	10^3
m-	milli-	10^{-3}
μ- (Greek mu)	micro-	10^{-6}
n-	nano-	10^{-9}
p-	pico-	10^{-12}
f-	femto-	10^{-15}

(Centi-, 10^{-2}, is used only in the centimeter.)

The Greek Alphabet

α	A	alpha	ν	N	nu
β	B	beta	ξ	Ξ	xi
γ	Γ	gamma	o	O	omicron
δ	Δ	delta	π	Π	pi
ϵ	E	epsilon	ρ	P	rho
ζ	Z	zeta	σ	Σ	sigma
η	H	eta	τ	T	tau
θ	Θ	theta	υ	Y	upsilon
ι	I	iota	ϕ	Φ	phi
κ	K	kappa	χ	X	chi
λ	Λ	lambda	ψ	Ψ	psi
μ	M	mu	ω	Ω	omega

Subatomic Particles

particle	**mass (kg)**	**radius (fm)**
electron	9.109×10^{-31}	$\lesssim 0.01$
proton	1.673×10^{-27}	~ 1.1
neutron	1.675×10^{-27}	~ 1.1

The radii of protons and neutrons can only be given approximately, since they have fuzzy surfaces. For comparison, a typical atom is about a million fm in radius.

Notation and Units

quantity	**unit**	**symbol**
distance	meter, m	$x, \Delta x$
time	second, s	$t, \Delta t$
mass	kilogram, kg	m
density	kg/m^3	ρ
velocity	m/s	$\mathbf{v}$
acceleration	m/s^2	$\mathbf{a}$
force	N = kg·m/s^2	$\mathbf{F}$
pressure	Pa=1 N/m^2	P
energy	J = kg·m^2/s^2	E
power	W = 1 J/s	P
momentum	kg·m/s	$\mathbf{p}$
angular momentum	kg·m^2/s or J·s	$\mathbf{L}$
period	s	T
wavelength	m	λ
frequency	s^{-1} or Hz	f
gamma factor	unitless	γ
probability	unitless	P
prob. distribution	various	D
electron wavefunction	m$^{-3/2}$	Ψ

Earth, Moon, and Sun

body	**mass (kg)**	**radius (km)**	**radius of orbit (km)**
earth	5.97×10^{24}	6.4×10^3	1.49×10^8
moon	7.35×10^{22}	1.7×10^3	3.84×10^5
sun	1.99×10^{30}	7.0×10^5	—

Fundamental Constants

gravitational constant	$G = 6.67 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$
Coulomb constant	$k = 8.99 \times 10^9\ \mathrm{N{\cdot}m^2/C^2}$
quantum of charge	$e = 1.60 \times 10^{-19}$ C
speed of light	$c = 3.00 \times 10^8$ m/s
Planck's constant	$h = 6.63 \times 10^{-34}$ J·s